

# Taming a Hydra of Singularities in Fourier Space

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September 20, 2023

## 1 Introduction

In [1] Bornemann and Schmelzer invited the reader to contemplate a remarkable limit problem which involves an oscillatory integral of extreme nature. Consider a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  that is integrable<sup>1</sup>. Does the sequence of violently oscillatory integrals

$$I_n[f] = \frac{1}{\pi} \int_0^\pi f(\tan^{[n]} x) dx, \quad n = 1, 2, 3, \dots, \quad (1)$$

have a limit as  $n$  approaches infinity? If so, what is the limit? Here  $\tan^{[n]} x = \tan \circ \tan \circ \dots \circ \tan x$  denotes the  $n$ -fold iteration of the  $\tan$  function.

It is the dramatic behavior of  $\tan^{[n]} x$  that inspired Bornemann to call this function the Hydra of singularities. The aggressive nature of this function is revealed already for small  $n$ . As  $x$  moves from 0 to  $\pi$ , the values of  $\tan x$  go all the way from 0 to  $+\infty$  and then from  $-\infty$  to 0 after passing the singularity at  $\pi/2$ . The singularity at  $\pi/2$  breeds infinitely many new singularities located at  $x_k$  where  $\tan x_k = (k + 1/2)\pi$  with  $k \in \mathbb{Z}$ . Note that the values  $x_k$  accumulate at  $\pi/2$ .

This process repeats for each further iteration of  $\tan$ . Each singularity breeds countably many new singularities which accumulate in their respective ancestor. In Figure 1 we have tried to illustrate this rather wild behavior of the  $\tan^{[n]} x$  iteration.

This paper is a brief companion for the original paper by Bornemann and Schmelzer. The approach taken here could probably be extended for a larger class of functions. This would require a careful analysis shading light away from the central ideas. However, Bornemann and Schmelzer [1] gave already an

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<sup>1</sup>A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is integrable if  $\int_{-\infty}^{\infty} f(x) dx$  exists. Bornemann and Schmelzer solved the problem for a larger class of functions. In their analysis it was sufficient for  $f$  to be bounded and continuous.

elementary proof for a large class of functions and therefore our focus is on a short and elegant analysis using tools from France, in particular Fourier analysis and Cauchy integrals.

## 2 Into Fourier space and back again

The Fourier transform of  $f$  exists but does not have to be integrable. This is an additional requirement for the analysis given here. If both  $f$  and its Fourier transform

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx, \quad \xi \in \mathbb{R}$$

are integrable then for almost every  $x$  (and for all  $x$  if  $f$  is continuous)  $f$  can be represented as the inverse transform of  $\hat{f}$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi x \xi} d\xi.$$

And therefore

$$f(\tan^{[n]} x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi \xi \tan^{[n]} x} d\xi.$$

We insert the this term into (1) and restate the problem as

$$I_n[f] = \frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi \xi \tan^{[n]} x} d\xi dx.$$

Since  $\hat{f}$  is integrable the integral

$$\frac{1}{\pi} \int_0^\pi \int_{-\infty}^{\infty} |\hat{f}(\xi) e^{2i\pi \xi \tan^{[n]} x}| d\xi dx$$

exists. We can therefore apply Fubini's theorem and get

$$I_n[f] = \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{1}{\pi} \int_0^\pi e^{2i\pi \xi \tan^{[n]} x} dx d\xi. \quad (2)$$



Figure 1: Graph of  $\tan^{[2]}(x)$  (left) and  $\tan^{[3]}(x)$  (right).

### 3 The inner Fourier integral

Still the problem does not look any simpler. The Hydra is lurking now in the inner integral

$$\frac{1}{\pi} \int_0^\pi e^{2i\pi\xi \tan^{[n]} x} dx \quad \xi \in \mathbb{R}. \quad (3)$$

The  $\tan$  function maps the upper halfplane into itself. Therefore for  $\xi \geq 0$  the integrand is bounded in the upper halfplane. We get using dominated convergence

$$\lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_0^\pi e^{2i\pi\xi \tan^{[n]}(x+\epsilon i)} dx = \frac{1}{\pi} \int_0^\pi e^{2i\pi\xi \tan^{[n]} x} dx \quad \xi \geq 0.$$

The function  $e^{2i\pi\xi \tan^{[n]} z}$  is analytic in the upper halfplane and therefore we can apply Cauchy's theorem. As a contour we choose the rectangle with corners  $(0, \epsilon i)$ ,  $(\pi, \epsilon i)$ ,  $(\pi, a i)$ ,  $(0, a i)$ . The contributions from both vertical vertexes vanish as  $\tan$  is periodic. And therefore

$$\lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_0^\pi e^{2i\pi\xi \tan^{[n]}(x+\epsilon i)} dx = \frac{1}{\pi} \int_0^\pi e^{2i\pi\xi \tan^{[n]}(x+ai)} dx \quad \xi \geq 0, a > 0.$$

Hence we can integrate on any parallel line above the real line. In the extreme case we can move  $a$  towards infinity and as

$$\lim_{a \uparrow \infty} \tan^{[n]}(x + ai) = i \tanh^{[n-1]} 1$$

we get

$$\frac{1}{\pi} \int_0^\pi e^{2i\pi \tan^{[n]} x \xi} dx = e^{-2\pi\xi \tanh^{[n-1]} 1} \quad \xi \geq 0.$$

For  $\xi \leq 0$  the same argument applied in the lower halfplane yields

$$\frac{1}{\pi} \int_0^\pi e^{2i\pi \tan^{[n]} x \xi} dx = e^{2\pi\xi \tanh^{[n-1]} 1} \quad \xi \leq 0.$$

For  $n \rightarrow \infty$  both integrals converge to 1 and hence

$$\lim_{n \uparrow \infty} I_n[f] = \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi = f(0).$$

In the last step we have applied Parseval's theorem.

### 4 From Fourier to Hardy

Comparing Equation (3) and Equation (1) reveals that the inner Fourier integral is just a special case of (1) with  $f(x) = e^{2i\pi\xi x}$ . It may seem that we have made use of special properties of this particular function  $f(x)$ , but in fact the results generalises for a larger class of functions  $f$ . To transfer our arguments from

above we need to assume that  $f(z)$  is analytic and bounded in the upper half-plane. However, the space of bounded analytic functions in the upper halfplane is the Hardy space  $H^\infty$ . So, let  $F \in H^\infty$ , then  $\|F\|_{H^\infty} = \sup_{\text{Im } z > 0} |F(z)| < \infty$ . The function  $f$  may be interpreted as the non-tangential limit of  $F$ , that is  $f(x) = F(x + iy)$  for  $y \downarrow 0$ . This implies  $f$  is bounded, too.

Assuming that  $f$  is continuous and bounded on the real line this resembles a kind of boundary value problem. However, the analytic extension of  $f$  is rarely bounded in the upper halfplane.

Now, from potential theory (see [2, Thms. 15.1a, 15.4d]) we know that there is a function  $F(z)$ , holomorphic in the upper complex half plane  $\text{Im } z > 0$ , such that the harmonic function  $\text{Re } F(z)$  is bounded and has boundary values given by  $f$ , that is,

$$\text{Re } F(x + iy) \rightarrow f(x), \quad x \in \mathbb{R}, \quad (4)$$

as the real number  $y$  approaches zero from above. The holomorphic function  $F$  is *unique* up to a purely imaginary additive constant. For the sake of simplicity of our presentation, we will further *assume that  $F$  itself, not just  $\text{Re } F$ , is bounded*; this additional assumption will be dropped in the elementary, real analysis proof given in [1].

Therefore

$$\frac{1}{\pi} \int_0^\pi f(\tan^{[n]} x) dx = \text{Re } \frac{1}{\pi} \int_0^\pi F(\tan^{[n]} x) dx = \text{Re } F(i \tanh^{[n-1]} 1).$$

Taking the limit for  $n \rightarrow \infty$  reveals:

$$\lim_{n \uparrow \infty} \frac{1}{\pi} \int_0^\pi f(\tan^{[n]} x) dx = f(0).$$

Arguably, there are simpler ways to evaluate  $f$  at 0. However, they all lack the drama, brutality and beauty of the Hydra.

## References

- [1] Folkmar Bornemann and Thomas Schmelzer, *Taming a hydra of singularities*, Amer. Math. Monthly **114** (2007), no. 8, 727–732.
- [2] Peter Henrici, *Applied and Computational Complex Analysis. Vol. 3*, Wiley, New York, 1986.