# Taming a Hydra of Singularities in Fourier Space

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## 1 Introduction

In [1] Bornemann and Schmelzer invited the reader to contemplate a remarkable limit problem which involves an oscillatory integral of extreme nature. Consider a function $f : \mathbb{R} \to \mathbb{R}$ that is integrable[1]. Does the sequence of violently oscillatory integrals

$$I_n[f] = \frac{1}{\pi} \int_0^\pi f(\tan^{[n]} x)\, dx, \qquad n = 1, 2, 3, \ldots, \tag{1}$$

have a limit as $n$ approaches infinity? If so, what is the limit? Here $\tan^{[n]} x = \tan \circ \tan \circ \cdots \circ \tan x$ denotes the n-fold iteration of the tan function.

It is the dramatic behavior of $\tan^{[n]} x$ that inspired Bornemann to call this function the Hydra of singularities. The aggressive nature of this function is revealed already for small $n$. As $x$ moves from 0 to $\pi$, the values of $\tan x$ go all the way from 0 to $+\infty$ and then from $-\infty$ to 0 after passing the singularity at $\pi/2$. The singularity at $\pi/2$ breeds infinitely many new singularities located at $x_k$ where $\tan x_k = (k + 1/2)\pi$ with $k \in \mathbb{Z}$. Note that the values $x_k$ accumulate at $\pi/2$.

This process repeats for each further iteration of tan. Each singularity breeds countably many new singularities which accumulate in their respective ancestor. In Figure 1 we have tried to illustrate this rather wild behavior of the $\tan^{[n]} x$ iteration.

This paper is a brief companion for the original paper by Bornemann and Schmelzer. The approach taken here could probably be extended for a larger class of functions. This would require a careful analysis shading light away from the central ideas. However, Bornemann and Schmelzer [1] gave already an elementary proof for a large class of functions and therefore our focus is on a

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[1] A function $f : \mathbb{R} \to \mathbb{R}$ is integrable if $\int_{-\infty}^{\infty} f(x)dx$ exists. Bornemann and Schmelzer solved the problem for a larger class of functions. In their analysis is was sufficient for $f$ to be bounded and continuous.

short and elegant analysis using tools from France, in particular Fourier analysis and Cauchy integrals.

## 2 Into Fourier space

The Fourier transform of $f$ exists but does not have to be integrable. This is an additional requirement for the analysis given here. If both $f$ and its Fourier transform

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i x \xi}\, dx, \qquad \xi \in \mathbb{R}$$

are integrable then for almost every $x$ (and for all $x$ if $f$ is continuous) $f$ can be represented as the inverse transform of $\hat{f}$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi x \xi}\, d\xi.$$

And therefore

$$f(\tan^{[n]} x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi \xi \tan^{[n]} x}\, d\xi.$$

We insert this term into (1) and restate the problem as

$$I_n[f] = \frac{1}{\pi} \int_0^{\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi \xi \tan^{[n]} x}\, d\xi\, dx.$$

Since $\hat{f}$ is integrable the integral

$$\frac{1}{\pi} \int_0^{\pi} \int_{-\infty}^{\infty} \left| \hat{f}(\xi) e^{2i\pi \xi \tan^{[n]} x} \right|\, d\xi\, dx$$

exists. We can therefore apply Fubini's theorem and get

$$I_n[f] = \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{1}{\pi} \int_0^{\pi} e^{2i\pi \xi \tan^{[n]} x}\, dx\, d\xi. \tag{2}$$

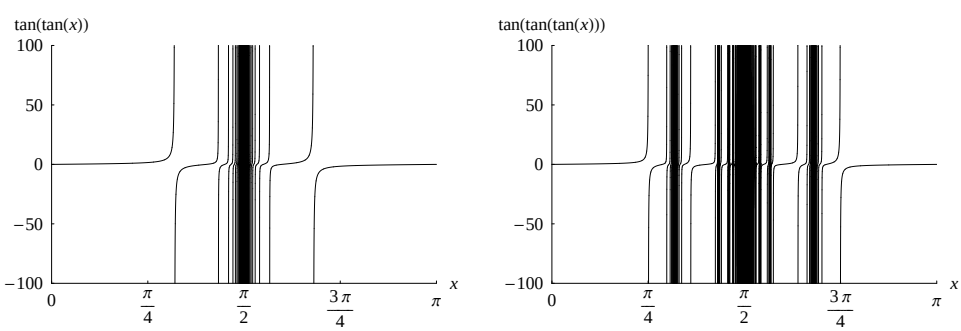


Figure 1: Graph of $\tan^{[2]}(x)$ (left) and $\tan^{[3]}(x)$ (right).

## 3 The inner Fourier integral

Still the problem does not look any simpler. The Hydra is lurking now in the inner integral

$$\frac{1}{\pi}\int_0^\pi e^{2i\pi\xi\tan^{[n]}x}\,dx \qquad \xi\in\mathbb{R}. \tag{3}$$

The tan function maps the upper halfplane into itself. Therefore, for $\xi \geqslant 0$ the integrand is bounded in the upper halfplane. We get using dominated convergence

$$\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_0^\pi e^{2i\pi\xi\tan^{[n]}(x+\epsilon\,i)}\,dx = \frac{1}{\pi}\int_0^\pi e^{2i\pi\xi\tan^{[n]}x}\,dx \qquad \xi\geqslant 0.$$

The function $e^{2i\pi\xi\tan^{[n]}z}$ is analytic in the upper halfplane. We can therefore apply Cauchy's theorem. As a contour we choose the rectangle with corners $(0,\epsilon\,i),(\pi,\epsilon\,i),(\pi,a\,i),(0,a\,i)$. The contributions from both vertical vertexes vanish as tan is periodic. And therefore

$$\lim_{\epsilon\downarrow 0}\frac{1}{\pi}\int_0^\pi e^{2i\pi\xi\tan^{[n]}(x+\epsilon\,i)}\,dx = \frac{1}{\pi}\int_0^\pi e^{2i\pi\xi\tan^{[n]}(x+a\,i)}\,dx \qquad \xi\geqslant 0, a>0.$$

Hence we can integrate on any parallel line above the real line. In the extreme case we can move $a$ towards infinity and as

$$\lim_{a\uparrow\infty}\tan^{[n]}(x+a\,i) = i\lim_{a\uparrow\infty}\tanh^{[n]}(a-i\,x) = i\tanh^{[n-1]}1$$

we get

$$\frac{1}{\pi}\int_0^\pi e^{2i\pi\tan^{[n]}x\xi}\,dx = e^{-2\pi\xi\tanh^{[n-1]}1} \qquad \xi\geqslant 0.$$

For $\xi\leqslant 0$ the same argument applied in the lower halfplane yields

$$\frac{1}{\pi}\int_0^\pi e^{2i\pi\tan^{[n]}x\xi}\,dx = e^{2\pi\xi\tanh^{[n-1]}1} \qquad \xi\leqslant 0.$$

For $n\to\infty$ both integrals converge to 1 and hence

$$\lim_{n\uparrow\infty} I_n[f] = \int_{-\infty}^{\infty}\hat{f}(\xi)\,d\xi = f(0).$$

In the last step we have applied Parseval's theorem.

## 4 Fourier at the boundary

Comparing Equation (3) and Equation (1) reveals that the inner Fourier integral is just a special case of (1) with $f(x)=e^{2i\pi\xi x}$. It may seem that we have made use of special properties of this particular function $f(x)$, but in fact the results generalises for a larger class of functions $f$. To transfer our arguments from

above we need to assume that $f(z)$ is analytic and bounded in the upper half-plane. However, the space of bounded analytic functions in the upper halfplane is the Hardy space $H^\infty$. So, let $F \in H^\infty$, then $\|F\|_{H^\infty} = \sup_{\operatorname{Im} z>0} |F(z)| < \infty$. The function $f$ may be interpreted as the non-tangential limit of $F$, that is $f(x) = F(x+iy)$ for $y \downarrow 0$. This implies $f$ is bounded, too.

Assuming that $f$ is continuous and bounded on the real line this resembles a kind of boundary value problem. However, the analytic extension of $f$ is rarely bounded in the upper halfplane.

Now, from potential theory (see [2, Thms. 15.1a, 15.4d]) we know that there is a function $F(z)$, holomorphic in the upper complex half plane $\operatorname{Im} z > 0$, such that the harmonic function $\operatorname{Re} F(z)$ is bounded and has boundary values given by $f$, that is,

$$\operatorname{Re} F(x+iy) \to f(x), \qquad x \in \mathbb{R}, \tag{4}$$

as the real number $y$ approaches zero from above. The holomorphic function $F$ is *unique* up to a purely imaginary additive constant. For the sake of simplicity of our presentation, we will further *assume that $F$ itself, not just* $\operatorname{Re} F$*, is bounded*; this additional assumption is dropped in the elementary, real analysis proof given in [1].

Therefore

$$\frac{1}{\pi}\int_0^\pi f(\tan^{[n]} x)\,dx = \operatorname{Re}\frac{1}{\pi}\int_0^\pi F(\tan^{[n]} x)\,dx = \operatorname{Re} F(i\tanh^{[n-1]} 1).$$

Taking the limit for $n \to \infty$ reveals:

$$\lim_{n\uparrow\infty}\frac{1}{\pi}\int_0^\pi f(\tan^{[n]} x)\,dx = f(0).$$

Arguably, there are simpler ways to evaluate $f$ at 0. However, they all lack the drama, brutality and beauty of the Hydra.

# References


[1] Folkmar Bornemann and Thomas Schmelzer, *Taming a hydra of singularities*, Amer. Math. Monthly **114** (2007), no. 8, 727–732.

[2] Peter Henrici, *Applied and Computational Complex Analysis. Vol. 3*, Wiley, New York, 1986.