TAMING A HYDRA OF SINGULARITIES IN FOURIER SPACE

THOMAS SCHMELZER

1. Introduction

In [1] Bornemann and Schmelzer invited the reader to contemplate a remarkable limit problem which involves an oscillatory integral of extreme nature. Consider a function $f : \mathbb{R} \to \mathbb{R}$ that is integrable[1]. Does the sequence of violently oscillatory integrals

$$I_n[f] = \frac{1}{\pi}\int_0^\pi f(\tan^{[n]} x)\,dx, \qquad n = 1, 2, 3, \ldots, \tag{1.1}$$

have a limit as n approaches infinity? If so, what is the limit? Here $\tan^{[n]} x = \tan \circ \tan \circ \cdots \circ \tan x$ denotes the n-fold iteration of the tan function.

It is the dramatic behavior of $\tan^{[n]} x$ that inspired Bornemann to call this function the Hydra of singularities. The aggressive nature of this function is revealed already for small n. As x moves from 0 to π, the values of $\tan x$ go all the way from 0 to $+\infty$ and then from $-\infty$ to 0 after passing the singularity at $\pi/2$. The singularity at $\pi/2$ breeds infinitely many new singularities located at x_k where $\tan x_k = (k + 1/2)\pi$ with $k \in \mathbb{Z}$. Note that the values x_k accumulate at $\pi/2$.

This process repeats for each further iteration of tan. Each singularity breeds countably many new singularities which accumulate in their respective ancestor. In Figure 1 we have tried to illustrate this rather wild behavior of the $\tan^{[n]} x$ iteration.

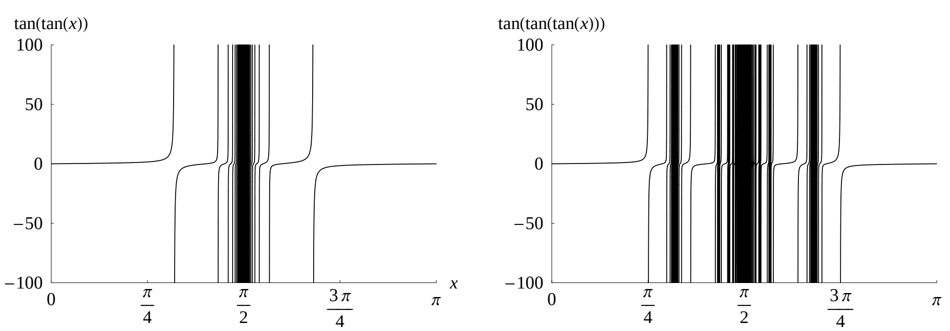

Figure 1. Graph of $\tan^{[2]}(x)$ (left) and $\tan^{[3]}(x)$ (right).

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[1]A function $f : \mathbb{R} \to \mathbb{R}$ is integrable if $\int_{-\infty}^{\infty} f(x)dx$ exists. Bornemann and Schmelzer solved the problem for a larger class of functions. In their analysis is was sufficient for f to be bounded and continuous.

This paper is a brief companion for the original paper by Bornemann and Schmelzer. The approach taken here could probably be extended for a larger class of functions. This would require a careful analysis shading light away from the central ideas. However, Bornemann and Schmelzer [1] gave already an elementary proof for a large class of functions and therefore our focus is on a short and elegant analysis using tools from Fourier analysis.

2. Into Fourier space and back again

The Fourier transform of f exists but does not have to be integrable. This is an additional requirement for the analysis given here. If both f and its Fourier transform

$$\hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)\; e^{-2\pi i x\xi}\, dx, \qquad \xi \in \mathbb{R}$$

are integrable then for almost every x (and for all x if f is continuous) f can be represented as the inverse transform of $\hat{f}$

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi x\xi}\, d\xi.$$

And therefore

$$f(\tan^{[n]} x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi\xi \tan^{[n]} x}\, d\xi.$$

We insert the this term into (1.1) and restate the problem as

$$I_n[f] = \frac{1}{\pi}\int_0^{\pi}\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi\xi \tan^{[n]} x}\, d\xi\, dx.$$

Since $\hat{f}$ is integrable the integral

$$\frac{1}{\pi}\int_0^{\pi}\int_{-\infty}^{\infty} \left|\hat{f}(\xi) e^{2i\pi\xi \tan^{[n]} x}\right|\, d\xi\, dx$$

exists. We can therefore apply Fubini's theorem and get

$$I_n[f] = \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{1}{\pi}\int_0^{\pi} e^{2i\pi\xi \tan^{[n]} x}\, dx\, d\xi. \tag{2.1}$$

3. The inner Fourier integral

Still the problem does not look any simpler. The Hydra is lurking now in the inner integral

$$\frac{1}{\pi}\int_0^{\pi} e^{2i\pi\xi \tan^{[n]} x}\, dx \qquad \xi \in \mathbb{R}. \tag{3.1}$$

The tan function maps the upper halfplane into itself. Therefore for $\xi \geqslant 0$ the integrand is bounded in the upper halfplane. We get using dominated convergence

$$\lim_{\epsilon \downarrow 0} \frac{1}{\pi}\int_0^{\pi} e^{2i\pi\xi \tan^{[n]}(x+\epsilon\, i)}\, dx = \frac{1}{\pi}\int_0^{\pi} e^{2i\pi\xi \tan^{[n]} x}\, dx \qquad \xi \geqslant 0.$$

The function $e^{2i\pi\xi \tan^{[n]} z}$ is analytic in the upper halfplane and therefore we can apply Cauchy's theorem. As a contour we choose the rectangle with corners $(0, \epsilon\, i), (\pi, \epsilon\, i), (\pi, a\, i), (0, a\, i)$. The contributions from both vertical vertexes vanish as tan is periodic. And therefore

$$\lim_{\epsilon\downarrow 0} \frac{1}{\pi}\int_0^\pi e^{2i\pi\xi \tan^{[n]}(x+\epsilon\, i)}\,dx = \frac{1}{\pi}\int_0^\pi e^{2i\pi\xi \tan^{[n]}(x+a\, i)}\,dx \qquad \xi \geqslant 0, a > 0.$$

Hence we can integrate on any parallel line above the real line. In the extreme case we can move a towards infinity and as

$$\lim_{a\uparrow\infty} \tan^{[n]}(x + a\, i) = i \tanh^{[n-1]} 1$$

we get

$$\frac{1}{\pi}\int_0^\pi e^{2i\pi \tan^{[n]} x\xi}\,dx = e^{-2\pi\xi \tanh^{[n-1]} 1} \qquad \xi \geqslant 0.$$

For $\xi \leqslant 0$ the same argument applied in the lower halfplane yields

$$\frac{1}{\pi}\int_0^\pi e^{2i\pi \tan^{[n]} x\xi}\,dx = e^{2\pi\xi \tanh^{[n-1]} 1} \qquad \xi \leqslant 0.$$

For $n \to \infty$ both integrals converge to 1 and hence

$$\lim_{n\uparrow\infty} I_n[f] = \int_{-\infty}^{\infty} \hat{f}(\xi)\,d\xi = f(0).$$

In the last step we have applied Parseval's theorem.

4. From Fourier to Hardy

Comparing Equation (3.1) and Equation (1.1) reveals that the inner Fourier integral is just a special case of (1.1) with $f(x) = e^{2i\pi\xi x}$. It may seem that we have made use of special properties of this particular function $f(x)$, but in fact the results generalises for a larger class of functions f. To transfer our arguments from above we need to assume that $f(z)$ is analytic and bounded in the upper halfplane. However, the space of bounded analytic functions in the upper halfplane is the Hardy space H^∞. So, let $F \in H^\infty$, then $\|F\|_{H^\infty} = \sup_{\operatorname{Im} z>0} |F(z)| < \infty$. The function f may be interpreted as the non-tangential limit of F, that is $f(x) = F(x+iy)$ for $y \downarrow 0$. This implies f is bounded, too.

Assuming that f is continuous and bounded on the real line this resembles a kind of boundary value problem. However, the analytic extension of f is rarely bounded in the upper halfplane.

Now, from potential theory (see [2, Thms. 15.1a, 15.4d]) we know that there is a function $F(z)$, holomorphic in the upper complex half plane $\operatorname{Im} z > 0$, such that the harmonic function $\operatorname{Re} F(z)$ is bounded and has boundary values given by f, that is,

$$\operatorname{Re} F(x+iy) \to f(x), \qquad x \in \mathbb{R}, \tag{4.1}$$

as the real number y approaches zero from above. The holomorphic function F is *unique* up to a purely imaginary additive constant. For the sake of simplicity of our presentation, we will further *assume that* F *itself, not just* $\operatorname{Re} F$*, is bounded*; this additional assumption will be dropped in the elementary, real analysis proof givne in [1].

Therefore

$$\frac{1}{\pi}\int_0^\pi f(\tan^{[n]} x)\,dx = \operatorname{Re}\frac{1}{\pi}\int_0^\pi F(\tan^{[n]} x)\,dx = \operatorname{Re} F(i\tanh^{[n-1]} 1).$$

Taking the limit for $n \to \infty$ reveals the Hydra is close to Dirac in spirit:

$$\lim_{n\uparrow\infty}\frac{1}{\pi}\int_0^\pi f(\tan^{[n]} x)\,dx = f(0).$$

References

1. Folkmar Bornemann and Thomas Schmelzer, *Taming a hydra of singularities*, Amer. Math. Monthly **114** (2007), no. 8, 727–732.
2. Peter Henrici, *Applied and Computational Complex Analysis. Vol. 3*, Wiley, New York, 1986.

Thomas Schmelzer
Winton Capital Management
Magdalen Centre, The Oxford Science Park, Oxford OX4 4GA, UK
Email address: thomas.schmelzer@gmail.com