TAMING A HYDRA OF SINGULARITIES IN FOURIER SPACE

THOMAS SCHMELZER AND FOLKMAR BORNEMANN AND EMMANUEL CANDES
AND EMMANUEL J. CANDES

1. Introduction

In [1] Bornemann and Schmelzer invited the reader to contemplate a remarkable limit problem which involves an oscillatory integral of extreme nature. Consider a function $f: \mathbb{R} \to \mathbb{R}$ that is integrable¹. Does the sequence of violently oscillatory integrals

(1.1)
$$I_n[f] = \frac{1}{\pi} \int_0^{\pi} f(\tan^{[n]} x) dx, \qquad n = 1, 2, 3, ...,$$

have a limit as n approaches infinity? If so, what is the limit? Here $\tan^{[n]} x = \tan \circ \tan \circ \cdots \circ \tan x$ denotes the n-fold iteration of the tan function.

It is the dramatic behavior of $\tan^{[n]} x$ that inspired Bornemann to call this function the Hydra of singularities. The aggressive nature of this function is revealed already for small n. As x moves from 0 to π , the values of $\tan x$ go all the way from 0 to $+\infty$ and then from $-\infty$ to 0 after passing the singularity at $\pi/2$. The singularity at $\pi/2$ breeds infinitely many new singularities located at x_k where $\tan x_k = (k+1/2)\pi$ with $k \in \mathbb{Z}$. Note that the values x_k accumulate at $\pi/2$.

This process repeats for each further iteration of tan. Each singularity breeds countably many new singularities which accumulate in their respective ancestor. In Figure 1 we have tried to illustrate this rather wild behavior of the $\tan^{[n]} x$ iteration.

This paper is a brief companion for the original paper by Bornemann and Schmelzer. The approach taken here could probably be extended for a larger class of functions. This would require a careful analysis shading light away from the central ideas. However, Bornemann and Schmelzer [1] gave already an elementary proof for a large class of functions and therefore our focus is on a short and elegant analysis using tools from Fourier analysis.

2. Into Fourier space and back again

The Fourier transform of f exists but does not have to be integrable. This is an additional requirement for the analysis given here. If both f and its Fourier

Date: September 20, 2023.

¹A function $f: \mathbb{R} \to \mathbb{R}$ is integrable if $\int_{-\infty}^{\infty} f(x) dx$ exists. Bornemann and Schmelzer solved the problem for a larger class of functions. In their analysis is was sufficient for f to be bounded and continuous.

transform

$$\widehat{f}(\xi) := \int_{-\infty}^{\infty} f(x) \ e^{-2\pi i x \xi} \ dx, \qquad \xi \in \mathbb{R}$$

are integrable then for almost every x (and for all x if f is continuous) f can be represented as the inverse transform of \hat{f}

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi x \xi} d\xi.$$

And therefore

$$f(\tan^{[n]} x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi\xi \tan^{[n]} x} d\xi.$$

We insert the this term into (1.1) and restate the problem as

$$I_n[f] = \frac{1}{\pi} \int_0^{\pi} \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2i\pi\xi \tan^{[n]} x} d\xi dx.$$

Since \hat{f} is integrable the integral

$$\frac{1}{\pi} \int_0^{\pi} \int_{-\infty}^{\infty} \left| \hat{\mathbf{f}}(\xi) e^{2i\pi \xi \tan^{[n]} x} \right| d\xi dx$$

exists. We can therefore apply Fubini's theorem and get

(2.1)
$$I_{n}[f] = \int_{-\infty}^{\infty} \hat{f}(\xi) \frac{1}{\pi} \int_{0}^{\pi} e^{2i\pi\xi \tan^{[n]}x} dx d\xi.$$

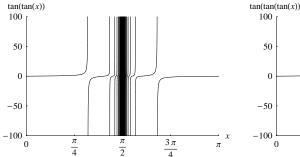
3. The inner Fourier integral

Still the problem does not look any simpler. The Hydra is lurking now in the inner integral

(3.1)
$$\frac{1}{\pi} \int_{0}^{\pi} e^{2i\pi\xi \tan^{[n]} x} dx \qquad \xi \in \mathbb{R}.$$

The tan function maps the upper halfplane into itself. Therefore for $\xi \geqslant 0$ the integrand is bounded in the upper halfplane. We get using dominated convergence

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_0^\pi e^{2i\pi\xi\tan^{[\pi]}(x+\varepsilon\,i)}\,dx = \frac{1}{\pi} \int_0^\pi e^{2i\pi\xi\tan^{[\pi]}x}\,dx \qquad \xi\geqslant 0.$$



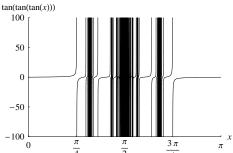


FIGURE 1. Graph of $tan^{[2]}(x)$ (left) and $tan^{[3]}(x)$ (right).

The function $e^{2i\pi\xi \tan^{[n]}z}$ is analytic in the upper halfplane and therefore we can apply Cauchy's theorem. As a contour we choose the rectangle with corners $(0, \epsilon i), (\pi, \epsilon i), (\pi, \alpha i), (0, \alpha i)$. The contributions from both vertical vertexes vanish as tan is periodic. And therefore

$$\lim_{\varepsilon\downarrow 0} \frac{1}{\pi} \int_0^{\pi} e^{2i\pi\xi \tan^{[\pi]}(x+\varepsilon i)} dx = \frac{1}{\pi} \int_0^{\pi} e^{2i\pi\xi \tan^{[\pi]}(x+\alpha i)} dx \qquad \xi \geqslant 0, \alpha > 0.$$

Hence we can integrate on any parallel line above the real line. In the extreme case we can move a towards infinity and as

$$\lim_{\alpha \uparrow \infty} \tan^{[n]}(x + \alpha i) = i \tanh^{[n-1]} 1$$

we get

$$\frac{1}{\pi}\int_0^\pi e^{2i\pi\tan^{[n]}x\xi}\,dx = e^{-2\pi\xi\tanh^{[n-1]}1} \qquad \xi\geqslant 0.$$

For $\xi \leqslant 0$ the same argument applied in the lower halfplane yields

$$\frac{1}{\pi} \int_0^{\pi} e^{2i\pi \tan^{[n]} x\xi} dx = e^{2\pi \xi \tanh^{[n-1]} 1} \qquad \xi \le 0.$$

For $n \to \infty$ both integrals converge to 1 and hence

$$\lim_{n\uparrow\infty} I_n[f] = \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi = f(0).$$

In the last step we have applied Parseval's theorem.

4. From Fourier to Hardy

Comparing Equation (3.1) and Equation (1.1) reveals that the inner Fourier integral is just a special case of (1.1) with $f(x)=e^{2i\pi\xi x}$. It may seem that we have made use of special properties of this particular function f(x), but in fact the results generalises for a larger class of functions f. To transfer our arguments from above we need to assume that f(z) is analytic and bounded in the upper halfplane. However, the space of bounded analytic functions in the upper halfplane is the Hardy space H^{∞} . So, let $F \in H^{\infty}$, then $\|F\|_{H^{\infty}} = \sup_{Im \, z>0} |F(z)| < \infty$. The function f may be interpreted as the non-tangential limit of F, that is f(x) = F(x+iy) for $y \downarrow 0$. This implies f is bounded, too.

Assuming that f is continuous and bounded on the real line this resembles a kind of boundary value problem. However, the analytic extension of f is rarely bounded in the upper halfplane.

Now, from potential theory (see [2, Thms. 15.1a, 15.4d]) we know that there is a function F(z), holomorphic in the upper complex half plane Im z > 0, such that the harmonic function Re F(z) is bounded and has boundary values given by f, that is,

(4.1) Re
$$F(x + iy) \rightarrow f(x), \quad x \in \mathbb{R},$$

as the real number y approaches zero from above. The holomorphic function F is *unique* up to a purely imaginary additive constant. For the sake of simplicity of our presentation, we will further *assume that* F *itself*, *not just* ReF, *is bounded*; this additional assumption will be dropped in the elementary, real analysis proof given in [1].

Therefore

$$\frac{1}{\pi} \int_0^{\pi} f(\tan^{[n]} x) \, dx = \text{Re} \, \frac{1}{\pi} \int_0^{\pi} F(\tan^{[n]} x) \, dx = \text{Re} \, F(i \tanh^{[n-1]} 1).$$

Taking the limit for $n \to \infty$ reveals the Hydra is close to Dirac in spirit:

$$\lim_{n\uparrow\infty}\frac{1}{\pi}\int_0^\pi f(tan^{[n]}\,x)\,dx=f(0).$$

REFERENCES

- Folkmar Bornemann and Thomas Schmelzer, Taming a hydra of singularities, Amer. Math. Monthly 114 (2007), no. 8, 727-732.
- 2. Peter Henrici, Applied and Computational Complex Analysis. Vol. 3, Wiley, New York, 1986.

THOMAS SCHMELZER STANFORD UNIVERSITY, USA ADIA, ABU DHABI, UAE

Email address: thomas.schmelzer@gmail.com

Current address: Stanford University, USA

Email address: candes@stanford.edu