Summing a Curious, Slowly Convergent Series

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1. INTRODUCTION AND HISTORY. The harmonic series

$$\sum_{s=1}^{\infty} \frac{1}{s}$$

diverges. Suppose we delete from the series all terms whose denominators, in base 10, contain the digit 9. Kempner [**13**] proved in 1914 that the remaining series converges. We also get convergent series by deleting terms whose denominators contain any digit or string of digits, such as "42", or "314159".

However, these series converge so slowly that calculating their sums directly is out of the question. Here, we describe an algorithm to compute these and related sums to high precision. For example, the sum of the series whose denominators contain no "314159" is approximately 2302582.33386. We explain why this sum is so close to $10^6 \log 10$ by developing asymptotic estimates for sums that omit strings of length n, as n approaches infinity.

At first glance, it seems counter-intuitive that merely omitting the terms $1/9$, $1/19$, $1/29, \ldots$ from the harmonic series would produce a convergent series. It appears that we are removing only every tenth term from the harmonic series. If that were the case, then the sum of the remaining terms would indeed diverge.

This series converges because in the long run, we in fact delete almost everything from the harmonic series. We begin by deleting $1/9$, $1/19$, $1/29, \ldots$. But when we reach $1/89$, we delete 11 terms in a row: $1/89$, then $1/90$ through $1/99$. When we reach $1/889$, we delete 111 terms in a row: $1/889$, then $1/890$ through $1/899$, and finally $1/900$ through $1/999$.

Moreover, the vast majority of integers of, say, 100 digits contain at least one "9" somewhere within them. Therefore, when we apply our thinning process to 100-digit denominators, we will delete most terms. Only $8 \times 9^{99}/(9 \times 10^{99}) \approx 0.003\%$ of terms with 100-digit denominators will survive our thinning process. Schumer [**14**] argues that *the problem is that we tend to live among the set of puny integers and generally ignore the vast infinitude of larger ones. How trite and limiting our view!*

We can paraphrase Kempner's argument as follows. There are $8 \times 9^{i-1}$ integers with i digits that do not contain a "9". Their reciprocals are all at most $1/10^{i-1}$, so the sum of their reciprocals is at most $8 \times (9/10)^{i-1}$. Summing these numbers over i gives a convergent geometric series that converges to 80. This is an upper bound of the sum of the reciprocals of integers not containing a "9".

Kempner's reasoning and convergence result (but not his upper bound) apply to any digit in any base. That is, if d is a digit in base B, then if we delete from the harmonic series all terms that contain the base-B digit d, we likewise get a convergent series. We can use this fact to show that deleting terms that contain any fixed string of digits also gives a convergent series.

Also, there is a connection between the set of numbers that contain the decimal string "42" and the set of numbers that, in base 100, have a digit equal to 42. The second is a proper subset of the first. For example, (decimal) 1942 and 4219 have a base-100 digit equal to 42, but 1429 does not.

Theorem 1. *Let X be a string of n base-10 digits. Then if we delete from the harmonic series all terms that contain X, the resulting series converges.*

Proof. We may interpret the n-digit string X as a single digit in base 10^n. Let $\mathcal{Y}$ be the set of all numbers in base 10^n that contain the digit X. Deleting the terms whose denominators are in $\mathcal{Y}$ gives a convergent series. But if we delete the terms whose decimal value contains X, we are deleting all elements of $\mathcal{Y}$, plus some additional terms. It therefore follows that if we delete terms from the harmonic series that contain the base-10 digit string X, the remaining series also converges. ■

Once a series is known to converge, the natural question is, "What is its sum?" Unfortunately, these series converge far too slowly to compute their sums directly [**12, 13**].

The problem has attracted wide interest through the years in books such as [**1**, p. 384], [**4**, pp. 81–83], [**9**, pp. 120–121] and [**10**, pp. 31–34]. The computation of these sums is discussed in [**2**], [**7**], and [**17**]. Fischer computed 100 decimals of the sum with "9" missing from the denominators, but his method does not readily generalize to other digits. His remarkable result is that the sum is

$$\beta_0 \ln 10 - \sum_{n=2}^{\infty} 10^{-n} \beta_{n-1} \zeta(n)$$

where $\beta_0 = 10$ and the remaining β_n values are given recursively by

$$\sum_{k=1}^{n} \binom{n}{k} (10^{n-k+1} - 10^k + 1)\beta_{n-k} = 10(11^n - 10^n).$$

Trott [**16**, pp. 1281–1282] has implemented Fischer's algorithm using *Mathematica*.

In 1979, Baillie [**2**] published a method for computing the ten sums that arise when we delete terms containing each of the digits "0" through "9". The sum with "9" deleted is about 22.92067. But the sum of all terms with denominators up to 10^{29} still differs from the final sum by more than 1.

In order to compute sums whose denominators omit strings of two or more digits, we must generalize the algorithm of [**2**]. We do that here. We will show how to compute sums of $1/s$ where s contains no odd digits, no even digits, or no strings like "42" or "314159" or even combinations of those constraints.

Recently the problem has attracted some interest again. The computation of the sum of $1/s$ where s does not contain "42" is a problem suggested by Bornemann et al. [**5**, p. 281]. Subsequently the problems related to those sums have been discussed in a German-speaking online mathematics forum.[1] In 2005 Bornemann presented his solution for the "42"-problem to Trefethen's problem solving squad at Oxford. His idea is very similar to the original approach of Baillie and is covered by our analysis.

Bold print indicates vectors, matrices, or tensors. Sets are in calligraphic print.

[1] At `http://matheplanet.com/matheplanet/nuke/html/viewtopic.php?topic=9875` a few participants discuss the computation of the sum of $1/n$ where n does not contain an even digit. It seems their solution combines a direct summation and Richardson extrapolation and is of limited accuracy.

2. RECURRENCE MATRICES. Let X be a string of $n \geq 1$ digits. Let $\mathcal{S}$ be the set of positive integers that do not contain X in base 10. We denote the sum by Ψ, that is,

$$\Psi = \sum_{s \in \mathcal{S}} \frac{1}{s}. \tag{1}$$

If X is the single digit m, Baillie's method partitions $\mathcal{S}$ into subsets $\mathcal{S}_i$. The ith subset consists of those elements of $\mathcal{S}$ that have exactly i digits. The following recurrence connects $\mathcal{S}_i$ to $\mathcal{S}_{i+1}$:

$$\mathcal{S}_{i+1} = \bigcup_{s \in \mathcal{S}_i} \{10s, 10s+1, 10s+2, \ldots, 10s+9\} \backslash \{10s+m\}$$

From this, a recurrence formula is derived that allows us to compute $\sum_{s \in \mathcal{S}_{i+1}} 1/s^k$ from the sums $\sum_{s \in \mathcal{S}_i} 1/s^k$. If $n > 1$, there is no simple recurrence relation between $\mathcal{S}_i$ and $\mathcal{S}_{i+1}$. However, we can further partition $\mathcal{S}_i$ into subsets $\mathcal{S}_i^j$ for $j = 1, 2, \ldots, n$ in a way that yields a recurrence between $\mathcal{S}_{i+1}^j$ and the sets $\mathcal{S}_i^1, \ldots, \mathcal{S}_i^n$. Once we have done this, we have

$$\Psi = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \sum_{s \in \mathcal{S}_i^j} \frac{1}{s}.$$

We let $\mathcal{S}_i^j$ be the set of all members of $\mathcal{S}_i$ whose last $j-1$ digits match the first $j-1$ digits of X, but whose last j digits do not match the first j digits of X. Notice that if $j < n$ and d is any digit, then there is a number k such that appending the digit d to an element of $\mathcal{S}_i^j$ leads to any element of $\mathcal{S}_{i+1}^k$. The same is true for $j = n$, except that if d is the last digit of X then appending d to an element of $\mathcal{S}_i^n$ gives a number that contains the string X, and is therefore not an element of $\mathcal{S}$. It is convenient to let $\mathcal{S}^j$ be the union of $\mathcal{S}_i^j$, over all i. We will represent the partition and the corresponding recurrence with an $n \times 10$ matrix $\mathbf{T}$. The (j, d) entry of $\mathbf{T}$ tells us which set we end up in when we append the digit d to each element of $\mathcal{S}^j$. In other words, if appending the digit d to an element of $\mathcal{S}^j$ leads to an element of $\mathcal{S}^k$, then $T(j, d) = k$. If the digit d cannot be appended because the resulting number would not be in $\mathcal{S}$, then we set $\mathrm{T}(j, d)$ to 0.

Here is an example that shows how to compute the matrix $\mathbf{T}$ for a given string. Let $\mathcal{S}$ be the set of integers containing no "314". We partition $\mathcal{S}$ into three subsets: $\mathcal{S}^1$ consists of the elements of $\mathcal{S}$ not ending in 3 or 31. $\mathcal{S}^2$ is the set of elements of $\mathcal{S}$ ending in 3. $\mathcal{S}^3$ is the set of elements of $\mathcal{S}$ ending in 31. The following matrix $\mathbf{T}$ shows what happens when we append the digits 0 through 9 to $\mathcal{S}^1$, $\mathcal{S}^2$, and $\mathcal{S}^3$.

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 3 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 1 & 2 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

When we append a 3 to an element of $\mathcal{S}^1$, we get an element of $\mathcal{S}^2$, so we set $\mathrm{T}(1, 3) = 2$. Appending any other digit to an element of $\mathcal{S}^1$ yields another element of $\mathcal{S}^1$, so for all other d, $\mathrm{T}(1, d) = 1$. Consider elements of $\mathcal{S}^2$. Appending a 1 yields an element of $\mathcal{S}^3$; appending a 3 yields another element of $\mathcal{S}^2$. Appending any other digit yields an element of $\mathcal{S}^1$. The only special feature of $\mathcal{S}^3$ is that if we append a 4, we get a number ending in 314, which is not in $\mathcal{S}$, so we set $\mathrm{T}(3, 4) = 0$.

Let us emphasize that the matrix $\mathbf{T}$ does not have to be induced by a string X. Indeed our approach is more general. We can also solve a puzzle stated by Boas [**3**] asking[2] for an estimate of the sum of $1/s$ where s has no even digits. Here it is enough to work with one set $\mathcal{S}$ but to forbid that an even integer can be attached. Hence $\mathbf{T}$ is

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

Many more interesting examples are discussed in Section 5.

The recursive relations between the sets $\mathcal{S}^1, \dots, \mathcal{S}^n$ may also be illustrated by means of a directed graph. There is a directed edge from $\mathcal{S}^i$ to $\mathcal{S}^j$ if by appending an integer d to elements of $\mathcal{S}^i$ we end up in $\mathcal{S}^j$; see Figure 1. For further analysis we assume only that the associated directed graph is strongly connected, that is, there are directed paths from $\mathcal{S}^i$ to $\mathcal{S}^j$ and $\mathcal{S}^j$ to $\mathcal{S}^i$ for any pair $i \neq j$. Graphs that are not strongly connected can be induced by more exotic constraints but are not discussed here.

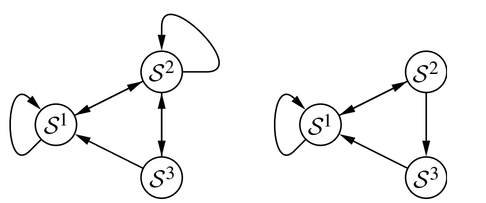

Figure 1. Directed, strongly connected graphs. Left: Graph for the partition induced by the string "314". Any other string with three distinct digits would have the same graph. Right: Graph of the partition induced by the string "333". Note that the strings "332" and "323" would induce two alternative graphs not shown here. The graph related to "233" would match the left graph.

In the next section, we show how $\mathbf{T}$ is used to compute $\sum_{s\in\mathcal{S}_{i+1}} 1/s^k$ from the values of $\sum_{s\in\mathcal{S}_i} 1/s^k$.

3. A RECURRENCE FORMULA.

It may seem a bit odd to introduce sums of s^{-k} although only the case $k = 1$ is desired, but they enable us to exploit the recurrence relations between the aforementioned sets $\mathcal{S}^1, \dots, \mathcal{S}^n$. The idea has been successfully applied in [**2**]. Let

$$\Psi^j_{i,k} = \sum_{s\in\mathcal{S}^j_i} \frac{1}{s^k}. \tag{2}$$

Therefore the sum of s^{-k} where s ranges over all i-digit integers is an upper bound for $\Psi^j_{i,k}$. There are $10^i - 10^{i-1}$ of these integers, each of which is at least 10^{i-1}. Therefore,

$$\Psi^j_{i,k} \le \left(10^i - 10^{i-1}\right)\frac{1}{10^{(i-1)k}} = \frac{9}{10^{(i-1)(k-1)}}. \tag{3}$$

[2] Actually Boas is pointing to problem 3555 published in *School Science and Mathematics*, April 1975.

Using the new notation the problem is to compute

$$\Psi = \sum_{j=1}^{n} \sum_{i=1}^{\infty} \Psi_{i,1}^{j}.$$

The recursive nature of the sets $\mathcal{S}_i^j$ is now used to derive recurrence relations for the sums $\Psi_{i,k}^j$. We introduce a tensor $\mathbf{f}$ of dimensions $n \times n \times 10$ with

$$f_{jlm} = \begin{cases} 1 & \text{if } \mathrm{T}(l, m) = j, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

This tells us in the sum below to either include a term ($f_{jlm} = 1$) or not include it ($f_{jlm} = 0$). Then

$$\Psi_{i,k}^{j} = \sum_{m=0}^{9} \sum_{l=1}^{n} f_{jlm} \sum_{s \in \mathcal{S}_{i-1}^{l}} (10s + m)^{-k}. \tag{5}$$

By construction of $\mathbf{f}$ the sum runs over all index pairs (l, m) such that $\mathrm{T}(l, m) = j$, which indicates that

$$\left\{10s + m \mid s \in \mathcal{S}_{i-1}^{l}\right\} \subset \mathcal{S}_i^j.$$

Although equation (5) is the crucial link to the recurrence matrices it is of no computational use. It is still a direct computation of $\Psi_{i,k}^j$. We should avoid summing over a range of integers in subsets $\mathcal{S}_{i-1}^l$. Using negative binomial series [**18**] we observe

$$(10s + m)^{-k} = (10s)^{-k} \sum_{w=0}^{\infty} (-1)^w \binom{k + w - 1}{w} \left(\frac{m}{10s}\right)^w$$

where $0^0 = 1$ and define:

$$c(k, w) = (-1)^w \binom{k + w - 1}{w}.$$

We replace the term $(10s + m)^{-k}$ in equation (5) and get

$$\begin{aligned} \Psi_{i,k}^{j} &= \sum_{m=0}^{9} \sum_{l=1}^{n} f_{jlm} \sum_{s \in \mathcal{S}_{i-1}^{l}} (10s)^{-k} \sum_{w=0}^{\infty} c(k, w) \left(\frac{m}{10s}\right)^w \\ &= \sum_{m=0}^{9} \sum_{l=1}^{n} f_{jlm} \sum_{w=0}^{\infty} 10^{-k-w} c(k, w) m^w \sum_{s \in \mathcal{S}_{i-1}^{l}} s^{-k-w} \end{aligned}$$

by reordering the sum. To simplify the notation we introduce

$$a(k, w, m) = 10^{-k-w} c(k, w) m^w$$

and write therefore

$$\Psi_{i,k}^{j} = \sum_{m=0}^{9} \sum_{l=1}^{n} f_{jlm} \sum_{w=0}^{\infty} a(k, w, m) \Psi_{i-1,k+w}^{l}. \tag{6}$$

Again it may seem odd that we have replaced the finite summation (5) in s by an infinite sum in w. But the infinite sum decays so fast in w that truncation enables us to approximate (6) much faster than evaluating the sums of equation (5).

4. TRUNCATION AND EXTRAPOLATION. We step into the numerical analysis of the problem. So far we have only reformulated the summation by introducing the partial sums $\Psi_{i,k}^{j}$. The ultimate goal is the efficient computation of $\Psi = \sum_{i,j} \Psi_{i,1}^{j}$. We use the following scheme:

For $i \leq 3$ the sums $\Psi_{i,k}^{j}$ ($k > 1$ is needed in the next step) are computed directly as suggested in equation (2).

For $i > 3$ a recursive evaluation of (6) is used. The indices i and w both run from 4 (resp. 0) to infinity. For w we use a simple truncation by neglecting all terms $\Psi_{i-1,k+w}^{l}$ smaller than an a priori given bound ε, or to be precise: we neglect all terms $\Psi_{i-1,k+w}^{l}$ where the estimate (3) is smaller than ε, that is, where $k + w$ is sufficiently large. We do not bound the resulting error as a function of ε in order to give a rigorous proof. Ignoring individual terms of a series once they are small is delicate. After all, the subject of this work is the harmonic series, which is everyone's favorite example of a series that diverges even though its terms approach 0. However, for a convergent series the truncated sum ignoring terms smaller than ε will converge towards the limit for $\varepsilon \to 0$. The limit may be estimated by using extrapolation, after computing the truncated sum for ε, ε^2, and ε^3 or even higher order.

For the same reason we can neglect for large i all contributions from terms $\Psi_{i-1,k+w}^{l}$ with $k + w > 1$. Once the algorithm has achieved that stage it is possible to apply extrapolation. Equation (6) with $w = 0$ and $k = 1$ reads in matrix form

$$\begin{pmatrix} \Psi_{i,1}^{1} \\ \vdots \\ \Psi_{i,1}^{n} \end{pmatrix} \approx \underbrace{\sum_{m=0}^{9} 10^{-1} \begin{pmatrix} f_{11m} & \cdots & f_{1nm} \\ \vdots & & \vdots \\ f_{n1m} & \cdots & f_{nnm} \end{pmatrix}}_{\mathbf{A}_n} \begin{pmatrix} \Psi_{i-1,1}^{1} \\ \vdots \\ \Psi_{i-1,1}^{n} \end{pmatrix} \tag{7}$$

since $a(1, 0, m) = 10^{-1}$ for all m.

The nonnegative matrix $\mathbf{A}_n$ is a contraction, that is, all its eigenvalues lie within the unit disc. The proof of this property reveals a link to graph theory.

Definition 1. The *associated digraph* of an $n \times n$ matrix $\mathbf{A}_n$ is the directed graph with vertices $1, \dots, n$ and an edge from i to j if and only if $\mathrm{A}(i, j) \neq 0$.

The associated digraph of $\mathbf{A}_n^{\mathsf{T}}$ with vertices $1, \dots, n$ representing the sets $\mathcal{S}^j$, $j = 1, \dots, n$ is exactly the graph illustrating the recurrence relations between the sets $\mathcal{S}^j$ as introduced above, see Figure 1, as $\mathrm{A}^{\mathsf{T}}(i, j) = \mathrm{A}(j, i) \neq 0$ if $\mathrm{T}(i, m) = j$ for some m.

We have assumed that this graph is strongly connected and hence $\mathbf{A}_n^{\mathsf{T}}$ and $\mathbf{A}_n$ are irreducible[3] and therefore the Perron-Frobenius Theorem [**11**, Theorem 8.4.4] applies:

[3]A matrix is reducible if and only if its associated digraph is not strongly connected [**8**, p. 163 ff.].

Theorem 2 (Perron-Frobenius). *Let* $\mathbf{A}$ *be a nonnegative*[4] *and irreducible matrix. Then*

- *there is an eigenvalue* λ_d *that is real and positive, with positive left and right eigenvectors,*
- *any other eigenvalue* λ *satisfies* $|\lambda| < \lambda_d$*,*
- *the eigenvalue* λ_d *is simple.*

The eigenvalue λ_d *is called the dominant eigenvalue of* $\mathbf{A}$.

It remains to show that the dominant eigenvalue λ_d of $\mathbf{A}_n$ is smaller than 1. Consider the lth column of the matrix

$$\begin{pmatrix} f_{11m} & \cdots & f_{1nm} \\ \vdots & & \vdots \\ f_{n1m} & \cdots & f_{nnm} \end{pmatrix}.$$

By definition of $\mathbf{f}$ (4) there is a 1 in row $\mathrm{T}(l, m)$ if $\mathrm{T}(l, m) > 0$. All other entries in the column are zero. Therefore $\|\mathbf{a}_l\|_1 \leq 1$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ denote the columns of $\mathbf{A}_n$.

The existence of columns which have no nonzero entry implies that there are columns of $\mathbf{A}_n$ with $\|\mathbf{a}_i\|_1 < 1$. Let $\mathbf{x}$ be the right eigenvector of $\mathbf{A}_n$ corresponding to λ_d. Applying the triangle inequality we conclude that

$$\lambda_d \, \|\mathbf{x}\|_1 = \|\mathbf{A}_n \mathbf{x}\|_1 \leq \sum_{i=1}^{n} |x_i| \, \|\mathbf{a}_i\|_1 < \sum_{i=1}^{n} |x_i| = \|\mathbf{x}\|_1 \tag{8}$$

and therefore $\lambda_d < 1$. Note that we have used the fact that the elements x_i of the positive vector $\mathbf{x}$ cannot vanish for any index i. If all columns of the nonnegative matrix $\mathbf{A}_n$ satisfy $\|\mathbf{a}_i\|_1 = 1$, the matrix $\mathbf{A}_n$ is called *stochastic* and $(1, \ldots, 1)$ is a left eigenvector with eigenvalue 1. In this case $\mathbf{A}_n$ is no longer a contraction. This situation occurs if $\mathbf{T}$ contains no zero and hence no integers are deleted at all. This may serve as an unusual explanation for the divergence of the harmonic series.

Having shown that the spectrum of $\mathbf{A}_n$ lies within the unit disk, for large K we can simplify

$$\sum_{i=1}^{\infty} \begin{pmatrix} \Psi^1_{i+K,1} \\ \vdots \\ \Psi^n_{i+K,1} \end{pmatrix} \approx \sum_{i=1}^{\infty} \mathbf{A}_n^i \begin{pmatrix} \Psi^1_{K,1} \\ \vdots \\ \Psi^n_{K,1} \end{pmatrix} \tag{9}$$

by a Neumann series

$$\sum_{k=1}^{\infty} \mathbf{A}_n^k = (\mathbf{I} - \mathbf{A}_n)^{-1} - \mathbf{I} =: \mathbf{B}_n^{\infty} \tag{10}$$

where $\mathbf{I}$ is an identity matrix of appropriate dimension. Hence

$$\Psi \approx \sum_{j=1}^{n} \sum_{i=1}^{K} \Psi^j_{i,1} + \left\| \mathbf{B}_n^{\infty} \begin{pmatrix} \Psi^1_{K,1} \\ \vdots \\ \Psi^n_{K,1} \end{pmatrix} \right\|_1 . \tag{11}$$

[4] A matrix is nonnegative if and only if every entry is nonnegative.

Using the same idea we can also estimate the result of truncating the series after, say, $M + K$ digits using

$$\sum_{k=1}^{M} \mathbf{A}_n^k = (\mathbf{I} - \mathbf{A}_n^{M+1})(\mathbf{I} - \mathbf{A}_n)^{-1} - \mathbf{I} =: \mathbf{B}_n^M. \tag{12}$$

5. EXAMPLES. We compute a few sums to a precision of 100 decimals, although if desired, more could easily be obtained. All of these sums have been computed to even higher precision, and in every case, the first 100 decimals agree.

First, let us compute the sum originally considered by Kempner [**13**], namely, where the digit "9" is missing from the denominators. Here, there is only one set $\mathcal{S} = \mathcal{S}^1$, namely, the set of integers that do not contain a "9". When we append a "9" to an element of $\mathcal{S}$, we get a number not in $\mathcal{S}$, so $\mathrm{T}(1, 9) = 0$. When we append any other digit, we get another element of $\mathcal{S}$, so for all other d, $\mathrm{T}(1, d) = 1$:

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right].$$

To 100 decimals, the sum is

22.92067 66192 64150 34816 36570 94375 93191 49447 62436 99848
15685 41998 35657 21563 38189 91112 94456 26037 44820 18989 . . . .

In Section 2 we mentioned the sum of $1/s$ where s has no even digits. The sum is

3.17176 54734 15904 95722 87097 08750 61165 67970 50708 39628
57241 64186 89843 71376 88585 61926 68852 31080 74715 60454 . . . .

Similarly, the sum over denominators with no odd digits can be found using the matrix

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right].$$

In this case the sum is

1.96260 84129 94616 98515 91542 64737 29435 67128 30665 51443
53546 71522 23586 65760 95274 32927 13468 24171 73826 12704 . . . .

In Section 2, we gave the matrix $\mathbf{T}$ that corresponds to the sum of $1/s$ where s has no string "314". The sum is:

2299.82978 27675 18338 45358 63536 11974 36784 61556 88394 19837
51645 98202 17625 43309 41712 63285 37992 24266 07454 90945 . . . .

We can also compute the sum of $1/s$ where s has no string "314159". Then the sum is:

2302582.33386 37826 07892 02375 60364 84435 61276 86862 90972 08627
80786 90557 30669 81792 73645 44979 47969 47311 14619 12012 . . . .

This sum demonstrates the power of the technique presented here. Using (12) we calculate that the partial sum of all terms whose denominators have 100000 or fewer digits is about 219121.34825 . . . . This is still only 1/10th as large as the final sum, and illustrates the futility of direct summation. Notice that this sum is close to $10^6 \log 10$. But there is nothing special about "314159"; we observe similar results for other strings of six digits. We say more about this in the next section.

Let $\mathcal{S}$ be the set of integers containing no "42". Computing Ψ in this case is the challenge recently posed by Bornemann et al. [**5**]. In this case

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

and the sum is given by

$$\begin{aligned}&228.44630\ 41592\ 30813\ 25414\ 80861\ 26250\ 58957\ 81629\ 27539\ 83036\\ &\quad 11859\ 13460\ 00045\ 28607\ 68650\ 21430\ 70480\ 46117\ 41443\ 21741\ldots.\end{aligned}$$

The next example combines various constraints and illustrates the flexibility of our approach. Let $\mathcal{S}$ be the set of integers containing no even digits, no "55", and no "13579". Then we define the following partition of $\mathcal{S}$. $\mathcal{S}^2$ is the subset of numbers ending in 5 (but not ending in 135), $\mathcal{S}^3$ is the set of numbers ending in 1, $\mathcal{S}^4$ is the set of numbers ending in 13, $\mathcal{S}^5$ is the subset of numbers ending in 135 and the elements of $\mathcal{S}^6$ end in 1357. All remaining elements of $\mathcal{S}$ are in $\mathcal{S}^1$. Following those rules the matrix $\mathbf{T}$ is given by

$$\mathbf{T} = \left[\begin{array}{c|cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \hline 1 & 0 & 3 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 \\ 2 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 3 & 0 & 3 & 0 & 4 & 0 & 2 & 0 & 1 & 0 & 1 \\ 4 & 0 & 3 & 0 & 1 & 0 & 5 & 0 & 1 & 0 & 1 \\ 5 & 0 & 3 & 0 & 1 & 0 & 0 & 0 & 6 & 0 & 1 \\ 6 & 0 & 3 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right].$$

Here the sum is

$$\begin{aligned}&3.09084\ 91496\ 53806\ 46825\ 46563\ 73157\ 80175\ 63888\ 91119\ 39765\\ &\quad 22149\ 64013\ 36906\ 53946\ 19395\ 87929\ 18235\ 63131\ 88124\ 97325\ldots.\end{aligned}$$

Our algorithm can be easily generalized for other bases than 10. The sum of $1/s$ where s has no "0" in base 100 is

$$\begin{aligned}&460.52520\ 26385\ 12471\ 14293\ 67535\ 66415\ 29497\ 12569\ 09908\ 47934\\ &\quad 06016\ 95672\ 87250\ 06818\ 86421\ 46967\ 22875\ 07176\ 27582\ 54794\ldots.\end{aligned}$$

All sorts of other experiments are possible. Interested readers might experiment by removing from the harmonic series their personal favorite numbers, such as their birthdays (as a single string, or as a set of three strings), their phone numbers, etc.

6. ASYMPTOTIC BEHAVIOR. We now discuss the asymptotic behavior of the sums that arise when we remove from the harmonic series terms whose denominators contain a string X_n of length n digits.

Data for several random strings of $n = 20$ digits are given in Table 1. It is striking that, for each random string X_n, the "normalized" sum, $\Psi_{X_n}/10^n$, is very close to $\log 10 = 2.30258\ 50929\ 94045\ 68401\ 79914\ldots$.

Table 1. Sums for several random 20-digit strings

n	String X_n	$\Psi_{X_n}/10^n$
20	21794968139645396791	2.30258 50929 94045 68397 52162
20	31850115459210380210	2.30258 50929 94045 68399 08824
20	67914499976105176602	2.30258 50929 94045 68401 09579
20	98297963712691768117	2.30258 50929 94045 68401 77079

Table 2 shows what happens for strings that consist of repeated patterns of substrings. If a string consists of shorter substrings repeated two or more times, we define the *period* of the string to be the length of that shortest substring. So, "11111" has period 1, while "535353" has period 2.

Table 2. Sums for strings of periods 1, 2, and 5

n	String X_n	$\Psi_{X_n}/10^n$
5	00000	2.55840 22969
10	0000000000	2.55842 78808 48652
15	000000000000000	2.55842 78811 04492 64603
20	00000000000000000000	2.55842 78811 04495 20443 88506
20	11111111111111111111	2.55842 78811 04495 20435 05433
20	44444444444444444444	2.55842 78811 04495 20442 19551
20	99999999999999999999	2.55842 78811 04495 20443 88506
4	4242	2.32542 92748
10	4242424242	2.32584 35278 62555
16	4242424242424242	2.32584 35282 76813 40798 19419
20	42424242424242424242	2.32584 35282 76813 82219 89695
20	09090909090909090909	2.32584 35282 76813 82221 85405
5	12345	2.30250 59575
10	1234512345	2.30260 81180 53596
15	123451234512345	2.30260 81190 75226 21998
20	12345123451234512345	2.30260 81190 75236 43628 01912

Here, the normalized sums appear to approach different limits. The limits do not depend on which digits comprise the strings, but instead depend on the periods. When all digits are identical (period 1), the limit of the normalized sum seems to be $(10/9)\log 10 = 2.55842\ldots$. When the period is 2, the limit seems to be $(100/99)\log 10 = 2.32584\ldots$. For period 5, it's $(100000/99999)\log 10 = 2.30260\ldots$.

We emphasize that we observe these same limits for other strings of digits. In the limit, the particular digits in a string do not matter. What matters is the structure of the strings.

Conjecture 1. *Let $(X_n)_{n\in p\mathbb{N}}$ be a sequence of strings, where each string X_n has n digits and has period p, where n is a multiple of p. Let Ψ_{X_n} be the sum of $1/s$ where s does not contain the substring X_n. Then*

$$\lim_{n\to\infty} \Psi_{X_n}/10^n = \frac{10^p}{10^p - 1}\log 10.$$

The lack of a periodic pattern may be interpreted as the limit as $p \to \infty$.

Conjecture 2. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of strings, where each string X_n has n digits with no periodic pattern. Let Ψ_{X_n} be the sum of $1/s$ where s does not contain the substring X_n. Then*

$$\lim_{n\to\infty} \Psi_{X_n}/10^n = \log 10.$$

Our numerical experiments strongly support these conjectures. Rather than attempt rigorous proofs, we prove a special case ($p = 1$) of Conjecture 1. The proof of the following Theorem reveals the strong link between the spectrum of $\mathbf{B}_n^\infty$ and the limits we observe in Tables 1 and 2.

Theorem 3. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of strings, where each string X_n has n digits that are all the same. Let Ψ_{X_n} be the sum of $1/s$ where s does not contain the substring X_n. Then*

$$\lim_{n\to\infty} \Psi_{X_n}/10^n = \frac{10}{9}\log 10.$$

We sketch the proof by a series of Lemmata analyzing the spectrum of the extrapolation matrices $\mathbf{A}_n$ given in (7) and $\mathbf{B}_n^\infty$ linked via (10). A string $X_n = d\ldots d$ gives rise to an $n \times n$ matrix $\mathbf{A}_n$ via (7) of the form

$$\mathbf{A}_n = \frac{1}{10}\begin{pmatrix} 9 & 9 & \cdots & \cdots & 9 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & 0 \end{pmatrix} \tag{13}$$

assuming that $\mathcal{S}^1$ is the set of integers not ending in the integer d, numbers in $\mathcal{S}^2$ end in d but not dd, numbers in $\mathcal{S}^3$ end in dd but not ddd, and so on.

The spectrum of $\mathbf{A}_n$ lies within the unit disc. We can say much more about the spectrum in this case. We need the Theorem of Gershgorin [**11**, Theorem 6.1.1] to make more precise statements:

Theorem 4 (Gershgorin). *Let $\mathbf{A}$ be a $n \times n$ matrix, and let*

$$R_i(\mathbf{A}) = \sum_{j=1, j\neq i}^{n} \left|a_{ij}\right|, \qquad 1 \le i \le n$$

denote the deleted absolute row sums of $\mathbf{A}$. Then all the eigenvalues of $\mathbf{A}$ are located in the union of the n discs

$$\{z \in \mathbb{C} \mid |z - a_{ii}| \le R_i(\mathbf{A})\}, \qquad 1 \le i \le n. \tag{14}$$

Furthermore, if a union of k of these n discs forms a connected region that is disjoint from all the remaining $n - k$ discs, then there are precisely k eigenvalues of $\mathbf{A}$ in this region.

Lemma 1. *The matrix* $\mathbf{A}_n$ *as defined in* (13) *is diagonalisable. The eigenvalues of the* $n \times n$ *matrix* $\mathbf{A}_n$ *are the solutions of the equation*

$$\lambda^n(1-\lambda) = 9/10^{n+1} \tag{15}$$

distinct from $1/10$. *There are exactly* $n-1$ *eigenvalues of* $\mathbf{A}_n$ *contained in a disc of radius* $1/10$ *centered at the origin. If* λ *is an eigenvalue of* $\mathbf{A}_n$ *then* $(\lambda^{n-1}, \lambda^{n-2}, \dots, \lambda, 1)^{\mathsf{T}}$ *is an eigenvector.*

Proof. Let $\mathbf{x}_n = (x_1, \dots, x_n)^{\mathsf{T}}$ be an eigenvector of $10\mathbf{A}_n$ corresponding to the eigenvalue λ of $10\mathbf{A}_n$. Then

$$10\mathbf{A}_n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 9\sum_{i=1}^n x_i \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}. \tag{16}$$

Hence $x_{n-1} = \lambda x_n$, which implies $x_{n-2} = \lambda^2 x_n$. Using induction we observe

$$x_{n-k} = \lambda^k x_n.$$

In particular $x_1 = \lambda^{n-1} x_n$. Both matrices $10\mathbf{A}_n$ and $\mathbf{A}_n$ share the same set of eigenvectors. Hence $(\lambda^{n-1}, \lambda^{n-2}, \dots, \lambda, 1)^{\mathsf{T}}$ is an eigenvector of $\mathbf{A}_n$ by choosing $x_n = 1$. The first row of (16) reveals that the characteristic equation of $10\mathbf{A}_n$ is

$$9\sum_{i=0}^{n-1} \lambda^i = \lambda^n.$$

Hence $\lambda = 1$ is not an eigenvalue of $10\mathbf{A}_n$. For $\lambda \neq 1$ we can write

$$9\sum_{i=0}^{n-1} \lambda^i = 9\frac{1-\lambda^n}{1-\lambda} = \lambda^n$$

or

$$\lambda^{n+1} - 10\lambda^n + 9 = 0. \tag{17}$$

The $n+1$ roots of this polynomial are all distinct as the roots of the derivative

$$(n+1)\lambda^n - 10n\lambda^{n-1}$$

are not roots of (17). That implies that each eigenvalue of $10\mathbf{A}_n$ is of algebraic multiplicity 1 and hence the matrix $10\mathbf{A}_n$ is diagonalisable. Clearly λ is an eigenvalue of $\mathbf{A}_n$ if and only if 10λ is an eigenvalue of $10\mathbf{A}_n$. Hence the eigenvalues of $\mathbf{A}_n$ are precisely the solutions other than $1/10$ of the equation

$$10^{n+1}\lambda^{n+1} - 10^{n+1}\lambda^n = -9$$

which yields (15). The roots of the polynomial in (17) are the eigenvalues of the $(n+1) \times (n+1)$ companion matrix

$$\mathbf{H}_{n+1} = \begin{pmatrix} 10 & 0 & \cdots & 0 & -9 \\ 1 & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & 1 & 0 \end{pmatrix}.$$

We write $\mathbf{H}_{n+1} = \mathbf{D}_{n+1} + \mathbf{B}_{n+1}$ where $\mathbf{D}_{n+1} = \operatorname{diag}(10, 0, \ldots, 0)$ and set $\mathbf{H}^{\varepsilon}_{n+1} = \mathbf{D}_{n+1} + \varepsilon\mathbf{B}_{n+1}$ for $\varepsilon \in [0, 1]$. For all $\varepsilon < 1$ the union of the Gershgorin circles of radius ε centered at the origin is disjoint from the disc centered at 10. Because the eigenvalues of $\mathbf{H}^{\varepsilon}_{n+1}$ are continuous functions of ε, there are n eigenvalues of $\mathbf{H}^{1}_{n+1}$ located within the closed unit disc, including the simple eigenvalue at 1. Hence there are $n - 1$ eigenvalues of $\mathbf{A}_n$ within the disc of radius $1/10$. Note that a direct application of the Theorem of Gershgorin to $\mathbf{A}_n$ does not yield the desired result. ∎

Lemma 2. *Let λ_n be the dominant eigenvalue of $\mathbf{A}_n$ defined in* (13). *Then*

$$\lim_{n\to\infty} \lambda_n^n = 1.$$

Proof. Since λ_n is an eigenvalue of $\mathbf{A}_n$, it is a solution of (15). The graph of $\lambda^n(1 - \lambda)$ is monotonically decreasing for $\lambda > n/(n + 1)$. But for $\lambda = (10^n - 1)/10^n$, $\lambda^n(1 - \lambda)$ is still larger than $9/10^{n+1}$, and hence $1 > \lambda_n > (10^n - 1)/10^n$. Taking the nth power of all terms yields the desired result. ∎

Lemma 3. *Let Λ_n be the dominant eigenvalue of* $1/10^n\mathbf{B}^{\infty}_n$ *defined in* (10) *using* $\mathbf{A}_n$ *given by* (13). *Then*

$$\lim_{n\to\infty} \Lambda_n = \frac{10}{9}. \tag{18}$$

Proof. The dominant eigenvalue of $\mathbf{B}^{\infty}_n$ is

$$(1 - \lambda_n)^{-1} - 1$$

where λ_n is the dominant eigenvalue of $\mathbf{A}_n$. Hence

$$\Lambda_n = \frac{(1 - \lambda_n)^{-1} - 1}{10^n}.$$

But $(1 - \lambda_n) = \frac{9}{10^{n+1}\lambda_n^n}$ as given by identity (15). Therefore

$$\Lambda_n = \frac{10^{n+1}\lambda_n^n}{9 \times 10^n} - \frac{1}{10^n}$$

Applying Lemma 2 finishes the proof. ∎

All other eigenvalues of $1/10^n\mathbf{B}^{\infty}_n$ are contained in a disc of radius $\frac{1}{9\times 10^n}$ corresponding to the eigenvalues of $\mathbf{A}_n$ contained in a disc of radius $1/10$.

The dominant eigenvector of $1/10^n\mathbf{B}^{\infty}_n$ is the dominant eigenvector of $10\mathbf{A}_n$. The dominant eigenvalue of $10\mathbf{A}_n$ is approaching 10 and hence the dominant eigenvector of $10\mathbf{A}_n$ is, using (16), converging towards $(1, 1/10, 1/100, \ldots, 1/10^{n-1})^{\mathsf{T}}$ where we have normalized this vector such that the first component is 1.

Although the matrix $1/10^n \mathbf{B}_n^\infty$ is invertible for any n we may regard the limit as an operator of rank 1. The matrix-vector product $1/10^n \mathbf{B}_n^\infty \mathbf{v}_n$ is in the limit a projection of $\mathbf{v}_n$ to $(1, 1/10, 1/100, \dots 1/10^{n-1})^\top$ followed by a multiplication with 10/9. If a vector $\mathbf{v}_n$ is already aligned with the eigenvector the application of $1/10^n \mathbf{B}_n^\infty \mathbf{v}_n$ boils down to a multiplication by 10/9.

Lemma 4. *Let $\mathbf{B}_n^\infty$ as defined in* (10) *using $\mathbf{A}_n$ given by* (13). *Then*

$$\left\| \frac{1}{10^n} \mathbf{B}_n^\infty \begin{pmatrix} \Psi_{n-1,1}^1 \\ \vdots \\ \Psi_{n-1,1}^n \end{pmatrix} \right\|_1 \xrightarrow[n\to\infty]{} \frac{10}{9} \log 10$$

where $\Psi_{n-1,1}^1, \dots, \Psi_{n-1,1}^n$ are the partial sums of reciprocals of $n-1$ digit numbers associated with the string $X_n = d \dots d$ of length n.

Proof. The norm of the vectors

$$\begin{pmatrix} \Psi_{n-1,1}^1 \\ \vdots \\ \Psi_{n-1,1}^n \end{pmatrix}$$

is exactly the sum of $1/s$ over all positive integers s with exactly $n-1$ digits. Not a single integer has been deleted at this stage yet. Hence

$$\left\| \begin{pmatrix} \Psi_{n-1,1}^1 \\ \vdots \\ \Psi_{n-1,1}^n \end{pmatrix} \right\|_1 \xrightarrow[n\to\infty]{} \int_{10^{n-1}}^{10^n} \frac{1}{t}\, dt = \log 10.$$

Next we claim that

$$\lim_{n\to\infty} \left\| \frac{1}{\Psi_{n-1,1}^1} \begin{pmatrix} \Psi_{n-1,1}^1 \\ \vdots \\ \Psi_{n-1,1}^n \end{pmatrix} - \begin{pmatrix} 1 \\ 1/10 \\ \vdots \\ 1/10^{n-1} \end{pmatrix} \right\|_1 = 0.$$

It is without doubt that the first entry of the difference is indeed 0. The second entry represents the ratio of the sum of the reciprocals of the $n-1$ digit numbers ending in d but not dd and the sum of the reciprocals of the $n-1$ digit numbers not ending in d. The ratio of the numbers of terms in the two sums is exactly $1/10$. The ratio of the sums approaches $1/10$ since the numbers ending in d are equally distributed amongst the numbers not ending in d. The same argument can be applied to any other row, too. The coarse relative approximation of $\Psi_{n-1,1}^n / \Psi_{n-1,1}^1$ by $1/10^{n-1}$ is irrelevant as the maximal entry of the last $n/2$ rows of both vectors is converging towards 0. Hence the vector

$$\begin{pmatrix} \Psi_{n-1,1}^1 \\ \vdots \\ \Psi_{n-1,1}^n \end{pmatrix}$$

is turning into the direction of the dominant eigenvector of $\mathbf{B}_n^\infty$ while approaching the norm $\log 10$. ■

Lemma 4 gives a new perspective on equation (11). There we fix the truncation parameter K to $n-1$ and introduce an error term e_n compensating this step. Hence (11) reads as

$$\frac{\Psi_{X_n}}{10^n} = \frac{1}{10^n}\left(\sum_{j=1}^{n}\sum_{i=1}^{n-1}\Psi_{i,1}^{j} + \left\|\mathbf{B}_n^\infty\begin{pmatrix}\Psi_{n-1,1}^1\\ \vdots \\ \Psi_{n-1,1}^n\end{pmatrix}\right\|_1 + e_n\right).$$

The largest terms collected in the error term are $\Psi_{n-1,2}^j$ for $j = 1, \dots, n$. Their sum, that is $\sum_{j=1}^n \Psi_{n-1,2}^j$, is smaller than $9/10^{n-2}$ as shown by equation (3). Hence the terms we neglect get exponentially smaller for increasing n. In the language of Section 4 we are neglecting terms smaller than or equal to $\varepsilon_n = \Psi_{n-1,2}^1 < 9/10^{n-2}$.

We observe that for increasing n the double sum multiplied by 10^{-n} converges to zero due to the slow growth of the sum. The error term scaled by 10^{-n} representing the neglected sums meets the same fate. So Lemma 4 implies

$$\lim_{n\to\infty}\frac{\Psi_{X_n}}{10^n} = \lim_{n\to\infty}\left\|\frac{1}{10^n}\mathbf{B}_n^\infty\begin{pmatrix}\Psi_{n-1,1}^1\\ \vdots \\ \Psi_{n-1,1}^n\end{pmatrix}\right\|_1 = \frac{10}{9}\log 10.$$

This proof of Theorem 3 may serve as a blueprint for a proof of the two conjectures. Such a proof following the footsteps of the Lemmata given above is ambitious as the strings X_n may give rise to a much larger class of matrices. The entries of the matrices $\mathbf{A}_n$ mildly depend on the order of the digits. We invite the readers of this MONTHLY to prove Conjectures 1 and 2.

7. CONCLUSIONS. We have derived an algorithm that allows us to efficiently sum the series that result when various constraints are used to delete terms from the harmonic series. The algorithm uses truncation and extrapolation and avoids futile direct summation. Embedding the problem in the language of linear algebra reveals an interesting link to graph theory. The Perron-Frobenius Theorem and Gershgorin circles enable us to present an asymptotic analysis for the special case when we delete integers having n identical digits.

Our *Mathematica* implementation can be downloaded from the webpage of the first author.[5] The core of the program fits on one page and produces 10 digits in less than 5 seconds. This is a model for good scientific computing that has recently been put forward by Trefethen [**15**].

This paper provides many opportunities for further exploration. It might be good fun to derive an algorithm for the inverse problem: what set of simple constraints might one apply in order to make the sum as close as possible to a given number?

Are these sums rational, irrational, algebraic, or transcendental? All we know is that a theorem of Borwein [**6**] implies that if the denominators that survive deletion consist of a single nonzero digit d in base B, then the resulting sum is irrational. The Nth partial sum of the harmonic series is never an integer [**10**, p. 24]. Is that true for the partial sums of the series we consider here?

[5] `http://web.comlab.ox.ac.uk/thomas.schmelzer`

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