Stochastic Machine Learning
Chapter 03 - Time series and LSTM

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Universal approximation theorems

We mention finally the two papers

- Kratsios (2021)
- Benth, Detering, and Galimberti (2023)

Time series

Dynamic phenomena are the classical frame for **time series**.

- A **time series** $Y = (Y_t)_{t \in \mathcal{T}}$ with at most countable $\mathcal{T}$ is a family of random variables.
- The state space can be quite general - for example $\mathbb{R}$, $\mathbb{R}^d$ or even a Hilbert/ Banach-space (functional time series).
- If $\mathcal{T}$ is not finite then we need to require that the state space is[1] **Polish**. Then the extension theorem of Kolmogorov says that the distribution of Y is already determined by all finite-dimensional marginal distributions (fidis), i.e. the distribution of

$$\left(Y_{t_1}, \dots, Y_{t_n}\right), \quad t_1, \dots, t_n \subset \mathcal{T}, n \in \mathbb{N}.$$

- If the random variables have densities we have that

$$f(y_1, \dots, y_n) = f(y_n | y_{n-1}, \dots, y_1) \cdots f(y_2 | y_1) \cdot f(y_1),$$

  i.e. the joint distribution is determined by the conditional probabilities. This is an important step for developing dynamic evolutions.

[1] A Polish space is a separable, completely metrizable space. If the space is not Polish, many things may go wrong since product sets become very badly behaved, see for example https://mathoverflow.net/questions/20919/polish-spaces-in-probability.

- If all fidis are Gaussian, then the process is called **Gaussian** or a **Gauss-process**.
- (In which spaces can we define Gauss distributions?)
- In this case, it is sufficient to study mean- and covariance function, i.e.

  $$m(t) = E[Y_t]$$

  and

  $$\gamma(t,h) = \mathrm{Cov}(Y_t, Y_t + h) = E\big[(Y_t - m(t)) \cdot (Y_{t+h} - m(t+h))\big].$$
- γ is called the **auto-covariance function**.
- The process Y is called **strictly stationary**, if all distributions $Y_{t+1}, \dots, Y_{t+n}$ and $Y_1, \dots, Y_n$ are identical for all $t \in \mathcal{T}$ and all $n \in \mathbb{N}$ (whenever this makes sense - this can also be formulated for general $\mathcal{T}$, but we stay a bit simpler).

Examples

- $Y_0, Y_1, \dots$ are i.i.d. Then they are also stationary. If the first two moments exist, we have that
$$m(t) = m(0) = m, \qquad \gamma(t,h) = 0$$
for any $h > 0$.
- Random walk: For (X_i) i.i.d., we define
$$Y_t = X_1 + \cdots + X_t,$$
$Y_0 = 0$. If $X_i \in \{-1, 1\}$ this is the **binomial tree**
- If we have a **linear trend** we think of
$$Y_t = m(t) + Z_t$$
where (Z_t) is a time-series with zero mean.

Trends

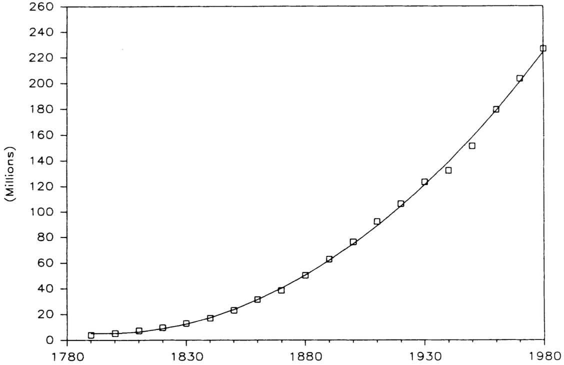


Figure 1.7. Population of the U.S.A., 1790–1980, showing the parabola fitted by least squares.

Population.png

Figure: Source: Brockwell and Davis (1991)

- If we look at this example we could propose a parametric trend:

$$Y_t = a_0 + a_1 t + a_2 t^2 + Z_t$$

- Estimation can then be done with least squares, just the the error terms now form a **time series**.

Smoothing

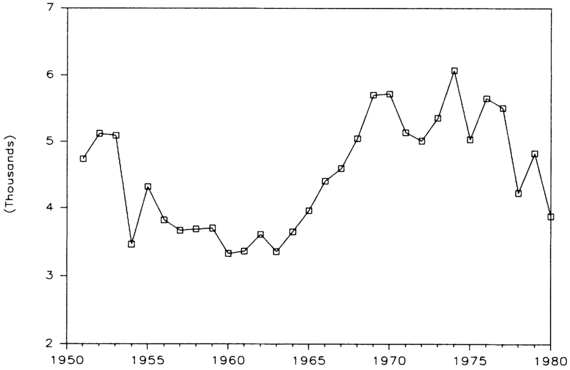


Figure 1.3. Strikes in the U.S.A., 1951–1980 (Bureau of Labor Statistics, U.S. Labor Department).

Figure: Source: Brockwell and Davis (1991)

- In this case we would like to smooth out the data
- We estimate the mean at time t with the average

$$\hat{m}(t) = \frac{1}{2q+1} \sum_{i=-q}^{q} Y_{t+i}$$

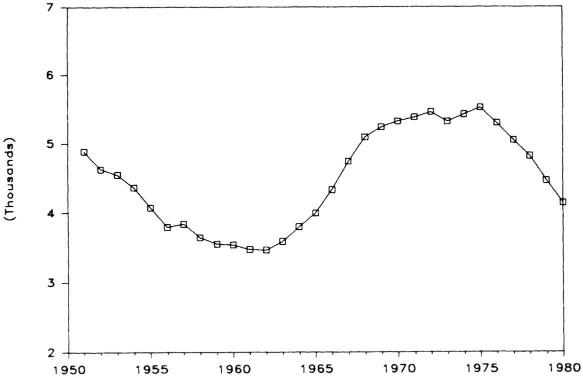


Figure 1.8. Simple 5-term moving average $\hat{m}_t$ of the strike data from Figure 1.3.

Figure: Source: Brockwell and Davis (1991)

- The smoothed time series
- The estimate $\hat{m}$ is a **linear filter** which in general looks like

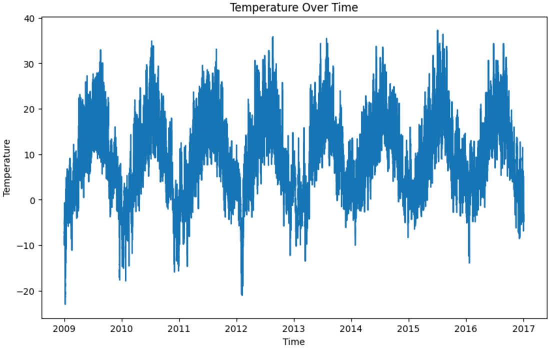
$$\sum_j a_j Y_{t-j}$$

- If q is sufficiently large then

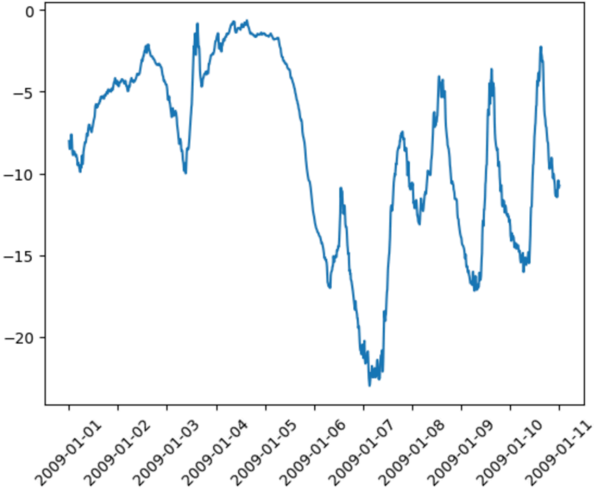
$$\hat{m} = \frac{1}{2q+1}\sum_{i=-q}^{q} m(t+i) + \frac{1}{2q+1}\sum_{i=-q}^{q} Z_{t+i} \approx m(t)$$

- The time series can also have other stylized facts.
- For example, energy prices have a strong seasonal component (estimation is non-trivial)
- Sometimes the time series has spikes which might also cause statistical difficulties to large values.
- Also effects which start at predictable times play a decisive role.

Jena dataset



- The Jena Climate dataset is a weather timeseries dataset recorded at the Weather Station of the Max Planck Institute for Biogeochemistry in Jena, Germany. `https://www.kaggle.com/datasets/mnassrib/jena-climate`



- Forecasting this time series is a classical challenge for neural networks.
- The external effects are difficult to capture obviously.

Properties

- Consider for simplicity $\mathcal{T} = \{0, 1, 2, \dots\}$. Assume that $f \in \mathcal{F}$ is a distribution determining class (for example Fourier transforms, or all continuous and bounded functions)
- The process Y is called **Markovian**, if
$$E[f(Y_t)|Y_{t-1}, \dots, Y_0] = E[f(Y_t)|Y_{t-1}]$$
for all $f \in \mathcal{F}$ and all $t \geq 1$.
- Intuitively, the distribution of Y_t does not depend on the full past but only on the previous value Y_t.
- This is often very useful and simplifies the setting - but it is also very often not true. Think of financial time series, etc.
- By enlarging the state space we can always make a process Markovian - how?

Autoregressive and moving average

- A process is called **autoregressive** of order k, if for all $t \geq 1$,
  $$E[Y_t | Y_{t-1}, \ldots, Y_1] = E[Y_t | Y_{t-1}, \ldots, Y_{t-k}].$$
- Here the expectation only depends on the last k values - however in any form. Typically one uses affine dependence only -
- The process is called **linear autoregressive**, if
  $$E[Y_t | Y_{t-1}, \ldots, Y_1] = a_0 + a_1 Y_{t-1} + \cdots + a_k Y_{t-k}.$$
  We denote the process class by **AR(k)**.
- We consider (Z_t) as i.i.d. with mean zero and variance σ^2 and would consider
  $$Y_t = \frac{1}{k} \sum_{i=1}^{k} Z_{t-k}$$
  as a moving average. This inspires the following definition.

Definition

A real-valued process Y is called **ARMA(p,q)** if it is strictly stationary and can be represented as

$$Y_t - \phi_1 Y_{t-1} - \cdots - \phi_p Y_{t-p} = c + Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}.$$

Here, $c \in \mathbb{R}$ and $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p)$ and $\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^p$ have no common factors.

We have the compact representation

$$\phi(L) Y_t = c + \theta(L) Z_t,$$

where L is the lag-operator.

Example (MA(1))

In this case we have

$$Y_t = c + Z_t + \theta_1 Z_{t-1}.$$

Moreover,

$$\gamma(t,h) = \begin{cases} \sigma^2 & h = 0 \\ \theta_1 \sigma^2 & h = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then, if Y_0 is distributed as Z_1 and independent of (Z), then Y is stationary.

Example (AR(1))

In this case, (for simplicity $c = 0$)

$$Y_t = \phi_1 Y_{t-1} + Z_t.$$

Then,

$$\gamma(t,h) = \begin{cases} \gamma(t,0) & h = 0 \\ \mathbb{E}(Y_t Y_{t+1}) = \mathbb{E}(Y_t^2 \phi_1) = \phi_1 \gamma(t,0) & h = 1 \\ \phi_1^h \gamma(t,0) & \text{otherwise.} \end{cases}$$

AR(1)

- Does there exist $Y = (Y_t)_{t\in\mathbb{Z}}$ such that Y is AR(1)?
- Let us look at the following equation

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + Z_t \\ &= Z_t + \phi_1 Z_{t-1} + \cdots + \phi_1^k Z_{t-k} + \phi_1^{k+1} Y_{t-k-1} \\ &\overset{?}{=} \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}. \qquad (1) \end{aligned}$$

In which sense do we have to understand the last equality? The answer resides on the following result (Brockwell and Davis (1991), Prop 3.1.1.):

Satz

Consider $(Z_t)_{t\in\mathbb{Z}}$ such that $\sup_{t\in\mathbb{Z}} E[Z_t^2] < \infty$ and $(\psi_i)_{i\in\mathbb{Z}}$ such that $\sum_{i\in\mathbb{Z}} |\psi_i| < \infty$. The the series

$$\psi(L)Z_t := \sum_{i\in\mathbb{Z}} \psi_i L^i Z_t$$

converges absolutely with probability one.

Since Y is stationary, $\gamma(t,0) = \gamma(0)$ and

$$\gamma(0) = \mathrm{Cov}(Y_t, Y_t) = \mathrm{Cov}(\phi_1 Y_{t-1} + Z_t, \phi_1 Y_{t-1} + Z_t) = \phi_1^2 \gamma(0) + \sigma^2.$$

We have the nice formula

$$\gamma(0) = \frac{\sigma^2}{1-\phi_1^2}.$$

- One consequence of this result is that operators as $\psi(L) = \sum_i \psi_i L^i$ inherit the properties of power series.
- In particular, if

  $$\psi_j = \sum_i \beta_{j-i}\alpha_i = \sum_i \alpha_{j-i}\beta_i$$

  (and all series converge absolutely) then

  $$\psi(L)X_t = \alpha(L)\beta(L)X_t.$$
- These manipulations will prove very useful in the following.

Existence

Proposition

A stationary solution Y for an ARMA(p,q)-process exists if and only if

$$\phi(z) \neq 0, \qquad \text{for all } z \in \mathbb{C} : |z| = 1.$$

It is interesting to have a causal solution which can be obtained if $\phi(z) \neq 0$ for all $z : |z| \leq 1$.

Proof.

With $\phi(z) \neq 0 \quad \forall |z| \leq 1$ we have

$$\frac{1}{\phi(z)} = \sum_{j=0}^{\infty} \zeta_j z^j =: \zeta(z) \qquad |z| \leq 1$$

Then

$$Y_t = \zeta(L)\theta(L)Z_t$$

and Y is causal. □

Estimation

- A weakly stationary process is determined by m and γ. We estimate these by moment methods.

  $$\hat{m}_n := \frac{1}{n}\sum_{i=1}^{n} Y_i.$$

  which is consistent and asymptotically normal.

- The estimator

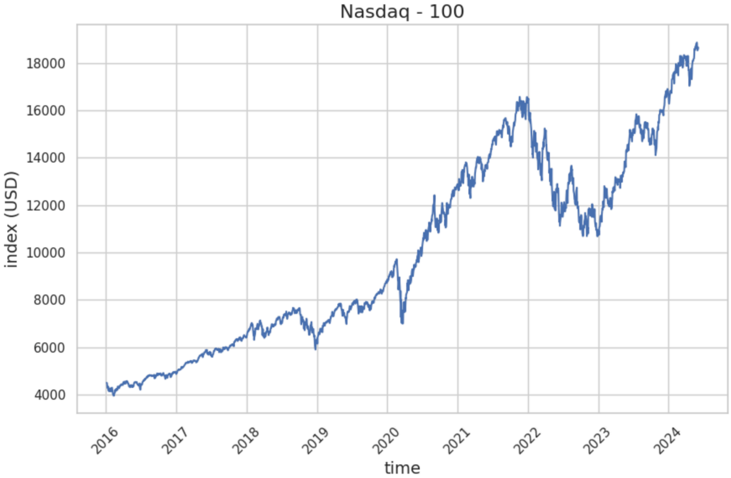
  $$\hat{\gamma}_n(h) := \frac{1}{n}\sum_{t=1}^{n-|h|}(Y_{t+|h|} - \hat{m}_n)(Y_t - \hat{m}_n)$$

  however has a number of difficulties, in particular when the sample size is to small. One estimates typically using the Yule-Walker equations which exploit better the properties of the time series.

Financial time series

- Financial time series have characteristics which differ from most other time series.
- Typically we have clusters of high and low volatility.
- It was the idea of Robert Engle to put forward a model which realizes this in a time series structure.
- **Homescedasticity** denotes a homogeneous variance, heteroscedasticity a time-varying variance.
- **Conditional heteroscdasticity**, i.e. a time varying variance conditional on the past observations is the key to model financial markets.

NASDAQ


Nasdaq - 100
index (USD)
18000
16000
14000
12000
10000
8000
6000
4000
2016
2017
2018
2019
2020
2021
2022
2023
2024
time

ARCH

- The simplest ARCH specification looks as follows: we consider (Z_t) iid with $E[Z_t] = 0$ and $\mathrm{Var}(Z_t) = 1$. For this we say Z is strict white noise and write $Z_t \sim SWN(0,1)$.
- Y is called ARCH(1) if

$$Y_t = \sigma_t Z_t \tag{2}$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2, \quad \alpha_0, \alpha_1 > 0. \tag{3}$$

- We have obviously conditional heteroscedasticity: clearly, $E[Y_t|Y_{t-1}] = 0$ and

$$\mathrm{Var}(Y_t|Y_{t-1}) = \sigma_t^2.$$

- Engle (1982) assumed additionally that $Z_t \sim \mathcal{N}(0,1)$. Then Y_t given Y_{t-1} is normally distributed.
- The unconditional distribution is obtained as mixture

$$\begin{aligned} &E\left[P(Y_t \leq x|Y_{t-1})\right] \\ &= E\left[\Phi\Big(\frac{x}{\sigma_t}\Big)\right] = \int \Phi\Big(\frac{x}{y}\Big) f_{\sigma_t}(y)dy, \end{aligned}$$

  where f_{σ_t} is the unconditional distribution of σ_t. The mixture distribution often has much fatter tails than the normal distribution, matching the observations in financial markets.

- Weak stationarity is equivalent to $\alpha_1 < 1$.

Definition

Let Z be strict white noise. Then Y is a **GARCH(p,q)** process if

(i) Y iis strictly stationary

(ii) $\forall\ t \in \mathbb{Z}$ and appropriate $\alpha_0 > 0, \alpha_i, \beta_j \geq 0 \quad i = 1, \dots, p\ j = 1, \dots, q$ it holds that

$$Y_t = \sigma_t Z_t$$

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_i \sigma_{t-i}^2.$$

Volatility

- A GARCH(p,0) process is an ARCH(p) process, of course
- The generalization is similar to AR → ARMA, but of course not the same.
- As expected Y^2 is an ARMA process:

$$Y_t^2 = E[Y_t^2|Y_{t-1}, Y_{t-2}, \ldots] + \underbrace{Y_t^2 - E[Y_t^2|Y_{t-1}, Y_{t-2}, \ldots]}_{=:V_t}$$

$$= \alpha_0 + \sum_{i=1}^{p} \alpha_i Y_{t-i}^2 + \sum_{j=1}^{q} \beta_i \sigma_{t-i}^2 + V_t.$$

- Here, $V_t = Y_t^2 - \sigma_t^2$. Inserting this, we obtain

$$Y_t^2 = \alpha_0 + \sum_{i=1}^{max(p,q)} (\alpha_i + \beta_i) Y_{t-i}^2 - \sum_{j=1}^{q} \beta_i V_{t-i}^2 + V_t,$$

  with $\alpha_i = 0$ and $\beta_j = 0$ for $i > p$ and $j > q$.
- The GARCH model leads hence to an ARMA model in the squares.

Existence and Asymptotics

- Nelson (1990) showed that for GARCH(1,1), a unique stationary solution exists if and only if $E[\log(\beta_1 + \alpha_1 Z_0^2)] < 0$.
- Bougerol and Picard in 1992 found conditions for the more general case, which however are also a bit more complicated (and skipped here)

Maximum-Likelihood

Define the **Likelihoodfunction** (conditional on Y_0) through

$$L(\boldsymbol{y};\theta) = \prod_{t=1}^{T} f_t(y_t|y_{t-1},\ldots,y_0;\theta).$$

As shorthand for the probably more precise notation

$$L_{Y_1,\ldots,Y_T|Y_0}(y_1,\ldots,y_T|y_0;\theta) = \prod_{t=1}^{T} f_t(y_t|y_{t-1},\ldots,y_0;\theta).$$

The ML estimator for θ is given by

$$\hat{\theta}_T := \arg\max_{\theta} L(\boldsymbol{Y};\theta) = \arg\max_{\theta} \log L(\boldsymbol{Y};\theta).$$

If the true conditional density f is replaced by a normal density, one calls the approach a **quasi-maximum likelihood estimator**.

- Let us be more precise: we additionally match the first two moments.
- Hence, we replace the density of $Y_t|Y_{t-1},\dots$ by a normal density with mean

  $$E[Y_t|Y_{t-1},\dots]$$

  and variance

  $$\mathrm{Var}(Y_t|Y_{t-1},\dots).$$

- Let us denote this density by $\tilde{f}_t(y_t|y_{t-1},\dots;\theta)$ and define the QMLE by

  $$\tilde{L}(y;\theta) := \prod_{t=1}^{T} \tilde{f}_t(y_t|y_{t-1},\dots,y_0;\theta)$$

  through

  $$\tilde{\theta}_T := \arg\max_{\theta} \tilde{L}(\boldsymbol{Y};\theta).$$

- Consistency and asymptotic normality can be obtained under fairly general conditions, see Berkes, Horváth, and Kokoszka (2003)
- For example, if a bit higher then second moments on Z_0 exist and Z_0 satisfies

  $$\lim_{x\to 0} \frac{P(Z_0^2 \leq x)}{x^c} = 0$$

  for some $c > 0$, then the QMLE is consistent, i.e. $\hat{\theta}_n \to \theta$ a.s.
- With a bit more than 4th moments, also asymptotic normality follows, i.e.

  $$\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathscr{L}} \mathcal{N}(0,\Sigma)$$

ARMA-GARCH

- For complex model we will mix the two approaches as follows.

Definition

Consider Z as $SWN(0, \sigma^2)$. A process Y is called ARMA (p_1, q_1)-process with GARCH (p_2, q_2)-errors, if

$$
\begin{aligned}
Y_t &= \mu_t + \sigma_t Z_t \\
\mu_t &= \mu + \sum_{i=1}^{p_1} \phi_i (Y_{t-i} - \mu) + \sum_{j=1}^{q_1} \beta_j \sigma_{t-j} Z_{t-j} \\
\sigma_t^2 &= \alpha_0 + \sum_{i=1}^{p_2} \alpha_i (Y_{t-i} - \mu)^2 + \sum_{j=1}^{q_2} \beta_j \sigma_{t-j}^2 .
\end{aligned}
$$

Here is $\alpha_0 > 0, \alpha_i, \beta_i \geq 0$ and $\sum \alpha_i + \beta_j < 1$.

There are many more approaches, asymmetric approaches and so on. We first discuss the estimation of the GARCH models.

Higher dimensions

- How do we come to higher dimensions ?
- We can build factor models
- We can assume component-wise GARCH and assume that the errors have a joint normal distribution or a more general distribution.
- Several multivariate extensions have been proposed.
- We therefore shortly dive into the concept of multivariate random variables.

- Benth, Fred Espen, Nils Detering, and Luca Galimberti (2023). „Neural networks in Fréchet spaces“. In: **Annals of Mathematics and Artificial Intelligence** 91.1, pp. 75–103.
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- Kratsios, Anastasis (2021). „The universal approximation property: characterization, construction, representation, and existence“. In: **Annals of Mathematics and Artificial Intelligence** 89.5, pp. 435–469.