5 Numerical Experiment for regular nodes distribution

In this section we will demonstrate the efficiency of the method on several typical example. All computations are done with 1000 nodes. Computation done with higher order method could be found in [8], thus we can convince ourselves that the method worked well.
I also recall that more results and some animations are available at https://github.com/tschmoderer/euler-prj.

5.1 Error and convergence

Because we don't know the analytic solution we adopt the *Manufactured Solution* strategy to check the convergence rate of the method :

1. We choose $q(x,t) \in \mathbb{R}^3$ as smooth as possible to be the solution of equations (1).
2. We put q in the Euler's equations and get, in general, a non zeros result (otherwise, it will mean we have an analytical solution, which is in general not possible). Let's call the rest $\mathcal{S}(x,t)$:
$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = \mathcal{S}$$
3. We now consider the modified problem :
$$\boxed{\begin{gathered}\frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} = \mathcal{S} \\ U(x,0) = q(x,0) \\ \text{Plus boundary conditions to be discusses bellow}\end{gathered}} \tag{9}$$
For which we know the solution to be q.
4. We apply our method to this problem. It requires minor modification in the code to add the source term.
5. We can now compare the numerical solution to the analytical one. And compute the convergence rate.

A common choice in this approach is to choose periodic boundary conditions in keep smooth solutions (which is not systemically the case with wall conditions). Two choice of q are presented bellow in order to be certain that the method is of order 2.

5.1.1 Case 1

This case could be found in [9] (example 3.3).
For this first case we want the solution to be :

$$q = \begin{cases} \rho(x,t) &= 1 + 0.2\sin(2\pi(x-t)) \\ u(x,t) &= 1 \\ P(x,t) &= 1 \end{cases}$$

The solution have to be periodic that's why the choice of the sin function is natural to get both smooth and periodic properties. Then a little computation using the equation of state gives the solution for the energy :

$$E(x,t) = \frac{\gamma+1}{2(\gamma-1)} + 0.1\sin(2\pi(x-t))$$

Hence the initial conditions are the following :

$$
\begin{aligned}
\rho(x,0) &= 1 + 0.2\sin(2\pi x) \quad \forall x \in \Omega \\
u(x,0) &= 1 \quad \forall x \in \Omega \\
P(x,0) &= 1 \quad \forall x \in \Omega
\end{aligned} \tag{10}
$$

In this particular case, the source term is identically zero :

$$
\mathcal{S}(x,t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

Thus, here we got an analytical solution to the problem (which enough remarkable).
We make computation for 100, 200, 300, 400, 600, 800, 1200, 1600, 2400, 3200, 4000 6400, 9600, 12800, 19200, 25600, 38400, 51200 nodes and up to 1000 iterations. Hence we are able to compare the exact solution for the density and the numerical approximation. Figures 7 and 8 show the log of the error for the L^1 and L^∞ norms against the log of the number of nodes.

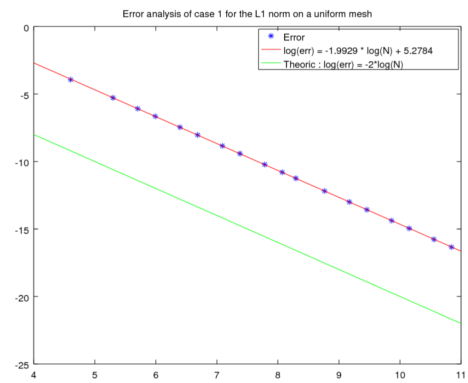


FIGURE 7 – Convergence rate for manufactured solution # 1 in L^1 norm

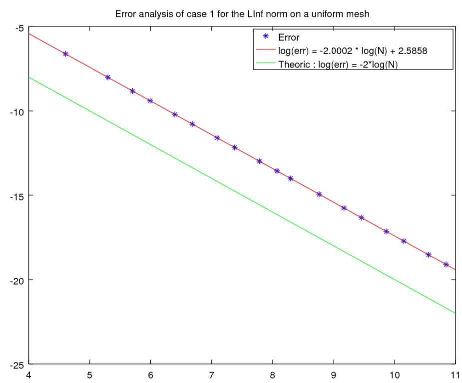


FIGURE 8 – Convergence rate for manufactured solution # 1 in L^∞ norm

Where the L^1 and L^∞ norms are the following :

$$
\|U^{ex} - U^a\|_1 = \sum_{j=1}^{N} |U^{ex}(x_j) - U^a_j| \qquad \|U^{ex} - U^a\|_\infty = \max_{j=1..N} |U^{ex}(x_j) - U^a_j|
$$

The log $error$ perfectly fit a -2 slope line. Hence as we hoped, the method seems to be of order 2.

5.1.2 Case 2

The idea behind this case comes mainly from [10]
The second case will comfort us in the fact tat the method is of order 2. We want the

solution to be :

$$q = \begin{cases} \rho(x,t) &=& 2 + 0.1\sin(2\pi(x-t)) \\ u(x,t) &=& 1 \\ E(x,t) &=& 2 + 0.1\cos(2\pi(x-t)) \end{cases}$$

Then, the initial conditions are the following :

$$\begin{aligned} u(x,0) &= 1 \quad \forall x \in \Omega \\ \rho(x,0) &= 2 + 0.1\sin(2\pi x) \quad \forall x \in \Omega \\ P(x,0) &= \frac{\gamma - 1}{20}\left(20 + 2\cos(2\pi x) - sin(2\pi x)\right) \quad \forall x \in \Omega \end{aligned} \tag{11}$$

And the source term is given by :

$$\mathcal{S}(x,t) = (1-\gamma)\pi(2\rho(x,t) + E(x,t) - 6)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{12}$$

After modifications of the code to handle the source term, we again make computation for 100, 200, 300, 400, 600, 800, 1200, 1600, 2400, 3200, 4000 6400, 9600, 12800, 19200, 25600, 38400, 51 nodes with step of 100 nodes up to 1000 iterations. Hence we are able to compare the exact solution and the numerical approximation. Figures 9 and 10 shows the log of the error against the log of the number of nodes.

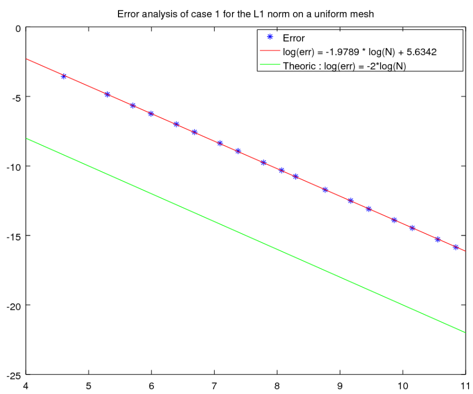


FIGURE 9 – Convergence rate for manufactured solution # 2 in L^1 norm

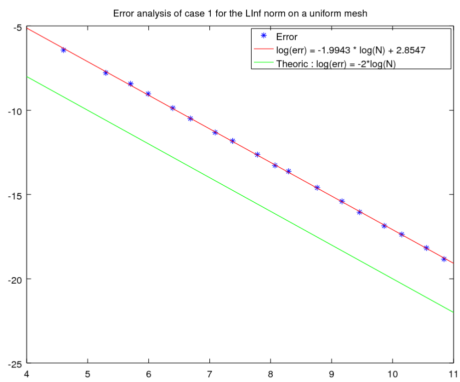


FIGURE 10 – Convergence rate for manufactured solution # 2 in L^∞ norm

Again the convergence rate is really near 2. We can conclude that the method it of order 2 as announced.

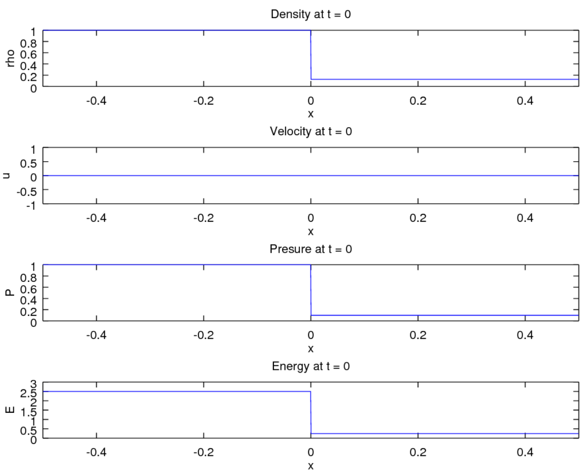
Having satisfying ourselves with the convergence rate we can now observe the method working on several typical example. For each example, I will try to give a real-life situation where the phenomenon describe could occur.

5.2 Sod shock tube

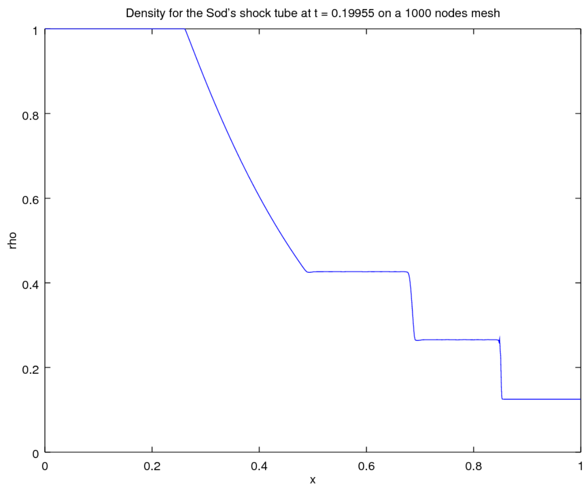
We begin our collection of test with a popular one. Imagine a pipe joining a gas tank from a factory (high pressure and density of gas) to a city (small pressure and density), at $t = 0$ the distribution center decide to open a valve to bring gas to the city.
The domain is $\Omega = [0, 1]$, and the initial conditions are :

$$\begin{aligned} u(x,0) &= 0 \qquad \forall x \in \Omega \\ \rho(x,0) &= \begin{cases} 1.0 & x \in [0, 0.5] \\ 0.125 & x \in [0.5, 1] \end{cases} \\ P(x,0) &= \begin{cases} 1.0 & x \in [0, 0.5] \\ 0.1 & x \in [0.5, 1] \end{cases} \end{aligned} \tag{13}$$

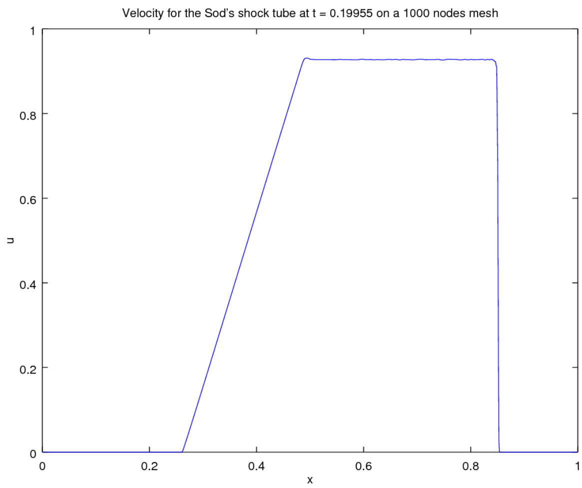
On this we apply wall boundary conditions and look at the result around $t = 0.2$.



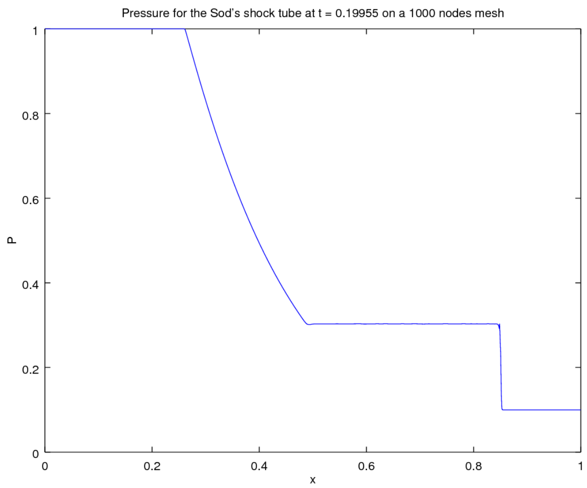
(a) Initial conditions



(b) Density



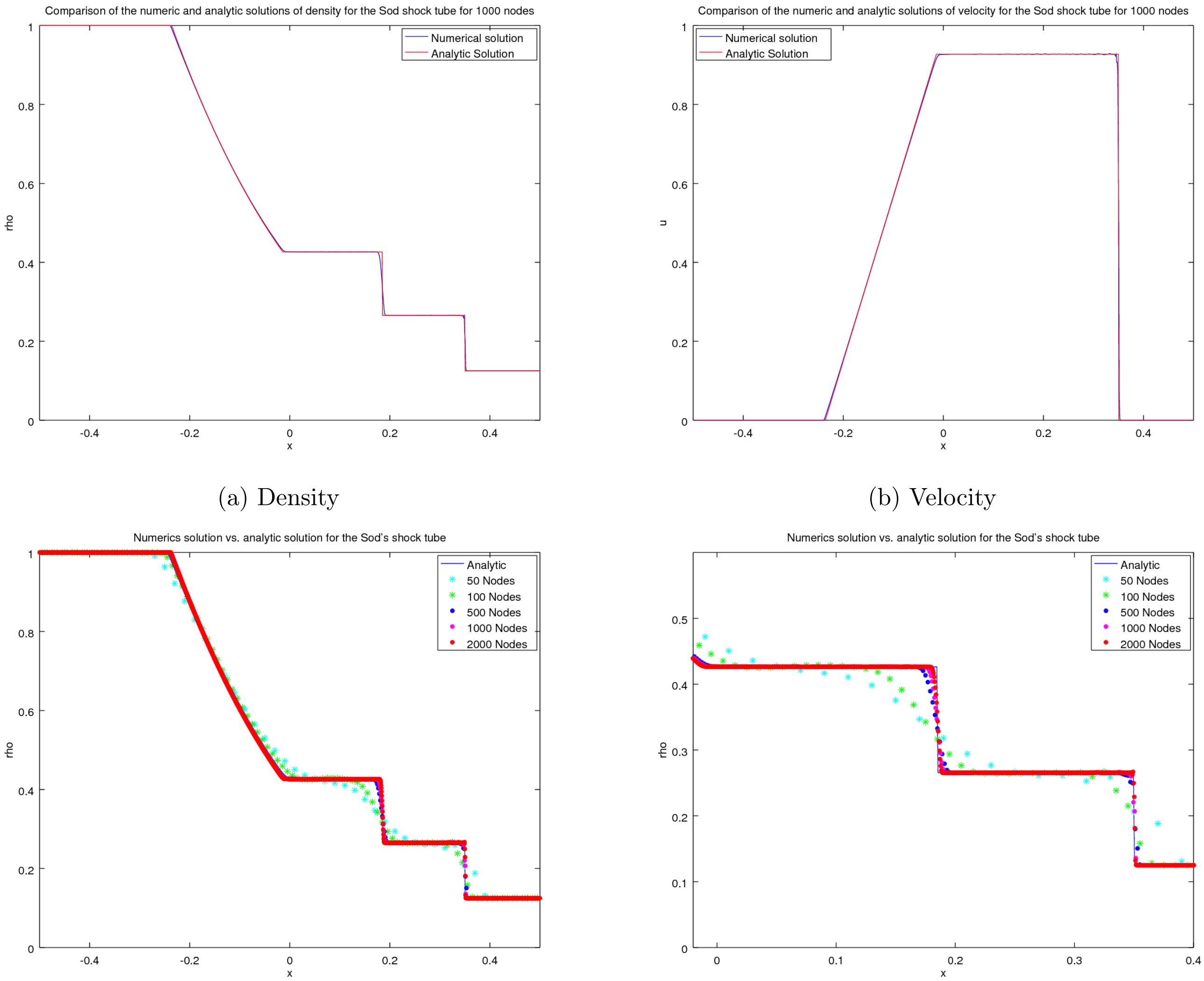
(c) Velocity



(d) Pressure

FIGURE 11 – Result for the Sod's shock tube

What we observe from this simulation, relative to our gas-factory problem is that a shock wave is going to the city and if (maybe) device are not calibrate strong enough this wave could damage some material. Another interesting fact is the rarefacting wave going backward to the factory.
Based on [11] (and the code in [12]) we can compute the analytic solution for this case. On the figure bellow we can see the difference between the analytic solution and the approximation we computed :



(a) Density

(b) Velocity

(c) Multi nodes

(d) Multi nodes zoom

FIGURE 12 – Result for the Sod's shock tube with the analytic solution and for different mesh size

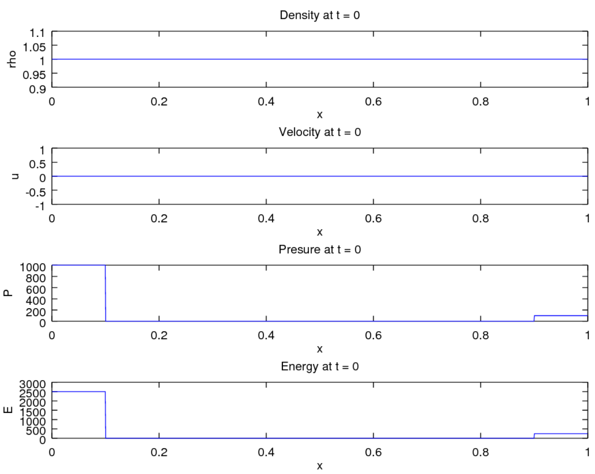
As we can see the method is not perfect. However the results are quiet really good and close to the analytic solution. We see on figures 12-(c) and 12-(d) that the more there is nodes, the closest is the numerical result to the analytic solution. This valid our intuition that the method converged.

5.3 Interacting blast wave

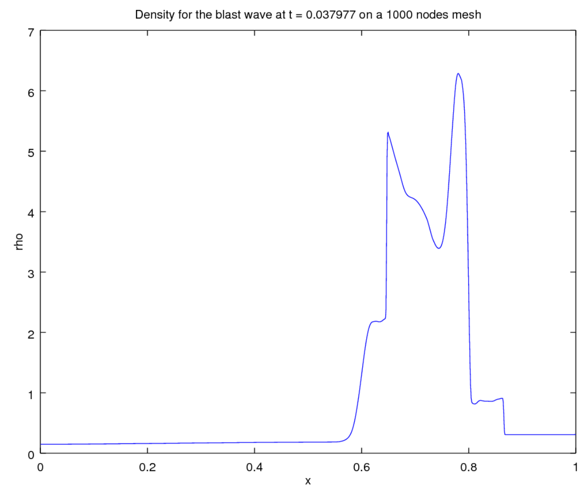
See [13] and [9] (example 3.7). Almost the same story as before, but this time there is two gas tank providing gas to the city, and two valves are open at $t = 0$.
The initial conditions are :

$$
\begin{aligned}
u(x,0) &= 0 \qquad \forall x \in \Omega \\
\rho(x,0) &= 1 \qquad \forall x \in \Omega \\
P(x,0) &= \begin{cases} 1000 & x \in [0, 0.1] \\ 0.01 & x \in [0.1, 0.9] \\ 100 & x \in [0.9, 1] \end{cases}
\end{aligned}
\tag{14}
$$

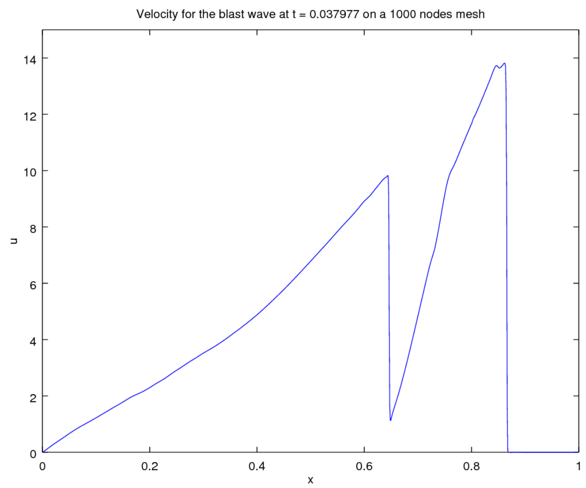
We use again solid wall boundary conditions and watch the result around $t = 0.038$.



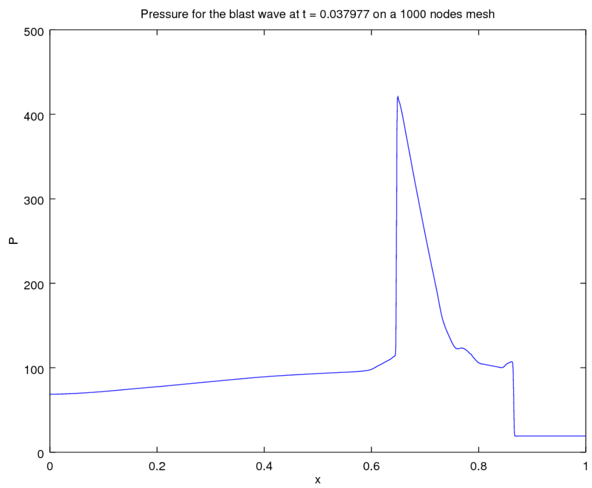
(a) Initial conditions



(b) Density



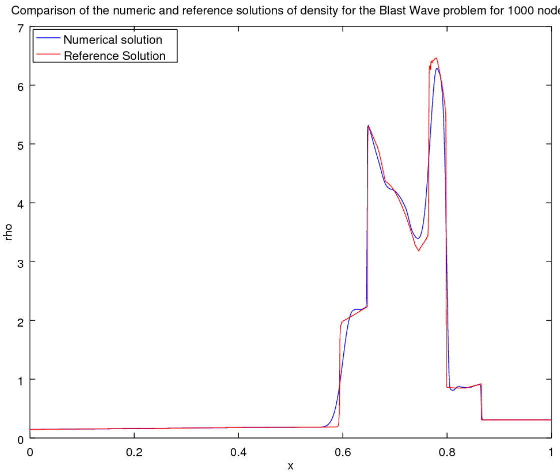
(c) Velocity



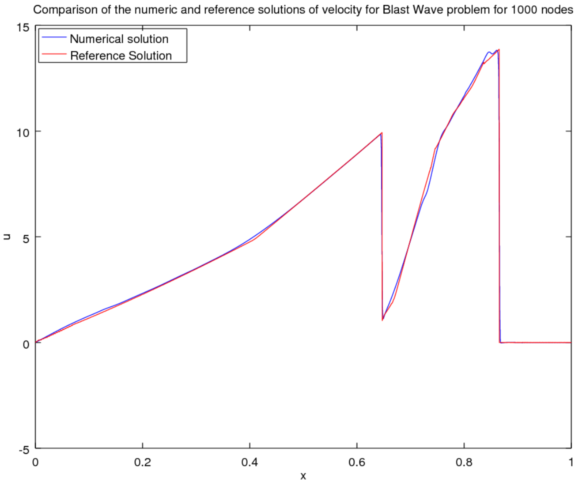
(d) Pressure

FIGURE 13 – Result for the blast wave

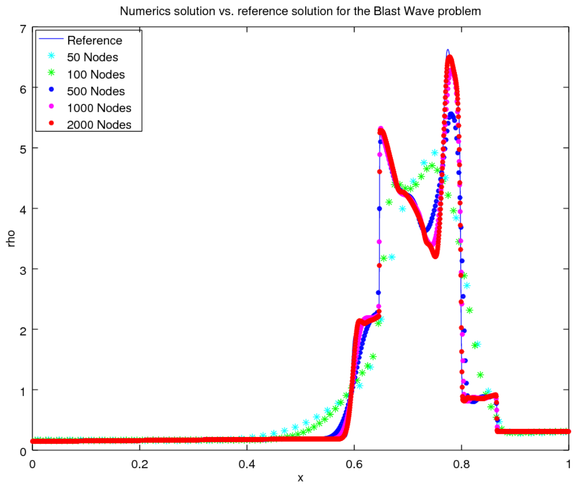
The conclusion of our story is that better not to put a weak device arroun $x = 0.6$. Unfortunately there is no analytic solution for this case. Hence, in order to observe the convergence of the method I used a WENO-5 method, from [14], to compute a reference solution with 16000 nodes.



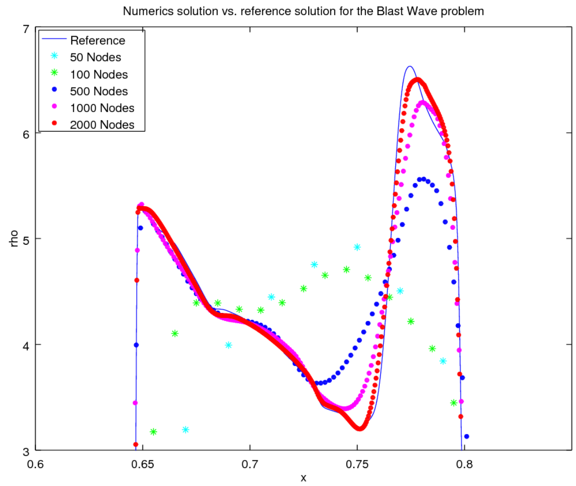
(a) Density



(b) Velocity



(c) Multi nodes



(d) Multi nodes zoom

FIGURE 14 – Result for the blast wave problem with the reference solution and for different mesh size

We can observe that the solution computed is close from the reference one. But we got a a bit more errors than in the previous case. This is probably due to the complexity of the case and the numerical pollution from Matlab execution. However we get the general forms of the solution and the shocks are well disposed so the method did not compute all wrong.

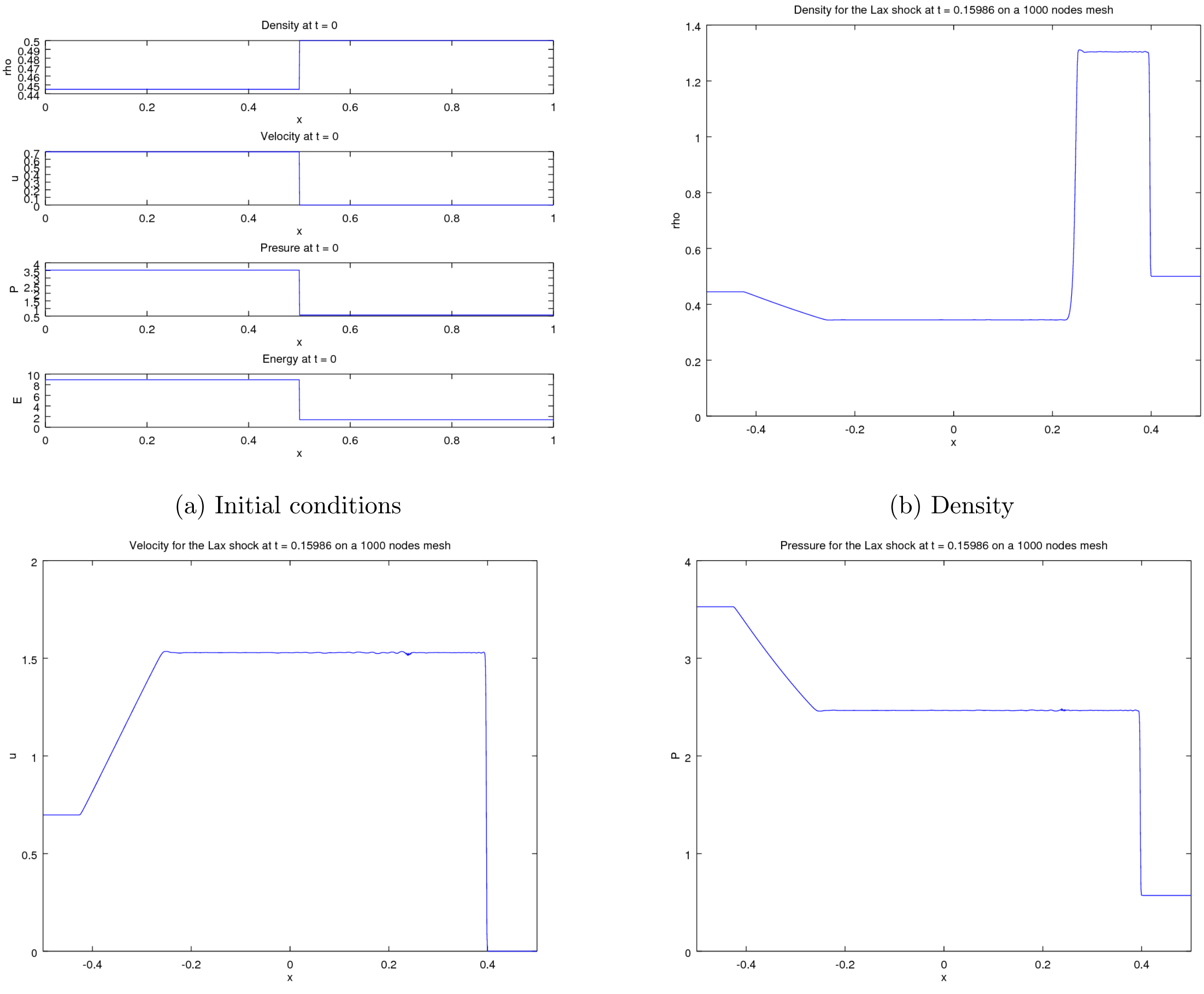
5.4 Lax's shock tube

See [9] (example 3.5). The initial conditions are :

$$
\begin{aligned}
u(x,0) &= \begin{cases} 0.698 & x \in [0,0.5] \\ 0 & x \in [0.5,1] \end{cases} \\
\rho(x,0) &= \begin{cases} 0.445 & x \in [0,0.5] \\ 0.5 & x \in [0.5,1] \end{cases} \\
P(x,0) &= \begin{cases} 3.528 & x \in [0,0.5] \\ 0.571 & x \in [0.5,1] \end{cases}
\end{aligned}
\tag{15}
$$

We use inlet conditions on the left and solid wall conditions in the right boundary and watch the result around $t = 0.16$. Inlet conditions are :

$$u(0,t) = 0.698 \quad \rho(0,t) = 0.445 \quad P(0,t) = 3.528$$



(a) Initial conditions

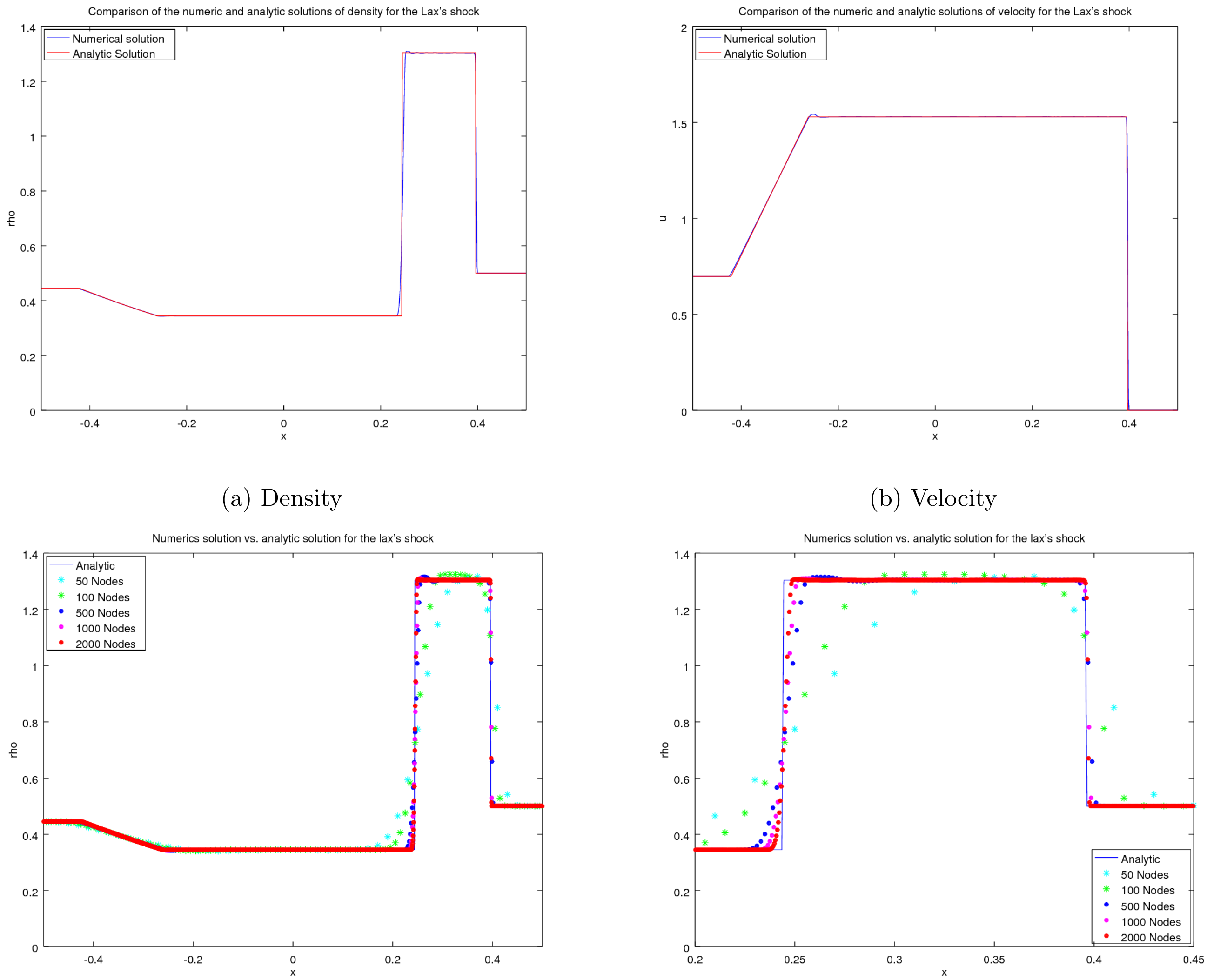
(b) Density

(c) Velocity

(d) Pressure

FIGURE 15 – Result for the Lax's shock tube

For this case, as for the Sod's shock tube, there exists an analytic solutions. Thus we can compare our method to this solution.



(a) Density

(b) Velocity

(c) Multi nodes

(d) Multi nodes zoom

FIGURE 16 – Result for the Lax's shock tube with the analytic solution and for different mesh size

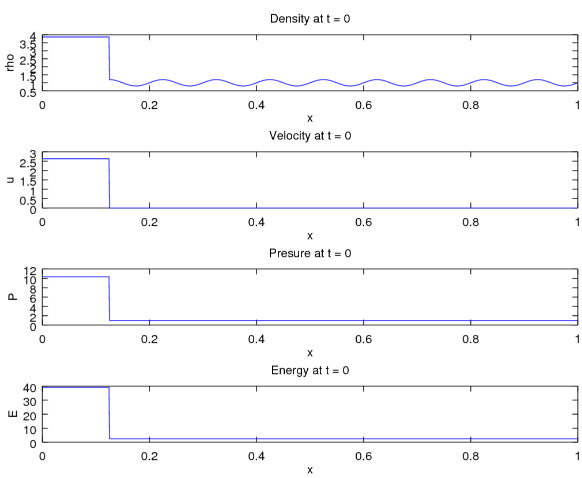
Again, we observe that the method fit well the analytic solution and captured the shocks.

5.5 Shu-Osher's problem

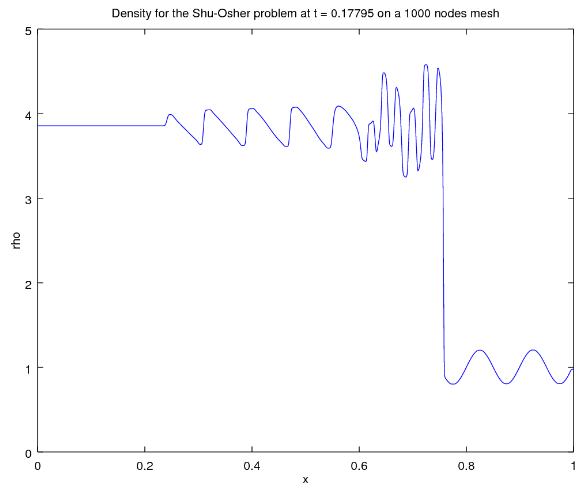
See [15] and [9] (example 3.6). The initial conditions are :

$$
\begin{aligned}
u(x,0) &= \begin{cases} 2.629369 & x \in [0, 0.125] \\ 0 & x \in [0.125, 1] \end{cases} \\
\rho(x,0) &= \begin{cases} 3.857143 & x \in [0, 0.125] \\ 1 + 0.2\sin(20\pi x) & x \in [0.125, 1] \end{cases} \\
P(x,0) &= \begin{cases} 31/3 & x \in [0, 0.125] \\ 1 & x \in [0.125, 1] \end{cases}
\end{aligned}
\tag{16}
$$

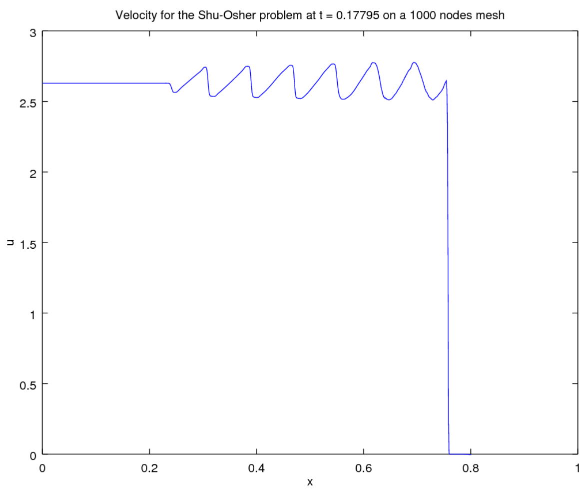
This time we use inlet conditions on the left and outflow conditions on the right. The inlet conditions are :



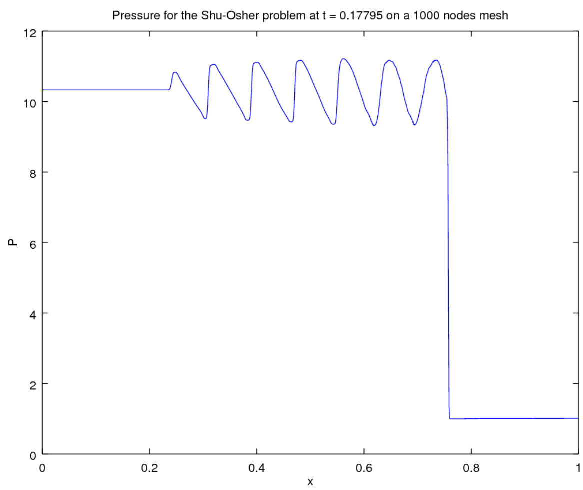
(a) Initial conditions



(b) Density



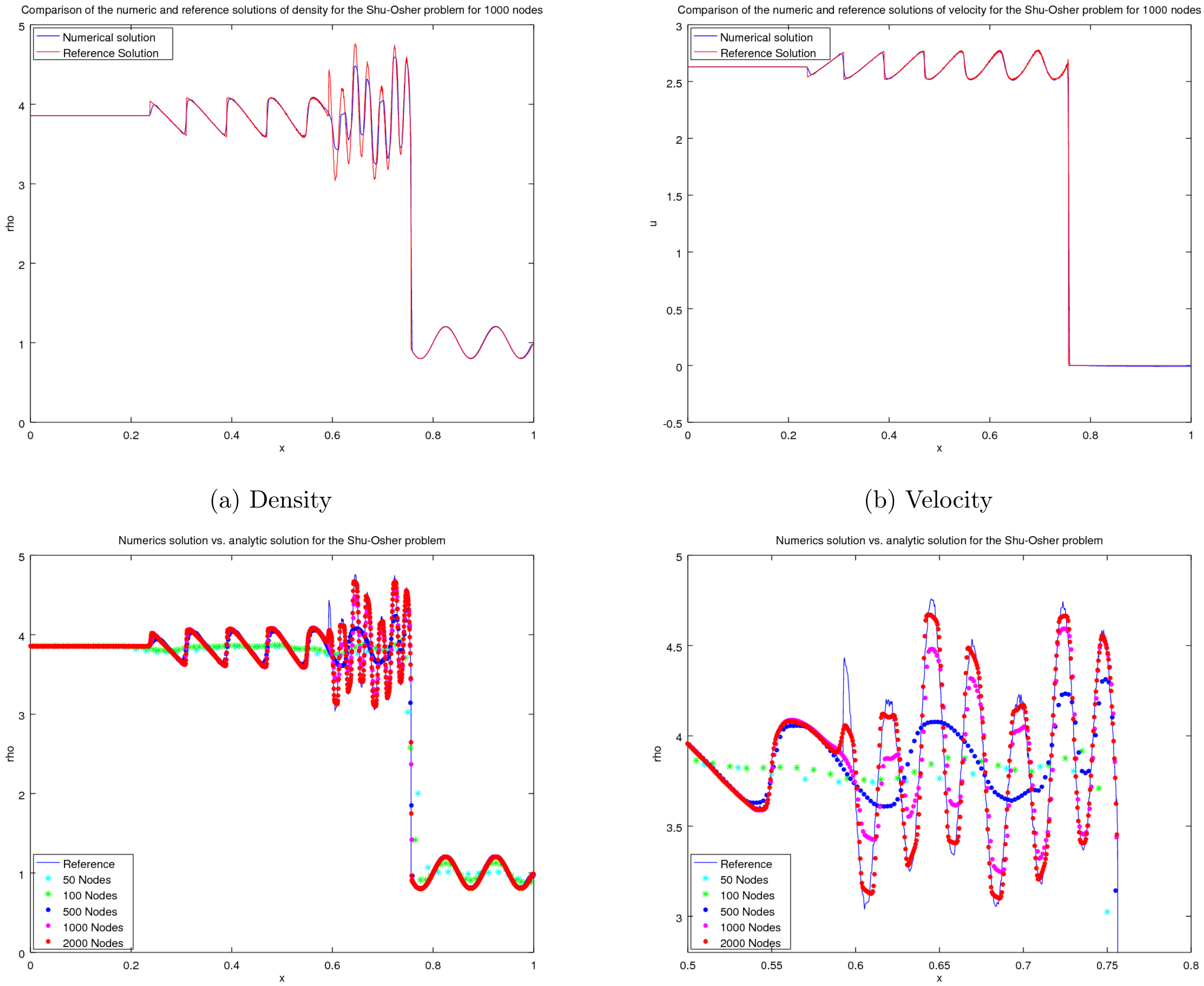
(c) Velocity



(d) Pressure

FIGURE 17 – Result for the Shu-Osher's problem

Again we don't know any analytic solution for this case, so the reference solution is given by the WENO-5 method on 16000 nodes.



(a) Density

(b) Velocity

(c) Multi nodes

(d) Multi nodes zoom

FIGURE 18 – Result for the Shu-Osher problem with the reference solution and for different mesh size

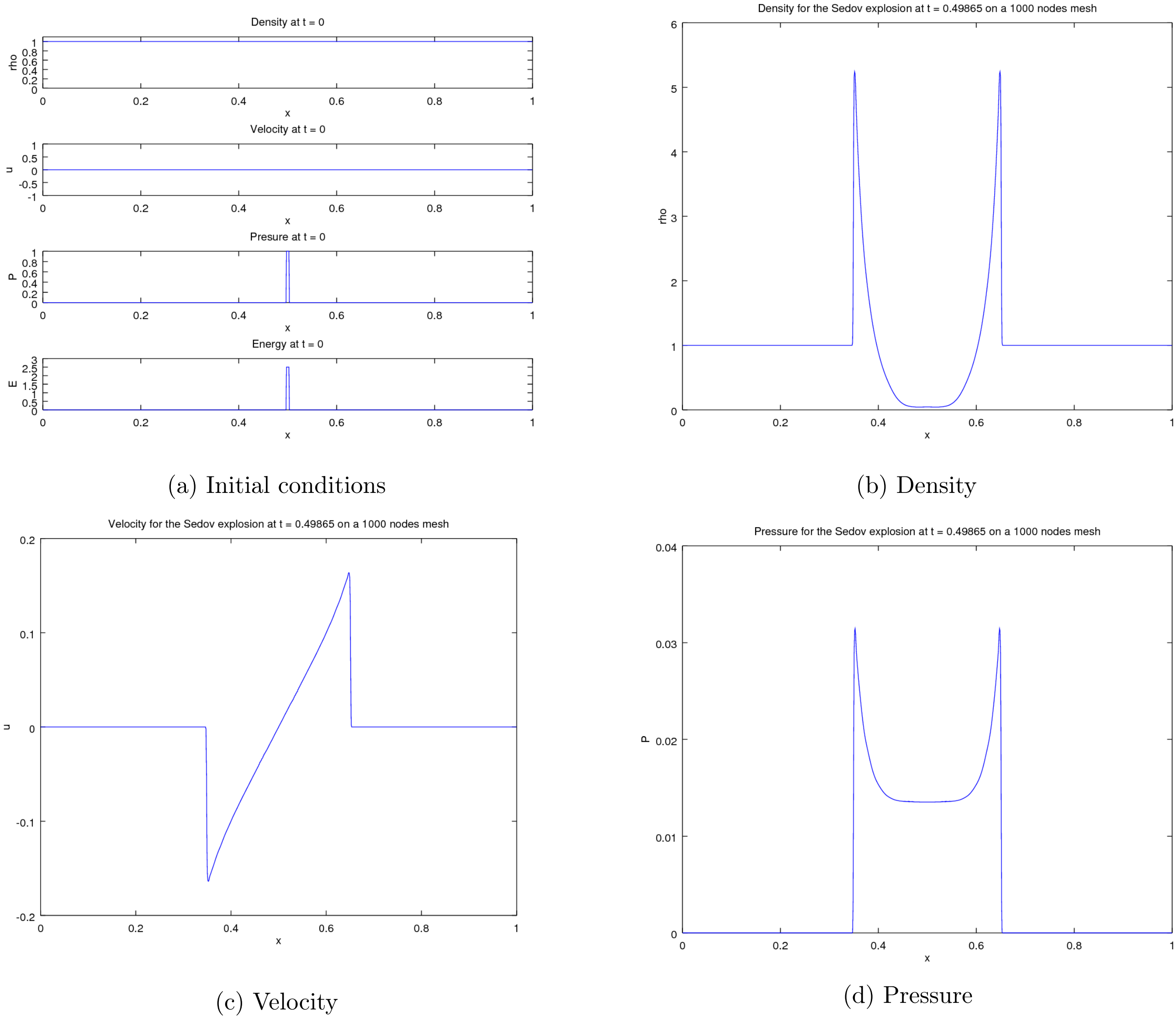
We definitely observe that the method converge to the reference solution.

5.6 Sedov explosion

The initial conditions are :

$$
\begin{aligned}
u(x,0) &= 0 \qquad \forall x \in \Omega \\
\rho(x,0) &= 1 \qquad \forall x \in \Omega \\
P(x,0) &= \begin{cases} 1 & x \in [0.5 - 3.5\dfrac{\Delta x}{2}, 0.5 + 3.5\dfrac{\Delta x}{2}] \\ 10^{-5} & \text{else} \end{cases}
\end{aligned}
\tag{17}
$$

We use again solid wall boundary conditions and watch the result around $t = 0.005$.



(a) Initial conditions

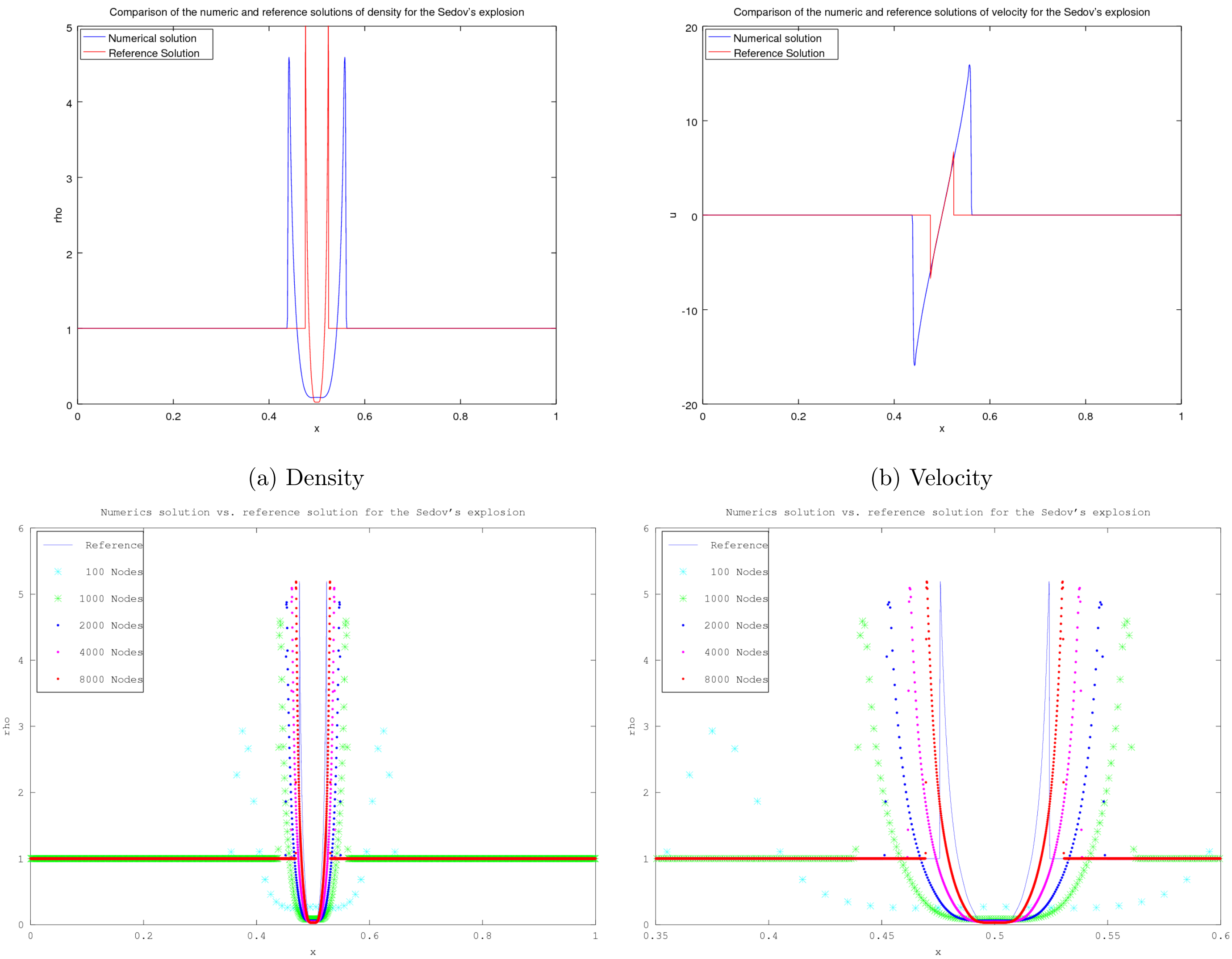
(b) Density

(c) Velocity

(d) Pressure

FIGURE 19 – Result for the Sedov's explosion

Again we don't know any analytic solution for this case, so the reference solution is given by the WENO-5 method on 16000 nodes. We watch the result at $t = 0.002$.



(a) Density

(b) Velocity

(c) Multi nodes

(d) Multi nodes zoom

FIGURE 20 – Result for the Sedov explosion with the reference solution and for different mesh size

This case is very special, because of the sharp discontinuities a lot more of nodes are needed to get a good approximation of the discontinuities.
It is also a good example of nodes-waste : to get a good approximation we should compute on a very coarse grid near the shock front but we don't need a lot of nodes in the extremities. Here an adaptive mesh would improve the execution time and with the same number of nodes (but better disposed) we could achieve greater accuracy.

7 Numerical experiment for non regular nodes distribution

In the next section we will compute the convergence rate of the method on the non uniform grid. Then we will see the method on only two examples : the Blast Wave 5.3 and the Shu-Osher problem 5.5, because the pictures are essentially the same. We do not present again the initial conditions which are the same as in section 5.
Before beginning, a preliminary remarks about Δt. As we saw in section 3.4, the time step depends on the mesh size. Because we don't have a regular mesh here, we use the following integration time step which guaranty that we don't exceed the Courant's number and let the method stable :

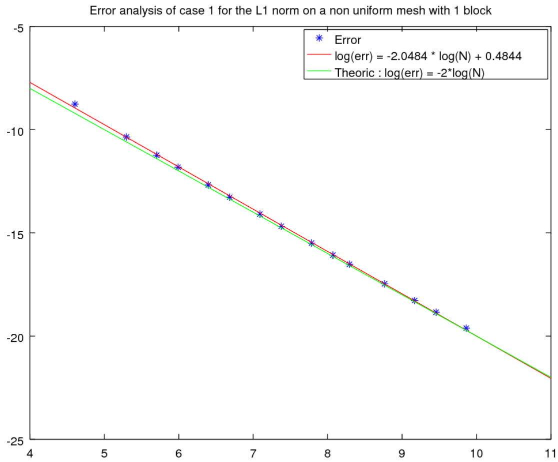
$$\Delta t^n = \frac{\min \Delta x}{\max a^n}$$

7.1 Error and convergence

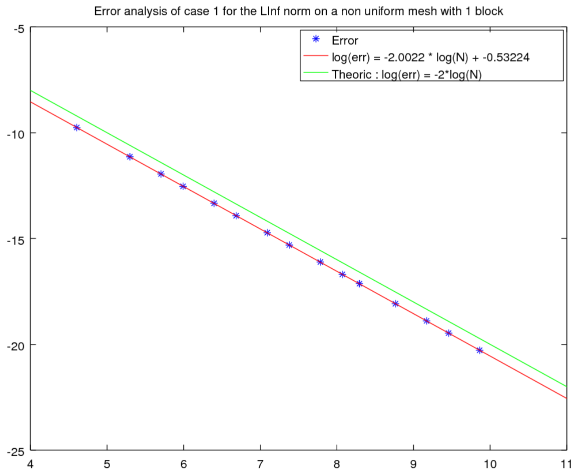
As before we computed the convergence rate with the method of the manufactured solutions. We take the two same previous cases :

$$\begin{aligned}
&\text{Case 1 : } \begin{cases} \rho(x,0) = 1 + 0.2\sin(2\pi x) \quad \forall x \in \Omega \\ u(x,0) = 1 \quad \forall x \in \Omega \\ P(x,0) = 1 \quad \forall x \in \Omega \end{cases} \\
&\mathcal{S}_1(x,t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
&\text{Case 2 : } \begin{cases} u(x,0) = 1 \quad \forall x \in \Omega \\ \rho(x,0) = 2 + 0.1\sin(2\pi x) \quad \forall x \in \Omega \\ P(x,0) = \dfrac{\gamma - 1}{20}\left(20 + 2\cos(2\pi x) - sin(2\pi x)\right) \quad \forall x \in \Omega \end{cases} \\
&\mathcal{S}_2(x,t) = (1-\gamma)\pi(2\rho(x,t) + E(x,t) - 6)\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}
\end{aligned} \tag{20}$$

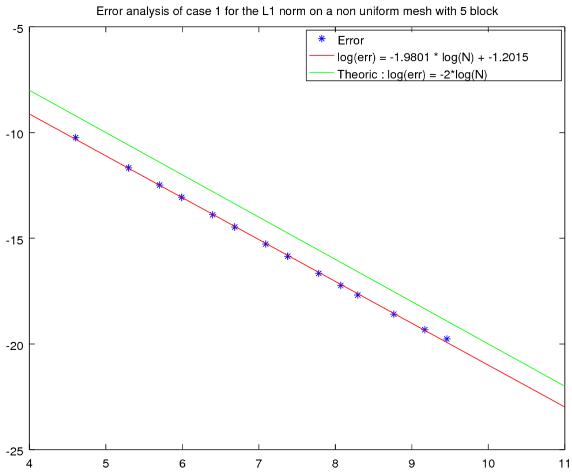
The results are quiet surprising. The convergence rate is the same as for the uniform case. It is surprising that we got such perfect fitting to straight lines. We could imagine that due to this particular distribution, the error would fluctuate more.
It also seems that the convergence rate doesn't depends on the number of blocks.



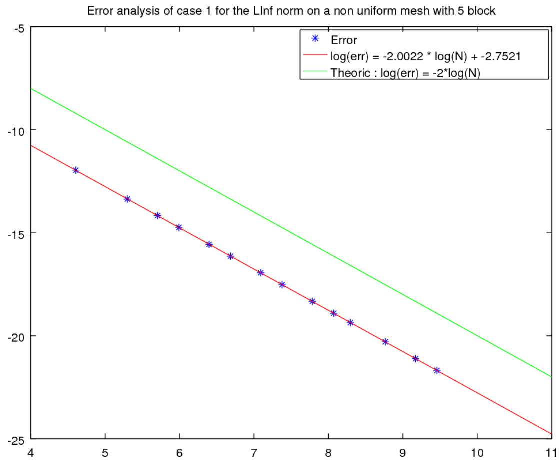
(a) Case 1 : Error L^1 with 1 block



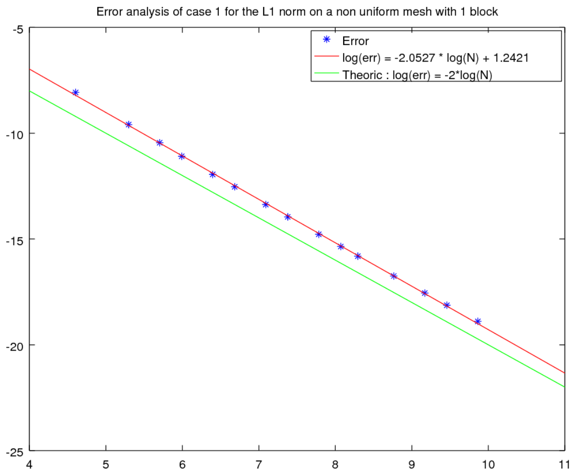
(b) Case 1 : Error L^∞ with 1 block



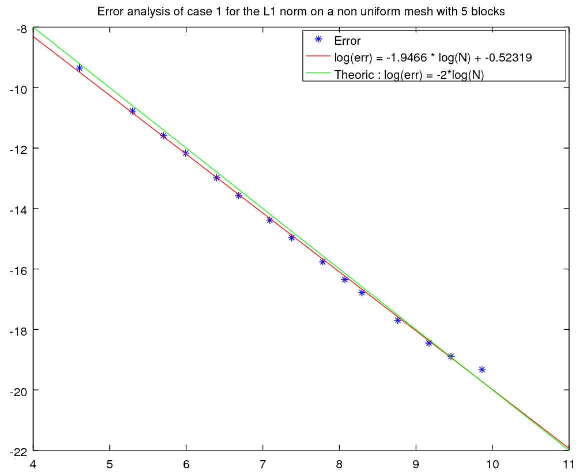
(c) Case 1 : Error L^1 with 5 blocks



(d) Case 1 : Error L^∞ with 5 blocks



(e) Case 2 : Error L^1 with 1 block



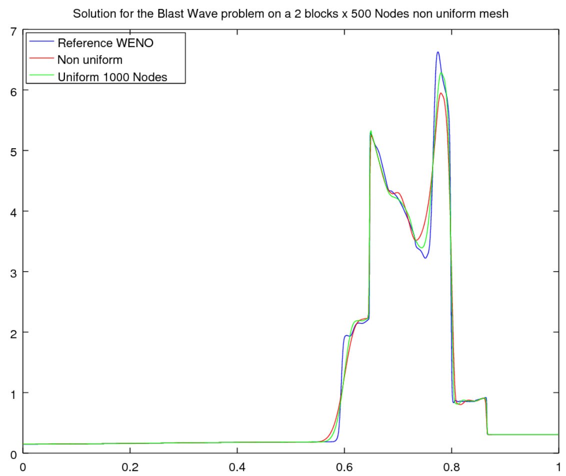
(f) Case 2 : Error L^1 with 5 blocks

FIGURE 22 – Error analysis for non uniform meshes

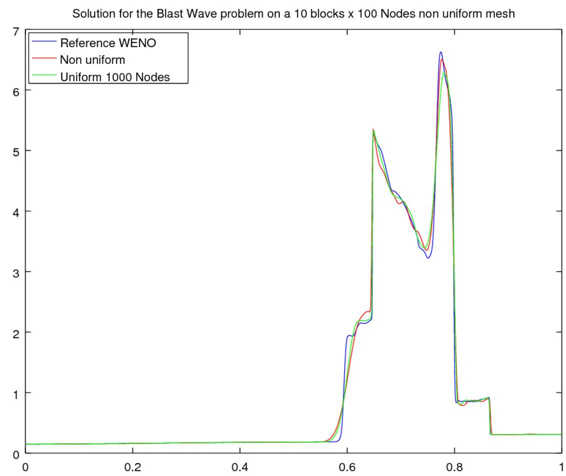
7.2 Interacting Blast Wave

As announced before, the next page will show the result computed for the interacting blast wave problem. We show the result for different number of blocks but we always kept the total number of nodes to be 1000.
On each figure are plotted the reference solution computed with WENO (16000 nodes), the solution computed on a 1000 uniform mesh and the solution on the non uniform mesh.

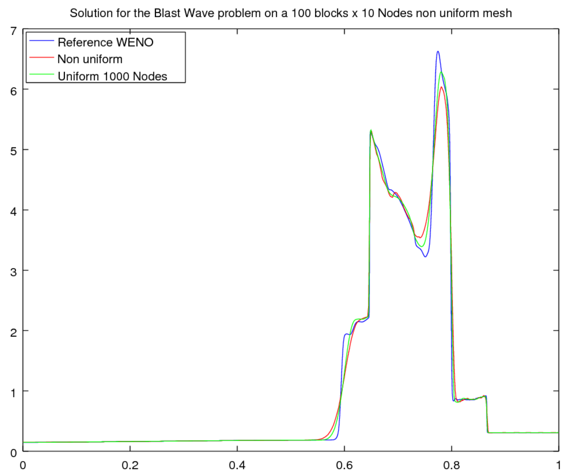
My observation are the following, the method converged well on this grid and the shock are well disposed. This solution seems to be smoother than the uniform-one. According to me, that's the reason why the non-uniform-solution is less accurate than the other (particularly on the right pick). Another little inconvenient of the method, is that the execution time is a bit longer due to the dense distribution on the extremities (the ore there is nodes in a block the most time it took) which give a small Δt.



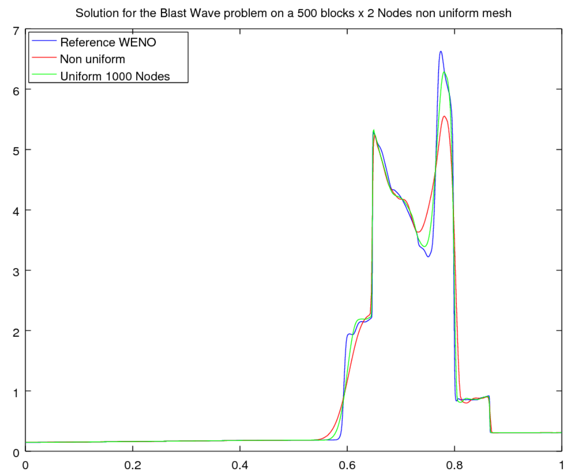
(a) Blast Wave with 2 blocks with 500 nodes each



(b) Blast Wave with 10 blocks with 100 nodes each



(c) Blast Wave with 100 blocks with 10 nodes each

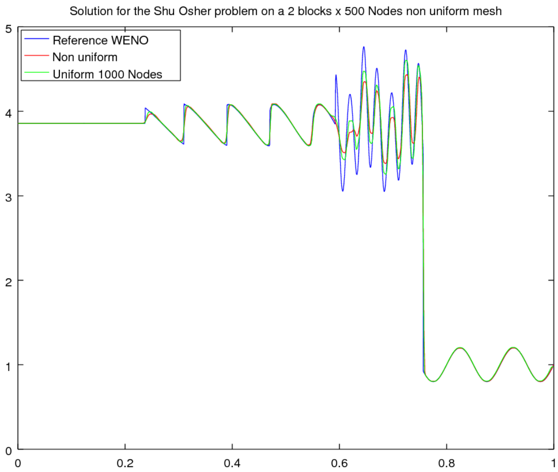


(d) Blast Wave with 500 blocks with 2 nodes each

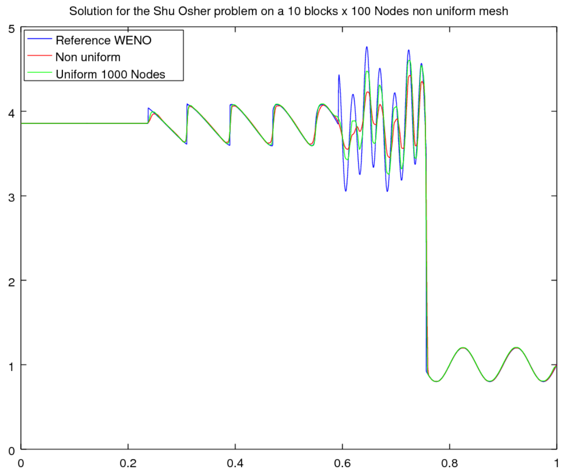
FIGURE 23 – Interacting blast wave on non uniform mesh

7.3 Shu Osher Problem

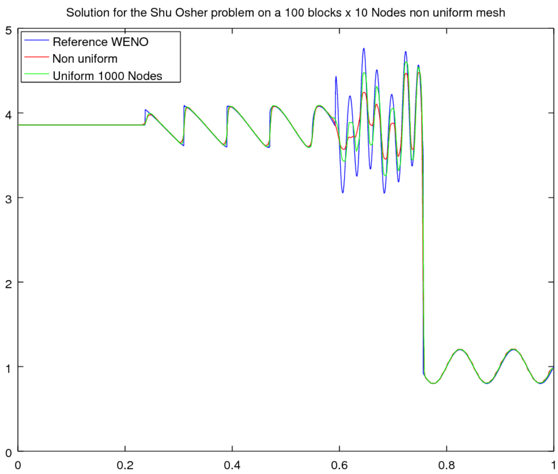
In this case we have almost the same observation as in the previous example. We can just add that the resolution in the left part of the front wave (the part where there is higher frequency oscillations), the resolution is actually not as good as in the uniform case. Moreover, the less nodes there is in a block, the less accurate is the solution.



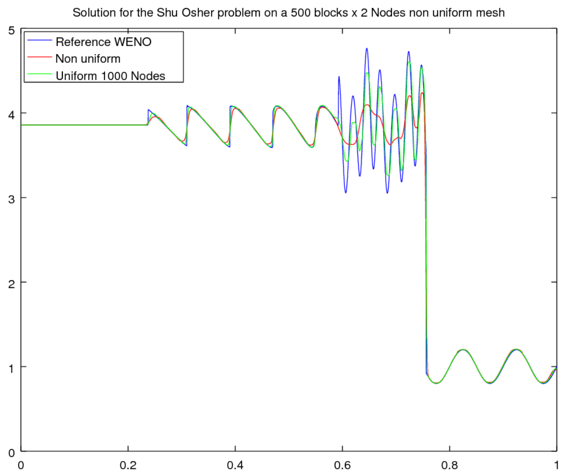
(a) Shu Osher problem with 2 blocks with 500 nodes each



(b) Shu Osher problem with 10 blocks with 100 nodes each



(c) Shu Osher problem with 100 blocks with 10 nodes each



(d) Shu Osher problem with 500 blocks with 2 nodes each

FIGURE 24 – Interacting blast wave on non uniform mesh