Implementation and Testing a Finite Difference Method for the Compressible Euler's Gas Equations

Timothée Schmoderer

12 juillet 2017

Contents

INSA INSTITUT NATIONAL DES SCIENCES APPLIQUÉES ROUEN

University of Cologne

1 Introduction

In this work we are interested in solving numerically the Euler's gas equations :

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u\\ P+\rho u^2\\ u(E+P)\end{pmatrix}=0 \qquad \forall x\in\Omega,\ \forall t\geq 0 \tag{1}$$

And the system is completed with the following equation of state :

$$E=\frac{P}{\gamma-1}+\frac{\rho u^2}{2}$$

We note $U=\begin{pmatrix}\rho\\ \rho u\\ E\end{pmatrix}$ the state vector and $f(U)=\begin{pmatrix}\rho u\\ \rho u^2+P\\ u(E+P)\end{pmatrix}$ the flux vector. Such as the system (1) reduces as :

$$\frac{\partial U}{\partial t}+\frac{\partial f(U)}{\partial x}=0$$

Here are the possible boundary conditions we might encountered (assume $\Omega=[a,b]$) :

1. **Periodic :** the easiest conditions. It is simply that what append at one side of the domain is reported at the other side :

$$\Phi(a,t)=\Phi(b,t) \qquad \Phi=u,\ \rho \text{ and } P \tag{2}$$

2. **Solid wall :** the most natural conditions. The fluid will behave like there is two solid wall at the two sides of the tube. In term of equation, it is :

$$\begin{aligned} u(a,t)=u(b,t)&=0 \\ \frac{\partial\Phi}{\partial x}(a,t)=\frac{\partial\Phi}{\partial x}(b,t)&=0 \qquad \Phi=\rho \text{ and } P\end{aligned} \tag{3}$$

3. **Inlet :** this condition prescribe the value of the flow at the boundary x_B. It is like there is some source of fluid :

$$\Phi(x_B,t)=\Phi_{in}(t) \qquad \Phi=u,\ \rho \text{ and } P$$

4. **Outflow :** All the fluids that reaches the boundary x_B is evacuate. In terms of equations :

$$\frac{\partial\Phi}{\partial x}(x_B,t)=0 \qquad \Phi=u,\ \rho \text{ and } P$$

The goal is to implement the scheme describe in [1] and to test it on different cases. Notice that this method focuses on moving boundary in two dimensions. But we forgot that and focus on the case the boundaries are fixed and the dimension is one.

In section 2, I will present the method, then in section 3 we will discuss convergence rate and see the method working on some test case.

2 Implementation

2.1 Cells and Nodes

Choose a integer N, and divide Ω into N equals grid "cells" (Figure 1), let x_i be the cell-centers and Δx be the cell's width.

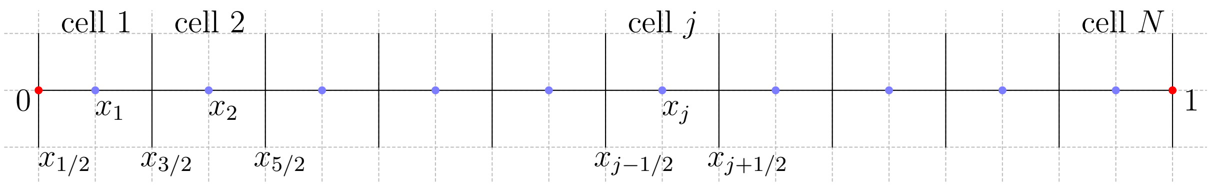


FIGURE 1 – Cells construction on Ω

We denote by $x_{j\pm 1/2}$ the cells interfaces : $x_j \pm \dfrac{\Delta x}{2}$. And we note U_j the approximation of $U(x_j, t)$.

2.2 Algorithm

The goal is to get the following equations :

$$\boxed{\frac{dU_j}{dt} = -\frac{\hat{f}_{j+1/2} - \hat{f}_{j-1/2}}{\Delta x} \qquad j = 1, \ldots, N \qquad (4)}$$

We compute the $\{\hat{f}_{j\pm 1/2}\}$ as follows :

1. Compute the spectral radius of the Jacobian at $x = x_j$. Analytic computation (see [2] for details) let us know that for compressible Euler equation this is :

$$a_j = |u_j| + c_j$$

2. We split the flux function f as :

$$f(U_j) = f_j^+ + f_j^- \qquad f_j^\pm = \frac{1}{2}\left(f(U_j) \pm a_j U_j\right)$$

3. We compute the slopes in each cells using the minmod function :

$$(f_x)_j^\pm = minmod\left(\theta\frac{f_j^\pm - f_{j-1}^\pm}{\Delta x}, \frac{f_{j+1}^\pm - f_{j-1}^\pm}{2\Delta x}, \theta\frac{f_{j+1}^\pm - f_j^\pm}{\Delta x}\right)$$

Where $\theta \in [1, 2]$ is a parameter that control the amount of numerical dissipation, it will be taken at 1.5.
The *minmod* function is defined by :

$$minmod(a, b, c) = \begin{cases} \min(a, b, c) & \text{if } a > 0,\ b > 0 \text{ and } c > 0 \\ \max(a, b, c) & \text{if } a < 0,\ b < 0 \text{ and } c < 0 \\ 0 & \text{else} \end{cases}$$

4. Construct f^E and f^W as :

$$f_j^E = f_j^+ + \frac{\Delta x}{2}(f_x)_j^+ \qquad f_j^W = f_j^- - \frac{\Delta x}{2}(f_x)_j^-$$

5. Finally :

$$\hat{f}_{j+1/2} = f_j^E + f_{j+1}^W \quad \hat{f}_{j-1/2} = f_{j-1}^E + f_j^W$$

2.3 Ghost Nodes

While presenting the scheme I haven't discuss on which nodes we have to apply the method. Obviously there is no problem when we are in the middle of the domain problems arises when we come close to the boundaries. In order to achieve the method two ghosts point are needed on each side of the domain, with value induced by the boundary conditions. Why two ? Remember that we want values at each cell's interfaces. Then for instance to get $\hat{f}_{1-1/2}$ we will need f_0^E so at this point we need one ghost cell left. Moreover to get f_0^E we will need to get the slopes for $j = 0$ then we might need $f_{-1}^{\pm}$ in the *minmod* function. So one more ghost cell.

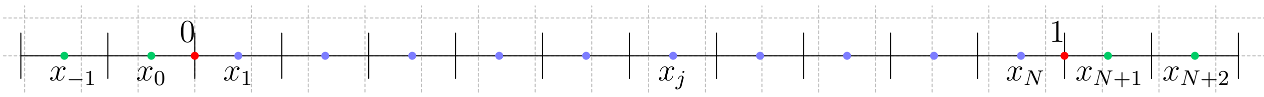


FIGURE 2 – Ω with ghost points

The whole method can be summarize in this array (red are ghost values, and dot show for what j we can compute the value) :

cell	-1	0	1	2	$\dots$	j	$\dots$	$N-1$	N	$N+1$	$N+2$
a_j	·	·	·	·	·	·	·	·	·	·	·
$f^{\pm}$	·	·	·	·	·	·	·	·	·	·	·
$f_j^{\pm} - f_{j-1}^{\pm}$		·	·	·	·	·	·	·	·	·	
$f_{j+1}^{\pm} - f_{j-1}^{\pm}$		·	·	·	·	·	·	·	·	·	
$f_{j+1}^{\pm} - f_j^{\pm}$		·	·	·	·	·	·	·	·	·	
$(f_x)^{\pm}$		·	·	·	·	·	·	·	·	·	
f^E		·	·	·	·	·	·	·	·	·	
f^W		·	·	·	·	·	·	·	·	·	
$\hat{f}_{j+1/2}$			·	·	·	·	·	·	·		
$\hat{f}_{j-1/2}$			·	·	·	·	·	·	·		
$\frac{dU_j}{dt}$			·	·	·	·	·	·	·		

2.4 Boundary condition

The boundary conditions are reflected in the value set in the ghost cells. I will describe how *wall* and *periodic* conditions are set, because, *inlet* conditions cause no problem and *outflow* is easy deduce from the wall conditions.

2.4.1 Periodic conditions

The periodic conditions describe the gas as if it is moving in a circle. As we want the value at $x = 0$ and $x = 1$ to coincide, the following figure (3) allow us to understand what happen.

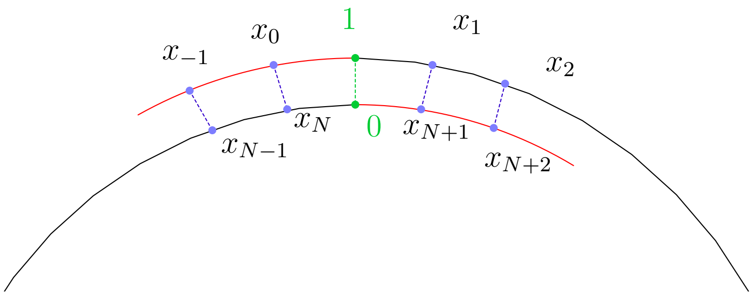


FIGURE 3 – Ω seen with periodic boundary conditions

Then it is easy to deduce the conditions :

$$\begin{aligned} \rho_0 &= \rho_N & u_0 &= u_N & P_0 &= P_N \\ \rho_{-1} &= \rho_{N-1} & u_{-1} &= u_{N-1} & P_{-1} &= P_{N-1} \end{aligned}$$

That is to say for the state vector :

$$U_0 = U_N \qquad U_{-1} = U_{N-1}$$

2.4.2 Wall conditions

From equations 3 we deduce the value in the value of ρ, u and P at the two ghost cells centers (I only describe them for left boundary the right case is the same.) :

$$\begin{aligned} \rho_0 &= \rho_1 & u_0 &= -u_1 & P_0 &= P_1 \\ \rho_{-1} &= \rho_2 & u_{-1} &= -u_2 & P_{-1} &= P_2 \end{aligned}$$

Which in term of U is the following :

$$U_0 = \begin{pmatrix} U_1^1 \\ -U_1^2 \\ U_1^3 \end{pmatrix} \qquad U_{-1} = \begin{pmatrix} U_2^1 \\ -U_2^2 \\ U_2^3 \end{pmatrix}$$

where the upperscript is the denotes the i-th component.

2.5 Integration of the semi-discrete system

Once we get the semi discrete scheme (4) we have to numerically integrate it. Based on [3, 4, 5] and as suggests by the author of the method I implemented the following 3rd order SSP Runge-Kutta method :

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \\ u^{(2)} &= \frac{3}{4}u^n + \frac{1}{4}u^{(1)} + \frac{1}{4}\Delta t L(u^{(1)}) \\ u^{n+1} &= \frac{1}{3}u^n + \frac{2}{3}u^{(2)} + \frac{2}{3}\Delta t L(u^{(2)}) \end{aligned} \tag{5}$$

Where the L operator is the one that gives us the second member.

Courant's number : In this work the best Courant's number we can choose is 1. Thus we get the bigger time step we could expect :

$$\Delta t = \frac{\Delta x}{\max_{x\in\Omega} a}$$

3 Numerical Experiment for regular nodes distribution

In this section we will show by experimentation that the method is of order 2 as announced and demonstrate the efficiency of the method on several typical example. All computations are done with 1000 nodes. Computation done with higher order method could be found in [6], thus we can convince ourselves that the method is working well.

3.1 Error and convergence

Because we don't know the analytic solution we adopt the *Manufactured Solution* strategy :

1. We choose $q(x,t) \in \mathbb{R}^3$ as smooth as possible to be the solution of equations (1).
2. We put q in these equations and get, in general, a non zeros result (otherwise, it will mean we have an analytical solution, which is in general not possible). Let's call the rest $\mathcal{S}(x,t)$:
$$\frac{\partial q}{\partial t} + \frac{\partial f(q)}{\partial x} = \mathcal{S}$$
3. We now consider the modified problem :
$$\boxed{\begin{gathered}\frac{\partial U}{\partial t} + \frac{\partial f(U)}{\partial x} = \mathcal{S} \\ U(x,0) = q(x,0) \\ \text{Plus boundary conditions to be discusses bellow}\end{gathered}} \tag{6}$$
For which we know the solution to be q.
4. We apply our method to this problem. It requires minor modification in the code to add the source term.
5. We can now compare the numerical solution to the analytical one. And compute the convergence rate.

A common choice in this approach is to choose periodic boundary conditions.

3.1.1 Case 1

This case could be found in [7] (example 3.3).
For this first case we want the solution to be :

$$\begin{aligned}\rho(x,t) &= 1 + 0.2\sin(2\pi(x-t)) \\ u(x,t) &= 1 \\ P(x,t) &= 1\end{aligned}$$

Then a little computation using the equation of state gives the solution for the energy :

$$E(x,t) = \frac{\gamma+1}{2(\gamma-1)} + 0.1\sin(2\pi(x-t))$$

Hence the initial conditions are the following :

$$\boxed{\begin{aligned} \rho(x,0) &= 1 + 0.2\sin(2\pi x) \quad \forall x \in \Omega \\ u(x,0) &= 1 \quad \forall x \in \Omega \\ P(x,0) &= 1 \quad \forall x \in \Omega \end{aligned}} \tag{7}$$

In this particular case, the source term is identically zero :

$$\mathcal{S}(x,t) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We make computation from 100 to 6400 nodes by step of 100 nodes and up to 1000 iterations. Hence we are able to compare the exact solution for the density and the numerical approximation. Figure 4 shows the log of the error against the log of the number of nodes.

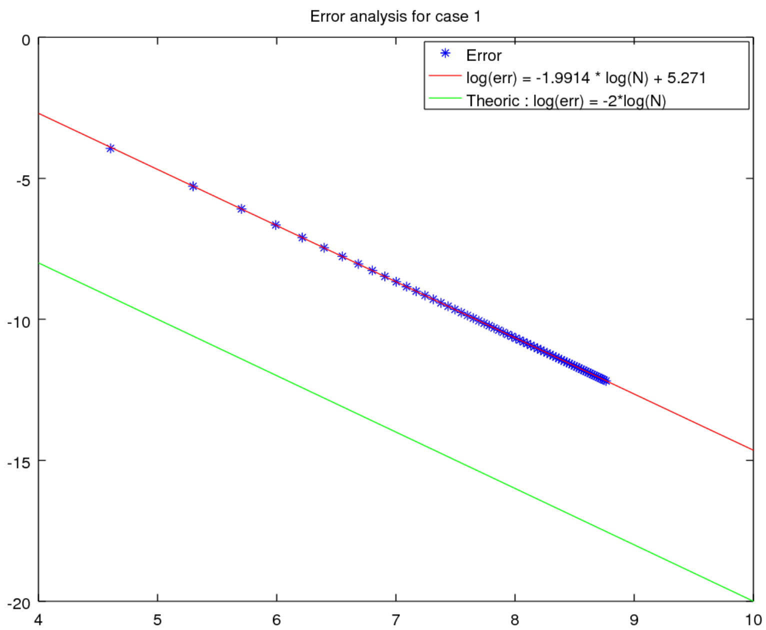


FIGURE 4 – Convergence rate for manufactured solution # 1

As we hoped the method seems to be of order 2.

3.1.2 Case 2

The idea behind this case comes mainly from [8]
The second case will comfort us in the fact tat the method is of order 2. We want the solution to be :

$$\begin{aligned} \rho(x,t) &= 2 + 0.1\sin(2\pi(x-t)) \\ u(x,t) &= 1 \\ E(x,t) &= 2 + 0.1\cos(2\pi(x-t)) \end{aligned}$$

Then, the initial conditions are the following :

$$\begin{aligned} u(x,0) &= 1 \quad \forall x \in \Omega \\ \rho(x,0) &= 2 + 0.1\sin(2\pi x) \quad \forall x \in \Omega \\ P(x,0) &= \frac{\gamma - 1}{20}\left(20 + 2\cos(2\pi x) - sin(2\pi x)\right) \quad \forall x \in \Omega \end{aligned} \tag{8}$$

And the source term is given by :

$$\mathcal{S}(x,t) = (1-\gamma)\pi(2\rho(x,t) + E(x,t) - 6)\begin{pmatrix}0\\1\\1\end{pmatrix} \tag{9}$$

We again make computation from 100 to 6400 nodes with step of 100 nodes up to 1000 iterations. Hence we are able to compare the exact solution and the numerical approximation. Figure 5 shows the log of the error against the log of the number of nodes.

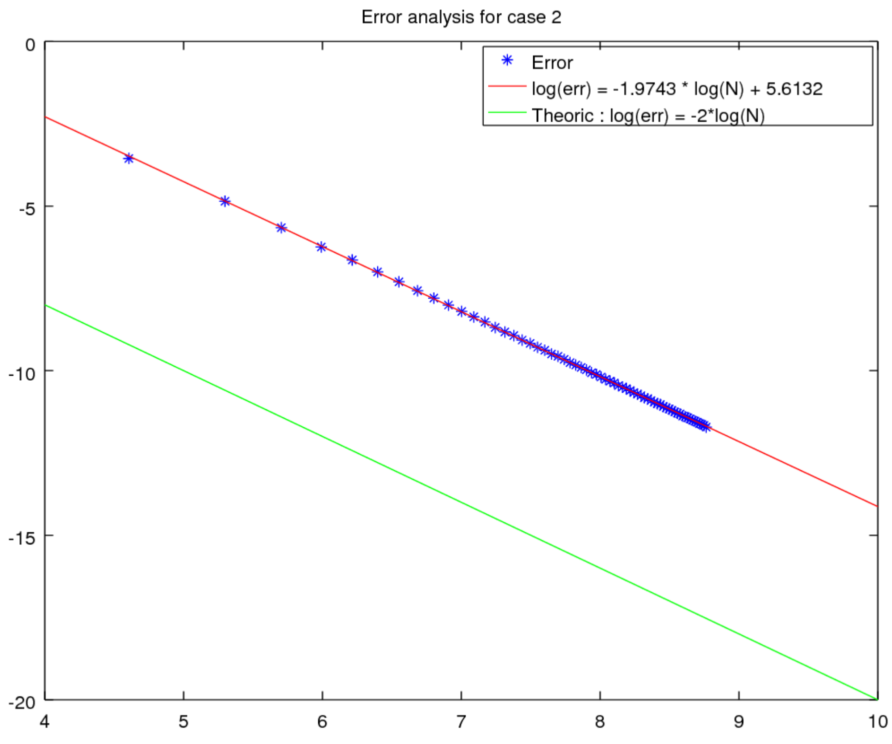


FIGURE 5

Again the convergence rate is really near 2. We can conclude that the method it of order 2 as announced.

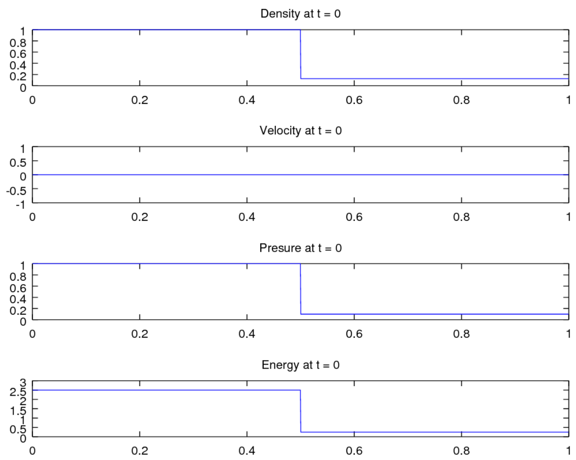
Having satisfying ourselves with the convergence rate we can now observe the method working on several typical example.

3.2 Sod shock tube

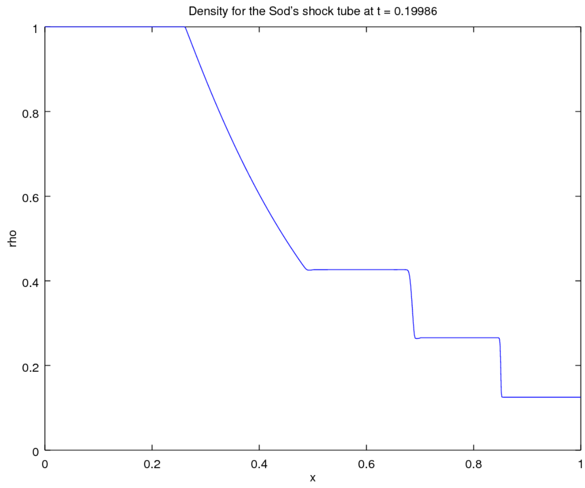
We begin our collection of test with a popular one. The domain is $\Omega = [0, 1]$, and the initial conditions are :

$$
\begin{aligned}
u(x,0) &= 0 \qquad \forall x \in \Omega \\
\rho(x,0) &= \begin{cases} 1.0 & x \in [0, 0.5] \\ 0.125 & x \in [0.5, 1] \end{cases} \\
P(x,0) &= \begin{cases} 1.0 & x \in [0, 0.5] \\ 0.1 & x \in [0.5, 1] \end{cases}
\end{aligned}
\tag{10}
$$

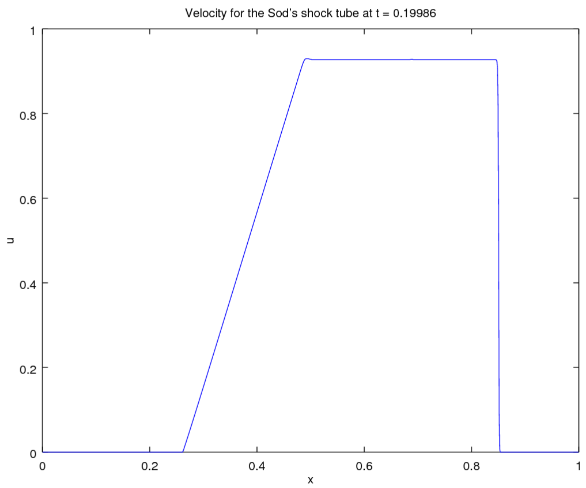
On this we apply wall boundary conditions and look at the result around $t = 0.2$.



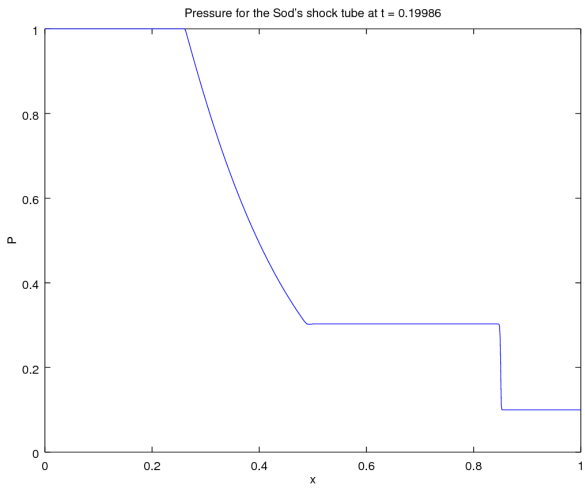
(a) Initial conditions



(b) Density



(c) Velocity



(d) Pressure

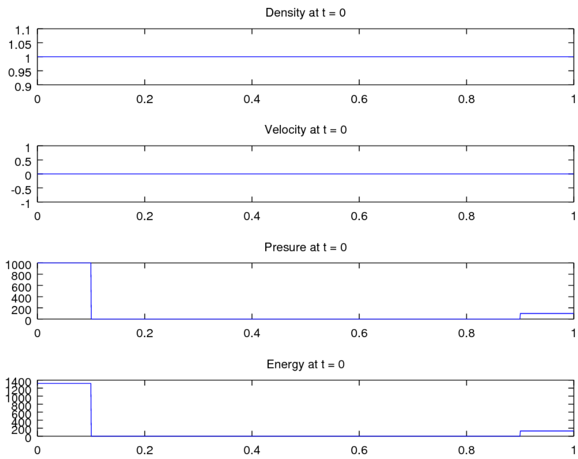
FIGURE 6 – Result for the Sod's shock tube

3.3 Interacting blast wave

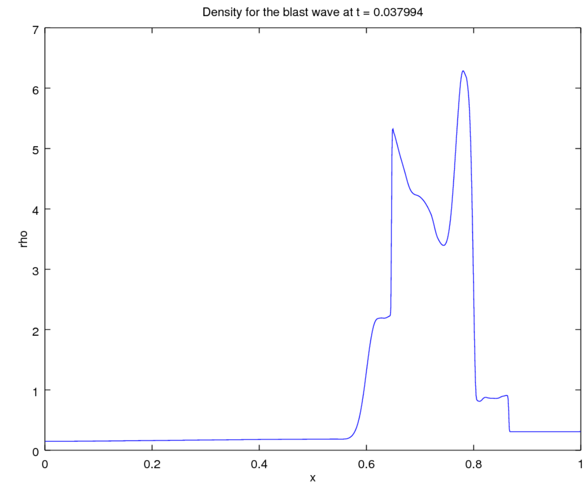
See [9] and [7] (example 3.7). Initial conditions :

$$
\begin{aligned}
u(x,0) &= 0 \qquad \forall x \in \Omega \\
\rho(x,0) &= 1 \qquad \forall x \in \Omega \\
P(x,0) &= \begin{cases} 1000 & x \in [0, 0.1] \\ 0.01 & x \in [0.1, 0.9] \\ 100 & x \in [0.9, 1] \end{cases}
\end{aligned}
\tag{11}
$$

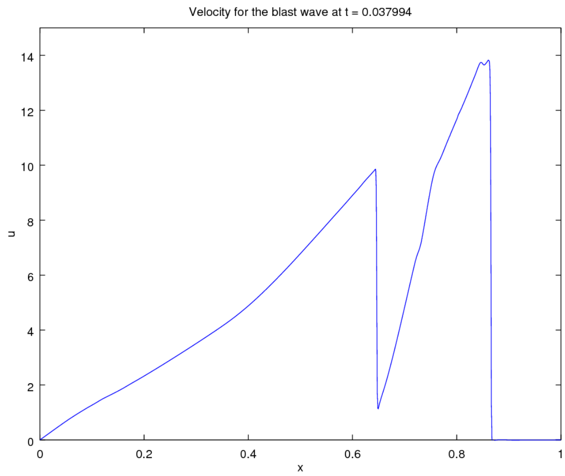
We use again solid wall boundary conditions and watch the result around $t = 0.038$.



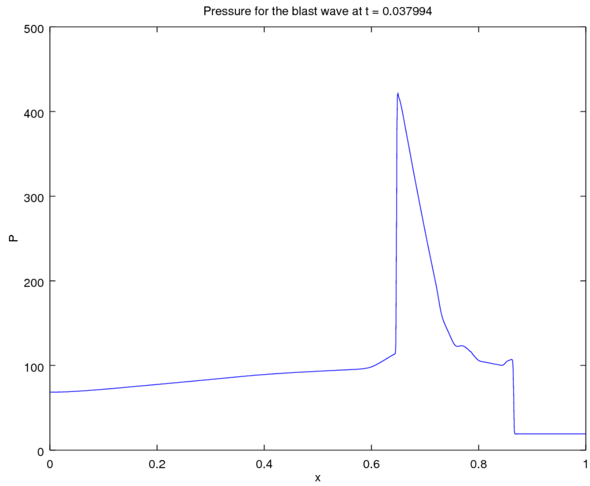
(a) Initial conditions



(b) Density



(c) Velocity



(d) Pressure

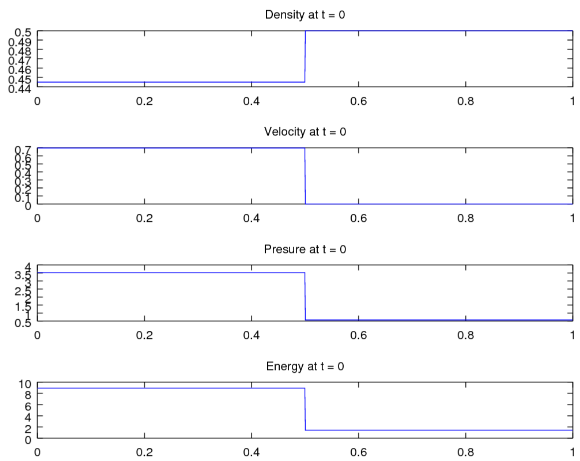
FIGURE 7 – Result for the blast wave

3.4 Lax's shock tube

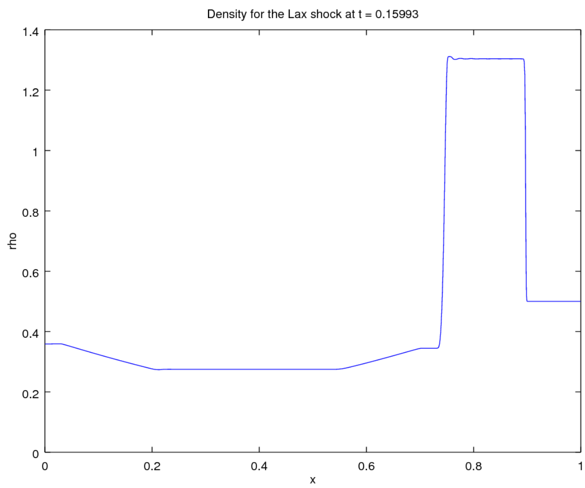
[7] (example 3.5). The initial conditions are :

$$
\begin{aligned}
u(x,0) &= \begin{cases} 0.698 & x \in [0, 0.5] \\ 0 & x \in [0.5, 1] \end{cases} \\
\rho(x,0) &= \begin{cases} 0.445 & x \in [0, 0.5] \\ 0.5 & x \in [0.5, 1] \end{cases} \\
P(x,0) &= \begin{cases} 3.528 & x \in [0, 0.5] \\ 0.571 & x \in [0.5, 1] \end{cases}
\end{aligned}
\tag{12}
$$

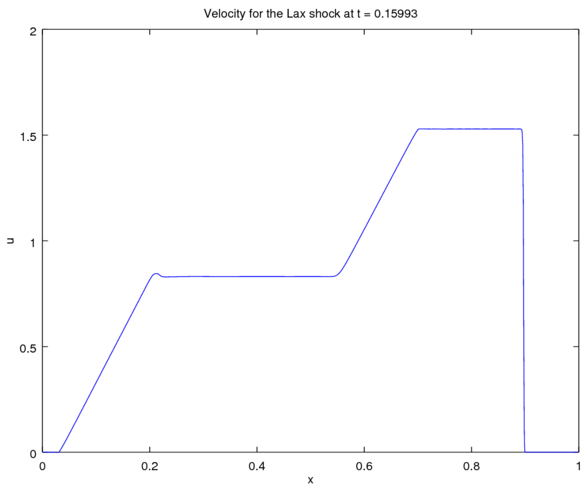
We use again solid wall boundary conditions and watch the result around $t = 0.16$.



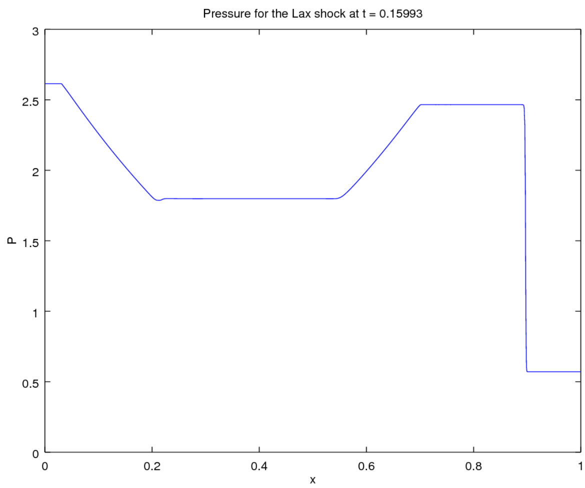
(a) Initial conditions



(b) Density



(c) Velocity



(d) Pressure

FIGURE 8 – Result for the Lax's shock tube

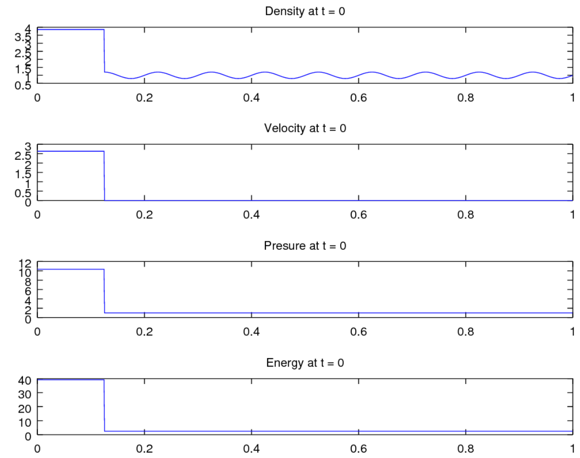
3.5 Shu-Osher's problem

[10] and [7] (example 3.6). The initial conditions are :

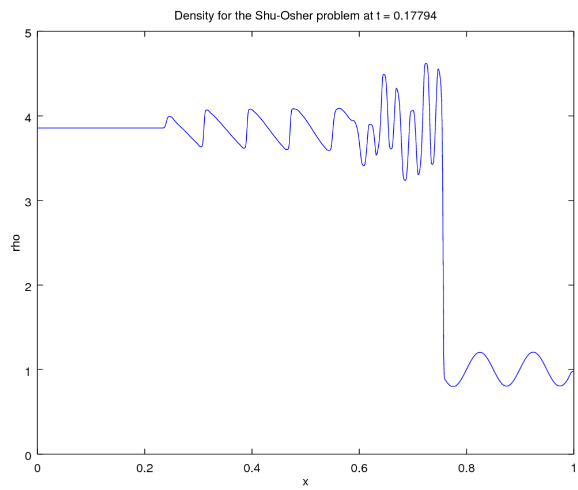
$$
\begin{aligned}
u(x,0) &= \begin{cases} 2.629369 & x \in [0, 0.125] \\ 0 & x \in [0.125, 1] \end{cases} \\
\rho(x,0) &= \begin{cases} 3.857143 & x \in [0, 0.125] \\ 1 + 0.2\sin(20\pi x) & x \in [0.125, 1] \end{cases} \\
P(x,0) &= \begin{cases} 31/3 & x \in [0, 0.125] \\ 1 & x \in [0.125, 1] \end{cases}
\end{aligned} \tag{13}
$$

This time we use inlet conditions on the left and outflow conditions on the right. The inlet conditions are :

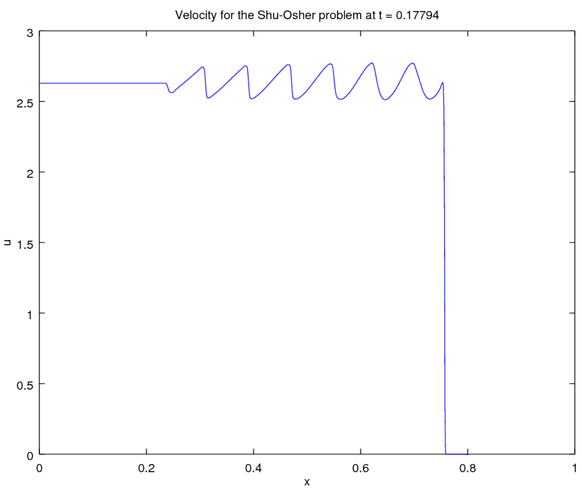
$$u(0,t) = 2.629369 \quad \rho(0,t) = 3.857143 \quad P(0,t) = 31/3$$



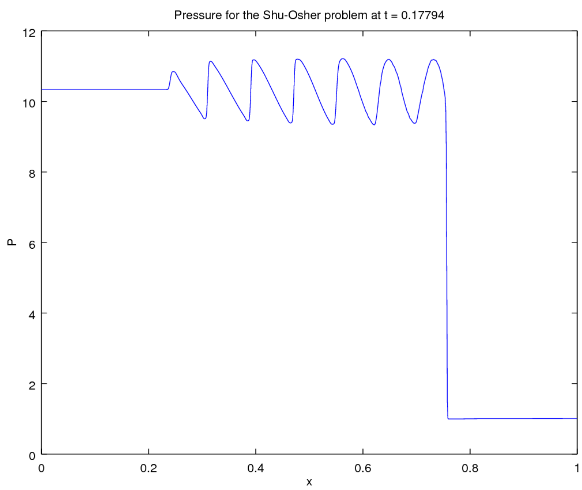
(a) Initial conditions



(b) Density



(c) Velocity



(d) Pressure

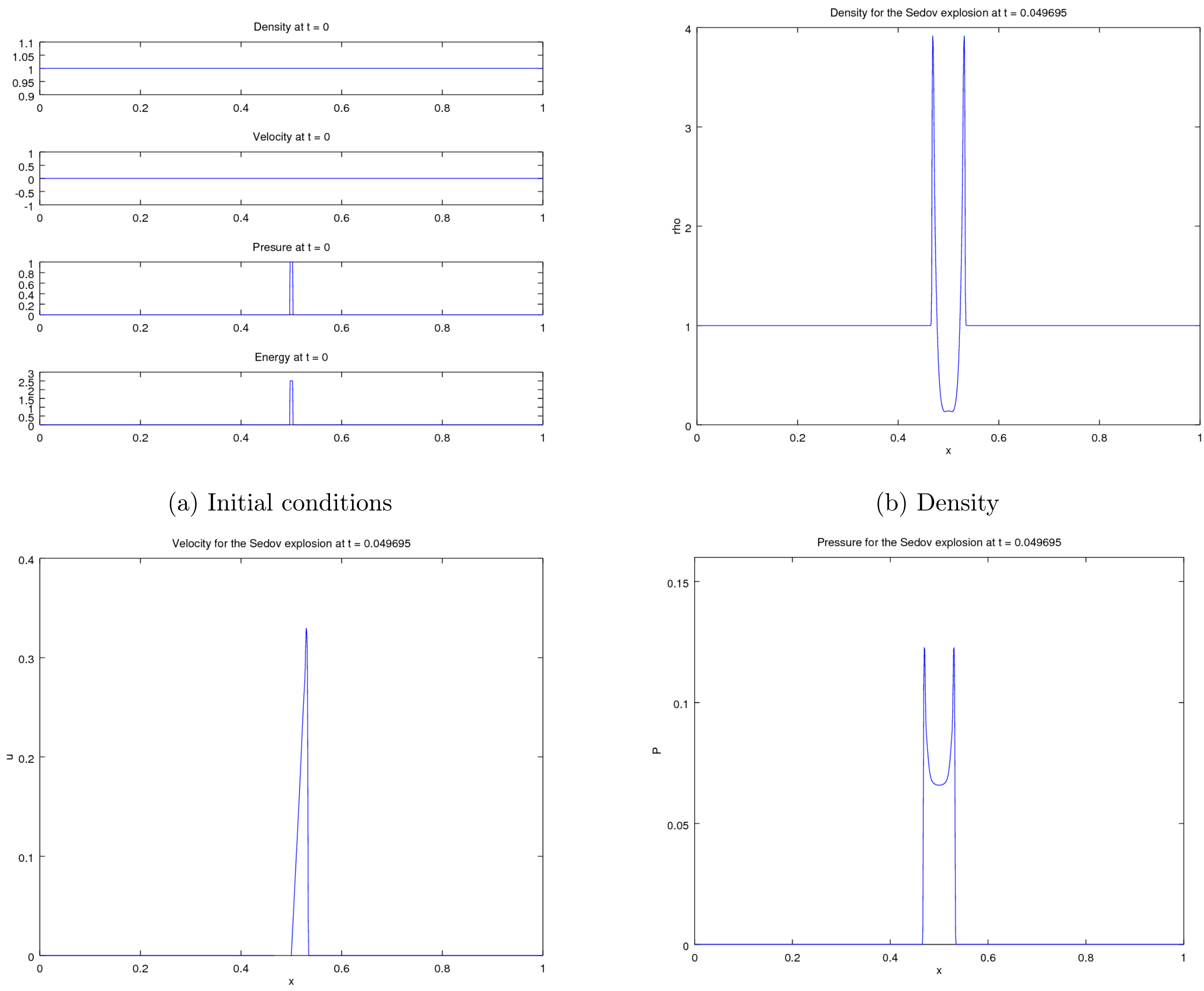
FIGURE 9 – Result for the Shu-Osher's problem

3.6 Sedov explosion

The initial conditions are :

$$
\begin{aligned}
u(x,0) &= 0 \qquad \forall x \in \Omega \\
\rho(x,0) &= 1 \qquad \forall x \in \Omega \\
P(x,0) &= \begin{cases} 1 & x \in [0.5 - 3.5\frac{\Delta x}{2}, 0.5 + 3.5\frac{\Delta x}{2}] \\ 10^{-5} & \text{else} \end{cases}
\end{aligned}
\tag{14}
$$

We use again solid wall boundary conditions and watch the result around $t = 0.038$.



(a) Initial conditions (b) Density

(c) Velocity (d) Pressure

FIGURE 10 – Result for the Sedov's explosion

References

[1] Kurganov Alexander Russo Giovanni Coco Armando, Chertock Alina. A second-order finite-difference method for compressible fluids in domains with moving boundaries. 2017.

[2] Michael Zingale. Notes on the euler equations. *hydro by example*, 2013.

[3] Xinghui Zhong. Strong stability-preserving (ssp) high-order time discretization methods, Sep 2009.

[4] Chi-Wang Shu and Stanley Osher. Efficient implementation of essentially non-oscillatory shock-capturing schemes. *Journal of Computational Physics*, 77(2) :439 – 471, 1988.

[5] Sigal Gottlieb, Chi-Wang Shu, and Eitan Tadmor. Strong stability-preserving high-order time discretization methods. *SIAM Review*, 43(1) :89–112, 2001.

[6] *FLASH User's Guide.*

[7] Jun Zhu and Jianxian Qiu. A new fifth order finite difference weno scheme for solving hyperbolic conservation laws. *Journal of Computational Physics*, 318 :110–121, 2016.

[8] Gregor J Gassner. A kinetic energy preserving nodal discontinuous galerkin spectral element method. *International Journal for Numerical Methods in Fluids*, 76(1) :28–50, 2014.

[9] A. Baeza, P. Mulet, and D. Zorío. High order boundary extrapolation technique for finite difference methods on complex domains with cartesian meshes. *Journal of Scientific Computing*, 66(2) :761–791, Feb 2016.

[10] Shu-osher shock tube problem. http://www.ttctech.com/Samples/shockwave/shockwave.htm .