ARTICLES

Roads and Wheels

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Introduction

San Francisco's Exploratorium contains an intriguing exhibit of a square wheel that rolls smoothly on a road made up of linked, inverted catenaries (see FIGURE 4). That exhibit inspired us to generate a computer animation of a rolling square and further explore the relationship between the shapes of wheels and roads on which they roll. In a sense, we are bringing up to date the paper by G. Robison [4], showing how much more can be done, both numerically and graphically, with modern computer hardware and software. The problem of the square wheel has been rediscovered and solved several times; see [**5**, **7**].

All the diagrams and animations were prepared in *Mathematica*. Our package that generated the diagrams and the associated animations (see Section 5) can be obtained by sending a Macintosh disk to one of the authors. It is noteworthy that some of the results of this paper, in particular the discovery of a cycloidal locus generated by a noncircular wheel, were discovered only after viewing certain graphics. *Mathematica* was also used to do all the symbolic integrations that occur. For further applications of symbolic and graphic computation to wheel/road problems, in particular, a complete discussion of the cycloid, see [**6**, Chapter 2].

The paper is organized as follows. Section 1 discusses the theory and the fundamental differential equation. Section 2 contains many closed-form examples. Section 3 shows how numerically approximating the solution to the differential equation is an excellent approach to diverse examples, even those solvable in closed form. Section 4 squares the circle by considering Fourier approximations to the catenary. And Section 5 discusses the *Mathematica* package that we built.

1. Building a wheel

Suppose we are given a road in the form of a rectifiable curve in the lower half-plane parametrized by $f(t) = (x(t), y(t))$, where $x(t)$ is increasing, $x(0) = 0$, $y(t) \leq 0$. By the *wheel* corresponding to the road we mean a curve that will roll smoothly on the road. More precisely, a wheel will be a curve given by a polar function $r = r(\theta)$ such that the axle of the wheel, which initially is at $(0, 0)$, stays on the x-axis directly above the wheel–road contact point as the curve rolls along the road. The wheel's axle may or may not coincide with the wheel's geometric center. The road is assumed to provide enough friction so that there is never any slipping of the wheel. The rolling motion can be described by a function $\theta = \theta(t)$ that describes the amount of angular rotation for the wheel to roll from $f(0)$ to $f(t)$. These functions must satisfy the following conditions (see FIGURE 1):

1. *Initial condition*. The initial contact point is at $f(0)$, directly under the origin, whence $\theta(0) = -\pi/2$.
2. *Rolling condition*. The amount unravelled on the wheel matches the distance travelled on the road: For any t, the arc length of f between $f(0)$ and $f(t)$ equals the arc length of the polar curve between $\theta(0)$ and $\theta(t)$.
3. *Radius condition*. The radius of the wheel matches the depth of the road at the corresponding point: For any $t, r(\theta(t)) = -y(t)$.

FIGURE 1 illustrates the formation of the wheel in the case when the road is given as $y = f(x)$, with $f(x)$ nonpositive, in which case the conditions simplify accordingly (that is, x can be used as the parameter, and so θ becomes a function of x).

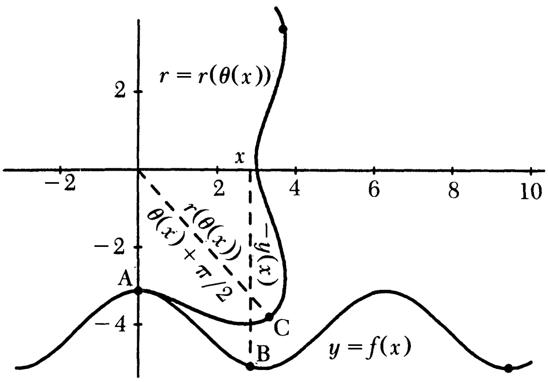


FIGURE 1
If a road is given by $y = f(x)$, then the relationship between θ and x is obtained from the equality of the arc lengths AB and AC and of the radius vector OC and the depth of the road (dashed lines). The road illustrated is given by $y = -\sqrt{17} + \cos x$, where $\sqrt{17}$ has been chosen so that the wheel closes up on itself (see Remark 5).

The key to getting a wheel is finding the function $\theta(t)$, since the radius condition will then yield $r(\theta)$. The first two conditions become a simple differential equation, which can lead to either a closed-form description of θ or a numerical approximation. The rolling condition is:

$$\int_0^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_{-\pi/2}^{\theta(t)} \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta .$$

Differentiating both sides with respect to t and squaring yields:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 \left(r(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2\right).$$

Now substitute $\frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{dy}{dt}$ (obtained by differentiating the radius condition) to get:

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r(\theta)^2 \left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2,$$

which simplifies to:

$$\frac{d\theta}{dt} = \pm\frac{dx}{dt}\frac{1}{-y(t)}.$$

Because $d\theta/dt$ is to be positive, the differential equation we seek is:

$$\frac{d\theta}{dt} = -\frac{dx}{dt}\frac{1}{y(t)},$$

with initial condition $\theta(0) = -\pi/2$.

Remarks.

1. If the road is given by $y = f(x)$, the differential equation for θ becomes simply $d\theta/dx = -1/f(x)$. In this case

$$\theta(x) = \int_0^x \frac{-1}{f(x)}\,dx - \frac{\pi}{2}.$$

2. If the function $\theta(t)$ can be inverted to $t(\theta)$ then the wheel is given by the polar equation $r = -y(t(\theta))$.
3. An alternative approach to characterizing $\theta(t)$ proceeds by matching slopes instead of arc lengths. The rolling condition then becomes: For any t, the slope of the road at $f(t)$ equals the slope of the tangent to the polar curve at $\theta(t)$, rotated clockwise through $\theta(t) + \pi/2$ radians. This leads to the same differential equation.
4. The inverse problem starts with $r(\theta)$, a polar representation of a wheel, and seeks the appropriate road. The preceding discussion implies that the road is given by $y(x) = -r(\theta(x))$, where θ satisfies $\theta(0) = -\pi/2$ and $d\theta/dx = 1/r(\theta)$. One can also deal with the case that the wheel is given parametrically by $(x(t), y(t))$; see Case 4 in Section 3.
5. Suppose the road $y = f(x)$ is periodic with period a. Then the corresponding wheel does not necessarily close up on itself to form a topological disk (see FIGURE 2). The condition for such closure—the *closed-wheel condition*—is that there exists a rational number r so that $2\pi r = \theta(a) - \theta(0) = \int_0^a -1/f(x)\,dx$. If $r = 1/n$, n a positive integer, then the wheel rolls over n periods of the road during each complete revolution. As an example, consider the road given by $y = d + \cos x$,

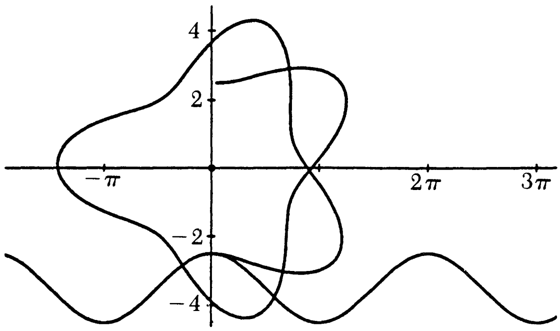


FIGURE 2
The wheel for a cosine road given by $-3.5 + \cos x$ winds around endlessly without closing up, because 3.5 is not one of the special values $\sqrt{1+n^2}$. This wheel was generated by the numerical technique discussed in Section 3.

where $d \leq -1$. For each positive integer n, there is a unique value of d, which turns out to be $-\sqrt{1+n^2}$, for which the wheel closes up into one that covers n periods per revolution. (See FIGURE 1 for the $n=4$ case and FIGURE 13 for the $n=1$ case.) To see that the values of d in the cosine case are as claimed, observe that $\int_0^{2\pi} -1/(d+\cos x)\,dx = 2\pi/\sqrt{d^2-1}$. It is easiest to integrate from 0 to π and then double. More generally, the road $y = d + b\cos(cx)$ yields a closed wheel that covers n periods when $d = -\sqrt{b^2+n^2/c^2}$. Cycloid roads and inverted cycloid roads provide two more examples in which the closed-wheel condition integral can be evaluated (see FIGURE 9(c) for the $n=0$ case, and FIGURE 19(d) for a fractional example, viz., $n=1/2$). The closed-wheel condition has a completely analogous form in the case of a parametrically defined road as well; see the cycloid example in Case 3 of Section 3.

2. Closed-form solutions

This section discusses several examples for which the differential equation can be solved in closed form. It is remarkable that, although the arc length of familiar curves is generally not solvable in closed form, the wheel–road problem is solvable for a wide variety of functions.

Polygonal wheels The roads corresponding to polygonal wheels are derived from the case of a wheel that is nothing more than a straight line. Consider the polar equation $r = -\csc\theta$, $-\pi < \theta < 0$, whose graph is a horizontal line one unit below the x-axis. The results of Section 1 show that the road on which this polar line will roll is given by $y = f(x) = -r(\theta(x))$ where $\theta(0) = -\pi/2$ and

$$\frac{d\theta}{dx} = \frac{1}{-y(x)} = \frac{1}{r(\theta(x))} = -\sin\theta.$$

The solution to this initial-value problem is $x = -\log(-\tan(\theta/2))$, or $\theta = -2\arctan e^{-x}$. Hence the road is given by:

$$\begin{aligned} y = f(x) &= \csc(-2\arctan e^{-x}) \\ &= \frac{-1}{\sin(2\arctan e^{-x})} = \frac{-1}{2\sin(\arctan e^{-x})\cos(\arctan e^{-x})} \\ &= -\frac{1+e^{-2x}}{2e^{-x}} = -\frac{e^{x}+e^{-x}}{2} = -\cosh x, \end{aligned}$$

whose graph is an inverted catenary. This means that the polar line will roll on the catenary so that the polar origin, which we imagine as attached to the line, stays on the x-axis.

Modifying the straight-line example yields roads for wheels that are regular polygons. Consider the square wheel. By simply truncating the catenary where its slope is ± 1—this occurs at $x = \pm\operatorname{arcsinh} 1$—and forming a periodic road by translating copies of the truncated catenary, the angle at the junctions will be 90° Hence a square will smoothly pass over the junction. FIGURE 4 shows several images from an animation of a rolling square on such a road, along with the locus of a vertex, which is related to the involute of the catenary.

The road appropriate for a regular n-gon may also be obtained from the catenary $y = -\cosh x$. If the catenary is truncated at $x = \pm\operatorname{arcsinh}[\tan(\pi/n)]$ then the angle at the road's cusp matches the interior angle of the n-gon, and the amount of rotation to get the wheel into the cusp is $\theta(x) + \pi/2$. This works out to be exactly $2\pi/n$ (the details, which involve a horrendous-looking identity involving tan, arcsinh, and arctan, are left to the reader). So the wheel corresponding to the road made of pieces of inverted catenaries closes up exactly into a regular n-gon. The case of a triangle is noteworthy in that the rolling cannot happen physically: Because the cusp angle is less than 90°, the triangle will crash into the road before the vertex gets into the cusp (FIGURE 5).

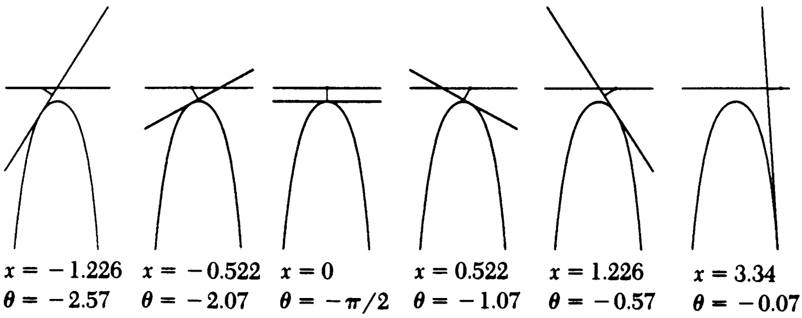


FIGURE 3

Some stills from an animation showing the polar line $r = -\csc\theta$, $-\pi < \theta < 0$, rolling over the catenary $y = -\cosh x$. The x-values and θ-values correspond to the x-coordinate of the point of tangency and the θ-value of the point of tangency viewed as a point on the polar line. Adding $\pi/2$ to the θ-value yields the amount of rotation of the horizontal line.

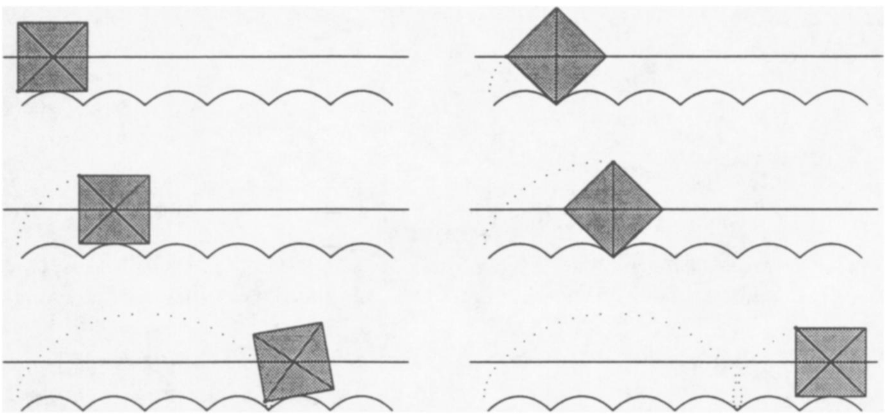


FIGURE 4

A road made up of pieces of an inverted catenary allows a square to roll smoothly. The dots are the locus of a vertex during the rolling. Note that the slope of the tangent to the locus has a discontinuity at the cusp.

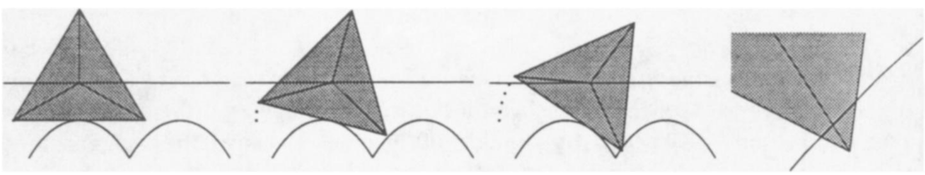
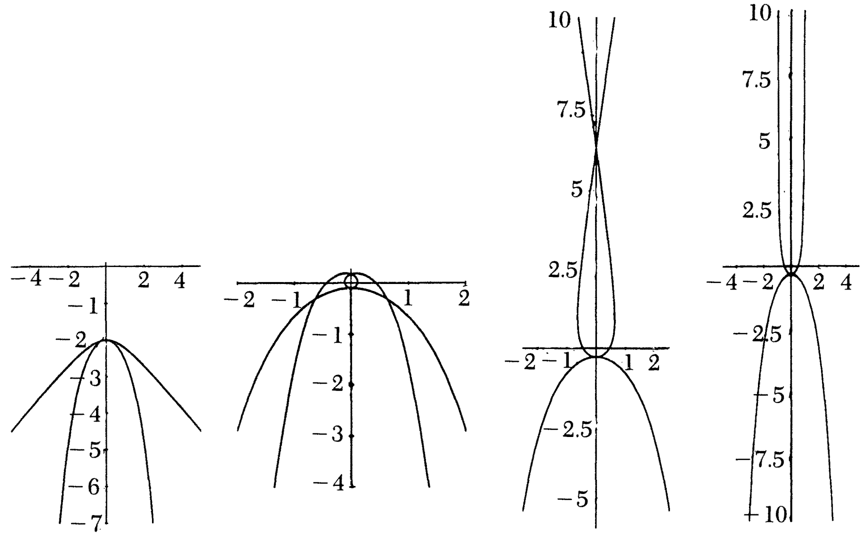


FIGURE 5

A vertex of a rolling triangle crashes into the road just before the vertex arrives at the cusp. The rightmost diagram is a close-up of the collision.

As we shall see several times in this paper, the depth of a road plays a crucial role in determining the shape of the wheel. If the catenary road $y = -\cosh x$ is raised or lowered, the shape of the wheel changes. Consider the family of roads $k - \cosh x$ where $k < 1$. If $k = 0$, then the wheel is a straight line, but other values of k yield radically different wheels, as shown in FIGURE 6. The closed-form solution, obtained with the help of *Mathematica*'s integrator, is given by $\theta(x) = \phi(x) - \phi(0) + \pi/2$ where

$$\phi(x) = \frac{-2\arctan\left(\dfrac{e^{-x} - k}{\sqrt{1-k^2}}\right)}{\sqrt{1-k^2}}.$$



roads: $y = -1 - \cosh x$ $\quad$ $0.9 - \cosh x$ $\quad$ $0.7 - \cosh x$ $\quad$ $0.676454 - \cosh x$

FIGURE 6

Raising or lowering a catenary road via $k - \cosh x$ leads to a variety of wheel shapes. Only the choice $k = 0$ leads to a straight-line wheel. The wheel in the last case has vertical asymptotes.

Tilted roads Let $f(x) = -1 - x, x \geq 0$, define a downward-sloping road. The solution to the initial-value problem for θ is then $\theta(x) = \log(x+1) - \pi/2$. Therefore the wheel for this oblique line is the equiangular spiral $r(\theta) = 1 + \exp(\theta + \pi/2) - 1 = \exp(\theta + \pi/2)$, which is shown in FIGURE 7. In the general case that the road is given by $-1 - mx$, the wheel is the spiral given by $r = \exp[(\theta + \pi/2)/m]$.

As with the catenary, we can turn the 45° road into a periodic function, in this case a sawtooth. In order to get a smoothly rolling wheel that covers four teeth of the road in one revolution, we need to find x such that $\theta(x) = -\pi/4$: $\theta(x) = -\pi/4$ if and only if $\log(x+1) - \pi/2 = -\pi/4$ if and only if $x = e^{\pi/4} - 1 = 1.19\ldots$. Now we can cut off the straight line at this point, generate a sawtooth road, and paste appropriately truncated pieces of the spiral together at right angles to get the sawtooth wheel shown in FIGURE 8. The pasting yields 90° angles because the tangent to the equiangular spiral makes a constant 45° angle with the radius vector; these angles plus their mirror images yield the right angles. A wheel that covers more teeth during each revolution can be obtained by truncating the line at a value of x that satisfies $\theta(x) = -\pi/n$ and pasting together more and shorter pieces of the spiral.

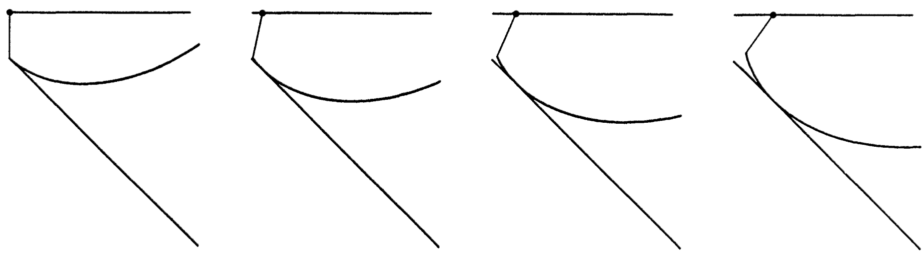


FIGURE 7
An exponential, or equiangular, spiral rolling along a tilted line.

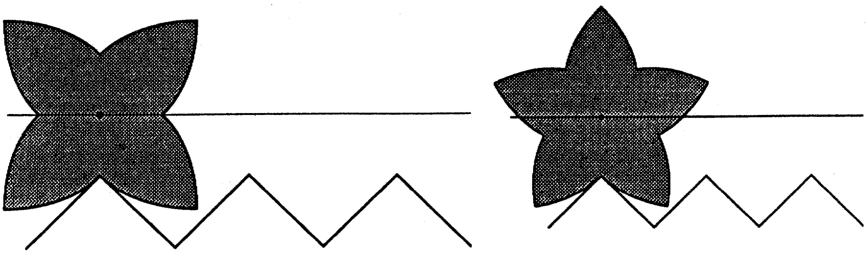


FIGURE 8
Pieces of an equiangular spiral can be pasted together to get a wheel that rolls on a sawtooth road. The examples shown cover four and five teeth per revolution, respectively.

Cycloidal roads The cycloid is the famous curve that is the path of a point on a traditional round wheel rolling on a straight road; inverting the cycloid leads to the parametric curve $f(t) = (t - \sin t, \cos t - 1)$. The wheel that rolls on an inverted cycloid can be found by solving the differential equation $d\theta/dt = (-1 + \cos t)/(\cos t - 1) = +1$, so $\theta(t)$ is simply $t - \pi/2$. Hence the wheel is given in polar form by $r = -y(t(\theta)) = 1 - \cos(\theta + \pi/2) = 1 + \sin\theta$, the polar form of a cardioid (FIGURE 9(a)). The cusp of the cardioid rolls over the cusp of the cycloid, at least in theory. In practice, there is a crash between the road and the cardioid, similar to the one that happens with a rolling triangle (FIGURE 5). As pointed out by Robison, a physical model can be built so as to avoid the cusp problem by introducing pieces of a catenary into the road and a straight line segment into the wheel so as to bypass the cusps. See [4, FIGURE 4] for details.

The locus of the top point of the cardioid seems to have the same general shape as a cycloid. As an exercise, the reader can verify that, indeed, this locus is a cycloid stretched vertically by a factor of 2. As further exercise, the reader can investigate the clover-like wheels that arise from lowering the cycloid so that the closed-wheel condition is met. *The case of $(t - \sin t, -13/5 - \cos t)$ is illustrated in FIGURE 9(b).* One can also consider the uninverted cycloid that is tangent to the x-axis from below — $(t - \sin t, -1 - \cos t)$ — for which the wheel is derived from the function:

$$\theta(x) = -x + \frac{2\sin x}{1 + \cos x} - \frac{\pi}{2}.$$

This leads to a spiral wheel that requires infinitely many revolutions to pass over the cycloid's high point (FIGURE 9(c)). Cycloidal roads are discussed further in Case 3 of Section 3.

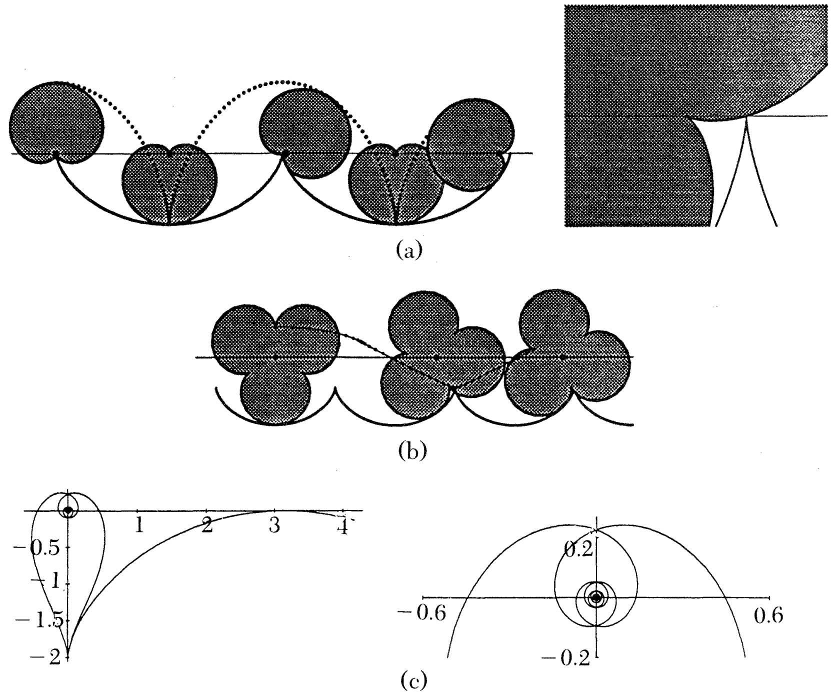


FIGURE 9
(a). A cardioid rolls on an inverted cycloid, and the locus of the point opposite the cardiod's cusp is a vertically scaled cycloid. The close-up view shows that, in actuality, a cardioid-shaped tire would be punctured by the cycloid's cusp. (b) Lowering the cycloid leads to the clover-shaped wheels. (c) A right-side-up cycloid touching the x-axis yields a wheel that takes infinitely many revolutions to pass over the cycloid's high point.

It turns out to be worth considering more general cycloidal roads, such as trochoidal roads represented by $(t - a\sin t, -1 + a\cos t)$. The usual computations show that the wheel is given by $r = 1 + a\sin\theta$, a limaçon that, when $a = 1$, is the just discussed cardioid. This limaçon intersects the positive y-axis at $(0, a \pm 1)$, points whose loci during the rolling are $(t \pm \sin t, (a \pm 1)\cos t)$. These loci are cycloids stretched in the y-direction. Note that there are two cases in which the stretching constant is 1—that is, the locus is an exact cycloid: The classical case $a = 0$, with

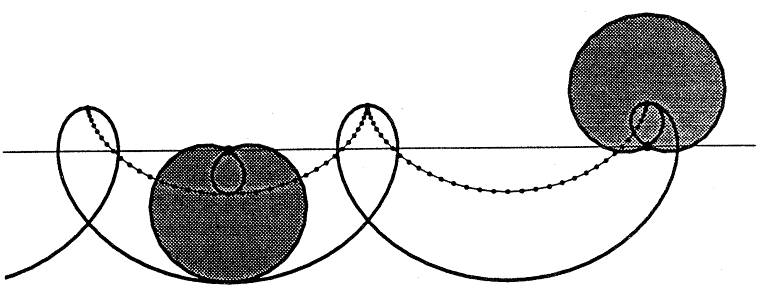


FIGURE 10
Rolling a limaçon along a trochoid yields an exact cycloidal locus.

straight road and round wheel, and the case when $a = 2$ (or -2), where the road is a trochoid, the wheel is a limaçon, and the locus of the top of the inner loop is an exact inverted cycloid (FIGURE 10). Are these the only two examples of road-wheel combinations for which a point on the wheel traces out an exact cycloid?

A road that is its own wheel Can there be a road for which the corresponding wheel is congruent to the road and for which the rolling motion matches points that correspond under the congruence? Consider the road that is given by the parabola $y = -x^2 - 1/4$. The differential equation yielding $\theta(x)$ is then $d\theta/dx = 1/(x^2 + 1/4)$, for which the solution is $\theta(x) = 2\arctan(2x) - \pi/2$, or $x = 1/2\tan(\pi/4 + \theta/2)$. The polar wheel then has the form $r(\theta) = -y(x(\theta))$, which simplifies to $r = 1/(2 - 2\sin\theta)$. This last is the graph of the parabola $y = x^2 - 1/4$, which is the reflection of the road in the line $y = -1/4$. FIGURE 11 shows this singular situation of a wheel rolling on itself. We leave the verification that corresponding points touch as an exercise. Robison [4] showed that this parabola is the only curve that has this property.

Note that raising or lowering the parabola changes the shape of the wheel dramatically (see FIGURE 11).

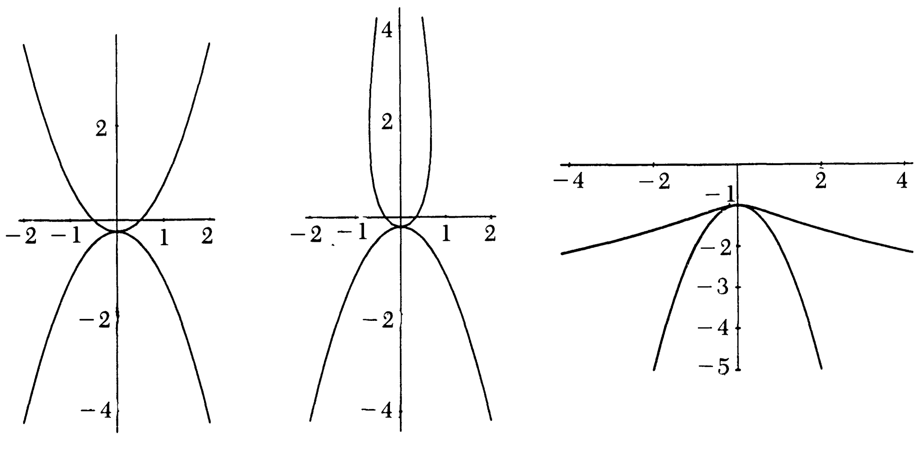


FIGURE 11

The wheel corresponding to the parabolic road given by $-x^2 - \frac{1}{4}$ is simply a reflection of the road itself. But the wheels for other parabolic roads are not parabolic.

Round wheels can roll on round roads A well-known puzzle can be interpreted as follows: What is the wheel corresponding to a road that is an upward-opening semicircle whose highest points are on the x-axis? Such a road is given by $f(t) = (\cos t, \sin t)$, $\pi \le t \le 2\pi$. The differential equation for $\theta(t)$ is simply $d\theta/dt = 1$, so $\theta(t) = t - \pi/2$ and $r(\theta) = -\sin(\theta + \pi/2) = -\cos\theta$, $\pi/2 \le \theta \le 3\pi/2$. Thus the wheel is a polar circle with geometrical center at $(-1/2, 0)$. The aforementioned puzzle is the one that asks for the locus of a point on the circumference of a circle that is rolling in a way tangent to the interior of a circle twice as large. The locus is a straight line, which shows itself in FIGURE 12 as the vertical lines in the arch-shaped locus.

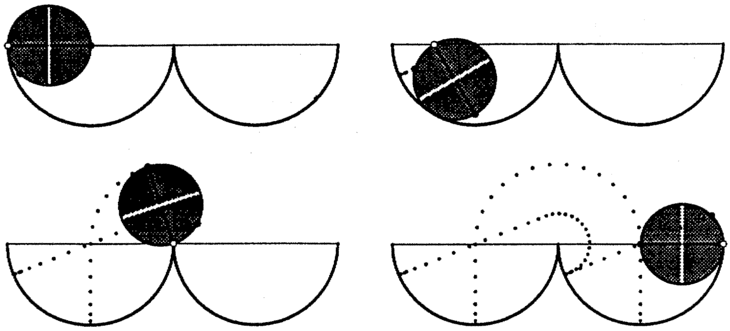


FIGURE 12
A circle rolls on inverted semicircle. The dotted paths are the loci of two points in the wheel's circumference.

Off-centered elliptical wheels Consider the ellipse given in polar form by $r = k\varepsilon/(1-\varepsilon \sin\theta)$, where $0 < \varepsilon < 1$ and $k > 0$; the origin, which corresponds to the axle of the wheel as it rolls, is one focus of the ellipse, the other focus is on the positive y-axis, ε is the eccentricity, and k is the distance from the origin to the corresponding directrix. Such an elliptical wheel rolls on the road $y = -(k\varepsilon/a^2)(1-\varepsilon\cos(cx))$, where $c = a/k\varepsilon$ and $a = \sqrt{1-\varepsilon^2}$. The derivation of this is slightly complex. Here's a sketch:

1. Solve the initial-value problem to get

$$\frac{ax}{2k\varepsilon} = \arctan\left(\frac{\tan(\theta/2)-\varepsilon}{a}\right) + \arctan\left(\frac{1+\varepsilon}{a}\right).$$

2. Take the tangent of the relationship in (1) and use some trig formulas to get

$$\frac{1+\varepsilon}{1-\varepsilon}\,\frac{1-\cos^2(cx)}{(1+\cos(cx))^2} = \frac{1+\sin\theta}{1-\sin\theta}.$$

3. Solve the preceding for $\sin\theta$, substitute into

$$y(x) = \frac{-k\varepsilon}{1-\varepsilon\sin\theta(x)},$$

and simplify to get the desired representation of the road as $(k\varepsilon/a^2)(1-\varepsilon\cos(cx))$.

If we set $k = 1$ and $\varepsilon = 1/\sqrt{2}$ then the road is just $y = -\sqrt{2} + \cos x$ (see FIGURE 13); this is a special case of one of the proper depths to lower the cosine so as to get a closed wheel (see Remark 5 in §1).

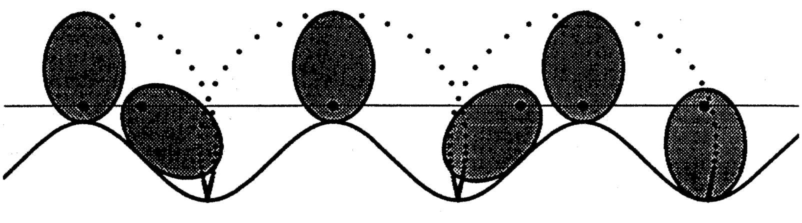


FIGURE 13
An ellipse rolls on a cosine curve. In the example shown the road is given by $y = -\sqrt{2} + \cos x$ and the ellipse has the polar form $r = 1/(\sqrt{2} - \sin\theta)$.

Centered elliptical wheels The preceding example involved an off-center elliptical wheel; that is, the axle is not at the ellipse's center. Let's now find the road on which an ellipse with centered axle will roll. The ellipse $(x/a)^2 + (y/b)^2 = 1$ has the polar representation $r = b/\sqrt{1 - m\cos^2\theta}$, where m abbreviates $1 - b^2/a^2$; in this representation the polar center—the axle of the wheel—coincides with the center of the ellipse. We can find the appropriate road by first finding the relationship between x and θ; the differential equation for θ separates to $\sqrt{1 - m\cos^2\theta}\,d\theta = (1/b)\,dx$ and the initial condition then leads to:

$$\int_{-\pi/2}^{\theta(x)} \frac{d\phi}{\sqrt{1 - m\cos^2\phi}} = \int_0^x \frac{dx}{b}.$$

Substituting $\psi = \phi + \pi/2$ yields $\sin\psi = \cos\theta$ and

$$\int_0^{\theta(x)+\pi/2} \frac{d\psi}{\sqrt{1 - m\sin^2\psi}} = \frac{x}{b}.$$

Now this is an incomplete elliptic integral of the first kind [**1**, 17.2.2, 17.2.17]. Therefore $x/b = F(\theta + \pi/2|m)$, which can be inverted by using what is known as the Jacobian elliptic function sn: $\sin(\theta + \pi/2) = \mathrm{sn}(x/b, m)$. The road is therefore given by $y = -r(\theta(x)) = -b/\sqrt{1 - m\,\mathrm{sn}^2(x/b, m)}$. The sn function is built into *Mathematica*, and so it is a simple matter to generate a diagram of the road and wheel. But some complications arise when one tries to use the closed form to generate an animation; they can be worked around by adding multiples of $\pi/2$ as x increases through the quarter-period value. But life is much simpler if we use the numerical approach of the next section to generate the animation. It is easiest to start with the ellipse and use the Case 2 discussion, as that avoids the computation of values of sn. The result of one such computation is shown in FIGURE 14 ($a = 1/2$, b = 1, eccentricity = 0.87). Ellipses with eccentricities greater than about 0.97 lead to a road-wheel crash similar to the one for the triangle (FIGURE 5).

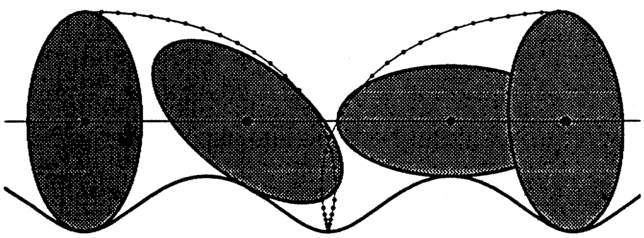


FIGURE 14
Four frames from an animation showing an elliptical wheel ($a = \frac{1}{2}, b = 1$) with its axle at its center rolling on a road defined using the elliptic sine function.

An elliptical road We can consider bounded roads as well as roads that protrude above the x-axis. As one example, consider the ellipse given parametrically by $(a\sin t, -b\cos t)$, where a and b are positive. The closed-wheel integral is $2\pi a/b$, and integer values of b/a lead to closed wheels that are familiar curves. The usual calculation shows that $t(\theta)$ is just $(b/a)(\theta + \pi/2)$, whence the wheel's polar form is $r(\theta) = -y(t) = b\cos[(b/a)(\theta + \pi/2)]$, a polar rose (FIGURE 15(a)). The locus here was a surprise to us as it turns out to be a piriform, which we had considered earlier in another context (§3, Case 4). If b/a is rational then the wheel is a rosette, as defined by Hall [**2**].

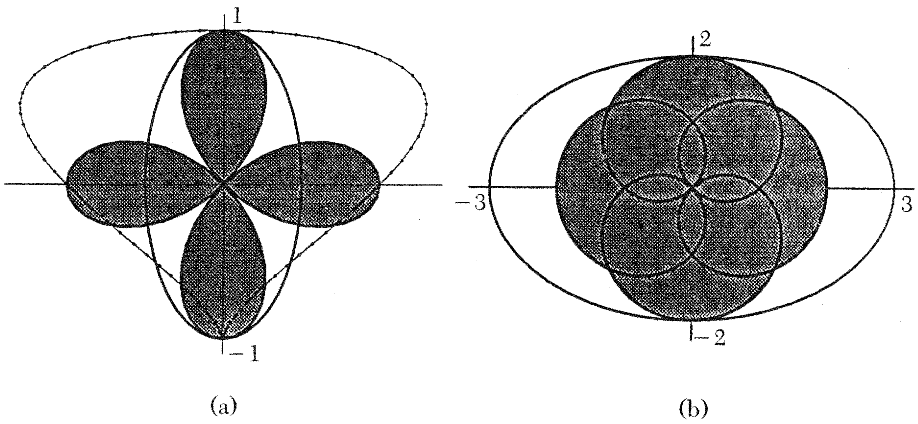


FIGURE 15
(a). A four-leafed rose rolls inside the ellipse $(\frac{1}{2}\sin t, -\cos t)$, and the locus of the tip of a petal is a piriform. (b) The wheels corresponding to more general ellipses [shown here is $(3\sin t, -2\cos t)$] are rosettes.

Vertical scaling We have seen in some of the preceding examples that raising or lowering the road usually changes the wheel significantly, and may destroy the closed-wheel condition. Another way to change the road is by scaling the y-coordinate: $y=f(x)$ becomes $y=kf(x)$, and $x=f_1(t)$, $y=f_2(t)$ becomes $x=f_1(t)$, $y=kf_2(t)$. The closed-wheel condition is affected as follows.

1. If the a-periodic road $y=f(x)$ has a closed wheel, then so does $y=kf(x)$ for any positive rational scaling factor k.
2. If the a-periodic road $y=f(x)$ does not have a closed wheel, the scaled road $y=kf(x)$ does have a closed wheel whenever k is a rational multiple of

$$\int_0^a \frac{-1}{2\pi f(x)}\,dx.$$

Similar results hold for roads defined parametrically.

As an example of scaling, consider the road $y=(\cos x)-b$, $b>1$. The scale factor $1/\sqrt{b^2-1}$ produces a closed wheel. Note that the scale factor goes to zero as b goes to infinity, and that the scaled road approaches $y=-1$. Since the wheels for these roads are ellipses, larger values of b correspond to wheels with smaller eccentricity.

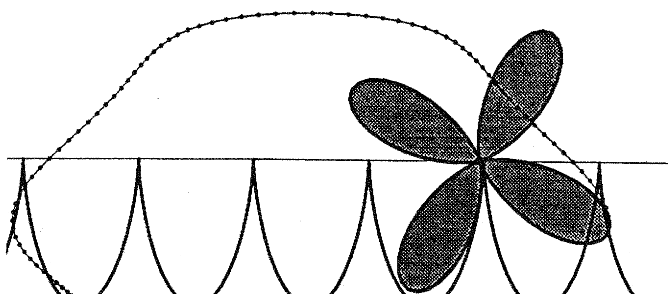


FIGURE 16
A scaled cycloidal road has a cuspitate rosette as its wheel.

Another case where scaling gives interesting road–wheel pairs is an inverted cycloid: $x = t + \sin t$, $y = k(-1 - \cos t)$. Assume k is a positive rational. Then the wheel is given by $r(\theta) = k(1 + \cos[k\theta + k\pi/2])$. For k a positive integer these wheels are cuspitate rosettes, curves similar in appearance to standard polar roses, but having cusps at their center. For more about cuspitate rosettes see [2] and the references therein.

Summary of some road–wheel relationships Here is a summary of road–wheel relationships including some not in closed form (12, 13) and some whose derivations are left as an exercise for the reader (14, 15, 16).

	Wheel	Form of Road	Locus
1.	Straight line	Catenary	
2.	Regular polygons	Piecewise catenaries	
3.	Circle	Horizontal line	Cycloid/Trochoid
4.	Circle, axle on circumference	Circle, radius doubled	Circular arch
5.	Equiangular spiral	Oblique line	
6.	Piecewise equiangular spiral	Sawtooth	
7.	Ellipse, axle at focus	Cosine	
8.	Ellipse, axle at center	$1/\sqrt{1 - (\text{elliptic sine})^2}$	
9.	Parabola $(x^2 - 1/4)$	Parabola $(-x^2 - 1/4)$	
10.	Cardioid; see FIGURE 9(a)	Inverted cycloid	Scaled cycloid
11.	Spirals, clovers; see FIGURE 9(b, c)	Lowered inverted cycloids	
12.	Pointed wheels; see FIGURE 19	Lowered cycloids	
13.	Piriform	See FIGURE 20	
14.	Hippopede	See FIGURE 18	
15.	Roses, rosettes	Ellipses	Piriform
16.	Limaçon	Trochoid	Cycloid
17.	Hyperbolic spiral $[r = 1/(\theta + \pi)]$	Exponential $[y = -(2/\pi)e^{-x}]$	
18.	Cuspitate rosettes	Scaled cycloids	

3. Generating solutions numerically

As we demonstrated in the last section, surprisingly many road–wheel problems are solvable in closed form. When one deals with arc length, however, functions that are not integrable in terms of elementary functions eventually show up. In this section we describe how to generate plots of roads and wheels numerically. Our procedures will be described for *Mathematica*, but the methods, with the possible exception of the animations, could be adapted to other software with graphics capabilities. What is needed is the ability to apply a numerical differential equation algorithm and plot points and lines.

The key step, as we have seen, is to solve the initial-value problem relating the road parameter and the polar angle. The differential equation involved is separable, but the resulting integrations can be daunting, even when feasible. Furthermore, we have often found this closed-form approach too slow to generate plots of the roads and wheels. A more efficient method is to solve the initial-value problem numerically using a standard Runge–Kutta algorithm; then a set of points defining the wheel or road can be generated with the help of the radius condition. The points are joined by lines to get the final image. The relationship between the road parameter and the polar angle is almost always nonlinear (an exception: a circular wheel rolling on a straight road), which means that constant velocity and constant angular velocity cannot be achieved simultaneously. The use of a fixed step-size in the numerical method can guarantee that, in an animation, one of these velocities is constant.

When animating one of these wheels rolling along its road, the relation between the polar angle θ and the horizontal coordinate x must be known. The rolling can be broken down into two parts: a rotation, given by the change in θ, and a translation, given by the corresponding change in x.

There are four cases to consider. Some stripped-down code for Case 1, the simplest case, is given in Section 5.

Case 1. Suppose the road is given by $y = f(x)$ and we want to generate the wheel. The initial-value problem is $d\theta/dx = -1/f(x)$, $\theta(0) = -\pi/2$. The Runge–Kutta method generates a table of ordered pairs (x, θ). This table and the radius condition are used to produce the wheel.

Example. Let $y = -1.887365 - (2/3)\cos x + \sin x - (1/2)\sin 2x$, where the constant term has been chosen so that the closed-wheel condition holds for one revolution per period. The wheel (FIGURE 17) is generated by using a Runge–Kutta method to solve the initial-value problem, taking the independent variable to be x. Thus, when animated, this wheel's axle moves with constant linear velocity. This example shows that wheels need not have an axis of symmetry.

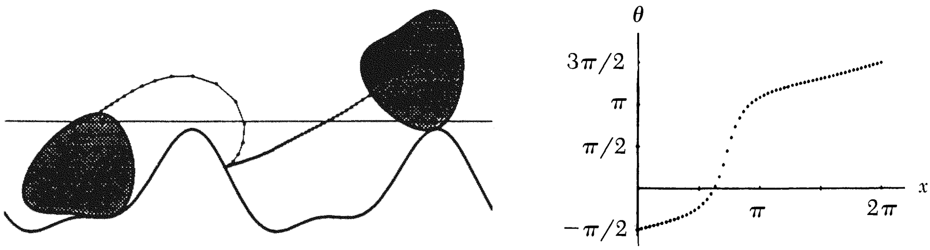


FIGURE 17
This example shows that the angular speed during rolling can vary a lot. The wheel rotates quickly when it is above the high bumps in the road, as illustrated by the steepness in the θ vs. x plot.

Case 2. Suppose the wheel is given by $r = g(\theta)$ and we want to generate the road. The initial-value problem is $dx/d\theta = g(\theta)$, $x(-\pi/2) = 0$. This time the numerical method gives pairs (θ, x), and the road can be generated using the radius condition.

Example. Let $r = 4\sqrt{5 - 4\sin^2\theta}$, a polar curve called a *hippopede* (see [3]). Again, Runge–Kutta is used to solve the initial-value problem, but this time it is convenient to take θ as the independent variable. Thus, when animated, the wheel (FIGURE 18) would exhibit constant angular velocity.

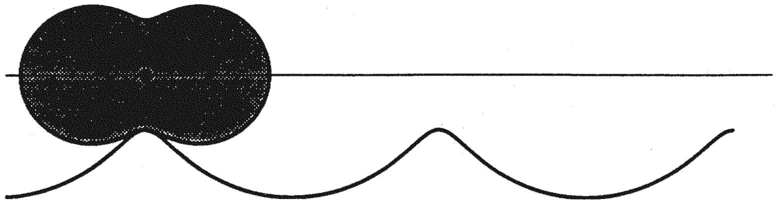


FIGURE 18
The example of a hippopedal wheel illustrates the case that the wheel is given in polar form and the road is found numerically.

Case 3. Suppose the road is given parametrically by $x = f_1(t)$, $y = f_2(t)$, where $f_1(0) = 0$. Now the initial-value problem relates the parameter t and the polar angle θ, and reduces to $d\theta/dt = -f_1'(t)/f_2(t)$, $\theta(0) = -\pi/2$. This time the numerical solution gives ordered pairs (t, θ), and the radius condition can be used to produce the wheel, very much as in Case 1.

Example. Let the road be a cycloid, lowered sufficiently so the closed-wheel condition (with an integer number, n, of periods per revolution) holds, and translated so the y-axis bisects one arch. The parametric equations are $x = t + \sin t$, $y = \cos(t) - d_n$. For this cycloid, the closed-wheel condition is

$$\int_0^{\pi} \frac{1 + \cos t}{d_n - \cos t}\, dt = \frac{\pi}{n}$$

for which the positive solutions are $d_n = 1 + 2n^2/(2n + 1)$. (One could also consider negative solutions: $d_1 = -1$ yields a road with a cardioid wheel similar to that in FIGURE 9.) Using the table of (t, θ) values obtained with the Runge-Kutta algorithm, and the radius condition, we get the wheels corresponding to different periods (see FIGURE 19). In this case both x and θ are given as functions of the parameter t, so in an animation neither will increase linearly. FIGURE 19 also shows the case of a period-1/2 roller; that is, $n = 1/2$, $d = 1\frac{1}{4}$ and the wheel rotates twice for each period of the cycloid. For the $n = 0$ case (infinitely many revolutions) see FIGURE 9(c).

Case 4. Suppose the wheel is given parametrically by $x = g_1(t)$, $y = g_2(t)$. In order to get a simple closed wheel, assume g_1 and g_2 are periodic with the same period. In terms of g_1 and g_2, the initial-value problem relating the x-coordinate of the road with the parameter t is:

$$\frac{dx}{dt} = \frac{g_1(t)g_2'(t) - g_1'(t)g_2(t)}{\sqrt{g_1(t)^2 + g_2(t)^2}}, \qquad x(0) = 0,$$

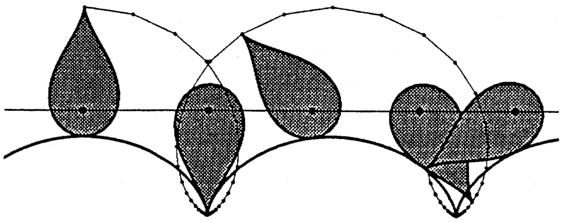
The numerical method produces pairs (t, x), and if care is taken to use the same t-values, the corresponding coordinates of the road are found from the radius condition to be given by: $y(t) = -\sqrt{g_1(t)^2 + g_2(t)^2}$.

To generate an animation of this case, we must also have the polar angle θ in terms of t. Unfortunately, there are complications involving branches of the arctangent function that prevent the direct use of $\theta = \arctan[g_2(t)/g_1(t)]$, so we generate another table of values, this time (t, θ), by applying Runge–Kutta to:

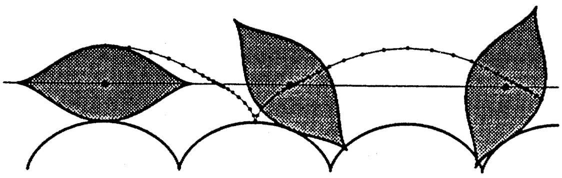
$$\frac{dx}{dt} = \frac{g_1(t)g_2'(t) - g_1'(t)g_2(t)}{\sqrt{g_1(t)^2 + g_2(t)^2}}, \qquad \theta(0) = -\frac{\pi}{2},$$

again being careful to use the same t-values as were used to generate the (t, x) pairs. Together, the (t, x) pairs and the (t, θ) pairs define a set of (x, θ) pairs that can be used to animate the wheel.

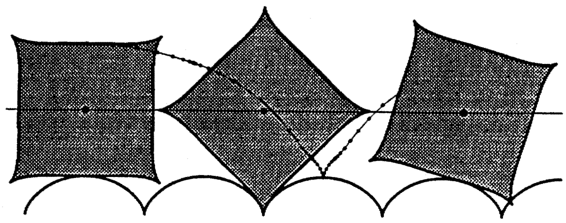
Example. Let the wheel be defined by $x = -\sin t + (1/2)\sin 2t$, $y = -\cos t$. This shape (FIGURE 20) is known as a *piriform* (again, see [3]). As in the example for Case 3, neither linear velocity nor angular velocity will be constant in an animation.



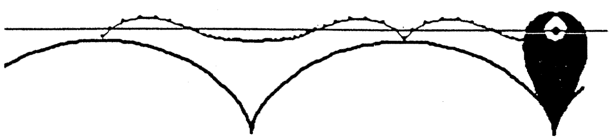
$n = 1$



$n = 2$



$n = 4$



$n = \frac{1}{2}$

FIGURE 19

Some wheels corresponding to a cycloidal road. These wheels can be given in closed form, but when generating images or animations it is much more convenient to ignore the closed form and just use the numerical approach. The top wheel covers one cycloid period during each revolution; the next covers two; the next covers four. The bottom wheel covers a half-period of the road during each revolution and corresponds to $n = \frac{1}{2}$ in the closed-wheel condition.

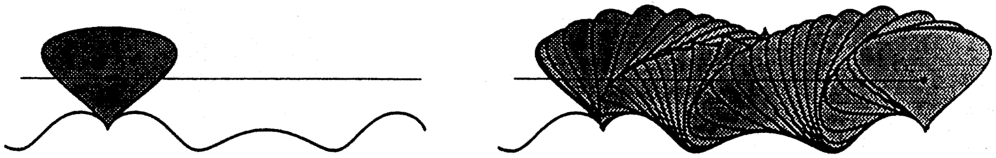


FIGURE 20

The piriform is an example of a parametrically given wheel. Its road is found, as in the other cases, by numerically solving the appropriate initial-value problem.

Remark. In all four cases, the choice of independent variable in the Runge–Kutta step is arbitrary. If an animation is planned, then this choice can be made to cause a particular quantity, usually x or θ, to change linearly in the animation.

4. Squaring the circle with Fourier series

A natural thought when dealing with periodic functions, such as the roads for closed wheels, is to look at the Fourier series. For wheels with an axis of symmetry through the axle, we can make the periodic function even, thus yielding a cosine series. The question then arises as to what shape the wheels for the various Fourier approximations will have. Clearly, for the approximation using only the constant term, the wheel will be a circle and, as the Fourier series more closely approximates the road, the wheels for the Fourier approximations will more closely approximate the original wheel. Thus if we begin with the road for a rolling square, the wheel for the Fourier approximations will "square the circle." Obviously, we are not restricting ourselves to Euclidean tools! There is one problem with this process, however: The Fourier approximations fail to satisfy the closed-wheel condition.

For example, suppose $p(x) = \cos x - \sqrt{5}$, which is a finite Fourier series already with period 2π, satisfying the closed-wheel condition. The constant term or 0th Fourier approximation, is $p_0(x) = -\sqrt{5}$, and so $\sqrt{5}$ is the radius of the corresponding wheel (a circle). It was shown earlier that the wheel for $p(x)$ traverses two periods of the road in one revolution, so the circumference of the circular wheel must be 4π (taking the period of the constant function the same as that of $p(x)$). But this makes the radius of the wheel equal to 2, a contradiction.

The case when $p(x)$ is the road for a square wheel is similar. Here, the circular wheel that rolls on the 0th Fourier approximation has circumference 7.12866 instead of $8 \operatorname{arcsinh} 1 \approx 7.05099$. And the closed-wheel condition fails here by 0.00138 for the two-term Fourier approximation road. Of course, as the trigonometric polynomials more closely approximate the original road, they will also come closer to satisfying the closed-wheel condition, but the condition fails nevertheless for each Fourier approximation.

What is needed, then, is a sequence of approximations, each satisfying the same closed-wheel condition as the road, which converges to the road. Such a sequence can be constructed by exploiting the orthogonality of the cosine functions in a Fourier series, along with the fact that the closed-wheel condition involves the reciprocal of the road function.

Road approximations having closed wheels Let $p(x)$ be a continuous, even, negative, periodic function with period T. Then $q(x) = 1/p(x)$ is also continuous, even, negative, and T-periodic, and so can be expanded in a Fourier cosine series. Denote the partial sums of the Fourier series for $q(x)$ by $q_n(x)$, $n = 0, 1, 2, \dots$. The q_n's converge to q, are continuous, even, T-periodic, and, for large enough n, negative. If, in addition, we assume that the derivative of q is piecewise continuous, the convergence is uniform. Finally, we shall assume that q is "nice enough" so that all the q_n's are negative, which is the case in all our examples. We now form the sequence $\{p_n(x)\}$, where $p_n(x) = 1/q_n(x)$, which converges uniformly to $p(x)$, and we shall use this sequence to approximate the road. The p_n's have the same period as p and satisfy the same closed-wheel condition because

$$\int_{-T/2}^{T/2} - \frac{dx}{p_n(x)} = \int_{-T/2}^{T/2} - q_n(x)\,dx,$$

and, because the definite integrals of the cosines in the Fourier series vanish,

$$\int_{-T/2}^{T/2} - q_n(x)\,dx = \int_{-T/2}^{T/2} - q(x)\,dx = \int_{-T/2}^{T/2} - \frac{dx}{p(x)}.$$

Approximations to the square wheel Recall that the road for a square wheel with side 2 is the periodic extension of $y = -\cosh x$, for $-\operatorname{arcsinh} 1 \le x \le \operatorname{arcsinh} 1$. The first seven Fourier coefficients for the reciprocal are: -0.891107, -0.12537, 0.0230828, -0.0100959, 0.0056363, -0.00359458, 0.00249146. These were found using the standard formulas and integrating numerically. The first three approximations to the road are:

$$p_0(x) = -1.1222$$
$$p_1(x) = 1/(-0.891107 - 0.12537\cos(3.56443x))$$
$$p_2(x) = 1/(-0.891107 - 0.12537\cos(3.56443x) + 0.0230828\cos(7.12886x))$$

For the wheel corresponding to the approximation $p_k(x)$, the polar angle as a function of x is described by the initial-value problem $d\theta/dx = -1/p_n(x)$, $\theta(0) = -\pi/2$. This can be integrated in closed form since $-1/p_k(x)$ is a trigonometric polynomial, but the numerical method described in Section 3 is more efficient, especially for producing animations.

FIGURE 21 shows two corresponding positions of the road-wheel pairs for $p_0(x)$, $p_1(x)$, $p_2(x)$, and $p_6(x)$. Note how the circle becomes square-like very quickly.

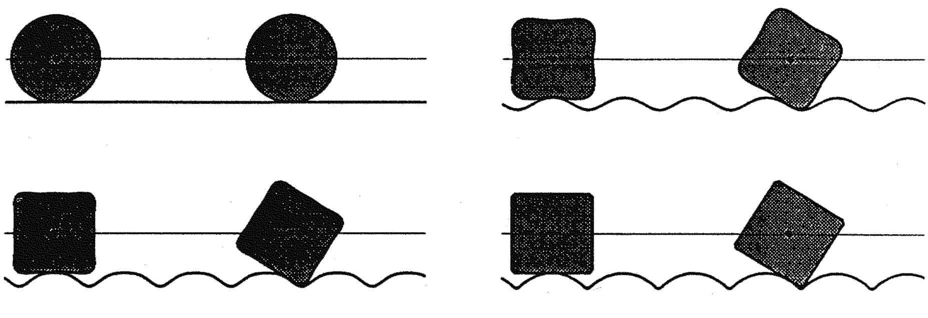


FIGURE 21
Approximating a catenary road with partial sums of its Fourier series yields wheels that transform a circle to a square.

5. A *Mathematica* wheel-building package

We have written a complete *Mathematica* package (version 1.2) that generates roads from wheels and vice versa. The package has options by which the user can generate stills or animations, with or without a locus, spokes, shading, and so on. The notebook can be obtained by sending a Macintosh disk to one of the authors. We include here a bare-bones version of the routine to give some idea of how such a program is written. This routine takes a function defining a road and displays the road and the wheel with its axle at the origin.

Even when a closed-form solution is available, it is often simpler to generate a diagram or animation by taking a numerical approach. Nevertheless, sometimes the closed form must be used. For example, rolling polygons are best generated directly, without numerical approximations. Some code for doing so can be found in the Appendix to [6].

```
RoadMovie[func_, {x_, xmax_}] := Module[{theta1, theta},
theta1[x_] = theta[x] /.
First @ NDSolve[
          {theta' [x] == -1/func, theta[0] == -Pi/2},
          theta[x], {x, 0, xmax}, PrecisionGoal->2];
ParametricPlot[{-func {Cos[theta1[x]], Sin[theta1[x]]}, {x, func}},
  {x, 0, xmax}, PlotRange->All, AspectRatio->Automatic] ]
```

For example, `RoadMovie[Cos[x]-Sqrt[10], {x, 6 Pi}]` will generate the period-3 closed wheel for a cosine road (similar to FIGURE 2, but with $n = 3$).

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If you ask mathematicians what they do, you always get the same answer. They think. They think about difficult and unusual problems. They do not think about ordinary problems: they just write down the answers.

M. Egrafov (translated from Russian), contributed by the late R.P. Boas, Jr.