Fast Marching Method

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Cours GM5 : "Equations de Hamilton-Jacobi et applications"

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Plan

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1. Level Set Method
2. La méthode Fast Marching classique

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3. Generalized Fast Marching Method

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4. Some other extension

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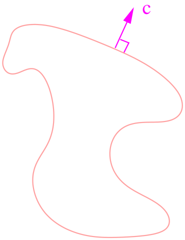
5. Simulations

Evolution d'un front



On s'intéresse à l'évolution d'un front Γ_t dans la **direction normale** et avec une vitesse $c = c(y) > 0, \; y \in \mathbb{R}^d$.

L'équation d'évolution de ce front s'écrit

$$\frac{d\Gamma_t}{dt} = c \, n_{\Gamma_t}$$

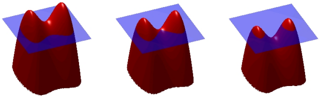
où n_{Γ_t} désigne la **normale unitaire** extérieure au front.

La méthode Level-Set

La méthode Level-Set est une méthode très populaire pour les problèmes de mouvements de fronts.

L'idée principale est de représenter le front par **la ligne de niveau zéro** d'une fonction u:

$$\Gamma_t = \{x,\ u(x,t) = 0\}.$$



La méthode Level-Set

L'inconvénient majeur est d'ajouter **une dimension supplémentaire** au problème, ce qui peut engendrer des coûts de calcul non négligeables.

Soit donc u une fonction telle que

$$\begin{cases} u(x,t) < 0 & \text{si } x \text{ est situé } \textbf{à l'intérieur} \text{ de } \Gamma_t, \\ u(x,t) = 0 & \text{si } x \text{ est situé } \textbf{appartient} \text{ à } \Gamma_t, \\ u(x,t) > 0 & \text{si } x \text{ est situé } \textbf{à l'extérieur} \text{ de } \Gamma_t. \end{cases}$$

On a

$$n_{\Gamma_t} = \frac{\nabla_x u(x,t)}{|\nabla_x u(x,t)|}$$

La méthode Level-Set

Rappelons que

$$\Gamma_t = \{x,\ u(x,t) = 0\}.$$

Formellement:

$$u(\Gamma_t, t) = 0 \Rightarrow u_t(x,t) + \nabla_x u(x,t) \cdot \frac{\partial \Gamma_t}{\partial t} = 0.$$

avec $\dfrac{\partial \Gamma_t}{\partial t} = c(x) n_{\Gamma_t},\ n_{\Gamma_t} = \dfrac{\nabla_x u(x,t)}{|\nabla_x u(x,t)|}$.

- On aboutit à **l'équation eikonale** suivante:

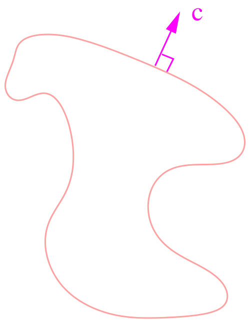
$$u_t(x,t) + c(x)|\nabla_x u(x,t)| = 0$$

Plan

La méthode Fast Marching

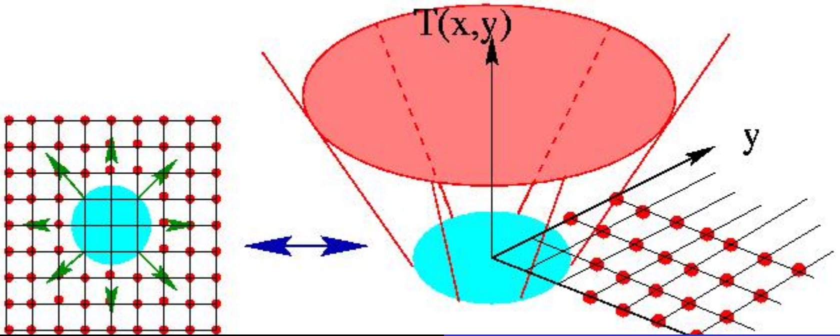
Le but de la méthode Fast Marching est de résoudre efficacement l'équation Eikonale

$$u_t(x,t) + c(x)|\nabla_x u(x,t)| = 0$$



Approche stationnaire

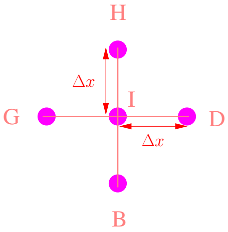
- On utilise une approche stationnaire en posant $u(x,t) = T(x) - t$
  $\Rightarrow |\nabla T(x)| = \dfrac{1}{c(x)}$.
- $T(x)$ représente le temps d'arriver du front au point x.



Description de la méthode

L'objectif est donc de **calculer numériquement** le temps d'arrivée T solution de **l'équation stationnaire**

$$|\nabla T(x)| = \frac{1}{c(x)}.$$



$$\max(T_I - T_G, T_I - T_D, 0)^2 + \max(T_I - T_H, T_I - T_B, 0)^2 = \left(\frac{\Delta x}{c_I}\right)^2 \quad (1)$$

Description de la méthode

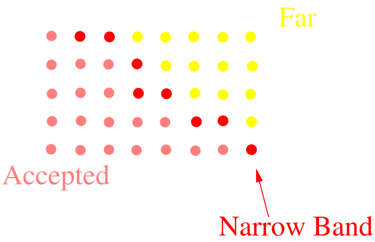
Une première idée serait d'appliquer un schéma itératif pour résoudre ce problème non linéaire et a attendre la convergence de l'algorithme à la précision souhaitée. **Ceci peut être très long!**

Il est en fait possible de calculer les valeurs T_I dans un ordre spécial qui permet d'obtenir la convergence en une seule itération. Cet ordre spécial correspond au **valeur croissante de** T_I.

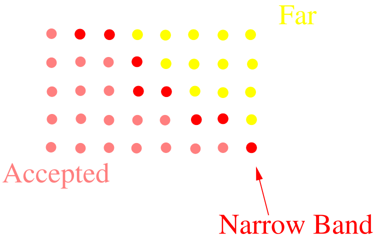
Introduction de la Narrow Band

Pour cela, on va répartir les points en trois régions:

- **Accepted points :** Il s'agit des points qui ont déjà été atteints par le front et pour lesquels on connait déjà la valeur de T_I
- **Narrow band :** Ce sont les points qui n'ont pas encore été atteints par le front mais qui sont sur le point de l'être, c'est à dire ayant un voisin qui a été atteint par le front.
- **Far away :** Ce sont les autres points.

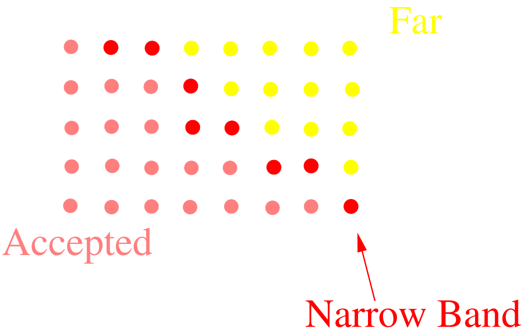


Algorithme de la méthode: initialisation



- On définit la région **accepted** au temps initial comme étant les points à l'intérieur de Γ_t et on initialise T à 0 sur ces points.
- On définit la **Narrow Band** comme étant l'ensemble des points qui ne sont pas accepté mais qui ont un voisin accepté.
- La région **Far away** est le reste des points.

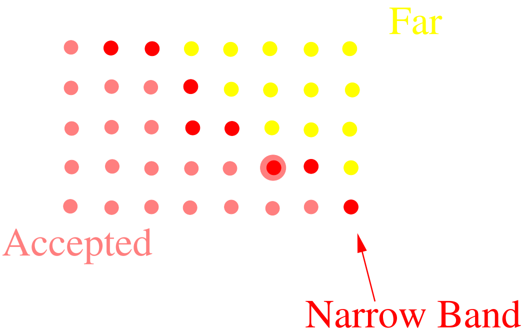
Algorithme de la méthode: boucle principale



- Calculer T_I sur la Narrow Band en résolvant:

$$\max(T_I - T_G, T_I - T_D, 0)^2 + \max(T_I - T_H, T_I - T_B, 0)^2 = \left(\frac{\Delta x}{c_I}\right)^2 \quad (2)$$

Algorithme de la méthode: boucle principale



- On cherche la plus petite valeur de T_I sur la Narrow Band.

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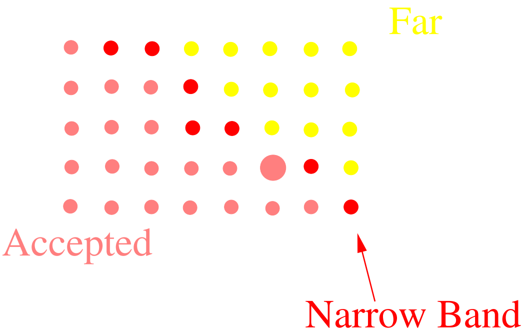
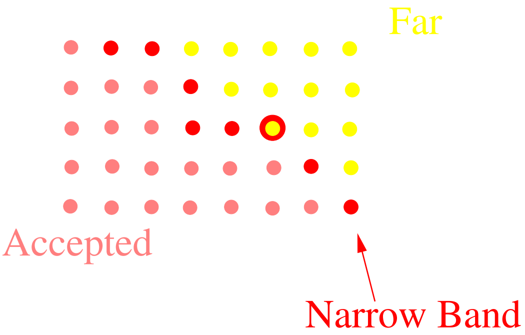


- On cherche la plus petite valeur de T_I sur la Narrow Band. Le point correspondant devient accepté.

Algorithme de la méthode: boucle principale



- On cherche la plus petite valeur de T_I sur la Narrow Band. Le point correspondant devient accepté.
- La nouvelle Narrow band est redéfinie comme le bord de la région acceptée.

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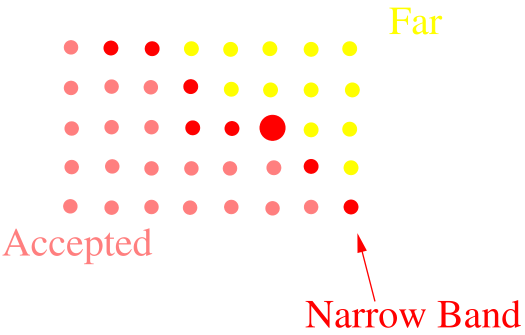

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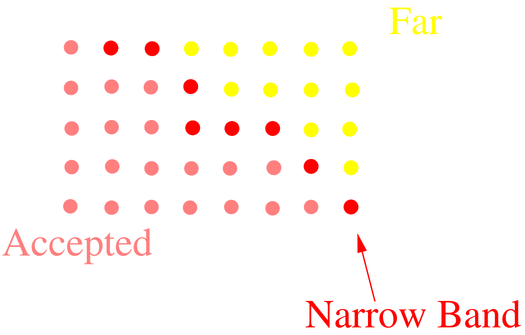

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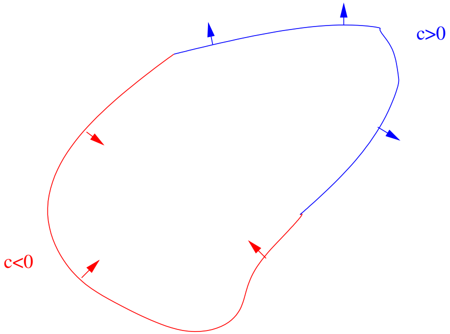
- La nouvelle Narrow band est redéfinie comme le bord de la région acceptée.

Complexité numérique

- Calcul du temps T_I : au plus 4 fois pour chaque noeud.
- Recherche du minimum: en utilisant un arbre binaire, le coût est en $O(\ln(N_{NB}))$.
- Le coût global est donc en $O(N \ln(N))$ (N représente le nombre total de point sur la grille).

General case: no stationnary representation

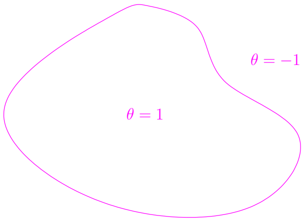
The notion of “Accepted” points is not adapted to generalize the algorithm.



Rewriting the classical FMM

Introduction of a field θ to represent the front:

$$\begin{cases} \theta = 1 & \text{inside} \\ \theta = -1 & \text{outside} \end{cases}$$



Rewriting the classical FMM

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- "Accepted" points at step $n = \{I,\ \theta_I^n = 1\}$
- Narrow Band at step $n = \{I,\ \exists\ J \in V(I), \theta_J^n = -\theta_I^n = 1\} = NB^n$
- Useful points at step n : if $I \in NB^n$,

$$\mathcal{U}^n(I) = \{J \in V(I),\ \theta_J^n = 1\}, \quad \mathcal{U}^n = \cup_{I \in NB^n} \mathcal{U}^n(I)$$

Algorithm

Initialization

1 $\theta_I^0 = \begin{cases} 1 & \text{if } x_I \in \Omega_0 \\ -1 & \text{otherwise} \end{cases}$

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Initialization

1. $\theta_I^0 = \begin{cases} 1 & \text{if } x_I \in \Omega_0 \\ -1 & \text{otherwise} \end{cases}$
2. $t_0 = 0$

   $T_I^0 = \begin{cases} 0 & \text{if } I \in \mathcal{U}^0 \\ +\infty & \text{otherwise} \end{cases}$

   $n = 1$

Algorithm

Loop

3 Computation of the candidate time $\tilde{T}_I$ for $I \in NB^{n-1}$:

$$\begin{array}{c} \max(\tilde{T}_I^{n-1} - T_G^{n-1}, \tilde{T}_I^{n-1} - T_D^{n-1}, 0)^2 \\ + \\ \max(\tilde{T}_I^{n-1} - T_H^{n-1}, \tilde{T}_I^{n-1} - T_B^{n-1}, 0)^2 \end{array} = \frac{(\Delta x)^2}{|c_I|^2}$$

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4 $t_n = \inf_{I \in NB^{n-1}} \tilde{T}_I^{n-1}$

Algorithm

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8 $n := n + 1$

Convergence result

We define

$$\theta^\varepsilon(x,t) = \theta_I^n \text{ if } x \in [x_I, x_I + \Delta x[,\ t \in [t_n, t_{n+1}[.$$

Theorem (Carlini, Falcone, F., Monneau)

Under regularity assumptions on Ω_0 and c,we have

$$\theta^\varepsilon \to \theta$$

solution of

$$\begin{cases} \theta_t = c(x)|\nabla\theta| \\ \theta(t=0,\cdot) = 1_{\Omega_0} - 1_{\Omega_0^c} \end{cases}$$

Case $c(x,t) > 0$

Loop

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If $t_n = t_{n-1} + \Delta t < \tilde{t}_n$, then go to 3 with $n := n+1$.

Case $c(x,t) > 0$

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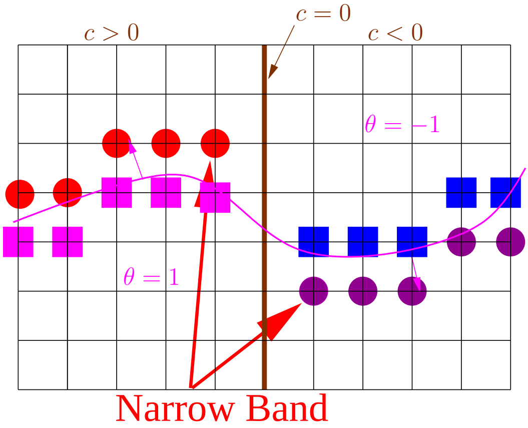
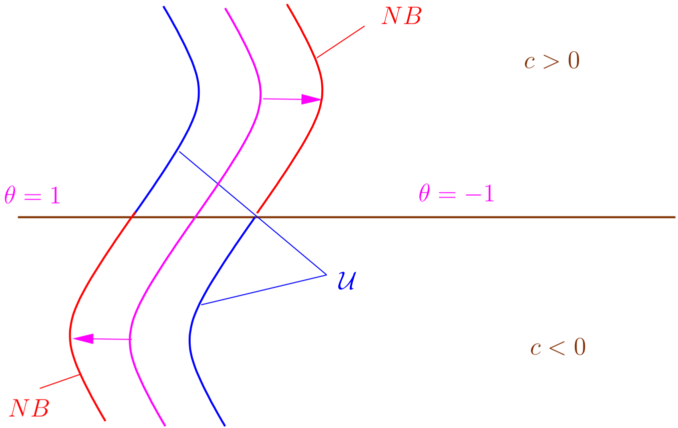
Plan

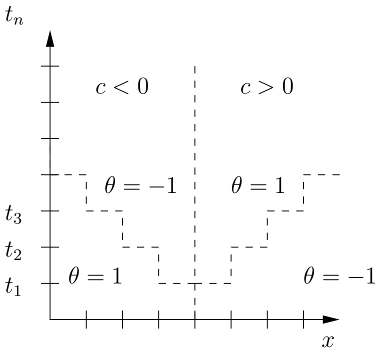
Idea of the GFMM



Schematic representation



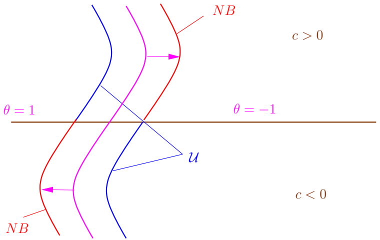
Regularisation of the speed



Definition

$$\hat{c}_I^n = \begin{cases} 0 & \text{if } \exists\, J \in V(I) \text{ t.q. } c_I^n c_J^n < 0 \text{ et } |c_I^n| < |c_J^n| \\ c_I^n & \text{otherwise} \end{cases}$$

Definition



- $NB^n = \{I,\ \exists\ J \in V(I),\ \theta_I^n = -\theta_J^n \text{ and } \hat{c}_I^n \theta_I^n < 0\}$

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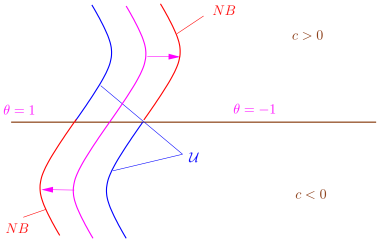

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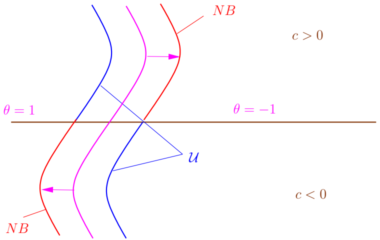

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  $\mathcal{U}^n = \cup_{I \in NB^n} \mathcal{U}^n(I)$
- $T_{J \to I}^n = \begin{cases} T_J^n & \text{if } J \in \mathcal{U}^n(I) \\ +\infty & \text{otherwise} \end{cases}$

Algorithm

Loop

3 Computation of the candidate time $\tilde{T}_I$ for $I \in NB^{n-1}$:

$$\begin{array}{c} \max(\tilde{T}_I^{n-1} - T_{G\to I}^{n-1}, \tilde{T}_I^{n-1} - T_{D\to I}^{n-1}, 0)^2 \\ + \\ \max(\tilde{T}_I^{n-1} - T_{H\to I}^{n-1}, \tilde{T}_I^{n-1} - T_{B\to I}^{n-1}, 0)^2 \end{array} = \frac{(\Delta x)^2}{|c_I^{n-1}|^2}$$

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8 $n := n + 1$

Convergence result

- Increasing sequence of time: $(t_{n_k})_{k\in\mathbb{N}}$ s.t.

$$t_{n_k} = t_{n_k+1} = ... = t_{n_{k+1}-1} < t_{n_{k+1}}$$

Convergence result

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$$t_{n_k} = t_{n_k+1} = ... = t_{n_{k+1}-1} < t_{n_{k+1}}$$

- We define

$$\theta^\varepsilon(x,t) = \theta_I^{n_{k+1}-1} \text{ if } x \in [x_I, x_I + \Delta x[,\ t \in [t_{n_k}, t_{n_{k+1}}[.$$

Convergence result

- Increasing sequence of time: $(t_{n_k})_{k\in\mathbb{N}}$ s.t.

$$t_{n_k} = t_{n_k+1} = \ldots = t_{n_{k+1}-1} < t_{n_{k+1}}$$

- We define

$$\theta^\varepsilon(x,t) = \theta_I^{n_{k+1}-1} \text{ if } x \in [x_I, x_I + \Delta x[,\ t \in [t_{n_k}, t_{n_{k+1}}[.$$

- half relaxed limits:

$$\overline{\theta}(x,t) = \limsup_{\varepsilon\to 0, y\to x, s\to t} \theta^\varepsilon(y,s), \quad \underline{\theta}(x,t) = \liminf_{\varepsilon\to 0, y\to x, s\to t} \theta^\varepsilon(y,s),$$

Convergence result

Theorem (Carlini, Falcone, F., Monneau)

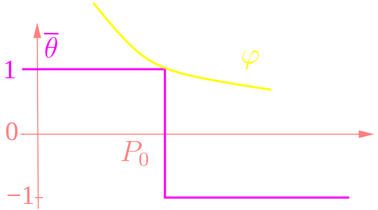
Under regularity assumptions on Ω_0 and c, we have that $\overline{\theta}$ is a sub solution and $\underline{\theta}$ is a super solution of

$$\begin{cases} \theta_t = c(x,t)|\nabla\theta| \\ \theta(t=0,\cdot) = 1_{\Omega_0} - 1_{\Omega_0^c} \end{cases} \tag{2}$$

In particular, if (2) satisfies a comparison principle, then $\overline{\theta} = (\underline{\theta})^*$ is the unique usc solution of (2)

Idea of the proof

- Assume that $\overline{\theta}^0$ is not a subsolution at P_0:

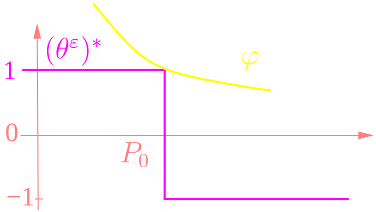


with

$$\varphi_t(P_0) = \bar{c}\,|\nabla\varphi(P_0)| > 0 \quad \text{and} \quad \bar{c} > c(P_0)$$

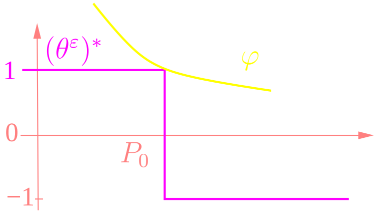
Idea of the proof

- Deduce at the ε-level that (with $P_\varepsilon = P_0$ to simplify)



Idea of the proof

- Deduce at the ε-level that (with $P_\varepsilon = P_0$ to simplify)



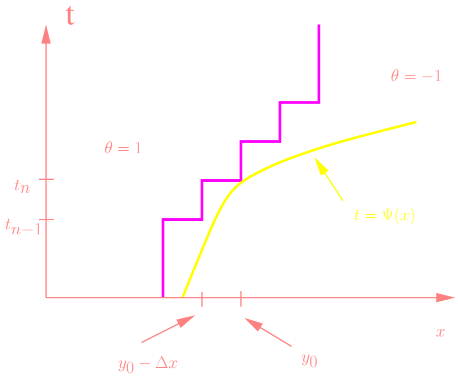
- Define ψ by

$$\varphi(x, \psi(x)) = 1$$

Idea of the proof

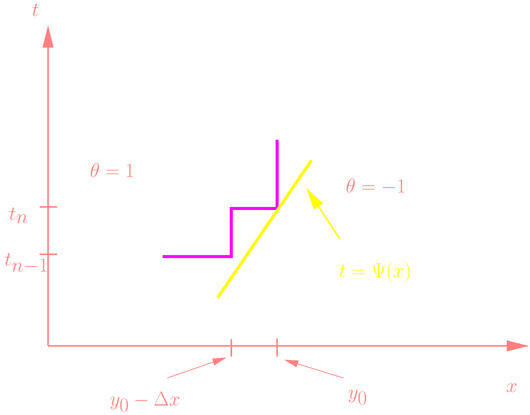
- To simplify, assume in 1D that

$$P_0 = (y_0, t_n) \quad \text{with} \quad \frac{y_0}{\Delta x} \in \mathbb{Z}$$

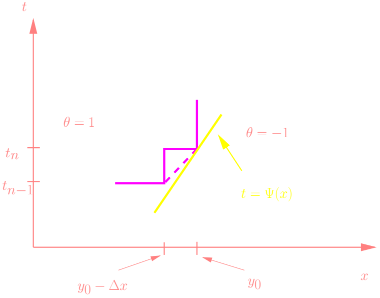


Idea of the proof

- To simplify, assume that φ is linear



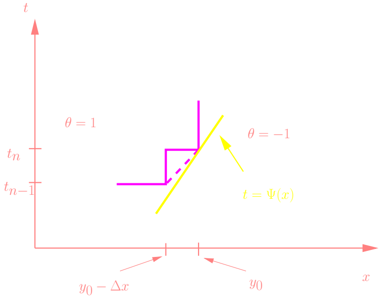
Idea of the proof



Then

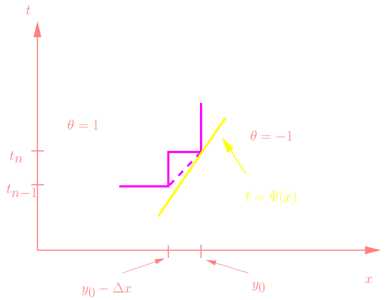
$$\psi' \geq \frac{t_n - t_{n-1}}{\Delta x}$$

Idea of the proof



$$\frac{1}{\overline{c}} = \psi' \geq \frac{t_n - t_{n-1}}{\Delta x} = \frac{1}{c(y_0 - \Delta x, t_{n-1})} = \frac{1}{c(P_0)} + o(1)$$

Idea of the proof



$$\frac{1}{\overline{c}} = \psi' \geq \frac{t_n - t_{n-1}}{\Delta x} = \frac{1}{c(y_0 - \Delta x, t_{n-1})} = \frac{1}{c(P_0)} + o(1)$$

Contradiction because $\overline{c} > c(P_0) > 0$

Numerical complexity

Assume that the velocity is constant in each intervall $[k\Delta T, (k+1)\Delta T)$ for some $\Delta T > 0$.

1. Constant in time velocity ($\Delta T = +\infty$): $O(N \ln N)$.
2. $O(\frac{1}{\sqrt{N}}) \leq \Delta T < +\infty$: $O(N \ln N)$.
3. $0 \leq \Delta T < O(\frac{1}{\sqrt{N}})$: $O(N^{\frac{3}{2}})$.

Comparison principle

For a GFMM algorithm slightly different, we have the following theorem :

Theorem (F.)

We consider 2 GFMM with speed c_u (θ_u) and c_v (θ_v). Assume that

$$\inf_{s\in[t-\Delta t,t]} c_v(x,s) \geq \sup_{s\in[t-\Delta t,t]} c_u(x,s).$$

If $\Omega_u^0 \subset \Omega_v^0$ then

$$\theta_u^\varepsilon(x,t) \leq \theta_v^\varepsilon(x,t).$$

Plan

Group Marching Method

- **Idea**: accept a group of points at every iteration [Kim]
- **Advantages**: $O(N)$ complexity.
- **Disadvantages**: introduction of some small errors and specifically design for the eikonal equations

Buffered Fast Marching Method

- **Anisotropic evolution**: a value T can depend on values greater than T
- **Idea**: do not accept the node with the minimum value BUT put it in a buffer. All the node of the buffer are recomputed until that their value is stabilized [Cristiani]
- **Advantages**: anisotropic evolution
- **Disadvantages**: very long and only for monotone evolution

GFMM for non local velocity

- GFMM for the dislocation dynamics:

$$c(x,t) = c_0 \star 1_{\Omega_t}.$$

- Convergence result obtained using the comparison principle for the GFMM [Carlini, F., Monneau]

Dislocation line dynamics

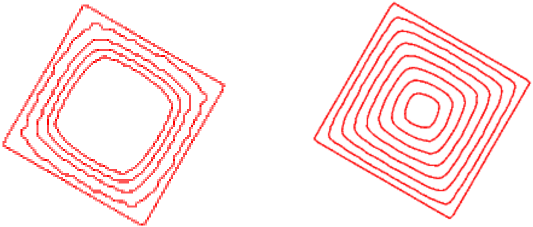
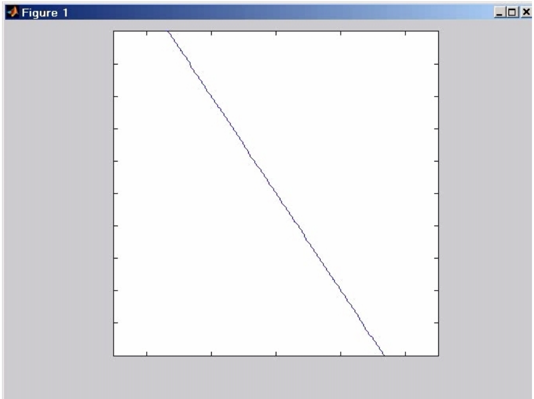


Figure : Finite difference (left) versus GFMM (right).

Plan

A straight line

$$c(x,t) = x_1$$



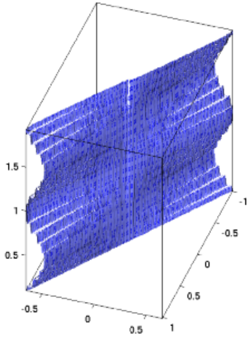
A straight line

	GFMM		**FD**	
Δx	$\mathcal{H}(\mathcal{C},\tilde{\mathcal{C}})$	CPU	$\mathcal{H}(\mathcal{C},\tilde{\mathcal{C}})$	CPU
0.04	$5.08 \cdot 10^{-2}$	$0.19s$	$4.10 \cdot 10^{-2}$	$1.82s$
0.02	$2.72 \cdot 10^{-2}$	$0.73s$	$2.05 \cdot 10^{-2}$	$13.2s$
0.01	$1.35 \cdot 10^{-2}$	$3.98s$	$1.03 \cdot 10^{-2}$	$102s$
0.005	$6.80 \cdot 10^{-3}$	$76s$	$2.60 \cdot 10^{-3}$	$810s$

Table : Hausdorff distance: GFMM versus Finite difference (FD)

A straight line with a velocity depended on time

$$c(x,t) = \sin(2\pi t)x_1$$



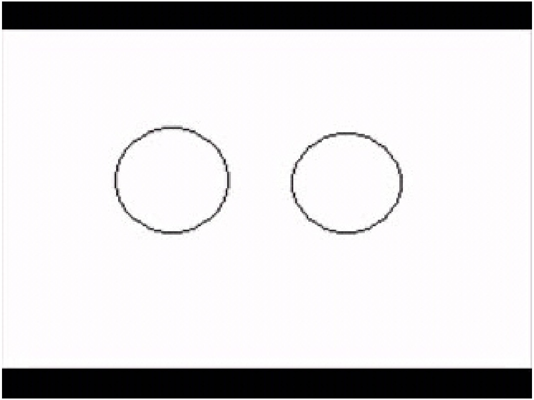
A straight line with a velocity depended on time

	GFMM		**FD**	
Δx	$\mathcal{H}(\mathcal{C},\tilde{\mathcal{C}})$	CPU	$\mathcal{H}(\mathcal{C},\tilde{\mathcal{C}})$	CPU
0.04	$5.21 \cdot 10^{-2}$	$0.52s$	$4.82 \cdot 10^{-2}$	$1.82s$
0.02	$3.07 \cdot 10^{-2}$	$1.71s$	$2.46 \cdot 10^{-2}$	$13.3s$
0.01	$1.54 \cdot 10^{-2}$	$10.5s$	$1.35 \cdot 10^{-2}$	$102s$
0.005	$9.00 \cdot 10^{-3}$	$130s$	$7.00 \cdot 10^{-3}$	$842s$

Table : Hausdorff distance: GFMM versus Finite difference (FD)

Two circles

$$c(x,t) = 1 - t$$



Application to image segmentation

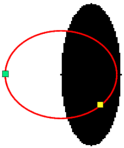
Chan-Vese model

We define the quantities

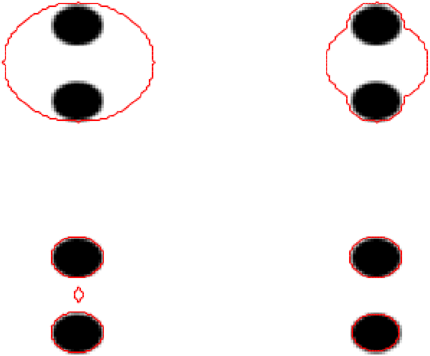
$$c_1(t) = \frac{\int_\Omega I(x)\frac{\theta(x,t)+1}{2}}{\int_\Omega \frac{\theta(x,t)+1}{2}} \quad c_2(t) = \frac{\int_\Omega I(x)\frac{1-\theta(x,t)}{2}}{\int_\Omega \frac{1-\theta(x,t)}{2}}$$

and the velocity

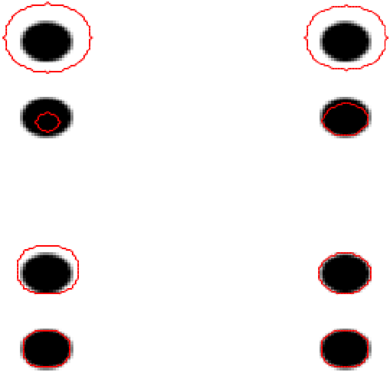
$$c(x,t) = (I(x) - c_2(t))^2 - (I(x) - c_1)^2$$



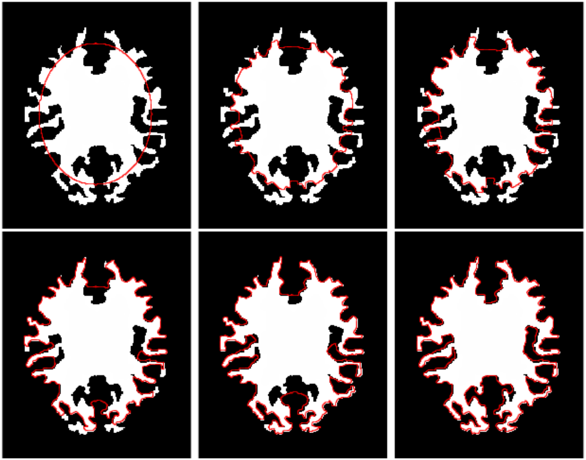
Segmentation



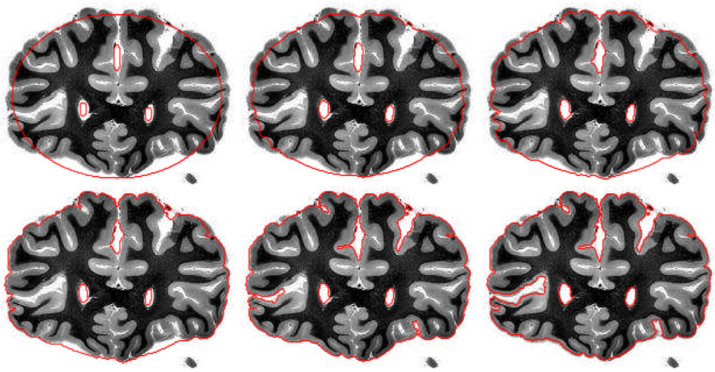
Segmentation



Segmentation



Segmentation



Medical data

Medical data

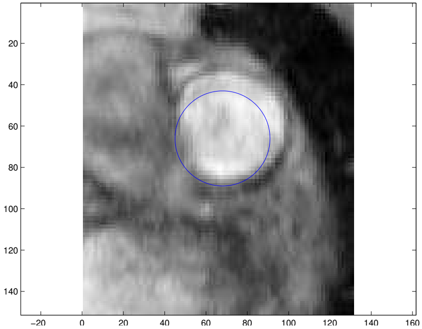


Figure : Initial data

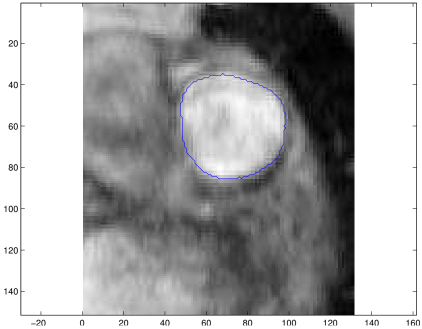
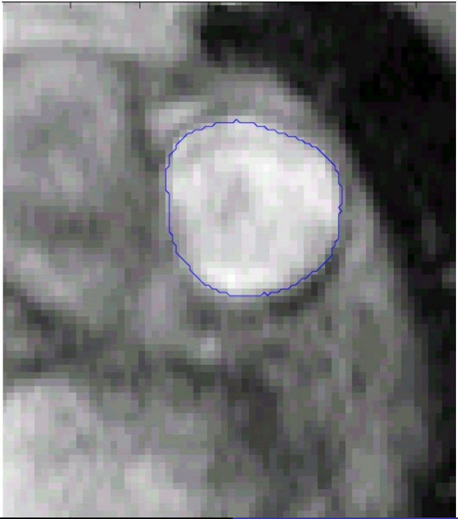


Figure : Final result

Calcul du débit d'une arthère



Main open problems

- (Non-monotone) anisotropic evolution
- Transport equation
- Mean curvature motion
- More general equations

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