Riemann-Roch Theorem

Travis Scholl

April 23, 2015

Abstract

*These notes were taken from a course by Ralph Greenburg in Spring 2015 [Gre15] on counting points on varieties over finite fields. As usual, any mistakes should be due to me.

1 Motivation: Riemann Manifolds

Let X be a connected, compact Riemann surface, f a non-zero meromorphic function on X, and P a point in X.

Definition 1.1. Let $\operatorname{ord}_P f$ be the *order vanishing of f at P*. In particular,

$$\operatorname{ord}_P f = \begin{cases} \text{order vanishing} & \text{if } f \text{ is holomorphic at } P \\ -(\text{order of the pole}) & \text{if } f \text{ has a pole at } P \end{cases}$$

Recall the standard fact from the theory of compact Riemann surfaces.

Fact 1.2. If $f \neq 0$ then $\operatorname{ord}_P f = 0$ for all but finitely many points P.

Next we define divisors and the divisor class group for X.

Definition 1.3. The *divisors* of X are formal $\mathbb{Z}$-linear combinations of points, i.e. $\sum_{P \in X} a_P P$ for some $a_P \in \mathbb{Z}$ and for all but finitely many P we have $a_P = 0$.

Given a divisor $D = \sum_{P \in X} a_P P$ define the *degree* of D to be $\deg(D) = \sum_{P \in X} a_P$. This is finite by definition.

For a non-zero meromorphic function f define the *divisor of f* to be

$$\operatorname{div} f = \sum_{P \in X} (\operatorname{ord}_P f) P.$$

Divisors of the form $\operatorname{div} f$ for some f are called *principle divisors*.

Another useful fact from general compact Riemann surfaces will allow us to define the class group.

Fact 1.4. If f is a non-zero meromorphic function on X, then $\deg(\operatorname{div} f) = 0$, i.e. the number of zeros equals the number of poles counting multiplicity.

Definition 1.5. The *divisor class group* of X is

$$\frac{\text{Group of degree 0 divisors}}{\text{Group of principle divisors}}.$$

Note that the divisor class group is well defined by Fact 1.4. Next we want to use divisors to count classes of functions.

Definition 1.6. Let $D = \sum_{P \in X} a_P$ be a divisor. We say D is *effective* if $D \geqslant 0$ meaning $a_P \geqslant 0$ for all P. Then define

$$\mathcal{L}(D) = \{f \mid \operatorname{div} f + D \text{ is effective}\} \cup \{0\}.$$

By Exercise 1.7 $\mathcal{L}(D)$ is a vector space over $\mathbb{C}$. Define $\ell(D)$ to be $\dim_{\mathbb{C}} \mathcal{L}(D)$.

Exercise 1.7. Show that $\mathcal{L}(D)$ is a complex vector space under the usual scaling action on functions. The work will be in showing that it is closed under addition. This requires some basic complex analysis.

Hint. Consider what happens to the poles and zeros of functions when you add them. Use the power series representations since you only have to look locally.

Fact 1.8. A non-constant meromorphic function f on X has at least one zero and one pole.

Example 1.9. Let D_0 be the zero divisor, i.e. $D_0 = \sum_{P \in X} 0 \cdot P$. By Fact 1.8, $\mathcal{L}(D_0)$ consists of only constant functions so $\ell(D_0) = 1$.

Proposition 1.10. *If* $\deg D = 0$ *then* $\mathcal{L}(D) = \{f \mid \operatorname{div} f = -D\} \cup \{0\}$ *and* $\ell(D)$ *is either* 0 *or* 1.

Proof. Since $\deg D = 0$ we have by definition $\mathcal{L}(D) = \{f \mid \operatorname{div} f \geqslant -D\} \cup \{0\}$. If $\operatorname{div} f > -D$ then $\deg \operatorname{div} f > \deg -D = 0$, contradicting Fact 1.4. The second statement follows from the fact that given two meromorphic functions f, g with $\operatorname{div} f = \operatorname{div} g$ then $\frac{f}{g}$ is a well defined meromorphic function on X with no zeros or poles. Hence it must be constant and so $\ell(D) \leqslant 1$. $\square$

Remark 1.11. If X has positive genus, then $\ell(D)$ is rarely 1 in the previous proposition. If X has genus 0 then $\ell(D) = 1$ for every divisor of degree 0. This follows from the fact that the divisor class group is trivial. The idea is that $X \cong \mathbb{P}^1(\mathbb{C})$, the Riemann sphere. Meromorphic functions on $\mathbb{P}^1(\mathbb{C})$ are just rational polynomials and we can describe any finite set of zeros and poles.

References

[Gre15] Ralph Greenberg. Math 583C: Counting Points on Varieties. Spring 2015.

Travis Scholl
Department of Mathematics, University of Washington, Seattle WA 98195
email: `tscholl2@uw.edu`