Parabolische Zylinderfunktionen

Differenzierbare komplexe Funktionen

Reminder (MathSem Woche 7):

$$\underbrace{F(z)}_{\text{komplexe Funktion}} = U(x,y) + jV(x,y)$$

⇒ wenn $F(z)$ differenzierbar, dann:

$$\mathrm{I} = \left[\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}\right] \quad \text{und} \quad \mathrm{II} = \left[\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}\right]$$

$$\mathrm{I} \cdot \frac{\partial U}{\partial x} \longrightarrow \frac{\partial^2 U}{\partial x^2} = \frac{\partial V}{\partial y}\frac{\partial U}{\partial x}$$

$$\mathrm{II} \cdot \left(-\frac{\partial U}{\partial y}\right) \longrightarrow \frac{\partial^2 U}{\partial y^2} = -\frac{\partial V}{\partial x}\frac{\partial U}{\partial y}$$

$$\Rightarrow \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

⇒ gleiche Herleitung gilt für V

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = \nabla^2 U(x,y) = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \nabla^2 V(x,y) = 0$$

Elektrische Feld

Reminder aus ELT „$\nabla^2 \phi = 0$" Maxwell

↳ „Knotenpotentialregel"

↳ keine Krümmung ohne Quelle/Senke

↳ diff. komplexe Funktionen sind Lösungen für Potential

↳ $F(z)$ beliebige differenzierbare Funktion

↳ $U(x,y)$ / $V(x,y)$ bilden Potential-/elektrisches Feld ab

↳ wählen was Randbedingungen beschreibt

Bsp. $F(z) = z^2 = (x+jy)^2 = x^2 - y^2 + j2xy$

$$U(x,y) = x^2 - y^2 \qquad V(x,y) = 2xy$$

$$\nabla^2 U(x,y) = \frac{\partial^2 (x^2-y^2)}{\partial x^2} + \frac{\partial^2 (x^2-y^2)}{\partial y^2} = 2 - 2 = 0$$

$$\nabla^2 V(x,y) = \frac{\partial^2 (2xy)}{\partial x^2} + \frac{\partial^2 (2xy)}{\partial y^2} = 0 + 0 = 0$$

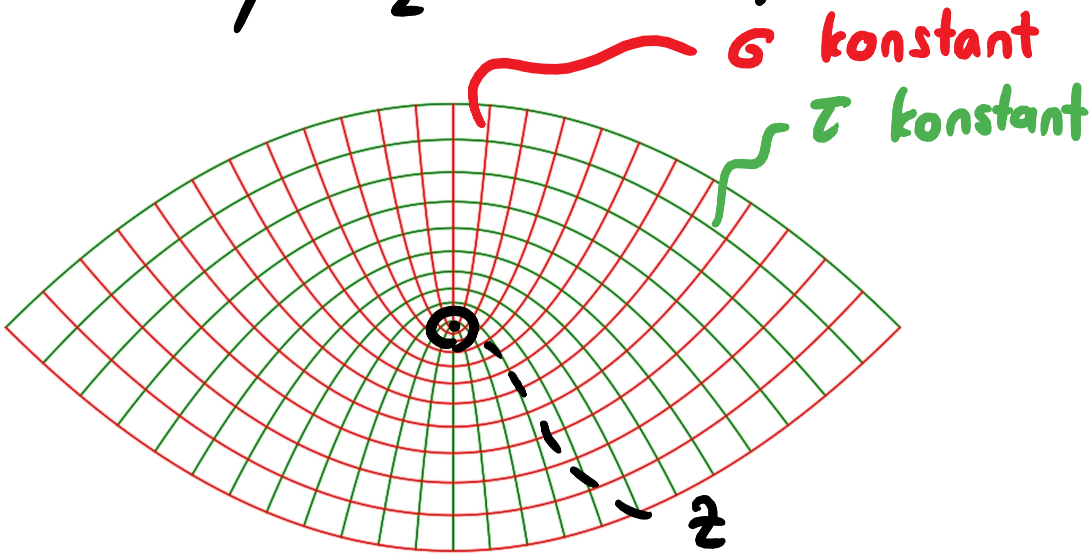
Bsp. $F(z) = \sqrt{z} = \sqrt{x+jy}$

Wolfram Alpha

$$F(z) = \underbrace{\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}}}_{U(x,y)} + j\underbrace{\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}}_{V(x,y)}$$

Parabolische Zylinderkoordinaten

$$x = \sigma\tau \qquad y = \frac{1}{2}(\tau^2 - \sigma^2) \qquad z = z$$



↳ alle Parabeln haben gleichen Brennpunkt

Helmholtz Gleichung $\longrightarrow \Delta f = \lambda f$

↳ Laplace Operator in parab Zylinderfunktionen

$$\Delta = \frac{1}{\sigma^2+\tau^2}\left(\frac{\partial^2}{\partial\sigma^2} + \frac{\partial^2}{\partial\tau^2}\right) + \underbrace{\frac{\partial^2}{\partial z^2}}_{\text{vernachlässigt}}$$

↳ Helmholtz Gleichung $\longrightarrow$ 2D

$$\frac{1}{\sigma^2+\tau^2}\left(\frac{\partial^2 f(\sigma,\tau)}{\partial\sigma^2} + \frac{\partial^2 f(\sigma,\tau)}{\partial\tau^2}\right) = \lambda f(\sigma,\tau)$$

$$\frac{\partial^2 f}{\partial\sigma^2} + \frac{\partial^2 f}{\partial\tau^2} = \lambda(\sigma^2+\tau^2) f$$

↳ Separation $\quad f(\sigma,\tau) = g(\tau)h(\sigma)$

$$\underbrace{\frac{\partial^2 h(\sigma)}{\partial\sigma^2}}_{h''(\sigma)} g(\tau) + \underbrace{\frac{\partial^2 g(\tau)}{\partial\tau^2}}_{g''(\tau)} h(\sigma) = \lambda(\sigma^2+\tau^2) g(\tau)h(\sigma)$$

→ Variabeln isolieren

$$\frac{h''(\sigma)}{h(\sigma)} + \frac{g''(\tau)}{g(\tau)} = \lambda(\sigma^2+\tau^2)$$

const.

$$\frac{h''(\sigma)}{h(\sigma)} - \lambda\sigma^2 = -\left(\frac{g''(\tau)}{g(\tau)} - \lambda\tau^2\right) = \mu$$

$$\boxed{\begin{aligned} h''(\sigma) - (\lambda\sigma^2+\mu)h(\sigma) &= 0 \\ g''(\tau) - (\lambda\tau^2-\mu)g(\tau) &= 0 \end{aligned}}$$

Lösung der Differenzialgleichung

→ Whittaker Funktion $W_{k,m}(z)$ löst folgende DGL:

$$\frac{d^2W_{k,m}(z)}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4}-m^2}{z^2}\right)W_{k,m}(z) = 0$$

→ Lineare DGL Ordnung 2 ⟶ 2 Lösungen

Lösung 1: $W_{k,m}(z) = e^{-\frac{z}{2}} z^{m+\frac{1}{2}} {}_1F_1\left(\frac{1}{2}+m-k, 1+2m; z\right)$

Lösung 2: $W_{k,-m}(z)$

→ Substitution von $w(z) = z^{-1/2} W_{k,\pm 1/4}\left(\frac{z^2}{2}\right)$ in DGL

$$\frac{d^2w(z)}{dz^2} + \left(2k - \frac{1}{4}z^2\right)w(z) = 0$$