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Math Camp Exercises

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1 Basic Topology and Linear Algebra

Question 1. *Show that the* Kullback-Leibler distance *is not a valid metric*

Proof. Note that a valid metric must have the property that: $d(f, g) = d(g, f)$. This is not true.

Let $f \sim exp(1), g \sim exp(2)$.

$$\begin{aligned} d(f,g) &= \int_{\mathbb{R}} \log\left(\frac{\exp(-x)}{2\exp(-2x)}\right)\exp(-x)dx \\ &= \int_{\mathbb{R}} x\exp(-x)\log\left(\frac{1}{2}\right)dx \\ d(g,f) &= \int_{\mathbb{R}} \log\left(\frac{2\exp(-2x)}{\exp(-x)}\right)2\exp(-2x) \\ &= \int_{\mathbb{R}} \frac{2\exp(-2x)}{x}\log(2)dx \end{aligned}$$

Clearly those integrands, and therefore the integrals are not equal. □

Question 2. 1. *Give an example of a set that is both open and closed*

2. *Give an example of a set that is neither open nor closed*

Proof. 1. $\mathbb{R}$ is both open and closed, as all convergent sequences in $\mathbb{R}$ have their limits in $\mathbb{R}$ by definition of being convergent, and clearly any neighborhood in $\mathbb{R}$ is a subset of $\mathbb{R}$.

2. Consider the set $E = \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$. We may first note that this set has a limit point of 0, but $0 \notin E$. Therefore E is not closed.

Consider any neighborhood of any point in E. This neighborhood contains a point not contained in E. As E is countable and any neighborhood is uncountable. Therefore it cannot be that the neighborhood is a subset of E.

□

Question 3. *Give an example to show that a set can be open in one metric space but not open in another.*

Proof. Consider the segment (a, b). This segment is clearly open in $\mathbb{R}$, however, if the embedded set is $\mathbb{R}^2$ it is not open, as any neighborhood would contain points with y coordinates that are not equal to zero, and therefore is not a subset of (a, b). □

Question 4. *Show that $C([0,1])$ is complete.*

Proof. Consider any Cauchy sequence $\{f_n\}$ in $C([0,1])$. For some $\epsilon > 0$, $d(f_m, f_n) < \epsilon \quad \forall m, n \geq N(\epsilon)$. For any $x_0 \in [0,1]$

$$\epsilon > \sup_{x \in [0,1]} |f_m(x) - f_n(x)| > |f_m(x_0) - f_n(x_0)|$$

Note that $f_n(x_0)$ is a sequence in $\mathbb{R}$ instead of $C([0,1])$. Since $\mathbb{R}$ is complete, this implies that $f_n(x_0)$ converges to a point in $\mathbb{R}$. Collect these limits into a function $f(x_0)$.

We must now show that $f(x)$ is continuous. Consider any point $x_0 \in [0,1]$. Choose $n \geq \lceil \sup_{x \in [0,1]} N(\frac{\epsilon}{3}, x) \rceil$. That is, let N be the supremum of N taken for all the convergent sequences $\{f_n(x)\}$ when $\epsilon = \frac{\epsilon}{3}$. Applying the definition of continuity of $f_n(x)$ at x_0 for $\frac{\epsilon}{3}$, for any x such that: $d(x, x_0) < \delta(\frac{\epsilon}{3}, x_0)$.

$$\begin{aligned}
|f(x) - f(x_0)| &= |f(x) - f(x_0) + f_n(x) - f_n(x) + f_n(x_0) - f_n(x_0)| \\
&= |[f_n(x_0) - f(x_0)] - [f_n(x_0) - f_n(x)] + [f(x) - f_n(x)]| \\
&\leq |f_n(x_0) - f(x_0)| + |f_n(x_0) - f_n(x)| + |f(x) - f_n(x)| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \\
&< \epsilon
\end{aligned}$$

As $[0,1]$ is compact, and f is continuous, it is compact valued, and therefore the supremum is obtained.

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \max_{x \in [0,1]} |f_n(x) - f(x)| < \epsilon$$

□

Question 5. *Let $(X, d(.))$ be a metric space, show that the collection of open sets in X is a topology.*

Proof. 1. The empty set is an open set vacuously.

Consider any point x_0 in X. A neighborhood of x_0 consists of points $x \in X$ such that $d(x, x_0) < \delta$. Clearly this is a subset of X and X is therefore open.

2. Consider a collection $\{O_k\}_{k=1}^n \in X$. Take any point $x \in \cap_{k=1}^n O_k$. As each O_k is open, there exists δ_k that defines a neighborhood of x such that $N_{\delta_k} \subset O_k$. Taking $\delta = \min\{\delta_k\}$ we have a neighborhood with a strictly positive radius, as it is the minimum of a finite set of real numbers. All the points in $N_\delta(x) \subset \cap_{k=1}^n O_k$ and therefore that set is open.

3. Consider a cover $\mathcal{F}$, Take any point $x \in \cup_{O \in \mathcal{F}} O$. x belongs to at least one of the sets $E \in \mathcal{F}$. As this set is open, there is a neighborhood of x that is a subset of E. As this neighborhood is contained in E, it is contained in $\cup_{O \in \mathcal{F}} O$ and therefore $\cup_{O \in \mathcal{F}} O$ is open.

□

Question 6. *Let $\succeq$ be a preference relation on $\mathbb{R}_+^n$ that is complete and transitive. Show that: $\{(x, y) \in \mathbb{R}_+^n x \mathbb{R}_+^n | x \succeq y\}$ is closed is equivalent to $\succeq$ is continuous.*

Proof. Assume that $\succeq$ is continuous. Then both its upper and lower contour sets are closed in $\mathbb{R}^n$. Approach by Contradiction. Assume that the above property is not true. Then $\exists\{(x_n, y_n)\} \to (x, y)$ such that $x_n \succeq y_n$ but $y \succ x$. Choose any neighborhood of (y, x), all but a finite number of (x_n, y_n) must be contained in this neighborhood.

Note that $x \notin \succeq (y)$. □

Question 7. *Prove:*

1. *f is continuous at $x_0 \in X$*
2. *For any $\{x_n\}$ such that $\{x_n\} \to x_0$, $\lim f(x_n) = f(x_0)$.*
3. *For any $\epsilon > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon$.*

Proof.

- (1) $\Rightarrow$ (3): Let f be continuous at x_0, Fix $\epsilon > 0$, take the neighborhood of $f(x_0)$ with radius ϵ. Since this set is open, the pre-image of $N_\epsilon(f(x_0))$, $f(N_\epsilon(f(x_0)))$ is an open set. Therefore it contains a neighborhood as a subset of itself. Let δ be the radius of this neighborhood. Every point in that neighborhood satisfies $d(x_0, y) < \delta$ and $d(f(x_0), f(y)) < \epsilon$.

- (3) $\Rightarrow$ (2). Fix $\epsilon > 0$, apply (3) to obtain a $\delta > 0$. Consider any sequence $\{x_n\} \to x_0$. Apply the definition of convergence of a sequence to obtain $N(\delta)$ such that $\forall n \geq N, d(x_n, x_0) < \delta$. By (3), for each of these $x_n, d(f(x_n), f(x_0)) < \epsilon$. Thus this $N(\delta)$ is exactly the N for $\{f(x_n)\}$ to converge, and (2) is satisfied.

- (2) $\Rightarrow$ (3). Approach by contrapositive. Assume that (3) is false, then there exists some $\epsilon > 0$ such that $\forall \delta > 0, \exists x \in E$ where $d(x, x_0) < \delta$ but $d(f(x), f(x_0)) \geq \epsilon$. This is true for all δ. So for $\delta_n = \frac{1}{n}$, apply this rule at each n to obtain an x_n. Clearly $x_n \to x_0$, but since $d(f(x_n), f(x_0)) > \epsilon \quad \forall n$, the sequence $f(x_n)$ cannot converge to $f(x_0)$.

- (3) $\Rightarrow$ (1). Consider any open set V. Take x_0 such that $f(x_0) \in V$. As V is open, $\exists \epsilon > 0$ such that $d(f(x_0), v) < \epsilon \Rightarrow v \in V$. Using (3), $\exists \delta > 0$ such that $d(f(x), f(x_0)) < \epsilon$ if $d(x, x_0) < \delta$. This implies that $x \in f(V)$. This is true for all x such that $d(x, x_0) < \delta$ and thus there is a neighborhood of x_0 that is a subset of $f(V)$.

□

Question 8. *Consider a Betrand competition game. That is, there are two firms, both of them are facing a market with perfectly inelastic demand and firm i has a marginal cost of production c_i. Suppose that firms are competing by setting lower prices. That is, if $p_i < p_j$ then firm i wins the whole market and gets profit $(p_i - c_i)$ and firm j loses and gets zero. Suppose also that whenever there is a tie, each gets half the market. Show:*

1. *When $c_1 = c_2 = c > 0$ there exists a unique pure strategy Nash equilibrium under which $p_1 = p_2 = c$.*

2. *When $0 < c_1 < c_2$ there is no pure strategy equilibrium. What went wrong? Is there any assumption you can make to get around this?*

Proof.

1. Approach by cases:

- Case: p_1 or $p_2 < c$. At least one firm is making a loss, and have an incentive to raise their prices to at least c.
- Case: $p_1 = p_2 > c$. Either firm has an incentive to reduce their price by $\epsilon > 0$ and obtain the entire market while still making profit. So this cannot be a Nash Equilibrium.
- $p_1, p_2 \geq c, p_1 \neq p_2$ The firm with the lower price wishes to raise its price to some number between p_1, p_2 in order to increase its profit. This cannot be a Nash equillibrium then.
- $p_1 = p_2 = c$. Both firms are indifferent between raising their prices and earning zero profit and maintaining prices and earning zero profit. Both are strictly averse to lowering their prices as they would then earn negative profit. This means that both prices are a best response to the other prices, and this is a Nash Equilibrium.

2. When $0 < c_1 < c_2$, Firm 1 now wishes to lower his from from c_2, and Firm 2 has no desire to drop their price below c_2 as they would earn negative profit. However Firm 1 would like to have as high of a profit as possible, but when $p_1 = p_2$ they will earn less profit than $p_1 = p_2 - \epsilon$. This means that there is no maximum to their profit function, and they can always raise their price and earn more profit, but must keep their price below c_2.

   One assumption to avoid this pitfall is to discretize the price space. If the prices are forced to be in whole cents, then Firm 1 will wish to charge $c_2 - .01$ and this will be the maximum profit that they are capable of earning.

$\square$

Question 9. *Let X be a compact topological space. Suppose that $W : C(X) \to C(X)$ is a self map on $C(X)$, the collection of continuous functions, under the norm*