Rudin Notes

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1 Chapter 1

1.1 Real Numbers

Rational Numbers are not quite descriptive enough:

There is no rational number p such that: $p^2 = 2$

Proof. Approach by Contradiction:

$p = \frac{m}{n}$ where m,n are irreducible integers. Then: $m^2 = 2n^2$. Thus m^2 is even, and therefore m is even. So m^2 is divisible by 4. This means $2n^2$ is divisible by 4, and n^2 is even, so n is even. Contradiction. $\square$

The point of the real numbers is to introduce a set of numbers that fills in these "gaps" and arises from the concepts of an ordered set and a field. As we can see later, the rational numbers are not complete, and do not satisfy some very important properties required for analysis.

For a set S, an order on S is a relation, denoted $<$.

1. If $x \in S$ and $y \in S$ then one of these is true: $x < y, x = y, y < x$
2. If $x, y, z \in S$, $x < y, y < z, \implies x < z$.

Ordered Set: An ordered Set is a set where there is a defined order.

If S is an ordered Set, and $E \subset S$, if $\exists \beta \in S, x \leq \beta \quad \forall x \in E$, E is bounded above with upper bound β.

Least Upper Bound (supremum): For an ordered set S, $E \subset S$ where E is bounded above. If $\exists \alpha \in S$

- α is an upper bound of E

- $\forall \gamma < \alpha$, γ is not an upper bound of E.

This is denoted: $\alpha = \sup E$, The infimum of a set, denoted $\delta = \inf E$ is defined in a similar manner.

The supremum of a set need not be contained within that set. Consider the set containing numbers of the form: $\frac{1}{N}, n \in \mathbb{Z}^{++}$. The sup of this set is 1, which is contained, but the inf of the set is 0, and is not contained within the set.

Least-Upper-Bound property: The supremum of every nonempty subset is contained in the original set.

- $E \subset S \neq \emptyset$.
- E is bounded above
- $\sup E$ exists in S.

Having the Least-Upper-Bound Property implies that you also have the Greatest-Lower-Bound Property:

Theorem 1.1. *If S is an ordered set with least-upper-bound property, $B \subset S, B \neq \emptyset$ and B is bounded below, let L be the set of all lower bounds of B. Then $\alpha = \sup L \in S$, and $\alpha = \inf B$.*

Proof. B is bounded below, so there is at least one element as a lower bound, and L is nonempty. L is all the lower bounds, so any element in B is an upper bound on L. This means L is bounded above, and thus L has a supremum contained in S. $\alpha = \sup L \in S$.

Since α is the least upper bound of L, any number $\gamma < \alpha$ is not an upper bound of L. Therefore $\gamma \notin B$ as if it were, $\delta > \gamma \in L$ could not be a lower bound of B. This means that $\forall x \in B, \quad \gamma < x$. Then $\alpha \leq x$. If this were not true: Let $\epsilon = \alpha - x > 0$. Choose $\gamma = \alpha - \frac{\epsilon}{2}$. Then $\gamma > x$ but this is an impossibility. Since α is an upper bound on L, any number larger than it cannot be contained in the set. Since $\alpha \in L$ it is a lower bound of B, but any number larger than it is not a lower bound, so it must be that: $\alpha = \inf B$. $\square$

Field: Set F with two operations that satisfy the following "field axioms".
(A) Axioms for addition

- If $x \in F$ and $y \in F$, then $x + y \in F$.
- $\forall x, y \in F \quad x + y = y + x$
- $\forall x, y, z \in F \quad (x + y) + z = x + (y + z)$
- $\exists 0 \in F$ such that: $0 + x = x$
- $\forall x \in F, \exists -x$ such that: $x + (-x) = 0$

(M) Axioms for Multiplication

- If $x \in F$ and $y \in F$ then their product $xy \in F$
- $\forall x, y \in F, xy = yx$
- $\forall x, y, z \in F, (xy)z = x(yz)$
- $\exists 1 \in F$ such that: $1x = x$
- $\forall x \in F, x \neq 0 \quad \exists \frac{1}{x} \in F$ such that: $x(\frac{1}{x}) = 1$

Distributive law:

- $\forall x, y, z \in F, x(y + z) = xy + xz$

The axioms above imply the following statements:

- $x + y = x + z \rightarrow y = z$
- $x + y = x \rightarrow y = 0$
- $x + y = 0 \rightarrow x = -y$
- $-(-x) = x$

Proof. This is filler

- $x + y = x + z$
  $(-x) + x + y = (-x) + x + z$
  $(-x + x) + y = (-x + x) + z$
  $y = z$

- Let z = 0 above.
- $x + y = 0 \to (-x) + x + y = -x \implies (-x + x) + y = -x \to y = -x$
- $-(-x) + (-x) = 0 \implies -(-x) + (-x) = x + -x \to -(-x) = x$

$\square$

Basically, cancellation laws hold, and the zero element in unique as well as the additive inverse of any element.

Similarly for multiplication: Note $x \neq 0$ for all results

- $xy = xz \implies y = z$
- $xy = x \implies y = 1$
- $xy = 1 \implies y = \frac{1}{x}$
- $\frac{1}{\frac{1}{x}} = x$

We also Get:

- $0x = 0$
- If $x \neq 0, y \neq 0$, then $xy \neq 0$
- $(-x)y = -(xy) = x(-y)$
- $(-x)(-y) = xy$

Proof:

- $0x = (-x + x)x = -x^2 + x^2 = 0$
- Assume $xy = 0$. Then $xy\frac{1}{y} = 0\frac{1}{y}$. By result above: $x = 0$ which is a contradiction.
- $-xy = xy - xy - xy = y(x - x - x) = y(-x)$
  $-xy = xy - xy - xy = x(y - y - y) = x(-y)$
- $(-x)(-y) = -(x(-y)) = -(-(xy)) = xy$

1. Ordered Field Ordered Field: A field F for which there is also an ordered set that satisfies:

    - $x + y < x + z$ if: $x, y, z \in F$ and $y < z$
    - $xy > 0$ if $x \in F, y \in F, x > 0, y > 0$

    Useful results in ordered Fields:

    - If $x > 0$ then $-x < 0$
    - If $x > 0$ and $y < z$ then $xy < xz$
    - If $x < 0$ and $y < z$ then $xy > xz$
    - If $x \neq 0$ then $x^2 > 0$, so $1 > 0$
    - If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$

    Proof:

    - $x > 0 \rightarrow x + (-x) > -x \rightarrow 0 > -x$
    - $y < z \rightarrow z - y > 0 \rightarrow x(z - y) > 0 \rightarrow zx > xy$
    - Repeat using -x
    - $x^2 = xx = (-x)(-x)$ One of which is the product of two positive numbers. Thus $x^2 > 0$.
    - $x < y \rightarrow x\frac{1}{x} < y\frac{1}{x} \rightarrow \frac{1}{y} < \frac{1}{x}$

Theorem 1.2. *There exists an ordered field $\mathbb{R}$ which has the least-upper-bound property. $\mathbb{R}$ contains $\mathbb{Q}$ as a sub-field.*

Theorem 1.3. *1. Archimedian Property: If $x > 0$ and $x, y \in \mathbb{R}$, then $\exists n \in \mathbb{Z}$ such that: $nx > y$*

2. $\mathbb{Q}$ is dense in $\mathbb{R}$: $x, y \in \mathbb{R}, x < y$ implies that $\exists p \in \mathbb{Q}$ such that: $x < p < y$.

Proof. 1. Let A be the set of all nx where $n \in \mathbb{Z}^{++}$. Assume the result is false, A is bounded above by y. A must have a supremum, let $\alpha = \sup A$. Note that: $\alpha - x < \alpha$ so $\alpha - x$ is not an upper bound for A. So there is an integer m such that $\alpha - x < mx$. However, this implies: $\alpha < (m + 1)x$ This is a contradiction since α is an upper bound of A.

2. Note: $y - x > 0$. By the Archimedian property: $n(y-x) > 1$. $m_1 > nx, m_2 > -nx$. This means that: $-m_2 < nx < m_1$. Since nx is bounded by two integers, there is a single integer m such that: $m-1 \leq nx < m$. Combining all these results:
$nx < m \leq 1 + nx < ny \to x < \frac{m}{n} < y$.

□

Some simple corollaries from this that can be created are: For any real number x there is an integer N such that $\frac{1}{N} < x$.

Theorem 1.4. *For every real $x > 0$ and integer $n > 0$ there is only one positive real y such that $y^n = x$*

Proof. Assume there are multiple distinct y values satisfying it: WLOG consider $y_1 < y_2$ Note then that: $y_1^n < y_2^n$ and they cannot be equal.
Let us consider the existence now. Let the set E contain all positive real numbers t such that: $t^n < x$. Note that E always contains: $\frac{x}{1+x}$, as this is less than one, and $t^n \leq t < x$. This ensures E is not empty.

$1 + x$ is an upper bound of t, as $t > 1 + x \to t > x$ and $t^n \geq t$. This means that E has a least upper bound.

Let $y = \sup E$. The book then follows some nasty algebraic identities to draw contradictions around $y^n > x$ and $y^n < x$ and then asserts that $y^n = x$. □

Extended Real Numbers: Real numbers with ∞ and $-\infty$ added to them. This still preserves the original ordering in $\mathbb{R}$. We declare that: $-\infty < x < \infty \quad \forall x \in \mathbb{R}$.

This provides an upper and lower bound to every subset of the extended reals, so as long as the subset is nonempty it will always have a least-upper-bound and a greatest-lower-bound. We are no longer a field, but we simply define the usual conventions with infinity.

1.2 Complex Field

A Complex number is an ordered pair of real numbers. For two complex numbers, $x = (a, b), y = (c, d)$, Addition and Multiplication are defined as such:

$$x + y = (a + c, b + d)$$
$$xy = (ac - bd, ad + bc)$$

These definitions ensure that the set of all complex numbers is a field, where $0 = (0, 0)$ and $1 = (1, 0)$. The axioms are then verified in the book.

One useful way to think about the complex numbers is as an extension of the real numbers, where the second ordered pair is equal to zero. All operations under the complex numbers follow the same properties.

We can now define the imaginary constant: $i = (0, 1)$. This is the number with the property: $i^2 = -1$. For two real numbers $a, b; (a, b) = a + bi$.

We define the *Conjugate* of a complex number $z = a+bi$ to be: $\overline{z} = a-bi$. For notational sake, define: $a = Re(z), b = Im(z)$

Theorem 1.5. *1.* $\overline{z + w} = \overline{z} + \overline{w}$

2. $\overline{zw} = \overline{z}\overline{w}$.

3. $z + \overline{z} = 2Re(z)$

4. $z - \overline{z} = 2iIm(z)$

5. $z\overline{z}$ *is real and positive, or zero if* $z = (0, 0)$

The Absolute value of a complex number: $|z|$. is the square root of $z\overline{z}$. $|z| = (z\overline{z})^{\frac{1}{2}}$ This number is unique and exists (from theorem 1.4). For real numbers this carries the traditional view of absolute value.

Theorem 1.6. *1.* $|z| > 0$ *unless* $z = 0, |0| = 0$

2. $|\overline{z}| = |z|$

3. $|zw| = |z|\,|w|$

4. $|Re(z)| \leq |z|$

5. $|z + w| \leq |z| + |w|$

Proof. 1. Follows directly from: $z\overline{z} > 0$ for $z \neq 0$

2. $|\overline{z}| = (a2 + (-b)^2)^{\frac{1}{2}} = (a^2 + b^2)\frac{1}{2} = |z|$

3. $|zw|^2 = (ac-bd)^2+(ad+bc)^2 = (a^2+b^2)(c^2+d^2) = |z|^2\,|w|^2 = (|z|\,|w|)^2$

4. $|Re(z)| = \sqrt{a^2} \leq \sqrt{a^2+b^2} = |z|$

5. $|z+w|^2 = z\overline{z} + z\overline{w} + \overline{z}w + w\overline{w}$
   $= |z|^2 + 2Re(z\overline{w}) + |w|^2$
   $\leq |z|^2 + 2\,|z\overline{w}| + |w|^2$
   $= (|z| + |w|)^2$

□

Theorem 1.7. *Let $a_1 ... a_n$ be complex numbers, and $b_1 ... b_n$ be complex numbers.*

$$\left|\sum_{j=1}^{n} a_j \overline{b_j}\right|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$$

Proof. Let $A = \sum |a_j|^2, B = \sum |b_j|^2, C = \sum a_j \overline{b_j}$. Note that if $B = 0$ then the result is trivial, let $B > 0$.

$$\begin{aligned}
\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\
&= B^2 \sum |a_j|^2 - B\overline{C} \sum a_j \overline{b_j} - BC \sum \overline{a_j} b_j + |C|^2 \sum |b_j|^2 \\
&= B^2 A - B\,|C|^2 \\
&= B(AB - |C|^2) \geq 0
\end{aligned}$$

Since it is known that $B > 0$ then we get $AB - |C|^2 \geq 0$. □

1.3 Euclidean Spaces

For a positive integer k, $\mathbb{R}^k$ is the set of of all ordered k-tuples. Elements in this space are called vectors, and denoted by boldface letters. Vector addition and scalar multiplication are defined as such:

$$\begin{aligned} \boldsymbol{x} + \boldsymbol{y} &= (x_1 + y_1, ..., x_k + y_k) \\ \alpha \boldsymbol{x} &= (\alpha x_1, ..., \alpha x_k) \end{aligned}$$

These operations are closed, and make $\mathbb{R}^k$ into a vector space over the real field. The zero element in this space is the point $\mathbf{0}$, whose coordinates are all 0.

The inner-product of two vectors as well as the norm is defined as:

$$\begin{aligned} \boldsymbol{x} \cdot \boldsymbol{y} &= \sum_{i=1}^{k} x_i y_i \\ |\boldsymbol{x}| &= \left(\sum_{i=1}^{k} x_i^2 \right)^{\frac{1}{2}} \end{aligned}$$

Theorem 1.8. *For* $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^k, \alpha \in \mathbb{R}$

1. $|\boldsymbol{x}| \geq 0$
2. $|\boldsymbol{x}| = 0 \Leftrightarrow \boldsymbol{x} = 0$
3. $|\alpha \boldsymbol{x}| = |\alpha| \, |\boldsymbol{x}|$
4. $|\boldsymbol{x} \cdot \boldsymbol{y}| \leq |\boldsymbol{x}| \, |\boldsymbol{y}|$
5. $|\boldsymbol{x} + \boldsymbol{y}| \leq |\boldsymbol{x}| + |\boldsymbol{y}|$

Proof. 1. Follows directly from the definition of $|\boldsymbol{x}|$

2. If $\boldsymbol{x} = 0, x_i = 0 \quad \forall i$. So $\sum x_i^2 = 0$. If $\boldsymbol{x} \neq 0, x_i \neq 0$ for some i. $x_i^2 > 0$ for at least one element i, and $\sum x_i^2 > 0$.
3. $|\alpha \boldsymbol{x}| = (\alpha \boldsymbol{x} \cdot \alpha \boldsymbol{x})\frac{1}{2} = |\alpha| \, |x|$
4. $|\boldsymbol{x} \cdot \boldsymbol{y}| = |\sum x_i y_i| \leq \left(\sum |x_i|^2 \sum |y_i|^2 \right)^{\frac{1}{2}} = |\boldsymbol{x}| \, |\boldsymbol{y}|$

5. $|\boldsymbol{x}+\boldsymbol{y}|^2 = (\boldsymbol{x}+\boldsymbol{y})\cdot(\boldsymbol{x}+\boldsymbol{y})$
   $= \boldsymbol{x}\cdot\boldsymbol{x} + 2\boldsymbol{x}\cdot\boldsymbol{y} + \boldsymbol{y}\cdot\boldsymbol{y}$
   $\leq |\boldsymbol{x}|^2 + 2\,|\boldsymbol{x}|\,|\boldsymbol{y}| + |\boldsymbol{y}|^2$
   $= (|x| + |y|)^2$

$\square$

1.4 Exercises

1.4.1 Question 1

Question. *If r is rational and not equal to zero, and x is irrational, prove that r+x and rx are irrational*

Proof. Let r+x be rational, then it can be written as a fraction of irreducible integers: $r+x = \frac{m}{n}$. Since r is rational, it can be written as $r = \frac{a}{b}$. Therefore x can be written as: $x = \frac{m}{n} - \frac{a}{b} = \frac{bm-an}{nb}$ This is a contradiction to x being irrational.

Let rx be rational. $rx = \frac{m}{n}$. Since r is rational it can be written as $r = \frac{a}{b}$. Then $x = \frac{\frac{m}{n}}{\frac{a}{b}} = \frac{mb}{an}$ which is again a contradiction $\square$

1.4.2 Question 2

Question. *Prove that there is no rational number whose square is 12*

Proof. Assume there is a rational number with square 12. $x = \frac{m}{n}, x^2 = 12$. $\frac{m^2}{n^2} = 12, m^2 = 12n^2$ This implies m^2 is divisible by 12. Assume m is not divisible by six. Then: $m = 6l + w$ and $m^2 = 36l^2 + 12lw + w^2$. Note that w^2 is not divisible by six for $w = \{1,2,3,4,5\}$. m is divisible by six. So $m^2 = 36k, 12n^2 = 36k$. Thus n^2 is divisible by 3. Assume n is not divisible by three. Then $n = 3l + w$ and $n^2 = 9l + 6lw + w^2$. Note that w^2 is not divisible by three for $w = \{1,2\}$. n is divisible by three, and m, n have a common factor, leading to a contradiction. $\square$

1.4.3 Question 3

Ommitted because of how trivial it is.

1.4.4 Question 4

Question. *Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E, and β is an upper bound of E. Prove that $\alpha \leq \beta$.*

Proof. Case: E is a singleton set. Let x be an element in E. Then $\alpha \leq x$ and $x \leq \beta$. So $\alpha \leq \beta$. □

1.4.5 Question 5

Question. *Let A be a nonempty set of real numbers which is bounded below. Let $-A$ bet he set of all numbers $-x$ where $x \in A$. Prove that:* $\inf A = -\sup(-A)$.

Proof. Let $\alpha = \inf A$. Then $\forall x \in A, \alpha \leq x$ and $\alpha + \epsilon$ is not an lower bound: $\exists y \in A, y < \alpha + \epsilon$. So: $-\alpha \geq -x$ and $-\alpha - \epsilon < y \quad \forall \epsilon > 0$. This means that $-\alpha$ is the least upper bound for $-A$. □

1.4.6 Question 8

Question. *Prove that no order can be defined in the complex field that turns it into a an ordered field.*

Proof. Assume that there is an order in the complex field: Note that this implies $1 > 0$. This means that $-1 < 0$ and for any $x \neq 0, x^2 > 0$. Take $x = i$. $i^2 = -1$ which is a contradiction. □

1.4.7 Question 12

Question. *If $z_1, ..., z_n$ are complex, prove that:* $|z_1 + z_2 + ... + z_n| \leq |z_1| + ... + |z_n|$

Proof. By Induction. Base Case: n = 1. $|z_1| = |z_1|$.
Inductive Step: $|z_1 + ...z_{k-1} + z_k| \leq |z_1 + ... + z_{k-1}| + |z_k| \leq |z_1| + ... + |z_{k-1}| + |z_k|$. The first step uses the triangle inequality, and the second uses the induction hypothesis. □

1.4.8 Question 13

Question. *If $x, y \in \mathbb{C}$, prove that:* $||x| - |y|| \leq |x - y|$

Proof.

$$\begin{aligned} ||x| - |y|| &= ||x - y + y| - |y|| \\ &\leq ||x - y| + |y| - |y|| = |x - y| \end{aligned}$$

□

1.4.9 Question 17

Question. *Prove that:*

$$|\boldsymbol{x} + \boldsymbol{y}|^2 + |\boldsymbol{x} - \boldsymbol{y}|^2 = 2\,|\boldsymbol{x}|^2 + 2\,|\boldsymbol{y}|^2$$

Proof.

$$\begin{aligned} |\boldsymbol{x} + \boldsymbol{y}|^2 + |\boldsymbol{x} - \boldsymbol{y}|^2 &= (\boldsymbol{x} + \boldsymbol{y}) \cdot (\boldsymbol{x} + \boldsymbol{y}) + (\boldsymbol{x} - \boldsymbol{y}) \cdot (\boldsymbol{x} - \boldsymbol{y}) \\ &= \boldsymbol{x} \cdot \boldsymbol{x} + 2\boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{y} \cdot \boldsymbol{y} + \boldsymbol{x} \cdot \boldsymbol{x} - 2\boldsymbol{x} \cdot \boldsymbol{y} + \boldsymbol{y} \cdot \boldsymbol{y} \\ &= 2\boldsymbol{x} \cdot \boldsymbol{x} + 2\boldsymbol{y} \cdot \boldsymbol{y} \\ &= 2\,|\boldsymbol{x}|^2 + 2\,|\boldsymbol{y}|^2 \end{aligned}$$

□

2 Basic Topology

2.1 Finite, Countable, and Uncountable Sets

2.1.1 Functions

A function (mapping) of a set A to B is an association where each element of A is associated with one element of B.

The set A is referred to as the *domain* of f, and the set B is called the *codomain* of f. The values of B that are mapped from A by f are called the *range*. This is also denoted as $f(A)$ and is referred to as the *image of A under f*. If the image is equal to the codomain, we say that f maps A *onto* B.

For some $E \subset B, f^{-1}(E)$ returns the set of all $x \in A$ such that $f(x) \in E$. We refer to f^{-1} as the *inverse image of E under f.* If the set returned is singleton for all values of E, then f is said to be *one-to-one*. We often express it as: $f(x_1) = f(x_2) \implies x_1 = x_2$.

A mapping that is 1-1 and onto is referred to as a *bijection*. If there is a bijection between A and B, we say that the two sets have the same cardinal number, or are *equivalent*, denoted $A \sim B$. This carries the properties of any equivalence relation:

- It is *reflexive*: $A \sim A$
- It is *symmetric*: If $A \sim B$ then: $B \sim A$
- It is *transitive*: $A \sim B, B \sim C \implies A \sim C$

Let $\mathbb{N}$ be the set of positive integers, and $\mathbb{N}_n$ be the set of positive integers which are less than or equal to n.

- A is *finite* if $A \sim \mathbb{N}_n$ for some n, or is empty.
- A is *infinite* if A is not finite.
- A is *countable* if $A \sim \mathbb{N}$.
- A is *uncountable* if A is neither finite nor countable
- A is at most countable if it is finite or countable.

This notion of equivalence loses its meaning of having the same number of elements when we consider infinite sets, but the notion of the bijection remains the same.

I finite set cannot be equivalent to one of its proper subsets, but this is possible for infinite sets, such as $\mathbb{R}$ and the unit interval.

2.1.2 Sequences

A *sequence* is a function f defined on $\mathbb{N}$. We often denote the sequence of f acting upon each integer as $\{x_n\}$. Each value of f is referred to as a *term* in the sequence. If all of $x_n \in A$, then $\{x_n\}$ is said to be a *sequence in A*.

Loosely said, the elements of any countable set can be arranged into a sequence. As anything that bijects $\mathbb{N}$ is countable.

Theorem 2.1. *Every infinite subset of a countable set A is countable*

Proof. Consider an infinite set $E \subset A$. Since A is countable, biject its elements into a sequence $\{x_n\}$.

Construct a new sequence $\{n_k\}$ where n_k is the smallest integer greater than n_{k-1} where $x_{n_k} \in E$.

This is a bijection between E and $\mathbb{N}$. □

Basically this is showing that countably infinite is the "smallest" infinity, anything infinite that is "smaller" is still countable.

Consider a Collection of sets denoted E_α. Instead of considering the set containing all of these as a set, we may consider the collection of sets defined by the *union*. This holds the standard definition, and can apply for infinite unions.

The intersection works the same way.

If $A \cap B$ is not empty, we say that A and B *intersect*; otherwise they are *disjoint*.

Every standard of unions and intersections like DeMorgan's law etc is true.

Theorem 2.2. *The union of countable sets over a countable iterator is itself countable*

Proof.

$$S = \bigcup_{n=1}^{\infty} E_n$$

This proof follows the diagonal nonsense usually used to prove that the Rationals are countable. Basically a Matrix is formed with the elements of E_n for the rows, and the iterator for the columns. Then we go along the diagonal to form a single sequence. $x_{11}; x_{21}, x_{12}; ....$ This sequence may repeat, but it shows that S is at most countable. Since E_1 is infinite, and S is a super-set of E_1, S is infinite, and therefore countable. □

As much as the proof is a bit of nonsense, it does rely on the countability of the iterator, which makes us the columns in order for us to be able to complete this process to include all the elements of S. An uncountably infinite union of countable sets can be uncountable.

Theorem 2.3. *Let A be a countable set, and let B_n be the set of tuples of size n of elements of A, not requiring distinction. B_n is countable.*

Proof. Obviously, B_1 is countable, as it is A. Proceed by induction: Let B_{n-1} be countable. Then each element of B_n can be written as a 2-tuple of (b, a) where $b \in B_{n-1}, a \in A$. If we fix b, then this tuple is countable, as A is countable. However, for fixed a, b is countable by inductive hypothesis, so this is a countable union of countable sets. By the above theorem it is countable. □

Since rational numbers are a subset of 2-tuples of the integers, and are certainly infinite, this implies that the rational numbers are countable.

Theorem 2.4. *Let A be the set of all sequences whose elements are the digits 0 and 1. This set is uncountable*

Proof. Consider a countable subset $E \subset A$. This contains a countable number of sequences, construct a sequence s by having the i^{th} element of s be equal to 1 minus the i^{th} element of the i^{th} sequence in E. This ensures that s is different from all the sequences contained in E for at least one point. Therefore s is not contained in E. This means E is a proper subset of A. A cannot be countable then, because A would be a proper subset of A. □

This proof is equivalent to stating that the set of all real numbers (in base 2) is uncountable.

2.2 Metric Spaces

For a set X, containing points is called a *metric space* if there is a function associated with each two points called the distance from p to q that is defined in such a manner:

- $d(p, q) > 0$ if $p \neq 0$. Otherwise $d(p, q) = 0$.
- $d(p, q) = d(q, p)$
- $d(p, q) \leq d(p, r) + d(r, q) \quad \forall r \in X$.

We call the function $d(.)$ a distance function, or a *metric*

In $\mathbb{R}^k$ the most common metric used is: $d(x, y) = |x - y|$ The conditions have already been shown to be satisfied in theorem 1.8.

One important note is that every subset of a metric space is a metric space on its own using the same distance function.

- A *segment* (a, b) is defined to mean all the real numbers x such that $a < x < b$.
- An *interval* $[a, b]$ is the set of all real numbers $a \leq x \leq b$.
- Half open intervals work in exactly the way one would think.
- A *k-cell* is a higher dimensional version of an interval, an interval in every dimension.
- An *open* or *closed ball* B with center $\boldsymbol{x}$ and radius r is the set of all the values $\boldsymbol{y} \in \mathbb{R}^k$ such that: $|\boldsymbol{y} - \boldsymbol{x}| < r$.
- A set $E \subset \mathbb{R}^k$ is convex if $\lambda\boldsymbol{x} + (1 - \lambda)\boldsymbol{y} \in E$ when $\boldsymbol{x} \in E, \boldsymbol{y} \in E, \lambda \in (0, 1)$.
- A *neighborhood* of p is a set $N_r(p)$ consisting of all $q \in X$ such that $d(p, q) < r$ for some r $>$ 0. r is called the *radius* of the neighborhood.
- A point p is a *limit point* of the set E if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.
- If $p \in E$ and p is not a limit point of E, then p is called an *isolated point* of E.
- E is *closed* if every limit point of E is a point of E.
- A point p is an *interior point* of E if there is a neighborhood N of a p such that $N \subset E$.
- E is *open* if every point E is an interior point of E.
- The *complement* of E (E^c) is the set of all points $p \in X$ such that $p \notin E$.
- E is *perfect* if E is closed and if every point of E is a limit point of E.

- E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p,q) < M \quad \forall p \in E$.
- E is *dense* in X if every point of X is a limit point of E, or a point of E.

Theorem 2.5. *Every neighborhood is an open set*

Proof. Consider a neighborhood $E = N_r(p)$ and let q be a point in E. Since $d(p,q) < r$, let h be the number such that: $d(p,q) = r - h$.

h is the distance from q to the edge of the neighborhood. Every point in the neighborhood around q with radius h is contained in E. This makes q an interior point of E. □

Theorem 2.6. *If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E*

Proof. Approach by Contradiction. Suppose there is a neighborhood N which contains only a finite number of points of E. Take the smallest distance from p and all of these finite distinct points. Since it is finite, the minimum is positive, and the neighborhood of this radius contains no points. This is a contradiction to it being a limit point. □

The most important Corollary of this result is that a finite point set has no limit points.

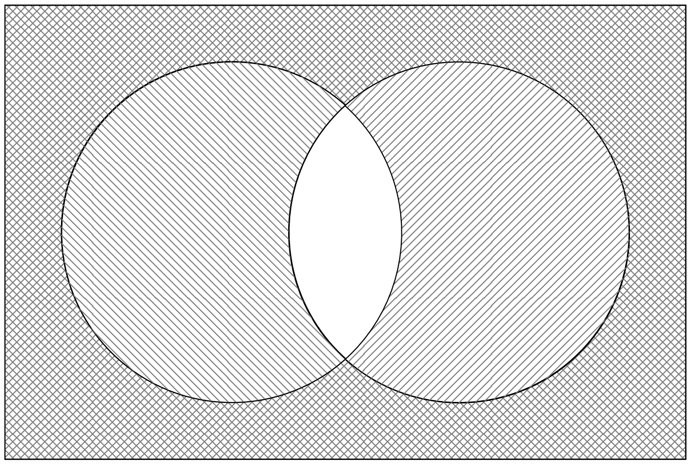
Here are examples of some sets and their properties:

	Closed	Open	Perfect	Bounded
$z \in \mathbb{C}, \lvert z\rvert < 1$	No	Yes	No	Yes
$z \in \mathbb{C}, \lvert z\rvert \leq 1$	Yes	No	Yes	Yes
Finite Set (nonempty)	Yes	No	No	Yes
$\mathbb{Z}$	Yes	No	No	No
$\frac{1}{n}$	No	No	No	Yes
$\mathbb{C}$	Yes	Yes	Yes	No
(a,b)	No	Both	No	Yes

Note that (a,b) would be open in $\mathbb{R}$ but is not open if it is a subset of $\mathbb{R}^2$.

Theorem 2.7. *For any collection of sets E_α, $(\cup_\alpha E_\alpha)^c = \cap_\alpha {E_\alpha}^c$*

This is the intuitive result one could see from a Venn diagram.



Theorem 2.8. *A set E is open if and only if its complement is closed*

Proof. Suppose E^c is closed. Choose $x \in E$. Then $x \notin E$, and x is not a limit point of E^c. So there is a neighborhood of x that is contained completely in E. This makes x an interior point of E, and E is open.

Suppose E is open. Let x be a limit point of E^c. Every neighborhood of x contains a point of E^c. This means that x is not an interior point of E, and since E is open, $x \in E^c$. Thus we see that E^c is closed □

This also implies that a set F is closed if and only if its complement is open.

Theorem 2.9. *Properties of unions and intersections of open and closed sets:*

- *For any collection of open sets, The union of them is open*
- *For any collection of closed sets, the intersection is closed*
- *For any finite collection of open sets, the intersection is open*
- *For any finite collection of closed sets, the union is closed.*

Proof. For any interior point of a set, it is also an interior point of a union of that set and another, so the union of open sets must be open. Using the complements one can show that the intersection of closed sets is closed by the above two theorems.

To see that the intersection of an infinite number of open sets doesn't have to be open, consider sets of intervals that are all subsets of each other, the intersection will be a single point, and therefore not be open. □

If X is a metric space, and $E \subset X$, The *closure* of E is the union of E and all of its limit points. It is denoted $\bar{E}$.

Theorem 2.10. *For a metric space X and $E \subset X$*

1. *$\bar{E}$ is closed*
2. *$E = \bar{E}$ if and only if E is closed*
3. *$\bar{E} \subset F$ for every closed set F where: $E \subset F$*

Proof.

1. Any point in $\bar{E}^c$ has a neighborhood that does not intersect E, and therefore $\bar{E}^c$ is open. This implies E is closed.
2. If E is closed, then the limit points of E is a subset of E, and $E = \bar{E}$.
3. If F is closed, than it contains its limit points, and therefore the limit points of E. So F is a super-set of the limit points of E as well as E, so it must be a super-set of the closure of E.

□

This is effectively stating that the closure of a set is the "smallest" closed set that contains the elements of the set.

Theorem 2.11. *Let E be a nonempty set of real numbers that is bounded above. The supremum of this set is contained in the closure.*

Proof. Let $y = \sup E$. If $y \in E$ trivially true.

For every $h > 0$, there is some $x \in E$ such that $y - h < x < y$ or $y - h$ would be a lower bound, and y couldn't be the least upper bound.

This is exactly a neighborhood of y, and we have shown that it contains an element of E. This makes y a limit point of E, and thus $y \in \bar{E}$. □

When we speak of subsets, open and closed are conditions that are relative to the super-set. As an interval can be open or not, depending on whether or not it is a subset of $\mathbb{R}$ or $\mathbb{R}^k$. To be clear on what the super-set is, we say *open relative to* a set.

Theorem 2.12. *Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X.*

For the interval example, $E = (a, b), Y = \mathbb{R}$ G is an open ball around the midpoint of (a, b) with diameter $b - a$.

2.3 Compact Sets

An *open cover* of a set E in some metric space X is a collection of open subsets of X such that:

$$E \subset \bigcup_{\alpha} G_\alpha$$

Basically, the union of the sets "covers" the set E.

A subset K of a metric space X is said to be *compact* if every open cover of K contains a finite sub-cover. This means that we could cover K using only a finite number of sets.

Clearly, every finite set is compact, and there are a large number of infinite compact sets. Compactness also does not rely on the super-set. This means we do not need to consider the embedding space.

Theorem 2.13. *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.*

Proof. Let K be compact relative to X. Let $\{V_\alpha\}$ be a collection of sets that are open relative to Y and cover K.

Since V is open relative to Y, each V can be written as: $V_\alpha = Y \cap G_\alpha$.

These sets G form a cover for K, and as K is compact relative to X, $K \subset G_{\alpha_1} \cup ... \cup G_{\alpha_n}$.

However since also: $K \subset Y$. We know that $K \subset V_{\alpha_1} \cup ... \cup V_{\alpha_n}$. Thus K is compact relative to Y.

Let K be compact relative to Y, and let $\{G_\alpha\} \subset X$ be a cover of K. Let $V_\alpha = Y \cap G_\alpha$. V_α forms a cover of K, and since K is compact in Y, there is a finite sub-cover of K. Since each G is a super-set of V, the particular Gs form a finite sub-cover of K in X as well. □

Theorem 2.14. *Compact subsets of metric spaces are closed*

Proof. Let K be a compact subset of a metric space X. We wish to show that its complement is open.

Let $p \in K^c$. Consider a point $q \in K$. Take neighborhoods V_q, W_q around p and q with radius less than $d(p, q)$.

Since K is compact, we need a finite number of q to cover K. If we take the intersection of all the neighborhoods of p, we have a neighborhood of p that does not intersect any W_q.

This makes p and interior point of K^c and thus K is closed. □

Theorem 2.15. *Closed subsets of compact sets are compact*

Proof. Honestly I couldn't understand this proof. It relies on the relationship between the compliment of a set and its cover. □

What this theorem does tell us though, is that if we have a closed set and a compact set, then the intersection of the two is compact. As the intersection between a closed and compact set is closed.

Theorem 2.16. *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space, where the intersection of every finite sub-collection is nonempty, then $\cap K_\alpha$ is nonempty.*

Proof. Assume that No point of K_1 belongs to $\cap K_\alpha$, Then all of K_1 is covered by the complement of the intersections. However since K_1 is compact, then there is a finite sub-cover of the complement that covers it, but this implies that the intersection over a finite number of the subsets is empty, which is a contradiction. □

This means that if we have a sequence of nonempty compact sets where each successive one is a subset, then the infinite intersection is nonempty.

Theorem 2.17. *If E is an infinite subset of a compact set K, then E has a limit point in K*

Proof. If this were not true, then every point $q \in K$ would have a neighborhood that contains at most one point of E (q if $q \in E$.) This means no finite sub-collection of these neighborhoods can cover E, as E is infinite. This must be true of K, as it is a super-set of E. This is a contradiction. □

Theorem 2.18. *If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$ such that $I_n \supset I_{n+1}$ then $\cap_1^\infty I_n$ is not empty.*

Proof. For each interval $I_n = [a_n, b_n]$, Let E be the smallest point in all the intervals. E is nonempty and bounded above by b_1. Let $x = \sup E$. Since each interval is nested we know that $x \leq b_m \quad \forall m$. Then $x \in I_m \quad \forall m$. □

In fact this result extends to all k-cells

Theorem 2.19. *Let k be a positive integer, if $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$ then $\cap_1^\infty I_n$ is not empty.*

Proof. Form a matrix of intervals, where the rows are each n, and the columns are the dimensions of the k-cells. Then we can simply follow the above theorem for each column, and choose the vector $x^* = (x_1^*, ..., x_k^*)$ □

Theorem 2.20. *Every k-cell is compact*

Proof. Let I be a k-cell, consistent of all points $\boldsymbol{x}$ such that $a_j \leq x_j \leq b_j$ Let $\delta = \left[\sum_{j=1}^k (b_j - a_j)^2\right]^{\frac{1}{2}}$ Then $|\boldsymbol{x} - \boldsymbol{y}| \leq \delta$.

Suppose that I is not compact, and let $c_j = \frac{a_j + b_j}{2}$ The intervals $[a_j, c_j]$ and $[c_j, b_j]$ then determine 2^k k-cells Q_i whose union is I. Since I is not compact, at least one of these cells is not covered by any finite sub-collection of an open cover G_α. Let this cell be denoted I_1. Continue this process on infinitely.

This gives us a sequence $\{I_n\}$ which has the following properties

1. $I \supset I_1 \supset I_2 \supset .... \supset I_n$
2. I_n is not covered by any finite sub-collection of $\{G_\alpha\}$
3. If $\boldsymbol{x} \in I_n$ and $\boldsymbol{y} \in I_n$ then $|\boldsymbol{x} - \boldsymbol{y}| \leq 2^{-n}\delta$

From (1) we know that there is a point $\boldsymbol{x}$ which lies in every I_n. For some α, $\boldsymbol{x}^* \in G_\alpha$ as G is a cover for I. Since G_α is open, there exists $r > 0$ such that $|\boldsymbol{y} - \boldsymbol{x}^*| < r$ implies that $\boldsymbol{y} \in G_\alpha$. Choose n to be large enough that $2^{-n}\delta < r$. This is possible from the Archimedian property. This means that (3) implies that $I_n \subset G_\alpha$ which is contradicting (2). □

Theorem 2.21. *If a set E in $\mathbb{R}^k$ has one of the following properties, then it has the rest:*

1. *E is closed and bounded*
2. *E is compact*
3. *Every infinite subset of E has a limit point in E*

Proof. If (1) holds, then $E \subset I$ for some k-cell I, and (2) follows immediately. We have shown already that (2) implies (3) so we only have to show that (3) implies (1).

If E is not bounded, then E contains points $\boldsymbol{x}_n$ with $|\boldsymbol{x}_n| > n$. The set S consisting of these points is infinite and has no limit points in $\mathbb{R}^k$, and therefore none in E. Thus E must be bounded.

If E is not closed, then there is a point $x_0 \in \mathbb{R}^k$ which is a limit point of E but not a point of E. There are points $x_n \in E$ such that $|x_n - x_0| < \frac{1}{n}$. Let S be the set of all of these points. Then S is infinite by the Archimedian property. S has x_0 as a limit point, and no other limit point. For some $\boldsymbol{y} \in \mathbb{R}^k, \boldsymbol{y} \neq \boldsymbol{x}$. $|x_n - \boldsymbol{y}| \geq |x_0 - \boldsymbol{y}| - |x_n - x_0| \geq |x_0 - \boldsymbol{y} - \frac{1}{n} \geq \frac{1}{2}|x_0 - \boldsymbol{y}|$. This means that $\boldsymbol{y}$ is not a limit point of S, and S has no limit point in E, so E must be closed for (3) to hold. □

In every metric space, (2) and (3) are equivalent, but (a) is only equivalent to the other two in Borel spaces.

Theorem 2.22 (Weierstrass)**.** *Every bounded infinite subset of* $\mathbb{R}^k$ *has a limit point in* $\mathbb{R}^k$*.*

Proof. Since the set is bounded, it is a subset of a k-cell I which is compact, and therefore the set has a limit point in I, which is in $\mathbb{R}^k$. □

2.3.1 Perfect Sets

Theorem 2.23. *Let P be a nonempty perfect set in* $\mathbb{R}^k$*. P is uncountable*

Proof. Since P has limit points, P must be infinite. Suppose P is countable and denote the points by $\{x\}$. Then do some nonsense where you assume that the closure of neighborhoods have some properties and use the intersection with the closure and P to get compact sets that are empty, contradicting the fact they need to be nonempty. □

This is useful to showing that $[a, b]$ is uncountable for $(a < b)$. In particular this means that the set of all Real numbers is uncountable.

The cantor set: An uncountable perfect set that contains no segment and has measure 0.

Form it the traditional way, removing the middle third of every interval, and splitting each one in half.

2.3.2 Connected Sets

Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty. This means no point of A is in the closure of B and no point of B is in the closure of A.

A set $E \subset X$ is said to be *connected* if E is not a union of two nonempty separated sets.

Separated is a stronger condition than disjoint, for example $[0, 1]$ and $(1, 2)$ are disjoint but not separated, as 1 is a limit point of $(1, 2)$.

Theorem 2.24. *A subset E of a the real line $\mathbb{R}$ is connected if and only if it has the following property: If $x \in E, y \in E, x < z < y \implies z \in E$*

Proof. If this were false, then there exists $x \in E, y \in E$ and some $z \in (x, y)$ such that $z \notin E$. Then $E = A_z \cup B_z$ where $A_z = E \cap (-\infty, z) B_z = E \cap (z, \infty)$. Since $x \in A_z$, $y \in B_z$ they must not be empty. They are clearly separated. Therefore E is not connected. □

2.4 Exercises

2.4.1 Question 1

Question. *Prove that the empty set is a subset of every set*

Proof. For a set to be a subset, for all elements of the subset, they must be elements of the super-set. The empty set has no elements so this is vacuously true. □

2.4.2 Question 2

Question. *Prove that the set of all algebraic numbers is countable*

Algebraic: There are integers $a_0, ..., a_n$ which are not all zero such that: $a_0 z^n + a_1 z^{n-1} + ... + a_{n-1} z + a_n = 0$. Hint: For every positive integer N, there are only finitely many equations with: $n + |a_0| + |a_1| + ... + |a_n| = N$.

Proof. Note that the set of all possible values for a_i is countable. Therefore the set of all possible coefficients is an n-tuple of a countable set, and therefore countable. The set of coefficients for algebraic numbers is a subset of these numbers, and is at most countable. Clearly this set is not finite, as there are infinite degrees, and always at least one solution where $z = 1, a_n = -\sum_{i=0}^{n-1} a_i$

There is clearly an onto map from the coefficients to the algebraic numbers, this indicates that the cardinality of the algebraic numbers is less than or equal to the cardinality of the coefficients, and the algebraic numbers are at most countable. Assume that the algebraic numbers are finite. Since they are finite there must be a maximum, denote it M. Consider $M' = \lfloor M \rfloor + 1$, Let $n = 1, a_0 = -1, a_n = M'$. Clearly M' must be an algebraic number, contradicting M being the maximal element. Therefore the algebraic numbers are not finite, and therefore countable.

□

2.4.3 Question 4

Question. *Is the set of all irrational real numbers countable?*

Proof. Denote the irrational numbers as: $\mathbb{I}$. Note that: $\mathbb{R} = \mathbb{I} \cup \mathbb{Q}$. Assume that $\mathbb{I}$ is countable. Then by theorem 2.12, The union of $\mathbb{I}$ and $\mathbb{Q}$ must be countable, this however contradicts the fact that $\mathbb{R}$ is not countable. □

2.4.4 Question 5

Question. *Construct a bounded set of real numbers with exactly three limit points*

Proof. Consider a sequence of numbers $\{x\}$ For each index k, using the division algorithm, generate numbers r, w such that: $k = 3r + w, w \in \{0, 1, 2\}$. Let $x_k = \frac{1}{r} + 2w$ This clearly has 3 limit points: $\{0, 2, 4\}$. □

2.4.5 Question 6

Question. *Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. Do E and E' always have the same limit points?*

Proof.

1. Assume that E' is not closed, let x be a limit point of E' that is not a limit point of E. Every neighborhood of x contains a limit point of E, and there is a number ρ such that the neighborhood of radius $r \leq \rho$ does not contain any point of E. Take $\epsilon = \frac{\rho}{3}$. For a neighborhood of radius ϵ there is a limit point contained in it, denote it y. Since y is a limit point of E, it contains a point in E within a neighborhood of radius ϵ. Therefore the distance from that point in E to x is at most 2ϵ

which is a contradiction to there being no points of E in a neighborhood of radius $r \leq \rho$.

2. It is clear that there is no element of E' that is not contained in $\bar{E}$. All that remains to be shown is that there is no limit point of $\bar{E}$ that is not in E'. Let x be such a limit point. This means that every neighborhood of x contains a point of E' and some do not contain points of E. The same contradiction as above shows this is impossible.

3. Yes by (1).

□

2.4.6 Question 8

Question. *Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.*

Proof. Assume there is a point in an open set E that is not a limit point of E. This means that it contains a neighborhood that is a subset of E, but not every neighborhood contains a point in E. This is impossible, as every smaller neighborhood remains a subset of E, and any larger neighborhood still contains at least one point from the neighborhood that is a subset of E.

No, consider the set $[0,1] \cup \{2\}$. Two is an isolated point of the set, and is not a limit point. More generally, any closed set that is not perfect will not have every point be a limit point. □

2.4.7 Question 10

Question. *Let X be an infinite set. For $p \in X, q \in X$ define:*

$$d(p,q) = \begin{cases} 1 & (\textit{ if } p \neq q) \\ 0 & (\textit{ if } p = q) \end{cases}$$

Prove that this is a metric. Which subsets of the result metric space are open? Which are closed? Which are compact?

Proof. The first two conditions are trivially true, the triangle inequality remains to be demonstrated. We will consider several cases:

Case: $d(p,q) = 0$. Then $d(p,r) + d(r,q) \geq 0$ and the result holds

Case: $d(p,q) = 1, d(p,r) = 0$ or $d(r,q) = 0$ Then $d(p,r) + d(r,q) = 1 \geq d(p,q)$

Case: $d(p,q) = 1, d(p,r) = 1, d(r,q) = 1$ Then $d(p,r) + d(r,q) = 2 \geq d(p,q)$

Note that it is impossible for $d(p,q) = 1, d(p,r) + d(r,q) = 0$.

In this metric space, there are no limit points, as neighborhoods with radius less than 1 contain only the center. This means that all sets are closed. Every point is also an interior point, so all subsets are open as well. Since there are no limit points in this set, no infinite subset is compact. Finite sets are still compact. □

2.4.8 Question 11

Question. *1.* $d(x,y)^1 = (x-y)^2$

2. $d(x,y)^2 = \sqrt{|x-y|}$

3. $d(x,y)^3 = |x^2 - y^2|$

4. $d(x,y)^4 = |x - 2y|$

5. $d(x,y)^5 = \frac{|x-y|}{1+|x-y|}$

Determine which of these is a metric space or not in $\mathbb{R}$

Proof. 1. Clearly the first two properties are satisfied. Now consider the triangle inequality: $(x-z)^2 = ((x-y)-(z-y))^2 = (x-y)^2 + (z-y)^2 - 2(x-y)(z-y)$ Note that if $y \in (x,z), 2(x-y)(z-y) < 0$ and the third inequality does not hold.

2. Clearly the first two properties are satisfied. Since it is known that $|x-y|$ is a metric: $|x-y| \leq |x-z| + |z-y|$. Thus: $\sqrt{|x-y|} \leq \sqrt{|x-z|+|z-y|}$. Note: $\sqrt{x+y} \leq \sqrt{x + 2\sqrt{x}\sqrt{y} + y} = \sqrt{(\sqrt{x}+\sqrt{y})^2} = \sqrt{x} + \sqrt{y}$. Thus: $\sqrt{|x-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$.

3. Consider the points $x = 2, y = -2$. $d(x,y)^3 = 0$ despite $x \neq y$. Clearly this is not a metric space.

4. Trivially the first two properties are satisfied. $|x-2y| = |(x-2z)+(z-2y)+z| \leq |(x-2z)+(z-2y)| \leq |x-2z| + |z-2y|$

5. We can see that: $d(x,x)^5 = 0$, and $d(x,y)^5 = d(y,x)^5$. Note that: $d(x,y) = 1 - \frac{1}{1+|x-y|}$. Clearly the third property holds as the fraction reverses the triangle inequality, and the negative keeps the inequality correct.

□

2.4.9 Question 14

Question. *Give an example of an open cover of the segment* $(0,1)$ *which has no finite sub-cover*

Proof. Consider the sequence: $\{x\}$ where $x_k = \frac{1}{k}$. Consider the sequence of segments: $\{s\}$ where $s_k = (x_k, x_{k+1})$. By the Archimedian property this sequence is a cover of $(0,1)$. However any finite sub-cover of it must not contain every element of $(0,1)$ by the Archimedian property. □

3 Numerical Sequences and Series

3.1 Convergence Sequences

A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property:

For every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_{n,}p) < \epsilon$.

In this case we also say that $\{p_n\}$ converges to p or that p is the limit of $\{p_n\}$. We write: $p_n \to p$ or $\lim\limits_{n\to\infty} p_n = p$

If $\{p_n\}$ does not converge, it is said to *diverge*.

It is useful to note that the notion of convergence relies upon the metric space that we have embedded ourselves in. When it is ambiguous, it will be specified *convergent in* X.

The range of a sequence is the set of all the different points p_n. It may be finite or infinite, and we say $\{p_n\}$ is bounded if its range is bounded.

Theorem 3.1. *Let* $\{p_n\}$ *be a sequence in a metric space* X*.*

1. $\{p_n\}$ *converges to* $p \in X$ *if and only if every neighborhood of* p *contains* p_n *for all but finitely many* n

2. *If* $p \in X, p' \in X$*, and if* $\{p_n\}$ *converges to* p *and* p'*, then* $p = p'$*.*

3. *If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.*

4. *If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.*

Proof. 1. Let $p_n \to p$ and let V be a neighborhood of p.

For some $\epsilon > 0$, the conditions $d(q,p) < \epsilon$ imply $q \in V$. Since $\{p_n\}$ is convergent, there exists N such that $n \geq N \implies d(p_n, p) < \epsilon$.

Thus $n \geq N \implies p_n \in V$. Conversely, if every neighborhood of p contains all but finitely many of the p_n.

Fix $\epsilon > 0$ and let V be the set of all $q \in X$ such that $d(p,q) < \epsilon$. Since all but finitely many lie in the neighborhood, there exists N (related to V) such that $p_n \in V$ if $n \geq N$. Thus $d(p_n, p) < \epsilon$ and $p_n \to p$.

2. Let $\epsilon > 0$ be given. There exists integers N, N' such that:

$$n \geq N \implies d(p_n, p) < \frac{\epsilon}{2}$$
$$n \geq N' \implies d(p_n, p') < \frac{\epsilon}{2}$$

Hence for all $n \geq \max\{N, N'\}$ we get: $d(p,p') \leq d(p,p_n) + d(p_n, p') < \epsilon$.

3. Suppose $p_n \to p$. There is an integer N such that for $n > N, d(p_n, p) < 1$. Let $r = \max\{1, d(p_1, p), ..., d(p_N, p)\}$ Then $d(p_n, p) \leq r \quad \forall n$.

4. For each positive integer n, there is a point $p_n \in E$ such that $d(p_n, p) < \frac{1}{n}$. For any $\epsilon > 0$ Choose $N = \lfloor \frac{1}{\epsilon} \rfloor + 1$ For all $n > N, d(p_n, p) < \epsilon$ and therefore $p_n \to p$

□

For sequences in $\mathbb{R}^k$ we can study the relation between convergence and algebraic operations.

Theorem 3.2. *Suppose $\{s_n\}, \{t_n\}$ are complex sequences, and $\lim\limits_{n\to\infty} s_n = s$, $\lim\limits_{n\to\infty} t_n = t$ Then:*

1. $\lim\limits_{n\to\infty} (s_n + t_n) = s + t$

2. $\lim_{n\to\infty} cs_n = cs$

3. $\lim_{n\to\infty} s_n t_n = st$

4. $\lim_{n\to\infty} \frac{1}{s_n} = \frac{1}{s}$ *Provided:* $s_n \neq 0, s \neq 0.$

Proof. 1. For some $\epsilon > 0$, there exists integers N_1, N_2 such that: $n \geq N_1 \implies |s_n - s| < \frac{\epsilon}{2}$ and $n \geq N_2 \implies |t_n - t| < \frac{\epsilon}{2}$. Choose $N = \max\{N_1, N_2\}$ then $n \geq N$ implies: $|s_n + t_n - s - t| \leq |s_n - s| + |t_n - t| < \epsilon$.

2. Trivial as c can be pulled out of $|cs_n - cs|$

3. $s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s)$ For some $\epsilon > 0$ Do the same N_1, N_2 for $\sqrt{\epsilon}$. Let $N = \max\{N_1, N_2\}$ so that: $n \geq N \implies |(s_n - s)(t_n - t)| < \epsilon$ and: $\lim_{n\to\infty}(s_n - s)(t_n - t) = 0$. Applying this, as well as the convergence of $\{s_n\} and \{t_n\}$ to the original identity yields: $\lim_{n\to\infty}(s_n t_n - st) = 0$.

4. Choose m such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$ Then $|s_n| > \frac{1}{2}|s|$ for $n \geq m$. For some $\epsilon > 0$ by convergence, there is an integer $N > m$ such that $n \geq N \implies |s_n - s| < \frac{1}{2}|s|^2 \epsilon$. Thus: $\left|\frac{1}{s_n} - \frac{1}{s}\right| = \left|\frac{s_n - s}{s_n s}\right| < \frac{2}{|s|^2}|s_n - s| < \epsilon$

□

That last proof made no sense to me either.

Theorem 3.3. *Suppose* $\{\boldsymbol{x}_n\} \in \mathbb{R}^k$ *and* $\boldsymbol{x}_n = (\alpha_{1,n}, ...\alpha_{k,n})$. *Then* $\{\boldsymbol{x}_n\}$ *converges to* $\boldsymbol{x} = (\alpha_1, ..., \alpha_k)$ *if and only if* $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$.

Proof. Let $\boldsymbol{x}_n \to \boldsymbol{x}$, Then from the definition of the norm in $\mathbb{R}^k$ we get that: $|\alpha_{j,n} - \alpha_j| \leq |\boldsymbol{x}_n - \boldsymbol{x}|$.

Now let $\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$. Then there is an integer N such that

$$n \geq N \implies |\alpha_{j,n} - \alpha_j| < \frac{\epsilon}{\sqrt{k}}$$

$$n \geq N \implies |\boldsymbol{x}_n - \boldsymbol{x}| = \left[\sum_{j=1}^{k} |\alpha_{j,n} - \alpha_j|^2\right]^{\frac{1}{2}} < \epsilon$$

This implies that $\boldsymbol{x}_n \to \boldsymbol{x}$. □

All the properties of sums, dot products and scalar products in convergence hold for sequences of vectors as well.

3.2 Subsequences

For some sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of strictly increasing positive integers. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

Theorem 3.4. *A sequence $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.*

Proof. Consider a convergent sequence $\{p_n\}$, For some $\epsilon > 0$ and some arbitrary subsequence, choose $N_k = \min\{n_i, n_i \geq N\}$. It is clear that for all $n \geq N_k, |p_{n_i} - p| < \epsilon$.

For the converse, assume that every subsequence is convergent to p. For some $\epsilon > 0$ Take $N = \max\{N_i\}$. Clearly for all $n \geq N, |p_n - p| < \epsilon$. □

Theorem 3.5. 1. *If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X*

2. *Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.*

Proof. 1. Let E be the range of $\{p_n\}$. If E is finite, then there is a sequence of indices such that $p_{n_1} = p_{n_2} = ... = p_{n_k} = p$. Clearly it converges.

If E is infinite, theorem 2.17 indicates that E has a limit point $p \in X$. Since it has a limit point, every neighborhood contains an element of $\{p_n\}$ Choose each n_i such that p_{n_i} lies in a neighborhood with radius $\frac{1}{i}$. This subsequence must converge to p.

2. Since every bounded subset of $\mathbb{R}^k$ lies in a compact subset of $\mathbb{R}^k$ this result follows immediately from part (1).

□

Theorem 3.6. *The sub-sequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.*

Proof. Let E^* be the set of all sub-sequential limits of $\{p_n\}$ and let q be a limit point of E^*. We wish to show that $q \in E^*$.

Choose n_1 so that $p_{n_1} \neq q$. If this is not possible, then the sequence has only one point and we are done. Let $\delta = d(q, p_{n_1})$. Since q is a limit point of E^*, we can choose n_i such that $d(q, p_{n_i}) < 2^{-i}\delta$. We can see that this sequence will converge to q and $q \in E^*$. □

3.3 Cauchy Sequences

A sequence $\{p_n\}$ in a metric space X is said to be a *Cauchy sequence* if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon$ if $n \geq N$ and $m \geq N$.

Let E be a nonempty subset of a metric space X, the *diameter* of a set is the supremum of $d(p, q)$ over all $p \in E, q \in E$.

For a sequence $\{p_n\} \in X$. Let E_N be the range of $\{p_n\}$. It is clear that $\{p_n\}$ is a Cauchy sequence if and only if: $\lim_{N\to\infty} \operatorname{diam} E_N = 0$

Theorem 3.7. *1. If $\bar{E}$ is the closure of a set E in a metric space X, then* $\operatorname{diam} \bar{E} = \operatorname{diam} E$

2. If $\{K_n\}$ is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ and $\lim_{n\to\infty} \operatorname{diam} K_n = 0$ *Then $\cap_1^\infty K_n$ consists of exactly one point.*

Proof. 1. Since $E \subset \bar{E}$, it is clear that: $\operatorname{diam} E \leq \operatorname{diam} \bar{E}$. Fix $\epsilon > 0$ and choose $p \in \bar{E}, q \in \bar{E}$. By the definition of $\bar{E}$, there are points p', q' in E such that $d(p, p') < \epsilon, d(q, q') < \epsilon$. Hence:

$$\begin{aligned} d(p,q) &\leq d(p,p') + d(p',q') + d(q',q) \\ &< 2\epsilon + d(p',q') \leq 2\epsilon + \operatorname{diam} E \end{aligned}$$

Since $\operatorname{diam} \bar{E}$ is the supremum over all p, q, and we know that $\bar{E}$ is closed, we may choose p, q such that $d(p, q) = \operatorname{diam} \bar{E}$. Then: $\operatorname{diam} \bar{E} \leq 2\epsilon + \operatorname{diam} E$ and since ϵ was arbitrary, we have $\operatorname{diam} E = \operatorname{diam} \bar{E}$.

2. Let $K = \cap_1^\infty K_n$, it is known that K is not empty. If K contains more than one point, $\operatorname{diam} K > 0$. But we know that since $K \subset K_n, \operatorname{diam} K \leq \operatorname{diam} K_n$ This contradicts $\operatorname{diam} K_n \to 0$ So K must contain only one point.

□

Theorem 3.8. *1. In any metric space X, every convergent sequence is a Cauchy sequence*

2. If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point of X.

3. In $\mathbb{R}^k$, every Cauchy sequence converges

Proof. 1. If $p_n \to p$ and if $\epsilon > 0$, there is an integer N such that $d(p, p_n) < \epsilon$ for all $n \geq N$. Hence: $d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < 2\epsilon$ if $n \geq N, m \geq N$. This means that $\{p_n\}$ is a Cauchy sequence.

2. Let $\{p_n\}$ be a Cauchy sequence in the compact space X. For $N \in \mathbb{N}$, let E_N be the set consisting of $p_N, p_{N+1}, ...$ Then: $\lim_{N\to\infty} \operatorname{diam} \bar{E_N} = 0$. Since $\bar{E_N}$ is a closed subset of a compact space, they are all compact. and $\bar{E_N} \supset \bar{E_{N+1}}$. Their infinite intersection therefore contains only one point by the above theorem. Call this point $p \in \bar{E_N}$. Since the limit of the diameter tends to zero, and p is in the sequence, $\{p_n\} \to p$.

3. Let $\{\boldsymbol{x}_n\}$ be a Cauchy sequence in $\mathbb{R}^k$. Define E_N the same as above. For some N, $\operatorname{diam} E_N < 1$. The range of $\{\boldsymbol{x}_n\}$ is the union of E_N and the finite set of the previous values of $\{\boldsymbol{x}_n\}$. This means that $\{\boldsymbol{x}_n\}$ is bounded. Every bounded subset of $\mathbb{R}^k$ has compact closure in $\mathbb{R}^k$, this result follows from above.

□

A metric space in which every Cauchy sequence converges is said to be *complete*.

The above theorem can be extended to now say: all compact metric spaces are complete, as well as all Euclidean spaces. It also implies that every closed subset of a complete metric space is complete.

3.4 Monotonic Sequences

A sequence $\{s_n\}$ of real numbers is said to be

- *monotonically increasing* if $s_n \leq s_{n+1}$
- *monotonically decreasing* if $s_n \geq s_{n+1}$

The class of monotonic sequences consists of the increasing and decreasing sequences.

Theorem 3.9. *Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.*

Proof. Let $\{s_n\}$ be a monotonically increasing sequence. Let E be the range. Assume $\{s_n\}$ is bounded, and let s be the least upper bound of E. It is clear that $s_n \leq s$ and for every $\epsilon > 0$ there is an integer N such that: $s - \epsilon < s_N \leq s$. This happens because s is the least upper bound. This is convergence to s.

The converse has already been established, as all convergent sequences are bounded. □

3.5 Upper and Lower Limits

Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of subsequential limits and $\{\infty, -\infty\}$ for divergent subsequences. Let $s^* = \sup E$ and $s_* = \inf E$. These numbers are called the *upper* and *lower* limits of $\{s_n\}$. The more common notation is:

$$\limsup_{n\to\infty} s_n = s^* \quad \liminf_{n\to\infty} s_n = s_*$$

Theorem 3.10. *Let $\{s_n\}$ be a sequence of real numbers, Using the above definitions for E and s^*. s^* is the only number with the following two properties*

1. $s^* \in E$

2. *If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$*

The same is true in opposite for s_*.

Proof. 1. If $s^* = \infty$, then E is not bounded above and $\{s_n\}$ is not bounded above, so there is a subsequence that converges to ∞.

   If s^* is real, then E is bounded above and at least one subsequential limit exists, and the result comes from theorem 3.6.

   If $s^* = -\infty$, then E contains only one element, so the result is trivial

2. Suppose there is a number $x > s^*$ such that $s_n \geq x$ for all values of n, that means there is a convergent subsequence that has a limit above s^* which contradicts it being the supremum.

3. uniqueness arises from the second property immediately driving uniqueness.

□

Theorem 3.11. *If $s_n \leq t_n$ for $n \geq N$ where N is fixed then:*

$$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n$$
$$\limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

Special Sequences

Theorem 3.12. *1. If $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$*

2. If $p > 0$ then $\lim_{N\to\infty} p^{\frac{1}{p}} = 1$

3. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

4. IF $p > 0$ and α is real, then $\lim_{N\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$

5. If $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$

Proof. 1. Take $n > \frac{1}{\epsilon}^{\frac{1}{p}}$ This is possible by the Archimedean property.

2. If $p > 1$ put $x_n = p^{\frac{1}{n}} - 1$ Then $x_n > 0$ and by the binomial theorem:

$$1 + nx_n \leq (1 + x_n)^n = p$$
$$0 < x_n \leq \frac{p-1}{n}$$

Then: $x_n \to 0$. If $p = 1$ the result is trivial. If $p \in (0, 1)$ simply take the reciprocal and repeat process for $p > 1$

3. Put $X_n = n^{\frac{1}{n}} - 1$ Then $x_n \geq 0$ and from the binomial theorem:

$$n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2$$
$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}} (n \geq 2)$$

4. Let k be an integer such that $k > \alpha, k > 0$. For $n > 2k$

$$(1+p)^n > \binom{n}{k} p^k > \frac{n^k p^k}{2^k k!}$$
$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha - k} (n > 2k)$$

5. Take $\alpha = 0$ in the above example.

□

Series

Series are omitted for the sake of time.

The Number e

Let $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ We can see that this series converges because of:

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + ... + \frac{1}{1 \cdot 2 \cdot 3 \cdot .... \cdot n} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} < 3 \end{aligned}$$

Theorem 3.13. $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

Proof. Let

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} \quad t_n = \left[1 + \frac{1}{n}\right]^n$$

From the binomial theorem:

$$t_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + ... + \frac{1}{n!}(1 - \frac{1}{n})...(1 - \frac{n-1}{n})$$

Hence: $t_n \leq s_n$ so that: $\limsup_{n \to \infty} t_n \leq e$. by theorem 3.11. Next if $n \geq m$

$$tn \geq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + ... + \frac{1}{m!}(1 - \frac{1}{n})...(1 - \frac{m-1}{n})$$

Letting $n \to \infty$ keeping m fixed. We get

$$\liminf_{n\to\infty} t_n \geq 1 + 1 + \frac{1}{2!} + ... + \frac{1}{m!}$$

so that: $s_m \leq \liminf_{n\to\infty} t_n$. Letting $m \to \infty$ we finally get $e \leq \liminf_{n\to\infty} t_n$ and we see that the two sequences converge to the same limit. □

Theorem 3.14. *e is irrational*

Proof. Suppose e is rational, then $e = \frac{p}{q}$ where $p, q \in \mathbb{N}$ and irreducible. Using the inequality: $0 < e - s_n < \frac{1}{n!n}$ we can plug in $n = q$ and arrive at:

$$0 < q!(e - s_q) < \frac{1}{q}$$

By assumption: $q!e$ is an integer, and $q!s_q = q!(1 + 1 + \frac{1}{2!} + ... + \frac{1}{q!})$ which is an integer. Clearly: $q!(e - s_q)$ is an integer.

However, we know that q is an integer so: $q \geq 1$ and $0 < q!(e - s_q) < 1$ which is a contradiction for an integer. □

In actuality, e is not even an algebraic number. However a proof is not supplied and can easily be googled.

Exercises

Question 1

Question. *Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?*

Proof. Let $\{s_n\}$ be a convergent sequence, then for $\epsilon > 0$ and for all $n \geq N$ $|s_n - s| < \epsilon$. Note that: $||s_n| - |s|| \leq |s_n - s| < \epsilon$ so clearly $|s_n|$ converges to: $|s|$.

The converse is not true, consider the sequence $s_k = (-1)^k$. This sequence is divergent, but its absolute value is a constant sequence. □

Question 2

Question. *Calculate* $\lim\limits_{n\to\infty} \sqrt{n^2 + n} - n$

Proof.

$$\begin{aligned}
\sqrt{n^2+n}-n &= \sqrt{n^2+n}-n\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} \\
&= \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} \\
\text{only holds with n} > 0 &= \frac{1}{\sqrt{1+\frac{1}{n}}+1} \\
\lim_{n\to\infty}\sqrt{n^2+n}-n &= \frac{1}{2}
\end{aligned}$$

□

Question 3

Question. *Let* $s_1 = \sqrt{2}, s_{n+1} = \sqrt{2+\sqrt{s_n}}$ *Prove that* $\{s_n\}$ *converges and that* $s_n < 2$.

Proof. Note that: s_n is positive for all n, so it is equivalent to consider the convergence of s_n^2. $s_{n+1}^2 = 2+\sqrt{s_n}$.

We wish to show that this sequence is increasing and bounded above.

Bounded above: Approach by induction. It is clear that $s_1 < 2$. Assume that $s_n < 2$, $s_{n+1}^2 = 2+\sqrt{s_n}$. From the inductive hypothesis we see that: $s_{n+1}^2 < 2+\sqrt{2} < 4$ and $s_{n+1} < 2$. By induction we can see that s_n is bounded above by 2.

Increasing:

$$\begin{aligned}
s_n - s_{n+1} &= s_n - \sqrt{2+\sqrt{s_n}} \\
&\leq s_n - \sqrt{2} - s_n^{\frac{1}{4}} \\
&< s_n - \sqrt{2} \\
&< 0 \text{ for } s_n < 2
\end{aligned}$$

□

Question 4

Question. *Find the upper and lower limits of the sequence* $\{s_n\}$ *defined by:*

$$s_1 = 0 \quad s_{2m} = \frac{s_{2m-1}}{2} \quad s_{2m+1} = \frac{1}{2} + s_{2m}$$

Proof. Clearly the lower limit of this sequence is zero, as neither operation can decrease the sequence.

Consider the subsequence made of only the odd elements: $s_0 = 0, s_k = \frac{s_{k-1}+1}{2} = \frac{2^k-1}{2^k}$.

The subsequence made of the even elements: $s_0 = 0, s_k = \frac{1+2s_{k-1}}{4} = \frac{2^{k+1}-2^k-1}{2^{k+1}}$

Any other convergent subsequence must be subsequences of these subsequences. Their limits are clearly $1, \frac{1}{2}$. The supremum of this set then is 1. □

Question 14

Question. *If* $\{s_n\}$ *is a complex sequence, its arithmetic mean is defined by:*

$$\sigma_n = \frac{s_0 + s_1 + \ldots + s_n}{n+1}$$

1. *If* $\lim_{n\to\infty} s_n = s$ *prove that* $\lim\limits_{n\to\infty} \sigma_n = s$
2. *Construct a sequence* $\{s_n\}$ *which does not converge although* $\lim \sigma_n = 0$
3. *Can it happen that* $s_n > 0$ *and that* $\limsup s_n = \infty$ *but* $lim\sigma_n = 0$
4. *Put* $a_n = s_n - s_{n-1}$. *Show that:*

$$s_n - \sigma_n = \frac{1}{n+1}\sum_{k=1}^{n} ka_k$$

Assume that $\lim(na_n) = 0$ *and that* $\{\sigma_n\}$ *converges to* σ.

Proof. 1. Note that since $\{s_n\}$ is convergent, it is bounded. Call its upper bound M

$$\begin{aligned} |\sigma_n - s| &= \left| \frac{s_0 + s_1 + ... + s_n}{n+1} - \frac{s(n+1)}{n+1} \right| \\ &= \left| \frac{(s_0 - s) + (s_1 - s) + ... + (s_n - s)}{n+1} \right| \\ &\leq \frac{|s_0 - s|}{n+1} + \frac{|s_1 - s|}{n+1} + ... + \frac{|s_n - s|}{n+1} \\ &< \frac{NM}{n+1} + \frac{(n+1-N)\epsilon}{n+1} \end{aligned}$$

It is clear that the first term can be made arbitrarily small, and the second term always remains less than ϵ. Clearly this sequence converges.

2. Consider the sequence $\{s_n\} = (-1)^n$. It is clearly divergent, however the limit of its arithmetic mean is 0.

3. No, for the sum of a strictly positive sequence divided by a strictly positive number to converge to zero, the sequence must converge to zero.

   Assume otherwise. Claim there is a number M such that infinitely many of $s_n \geq M$. Then $s_0 + s_1 + ... + s_n \geq n \min\{s_i\}$ for some $n > 0$. and when divided by n, this tends to the minimum, which is not zero.

   However if the sequence converges to zero, then no subsequence can be divergent, and it is impossible for $\limsup s_n = \infty$.

4.

$$\begin{aligned}
s_n - \sigma_n &= \frac{(n+1)s_n - s_0 - s_1 - ... - s_n}{n+1} \\
\sum_{k=1}^{n} ka_k &= \sum_{k=1}^{n} k(s_n - s_{n-1}) \\
&= \sum_{k=1}^{n} -s_k + ns_n \\
s_n - \sigma_n &= \frac{1}{n+1} \sum_{k=1}^{n} ka_k
\end{aligned}$$

We wish to show that: $\frac{1}{n+1} \sum_{k=1}^{n} [ka_k] + \sigma_n$ converges.

□

4 Limits and Continuity

4.1 Limits

For metric spaces X and Y, Let $E \subset Y$, f maps E into Y and p be a limit point of E. We write: $f(x) \to q$ as $x \to p$ or $\lim_{x \to p} f(x) = q$ if there is a point $q \in Y$ with the following property:

For every $\epsilon > 0$ there exists a $\delta > 0$ such that: $d(f(x), q) < \epsilon$ for all points $x \in E$ for which $0 < d(x, p) < \delta$.

Note that although $p \in X$, p does not need to be a point of E, and even if p is a point in E, the limit does not need to equal the function at that point.

Since this definition is concerned with limit points, which can be viewed as limits of sequences, there is an equivalent definition for a limit using sequences:

We say that: $\lim_{x \to p} f(x) = q$ if and only if $\lim_{n \to \infty} f(p_n) = q$ for every sequence $\{p_n\}$ in E such that: $p_n \neq p$, $\lim_{n \to \infty} p_n = p$

Proof. Let f be a function such that: $\lim_{x \to p} f(x) = q$, and let $\{p_n\}$ be a sequence in E such that $p_n \neq p$, $\lim_{n \to \infty} p_n = p$. For some $\epsilon > 0$, there exists $\delta > 0$ such that: $d(f(x), q) < \epsilon$ if $x \in E$ and $0 < d(x, p) < \delta$.

For a the sequence $\{p_n\}$, there exists some N such that $n > N$ implies $0 < d(p_n, p) < \delta$. So taking $x = p_n$, $d(f(p_n), q) < \epsilon$ and shows that $Lim_{n\to\infty} f(p_n) = q$.

Now suppose that f does not have limit q as $x \to p$. For some $\epsilon > 0$, for every $\delta > 0$ there is some point $x \in E$ for which $d(f(x), q) \geq \epsilon$, but $0 < d(x, p) < \delta$.

Let $p_n = p + / - \frac{1}{n}$, or its projection into E, then $p_n \neq p$ as p is a limit point, and $\lim_{n\to\infty} p_n = p$. However, $\lim_{n\to\infty} f(p_n) \neq q$. □

Since limits of sequences have already been proved to be unique, this definition tells us that if f has a limit at p, then this limit is unique.

Theorem 4.1. *Suppose $E \subset X$ is a metric space, p is a limit point of E and f, g are complex functions on E such that: $\lim_{x\to p} f(x) = A, \lim_{x\to p} g(x) = B$. Then:*

1. $\lim_{x\to p}(f + g)(x) = A + B$
2. $\lim_{x\to p}(fg)(x) = AB$
3. $\lim_{x\to p}(\frac{f}{g})(x) = \frac{A}{B}$ *for* $B \neq 0$

These follow immediately form their proofs of the sequences.

If f, g are mapping E into $\mathbb{R}^k$ then instead of multiplication and division, the limit property of the scalar product remains true:
$\lim_{x\to p}(f \cdot g)(x) = A \cdot B$.

4.2 Continuous functions

Let X and Y be metric spaces, $E \subset X, p \in E$ and f maps E into Y. Then f is said to be *continuous at* p if for every $\epsilon > 0$ there exists a $\delta > 0$ such that: $d(f(x), f(p)) < \epsilon$.

If f is continuous at every point of E, then f is said to be *continuous on* E.

Note that f has to be defined at the point p in order to be continuous at p. However there is no requirement that p is a limit point, continuity is defined for isolated points as well. A function is always continuous at isolated points, as there is a neighborhood that contains no points of E besides p.

Theorem 4.2. *When p is a limit point of E, f is continuous at p if and only if* $\lim_{x \to p} f(x) = f(p)$.

Theorem 4.3. *Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps $f(E)$ into Z and h is the mapping of E into Z defined by $h(x) = g(f(x))$*

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$ then h is continuous at p.

This function h is called the composition *of f and g and is denoted: $h = f \circ g$*

Proof. Let $\epsilon > 0$, since g is continuous at $f(p)$ there exists $\eta > 0$ such that: $d\left(g(y), g(f(p))\right) < \epsilon$ if $d\left(y, f(p)\right) < \eta$ and $y \in f(E)$.

Since f is continuous at p, there exists $\delta > 0$ such that: $d\left(f(x), f(p)\right) < \eta$ if $d\left(x, p\right) < \delta$ and $x \in E$. Thus we can see that: $d\left(h(x), h(p)\right) = d\left(g(f(x)), g(f(p))\right) < \epsilon$ For $d\left(x, p\right) < \delta$ and $x \in E$. Thus h is continuous at p. □

Theorem 4.4. *A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.*

Proof. Suppose f is continuous on X and V is an open set in Y. We wish to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$.

Suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\epsilon > 0$ such that $y \in V$ if $d\left(y, p\right) < \epsilon$. Since f is continuous at p, there exists $\delta > 0$ such that $d\left(f(x), f(p)\right) < \epsilon$ if $d\left(x, p\right) < \delta$. Thus: $x \in f^{-1}(V)$ if $d\left(x, p\right) < \delta$. This gives a neighborhood of p that is a subset of $f^{-1}(V)$ and thus it is open.

Conversely: Suppose that $f^{-1}(V)$ is open in X for every open set V in Y. Fix $p \in X, \epsilon > 0$ and let V be the set of all $y \in Y$ such that $d\left(y, f(p)\right) < \epsilon$. Then V is open, and $f^{-1}(V)$ is open by assumption.

This gives rise to a $\delta > 0$ such that $x \in f^{-1}(V)$ when $d\left(p, x\right) < \delta$. But if $x \in f^{-1}(V)$, then $f(X) \in V$ so that $d\left(f(x), f(p)\right) < \epsilon$. □

This also implies that a mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.

This arises from the above theorem, and from the complement of open sets being closed as well as: $f^{-1}(E^c) = \left[f^{-1}(E)\right]^c$ for every $E \subset Y$

Theorem 4.5. *Let f, g be complex continuous functions on a metric space X. Then $f+g, fg, \frac{f}{g}$ are continuous on X. We require that $g(x) \neq 0$ for the quotient identity.*

Proof. There is nothing to prove at isolated points, and the result follows immediately for limit points from theorems 4.1 and 4.2. □

Theorem 4.6. 1. *Let $f_1, ...f_k$ be real functions on a metric space X, and let $\boldsymbol{f}$ be the mapping of X into $\mathbb{R}^k$ defined by:*

$$\boldsymbol{f}(x) = (f_1(x), ..., f_k(x))$$

then $\boldsymbol{f}$ is continuous if and only if each of the functions $f_1, ...f_k$ are continuous.

2. *If $\boldsymbol{f}, \boldsymbol{g}$ are continuous mappings of X into $\mathbb{R}^k$, then $\boldsymbol{f}+\boldsymbol{g}$ and $\boldsymbol{f}\cdot\boldsymbol{g}$ are continuous on X.*

we call the functions $f_1, ..., f_k$ the components *of $\boldsymbol{f}$. Note that $\boldsymbol{f}+\boldsymbol{g}$ is a mapping into $\mathbb{R}^k$, whereas $\boldsymbol{f}\cdot\boldsymbol{g}$ is a real function on X.*

Proof. 1.

$$|f_j(x) - f_j(y)| \leq |\boldsymbol{f}(x) - \boldsymbol{f}(y)| = \left[\sum_{i=1}^{k} |f_i(x) - f_i(y)|^2\right]^{\frac{1}{2}}$$

Clearly if $\boldsymbol{f}$ is continuous, then each of the component functions are continuous.

If we apply the definition for $\eta < \epsilon\sqrt{k}$ for each of the component functions, then we can see that $|\boldsymbol{f}(x) - \boldsymbol{f}(y)| < \epsilon$.

2. Note that $\boldsymbol{f}\cdot\boldsymbol{g}$ is a sum of component functions, each of which are continuous, so it must be continuous by theorem 4.5.

□

One important factor of continuous functions is that even though they are defined for subsets of metric spaces, the subset plays no role in the definition. This means that it is equivalent to consider continuous mappings of entire metric spaces without losing any information.

4.3 Continuity and Compactness

A mapping $\boldsymbol{f}$ of a set E into $\mathbb{R}^k$ is said to be *bounded* if there is a real number M such that $|\boldsymbol{f}(x)| \leq M$ for all $e \in E$.

Theorem 4.7. *Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.*

Proof. Consider $\{V_\alpha\}$ which is an open cover of $f(X)$. Since f is continuous, theorem 4.4 shows that each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, there are finitely many indices such that:

$$X \subset f^{-1}(V_{\alpha_1}) \cup ... \cup f^{-1}(V_{\alpha_n})$$

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, then

$$f(X) \subset V_{\alpha_1} \cup ... \cup V_{\alpha_n}$$

□

This used the fact that $f(f^{-1}(E)) \subset E$ which is valid as long as $E \subset Y$. The other identity of use for inverses is: $f^{-1}(f(E)) \supset E$. For neither case is equality guaranteed.

Theorem 4.8. *If $\boldsymbol{f}$ is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $\boldsymbol{f}(X)$ is closed and bounded. Thus $\boldsymbol{f}$ is bounded.*

Proof. This follows directly from theorem 2.21. □

Theorem 4.9. *Suppose f is a continuous real function on a compact metric space X. Let*

$$M = \sup_{p \in X} f(p) \quad m = \inf_{p \in X} f(p)$$

Then there exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

This is basically saying that a function defined on a compact metric space has a minimum and a maximum and obtains both.

Proof. Theorem 4.8 shows that $f(X)$ is closed and bounded, and therefore it contains its supremum and infimum by theorem 2.11. □

Theorem 4.10. *Suppose f is a continuous $1-1$ mapping of a compact metric space X onto a metric space Y. Then the inverse mapping f^{-1} defined on Y is a continuous mapping of Y onto X.*

Proof. We need only show that $f(V)$ is an open set in Y for every open set $V \subset X$. Fix such a set V.

The complement of V, V^c is closed in X. Theorem 2.15 implies that it is compact. Therefore $f(V^c)$ is a compact subset of Y by theorem 4.7. This implies that it is closed in Y. Since f is $1-1$ and onto, $f(V)$ is the complement of $f(V^c)$ and therefore is open. □

Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\epsilon > 0$ there exists $\delta > 0$ such that:

$$d\left(f(p), f(q)\right) < \epsilon \quad \forall p, q \in X : d\left(p, q\right) < \delta$$

Note that uniform continuity is a property of a function on a set rather than a definition at a single point. Note that for a continuous function, δ is a function that depends on both ϵ and the point p. In the case of uniform continuity, δ only depends on ϵ, and must do so for all points $p \in X$.

Clearly every uniformly continuous function is continuous.

Theorem 4.11. *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X*

Proof. Let $\epsilon > 0$. Since f is continuous, for each point $p \in X$ there is a positive number $\phi(p)$ such that:

$$q \in X, d\left(p, q\right) < \phi(p) \implies d\left(f(p), f(q)\right) < \frac{\epsilon}{2}$$

Let $J(p)$ be the set of all $q \in X$ for which $d\left(p, q\right) < \frac{1}{2}\phi(p)$. For each $p, J(p)$ contains this p, so the set of all $J(p)$ is an open cover of X. Since X is compact, we can find a finite set of points $p_1, ..., p_n \in X$ such that: $X \subset J(p_1) \cup ... \cup J(p_n)$. Let $\delta = \frac{1}{2}\min\left[\phi(p_1), ...\phi(p_n)\right]$ Because there are finite p_n we know that $\delta > 0$.

Let $q, p \in X$ be points such that: $d\left(p, q\right) < \delta$. There is some integer m such that $p \in J(p_m)$ so that: $d\left(p, p_m\right) < \frac{1}{2}\phi(p_m)$. This tells us:

$$d\left(q, p_m\right) \leq d\left(p, q\right) + d\left(p, p_m\right) < \delta + \frac{1}{2}\phi(p_m) \leq \phi(p_m)$$

as well as:

$$d\left(f(p), f(q)\right) \leq d\left(f(p), f(p_m)\right) + d\left(f(q), f(p_m)\right) < \epsilon$$

□

Theorem 4.12. *Let E be a non-compact set in $\mathbb{R}$. Then:*

1. *There exists a continuous function on E which is not bounded.*
2. *There exists a continuous and bounded function on E which has no maximum.*
3. *If E is bounded, then there exists a continuous function on E which is not uniformly continuous.*

Proof.

Consider: $f(x) = x$ for the set: $E = \mathbb{R}$.

Let: $h(x) = \frac{x^2}{1+x^2}$ Note that: $\sup_{x \in E} h(x) = 1$ and: $h(x) < 1 \quad \forall x \in E$.

Consider: $E = \mathbb{R}$

$\{x_0\}$. Let $f(x) = \frac{1}{x - x_0}$ This function is continuous on E but unbounded. Note that it is not uniformly continuous as by getting closer to x_0 for some δ neighborhood you can always make the difference greater than ϵ. □

4.4 Continuity and Connectedness

Theorem 4.13. *If f is a continuous mapping of a metric space X into a metric space Y, and if E is a connected subset of X, then $f(E)$ is connected.*

Proof. Assume that $f(E) = A \cup B$ where A, B are nonempty separated subsets of Y. Let $G = E \cap f^{-1}(A), H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, and neither G nor H is empty.

Note: $G \subset f^{-1}(\bar{A})$, since f is continuous: G is closed. Therefore: $\bar{G} \subset f^{-1}(\bar{A})$ and $f(\bar{G}) \subset \bar{A}$. Since $f(H) = B$ and $\bar{A} \cap B$ is empty as the sets are separated, $\bar{G} \cap H$ is empty. This same argument shows that $G \cap \bar{H}$ is empty. This means that G and H are separated. This contradicts E being connected. □

Theorem 4.14. *Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that: $f(a) < c < f(b)$ then there exists a point $x \in (a, b)$ such that: $f(x) = c$.*

Proof. From theorem 2.24, $[a,b]$ is connected. Thus by theorem 4.13, $f([a,b])$ is a connected subset of $\mathbb{R}$. Thus theorem 2.24 implies the result. $\square$

4.5 Discontinuities

If x is a point in the domain of the function f, where f is not continuous, we say f is *discontinuous* at x. This is equivalent to saying that f has a *discontinuity* at x.

If f is defined on an interval or on a segment, it is customary to divide discontinuities into two types based on the right and left hand limits of f at x.

Consider the interval (a,b) and the function f defined on this interval. Let x be such that: $a \leq x < b$. We write: $f(x+) = q$ if $f(t_n) \to q$ as $n \to \infty$ for all sequences $\{t_n\}$ in (x,b) such that: $t_n \to x$. For $f(x-)$ we use the same result for $a < x \leq b$.

Clearly we can see that at any point $x \in (a,b), \lim\limits_{t \to x} f(t)$ exists if and only if $f(x+) = f(x-) = \lim\limits_{t \to \infty} f(t)$.

If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the*first kind*, also called a *simple discontinuity* at x. Otherwise we say that the discontinuity is of the *second kind*.

For example, the Dirichlet function has a discontinuity of the second kind at every point.

4.6 Monotonic Functions

Let f be real on (a,b). Then f is said to be *monotonically increasing* on (a,b) if $a < x < y < b \implies f(x) \leq f(y)$. If the last inequality is reversed, we obtain the definition of a *monotonically decreasing* function. The class of monotonic functions consists of both the increasing and the decreasing functions.

Theorem 4.15. *Let f be monotonically increasing on (a,b). Then $f(x+)$ and $f(x-)$ exist at every point of x of (a,b). Precisely:*

$$\sup_{a<t<x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x<t<b} f(t)$$

Furthermore, if $a < x < y < b$ then $f(x+) \leq f(y-)$.

Proof. By hypothesis, $f(t)$ where $a < t < x$ is bounded above by $f(x)$. Therefore it has a least upper bound denoted A. □

As a result, monotonic functions have no discontinuities of the second kind.

Theorem 4.16. *Let f be monotonic on (a, b). Then the set of points (a, b) at which f is discontinuous at most countable.*

Proof. Without loss of generality, suppose that f is increasing, and let E be the set of points at which f is discontinuous. □

4.7 Exercises

Question. *1. Suppose f is a real function defined on $\mathbb{R}$ which satisfies:*

$$\lim_{h \to 0} [f(x + h) - f(x - h)] = 0$$

for every $x \in \mathbb{R}$. Does this imply that f is continuous?

Proof. No, consider the function where $f(x) = 0$ if $x \neq 0$ and $f(x) = 1$ if $x = 1$. Clearly this function satisfies the above property. However it is not continuous at zero. To see this consider the sequences: $\{x_n\} = \frac{1}{n}$ and $\{p_n\} = 0$ Both of these converge to zero, but $\lim_{n \to \infty} f(x_n) = 0$ and $\lim_{n \to \infty} f(p_n) = 1$. □

Question. *If f is a continuous mapping of a metric space X into a metric space Y, prove that: $f(\bar{E}) \subset f(\bar{E})$ for every set $E \subset X$. Show by example that this can be a proper subset.*

Proof. Let x be an element of $\bar{E}$. If $x \in E$ then clearly $f(x) \in f(\bar{E})$. Let $x \notin E$. x is a limit point of E. We wish to show that $f(x)$ is a limit point of $f(E)$. Since f is continuous, and any neighborhood of $f(x)$ is an open set, so $f^{-1}(N_r(f(x)))$ is open as well.

Since it is open, it contains a neighborhood as a subset, and that neighborhood contains a point of E, as x is a limit point. This implies that this neighborhood of $f(x)$ contains a point of $f(E)$.

□

Question. *Let f be a continuous real function on a metric space X. Let $Z(f)$ be the set of all $p \in X$ for which $f(p) = 0$. Prove that $Z(f)$ is closed.*

Proof. Note that the set: $\{0\}$ is closed. Since f is continuous, $f^{-1}(\{0\})$ must also be closed. □

Question. *Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that: $g(p) = f(p)$ for all $p \in X$.*

E is dense *in X if every point X is a limit point of E or a point of E (or both).*

Proof. Assume that $f(E)$ is not dense in $f(X)$. Then there is a point of $f(X)$ that is not in $f(E)$ and is not a limit point of that set. Call this point p. There is a neighborhood of p where no point of the neighborhood is a point in $f(E)$. The radius of this neighborhood is r.

Take $\epsilon = r$, and apply the definition of continuity to the set X for any point that maps into p. This gives us a neighborhood $N_\delta \in X$ where each point in that neighborhood maps into the neighborhood of p.

Every point of this neighborhood by assumption must be a limit point of E. If it were a point of E, then p would be a limit point of $f(E)$. □