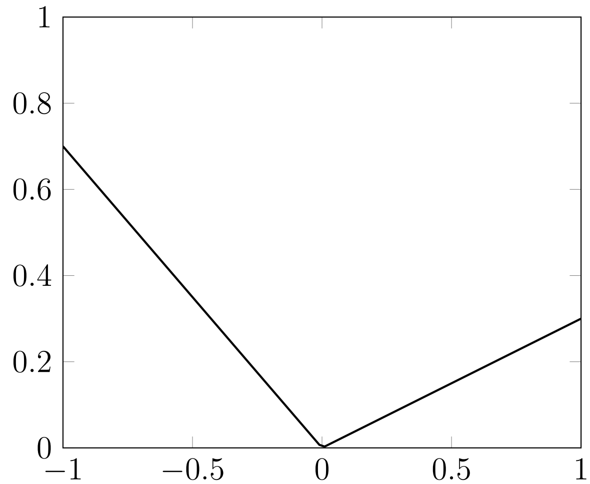
Question 1

Provide the intuition as to why the solution to (1) is the τ^{th} quantile of Y. Think about the shape of ρ_τ, where the loss is greatest and how that depends on τ.

We can think of the check function ρ_τ as a function that weighs over and under-estimation differently. The loss is linear to both the left and right of zero. We can see a plot of the check function for $\tau = .3$ below

Fig. 1: $\rho_\tau, \tau = .3$



If we consider the loss function of

$$\min_q \mathbb{E}\left[\rho_\tau(Y - q)\right]$$

This means that underestimates (where $q < Y$) are "punished" differently from overestimates. An overestimate is punished by τ and an overestimate is punished by $1 - \tau$. This means we want to balance the mass of the distribution by these amounts on either side of q. For example if $\tau = .1$, then we want 90% of the mass above q, and 10% below q. This is exactly the 10^{th} quantile of the distribution of Y.

Question 2

What would be the interpretation of the coefficient estimates from running OLS of Y on $\boldsymbol{X}$? Likewise, what would be the interpretation of the coefficient estimates of a quantile regression (QR) of Y on $\boldsymbol{X}$, as in β_τ? How do the two interpretations differ?

If we believe that our OLS model is correctly specified, then we can interpret β as the change in Y caused by a single unit increase of X locally.

For quantile regression, β_τ can be interpreted as the change in the portion of Y at the τ^{th} quantile when there is a single unit change in β locally. Rather than looking at how β

changes the entire distribution, the QR estimate only examines how changes in β affect the distribution around that particular quantile. This is useful if we believe that our co-variates affect the distribution of Y differently in different places.

One such example would be unions. If we believe that the benefit of unions depends upon the quantiles of the wage (skill) distribution, then we should expect very different estimates for a unionization dummy variable. In this case, our interpretation of the β is the benefit of being unionized for this quantile of wage-earners.

Question 3

Suppose you wished to make a causal interpretation of the regression model. What assumptions are required for OLS? Will those assumptions differ for QR? If so, how?

To make a causal interpretation of OLS, we need to ensure that our covariates are exogenous to the unobserved error or that there is a set of valid instruments, and that the model is specified such that $\frac{\partial Y}{\partial X_j}$ is constant. For technical reasons, we require that there is no perfect colinearity in X as well. Under these assumptions, the reduced-form linear model has a causal interpretation for β. If we wished to examine causality under heterogeneity, such as an average treatment effect, we would also need that it would be impossible to reject treatment and treatment is given randomly.

For the quantile regression model, it is clear that the specification assumption must be made stronger, and that we must say that for the specified quantile, $\frac{\partial Y}{\partial X_j}$ must be constant, which may be difficult to believe. No perfect colinearity will still be required to ensure that the solution for β occurs at a unique point, and not along the face of the hyper-plane. We should not believe that there will be any different requirement for exogeneity or instruments, as long as they are valid for the specified quantile rather than the entire distribution.

One thing that is important to be able to do causal inference in quantile regression is that individuals are not able to "jump" quantiles between treatments. That is, if the individuals in the 20^{th} percentile for the treatment group were very different from the individuals in the 20^{th} percentile for the control group, then we could conclude nothing from a causal perspective.

Question 4

Tab. 1: OLS Output

	Dependent variable:
	birthweight
boy	105.198*** (12.159)
married	22.657 (17.003)
black	−215.115*** (15.834)
age	38.736*** (8.810)
highschool	40.686** (17.715)
somecollege	70.030*** (20.583)
college	19.815 (24.029)
prenone	−158.701*** (56.808)
presecond	49.998*** (17.920)
prethird	68.862* (38.143)
smoker	200.895*** (32.739)
cigsdaily	−4.351** (2.122)
weightgain	19.829*** (1.004)
age^2	−0.641*** (0.159)
$weightgain^2$	−0.164*** (0.009)
Constant	2,103.312*** (118.914)
Observations	9,800
R^2	0.109
Adjusted R^2	0.108
Residual Std. Error	601.110 (df = 9784)
F Statistic	80.124*** (df = 15; 9784)
Note:	*p<0.1; **p<0.05; ***p<0.01

At the first glance, the positive coefficient on the smoker indicator seems quite puzzling, but it is also interacting with the *cigsdaily* covariate. We are not given enough information about the number of cigarettes smoked daily by the mothers in the sample to determine if this interaction is a net positive or negative, but it would be very strange if it was a net positive.

The age and weight-gain affects are both positive in the linear term, and negative for the quadratic term, indicating that for smaller increases in both it is healthy, but for large increases, it can be unhealthy for the baby. This makes economic sense as we expect young and old mothers to be the ones that are most likely to have unhealthy babies.

The *presecond* and *prethird* indicators show the deviations from having prenatal care in the first trimester, and it is interesting that later prenatal care appears to be better for the health of the baby. Selection bias is not controlled for here however, and it is likely that those that are more at risk or health problems for their baby would seek earlier care.

Question 5

Tab. 2: Quantile Regression Coefficients

	Dependent variable:		
	birthweight		
	$\tau = .15$	$\tau = .3$	$\tau = .45$
	(1)	(2)	(3)
Constant	1,354.523*** (151.264)	2,099.697*** (145.188)	2,237.932*** (131.663)
boy	82.553*** (16.405)	94.133*** (13.768)	109.468*** (12.671)
married	31.482 (22.702)	33.702* (19.021)	7.845 (17.857)
black	−226.487*** (23.651)	−197.737*** (18.911)	−194.927*** (16.678)
age	53.608*** (10.835)	24.179** (11.112)	27.083*** (9.979)
highschool	41.458* (23.699)	56.252*** (19.778)	45.110** (19.737)
somecollege	101.635*** (30.179)	85.615*** (23.218)	85.765*** (22.527)
college	42.021 (33.090)	55.814** (27.771)	60.843** (26.691)
prenone	−332.879** (137.444)	−50.595 (52.144)	−21.510 (57.304)
presecond	46.023** (19.211)	16.819 (19.866)	30.375 (20.538)
prethird	97.283*** (36.520)	25.659 (44.377)	28.664 (34.109)
smoker	226.907*** (50.526)	222.487*** (36.030)	243.871*** (36.020)
cigsdaily	−1.586 (2.884)	−1.953 (2.409)	−1.134 (2.271)
weightgain	25.731*** (1.731)	18.188*** (1.270)	15.912*** (1.106)
age^2	−1.034*** (0.192)	−0.412** (0.207)	−0.398** (0.183)
$weightgain^2$	−0.226*** (0.018)	−0.149*** (0.012)	−0.126*** (0.011)
Observations	9,800	9,800	9,800
Note:			*p<0.1; **p<0.05; ***p<0.01

Tab. 3: Quantile Regression Coefficients

	Dependent variable:		
	birthweight		
	$\tau = .6$	$\tau = .75$	$\tau = .9$
	(1)	(2)	(3)
Constant	2,482.852*** (115.623)	2,648.460*** (123.414)	2,583.119*** (188.183)
boy	119.489*** (11.441)	120.179*** (12.536)	141.591*** (18.841)
married	4.259 (16.003)	1.017 (17.154)	37.376 (26.310)
black	−204.335*** (15.218)	−196.371*** (17.064)	−182.540*** (24.637)
age	24.696*** (8.637)	33.443*** (8.971)	59.731*** (14.241)
highschool	24.110 (17.248)	19.919 (19.519)	−23.160 (25.882)
somecollege	60.361*** (19.706)	42.008* (22.830)	−6.022 (33.196)
college	26.775 (23.477)	17.457 (24.445)	−96.917*** (37.271)
prenone	−51.263 (90.063)	−95.344*** (19.259)	−107.646 (162.621)
presecond	34.337** (17.384)	50.466*** (17.899)	56.693 (34.814)
prethird	24.499 (31.765)	15.043 (43.655)	−18.916 (46.658)
smoker	242.045*** (29.348)	164.092*** (31.585)	151.351** (59.975)
cigsdaily	−0.703 (1.633)	−5.533*** (1.337)	−9.181*** (3.550)
weightgain	14.595*** (1.008)	13.257*** (1.155)	12.879*** (1.435)
age^2	−0.339** (0.156)	−0.471*** (0.160)	−0.875*** (0.266)
$weightgain^2$	−0.113*** (0.010)	−0.096*** (0.011)	−0.091*** (0.013)
Observations	9,800	9,800	9,800
Note:			*p<0.1; **p<0.05; ***p<0.01

We see that the patterns for age and birth-weight appear in the quantile regression, with age and birth-weight having a much larger affect on the extremes of the quantile than nearer to the median of the distribution.

prethird's affect on the weight of the baby seems to be decreasing as we go higher in the quantiles of the distribution, becoming negative as we reach the 90^{th} quantile of the distribution. This seems to support the belief that there is endogeneity in this relationship, as other health concerns unobserved may lead someone to receive late prenatal care, so it is more beneficial to the smaller babies.

Interesting, the education factors which are noisy but all positive for the lower levels of τ tend downwards for the upper quantiles of the distribution.

Question 6

The biggest difference between the two estimates is the behavior of age and weight-gain for the different quantiles of the distribution, as well as the declining value of education in τ. We cannot find these sorts of patterns through OLS regression, but the fact that in larger babies, education decreases the birth weight is not intuitive.

For many of the coefficients, OLS and the quantile regressions are nearly similar. The indicator for *black* has nearly the same effect throughout the entire distribution, and as such, the OLS estimate is nearly the same as the quantile regression estimates. Most of the variation between the two can be accounted by the fact that OLS is much more easily corruptible by outliers and is much more sensitive to them than quantile regression is.

Quantile regression is able to summarize the relationship better because it is one step closer to non-parametric estimation. However, it comes at the cost of being much harder to make causal inference due to the nature of the linear conditional of each of the quantiles. We gain predictive power but this comes at the cost of a causal interpretation.

Question 7

Let $\hat{\epsilon}_i$ denote the residuals from your OLS regression in question 4. What is $\sum_{i=1}^{n} \hat{\epsilon}_i$? Also, how many of the residuals are 0? Provide an explanation for your findings.

The sum of the residuals is zero, and this is one of the first-order conditions for OLS. The condition used is that: $\mathbb{E}\left[XU\right] = 0$, for which the sample analog is: $\frac{1}{n}\sum_{i=1}^{n} X_i\hat{\epsilon}_i = 0$ and since X contains a column of only ones, this implies that $\sum_{i=1}^{n} \hat{\epsilon}_i = 0$

None of the residuals are zero, but it is possible in OLS for there to be residuals that are exactly zero. But nothing in the estimate requires that a single one be equal to zero.

Question 8

Re-run your QR from question 5 for a τ of your choice. Let $\tilde{\epsilon}_i$ denote the residuals. What is $\sum_{i=1}^{n} \tilde{\epsilon}_i$? Also, how many of the residuals are (approximately) 0?

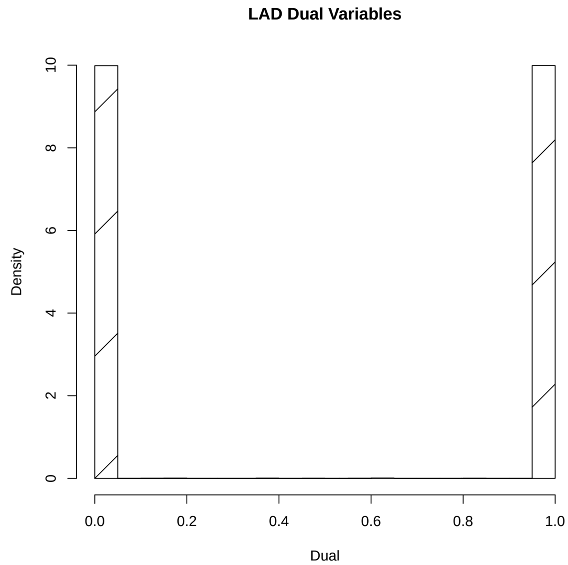
When we sum the residuals of the Least Absolute Deviations estimator ($\tau = .5$) we find that the sum of the residuals is equal to -481228.6. However, when we calculate the number of residuals that are equal to zero, we find that there are sixteen measurements with zero residual value. This does not depend on the value of τ, and for all tested values of τ I get 16 measurements which have zero residual. This is exactly the number of covariates in the model.

This is different from the results from question 7 as we find that the fit (expressed as the sum of the residuals) is worse, but that there are closer fits for certain data points. Both sets of residuals have nearly the same standard deviations, it is just that the LAD estimate is biased.

Question 9

Access the results of the dual problem. How many of these values are strictly between 0 and 1?

We find that almost all of the values are between 0 and 1, and in fact all except for 16 of the measurements are equal to 0 or 1. Each of these 16 measurements are between zero and one.



This makes sense, as we can think of the dual values as the cost to the objective function for relaxing the constraint to which they represent. The constraints for the quantile regression primal are that $Y_i - X\beta + u_i - v_i = 0$. If this constraint is relaxed say by increasing Y_i, the objective function benefits if $u_i > 0$, and does not benefit if $v_i > 0$. We can see this because if $u_i > 0$, then $Y_i - X\beta < 0$. Increasing Y_i improves the fit, and allows for a smaller value of u_i to meet the constraint. This gives a benefit of τ. However if $v_i > 0$, then $Y_i - X\beta > 0$ and we must increase v_i in order to maintain the equality of the constraint. When this occurs, the dual variable corresponding to the constraint will be 1, and when $u_i > 0$, we will see that the dual variable is 0. The constraints for which $u_i = v_i = 0$ will have dual variables between 0 and 1.

Question 10

Given your findings from question 8, can you think of a way to recover β_τ for some τ using only h observations from the data set, where h is defined as in question 8.

If we know the 16 points for which there is no residual value, and there are 16 dimensions to β, then there is a perfectly identified system for β. Define X_h to be the matrix of covariates containing only the 16 rows that are in h. Likewise define Y_h.

We know that $Y_h = X_h\beta$ as all of these points have zero residuals. But X_h is a 16×16 matrix by construction, and by assumption contains no perfect colinearity so we may invert this matrix. Then $\beta = {X_h}^{-1}Y$.