Optimization Conscious Econometrics PS3.3

Timothy Schwieg

1 1.3

We may write the function f as a piecewise-linear convex problem.

$$f(x) = \max\{-(x-1), 0, 2(x-2)\}$$

$$\begin{aligned} \min_{x,z} \quad & c'\boldsymbol{x} + z \\ \text{s.t.} \quad & z \geq -(d'\boldsymbol{x} - 1) \\ & z \geq 0 \\ & z \geq 2(d'\boldsymbol{x} - 2) \\ & A\boldsymbol{x} \geq b \end{aligned}$$

Putting this in standard form, by adding slack variables $\boldsymbol{s}, e_1, e_2$, and defining $x = x^+ - x^-$

$$\begin{aligned} \min_{x^+, x^-, s, e_1, e_2, z} \quad & c'(x^+ - x^-) + z \\ \text{s.t.} \quad & z + d'(x^+ - x^-) - e_1 = 1 \\ & z - 2d'(x^+ - x^-) - e_2 = -4 \\ & A(x^+ - x^-) - \boldsymbol{s} = b \\ & x^+, x^-, z, e_1, e_2, \boldsymbol{s} \geq 0 \end{aligned}$$

The matrix form of this standard form is omitted to save myself the hell of writing out this matrix in $\LaTeX$. But can easily be done.

2 1.4

Consider the problem

$$\begin{aligned} \min_{x_1, x_2} \quad & 2x_1 + 3\left|x_2 - 10\right| \\ \text{s.t.} \quad & \left|x_1 + 2\right| + \left|x_2\right| \leq 5 \end{aligned}$$

reformulate it as a linear program.

We make the following substitutions:

$$
\begin{aligned}
x_2 &= s_2^+ - s_2^- \\
x_1 + 2 &= s_1^+ - s_1^-
\end{aligned}
$$

We define a new variable $z_1 \geq 0$ such that $z_1 \geq \max\{x_2 - 10, 10 - x_2\} = |x_2 - 10|$.

$$
\begin{aligned}
\min \quad & 2(s_1^+ - s_1^- - 2) + 3z_1 \\
\text{s.t.} \quad & s_1^+ + s_1^- + s_2^+ + s_2^- \leq 5 \\
& z_1 \geq s_2^+ - s_2^- - 10 \\
& z_1 \geq 10 - s_2^+ + s_2^- \\
& z_1, s_2^+, s_2^-, s_1^+, s_1^- \geq 0
\end{aligned}
$$

Rewriting the expression in standard form by adding slack variables e_1, e_2, e_3

$$
\begin{aligned}
\min \quad & 2s_1^+ - 2s_1^- + 3z_1 \\
\text{s.t.} \quad & s_1^+ + s_1^- + s_2^+ + s_2^- + e_1 = 5 \\
& z_1 - s_2^+ + s_2^- - e_2 = 10 \\
& z_1 + s_2^+ - s_2^- - e_3 = 10 \\
& z_1, s_2^+, s_2^-, s_1^+, s_1^-, e_1, e_2, e_3 \geq 0
\end{aligned}
$$

3 1.12

It is clear we want to choose the radius as well as the center of the ball.

That is we want to the center of the ball plus any vector with length less than r to be inside the polyhedron. Being inside the polyhedron is equivalent to satisfying the constraint for all i.

$$a_i'(y + c) \leq b_i \quad \|c\| \leq r$$

We also know that $a_i'c \leq r\|a_i\|$

$$
\begin{aligned}
\max_{y,r} \quad & r \\
\text{s.t.} \quad & a_i'y + r\|a_i\| \leq b_i
\end{aligned}
$$

To rewrite this in its standard form:

$$
\begin{aligned}
\max_{y^+, y^-, r, e} \quad & r \\
\text{s.t.} \quad & a_i'(y^+ - y^-) + r\|a_i\| + e_i = b_i \\
& y^+, y^-, r, e_i \geq 0
\end{aligned}
$$

4 1.17

He wishes to maximize his expected portfolio value next year, which is given by $\sum_{i=1}^{n} r_i(s_i - x_i)$ where x_i is the amount of share i that he elects to sell.

He is required to raise K capital in the current period, and when he sells his stock at the current price, he pays 30% in taxes, and 1% in transaction fees, leaving him with 69% leftover.

His constraint then becomes: $.69 \times \sum_{i=1}^{n} q_i x_i = K$ He is also constrained by not being able to sell more shares than he currently owns. This is assuming that there is no derivative market, and no ability to short.

His problem then can be written as:

$$\begin{aligned} \max \sum_{i=1}^{n} r_i(s_i - x_i) & \\ \text{s.t. } x_i \leq s_i & \\ .69 \sum_{i=1}^{n} q_i x_i = K & \\ x_i \geq 0 & \end{aligned}$$

5 1.20

1. Let $S = \{Ax | x \in \mathbb{R}^n\}$, show that S is a subspace of $\mathbb{R}^l$

   We know that $A0 = 0$, so it is clear that $0 \in S$.

   Let $y_1, y_2 \in S$, Note that there is at least one x_1, x_2 such that $y_1 = Ax_1$ and $y_2 = Ax_2$. Therefore we can write $y_1 + y_2$ as $Ax_1 + Ax_2 = A(x_1 + x_2)$ and S is therefore closed under addition.

   Let $y \in S$. There exists x such that $y = Ax$. Therefore $cy = cAx = A(cx) \in S$. So we know that S is closed under scalar multiplication. These three properties verify that S is a subspace.

2. Assume that S is a proper subspace of $\mathbb{R}^n$. Show that there exists a matrix B such that $S = \{y \in \mathbb{R}^n | By = 0\}$.

   Since S is a proper subspace of $\mathbb{R}^n$, then we know that the dimension of S is less than n. Therefore the dimension of the orthogonal complement is at least one. Consider any basis $b_1, ..., b_k$ of this space. By definition, for any vector $y \in S$, $\sum_{i=1}^{k} \lambda_i b_i' y = 0$ as each vector b is orthogonal to y. Therefore we may construct a matrix using the vectors b_i as the columns of B. This matrix has the property that $By = 0$, which is exactly what we are looking for.

   It remains to be shown that all vectors such that $Bx = 0$ live in S. It is obvious that $0 \in S$, so consider $x \neq 0$. By definition, any vector x lives in S or its orthogonal complement. If x is in the orthogonal complement of S, then $Bx \neq 0$ as it can be written as a non-trivial linear combination of the basis vectors b_i. So it must be that if $Bx = 0$, $x \in S$.

3. Suppose that V is a m-dimensional affline subspace of $\mathbb{R}^n$ with $m < n$. Show that there exist linearly independent vectors $a_1, ..., a_{n-m}$ and scalars $b_1, ..., b_{n-m}$ such that:

$$V = \{y | a_i' y = b_i, i = 1, 2, ..., n - m\}$$

Abusing notation, $V - b$ will denote the elements of V minus b. Since V is an affline subspace, $V - b$ is a proper subspace. Therefore there exists a matrix B such that $V - b = \{y | Ay = 0\}$ This can then be written as: $V = \{y + b | Ay = 0\}$, or alternatively as: $V = \{y | Ay = b\}$ This is exactly equivalent to saying that $V = \{y | a_i' y = b_i, i = 1, 2, ...n - m\}$ as the dimension of the orthogonal complement will be $n - m$, and since A is composed of basis vectors, they are linearly independent.

6 2.3

Consider a polyhedron defined by the constraints $A\boldsymbol{x} = \boldsymbol{b}$ and $0 \leq \boldsymbol{x} \leq u$ and assume that the matrix A has linearly independent rows. Provide a procedure analogous to the one in section 2.3 for constructing basic solutions and prove an analog of theorem 2.4

Consider the vector $s = u - \boldsymbol{x}$. The existing polyhedron is equivalent to doubling the state space, and adding the constraints that $s_i + x_i = u$. Define

$$A_{x,s} = \begin{pmatrix} A & 0 \\ I & I \end{pmatrix} \quad b_{x,s} = \begin{pmatrix} b \\ u \end{pmatrix}$$

Now we have a new polyhedron defined by $A_{x,s}(x, s)' = b_{x,s}$ and $0 \leq (x, s)'$. This now fits the standard form, and we may follow the procedure used in section 2.3. Note that the dimensions of $A_{x,s}$ are $m + n \times 2n$.

We can choose $m + n$ linearly independent columns of $A_{x,s}$ and let $(x, s)_i' = 0$ for all of the columns not chosen, and simply invert this system to obtain the values of x and s. Note that since we have taken $m + n$ columns, it is impossible for both x_i and s_i to equal zero, allowing this procedure to make sense.

The equality constraints holding implies that $Ax = b$ and n linearly independent active inequality constraints implies that for n values of x_i, $x_i = 0$ or $x_i = u$.

In the system with x, s as the state variables, the system is in standard form, and therefore theorem 2.4 applies directly. Choose $B_1, ...B_{m+n}$ linearly independent columns of $A_{x,s}$.

If we do not select the i^{th} column, we must select the $n + i$ column in order to maintain linear independence, but there will be times when both are selected. It is still true that when $i \notin B_1, ...B_{m+n}$ that $x_i = 0$, it is also true that when $i + n \notin B_1, ..., B_{m+n}$ then $x_i = u$. The columns of $A_{x,s}^{B_1,...,B_m}$ remain linearly independent, but this will not be true for A. Consider the example:

$$
\begin{aligned}
x_1 + x_3 &= 1 \\
x_2 + x_3 &= 1 \\
0 \leq x &\leq u
\end{aligned}
$$

When we move to the slack-variable state, a valid basic solution, by selecting the first five columns according to the algorithm reveals:

$$x_1 = 1 - u \quad x_2 = 1 - u \quad x_3 = u \quad s_1 = 2u - 1 \quad s_2 = 2u - 1 \quad s_3 = 0$$

One can verify that taking only the values for x, this is a basic solution for the original system as well.

The columns of A that correspond to this basic solution are all of the columns of A, which are clearly not linearly independent, as it includes all three of the columns.

7 2.5

A mapping f is called affline if it is of the form $f(x) = Ax + b$. Two polyhedra P, Q are isomorphic if there exists affline mappings $f : P \to Q$, $g : Q \to P$ such that: $f(g(x)) = x$ and $g(f(y)) = y$.

If P and Q are isomorphic, show that there exists a one-to-one correspondence between their extreme points. i.e. Show that x is an extreme point of P if and only if $f(x)$ is an extreme point of Q.

Let x be an extreme point of P. Assume that $f(x)$ is not an extreme point of Q. Then there exists points $y, z \in Q$ and $\lambda \in (0, 1)$ such that $f(x) = \lambda y + (1 - \lambda)z$. Applying g to both sides of the equation,

$$x = g(\lambda y + (1 - \lambda)z) = \lambda g(y) + (1 - \lambda)g(z)$$

using the affline property of g. However, this shows that x can be written as a convex combination of two points in P. This is a contradiction to x being an extreme point of P,so $f(x)$ must in fact be an extreme point of Q.

Let $f(x)$ be an extreme point of Q. Assume that x is not an extreme point of P. Therefore there exists points $y, z \in P$ and $\lambda \in (0, 1)$ such that $x = \lambda y + (1 - \lambda)z$. Applying f and using its affline properties:

$$f(x) = \lambda f(y) + (1 - \lambda)f(z)$$

This is a contradiction, as we assumed that $f(x)$ was an extreme point. Therefore it must be the case that x is an extreme point of P.

Let

$$P = \{x \in \mathbb{R}^n | Ax \geq b, x \geq 0\}$$
$$Q = \{(x, z) \in \mathbb{R}^{n+k} | Ax - z = b, x \geq 0, z \geq 0\}$$

Show that P and Q are isomorphic.

Define $z = Ax - b$. it is clear that $Ax - z = b$. We can see that $Ax \geq b \Leftrightarrow z \geq 0$. We wish to transform $x \mapsto (x, Ax - b)$ and $(x, z) \mapsto x$

$$f(x) = \begin{pmatrix} I_n \\ A \end{pmatrix} x + \begin{pmatrix} 0 \\ -b \end{pmatrix}$$
$$g(y) = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} y$$

These two affline transforms satisfy our mapping restrictions, now we must show that they satisfy the identity restrictions.

$$g(f(x)) = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} I_n \\ A \end{pmatrix} x + \begin{pmatrix} 0 \\ -b \end{pmatrix} \right] = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ Ax - b \end{pmatrix} = x$$
$$f(g(y)) = \begin{pmatrix} I_n \\ A \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} y + \begin{pmatrix} 0 \\ -b \end{pmatrix} = y$$

This establishes the isomorphism between the two polyhedra.

8 2.6

Let $A_1, ..., A_n$ be vectors in $\mathbb{R}^m$

$$C = \left\{ \sum_{i=1}^{n} \lambda_i A_i \middle| \lambda_1, ..., \lambda_n \geq 0 \right\}$$

Show that any element of C can be written in the form $\sum_{i=1}^{n} \lambda_i A_i$ with $\lambda_i \geq 0$ and at most m of the coefficients being nonzero.

If we let $m \geq n$ this is trivial, so assume that $m < n$. Note that there are at most m linearly independent vectors A. Choose m of these vectors, and note that any vector in the non-negative span of A_i can be written as a non-negative linear combination of these m vectors. Set $\lambda_i = 0$ for all vectors not in this list, and then we have that any element in C, which is the non-negative span of A, can be written as a linear combination of at most m vectors in A.

Let P be the convex hull of the set of vectors A. Show that any element of P can be expressed as $\sum_{i=1}^{n} \lambda_i A_i$ where $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ with at most $m + 1$ of $\lambda_i \neq 0$.

Assuming that $m < n$, there must be at least one linearly dependent vector in A. Without loss of generality, reorder the set A such that the dimension of the set $A_1, ..., A_n$ is the same as the set $A_2, ..., A_n$. Now consider the set $A_2 - A_1, ...A_n - A_1$. This set has the same dimension as $A_1, ..., A_n$ by construction.

Consider any point $x \in P$. We may write that $x = \sum_{i=1}^n \lambda_i A_i = A_1 + \sum_{i=1}^n \lambda_i (A_i - A_1)$. However, since $A_2, ..., A_n$ has dimension of at most m, we may rewrite this expression as:

$$x = A_1 + \sum_{i=2}^{n} \lambda'_i (A_i - A_1)$$

where $\sum_{i=2}^n \lambda'_i \leq 1$ and at most m of λ'_i are non-zero. Simplifying this expression we arrive at:

$$x = (1 - \sum_{i=2}^{n} \lambda'_i) A_1 + \sum_{i=2}^{n} \lambda'_i A_i = \sum_{i=1}^{n} \lambda'_i A_i$$

Where $\lambda'_1 = 1 - \sum_{i=2}^n \lambda'_i$ and at most $m+1$ of the λ'_i are non-zero, as well as $\sum_{i=1}^n \lambda'_i = 1$.

9 2.8

Consider the standard polyhedron $\{x | Ax = b, x \geq 0\}$ and assume that the rows of A are linearly independent. Let x be a basic solution, and let $J = \{i, x_i \neq 0\}$. Show that a basis is associated with basic solution x if and only if every column A_i, $i \in J$ is in the basis.

We say that a basis h is associated with a basic solution x if we can write $x = {A_h}^{-1} b$ where A_h is a matrix with the basis vectors as its columns.

Assume that there is a basis which contains every column of $A_i, i \in J$. This means that each of these columns A_i are linearly independent, and therefore the matrix A_J formed by the columns is invertible. From this we can obtain: $x_j = {A_J}^{-1} b$ and then set $x_i = 0$ for $i \notin J$. This fits all the conditions required by theorem 2.4 to show that x which is the combination of $x_i = {A_J}^{-1} b$ for $i \in J$ and $x_i = 0$ for $i \notin J$ is a basic solution.

For the converse, assume that x is a basic solution for A. By theorem 2.4 there are m columns of A that are linearly independent, and that $x_i = 0$ for all i not in these m. Consider the m dimensional subspace spanned by these columns. Since there is a basis associated with this solution, there are m vectors in that form this basis.

Assume that there is a column A_i for which $x_i \neq 0$ but is not contained in the basis. Then there are columns for which $x = {A_h}^{-1} b$ for the m non-zero components of x. However, this requires that A_h be a full rank $m \times m$ matrix, and therefore have linearly independent columns. This is m columns that are linearly independent, and therefore form a basis. Therefore there is a basis associated with x that contains all the columns A_i for $i \in J$.

10 2.10

Consider the standard polyhedron $P = \{x | Ax = b, x \geq 0\}$ and assume that the rows of A are linearly independent. Answer true or false, providing a proof or counter-example.

1. If $n = m + 1$ then P has at most two basic feasible solutions.

   We can count the number of possible basic solutions of a polyhedron by setting m components equal to zero. The number of basic solutions for a polyhedron is given by $\binom{n}{m}$. Substituting $n = m + 1$ we get that number of solutions is: $\frac{(m+1)!}{m!} = m + 1$. If we consider a polyhedron where all basic solutions are feasible (it is in standard form with all basic points non-negative), then this will clearly not be equal to 2.

2. The set of all optimal solutions is bounded.

   False. If the optimal cost is infinite, then there is no optimal solution. If it is finite, then there is an optimal solution. However, for this to occur at an extreme point, we require that there is an extreme point of P. If it has no extreme points, the set of optimal solutions need not be bounded. Consider the problem: $\min_{x_1,x_2} x_1$ subject to: $x_1 = 1$. The set of all solutions is given by: $(1, x)$ and is unbounded.

3. At every optimal solution, no more than m variables can be positive.

   False, there can be optimal solutions that are not extreme points (the polyhedron does not have any extreme points.) In this case, there are no restrictions placed on the number of positive terms at the optimal.

4. If there is more than one optimal solution, then there are uncountably many optimal solution.

   True. If there is more than one solution, it occurs along an edge of the polyhedron, which can be bijected to an interval. Since the cardinality of an interval is uncountably infinite, the same must be true of the optimal solution set.

5. If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.

   True. Basic feasible solutions are vertices of the polyhedron, and if there are several optimal solutions, the solutions occur on an edge of the polyhedron, which is a convex combination of the vertices. This means that the vertices will also be optimal. These are two basic-feasible solutions.

6. Consider the problem of minimize $\max\{c'x, d'x\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

   False. Consider the question of $\max\{x, -x\}$ subject to $x \in [-1, 1]$. The minimum of this is given at $x = 0$, but the extreme points are $x = -1, 1$

11 2.13a

Consider the standard polyhedron $P = \{x|Ax = b, x \geq 0\}$ and assume that the rows of A are linearly independent. Let A be $m \times n$ and that all feasible solutions are non-degenerate. Let x be an element of P that has exactly m positive components. Show that x is a basic feasible solution.

Since we know that $x \in P$ it is sufficient to show that x is a basic solution. Define B to be the matrix formed by the columns of A that correspond to the non-zero elements of x.

All of the equality constraints of P are satisfied by any point of P trivially, and if there are only m positive components, it must be that there are $n-m$ components that are zero. There are m active constraints from the matrix equality, and $n-m$ active constraints from the positive inequality, therefore there are n active constraints, and all that remains is to show that these are linearly independent. There is no degeneracy on any of the basic feasible solutions, and we know that the rows are linearly independent, and must be independent of the binding constraints on $x \geq 0$. Therefore all the rows are linearly independent, so the determinant of the system is non-zero, and therefore the columns are linearly independent. This satisfies the conditions for a basic solution, and since it is feasible, it must be a basic feasible solution.

12 2.14

Let P be a bounded polyhedron in $\mathbb{R}^n$, let a be a vector in $\mathbb{R}^n$ and let b be some scalar. Let $Q = \{x \in P | a'x = b\}$. Show that every extreme point of Q is either an extreme point of P or a convex combination of two adjacent extreme points of P.

Q is a hyper-plane intersecting with P. There are three possible geometric shapes of this object. First is that the set is empty as P does not overlap, then there are no extreme points and this is true vacuously.

Second the intersection contains only one point. This point is therefore an extreme point of Q. Call this point x_0, we wish to show that this point is a vertex of P. Since Q contains only x_0, it must be the case that $a'x > b$ or $a'x < b$ for all $x \in P \neq x_0$. Using either a or $-a$, we have that $c'x_0 < c'x$ for all $x \in P, x \neq x_0$. This implies that x_0 is a vertex of P, and therefore an extreme point.

Consider an extreme point of Q. This point lies in P, a bounded polyhedron, so it can be written as a convex combination of extreme points of P. However it cannot be written as a convex combination of any points of Q.

Since Q is the intersection of a hyper-place and a polyhedron, its extreme points must occur ether on the vertex or on the edge of the polyhedron. If it were in the interior of P, it could be written as a convex combination of points in Q, and therefore could not be an extreme point. If it were on a face, but not an edge of P, then it could still be written as a convex combination of other points along that face in Q, so it must occur at the edges or vertices. However, we define edges as convex combinations of adjacent vertices, and therefore the extreme points of Q lie either on the extreme points of P, or as a convex combination of two adjacent extreme points of P.

13 2.15

Consider the polyhedron $P = \{x \in \mathbb{R}^n | a_i'x \geq b_i\}$ Suppose that u, v are distinct feasible basic solutions that satisfy $a_i'u = a_i'v = b_i$ for $i = 1, ..., n-1$, and that $a_i, ...a_{n-1}$ are linearly independent. Let $L = \{\lambda u + (1-\lambda)v | \lambda \in [0,1]\}$ Let $Q = \{z \in P | a_i z = b_i, i = 1, ..., n-1\}$. Prove that $L = Q$.

Let x be a convex combination of u and v, $x = \lambda u + (1-\lambda)v$, For any $i = 1, ..., n-1$, we know that $a_i' x = \lambda a_i' u + (1-\lambda) a_i' v = b_i$. This tells us that $x \in Q$.

Let x be such that it cannot be written as a convex combination of u and v. Either $\lambda \notin [0,1]$ or it does not lie on the line connecting u and v. We wish to show that $x \notin Q$.

Case $\lambda \notin [0,1]$: Then $x \notin P$ and therefore $x \notin Q$.

Case x is not on the line connecting u and v. Then it can be written as a point on the line plus a term orthogonal to this line. That is, $x = \lambda u + (1-\lambda)v + y$ where $y'(u-v) = 0$

$$a_i' x = \lambda a_i' u + (1-\lambda) a_i' v + a_i' y = b_i + a_i' y$$

Clearly, x is in Q if $a_i' y = 0 \forall i = 1, ..., n-1$. That is, y is orthogonal to A. We know that $y'u = y'v$, so y can be written as a linear combination of a_i. It is therefore impossible for $a_i' y = 0$ for all a_i as y is an element of their span. This implies that $x \notin Q$, and completes the proof.

14 2.17

Consider the polyhedron $\{x \in \mathbb{R}^n | Ax \le b \quad x \ge 0\}$ and a non-degenerate feasible basic solution x^*. Introduce slack variables and construct a corresponding polyhedron: $\{(x, z) | Ax + z = b, x \ge 0, z \ge 0\}$ in standard form. Show that $(x^*, b - Ax^*)$ is a non-degenerate basic feasible solution for the new polyhedron.

From exercise 2.5b, we know that polyhedron of this form are isomorphic. (Apply -A and -b to change the signs). Since these two polyhedra are isomorphic, from exercise 2.5a, we know that if x^* is an extreme point of the first, it must be an extreme point of the second, following the isomorphism described in exercise 2.5b.

Note that x^* being a feasible basic solution implies that it is an extreme point of the first polyhedron, so therefore $(x^*, b - Ax^*)$ must be an extreme point of the second polyhedron, and therefore a feasible basic solution for the new polyhedron.

15 2.19ab

Let $P \subset \mathbb{R}^n$ be a polyhedron in standard form whose definition involves m linearly independent equality constraints. Its dimension is defined as the smallest integer k such that P is contained in some k-dimensional affine subspace of $\mathbb{R}^n$

1. Explain why the dimension of P is at most $n - m$.

   Consider the polyhedron P' defined by only the equality constraints of P. The equality constraints of the polyhedron can be written in the form $Ax = b$. Where A is an $m \times n$ matrix of rank m. Consider the affine space $V = P' - b$. Any element of V can then be expressed by a vector x such that $Ax = 0$. This means that all the vectors contained in V are in the Null Space of A. The dimension of the column space of A plus the dimension of the Null space equals n. Therefore the dimension of P' is $n - m$.

The intersection of the inequality constraints and the Null space of A can only be a subset of the Null space of A, and therefore the dimension can only be decreased. That is $dimP \leq dimP'$. This establishes an upper bound on the dimension of $dimP$.

2. Suppose that P has a non-degenerate basic feasible solution. Show that the dimension of P is equal to $n - m$

   Having a non-degenerate basic feasible solution means that there is a basic solution that meets all constraints, with all equality constraints active, and has n linearly independent active constraints. Since it is non-degenerate, there are exactly $n - m$ elements of x that are equal to zero. Therefore there are m elements that are non-zero, and in the Null Space of the affline subspace $P - b$. This tells us that the Null Space of affline subspace $P - b$ has dimension of m. Therefore the dimension of $P - b$ must be equal to $n - m$.