**Problem Set #5 (Due November 14, 2018)**

1. (Constrained Least Squares and a Lagrange Multiplier Test) Let $(Y, X, U)$ satisfy

$$Y = X'\beta + U \ ,$$

where $Y$ and $U$ take values in $\mathbf{R}$, $X = (1, X_1, \ldots, X_k)'$ takes values in $\mathbf{R}^{k+1}$, $\beta = (\beta_0, \ldots, \beta_k)'$. Suppose $E[XU] = 0$, $E[XX'] < \infty$, $\text{Var}[XU]$ is non-singular, and there is no perfect colinearity in $X$. Suppose further that $R\beta = c$, where $R$ is a $p \times (k+1)$ matrix such that the rows of $R$ are linearly independent. Let $(Y_1, X_1), \ldots, (Y_n, X_n)$ be an i.i.d. sample from the distribution of $(Y, X)$. Define the *constrained least squares* (CLS) estimator of $\beta$, $\tilde{\beta}_n$, as the solution to

$$\min_{b \in \mathbf{R}^{k+1} : Rb = c} \frac{1}{n} \sum_{1 \leq i \leq n} (Y_i - X_i' b)^2 \ .$$

(a) Consider the Lagrangian

$$\mathcal{L}(\beta, \lambda) = \frac{1}{2} \frac{1}{n} \sum_{1 \leq i \leq n} (Y_i - X_i' b)^2 + \lambda'(Rb - c) \ .$$

(The $\frac{1}{2}$ out front just makes the algebra work out a bit more nicely.) Compute $\frac{\partial}{\partial \beta} \mathcal{L}(\beta, \lambda)$ and $\frac{\partial}{\partial \lambda} \mathcal{L}(\beta, \lambda)$. Let $\tilde{\beta}_n$ and $\tilde{\lambda}_n$ be such that these two derivatives are equal to zero.

(b) Show that

$$\tilde{\lambda}_n = \left( R \left( \frac{1}{n} \sum_{1 \leq i \leq n} X_i X_i' \right)^{-1} R' \right)^{-1} \left( R\hat{\beta}_n - c \right) \ ,$$

where $\hat{\beta}_n$ is the OLS estimator of $\beta$.

(c) Show that $\tilde{\lambda}_n \xrightarrow{P} 0$ when $R\beta = c$.

(d) Derive the asymptotic distribution of $\sqrt{n}\tilde{\lambda}_n$ when $R\beta = c$.

(e) Use the preceding exercises to suggest a test based on the distance of $\tilde{\lambda}_n$ to zero for $H_0 : R\beta = c$ versus $H_1 : R\beta \neq c$ at level $\alpha$. Show that your test is consistent in level.

(f) How does your test compare with the Wald test studied in class?

2. (Frisch Bounds) Let $(Y, X, U)$ satisfy

$$Y = \beta_0 + \beta_1 X + U \ ,$$

where $Y$, $X$ and $U$ all take values in $\mathbf{R}$. Suppose $E[XU] = E[U] = 0$. Suppose $X$ is unobserved, but $\hat{X} = X + V$ is observed, where $E[V] = E[XV] = E[UV] = 0$.

   (a) Show that

$$\begin{aligned} \mathrm{Var}[\hat{X}] &= \mathrm{Var}[X] + \mathrm{Var}[V] \\ \mathrm{Var}[Y] &= \beta_1^2 \mathrm{Var}[X] + \mathrm{Var}[U] \\ \mathrm{Cov}[\hat{X}, Y] &= \beta_1 \mathrm{Var}[X] \ . \end{aligned}$$

   (b) If $\beta_1 \geq 0$, show that

$$\frac{\mathrm{Cov}[\hat{X}, Y]}{\mathrm{Var}[\hat{X}]} \leq \beta_1 \leq \frac{\mathrm{Var}[Y]}{\mathrm{Cov}[\hat{X}, Y]} \ .$$

   Interpret the upper and lower bounds in terms of coefficients from a regression.

   (c) Derive an analogous result for the case when $\beta_1 \leq 0$.

(Generalizations of this result are developed in Klepper and Leamer (1984).)

3. Let $X$ and $Z$ be a $k+1$-dimensional random vectors. Suppose that the rank of $E[ZX']$ is $k+1$. Show that there is no perfect colinearity in $Z$.

4. (Solving for $\beta$ using a Control Variable) Let $(Y, X, U)$ be such that

$$Y = X'\beta + U \ ,$$

where $Y$ and $U$ take values in $\mathbf{R}$, $X = (1, X_1, \ldots, X_k)'$ takes values in $\mathbf{R}^{k+1}$, $\beta = (\beta_0, \ldots, \beta_k)'$. Suppose some components of $X$ are endogenous, but there is an $\ell+1$-dimensional vector of instruments $Z$. Suppose $E[ZZ']$ and $E[ZX']$ exist, that the rank of $E[ZX']$ is $k+1$, that $E[ZU] = 0$, and that there is no perfect colinearity in $Z$.

   (a) Let $V = X - \mathrm{BLP}(X|Z)$ and define $V'\gamma = \mathrm{BLP}(U|V)$. Rewrite the model above as

$$Y = X'\beta + V'\gamma + \tilde{U} \ ,$$

   where $\tilde{U} = U - V'\gamma$. Show that in this model $X$ and $V$ are exogenous. (The variable $V$ is called a *control variable*.)

(b) Use the results on solving for sub-vectors in linear regression to derive an expression for $\beta$. Show that this expression is equal to the one derived in class.

5. (Solving for $\beta$ using the "Reduced Form") Let $(Y, X, U)$ be such that

$$Y = X'\beta + U \ ,$$

where $Y$ and $U$ take values in $\mathbf{R}$, $X = (1, X_1, \ldots, X_k)'$ takes values in $\mathbf{R}^{k+1}$, $\beta = (\beta_0, \ldots, \beta_k)'$. Suppose some components of $X$ are endogenous, but there is an $\ell+1$-dimensional vector of instruments $Z$. Suppose $E[ZZ']$ and $E[ZX']$ exist, that the rank of $E[ZX']$ is $k+1$, that $E[ZU] = 0$, and that there is no perfect colinearity in $Z$.

(a) Let $Z'\lambda = \text{BLP}(Y|Z)$ and $\Gamma' Z = \text{BLP}(X|Z)$. Write

$$\begin{aligned} Y &= Z'\lambda + \epsilon \\ X &= \Gamma' Z + \eta \ . \end{aligned}$$

Show that $\lambda = \Gamma\beta$ and $\epsilon = \eta'\beta + U$. (These equations are sometimes called the *reduced form* equations for $Y$ and $X$.)

(b) Use the preceding results to derive an expression for $\beta$. Show that this expression is equal to the one derived in class.

6. Consider the following model of the determinants of wages

$$\log(Y) = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + U \ ,$$

where

$$\begin{aligned} Y &= \text{hourly wage} \\ X_1 &= \text{years of education} \\ X_2 &= \text{years of work experience} \ . \end{aligned}$$

Suppose $X_2$ is exogenous, $X_1$ is endogenous, that $Z_1$ is an instrument for $X_1$. Define $X = (1, X_1, X_2)'$ and $Z = (1, Z_1, X_2)'$. Suppose $E[ZZ']$, $E[ZX']$ and $\text{Var}[XU]$ exist, that $\text{Var}[ZU]$ is invertible, and that there is no perfect colinearity in $Z$. Assume further that instrument relevance and exogeneity both hold. Let

$$(Y_1, X_1, Z_1), \ldots, (Y_n, X_n, Z_n)$$

be an i.i.d. sample from the distribution of $(Y, X, Z)$.

(a) Can we assume that $E[U] = 0$ without loss of generality? Explain briefly.

(b) Define exogenous and endogenous. Explain why we might suspect that $E[UX_1] \neq 0$.

(c) Define instrument relevance and instrument exogeneity.

(d) Describe how you would estimate $\beta$.

(e) Suppose you wished to test the null hypothesis that $Z$ is not a relevant instrument versus the alternative that it is a relevant instrument at level $\alpha$.

   i. Formally state the null and alternative hypotheses.

   ii. Describe how you would perform the test.

(f) Suppose $\mathrm{Cov}[X_1, X_2] \neq 0$. Can the researcher use $X_2$ as an instrument for $X_1$? Explain briefly.