Empirical Analysis I - Problem Set 2

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Question 1

a

Because $Var[X_i]$ and $Var[Y_i]$ both exist and $Var[X_i] = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2$ (similar for Y_i), then $\mathbb{E}[X_i]$ and $\mathbb{E}[Y_i]$ also exist. Thus, with i.i.d. observations, we can apply the WLLN to conclude that $\overline{X}_n \xrightarrow{P} \mathbb{E}[X_i]$ and $\overline{Y}_n \xrightarrow{P} \mathbb{E}[Y_i]$. Since marginal convergence in probability implies joint convergence, we have $(\overline{X}_n, \overline{Y}_n) \xrightarrow{P} (\mathbb{E}[X_i], \mathbb{E}[Y_i])$.

Now, consider the function $g(a, b) = \frac{a}{b}$, if $b \neq 0$, and $g(a, b) = 0$ otherwise (continuous everywhere except at $b = 0$). We can apply the continuous mapping theorem (because $\mathbb{E}[Y_i] \neq 0$) to conclude that $g(\overline{X}_n, \overline{Y}_n) \xrightarrow{P} g(\mathbb{E}[X_i], \mathbb{E}[Y_i])$, which means:

$$\hat{\theta}_n = \frac{\overline{X}_n}{\overline{Y}_n} \xrightarrow{P} \frac{\mathbb{E}[X_i]}{\mathbb{E}[Y_i]} = \theta \tag{1}$$

Therefore, we have that $\hat{\theta}_n$ is consistent for θ.

b

First without using the Delta method:

Define $\mu_X := \mathbb{E}[X_i]$ and $\mu_Y := \mathbb{E}[Y_i]$. First notice that, by the Central Limit Theorem (we can use, because of iid and finite variance), $\sqrt{n}(\overline{X}_n - \mu_X) \xrightarrow{d} N(0, \sigma_X^2)$ and $\sqrt{n}(\overline{Y}_n - \mu_Y) \xrightarrow{d} N(0, \sigma_Y^2)$, with σ_X^2 and σ_Y^2 being the variances of X and Y. Due to Slutsky and the properties of the normal distribution, we can also conclude that:

$$\begin{aligned} \sqrt{n}(\overline{X}_n\mu_Y - \mu_X\mu_Y) = \mu_Y\sqrt{n}(\overline{X}_n - \mu_X) \xrightarrow{d} \mu_Y N(0, \sigma_X^2) = N(0, \sigma_X^2\mu_Y^2) \\ \sqrt{n}(\overline{Y}_n\mu_X - \mu_Y\mu_X) = \mu_X\sqrt{n}(\overline{Y}_n - \mu_Y) \xrightarrow{d} \mu_X N(0, \sigma_Y^2) = N(0, \sigma_Y^2\mu_X^2) \end{aligned} \tag{2}$$

Now, because of the independence of X_i and Y_i, we can subtract the second normal from the first and conclude that:

$$
\begin{aligned}
\sqrt{n}(\overline{X}_n\mu_Y - \mu_X\mu_Y - \overline{Y}_n\mu_X + \mu_Y\mu_X) &\xrightarrow{d} N(0, \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2) \\
\sqrt{n}(\overline{X}_n\mu_Y - \overline{Y}_n\mu_X) &\xrightarrow{d} N(0, \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2) \\
\frac{1}{\overline{Y}_n\mu_Y}\sqrt{n}(\overline{X}_n\mu_Y - \overline{Y}_n\mu_X) &\xrightarrow{d} \frac{1}{\mu_Y^2}N(0, \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2) \\
\sqrt{n}(\frac{\overline{X}_n}{\overline{Y}_n} - \frac{\mu_X}{\mu_Y}) &\xrightarrow{d} N(0, \frac{\sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2}{\mu_Y^4}),
\end{aligned} \tag{3}
$$

where the third line is possible because: $\overline{Y}_n\mu_Y \xrightarrow{P} \mu_Y^2$ (due to slutsky); applying the continuous mapping theorem with the function $g(a) = 1/a$ (not-continuous only on 0, and $\mu_Y^2 \neq 0$), we can also conclude that $\frac{1}{\overline{Y}_n\mu_Y} \xrightarrow{P} \frac{1}{\mu_Y^2}$; thus, by slutsky again, the third line follows.

Now, using the Delta method:

From the Central Limit Theorem, we know that $\sqrt{n}((\overline{X}_N, \overline{Y}_N) - (\mathbb{E}[X], \mathbb{E}[Y])) \xrightarrow{d} \mathcal{N}(0, \Sigma(X,Y))$, because both variances existing implies $\Sigma(X,Y)$ exists. We can now apply the delta method with the function $g(x,y) = \frac{x}{y}$, since it is continuous and differentiable at $\theta = \frac{\mathbb{E}[X_i]}{\mathbb{E}[Y_i]}$.

Note that

$$\nabla g(x,y) = \left(\frac{1}{y}, \frac{-x}{y^2}\right)'$$

Thus, by the delta method, we have:

$$
\begin{aligned}
\sqrt{n}\left(\widehat{\theta}_N - \theta\right) &\xrightarrow{d} \mathcal{N}\left(0, \nabla g'\Sigma\nabla g\right) \\
&= \mathcal{N}\left(0, \frac{\mathbb{V}(X)\mathbb{E}[Y]^2 + \mathbb{V}(Y)\mathbb{E}[X]^2}{\mathbb{E}[Y]^4}\right)
\end{aligned}
$$

Question 2

First, we recall that the mean of a χ_n^2 (chi-square with n degrees of freedom) is n, and its variance is $2n$. Also, we know that a χ_n^2 random variable is the sum of the square of n independent standard normal random variables. Therefore, since $X_n \sim \chi_n^2$, we have:

$$X_n = \sum_{i=1}^{n} Z_i^2, \tag{4}$$

where $Z_i \sim N(0,1)$.

Applying the Central Limit Theorem to $\sqrt{n}(\frac{X_n}{n} - \mathbb{E}[Z_i^2])$ (we can, because the fourth moment of a standard normal exists and the Z_i^2 are iid), we get:

$$
\begin{aligned}
(\frac{X_n - \mathbb{E}[X_n]}{\sqrt{n}}) &= (\frac{X_n - n}{\sqrt{n}}) = \sqrt{n}(\frac{X_n}{n} - 1) \\
&= \sqrt{n}(\frac{X_n}{n} - \mathbb{E}\left[Z_i^2\right]) \xrightarrow{d} N(0, Var(Z_i^2)) = N(0,2)
\end{aligned} \tag{5}
$$

Now slutsky gives us that $X_n - \mathbb{E}[X_n] = \sqrt{n}(\frac{X_n - \mathbb{E}[X_n]}{\sqrt{n}}) \xrightarrow{d} N(0, 2n)$, and slutsky again gives us that $X_n = X_n - \mathbb{E}[X_n] + \mathbb{E}[X_n] \xrightarrow{d} N(n, 2n)$. Thus this is the sense in which a χ^2_n random variable is approximately normal, specifically $N(n, 2n)$, when n is large.

Question 3

Let $T_n \xrightarrow{d} T$ and $C_n \xrightarrow{p} c$. Let $\Pr(T \leq t)$ be continuous at c.

We know that $C_n \xrightarrow{d} c$, so by Slutky's lemma, we know that $T_n - C_n \xrightarrow{d} T - c$. Since it is known that $\Pr(T \leq t)$ is continuous at c, this tells us that: $\Pr(T - c \leq x)$ is continuous at $x = 0$. Convergence in Distribution then means: $\Pr(T_n - C_n \leq 0) \to \Pr(X - c \leq 0)$. Equivalently: $\Pr(T_n \leq C_n) \to \Pr(X \leq c)$

Let $C_n = c - \frac{1}{n}$ w.p. 1. We know that $C_n \xrightarrow{p} c$. Let $T_n = c = T$ for every n. Then, although we have $T_n \xrightarrow{d} T$, we have that $\Pr(T_n = c \leq C_n = c - 1/n) = 0 \neq \Pr(T \leq c) = 1$ for every n. Thus they cannot converge.

Question 4

We wish to show that $\limsup_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{\{\sqrt{N}\frac{\overline{X}_N}{S_N} > t_{n-1,1-\alpha}\}}\right] \leq \alpha$ is true under the Null.

Under the Null hypothesis, and using the fact that the variance is finite:

$$\begin{aligned}
\Pr\left(\sqrt{N}\frac{\overline{X}_N}{S_n} > t_{n-1,1-\alpha}\right) &= \Pr\left(\sqrt{N}\frac{\overline{X}_N - \mu(P)}{S_n} + \sqrt{N}\frac{\mu(P)}{S_n} > t_{n-1,1-\alpha}\right) \\
&\leq \Pr\left(\sqrt{N}\frac{\overline{X}_N - \mu(P)}{S_n} > t_{n-1,1-\alpha}\right) \xrightarrow{d} Pr\left(Z > t_{n-1,1-\alpha}\right)
\end{aligned}$$

where $Z \sim \mathcal{N}(0, 1)$

Since $t_{n-1,1-\alpha} \to z_{1-\alpha}$, and neither are random, $t_{n-1,1-\alpha} \xrightarrow{p} z_{1-\alpha}$ trivially. We may also note that the cdf of the normal random variable is continuous on $\mathbb{R}$. Applying the result from question 3:

$$\begin{aligned}
\limsup_{N\to\infty} \mathbb{E}\left[\mathbb{1}_{\{\sqrt{N}\frac{\overline{X}_N}{S_N} > t_{n-1,1-\alpha}\}}\right] &= \lim_{N\to\infty} \Pr\left(\sqrt{N}\frac{\overline{X}_N}{S_N} > t_{n-1,1-\alpha}\right) \\
&\leq \lim_{N\to\infty} \Pr\left(\sqrt{N}\frac{\overline{X}_N - \mu(P)}{S_n} > t_{n-1,1-\alpha}\right) \\
&= \Pr\left(Z > z_{1-\alpha}\right) = \alpha
\end{aligned}$$

Question 5

From the Central Limit Theorem, we know that $\sqrt{N}\left(\overline{X}_N - \mu(P)\right) \xrightarrow{d} \mathcal{N}(0, \Sigma(P))$. As $\Sigma(P)$ is invertible, we know that

$$N(\overline{X}_N - \mu(P))'\Sigma(P)^{-1}(\overline{X}_N - \mu(P)) \xrightarrow{d} z'\chi^2(k)z \sim \chi^2(k)$$

Let us define an estimator of $\Sigma(P)$ as $\widehat{\Sigma(P)}_N = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \overline{X}_N)(X_i - \overline{X}_N)'$.

$$\begin{aligned}
\widehat{\Sigma(P)}_N &= \frac{N}{N-1}\left[\frac{1}{N}\sum_{i=1}^{N} X_i X_i' - \overline{X}_N \overline{X}_N'\right] \\
&= g(\frac{N}{N-1}, \frac{1}{N}\sum_{i=1}^{N} X_i X_i', \overline{X}_N \overline{X}_N')
\end{aligned}$$

This is a continuous function of those three parameters defined as $g(a, b, c) = a(b - c)$. We may also note that:

$$\begin{aligned}
\frac{N}{N-1} &\to 1 \\
\frac{1}{N}\sum_{i=1}^{N} X_i X_i' &\to \mathbb{E}\left[XX'\right] \\
\overline{X}_N &\to \mathbb{E}\left[X\right]
\end{aligned}$$

The continuous mapping theorem tells us that

$$\overline{X}_N \overline{X}_N' \xrightarrow{p} \mathbb{E}\left[X\right]\mathbb{E}\left[X\right]'$$

Applying the Continuous mapping theorem to g:

$$\widehat{\Sigma(P)}_N \xrightarrow{p} \mathbb{E}\left[XX'\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[X\right]' = \Sigma(P)$$

Since $\widehat{\Sigma(P)}_N$ is a consistent estimator of $\Sigma(P)$, any continuous function of it should converge in probability to that function of $\Sigma(P)$. This occurs because of the continuous mapping theorem and the special case where convergence in distribution to a constant implies convergence in probability to that constant.

In particular, we may take the inverse of the matrix as a continuous function. We can see that the inverse of a matrix is continuous by considering the fact that it can be calculated using elementary row operations all of which are continuous functions. This tells us that $\widehat{\Sigma(P)}_N^{-1} \xrightarrow{p} \Sigma(P)^{-1}$.

Via Slutsky's lemma:

$$N(\overline{X}_N - \mu(P))'\widehat{\Sigma(P)}_N^{-1}(\overline{X}_N - \mu(P)) \xrightarrow{d} \chi^2(k)$$

We can then define a confidence interval for $\mu(P)$ by

$$C_N = \{t \in R^k | N(\overline{X}_N - t)' \widehat{\Sigma(P)}_N^{-1} (\overline{X}_N - t) < \chi^2_{1-\alpha}(k)\}$$

where $\chi^2_{1-\alpha}(k)$ is the critical value for the $\chi^2(k)$ distribution with error level α.

It is clear that this is consistent in level, as we choose $\chi^2_{1-\alpha}(k)$ as precisely the value for which $\Pr(\mu(P) \notin C_n) \to \alpha$, thus $\Pr(\mu(P) \in C_n) \to 1 - \alpha$

Question 6

a

Let C denote the fair coin flipped.

$$\begin{aligned}
\Pr(X_i = 1) &= \Pr(X_i = 1 | C = H) \Pr(C = H) + \Pr(X_i = 1 | C = T) \Pr(C = T) \\
&= \theta \frac{1}{2} + .1 \frac{1}{2} \\
&= \frac{\theta + .1}{2}
\end{aligned}$$

b

Denote $Y = \mathbb{1}_{\{X_i=1\}}$. Note that $\mathbb{E}[Y] = \Pr(X_i = 1)$. Let $\widehat{\theta}_N = 2\overline{Y}_N - 0.1$.

From the weak law of large numbers, we know that $\overline{Y}_N \xrightarrow{p} \frac{\theta+.1}{2}$. We can see $\widehat{\theta}_N$ as a continuous function of $\mathbb{E}[Y]$, $g(y) = 2y - 0.1$. Thus, by the continuous mapping theorem, we know that $2\overline{Y}_N - 0.1 \xrightarrow{p} 2\frac{\theta+.1}{2} - 0.1$, so $\widehat{\theta}_N \xrightarrow{p} 2\frac{\theta+.1}{2} - 0.1 = \theta$.

c

To be able to apply the Central Limit Theorem, we must verify that the expected value and the variance of Y are finite.

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}\left[\mathbb{1}_{\{X_i=1\}}\right] = \Pr(X_i = 1) = \frac{\theta + .1}{2} \\
\mathbb{V}(Y) &= \Pr(X_i = 1)(1 - \Pr(X_i = 1)) = \left(\frac{10\theta + 1}{20}\right)\left(\frac{19 - 10\theta}{20}\right)
\end{aligned}$$

By the central limit theorem: $\sqrt{N}(\overline{Y}_N - \frac{\theta+.1}{2}) \xrightarrow{d} \mathcal{N}(0, \mathbb{V}(Y))$.

Define:

$$g(x) = 2x - .1$$

Note that this is a continuous function on $\mathbb{R}$ and that $\widehat{\theta}_N = g(\overline{Y}_N)$. $g(\mathbb{E}[Y]) = \theta$.

By the delta-method we have that:

$$\sqrt{N}\left(\widehat{\theta}_N - \theta\right) \xrightarrow{d} \mathcal{N}(0, 4\mathbb{V}(Y)) = \mathcal{N}(0, 4\left(\frac{10\theta + 1}{20}\right)\left(\frac{19 - 10\theta}{20}\right))$$

Define $h(x) = 4\left(\frac{10x+1}{20}\right)\left(\frac{19-10x}{20}\right)$. This is a continuous function as well, and as we know that $\widehat{\theta}_N \xrightarrow{p} \theta$ then $h(\widehat{\theta}_N) \xrightarrow{p} h(\theta)$ by the continuous mapping theorem. Then, because $h(\theta)$ is constant, using slutsky, we have:

$$\sqrt{N}\left(\frac{\widehat{\theta}_N - \theta}{\sqrt{h(\widehat{\theta}_N)}}\right) \xrightarrow{d} \mathcal{N}(0,1)$$

Therefore we can define

$$C_n = \left\{\theta | \sqrt{N}\left(\frac{|\widehat{\theta}_N - \theta|}{\sqrt{h(\widehat{\theta}_N)}}\right) < z_{1-0.025}\right\}$$

$$C_n = \left[\widehat{\theta}_N - z_{1-0.025}\sqrt{\frac{h(\widehat{\theta}_N)}{N}}, \widehat{\theta}_N + z_{1-0.025}\sqrt{\frac{h(\widehat{\theta}_N)}{N}}\right]$$

From the convergence in distribution above ($\Pr(\theta \in Cn) \xrightarrow{d} \Pr(|Z| \leq z_{1-0.025} = 0.95)$, it is clear that $\Pr(\theta \in C_n) \to .95$

d

We know that $\widehat{\theta}_N = 2\overline{Y}_N - .1$. Thus:

$$\begin{aligned}
\mathbb{V}\left(\widehat{\theta}_N\right) &= 4\mathbb{V}\left(\overline{Y}_N\right) \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\mathbb{V}\left(\mathbb{1}_{\{X_i=1\}}\right) \\
&\leq \frac{4}{N}\frac{1}{4} = \frac{1}{N}
\end{aligned}$$

Using Markov's inequality we know that:

$$\begin{aligned}
\Pr(\left|\widehat{\theta}_N - \theta\right| > \epsilon) &\leq \frac{\mathbb{V}\left(\widehat{\theta}_N\right)}{\epsilon^2} \\
&\leq \frac{1}{N\epsilon^2} \\
&\iff \\
\Pr(\left|\widehat{\theta}_N - \theta\right| \leq \epsilon) &\geq 1 - \frac{1}{N\epsilon^2}
\end{aligned}$$

Letting $\epsilon = \sqrt{\frac{1}{.05N}}$ and $C_n = [\widehat{\theta}_N - \epsilon, \widehat{\theta}_N + \epsilon]$ we set that:

$$\Pr(\theta \in C_n) = \Pr(\left|\widehat{\theta}_N - \theta\right| \leq \epsilon) \geq 1 - \frac{1}{N\epsilon^2} = .95$$

e

The length of the Markov's inequality Confidence region is $2\epsilon = \frac{2}{\sqrt{.05N}}$. The confidence interval based upon the Central Limit Theorem has length:

$$\widehat{\theta}_N + z_{1-0.025}\sqrt{\frac{h(\widehat{\theta}_N)}{N}} - \widehat{\theta}_N + z_{1-0.025}\sqrt{\frac{h(\widehat{\theta}_N)}{N}} = 2z_{1-0.025}\sqrt{\frac{h(\widehat{\theta}_N)}{N}}$$

Thus, the ratio of the length of the confidence region from the markov inequality over the CLT region (which is $\frac{1}{\sqrt{.05}z_{1-0.025}\sqrt{h(\widehat{\theta}_N)}}$) depends on n only through the variance estimate $h(\widehat{\theta}_N)$. Because it converges to the true variance as $n \to \infty$ (which is at most 1, recall the expression for $h()$ and the properties of a bernoulli), we have that $\frac{1}{\sqrt{.05}z_{1-0.025}\sqrt{h(\widehat{\theta}_N)}} \to X$ such that $X \geq \frac{1}{\sqrt{.05}z_{1-0.025}}$. To reach further conclusions, we need to know the value of $z_{1-0.025}$. Since it is approx. 1.96, we have that $X > \frac{1}{1.96\sqrt{.05}} > 1$. Thus the confidence interval of the CLT gets smaller as $n \to \infty$.

f

Running the Monte-Carlo Simulations in Julia:

```julia
using Distributions

srand(235711)

theta = [.001,.1,.25,.5]
N = [5,20,50,100,500,1000]

critVal = -quantile( Normal( 0,1), .025)

M = 1000
confCheck = zeros(M,length(theta), length(N))
NconfCheck = zeros(M,length(theta), length(N))
for m in 1:M
    for t in theta
        for n in N
            Yes = Vector{Int64}(n)
            coinFlips = rand(Uniform(),n)
            questions = rand(Uniform(),n)
            for i in 1:n
                if( coinFlips[i] > .5)
                    Yes[i] = questions[i] <= .1
                else
                    Yes[i] = questions[i] <= t
                end
```

```
            end

            yBar = mean(Yes)
            thetaHat = 2*yBar - .1

            MarkovBound = sqrt(1/(.05*n))
            confCheck[m,findfirst(theta,t),findfirst(N,n)] = abs(thetaHat - t) < MarkovBound
            vHat = (4/n)*((1.0+10.0*thetaHat)/20.0)*((19.0 - thetaHat*10.0)/20.0)

            nBound = critVal*sqrt(vHat)
            NconfCheck[m,findfirst(theta,t),findfirst(N,n)] = abs(thetaHat - t) < nBound
        end
    end
end
```

The Results can be summarized as:

θ	5	20	50	100	500	1000
.001	1.0	1.0	1.0	1.0	1.0	1.0
.1	1.0	1.0	1.0	1.0	1.0	1.0
.25	1.0	1.0	1.0	1.0	1.0	1.0
.5	1.0	1.0	1.0	1.0	1.0	1.0

θ	5	20	50	100	500	1000
.001	.234	.638	.925	.869	.94	.941
.1	.396	.881	.888	.943	.948	.948
.25	.602	.868	.94	.936	.942	.942
.5	.8	.95	.933	.947	.948	.945

Question 7

a

To show that $\hat{F}_n(x)$ is a consistent estimator of $F(x)$, we need to show that $\hat{F}_n(x) \xrightarrow{P} F(x)$. This can be seen using the Weak Law of Large Numbers. Because observations are i.i.d. (and thus $\mathbb{1}_{\{X_i \leq x\}}$ are i.i.d as well), $\mathbb{E}\left[\mathbb{1}_{\{X_i \leq x\}}\right] = \Pr(X_i \leq x) = F(x)$, $F(x)$ is finite, and $\hat{F}_n(x)$ is just an average of the random variables $\mathbb{1}_{\{X_i \leq x\}}$, we can apply the WLLN and conclude that $\hat{F}_n(x) \xrightarrow{P} F(x)$.

b

To obtain the result, we can apply the Central Limit Theorem. We can do that because, as seen in item (1) above, observations are i.i.d., $\mathbb{E}\left[\mathbb{1}_{\{X_i \leq x\}}\right] = F(x)$, which is finite, $\hat{F}_n(x)$

being just an average of $\mathbb{1}_{\{X_i \leq x\}}$, and also because $\sigma^2(x) = Var(\mathbb{1}_{\{X_i \leq x\}}) = \Pr(X_i \leq x)(1 - \Pr(X_i \leq x)) = F(x)(1-F(x)) < \infty$. Thus we can conclude that $\sqrt{n}(\hat{F}_n(x) - F(x)) \xrightarrow{d} N(0, F(x)(1-F(x))$. This also gives us a theoretical expression for $\sigma^2(x) = F(x)(1-F(x))$.

c

Because $g(p) = p(1-p)$ is a continuous function of p, and we know that $\hat{F}_n(x) \xrightarrow{P} F(x)$, by the continuous mapping theorem we have that $\hat{F}_n(x)(1-\hat{F}_n(x)) \xrightarrow{P} F(x)(1-F(x))$. Thus, $\hat{F}_n(x)(1-\hat{F}_n(x))$ is a consistent estimator of $\sigma^2(x)$.

d

$\theta = \inf\{x \in \mathbb{R} | F(x) \geq 0.5\}$. We want to show that $F(\theta) = 0.5$.

First, suppose that $F(\theta) > 0.5$. Then, we can take N large enough, such that $F(\theta - \frac{1}{n}) > 0.5$ for all $n > N$, because continuity of F and $\theta - \frac{1}{n} \to \theta$ implies $F(\theta - \frac{1}{n}) \to F(\theta)$ (i.e., they get arbitrarily close). But then this contradicts θ being a lower bound of $\{x \in \mathbb{R} | F(x) \geq 0.5\}$.

Now assume that $F(\theta) < 0.5$; similarly we can take N large enough, such that $F(\theta + \frac{1}{n}) < 0.5$ for all $n > N$, because continuity of F (in this case, right continuity would be enough) and $\theta + \frac{1}{n} \to \theta$ implies $F(\theta + \frac{1}{n}) \to F(\theta)$. But this contradicts θ being the greatest lower bound.

Thus, we can conclude that $F(\theta) = 0.5$ must be valid.

e

The CLT gives us that $\sqrt{n}(\hat{F}_n(x) - F(x)) \xrightarrow{d} N(0, F(x)(1-F(x))$ for every x, in particular for $x = \theta_0$. Also, $\hat{F}_n(\theta_0)(1-\hat{F}_n(\theta_0)) \xrightarrow{P} F(\theta_0)(1-F(\theta_0))$. Thus, as long as $F(\theta_0)(1-F(\theta_0)) > 0$ (which is the case if the null $H_0 : \theta = \theta_0$ is true, for instance), we can use the continuous mapping theorem and get that $\frac{1}{\sqrt{\hat{F}_n(\theta_0)(1-\hat{F}_n(\theta_0))}} \xrightarrow{P} \frac{1}{\sqrt{F(\theta_0)(1-F(\theta_0))}}$ (because the function $g(a) = 1/\sqrt{a}$ is continuous except when a is zero).

Using Slutsky and the continuous mapping theorem (because $|.|$ is continuous), we can obtain that: $\frac{\sqrt{n}(|\hat{F}_n(\theta_0) - 0.5)|}{\sqrt{\hat{F}_n(\theta_0)(1-\hat{F}_n(\theta_0))}} \xrightarrow{d} |N(0,1)|$ under the null hypothesis, because, if true, $F(\theta_0) = 0.5$. This gives us a test statistic $T_n = \frac{\sqrt{n}(|\hat{F}_n(\theta_0) - 0.5)|}{\sqrt{\hat{F}_n(\theta_0)(1-\hat{F}_n(\theta_0))}}$ (the test is to reject if $T_n \geq z_{1-\frac{\alpha}{2}}$, i.e., greater than the $1 - \frac{\alpha}{2}$ quantile of the standard normal distribution) and also:

$$\begin{aligned} \Pr(T_n > z_{1-\frac{\alpha}{2}}) &= \Pr(\frac{\sqrt{n}(|\hat{F}_n(\theta_0) - 0.5)|}{\sqrt{\hat{F}_n(\theta_0)(1-\hat{F}_n(\theta_0))}} > z_{1-\frac{\alpha}{2}}) \\ &\iff \\ \limsup \Pr(T_n > z_{1-\frac{\alpha}{2}}) &\leq \limsup \Pr(\frac{\sqrt{n}(|\hat{F}_n(\theta_0) - 0.5)|}{\sqrt{\hat{F}_n(\theta_0)(1-\hat{F}_n(\theta_0))}} \geq z_{1-\frac{\alpha}{2}}) \\ &\leq \Pr(|z| \geq z_{1-\frac{\alpha}{2}}) = \alpha, \end{aligned} \tag{6}$$

where we used portmanteu lemma (after closing the set and obtaining the inequality in the second line) and the fact that T_n converges to a $|N(0,1)|$. Thus, we can conclude that the test is consistent in level α, because the $\limsup$ of the probability of type 1 error is less than or equal α.

f

The $p-value$ is defined as $\inf\{\alpha \in (0,1) | T_n > z_{1-\frac{\alpha}{2}}\}$, where T_n is the test statistic developed above. Thus, we have:

$$\begin{aligned} \inf\{\alpha \in (0,1) | T_n > z_{1-\frac{\alpha}{2}}\} &= \inf\{\alpha \in (0,1) | \phi(T_n) > \phi(z_{1-\frac{\alpha}{2}})\} \\ &= \inf\{\alpha \in (0,1) | \phi(T_n) > 1 - \frac{\alpha}{2}\} \\ &= \inf\{\alpha \in (0,1) | \alpha > 2(1-\phi(T_n))\} = 2(1-\phi(T_n)) \end{aligned} \tag{7}$$

Question 8

a

First let's apply a continuous function to $\tilde{X}_n$ and define $\overline{lnX}_n := \ln \tilde{X}_n = \ln(\prod_i X_i)^{\frac{1}{n}} = \frac{1}{n}(\sum_i \ln X_i)$. In words, $\ln \tilde{X}_n$ is the average of the random variable $\ln X_i$ (which is also i.i.d., although possibly with a different distribution than X_i).

Because we have an average, if $\mathbb{E}[\ln X_i]$ is finite, we can apply the Weak Law of Large Numbers to $\overline{lnX}_n$. Through integration by parts we can obtain that $\mathbb{E}[\ln X_i]$ is finite:

$$\begin{aligned} \mathbb{E}[\ln X_i] &= \int_0^b \ln x \frac{1}{b} dx = \frac{1}{b}\int_0^b \ln x dx \\ &= \frac{1}{b}(b \ln b - \lim_{x \to 0} x \ln x - \int_0^b x \frac{1}{x} dx) \\ &= \frac{1}{b}(b \ln b - b) = \ln b - 1 \end{aligned} \tag{8}$$

Thus, since $\ln b - 1$ is finite, by the WLLN, we can conclude that $\overline{lnX}_n = \ln \tilde{X}_n \xrightarrow{P} \ln b - 1$. Because $g(x) = e^x$ is a continuous function, the continuous mapping theorem gives us that $\tilde{X}_n = e^{\ln \tilde{X}_n} \xrightarrow{P} e^{\ln b - 1} = \frac{e^{\ln b}}{e} = \frac{b}{e}$. Define $\lambda(b) = \frac{b}{e}$ and we have the desired result.

b

First, we will show that $\sqrt{n}(\ln \tilde{X}_n - (\ln b - 1)) \xrightarrow{d} N(0, \sigma^{2,*}(b))$, with $\sigma^{2,*}(b)$ the variance of $\ln X_i$, and then we will be able to apply the delta method to achieve the desired result. To show the above convergence we need to check that the variance of $\ln X_i$ is finite. Because $Var(\ln X_i) = \mathbb{E}[(\ln X_i)^2] - (\mathbb{E}[\ln X_i])^2$, and we know the second term is finite, we need only check the first term:

$$\begin{aligned}
\mathbb{E}\left[(\ln X_i)^2\right] &= \int_0^b (\ln x)^2 \frac{1}{b} dx = \frac{1}{b}\int_0^b (\ln x)^2 dx \\
&= \frac{1}{b}(b(\ln b)^2 - \lim_{x\to 0} x(\ln x)^2 - \int_0^b x 2\ln x \frac{1}{x} dx) \\
&= \frac{1}{b}(b(\ln b)^2 - \lim_{x\to 0} x(\ln x)^2 - 2(b\ln b - b)) \\
&= \frac{1}{b}((b(\ln b)^2 - 2b(\ln b - 1)) \\
&= (\ln b)^2 - 2(\ln b - 1)
\end{aligned} \tag{9}$$

where the fact that $\lim_{x\to 0} x(\ln x)^2 = 0$ can be seen by applying L'Hopital rule tow times.

Because $(\ln b)^2 - 2(\ln b - 1)$ is finite, we can apply the Central Limit Theorem and conclude that $\sqrt{n}(\ln \tilde{X}_n - (\ln b - 1)) \xrightarrow{d} N(0, \sigma^{2,*}(b))$, where $\sigma^{2,*}(b) = (\ln b)^2 - 2(\ln b - 1) - (\ln b)^2 + 2\ln b - 1 = 1$.

Now we can apply the delta method with $g(x) = e^x$ (continuous and differentiable at $(\ln b - 1)$) to conclude that:

$$\begin{aligned}
\sqrt{n}(\exp\{\ln \tilde{X}_n\} - \exp\{(\ln b - 1)\}) &= \sqrt{n}(\tilde{X}_n - \frac{b}{e})) \\
&\xrightarrow{d} e^{(\ln b - 1)} N(0,1) = N(0, e^{2(\ln b - 1)})
\end{aligned} \tag{10}$$

Therefore, we obtained that $\sqrt{n}(\tilde{X}_n - \lambda(b))) \xrightarrow{d} N(0, \sigma^2(x))$, with $\lambda(b) = \frac{b}{e}$ and $\sigma^2(b) = e^{2(\ln b - 1)}$, as desired.