Ps3 questions $\in \{4i\}_{i=1}^{4}$

October 2018

q4

Suppose $\tau_n \uparrow \infty$ and for all $\epsilon > 0$ and some $B > 0$, we have

$$\inf_n Pr(|\tau_n(\hat{\theta} - \theta)| \leq B) \geq 1 - \epsilon$$

Equivalently, we have $\inf_n Pr(|\hat{\theta} - \theta| \leq \frac{B}{|\tau_n|}) \geq 1 - \epsilon$. Now, we can choose some $\mathrm{N} \in \mathbb{N}$ such that, for all $n > N$, $\frac{M}{\tau_n} < \epsilon$ as M is a constant and $\tau_n \uparrow \infty$. Thus

$$\begin{aligned} 1 - \epsilon &\leq \inf(Pr(|\hat{\theta} - \theta)| \leq \frac{M}{|\tau_n|}) \\ &\leq \inf(Pr(|\hat{\theta} - \theta)| \leq \delta) \end{aligned}$$

q8

0.1 a

Noting that $f(y|x) = 0$ if $f_X = 0$, we know the integral over $\mathbb{R}^{\Bbbk} \times \mathbb{R}$ simplifies to the integral over the area where $f_X(x) > 0$ (as it is 0 everywhere else).

$$\begin{aligned} E[m^{*2}(X)] &= \int (\int y f(y|x) dy)^2 f_X(x) dx \\ &\leq \int (\int |y| \frac{f(y,x)}{f_X(x)} dy)^2 f_X(x) dx \end{aligned}$$

Knowing that $\int \frac{f(y,x)}{f_X(x)} dy = 1$, we know (i.e by Cauchy -Schwartz):

$$
\begin{aligned}
E[m^{*2}(X)] &\leq \int (\int y^2 \frac{f(y,x)}{f_X(x)}) f_X(x) dx \\
&= \int \int (y^2 \frac{f(y,x)}{f_X(x)} f_X(x)) dy dx \\
&\leq \int \int y^2 f(x,y) dy dx = E(Y^2) < \infty
\end{aligned}
$$

0.2 b

Recall, from class that

$$
\begin{aligned}
E[(y - m(x))^2] =& E[(y - m(x) + m^*(x) - m^*(x))^2] \\
=& E[(y - m^*(x))^2] + 2E[(y - m^*(x))(m^*(x) - m(x))] + E[(m^*(x) - m(x))^2] \\
\geq & E[(Y - m^*(X))^2]
\end{aligned}
$$

Thus, we found that $\min E[(Y - m^*(X))] \Leftrightarrow E[(Y - m^*(X))m(X)] = 0$ Now, see that

$$
\begin{aligned}
E[(y - m^*(x))m(x)] &= \int \int (y - m^*(x)) m(x) f(y,x) dw dx \\
&= \int (\int (y - m^*(x)) m(x) f(y,x) dy) dx \\
&= \int m(x) f_X(x) (\int y f(y|x) - m^*(x) f(|x) dy dx) \\
&= m(x) m*(x) f_X(x) dx - \int m(x) m^*(x) (\int f(y|x) dy) f_X(x) dx \\
&= \int m(x) m^*(x) f_X d(x) - \int m(x) m^*(x) f_X(x) dx = 0
\end{aligned}
$$

q12

Consider

$$
\begin{aligned}
\beta_1 =& \frac{Cov(X,Y)}{\sigma_X} \\
=& \rho_{X,Y} \frac{\sigma_Y}{\sigma_X}
\end{aligned}
$$

Thus, the $|\beta_1| < 1$ does not necessarily mean $\frac{Var(X)}{Var(Y)} < 1$ as we need $\frac{\beta_1}{\rho_{X,Y}} < 1$

0.3 b

As $\sigma_X = \sigma_Y$, the above equation implies that we have $\beta_1 = \rho_{X,Y}$, so $\beta_1 = 1$ iff $\rho_{X,Y} = 1$. Also, as $\sigma_Y^2 = \beta_1^2\sigma_X^2 + \sigma_U^2$, we requite that $\sigma_U^2 = 0$

0.4 c

Again, as we have

$$\begin{aligned}\beta_1 &= \rho_{X,Y}\frac{\sigma_Y}{\sigma_X}\\ &= \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}\\ &= \alpha_1\end{aligned}$$

as the distributions are equal. This equality requires, either $\rho_{X,Y} = 0$ or $\sigma_X = \sigma_Y$

q16

Intuitively, we have that since E(V) = 0 and $V \in \{0, 1\}$, we cannot have the measurement error to "cancel out" as as V can never be below 0.
Note that if E(V) = 0,

$$\begin{aligned}Cov(X,V) &= E((X - E(X))(V - E(V)))\\ &= E(XV) - E(X)E(V)\\ &= E(XV)\end{aligned}$$

Now, looking at variance of $\hat{X}$, we see that if $Cov(X, V) = E(XV) = 0$, $Var(\hat{X}) = Var(X)$

$$\begin{aligned}Var(\hat{X}) =&E(X^2) + E(V^2) + E(XV) - E(\hat{X})^2\\ =&E(X^2) - E(X)^2 + E(V^2)\\ =&Var(X) + Var(V)\end{aligned}$$

Herein, the $Var(V) = 0$ so V= 0 and $\hat{X}$ is just X.