Question 1

First, we need to show that:

$$\rho^2 = \frac{(\mathbf{Cov}[X,Y])^2}{\mathbf{Var}[X]\mathbf{Var}[Y]} = 1 - \frac{\mathbf{Var}[U]}{\mathbf{Var}[Y]}.$$

Further, given $Y = \beta_0 + \beta_1 X + U$,

$$\begin{aligned}
\mathbf{Var}[Y] =& \mathbf{Var}[\beta_0 + \beta_1 X + U] \\
=& \beta_1^2 \mathbf{Var}[X] + \mathbf{Var}[U] + 2\mathbf{Cov}[X,U] \\
=& \left(\frac{\mathbf{Cov}[X,Y]}{\mathbf{Var}[X]}\right)^2 \mathbf{Var}[X] + \mathbf{Var}[U] \\
=& \frac{(\mathbf{Cov}[X,Y])^2}{\mathbf{Var}[X]} + \mathbf{Var}[U].
\end{aligned}$$

This implies that:

$$\mathbf{Var}[U] = \mathbf{Var}[Y] - \frac{(\mathbf{Cov}[X,Y])^2}{\mathbf{Var}[X]}$$

Since we are interested in ρ^2, and since $\rho^2 = \frac{\mathbf{Var}[U]}{\mathbf{Var}[Y]}$, we need to divide by $\mathbf{Var}[Y]$.

$$\begin{aligned}
\frac{\mathbf{Var}[U]}{\mathbf{Var}[Y]} =& 1 - \frac{(\mathbf{Cov}[X,Y])^2}{\mathbf{Var}[X]} \\
=& 1 - \rho^2 \\
\rho^2 =& 1 - \frac{\mathbf{Var}[U]}{\mathbf{Var}[Y]}.
\end{aligned}$$

Next, we need to actually determine $\mathbf{Var}[U]$ and $\mathbf{Var}[Y]$. We have shown earlier in lectures that $\beta = \mathbf{E}[XX']\mathbf{E}[XY]$. Given the vector $(1, X)$, and $Y = \gamma X + X^2]$, we can say:

$$\begin{aligned}
\beta = \mathbf{E}\left[\begin{pmatrix} 1 \\ X \end{pmatrix} \begin{pmatrix} 1 & X \end{pmatrix}\right] \mathbf{E}\left[\begin{pmatrix} Y \\ YX \end{pmatrix}\right] \\
= \mathbf{E}\left[\begin{pmatrix} 1 & X \\ X & X^2 \end{pmatrix}\right] \mathbf{E}\left[\begin{pmatrix} \gamma X + X^2 \\ \gamma X^2 + X^3 \end{pmatrix}\right].
\end{aligned}$$

Since we know $\mathbf{E}[1] = 1$, $\mathbf{E}[X] = 0$, $\mathbf{E}[X^2] = 1$, and $\mathbf{E}[X^3]$, the above is equal to:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \begin{pmatrix} 1 \\ \gamma \end{pmatrix} = \beta.$$

It is given that $U = Y - \beta_0 + \beta_1 X$. Using the β above, we can plug in for $U = \gamma X + X^2 - 1 - \gamma X = X^2 - 1$. Thus, we have

$$\mathbf{Var}[U] = \mathbf{Var}[X^2 - 1] = 2.$$

Next, consider $Y = \gamma X + X^2$.

$$\mathbf{Var}[Y] = \gamma \mathbf{Var}[X] + \mathbf{Var}[X^2] + \gamma \mathbf{Cov}[X, X^2].$$

$\mathbf{Cov}[X, X^2] = 0$, since $\mathbf{Cov}[X, X^2] = \mathbf{E}[X^3] - \mathbf{E}[X]\mathbf{E}[X^2]$, and we know that both right hand side terms are zero. Further, since $\mathbf{Var}[X] = 1$, and $\mathbf{Var}[X^2] = 2$, we have

$$\mathbf{Var}[Y] = \gamma^2 + 2.$$

Using our previous result that $\rho^2 = 1 - \frac{\mathbf{Var}[U]}{\mathbf{Var}[Y]}$ and substituting in from above,

$$\rho^2 = 1 - \frac{\mathbf{Var}[U]}{\mathbf{Var}[Y]} = 1 - \frac{2}{\gamma^2 + 2} = \frac{\gamma^2}{\gamma^2 + 2}.$$

Question 5

A)

Simply by the Delta Method we get:

$$n^{\frac{1}{2}}(f(\hat{\beta}_n) - f(\beta)) \xrightarrow{d} N(0, D_\beta f(\beta) \Omega D_\beta f(\beta)').$$

B)

With $\hat{\beta}_n$ and $\hat{\Omega}_n$ being a consistent estimators of β and Ω respectively, using the CMT, we have:

$$\frac{\sqrt{n}(f(\hat{\beta}_n) - f(\beta))}{\sqrt{D_{\hat{\beta_n}} f(\hat{\beta}_n) \Omega D_{\hat{\beta_n}} f(\hat{\beta}_n)'}} \xrightarrow{d} N(0, 1).$$

Since our test is one-sided we only want to reject the null (H_0: $f(\beta) \leq 0$) in one direction. The critical value is based on the standard normal distribution:

$$c_n := \Phi^{-1}(1 - \alpha) := z_{1-\alpha},$$

where Φ is the CDF of $N(0, 1)$. We want our c_n to be such that the probability of z being less than c_n is $1 - \alpha$. Thus, out test is:

$$\Phi_n = \mathbf{1}_{\{T_n > c_n\}}.$$

To show that this test is consistent in level, we have to show that, under the null:

$$\lim_{n\to\infty} \sup \mathbf{E}_P[\Phi_n] \leq \alpha$$

Consider,

$$\mathbf{E}_P[\Phi_n] = \mathbf{Pr}(T_n > c_n) = \mathbf{Pr}\left(\frac{\sqrt{n}f(\hat{\beta}_n)}{\sqrt{D_{\hat{\beta}_n}f(\hat{\beta}_n)\hat{\Omega}_n D_{\hat{\beta}_n}f(\hat{\beta}_n)'}} > z_{1-\alpha}\right).$$

Add and subtract $f(\beta)$,

$$\mathbf{Pr}\left(\frac{\sqrt{n}(f(\hat{\beta}_n) - f(\beta))}{\sqrt{D_{\hat{\beta}_n}f(\hat{\beta}_n)\hat{\Omega}_n D_{\hat{\beta}_n}f(\hat{\beta}_n)'}} + \frac{\sqrt{n}f(\beta)}{\sqrt{D_{\hat{\beta}_n}f(\hat{\beta}_n)\hat{\Omega}_n D_{\hat{\beta}_n}f(\hat{\beta}_n)'}} > z_{1-\alpha}\right).$$

Under the null, we have that $f(\hat{\beta}_n) \leq 0$, and so

$$\mathbf{E}[\Phi_n] \leq \mathbf{Pr}\left(\frac{\sqrt{n}(f(\hat{\beta}_n) - f(\beta))}{\sqrt{D_{\hat{\beta}_n}f(\hat{\beta}_n)\hat{\Omega}_n D_{\hat{\beta}_n}f(\hat{\beta}_n)'}} > z_{1-\alpha}\right) \leq \mathbf{Pr}\left(\frac{\sqrt{n}(f(\hat{\beta}_n) - f(\beta))}{\sqrt{D_{\hat{\beta}_n}f(\hat{\beta}_n)\hat{\Omega}_n D_{\hat{\beta}_n}f(\hat{\beta}_n)'}} \geq z_{1-\alpha}\right),$$

where the weak inequality is so that we can apply the Portmanteau Lemma. Thus, taking lim sup of both sides,

$$\lim_{n\to\infty} \sup \mathbf{E}[\Phi_n] \leq \lim_{n\to\infty} \sup \mathbf{Pr}\left(\frac{\sqrt{n}(f(\hat{\beta}_n) - f(\beta))}{\sqrt{D_{\hat{\beta}_n}f(\hat{\beta}_n)\hat{\Omega}_n D_{\hat{\beta}_n}f(\hat{\beta}_n)'}} \geq z_{1-\alpha}\right)$$

We already know that the inside the probability on the RHS converges in distribution to a standard normal. Thus,

$$\begin{aligned}
\lim_{n\to\infty} \sup \mathbf{E}[\Phi_n] &\leq \mathbf{Pr}\left(Z \geq z_{1-\alpha}\right) \\
&= 1 - \mathbf{Pr}(Z < z_{1-\alpha}) \\
&= 1 - \Phi(z_{1-\alpha}) \\
&= 1 - (1 - \alpha) \\
&= \alpha.
\end{aligned}$$

Our test is consistent in level.

C)

We can easily construct a confidence region with the result from **B)**.

$$C_n := \left\{ x \in \mathbb{R} | \mathbf{Pr} \left(\frac{\sqrt{n}(f(\hat{\beta}_n) - x)}{\sqrt{D_{\hat{\beta}_n} f(\hat{\beta}_n) \hat{\Omega}_n D_{\hat{\beta}_n} f(\hat{\beta}_n)'}} \geq z_{1-\alpha} \right) \right\}.$$