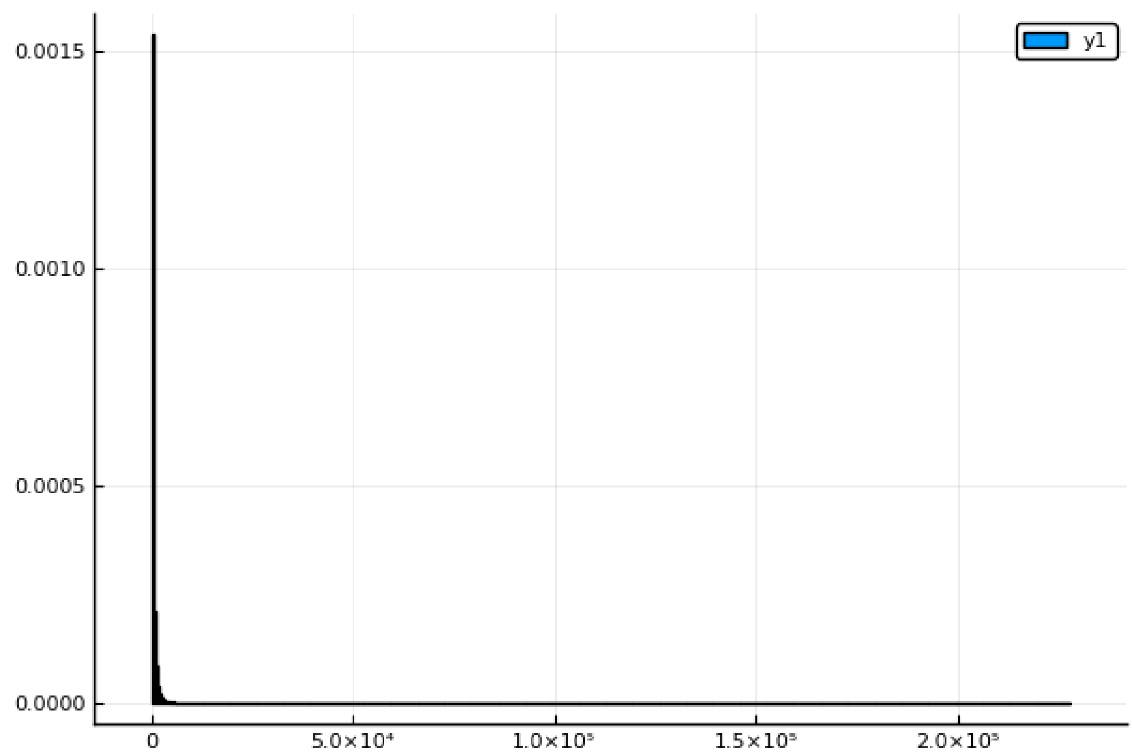
Structural Estimation Pset 2

Timothy Schwieg

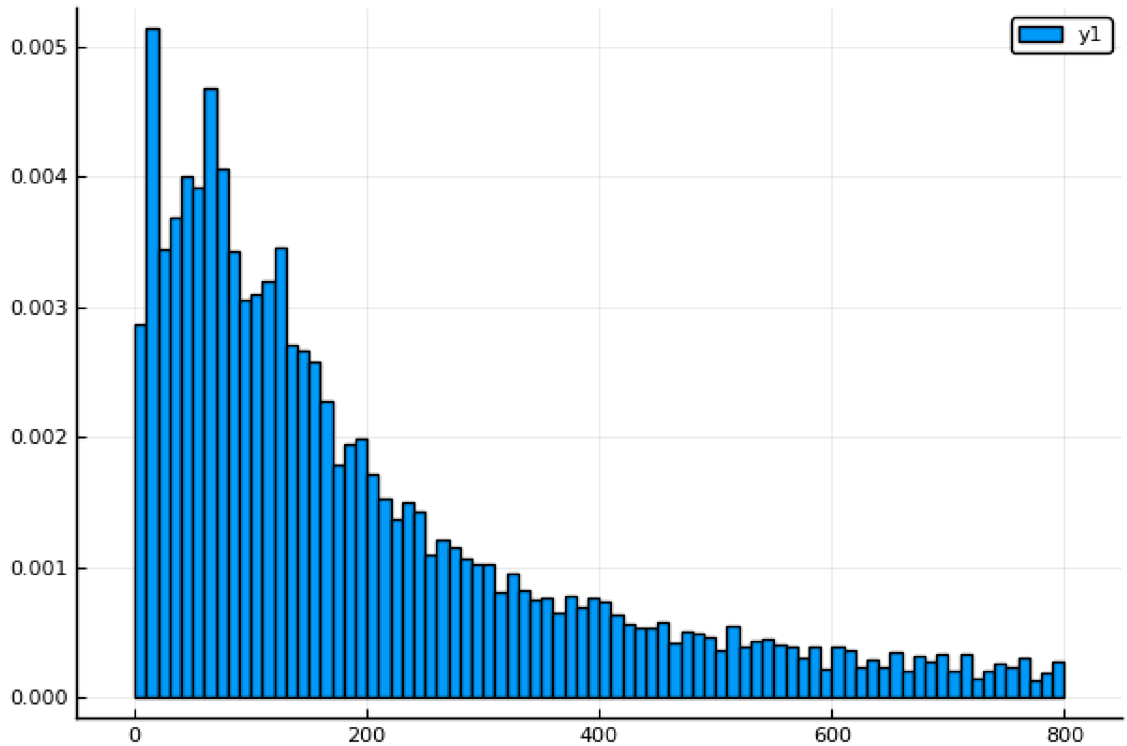
1 Question One

1.1 a

```
healthClaims = CSV.read( "clms.txt", header=[:A] )
describe( healthClaims )

println( "Standard Deviation: ", std(healthClaims[:A]))

histogram( healthClaims[:A], bins=1000, normalize = true)
savefig("histOne.png")
```




```
truncatedHealthClaims = healthClaims[healthClaims[:A] .<= 800, 1]


histogram( truncatedHealthClaims, bins = 100, normalize = true)
savefig("histTwo.png")
```



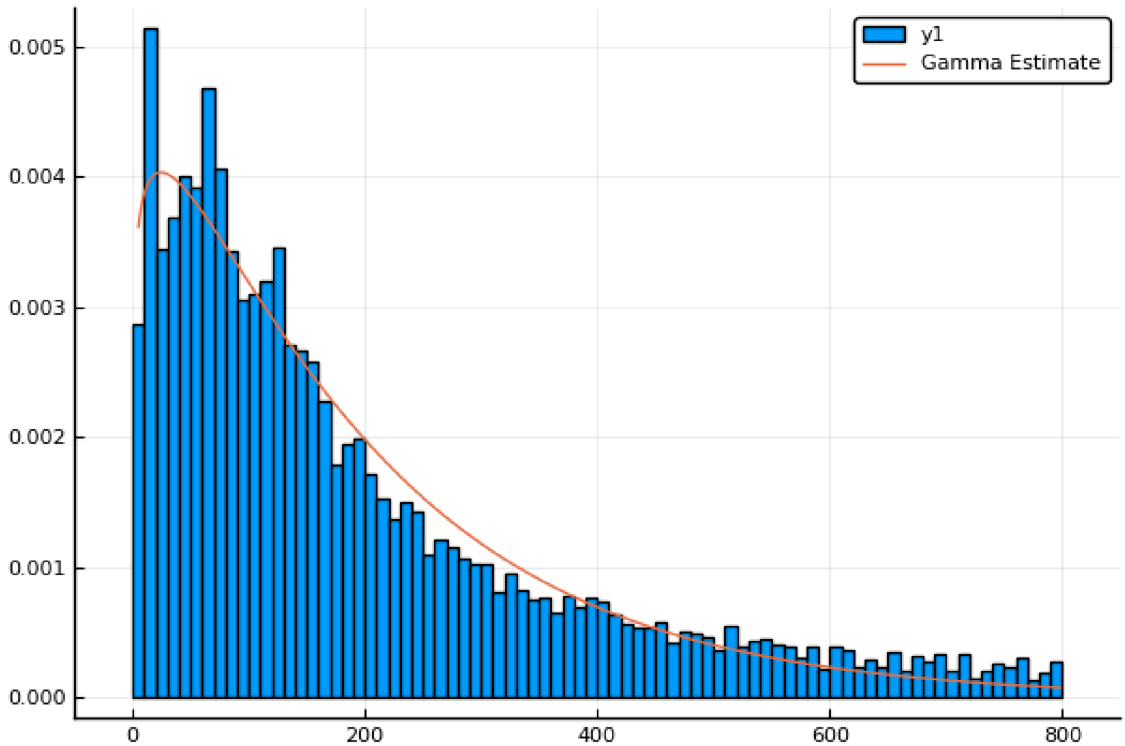
1.2 b

```
function GammaLogLikelihood( x::Vector{Float64}, α::Float64, β::Float64)
    #Yes I know I could get this using Distributions.jl which could
    #even do the MLE estimate But thats pretty much cheating, and
    #gamma is in the exponential family so using Newton's method will
    #cause no issues.

    #Pdf is: $\frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left(-\frac{x}{\beta}\right)$
    #Log-likelihood is: $-\alpha\log(\beta) - \log(\Gamma(\alpha)) + (\alpha - 1)\log x - \frac{x}{\beta}$

    return -α*log( β) - lgamma(α) + (α - 1)*mean(log.(x)) - mean(x) / β
end

function GammaGradient( x::Vector{Float64}, α::Float64, β::Float64)
    delA = -log(β) - digamma(α) + mean(log.(x))
    #delB = mean(x) / β - α
    delB = mean(x) / β^2 - α / β
    return [delA,delB]
end

function GammaHessian( x::Vector{Float64}, α::Float64, β::Float64)
    delAA = -trigamma(α)
    delAB = -1 / β
    delBB =( α / (β*β)) - ((2* mean(x)) / (β*β*β))
    return [delAA delAB; delAB delBB]
end

function GammaPDF( α::Float64, β::Float64, x::Float64)
    return  (1 / (gamma(α)*β^α))*x^(α-1)*exp( -x/β)
end

function EstimateGammaParameters( data::Vector{Float64}, guess::Vector{Float64}, gradientFun, hessianFun)

```

```julia
    θ = guess
    tol = 1e-10
    maxLoops = 100

    grad = gradientFun( data, θ... )
    hess = hessianFun( data, θ... )

    loopCounter = 0
    while( loopCounter < maxLoops && norm(grad) >= tol)
        θ = θ - hess \ grad
        grad = gradientFun( data, θ... )
        hess = hessianFun( data, θ... )

        loopCounter += 1
        # println( norm(grad))
        # println( θ)
        # println( " ")
    end
    #println( loopCounter)
    return θ
end
healthCosts = convert(  Vector{Float64}, truncatedHealthClaims )#healthClaims[:A] )

β₀ =  var(healthCosts) / mean(healthCosts)
α₀ = mean(healthCosts) / β₀

(Gamma_α̂, Gamma_β̂) = EstimateGammaParameters( healthCosts, [α₀, β₀], GammaGradient, GammaHessian)

likelihood = GammaLogLikelihood(  healthCosts, Gamma_α̂, Gamma_β̂)

result = [["\$\\est{\\alpha}\$: ", "\$\\est{\\beta}\$: ", "Likelihood: " ] [ Gamma_α̂,  Gamma_β̂, likelihood]]
```

$\widehat{\alpha}_n$:	1.1397564780585858
$\widehat{\beta}_n$:	174.8688733959653
Likelihood:	-6.28964508639924

```julia
histogram( truncatedHealthClaims, bins = 100, normalize = true)
pdfXVal = range( minimum(truncatedHealthClaims)+5, maximum(truncatedHealthClaims))
#pdfXVal = linspace( minimum(truncatedHealthClaims), maximum(truncatedHealthClaims))
pdfYVal = [GammaPDF( Gamma_α̂, Gamma_β̂, x ) for x in pdfXVal]


plot!( pdfXVal, pdfYVal, label="Gamma Estimate" )
savefig("histPDF_Gamma.png")
```



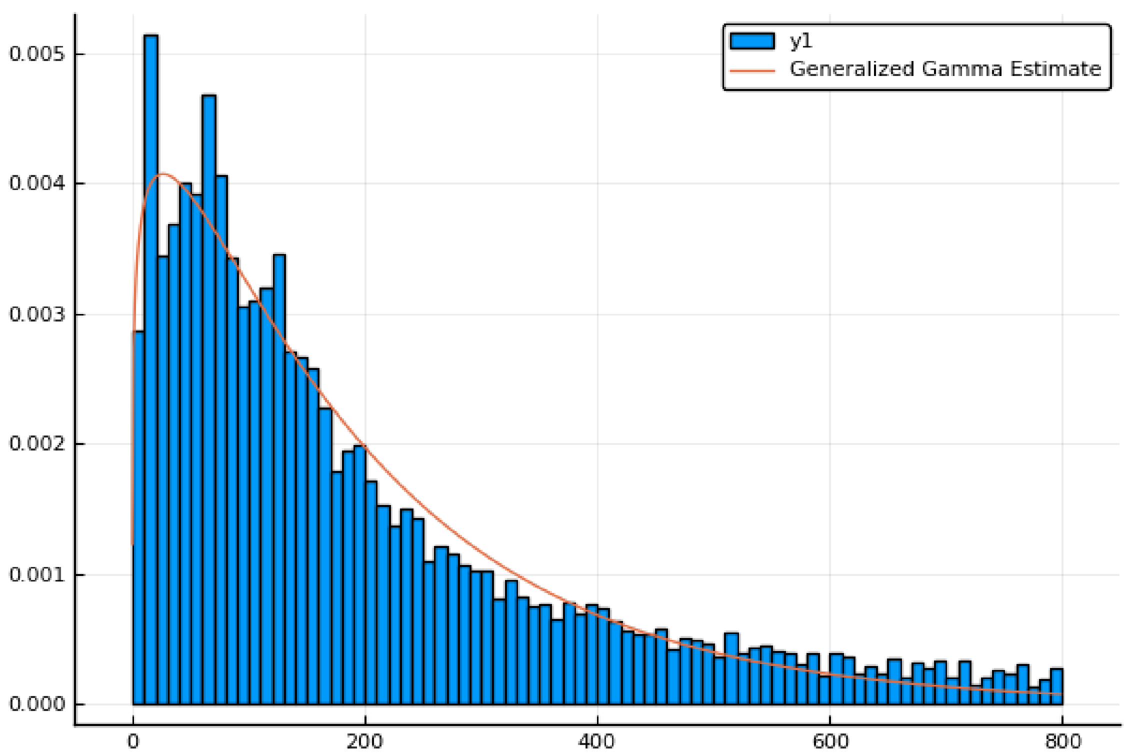
2 c

```
#I don't think this is the correct pdf?
function GGammaPDF( α::Float64, β::Float64, m::Float64, x::Float64)
    return ( (m / β^α) * x^(α-1) * exp( - (x / β)^m) ) / gamma( α / m)

    #return (m * x^(m*β - 1) * exp( - (x / α)^m))/ (α^(m*β) * gamma( β ) )
end


function GGammaLikelihood( x::Vector{Float64}, α::Real, β::Real, m::Real)
    return log(m) - α*log(β) + (α - 1)*mean(log.(x)) - mean( (x ./ β).^m  ) - lgamma( α / m )
end

function EstimateGG( data::Vector{Float64}, guess::Vector{Float64})
    #To hard enforce that all of our parameters are positive, we
    #exponentiate them
    θ = log.(guess)
    fun(x::Vector) = -GGammaLikelihood( data, exp.(x)... )



    result = optimize(fun, θ, ConjugateGradient(), autodiff=:forward)
end

sln = EstimateGG( healthCosts, [Gamma_α̂, Gamma_β, 1.0])

GG_α̂ = exp(sln.minimizer[1])
GG_β = exp(sln.minimizer[2])
GG_m̂ = exp(sln.minimizer[3])
GG_LogLikelihood = -sln.minimum

println( "GG α̂ = ", GG_α̂)
println( "GG β = ", GG_β )
println( "GG m̂ = ", GG_m̂ )
```

```
println( "Likelihood Value: ", GG_LogLikelihood )

result = [["GG \$\\est{\\alpha}\$: ", "GG \$\\est{\\beta}\$: ", "GG \$\\est{m}\$: ","GG Likelihood: " ] [ GG_α̂,  GG_β,  GG_m̂, GG_LogLikelihood]]
```

GG $\widehat{\alpha}_n$:	1.1755020098846642
GG $\widehat{\beta}_n$:	156.18446475134172
GG $\widehat{m}_n$:	0.9498167064643459
GG Likelihood:	-6.289560051458711

```
histogram( truncatedHealthClaims, bins = 100, normalize = true)
pdfXVal = range( minimum(truncatedHealthClaims), maximum(truncatedHealthClaims))
#pdfXVal = linspace( minimum(truncatedHealthClaims), maximum(truncatedHealthClaims))
pdfYVal = [GGammaPDF( GG_α̂, GG_β, GG_m̂, x ) for x in pdfXVal]

plot!( pdfXVal, pdfYVal, label="Generalized Gamma Estimate" )
savefig( "histPDF_GG.png" )
```



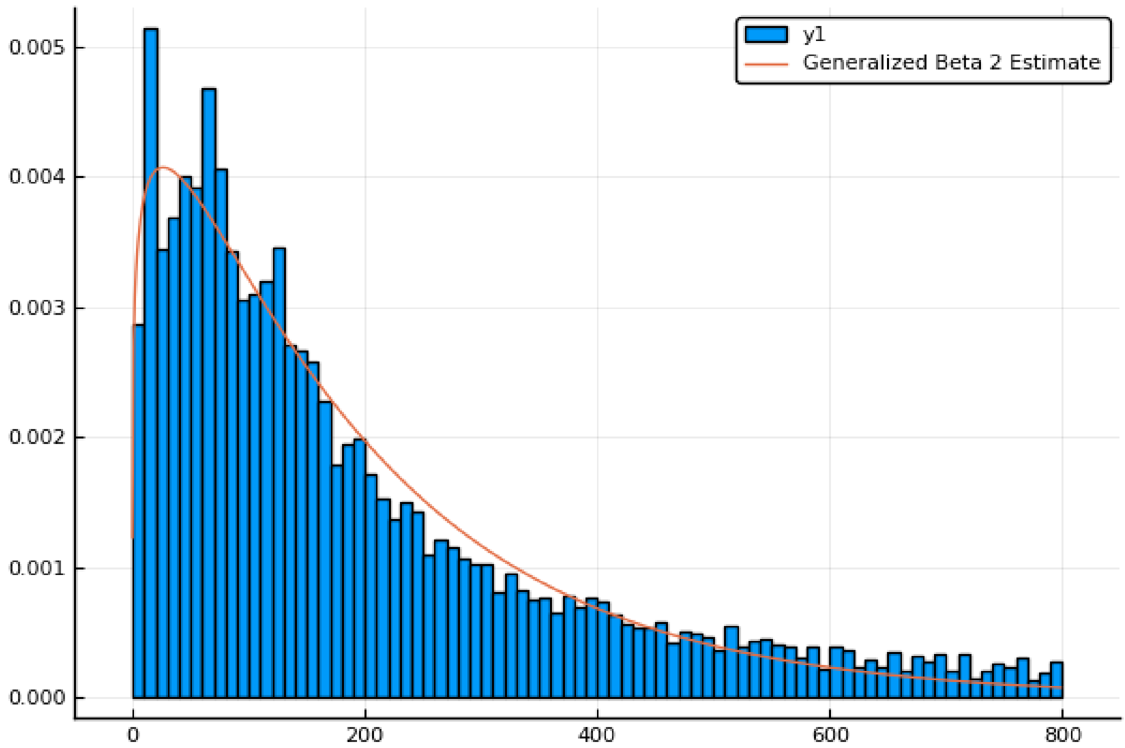
2.1 d

```
function GBetaTwoPDF( x::Float64, a::Real, b::Real, p::Real, q::Real)
    #We require all parameters to be positive, so abs(a) = a
    return a*x^(a*p -1) / (b^(a*p) *beta(p,q)*(1+(x/b)^a)^(p+q))
end

function GBetaTwoLikelihood( x::Vector{Float64}, a::Real, b::Real, p::Real, q::Real)
    return log( a) + (a*p -1)*mean(log.(x)) - (a*p)*log(b) - log(beta(p,q)) - (p+q)*mean( log.( 1 .+(x ./ b).^a ))
end

function EstimateGBetaTwo( data::Vector{Float64}, guess::Vector{Float64})
```

```
      #To hard enforce that all of our parameters are positive, we
      #exponentiate them
    θ = log.(guess)
    #θ = guess
    fun(x::Vector) = -GBetaTwoLikelihood( data, exp.(x)... )


    #This guy is being fickle, and Newton() would not converge
    #LBFGS converges, but to a higher value than Newton()
    result = optimize(fun, θ, NewtonTrustRegion(), autodiff=:forward, Optim.Options(iterations=2000) )
end

sln = EstimateGBetaTwo( healthCosts, [GG_α̂, GG_β, GG_m̂, 10000])

GB2_α̂ = exp( sln.minimizer[1])
GB2_β = exp( sln.minimizer[2])
GB2_p̂ = exp( sln.minimizer[3])
GB2_q̂ = exp( sln.minimizer[4])
GB2_LogLikelihood = -sln.minimum

result = [["GB2 \$\\est{\\alpha}\$: ", "GB2 \$\\est{\\beta}\$: ", "GB2 \$\\est{p}\$: ","GB2 \$\\est{q}\$: ","GB2 Likelihood: " ] [GB2_α̂, GB2_β,  GB2_p̂,  GB2_q̂, -sln.minimum]]
```

GB2 $\widehat{\alpha}_n$:	0.9498180950429491
GB2 $\widehat{\beta}_n$:	1.0983701276884081 (9)
GB2 $\widehat{p}_n$:	1.2376067626960379
GB2 $\widehat{q}_n$:	3.187929333688613 (6)
GB2 Likelihood:	-6.289560054356965

```
histogram( truncatedHealthClaims, bins = 100, normalize = true)
pdfXVal = range( minimum(truncatedHealthClaims), maximum(truncatedHealthClaims))
#pdfXVal = linspace( minimum(truncatedHealthClaims), maximum(truncatedHealthClaims))
pdfYVal = [GBetaTwoPDF( x, GB2_α̂, GB2_β, GB2_p̂, GB2_q̂ ) for x in pdfXVal]

plot!( pdfXVal, pdfYVal, label="Generalized Beta 2 Estimate" )
savefig( "histPDF_GB2.png" )
```



2.2 e

Since the likelihood function values at the optimum for parts (b) and (c) are the constrained maximum likelihood estimators, the likelihood ratio test is simply:

$$2\left(f(\widehat{\theta}_n - \tilde{\theta}_n)\right) \sim \chi^2_p$$

Where p is the number of constraints in the estimation procedure.

```
# Gamma Has Two restrictions
tStatGamma = 2*(GB2_LogLikelihood - likelihood)
# Generalized Gamma Has One Restriction
tStatGG = 2*(GB2_LogLikelihood - GG_LogLikelihood)

results = [["", "Gamma", "Generalized Gamma"] [ "\$\\chi^{2}\$", tStatGamma, tStatGG] ["p-value",
↪  cdf(Chisq(2),tStatGamma), cdf( Chisq(1),tStatGG) ] ]
```

	χ^2	p-value
Gamma	0.00017006408454989241	8.502842715330726 (-5)
Generalized Gamma	-5.796508162347891 (-9)	0.0

2.3 f

The Probability that someone has a health care claim of more than \1000$isgivenby$:

$$
\begin{aligned}
\Pr(X > 1000) &= 1 - \Pr(X \leq 1000) \\
&= \int_0^{1000} f_X dx
\end{aligned}
$$

However, since the integral of a Generalized Beta 2 Distribution is quite nasty, we will compute it numerically.

```
f(x) = GBetaTwoPDF( x, GB2_α̂, GB2_β̂, GB2_p̂, GB2_q̂ )
area = quadgk( f, 0, 1000 )[1]
output = ["Probability of Having > 1000: " (1-area)]
```

$$\text{Probability of Having} > 1000: \quad 0.005078296924289\,96$$

3 Question 2

3.1 a

Equations (3) and (5) tell us that

$$
\begin{aligned}
w_t - (1-\alpha)exp(z_t)(k_t)^{\alpha-1} &= 0 \\
z_t &= \rho z_{t-1} + (1-\rho)\mu + \epsilon_t
\end{aligned}
$$

Note that: $z_0 = \mu$ Therefore:

$$
\begin{aligned}
z_1 &= \mu + \epsilon_1 \\
z_2 &= \mu + \rho\epsilon_1 + \epsilon_2 \\
z_t &= \mu + \sum_{i=0}^{t-1} \rho^i \epsilon_{t-i}
\end{aligned}
$$

Combining these two together:

$$w_t - (1-\alpha)exp\left(\mu + \sum_{i=0}^{t-1}\rho^i\epsilon_{t-i}\right)k_t^{\alpha} = 0$$

Taking logs and isolating the random component:

$$\log w_t - \log(1-\alpha) - \mu - \alpha\log k_t = \sum_{i=0}^{t-1}\rho^i\epsilon_{t-i}$$

Note that the sum of iid distributed normal random variables is distributed normal, where the variance is given by the sum of the variances.

Thus

$$\sum_{i=0}^{t-1}\rho^i\epsilon_{t-i} \sim \mathcal{N}(0, \sigma^2\sum_{i=0}^{t-1}\rho^{2i}) = \mathcal{N}\left(0, \sigma^2\frac{1-\rho^{2i}}{1-\rho}\right)$$

We may now estimate this model using Maximum Likelihood Estimation

```julia
#$\log w_t - \log(1-\alpha) - \mu - \alpha \log k_t = \sum_{i=0}^{t-1} \rho^i \epsilon_{t-i}$
# Variance of error: $\sigma^2 \frac{1-\rho^{2i}}{1-\rho}$

#Clean it up when it exists, comes in the order: (c, k, w, r)
macroData = CSV.read( "MacroSeries.txt", header=[:C,:K,:W,:R])

w = convert( Vector{Float64}, macroData[:W] )
k = convert( Vector{Float64}, macroData[:K] )

function LogLikelihood( N, w::Vector{Float64}, k::Vector{Float64}, α::Real, ρ::Real, μ::Real, σ²::Real  )
    #The pdf of a normal: $\frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$
    #Log Likelihood: $-\frac{1}{2}\log\sigma^2 - \frac{(x-\mu)^2}{2\sigma^2}$

    logLik = 0.0
    #Note the way that the model is structured is: F(...) = 0, so we
    #are maximizing the likelihood of getting a 0 returned for all the
    #moments

    #Note we do not have the -.5*log(2*pi)
    #Because that does not matter at all for MLE estimation.
    for i in 1:N
        mean = log(w[i]) - log( 1 - α) - μ - α*log( k[i])
        var = σ² * ( 1 - ρ^(2*i)) / ( 1 - ρ)
        logLik += -.5*log( σ² ) - (  mean*mean / (2*σ²))
    end
    return logLik
end

N = length(w)

α₀ = .5
β = .99
μ₀ = 1.0
σ₀ = 1.0
ρ₀ = 0.0

#We parameterize each of the variables so that they meet their constraints.
# tanh is used to ensure that $\rho \in (-1,1)$
θ = zeros(4)
θ[1] = log( α₀ / ( 1 - α₀) )
θ[2] = atanh( ρ₀)
θ[3] = log( μ₀ )
θ[4] = log( σ₀)


fun(x::Vector) = -LogLikelihood( N, w, k, exp(x[1]) / (1 + exp(x[1])), tanh(x[2]), exp(x[3]), exp(x[4])  )

result = optimize(fun, θ, LBFGS(), autodiff=:forward)

model_θ = result.minimizer

model_α̂ = exp(model_θ[1]) / (1 + exp(model_θ[1]))
model_ρ̂ = tanh(model_θ[2])
model_μ̂ = exp(model_θ[3])
model_σ̂ = exp(model_θ[4])

output = [["\$\\est{\\alpha}\$:", "\$\\est{\\rho}\$:", "\$\\est{\\mu}\$:", "\$\\est{\\sigma^{2}}\$:"]  [model_α̂,
↪  model_ρ̂, model_μ̂, model_σ̂]]
```

$\widehat{\alpha}_n$:	0.9999999999985967
$\widehat{\rho}_n$:	0.0
$\widehat{\mu}_n$:	27.626774841787046
$\widehat{\sigma^2}_n$:	0.010037258768121 15

4 b

Equations (4) and (5) read:

$$r_t - \alpha \exp(z_t) k_t^{\alpha-1} = 0$$
$$z_t = \rho z_{t-1} + (1-\rho)\mu + \epsilon_t$$
$$\epsilon_t \sim \mathcal{N}(0, \sigma^2)$$

From part (a) we know that (5) can be recursively solved to yield:

$$z_t \sim \mathcal{N}\left(\mu, \sigma^2 \frac{1-\rho^{2i}}{1-\rho}\right)$$

Solving for r_t then taking logs in equation (4)

$$\log r_t = \log \alpha + z_t + (\alpha - 1)\log k_t$$

This can be written as:

$$F(r_t, k_t, \alpha, \mu, \sigma, \rho) = 0$$

where the variance of the random variable described by F is known, and the same as the variance of z_t. Thus this system can be estimated by MLE.

```julia
r = convert( Vector{Float64}, macroData[:R] )
k = convert( Vector{Float64}, macroData[:K] )

#log r_t - log α - z_t - (α - 1) log k_t = 0

function LogLikelihood( N, w::Vector{Float64}, k::Vector{Float64}, α::Real, ρ::Real, μ::Real, σ²::Real  )
    #The pdf of a normal: 1/sqrt(2πσ^2) exp(-(x-μ)^2/(2σ^2))
    #Log Likelihood: -1/2 log σ^2 - (x-μ)^2/(2σ^2)

    logLik = 0.0
    #Note the way that the model is structured is: F(...) = 0, so we
    #are maximizing the likelihood of getting a 0 returned for all the
    #moments

    for i in 1:N
        mean = log(r[i]) - log( α) - μ - (α - 1)*log( k[i])
        var = σ² * ( 1 - ρ^(2*i)) / ( 1 - ρ)
        logLik += -.5*log( σ² ) - (  mean*mean / (2*σ²))
    end
    return logLik
end

N = length(w)

α₀ = .5
β = .99
μ₀ = 1.0
σ₀ = 1.0
ρ₀ = .99

#We parameterize each of the variables so that they meet their constraints.
# tanh is used to ensure that ρ ∈ (-1, 1)
θ = zeros(4)
```

```julia
θ[1] = log( α₀ / ( 1 - α₀) )
θ[2] = atanh( ρ₀)
θ[3] = log( μ₀ )
θ[4] = log( σ₀)


fun(x::Vector) = -LogLikelihood( N, w, k, exp(x[1]) / (1 + exp(x[1])), tanh(x[2]), exp(x[3]), exp(x[4])  )

result = optimize(fun, θ, Newton(), autodiff=:forward)

model_θ = result.minimizer

model_α̂ = exp(model_θ[1]) / (1 + exp(model_θ[1]))
model_ρ̂ = tanh(model_θ[2])
model_μ̂ = exp(model_θ[3])
model_σ̂ = exp(model_θ[4])

output = [["\$\\est{\\alpha}\$:", "\$\\est{\\rho}\$:", "\$\\est{\\mu}\$:", "\$\\est{\\sigma^{2}}\$:"]  [model_α̂,
  model_ρ̂, model_μ̂, model_σ̂]]
```

$\widehat{\alpha}_n$:	0.8887650406380285
$\widehat{\rho}_n$:	0.99
$\widehat{\mu}_n$:	1.8877579805483233
$\widehat{\sigma^2}_n$:	0.009515136163054447

4.1 c

From the derivation of the distribution of $\log r_t$ in part (b):

$$\begin{aligned}
\Pr(r_t > 1) &= \Pr(\log r_t > 0) \\
&= \Pr(\log \alpha + z_t + (\alpha - 1)\log k_t > 0) \\
&= \Pr(\log \alpha + \rho z_{t-1} + (1-\rho)\mu + \epsilon_t + (\alpha - 1)\log k_t > 0) \\
&= \Pr(\log(\alpha) + \rho z_{t-1} + (1-\rho)\mu + \frac{Z}{\sigma} + (\alpha - 1)\log k_t > 0) \\
&= \Pr(Z > -\sigma(\log(\alpha) + \rho z_{t-1} + (1-\rho)\mu + (\alpha - 1)\log k_t)) \\
&= 1 - \Pr(Z \leq -\sigma(\log(\alpha) + \rho z_{t-1} + (1-\rho)\mu + (\alpha - 1)\log k_t)) \\
&= \Phi^{-1}(-\sigma(\log(\alpha) + \rho z_{t-1} + (1-\rho)\mu + (\alpha - 1)\log k_t)) \\
&\approx \Phi^{-1}(-\widehat{\sigma}_n(\log \widehat{\alpha}_n + \widehat{\rho}_n 10 + (1-\widehat{\rho}_n)\widehat{\mu}_n + (\widehat{\alpha}_n - 1)\log(7,500,000)))
\end{aligned}$$

```julia
  prob = cdf( Normal(), -sqrt(model_σ̂)*( log(model_α̂) + model_ρ̂*10 + (1-model_ρ̂)*model_μ̂ + (model_α̂-1)*log( 7500000)))
result = ["Prob" prob]
```

Prob 0.21644022445230773