Operations Research Test 2

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Question 1

$$
\begin{aligned}
&-x_1 + 2x_2 - x_3 - 2x_4 \leq 0 \\
&2x_1 - 3x_2 - x_3 + 3x_4 \leq 0 \\
&-3x_1 + 4x_2 + 3x_3 - 4x_4 < 0
\end{aligned}
$$

We may first note that the third inequality is strict, so we may add a new variable: $\epsilon > 0$ to the third constraint to make it non-strict.

$$
\begin{aligned}
&-x_1 + 2x_2 - x_3 - 2x_4 \leq 0 \\
&2x_1 - 3x_2 - x_3 + 3x_4 \leq 0 \\
&-3x_1 + 4x_2 + 3x_3 - 4x_4 + \epsilon \leq 0 \\
&\epsilon > 0
\end{aligned}
$$

This can be written as:

$$
\begin{pmatrix} -1 & 2 & -1 & -2 & 463487 \\ 2 & -3 & -1 & 3 & 0 \\ -3 & 4 & 3 & -3 & 1 \end{pmatrix} \begin{pmatrix} x \\ \epsilon \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ and: } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} x \\ \epsilon \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

By Farkas' Lemma, this does not have a solution if $A^T y = c$ has a solution for $y \geq 0$.

$$
\begin{pmatrix} -1 & 2 & -1 & -2 & 0 \\ 2 & -3 & -1 & 3 & 0 \\ -3 & 4 & 3 & -3 & 1 \end{pmatrix}^T y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

We can solve this system and we arrive at:

$$
y = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \geq 0
$$

Since we found a weakly positive solution for y, Farkas' Lemma informs us that there is no solution to the inequality system.

Question 2

$$\begin{aligned} \max\ & 5x_1 + 4x_2 \\ \text{s.t.}\ \ & 6x_1 + 4x_2 \leq 24 \\ & x_1 + 2x_2 \leq 6 \\ & 0x_1 + x_2 \leq 2 \\ & -x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Putting in Tableau form:

$$\left[\begin{array}{ccccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & RHS \\ \hline 1 & -5 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 4 & 1 & 0 & 0 & 0 & 24 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 6 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

This becomes:

$$\left[\begin{array}{ccccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & s_4 & RHS \\ \hline 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & 0 & 0 & 21 \\ 0 & 1 & 0 & \frac{1}{4} & \frac{-1}{2} & 0 & 0 & 3 \\ 0 & 0 & 1 & \frac{-1}{8} & \frac{3}{4} & 0 & 0 & \frac{3}{2} \\ 0 & 0 & 0 & \frac{1}{8} & \frac{-3}{4} & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{3}{8} & \frac{-5}{4} & 0 & 1 & \frac{5}{2} \end{array}\right]$$

Taking a Gomory Cut based on the basic variable s_3.

$$\begin{gathered} \frac{1}{8}s_1 - \frac{3}{4}s_2 + s_3 = \frac{1}{2} \\ (0 + \frac{1}{8})s_1 + (-1 + \frac{1}{4})s_2 + (1 + \frac{0}{1})s_3 = (0 + \frac{1}{2}) \\ \frac{1}{8}s_1 + \frac{1}{4}s_2 \geq \frac{1}{2} \\ s_1 + 2s_2 \geq 4 \end{gathered}$$

Applying this cut we get:

$$\begin{aligned}
\max \quad & 5x_1 + 4x_2 \\
\text{s.t.} \quad & 6x_1 + 4x_2 + s_1 = 24 \\
& x_1 + 2x_2 + s_2 = 6 \\
& 0x_1 + x_2 \leq 2 \\
& -x_1 + x_2 \leq 1 \\
& s_1 + 2s_2 \geq 4 \\
& x_1, x_2 \geq 0
\end{aligned}$$

This yeilds the solution:Z $= 20$ located at $x_1 = 4, x_2 = 0, s_1 = 0, s_2 = 2$ Note that this is an integer solution and the problem states that two iterations are to be done only "IF NEEDED," as they are not needed they will not be done.

Question 3.

$$\begin{aligned}
\min\ & x_1^2 + x_2^2 + x_3^2 \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 5 \quad (1) \\
& x_1 + x_3 \leq 2 \quad (2) \\
& x_1 \geq 1 \quad (3) \\
& x_2 \geq 2 \quad (4) \\
& x_3 \geq 0 \quad (5)
\end{aligned}$$

Our point to verify is: $x = (1, 2, 0)$ First we can check primal feasibility:

$$\begin{aligned}
&(1) \quad 2(1) + 2 = 4 \leq 4 \\
&(2) \quad 1 + 0 = 1 \leq 2 \\
&(3) \quad 1 \geq 1 \\
&(4) \quad 2 \geq 2 \\
&(5) \quad 0 \geq 0
\end{aligned}$$

Now we can consider dual feasibility: Note that only constraints (3),(4),(5) bind. Checking dual feasibility:

$$\nabla f = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{pmatrix} \quad \nabla g_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad \nabla g_4 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \quad \nabla g_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$

It is obvious that the constraints g are linearly independent, as they are orthogonal.

$$\begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} -u_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -u_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -u_5 \end{pmatrix} = 0$$

These solutions imply that: $u_3 = 2, u_4 = 4, u_5 = 0$. It is clear that the first two constraints do not bind, so $u_1 = u_2 = 0$ Immedietly complementary slackness is satisfied in the first two constraints and the fifth, so we need only verify (3) and (4)

$$\begin{array}{ll} (1) & 2(1 - x_1) = 2(1 - 1) = 0 \\ (2) & 4(2 - x_2) = 4(2 - 2) = 0 \end{array}$$

Since all the u's are all weakly positive, it is a KKT point.

Question 4

Let X_i indicate the number of officers that begin their double shift at time index i.

$$\begin{aligned} &\min\ x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \\ &\text{s.t.} \qquad\qquad\qquad\qquad x_5 + x_0 \geq 8 \\ &\qquad\qquad\qquad\qquad\qquad x_0 + x_1 \geq 7 \\ &\qquad\qquad\qquad\qquad\qquad x_1 + x_2 \geq 6 \\ &\qquad\qquad\qquad\qquad\qquad x_2 + x_3 \geq 6 \\ &\qquad\qquad\qquad\qquad\qquad x_3 + x_4 \geq 5 \\ &\qquad\qquad\qquad\qquad\qquad x_4 + x_5 \geq 4 \\ &\qquad\qquad\qquad\qquad\qquad x_i \in \mathbb{Z}_+ \end{aligned}$$

b

Solving the relaxed Linear Program:

$$\begin{aligned} &\min\ x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \\ &\text{s.t.} \qquad\qquad\qquad\qquad x_5 + x_0 \geq 8 \\ &\qquad\qquad\qquad\qquad\qquad x_0 + x_1 \geq 7 \\ &\qquad\qquad\qquad\qquad\qquad x_1 + x_2 \geq 6 \\ &\qquad\qquad\qquad\qquad\qquad x_2 + x_3 \geq 6 \\ &\qquad\qquad\qquad\qquad\qquad x_3 + x_4 \geq 5 \\ &\qquad\qquad\qquad\qquad\qquad x_4 + x_5 \geq 4 \\ &\qquad\qquad\qquad\qquad\qquad x_i \in \mathbb{R}_+ \end{aligned}$$

This yeilds solution Z = 19 located at x=(2,5,1,5,0,6). Since this has all integer values, it is optimal for the integer program. Again, since branch and bound steps are not necessary, and the problem states "if needed" we will not do any steps.

Question 5.

Firstly we may note that this is an M/M/1/25 Queue with parameters: $\lambda = \frac{1}{2}, \mu = \frac{1}{3}, \rho = \frac{3}{2}$

$$\begin{aligned}
p_0 &= \frac{1-\rho}{1-\rho^{26}} = \frac{2^{25}}{3^{26}-2^{26}} && \approx 0.00001320 \\
p_{25} &= \rho^{25} p_0 = \frac{3^{25}}{3^{26}-2^{26}} && \approx 0.333342 \\
\lambda_{eff} &= \lambda(1-p_{25}) \approx \frac{1}{2}(1-.333342) && \approx 0.3333289 \\
L_s &= \frac{\rho[1-26\rho^{25}+25\rho^{26}]}{(1-\rho)(1-\rho^{26})} && \approx 23.000686 \\
W_s &= \frac{L_s}{\lambda_{eff}} && \approx 69.002977 \\
W_q &= W_s - \frac{1}{\mu} && \approx 66.00297710 \\
L_q &= \lambda_{eff} W_q && \approx 22.0007
\end{aligned}$$

a

$$p_0 \approx 0.00001320$$

b

$$L_q \approx 22.0007$$

c

$$W_s \approx 69.002977$$

d

Now the problem has become an M/M/2/25 Queue.

$$\lambda = \frac{1}{2}, \mu = \frac{1}{3}, \rho = \frac{3}{2}, N = 25, c = 2$$

$$p_0 = (1 + \rho + \frac{\rho^s(1 - \frac{\rho}{c}^{N-c+1})}{c!(1 - \frac{\rho}{c})})^{-1} \approx .14295$$

$$p_{25} = p_0 \frac{\rho^{25}}{c!c^{N-c}} = \frac{3^{25}}{2^{49}} p_0 \approx .00021515$$

$$L_q = \frac{\rho^{c+1}}{(c-1)!(c-\rho)^2} [1 - \frac{\rho}{c}^{N-c+1} - (N-c+1)(1 - \frac{\rho}{c})(\frac{\rho}{c})^{N-c}] p_0 \approx 1.9124$$

$$\lambda_{eff} = (1 - p_{25})\lambda \approx .49989$$

$$W_q = \frac{L_q}{\lambda_{eff}} \approx 3.8256$$

$$W_s = W_q + \frac{1}{\mu} \approx 6.8256$$

Question 6.

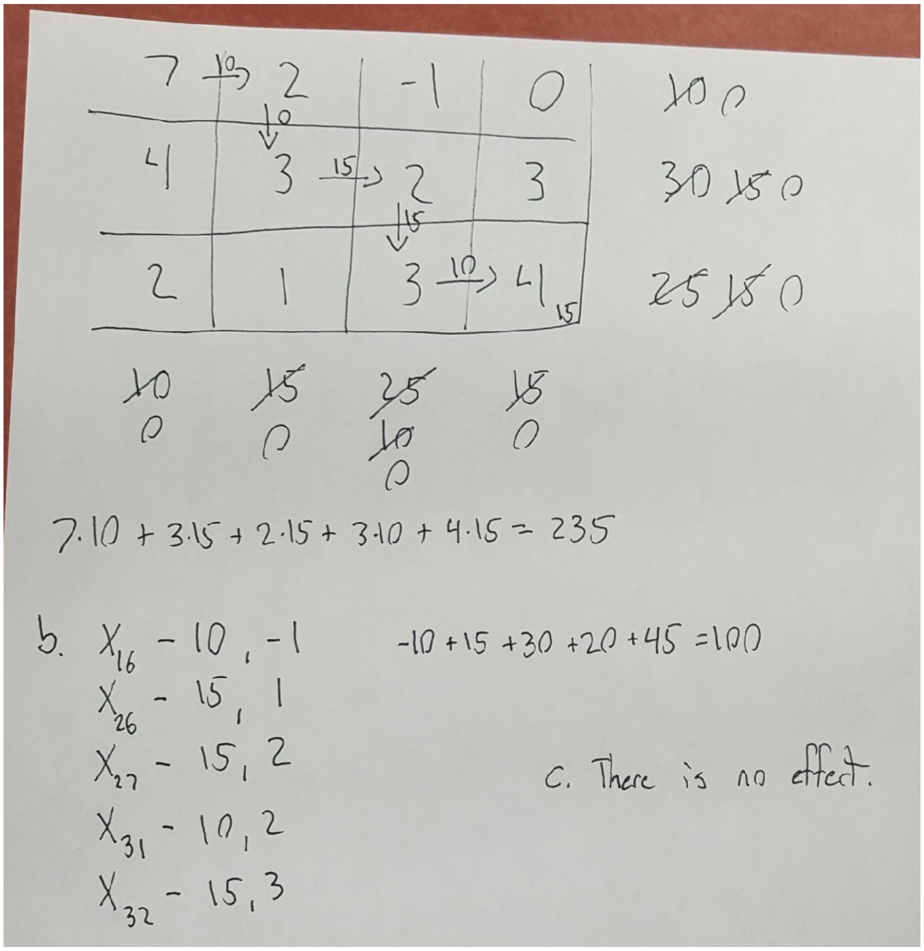

Fig. 1: Question 6