HW1

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Question 1

a.

$$
\begin{aligned}
cov(U_1, U_2) &= \mathbb{E}[(U_1 - \mu_1)(U_2 - \mu_2)] \\
\mathbb{E}[U_1U_2 - \mu_1U_2 - \mu_2U_1 + \mu_1\mu_2] \\
\mathbb{E}[U_1]\mathbb{E}[U_2] - \mu_1\mathbb{E}[U_2] - \mu_2\mathbb{E}[U_1] + \mu_1\mu_2 \\
\mu_1\mu_2 - \mu_1\mu_2 - \mu_2\mu_1 + \mu_1\mu_2 = 0
\end{aligned}
$$

b

No, while zero correlation is implied by independence, the reverse is not implied. Consider the example of The random variable X which is uniform on the interval (-1,1), and Y which is given by X^2. We can see that $Cov(X,Y) = \mathbb{E}[X(Y - \mu_Y)] = \mathbb{E}[XY] - \mu_Y\mathbb{E}[X] = \mathbb{E}[XY] = 0$ as they are symmetric. However, we can clearly see that X and Y are not independent, as Y is determined completely by X.

c

$$
\begin{aligned}
S'(\mu) = \sum_{n=1}^{N} 2(Y_n - \mu)(-1) = 0 \\
\sum_{n=1}^{N} Y_n - N\mu = 0 \\
\mu = \frac{1}{N}\sum_{n=1}^{N} Y_n
\end{aligned}
$$

Verifying that it is a minimum:

$$S''(\mu) = 2N > 0$$

Since this is positive, our value of μ is a minimum.

d

$$\mathbb{E}[B] = \mathbb{E}[\sum_{n=1}^{N} k_n Y_n]$$
$$\sum_{n=1}^{N} k_n \mathbb{E}[Y_n] = \sum_{n=1}^{N} k_n \mu_n$$
$$Var(B) = Var(\sum_{n=1}^{N} k_n Y_n) =$$
$$\sum_{n=1}^{N} k_n^2 Var(Y_n) + 2\sum_{i<j}\sum k_i k_j Cov(Y_i, Y_j) =$$
$$\sum_{n=1}^{N} k_n^2 \sigma^2 + \sum_{i<j}\sum k_i k_j \sigma^2$$

Since B is a linear combination of jointly normal random variables, it is normally distributed, with mean and variance given above.

e

i

Note that since Y_n is normally distributed, and $\bar{Y_N}$ is a linear combination at Y_n, thus $\bar{Y_N}$ is normally distributed. By d, we see that its mean is μ and its variance is σ^2. Thus
$$\frac{\bar{Y_N} - \mu}{\sigma} \sim N(0, 1)$$

ii

First we note that:

$$\frac{\sum_{n=1}^{N}(Y_n - \bar{Y_N})^2}{\sigma^2} = \frac{\sum_{n=1}^{N}(Y_n - \mu + \mu - \bar{Y_N})^2}{\sigma^2}$$
$$\frac{\sum_{n=1}^{N}((Y_n - \mu) - (\bar{Y_N} - \mu))^2}{\sigma^2} = \frac{\sum_{n=1}^{N}((Y_n - \mu)^2 - (\bar{Y_n} - \mu)^2)}{\sigma^2}$$
$$\sum_{n=1}^{N}(\frac{Y_n - \mu}{\sigma})^2 - N\frac{(\bar{Y_N} - \mu)}{\sigma}^2$$

Since $\frac{Y_n - \mu}{\sigma}$ is distributed $\mathcal{N}(0, 1)$, its square is distributed $\chi^2(1)$, and $\sum_{n=1}^{N}(\frac{Y_n - \mu}{\sigma})^2 \sim \chi^2(N)$. We can also note that $(\frac{\bar{Y_N} - \mu}{\sigma})^2 \sim N(0, \frac{1}{N})$. So multiplying it by $\sqrt{N}$

should ensure that it is distributed $\mathcal{N}(0,1)$. Thus $N(\frac{\bar{Y_N}-\mu}{\sigma})^2 \sim \chi^2(1)$. The difference between two χ^2 is χ^2 itself so:

$$\frac{\sum_{n=1}^{N}(Y_n - \bar{Y_N})^2}{\sigma^2} \sim \chi^2(N-1)$$

iii

$$\frac{\sqrt{N}(\bar{Y_N}-\mu)}{\sqrt{\frac{\sum_{n=1}^{N}(Y_n-\bar{Y_N})^2}{N-1}}} = \sqrt{N}\frac{(\bar{Y_N}-\mu)}{\sigma}\sqrt{\frac{\sigma^2(N-1)}{\sum_{n=1}^{N}(Y_n-\bar{Y_N})^2}}$$

We may note that this can be written in the form of:

$$\frac{Z}{\sqrt{\frac{\chi^2(N-1)}{N-1}}}$$

since it is a normal distribution divided by the square root of an independent chi-squared divided by its degrees of freedom, this is a t-distribution with N-1 degrees of freedom.

f

i

This point estimator, $\bar{Y_N}$ converges by the law of large numbers to $\mathbb{E}[Y] = \theta$.

ii

Since $Y \sim$ Bernoulli, $\mathbb{E}[Y] = \theta, Var(Y) = \theta(1-\theta)$ This tells us that $\mathbb{E}[\bar{Y_N}] = \theta, Var(\bar{Y_N}) = \frac{\theta(1-\theta)}{N}$. We may rewrite $Z_N = \frac{(\bar{Y_N}-\mathbb{E}[\bar{Y_N}])}{\sqrt{Var(\bar{Y_N})}}$ We may apply the DeMoivre-Laplace Central Limit theorem, which tells us that Z_N converges in distribution to $\mathcal{N}(0,1)$. This is in contrast to $\bar{Y_N}$ which converges to a point.