Behavioral Homework 3

Timothy Schwieg

Question 1

$$F_V(v) = v \quad \forall v \in [0,1]$$
$$f_v(v) = 1 \quad \forall v \in [0,1]$$

a

$$\sigma(v) = v - \frac{\int_0^v w^{N-1} dw}{v^{N-1}}$$
$$\sigma(v) = v - \frac{\frac{w^N}{N}\Big|_0^v}{v^{N-1}}$$
$$\sigma(v) = v - \frac{v^N}{N v^{N-1}} = \frac{(N-1)v}{N}$$

b

Since Y is the second order statistic coming from an iid distribution, it is known that its distribution function is given by:

$$f_Y = N(N-1)F_V(y)^{N-2}(1 - F_v(y))f_v(y)$$
$$f_Y = N(N-1)v^{N-2}(1-v)$$

c

$$
\begin{aligned}
F_Y(y) = \int_0^y f_Y(v)dv = \int_0^y N(N-1)v^{N-2}(1-v)dv \\
N(N-1)\big[\int_0^y v^{N-2}dv - \int_0^y v^{N-1}dv\big] \\
N(N-1)\big(\frac{y^{N-1}}{N-1} - \frac{y^N}{N}\big) \\
N(N-1)\big(\frac{y^{N-1}N}{N(N-1)} - \frac{y^N(N-1)}{N(N-1)}\big) \\
y^{N-1}(N - yN + y) \\
Ny^{N-1} - y^N(N-1)
\end{aligned}
$$

d

$$
\begin{aligned}
\mathbb{E}[Y] = \int_0^1 y f_Y(y)dy = \int_0^1 N(N-1)y^{N-1}(1-y)dy \\
N(N-1)(\int_0^1 y^{N-1}dy - \int_0^1 y^N dy) = N(N-1)\big(\frac{1}{N} - \frac{1}{N+1}\big) \\
N(N-1)\big(\frac{N+1}{N(N+1)} - \frac{N}{N(N+1)}\big) = \frac{N-1}{N+1}
\end{aligned}
$$

e

$$\mathbb{E}[Y^2] = \int_0^1 y^2 f_Y(y)dy = \int_0^1 N(N-1)y^N(1-y)dy$$
$$N(N-1)(\int_0^1 y^N dy - \int_0^1 y^{N+1}dy) = N(N-1)\big(\frac{1}{N+1} - \frac{1}{N+2}\big)$$
$$N(N-1)\big(\frac{N+2}{(N+2)(N+1)} - \frac{N+1}{(N+2)(N+1)}\big) = \frac{N(N-1)}{(N+1)(N+2)}$$

$$\mathbb{V}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2$$
$$\mathbb{V}(Y) = \frac{N(N-1)}{(N+1)(N+2)} - \frac{(N-1)^2}{(N+1)^2}$$
$$\frac{N(N-1)(N+1)}{(N+1)^2(N+2)} - \frac{(N+2)(N-1)^2}{(N+1)^2(N+2)}$$
$$\frac{2N-2}{(N+1)^2(N+2)}$$

f

$$F_Z(z) = F_V(z)^N = z^N \forall v \in [0,1]$$

g

$$f_Z(z) = \frac{\partial}{\partial z}F_Z(z) = N(z^{N-1})$$

h

$$\mathbb{E}[Z] = \int_0^1 z f_Z(z) = N\int_0^1 z^N dz = \frac{N}{N+1}$$

i

$$\mathbb{E}[Z^2] = \int_0^1 Nz^{N+1} = \frac{N}{N+2}z^{N+2}\big|_0^1 = \frac{N}{N+2}$$
$$Var(Z) = \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 = \frac{N}{N+2} - (\frac{N}{N+1})^2 =$$
$$\frac{N(N+1)^2}{(N+2)(N+1)^2} - \frac{N^2(N+2)}{(N+2)(N+1)^2} = \frac{N}{(N+2)(N+1)^2}$$

j

First we must verify that $\sigma(v)$ is a monotonic transformation.

$$\frac{\partial}{\partial v}\sigma(v) = \frac{N-1}{N} > 0$$

Since it is a monotonic transformation, as the probability that the first order statistic is zero is 0 almost surely, we may proceed with the method of transformations for finding the pdf of W.

$$\begin{aligned}\sigma^{-1}(w) &= \frac{N}{N-1}w \\ \frac{\partial\sigma^{-1}}{\partial w} &= \frac{N}{N-1} \\ f_W(w) &= f_{V_{(1:N)}}(\sigma^{-1}(w))\Big|\frac{\partial\sigma^{-1}}{\partial w}\Big| \\ f_W(w) &= N(\frac{N}{N-1}w)^{N-1}\frac{N}{N-1} \\ f_W(w) &= \frac{N^{N+1}w^{N-1}}{(N-1)^N}\end{aligned}$$

k

$$\begin{aligned}\mathbb{E}[W] = \mathbb{E}[\frac{(N-1)v}{N}] &= \frac{N-1}{N}\mathbb{E}[V_{(1:N)}] \\ \frac{N-1}{N}\frac{N}{N+1} &= \frac{N-1}{N+1}\end{aligned}$$

l

$$\begin{aligned}\mathbb{V}(W) = \frac{(N-1)^2}{N^2}\mathbb{V}(V_{(1:N)}) &= \\ \frac{(N-1)^2}{N^2}\frac{N}{(N+2)(N+1)^2} &= \frac{(N-1)^2}{N(N+2)(N+1)}\end{aligned}$$

Question 2

a

Logically, the bidder will wish to not place any bid if his valuation is below the reserve price, so we shall assume that the new bids are uniform on the interval [r,1] and there will be M = $\sum_{n=1}^{N} 1_{v_n \geq r}$ participants. This will be considered this exogenous.

$$\begin{aligned}
\max \mathbb{E}[(v-s)P(win|s)] \\
\max(v-s)F_v(\sigma^{-1}(s_m))^{M-1} \\
\max(v-s)(\frac{\sigma^{-1}(s_m)-r}{1-r})^{M-1} \\
(v-s)(M-1)(\frac{\sigma^{-1}(s_m)-r}{1-r})^{M-2}\frac{1}{1-r}\frac{1}{\sigma'(v)} - (\frac{\sigma^{-1}(s_m)-r}{1-r})^{M-1} = 0 \\
(v-\sigma(v))(M-1)(\frac{v-r}{1-r})^{M-2}(\frac{1}{1-r})\frac{1}{\sigma'(v)} - (\frac{v-r}{1-r})^{M-1} = 0 \\
\sigma'(v)(\frac{v-r}{1-r})^{M-1} + \sigma(v)(\frac{v-r}{1-r})^{M-2}(M-1)\frac{1}{1-r} = v(M-1)(\frac{v-r}{1-r})^{M-2}\frac{1}{1-r} \\
\sigma'(v) + \sigma(v)\frac{M-1}{v-r} = \frac{v(M-1)}{v-r} \text{ Applying } \mu = \exp(\int \frac{M-1}{v-r}) = (v-r)^{M-1} \\
(\sigma(v)(v-r)^{M-1})' = v(M-1)(v-r)^{M-2} \\
\sigma(v)(v-r)^{M-1} = v(v-r)^{M-1} - \frac{(v-r)^M}{M} + C \\
\sigma(v) = \frac{M-1}{M}v + \frac{r}{M} + C(v-r)^{1-M} \\
\sigma(v) = \frac{M-1}{M}v + \frac{r}{M}
\end{aligned}$$

b

Since M $= \sum_{n=1}^{N} 1_{v_n \geq r}$ we can see that it is the sum of bernoulli random variables and thus is binomial. The probability that each event occurs is $P(v_n \geq r) = 1 - F_V(r)$. Thus $M \sim binom(N, 1 - F_V(r))$.

c

The optimal reserve price can be found by solving the equation

$$\begin{aligned}
r^* &= v^0 - \frac{1-F_V(r^*)}{f_V(r^*)} \\
r^* &= v^0 + \frac{1-r^*}{1} \\
2r^* &= v^0 + 1 \\
r^* &= \frac{v^0+1}{2}
\end{aligned}$$