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Game Theory Primer

Much of the modeling that economists do involves, in some form or another, perfectly-competitive markets. A variety of reasons for this decision exists, some having to do with the reality of actual markets, others having to do with computational simplicity. Economic agents in models of perfectly-competitive markets have no power, are atoms. Consequently one need not worry that the decisions of any one agent will directly affect those of the others. For, even though in equilibrium the decisions of all agents will be aggregated, each agent's decision makes up a relatively small portion of the total. Thus, from an analytic perspective, examining behavior within the perfectly-competitive paradigm is attractive.

Obviously the simplifying assumption of perfect competition need not apply in all situations. In some circumstances economists assume alternatively that one agent has all of the power, as in the case of a monopolist. While the simplifying assumption of monopoly can be useful in certain situations, it clearly does not apply in all either. What to do?

Game theory is concerned with the analysis of strategic interaction when all agents involved have some power, often situations in which a small number of agents exists. Game theory was invented by the polymath John von Neumann and the economist Oskar Morgenstern; their findings were published by Princeton University Press in 1944 in a famous book entitled *Theory of Games and Economic Behavior*.

As developed by von Neumann and Morgenstern, two branches of game theory exist: 1) the *strategic* (or *noncoöperative*) approach in which the analyst specifies in detail what the economic agents (often referred to as "players") in the game can and cannot do, and then devises what is optimal for each player to do; and 2) the *coalitional* (or *coöperative*) approach in which the analyst describes optimal behavior in games that often have many (but not necessarily infinitely many) players. In these notes we shall focus on noncoöperative game theory. In a separate handout, entitled "Bargaining," we shall consider the analysis of a particular problem in coöperative game theory.

1. Noncoöperative Game Theory

Within the theory of noncoöperative games two branches exist, one involving the study of games in *normal* form and the other involving the study of games in *extensive* form. Because normal-form games are typically easier to define and to analyze than extensive-form games, we shall begin by investigating them.

1.1. Vocabulary and Notation

Before analyzing some classic normal-form games we introduce a vocabulary and some notation to make concrete the concepts and notions on which any analysis of strategic interaction rests. The essential ingredients of any noncoöperative game include the following:

1) a set of decision makers, players, $\mathcal{P}$;
2) a description of the rules of the game $\mathcal{R}$;
3) a set of feasible actions, strategies, for each player $\mathcal{S}$;
4) an information structure describing what is known to each player and when this is known $\mathcal{I}$;
5) a payoff function for each player which maps the strategies of all players into the payoff for that player $\mathcal{U}$;
6) a concept of equilibrium.

We shall discuss each of these in turn.

1.1.1. Players

In most situations the set of players is obvious. In the analysis of strategic behavior within the family $\mathcal{P}$ would be $\{\text{husband}, \text{wife}, \text{child}_1, \text{child}_2, \ldots\}$. In the analysis of union-firm interactions $\mathcal{P}$ would be $\{\text{union}, \text{firm}\}$. In applications of game theory to the study of industrial organization the firms in a particular industry would be contained in $\mathcal{P}$.

1.1.2. Rules

The rules of the game $\mathcal{R}$ are typically dictated by the legal reality of the situation. For example, at auctions the type of bidding (*e.g.*, lump-sum or per-unit) imposed by the seller is one rule of the game. Collective bargaining is a legal feature of games played between firms and unions. The rules of the game are typically important in determining the strategies available to the players. In some of the economics literature the rules of the game are choice variables as in the literature concerned with mechanism design. This literature examines the optimal way in which to set up systems when strategic behavior on the part of a small number of economic actors is important.

1.1.3. Strategies

The strategy sets of players $\mathcal{S}$ are dictated as much by the rules of the game as by the physical reality of a situation. For example, firms often can only produce non-negative amounts of output, workers can only supply non-negative amounts of labor, production functions determine feasible input-output combinations, *etc.*.

1.1.4. Information

One of the most important aspects of games is information $\mathcal{I}$: who knows what when is critical to solving games. Two branches exist. The easiest to understand and to use involves games of *complete* information in which each player knows everything about anything relevant to the game being played. We shall investigate games of complete information first and then examine one particular game of *incomplete* information later in this handout.

1.1.5. Payoffs

The payoff functions $\mathcal{U}$ (often just referred to as the "payoffs") map the strategies of all players into monetary measures for the players. A number of ways exist in which to represent payoffs. We shall examine a few in this handout, but leave others for future study.

1.1.6. Equilibrium

The notion of equilibrium behavior is a principal feature that distinguishes economics from other disciplines in the social sciences. Roughly speaking equilibrium obtains when things settle down. We shall examine two equilibrium concepts.

1.2. Some Examples of Games in Normal Form

To make concrete the structure of normal-form games in this subsection we shall introduce the rules of and payoffs in some classic games. While the descriptions of these games may appear quite stylized and specific, the games presented below have been widely and successfully used to investigate a variety of different phenomena by researchers in economics as well as in other disciplines.

1.2.1. Prisoner's Dilemma

Perhaps the most well-known game is the "Prisoner's Dilemma." In one description of this game two individuals have been apprehended. The police "know" the two have committed a crime, but prosecuting them will be made easy if at least one of them confesses to their joint crime. The police isolate the two suspects in separate rooms and question each without the other's knowing what his partner is saying.

In this game the players are the prisoners; denote them by 1 and 2 so $\mathcal{P} = \{1, 2\}$. The strategies available are to confess (C) or to remain quiet (Q) so $\mathcal{S} = \{\mathsf{C}, \mathsf{Q}\}$. Suppose that if both confess then each will go to prison for a medium-length term, while if both remain silent then there is a good chance (although it is not a certainty) that neither will be jailed. On the other hand, if 1 confesses when 2 remains silent then 1 will be given immunity while 2 will be given a harsh sentence and *vice versa*.

In order to make game theory operational the analyst must be able to represent the payoffs to various strategies by real numbers. However, these real numbers need not be unique. For example, if the pair of numbers (u_i^1, u_j^2) denote the payoffs to 1 and 2 of their strategies $(\mathsf{S}_i^1, \mathsf{S}_j^2)$, then so does $(u_i^1 - 9, u_j^2 - 9)$, provided all other payoffs have nine subtracted from them. Thus, while the payoffs described below may seem arbitrarily chosen, the framework is relatively flexible.

A common way to represent payoffs is to tabulate the numerical values in a matrix which is often referred to as the "payoff matrix." One representation of the payoffs in the above game is

$$
\begin{array}{ccc}
 & & \text{Suspect 2} \\
 & & \begin{array}{cc} \mathsf{C} & \mathsf{Q} \end{array} \\
\text{Suspect 1} & \begin{array}{c} \mathsf{C} \\ \mathsf{Q} \end{array} & \begin{pmatrix} (-2,-2) & (\ \ 0,-4) \\ (-4,\ \ 0) & (-1,-1) \end{pmatrix}.
\end{array}
$$

Here the first term in parentheses in any cell is the payoff to suspect 1 from choosing the strategy in that row of the matrix, while the second term in parentheses in any cell is the payoff to suspect 2 from choosing the strategy in that column of the matrix. Note that the following payoff matrix:

$$
\begin{array}{ccc}
 & & \text{Suspect 2} \\
 & & \begin{array}{cc} \mathsf{C} & \mathsf{Q} \end{array} \\
\text{Suspect 1} & \begin{array}{c} \mathsf{C} \\ \mathsf{Q} \end{array} & \begin{pmatrix} (2,2) & (4,0) \\ (0,4) & (3,3) \end{pmatrix},
\end{array}
$$

which has 4 added to each element of the previous payoff matrix, would be an equivalent representation of the payoffs.

1.2.2. Coördination Games

Another interesting normal-form game is the "Coördination Game" for which a number of useful applications exists. For example, suppose we have two drivers who live on an island with one road. Each can drive either on the left-hand side (L) or on the right-hand side (R) of the road. If both drive on either the left or the right, then no crashes will likely occur and each can drive safely, but if one chooses left and the other chooses right, then they risk a crash when driving in opposite directions.

One representation of the payoff matrix of the above game is

$$
\begin{array}{ccc}
 & & \text{Driver 2} \\
 & & \begin{array}{cc} \mathsf{L} & \mathsf{R} \end{array} \\
\text{Driver 1} & \begin{array}{c} \mathsf{L} \\ \mathsf{R} \end{array} & \begin{pmatrix} (2,2) & (0,0) \\ (0,0) & (2,2) \end{pmatrix}.
\end{array}
$$

Another example of a Coördination Game involves the choice of an operating system for a personal computer. For personal computers having the *Intel* chip at least two alternative operating systems exist: Microsoft's *Windows* (W) and a Unix-like system known generically as *Linux*. Now *Linux* (L) is a better operating system than *Windows*, but more people use *Windows* than use *Linux*, so there are advantages to having a technology which is compatible with other personal-computer users.

A personal-computer user 1 who is simultaneously making a choice with a representative user from the rest of the industry 2 might face the following payoff matrix:

$$\text{User 1} \quad \begin{array}{c} \\ \mathsf{L} \\ \mathsf{W} \end{array} \!\!\begin{array}{c} \text{Industry 2} \\ \begin{array}{cc} \mathsf{L} & \mathsf{W} \end{array} \\ \begin{pmatrix} (4,4) & (1,3) \\ (3,1) & (2,2) \end{pmatrix} \end{array}.$$

1.2.3. Matching Pennies

Game theory grew out of people's desire to make better decisions in strategic situations. Parlor games are simple examples of strategic situations involving only a few decision makers where the behavior of others can greatly influence any particular player's rewards. Thus parlor games have been studied extensively by game theorists. One commonly-studied parlor game is called "Matching Pennies." In this game player 1 gets a dollar from player 2 if each chooses heads $(\mathsf{H}^1, \mathsf{H}^2)$ or each chooses tails $(\mathsf{T}^1, \mathsf{T}^2)$ where the superscripts on the letters H (for heads) and T (for tails) denote the strategy of players 1 and 2, respectively. If the pennies do not match, so either $(\mathsf{H}^1, \mathsf{T}^2)$ or $(\mathsf{T}^1, \mathsf{H}^2)$ obtain, then player 1 pays player 2 a dollar. The payoff matrix for this game is then

$$\text{Player 1} \quad \begin{array}{c} \\ \mathsf{H} \\ \mathsf{T} \end{array} \!\!\begin{array}{c} \text{Player 2} \\ \begin{array}{cc} \mathsf{H} & \mathsf{T} \end{array} \\ \begin{pmatrix} (\ \ 1,-1) & (-1,\ \ 1) \\ (-1,\ \ 1) & (\ \ 1,-1) \end{pmatrix} \end{array}.$$

1.3. Equilibrium Concepts

In the games described above perhaps the most interesting question to ask is "What will happen?" To be specific, given the set of players $\mathcal{P} = \{1, 2\}$ who operate under the rules $\mathcal{R}$, having strategy sets $\mathcal{S}^i = \{\mathsf{S}_1^i, \mathsf{S}_2^i, \ldots\}$ for $i \in \mathcal{P}$, facing information $\mathcal{I}$, and receiving the payoffs $\mathcal{U} = \{[u_1(\mathsf{S}_i^1, \mathsf{S}_j^2), u_2(\mathsf{S}_i^1, \mathsf{S}_j^2)]\}_{\mathsf{S}_i^1 \in \mathcal{S}_1, \mathsf{S}_j^2 \in \mathcal{S}_2}$, if economic agents act purposefully in their own self interest, then which outcomes will obtain in equilibrium? To answer questions like these we need to have some concept of equilibrium. Two common equilibrium concepts are called "dominant" and "Nash," but others exist too.

To illustrate how to find equilibria in normal-form games in particular and noncoöperative games in general we first introduce the "Best Response" correspondence $\mathcal{BR}(\cdot)$; a correspondence is like a function so we shall often refer to $\mathcal{BR}(\cdot)$ as the "best-response function." Suppose player 2 has adopted strategy S_j^2 from the strategy set $\mathcal{S}^2$. We denote player 1's best response $\mathcal{BR}^1(\mathsf{S}_j^2)$ by $\hat{\mathsf{S}}^1$ which must be contained in his strategy set $\mathcal{S}^1$.

1.3.1. Dominant Strategies

An equilibrium strategy is a *dominant* strategy if a player would choose this strategy regardless of the strategies chosen by the other players. In terms of the best-response function, the strategy $\hat{\mathsf{S}}^1$ is a dominant one if $\mathcal{BR}^1(\mathsf{S}_j^2)$ is the same $\hat{\mathsf{S}}^1$ for any $\mathsf{S}_j^2 \in \mathcal{S}^2$.

To illustrate a dominant strategy consider the Prisoner's Dilemma game above: Start with player 1 and focus on the first column of the payoff matrix. When player 2 plays C player 1 should play C since -2 is better than -4. Still consider player 1, but now focus on the second column of the payoff matrix. When player 2 plays S player 1 should play C since 0 is better than -1.

Now start with player 2 and focus on the first row of the payoff matrix. When player 1 plays C player 2 should play C since -2 is better than -4. Still consider player 2, but now focus on the second row of the payoff matrix. When player 1 plays S player 2 should play C since 0 is better than -1.

Thus the dominant-strategy equilibrium to this formulation of the Prisoner's Dilemma game is $(\mathsf{C}^1, \mathsf{C}^2)$, both suspects confess. Notice that purposeful, self-interested behavior on the part of each agent leads to an equilibrium at which both are worse off than they would be were they to coöperate, remain quiet, choose $(\mathsf{Q}^1, \mathsf{Q}^2)$! This is a striking conclusion, especially in light of the First Welfare Theorem in the perfectly-competitive paradigm where purposeful, self-interested behavior leads to a Pareto-efficient allocation.

1.3.2. Nash Equilibrium

Dominant strategies are rare in normal-form games in particular and in any game in general. In fact, neither of the other two games considered above has a dominant-strategy equilibrium. (Check this claim.) Do other types of equilibrium exist?

For normal-form games with a finite number of players each having a finite number of strategies, the economics Nobel prize-winning mathematician John F. Nash Jr. has proven that a particular type (weaker form) of equilibrium (now called the "Nash equilibrium" which we shall abbreviate by NE) exists. This result is stated in the following remarkable and useful theorem:

> *Nash's Theorem:* Every normal-form game with a finite number of players each having a finite number of strategies has at least one NE.

Such an "existence theorem" is important because, if an analyst is going to embark on a search for the NEs of normal-form games, it would be handy if they existed. Nash's theorem guarantees they do.

But what is a NE and how can we find one? A NE is a best response to the best response of a player's opponent(s). In terms of a two-player game, a pair of strategies $(\hat{\mathsf{S}}^1, \hat{\mathsf{S}}^2)$ is a NE if the following holds:

$$\hat{\mathsf{S}}^1 = \mathcal{BR}^1(\hat{\mathsf{S}}^2)$$
$$\hat{\mathsf{S}}^2 = \mathcal{BR}^2(\hat{\mathsf{S}}^1).$$

In words, $\hat{\mathsf{S}}^1$ is player 1's best response to $\hat{\mathsf{S}}^2$ and $\hat{\mathsf{S}}^2$ is player 2's best response to $\hat{\mathsf{S}}^1$. Thus we have a constructive way of deciding whether any pair of strategies $(\mathsf{S}^1_i, \mathsf{S}^2_j)$ is, in fact, a NE. Finding the NEs to a normal-form game simply involves examining the best-response functions of the players in each cell of the payoff matrix. For large games this can be a tedious exercise, but it is guaranteed to converge to an answer because only a finite number of cells exists.

Let's try this out on the games we specified above. Consider again the Prisoner's Dilemma game. Note that

$$\mathsf{C}^1 = \mathcal{BR}^1(\mathsf{C}^2)$$

$$\mathsf{C}^2 = \mathcal{BR}^2(\mathsf{C}^1),$$

so $(\mathsf{C}^1, \mathsf{C}^2)$ is a dominant-strategy NE.

Consider next the first Coördination Game. Start with player 1 and focus on the first column of the payoff matrix. When player 2 plays L player 1 should play L since 2 is better than 0. Still consider player 1, but now focus on the second column of the payoff matrix. When player 2 plays R player 1 should play R since 2 is better than 0.

Now start with player 2 and focus on the first row of the payoff matrix. When player 1 plays L player 2 should play L since 2 is better than 0. Still consider player 2, but now focus on the second row of the payoff matrix. When player 1 plays R player 2 should play R since 2 is better than 0.

Thus two NEs exist: $(\mathsf{L}^1, \mathsf{L}^2)$ and $(\mathsf{R}^1, \mathsf{R}^2)$. This may not be all that surprising since in North America as well as in most of Europe people drive on the right-hand side of the road while in Australia, New Zealand, Japan, and the United Kingdom people drive on the left-hand side of the road, typically without incident. Obviously the peoples in these lands have "coördinated" on a convention. Given the payoffs in this game it is irrelevant which pair they choose. Such is not the case in the second Coördination Game.

What are the NEs of the second Coördination Game? Start with player 1 and focus on the first column of the payoff matrix. When player 2 plays L player 1 should play L since 4 is better than 3. Still consider player 1, but now focus on the second column of the payoff matrix. When player 2 plays W player 1 should play W since 2 is better than 1.

Now start with player 2 and focus on the first row of the payoff matrix. When player 1 plays L player 2 should play L since 4 is better than 3. Still consider player 2, but now focus on the second row of the payoff matrix. When player 1 plays W player 2 should play W since 2 is better than 1.

Again two NEs exist: $(\mathsf{L}^1, \mathsf{L}^2)$ and $(\mathsf{W}^1, \mathsf{W}^2)$, but it is relevant which pair is chosen. When $(\mathsf{W}^1, \mathsf{W}^2)$ is chosen everyone is worse off than when $(\mathsf{L}^1, \mathsf{L}^2)$ is chosen. Moreover, no individual incentive exists to move to the Pareto-superior equilibrium $(\mathsf{L}^1, \mathsf{L}^2)$ if $(\mathsf{W}^1, \mathsf{W}^2)$ has been chosen initially. This too is a striking result because it is a counter-example to Adam Smith's "Invisible Hand" which is supposed to guide Mankind to Pareto-superior allocations.

What about the game of Matching Pennies? Well, if you go through each cell of the payoff matrix you will find that none of the cells is a NE. But doesn't Nash's theorem say that at least one NE must exist? Have we found a counter-example to Nash's theorem? Alas, two kinds of Nash strategies exist: pure and mixed. For the game Matching Pennies no pure-strategy NE exists, but a mixed-strategy NE does. What are pure and mixed strategies?

1.3.2.1. Pure Strategies

A pure strategy is a rule like "always play H." A pure strategy is deterministic; it never changes.

1.3.2.2. Mixed Strategies

A mixed strategy is a rule like "play H with probability one third and play T with probability two thirds." It is random, depending on the toss of an appropriately-weighted coin or a die having certain probabilistic characteristics.

The key insight to mixing in games like Matching Pennies is that a deterministic strategy can be defeated by an appropriately-chosen strategy by a player's opponent(s). If player 1 is certain to choose H^1, then player 2 need only choose T^2 to be certain of winning. Thus a player wants to choose a strategy with some probability. By behaving randomly each player puts his opponent(s) on edge.

But how should the probability of a strategy S_i^1 for player 1 given the strategy of player S_j^2 for player 2 be chosen? To begin let us introduce the notation $\sigma^i(\mathsf{S}_j)$ to denote the probability that player i chooses strategy S_j. In the Matching Pennies game only two possibilities exist H and T so, by the theorem of total probability, $\sigma^i(\mathsf{H}^i) + \sigma^i(\mathsf{T}^i)$ equals one or $\sigma^i(\mathsf{T}^i)$ equals $[1 - \sigma^i(\mathsf{H}^i)]$ where $0 \leq \sigma^i(\mathsf{H}^i),\ \sigma^i(\mathsf{T}^i) \leq 1$. To reduce the number of letters written, collect all of the probabilities for all of the players in the letter $\boldsymbol{\sigma}$.

Now player 1 in choosing $\sigma^1(\mathsf{H}^1)$ and $\sigma^1(\mathsf{T}^1)$ should be indifferent between the *expected* (average) return to the strategies chosen by player 2 which are $\sigma^2(\mathsf{H}^2)$ and $\sigma^2(\mathsf{T}^2)$, and *vice versa*. Calculating the mixed-strategy NE is often aided by graphing the best-response functions of the players.

In Figure 1 is presented a graph with H^1 and T^1 on the axis of abscissae. When H^1 is chosen $\sigma^1(\mathsf{T}^1)$ is zero while $\sigma^1(\mathsf{H}^1)$ is one. On the other hand when T^1 is chosen $\sigma^1(\mathsf{H}^1)$ is zero while $\sigma^1(\mathsf{T}^1)$ is one. Any number on the axis of abscissae between H^1 and T^1 is between zero and one, the farther to the right, the lower is $\sigma^1(\mathsf{H}^1)$ and the higher is $\sigma^1(\mathsf{T}^1)$.

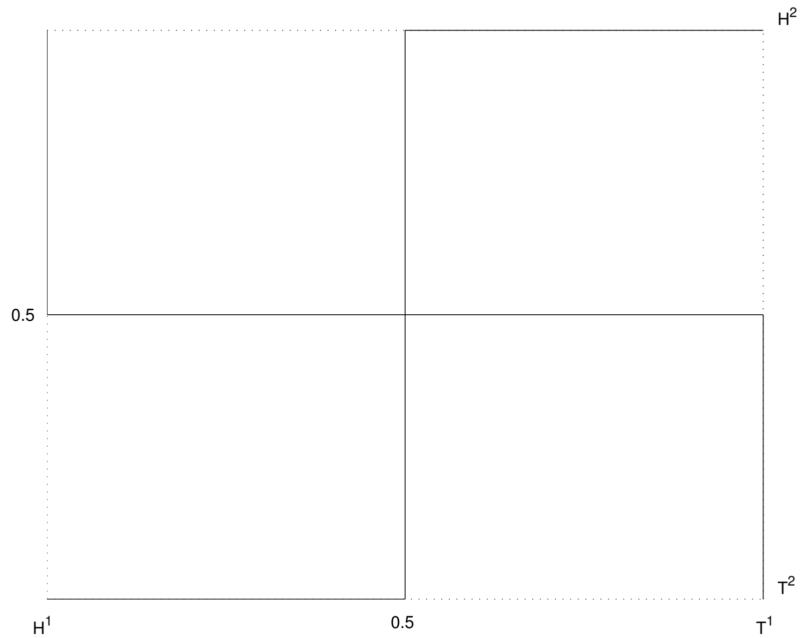
Graphed on the axis of ordinates and at the far right of Figure 1 are H^2 and T^2. When H^2 is chosen $\sigma^2(\mathsf{T}^2)$ is zero while $\sigma^2(\mathsf{H}^2)$ is one. Similarly, when T^2 is chosen $\sigma^2(\mathsf{H}^2)$ is zero while $\sigma^2(\mathsf{T}^2)$ is one. Any number on the axis of ordinates between H^2 and T^2 is between zero and one, the farther down, the lower is $\sigma^2(\mathsf{H}^2)$ and the higher is $\sigma^2(\mathsf{T}^2)$.

Now consider the optimal response of player 1, conditional on a particular strategy by player 2. Notice that if $\sigma^2(\mathsf{H}^2)$ is greater than one half, player 1 should always play H^1 while if $\sigma^2(\mathsf{H}^2)$ is less than one half, player 1 should always play T^1. On the other hand, if $\sigma^1(\mathsf{H}^1)$ is greater than one half, player 2 should always play T^2 while if $\sigma^1(\mathsf{H}^1)$ is less than one half, player 2 should always play H^1. For either player a $\sigma(\mathsf{H})$ of one half is optimal. Thus the intersection of the best-response functions characterizes the mixing proportions: the mixed-strategy NE for the Matching Pennies game is for both players to choose Hs and Ts with probability one half.

1.4. Some Examples of Games in Extensive Form

Extensive-form games are richer in structure than normal-form games. Normal-form games highlight the interaction between strategies and payoffs as well as the fact that the "best"

Figure 1
Solving for the Mixed Strategies in a Matching Pennies Game



outcome for all players need not be an equilibrium. Extensive-form games do this and more. Whereas normal-form games are either explicitly or implicitly simultaneous-move games, extensive-form games admit sequential play and thus can be dynamic in nature. Extensive-form games are also rich enough to accomodate *incomplete* information on the part of the players.

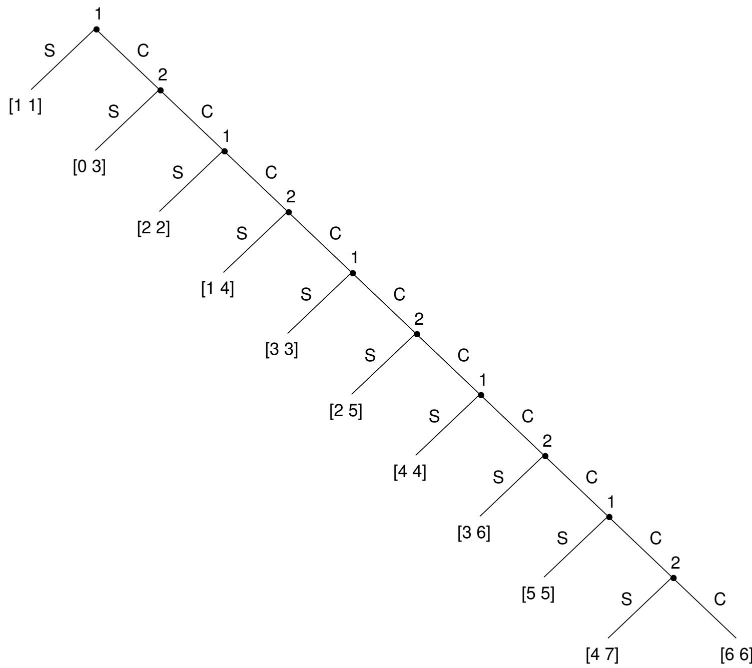
The hallmark of an extensive-form game is the decision tree it induces. Rather than define a decision tree formally we shall describe several classic extensive-form games and then illustrate the games using decision trees.

1.4.1. Centipede Game

The "Centipede Game" can be described as follows: Imagine two players, 1 and 2, who are trying to decide how to divide an expanding pie whose initial size is two. Player 1 starts the game by deciding whether to take one and to give one to player 2, or to wait until the second period when the pie is three. In the second period player 2 can decide whether to stop the game and keep the three or continue into the next period when the pie is four. This sequence of events continues on into the future.

In Figure 2 is a graph of the decision tree induced by the extensive-form centipede game. The • with a `1` at the top left indicates that player 1 begins the game and he can choose to stop (S) the game or to continue (C). If player 1 continues, plays C, then in the next period it is player 2's move and she can choose to stop (S) the game or to continue

Figure 2
Decision Tree of Centipede Game



(C).[1] The numbers in square brackets (*e.g.*, [4 1]) denote the payoffs to player 1 and 2, respectively; *i.e.*, player 1 gets 4 while player 2 gets 1.

1.4.2. Ultimatum Game

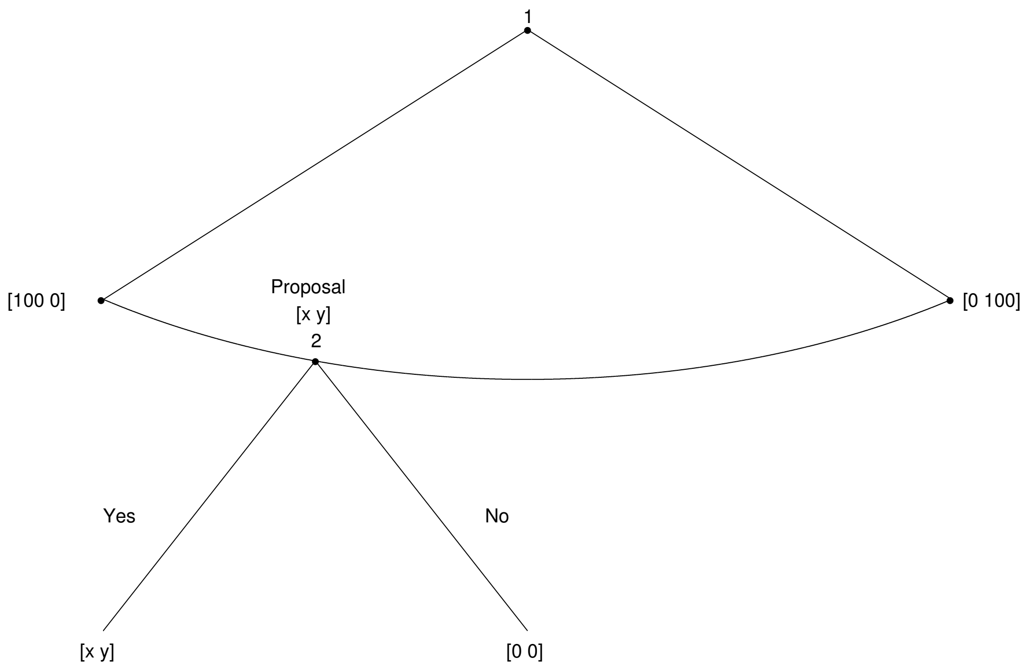
The "Ultimatum Game" is another example of an extensive-form game. In this game two players are trying to decide how to divide a sum of \$100. One player is chosen at random to go first. Without loss of generality call this player 1. Player 1 must propose a division of the \$100 which we shall denote [x y] where x denotes the amount that player 1 will get and y is the amount that player 2 will get, provided player 2 agrees to 1's proposal. In the second round of the game player 2 must decide whether to accept or to reject player 1's proposal. If player 2 rejects the proposal, then both players get nothing. The decision tree of this game is illustrated in Figure 3 where the arc between the • at [100 0] and the • at [0 100] denotes a continuum of possibilities from which player 1 can choose.

1.4.3. Entry Game

A third example of an extensive-form game involves entry into an industry. Consider an industry in which the incumbent is currently the sole producer of a good, a monopolist. Because the monopolist is garnering unusual profits, another firm is considering entering

[1] To reduce ambiguity, in what follows we shall adopt the convention that player 1 is a male while player 2 is a female.

Figure 3
Decision Tree of Ultimatum Game



the industry. Of course the monopolist is none too pleased about this potential competitor and threatens a "scorched-earth" policy of low prices if the other firm actually enters the industry. In Figure 4 is illustrated a decision tree in which payoffs to the entrant and the incumbent are listed where High and Low denote high and low prices, respectively. Note too that [2 3] indicates that the entrant will earn profits of 2 and the incumbent will earn profits 3 if the entrant actually enters the industry and high prices are charged, while $[-1\ 0]$ indicates that the entrant will earn a loss and the incumbent no profits if the entrant actually enters and low prices are charged.

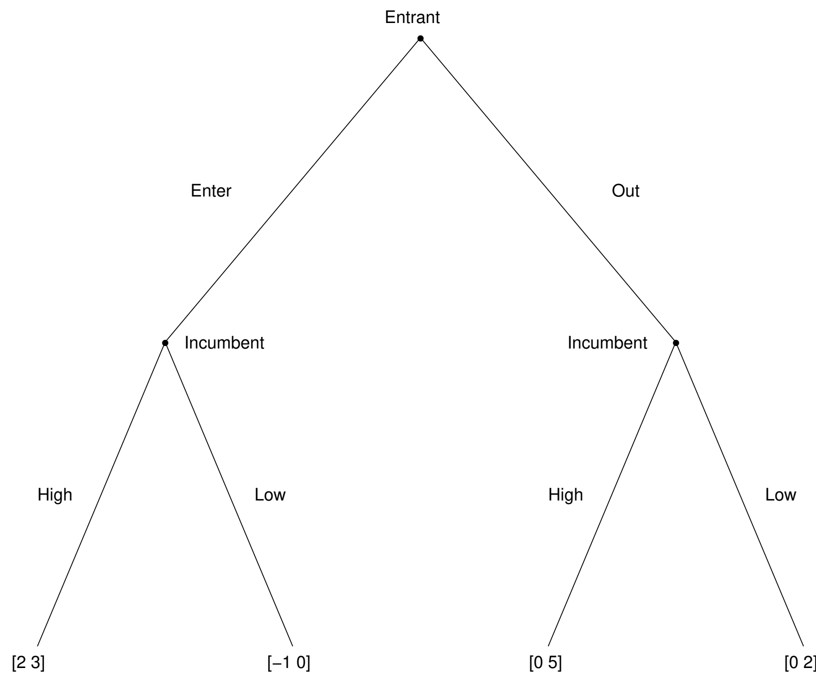
1.5. Equilibrium Refinement: Subgame Perfection

In extensive-form games the concept of NE typically yields far too many equilibria, some of them implausible, so researchers have sought refinements of the NE. Perhaps the most famous refinement is due to the economics Nobel prize-winning mathematician Reinhard Selten.

Because the concept of NE puts no restrictions on what players can do out of equilibrium, the analyst needs to eliminate outlandish threats off the equilibrium path. Thus the concept of subgame perfection requires additionally that players adopt Nash behavior off the equilibrium path. This refinement has important implications which we state in the following remarkable and useful theorem:

> *Selten's Theorem:* The subgame perfect (SGP) NE to an extensive-form game is generically unique.

Figure 4
Decision Tree of Entry Game



How does one calcuate the SGP NE of an extensive-form game? One suggestion would be to collapse the payoffs of the extensive-form game into a payoff matrix for a normal-form game and then proceed as before. This, in fact, is a *bad* idea because different extensive-form games can have the same normal-form payoff matrix. Because a strategy in an extensive-form game is a complete contigent plan of a form similar to the following:

$$\textit{if } \mathsf{E} \textit{ then do } \mathsf{X} \textit{ while if } \mathsf{O} \textit{ then do } \mathsf{Y}$$

a different algorithm is required.

1.5.1. Backward Induction

The main algorithmic strategy used to compute the SGP NE of an extensive-form game is *backward induction.* What the analyst does is go to the bottom of the decision tree and, at each node of the tree, calculates what the optimal strategy would be. By working backward (up) through the limbs of the tree, the analyst constructively calculates the equilibrium to the game.

To see how backward induction works examine first the Centipede game. Go the end of the decision tree and consider the choice of player 2. Does player 2 continue (C) or stop (S)? Since 7 is greater than 6 she chooses to stop S_5^2 where the subscript 5 denotes that this is player 2's fifth move. Conditional on S_5^2 player 1 chooses to stop S_5^1 since 5 is greater than 4. Conditional on S_5^1 player 2 choose to stop S_4^2 also since 6 is greater than 5 and so forth up the decision tree. Thus, seeing optimal play to the end of the game, player 1 at the first node chooses to stop S_1^1, yielding one unit for him and one unit for player 2.

Backward induction works in the same way in the Ultimatum game. For any choice [x y] made by player 1, provided y is greater than zero, player 2 is better off accepting the proposal, choosing Yes. Thus the equilibrium to the game is for player 1 to propose $[100 - \varepsilon,\ \varepsilon]$ for some ε just greater than zero.

Where backward induction really shows its might is in analyzing the Entry Game. Some researchers in the field of industrial organization used to think that the monopolist's scorched-earth threat was credible and would deter the other firm from entering the industry. By applying backward induction we can see they were wrong, at least in this simple model. Go to the last node in each branch of the decision tree. For the Out branch it is optimal for the incumbent to choose the High strategy and for the Enter branch it is optimal for the incumbent to choose the High strategy. Conditional on the strategy High for the incumbent the optimal strategy for the Entrant is to choose Enter. Using backward induction we see immediately that, once the other firm has entered the industry, it is in the incumbent's best interest to choose the strategy High. Thus, threats made off the equilibrium path are not credible.

2. An Economic Application: An Analysis of Duopoly

Consider an industry in which two firm exist: 1 and 2. Firm 1 produces output q_1 according to the following total cost function:

$$C_1(q_1) = \gamma_1 q_1 \quad \gamma_1 > 0,$$

while firm 2 produces output q_2 according to the total cost function

$$C_2(q_2) = \gamma_2 q_2 \quad \gamma_2 > \gamma_1 > 0.$$

Thus firm 2 is less efficient than firm 1 in production of output q. Suppose that the industry's inverse demand curve has the following functional form:

$$p = \alpha - \beta Q = \alpha - \beta(q_1 + q_2) \quad \alpha > \gamma_2 > \gamma_1 > 0,\ \beta > 0$$

where Q is total industry output and p is the price of that output.

Initially consider the case where the two firms have merged into one producer, a monopolist, having two plants. The optimal behavior of this monopolist would be to shut down the inefficient, more expensive, plant which was previously firm 2, and produce everything at the plant which was previously firm 1. The profit function for the monopolistic firm would then be

$$\pi_m(Q) = pQ - \gamma_1 Q = (\alpha - \beta Q)Q - \gamma_1 Q.$$

As we know from previous handouts, the optimal total output Q_m^* for this monopolist would be

$$Q_m^* = q_{1,m}^* = \frac{(\alpha - \gamma_1)}{2\beta} < \frac{(\alpha - \gamma_1)}{\beta} = Q_c^*$$

which is less than the economically-efficient amount (determined by setting price p equal to marginal cost γ_1) that should be produced Q_c^*. Thus the price paid by consumers to the monopolist would be

$$p_m^* = \frac{(\alpha + \gamma_1)}{2} > p_c^* = \gamma_1,$$

which is greater than the economically-efficient price p_c^* which is marginal cost γ_1.

Now suppose that the two firms compete in a noncöoperative way against one another. How can we use game theory to analyze this situation? Let us first outline the elements of the game. The players in this game are firms 1 and 2; each firm can choose any amount of output from the interval $[0, \infty)$; each firm knows its output as well as that of its opponent, so information is complete; firms seek to maximize profit; and we shall impose the NE concept. Given these elements, we can now begin.

Let Q again denote total industry output which, in this case, would equal $(q_1 + q_2)$. The profit function for firm 1 is

$$\pi_1(q_1; q_2) = pq_1 - \gamma_1 q_1 = [\alpha - \beta(q_1 + q_2)]q_1 - \gamma_1 q_1,$$

while the profit function for firm 2 is

$$\pi_2(q_2; q_1) = pq_2 - \gamma_2 q_2 = [\alpha - \beta(q_1 + q_2)]q_2 - \gamma_2 q_2$$

whence come the following first-order conditions for profit maximization:

$$\frac{\Delta \pi_1(q_1^*; q_2)}{\Delta q_1} = (\alpha - 2\beta q_1^* - \beta q_2) - \gamma_1 = 0$$
$$\frac{\Delta \pi_2(q_2^*; q_1)}{\Delta q_2} = (\alpha - \beta q_1 - 2\beta q_2^*) - \gamma_2 = 0,$$

so

$$q_1^* = \frac{\alpha - \gamma_1 - \beta q_2}{2\beta}$$
$$q_2^* = \frac{\alpha - \gamma_2 - \beta q_1}{2\beta}.$$

Note that firm 1's optimal solution q_1^* depends on q_2, and firm 2's optimal solution q_2^* depends on q_1. What to do?

By imposing the concept of NE (*viz.*, that firm 1 responds to firm 2's optimal choice q_2^* and firm 2 responds to firm 1's optimal choice q_1^*), we get the following solution:

$$q_1^* = \frac{(\alpha - 2\gamma_1 + \gamma_2)}{3\beta}$$
$$q_2^* = \frac{(\alpha - 2\gamma_2 + \gamma_1)}{3\beta}.$$

Thus total industry supply Q_d^* is

$$Q_d^* = \frac{(2\alpha - \gamma_1 - \gamma_2)}{3\beta}$$

which is greater than Q_m^* but less than Q_c^*. To wit the duopolists produce more in aggregate than the monopolist, but less than the efficient amount. The duopolists also do it at higher cost. Is society better off? Well it depends on how one wants to trade-off consumer and producer surplus. The Theory of Second Best rears its ugly head.

3. Auctions: An Example of a Game of Incomplete Information

How can the owner of an object choose its selling price when potential buyers have better information concerning the object's value than does the seller? One option for the seller would be to ask the potential buyers their opinions, but these buyers would have no incentive to reveal their private valuations, especially if they really wanted the object for sale. To get around such an informational asymmetry, the seller can use an auction to induce potential buyers to reveal their private information through their bidding behavior. In the course of this bidding, a winning price will obtain which, under certain conditions, can provide a good estimate of the opportunity cost of the object for sale, the *economic* value of this object.[2]

Economists are extremely interested in prices because prices are signals of how resources should be allocated in an economy. In many economic models, however, the determination of prices is described only vaguely. Auctions are formal processes for determining prices. Auctions are ubiquitous in market economies; they are also ancient, their durability suggesting that they serve an important allocational role.

One of the most successful applications by economists of game theory has been to the analysis of auctions. One can see why because the number of bidders at an auction is typically small and the rules of an auction are usually written down prior to the sale, so such real-world situations are natural environments in which to apply game theory. Over the past forty years economists have made considerable progress in understanding the factors influencing prices realized from goods sold at auction. For example, they have found that the average selling price depends on the auction format employed (*e.g.* oral or sealed-bid), the type of bidding (*e.g.* lump-sum or per-unit), the information available to potential buyers, the attitudes of potential buyers toward risk, and the number of potential buyers (competition). In the remainder of this section, we shall review briefly the application of game theory to the study of auctions.

The game-theoretic elements of an auction game are as follows: the potential bidders at the auctions are the players; the rules of the auction determine the rules of the game; the strategies of the players are their bids; the information structure is what the players know about the object for sale; the objective functions of the players are profits; and two concepts of equilibrium are used, dominance and an extension of Nash called "Bayes-Nash."

3.1. Auction Formats

At least four different auction formats exist. The most well-known format is the oral, ascending-price auction, often referred to as the "English" auction. At such auctions, prices

[2] In economics the term "opportunity cost" is used to refer to the resources used in a current activity but foregone to the next-best competing activity. Because these resources are what agents in the economy give up by choosing the current activity over the next-best alternative, these resources represent the opportunity lost to (opportunity cost of) the current activity. For example, when a self-employed contractor, who has ample work to choose from, goes golfing the opportunity cost of this activity is the income he could have earned working rather than playing. The total costs of playing golf are the opportunity cost plus the out-of-pocket costs.

are called out more or less continuously. Using a variety of different behaviors, potential buyers affirm their willingness to pay the current price. The last potential buyer willing to accept the current price wins the auction and pays that last-cried price. For a variety of technical reasons, which will be made clear later, English auctions are sometimes referred to as "second-price" auctions.

The next most common auction format is the first-price, sealed-bid auction. At such auctions, potential buyers submit their bids in sealed envelopes which are opened more or less simultaneously; the potential buyer who submits the highest bid wins the object and the price she pays is her bid.[3]

Another auction format is the oral, descending-price auction, often referred to as the "Dutch" auction, perhaps because it is used in the Netherlands to sell flowers at auction in Aalsmeer, near Amsterdam. At such auctions the current price is set very high on a thermometer-type clock; the clock then proceeds to fall. The first potential buyer to cry out wins the object and pays the current price on the clock.

The least-used of the four commonly-known auction formats is the second-price, sealed-bid auction. Second-price, sealed-bid auctions are often referred to as "Vickrey" auctions in honor of the economics Nobel prize-winning economist William S. Vickrey who first proposed this auction format in 1961. Like potential buyers at first-price, sealed-bid auctions potential buyers at Vickrey auctions submit their bids in sealed envelopes. Again, the potential buyer who submits the highest bid wins the object, but in this case the price she pays is the bid of her nearest opponent, the second-highest bid submitted. If the winner has no opponents, then she pays the reserve price, the minimum price which must be bid at the auction. The reason these auctions are referred to as "second-price" auctions is obvious from their description.

3.2. Types of Bidding

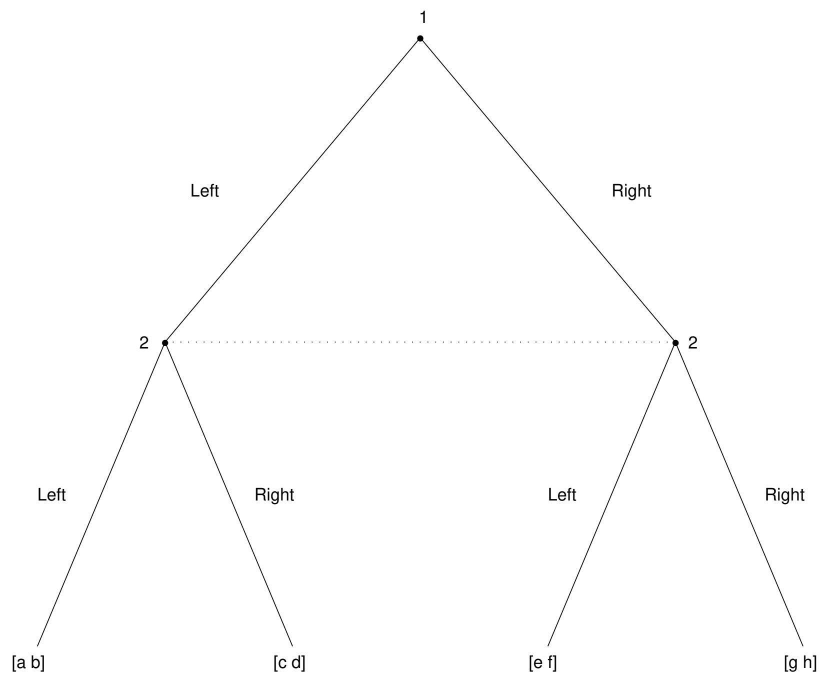
Lump-sum and per-unit are the two most common types of bidding, payment schemes, used at the four auctions listed above. Lump-sum payment schemes require the buyer to pay a fixed sum of money for the object, independent of any other considerations. Per-unit schemes require the buyer to pay a price for each unit of the object purchased. Obviously when only one object exists the two types of bidding are identical.

3.3. Two Informational Paradigms

Following the economics Nobel prize-winning economist John C. Harsanyi, researchers modeling behavior at auctions have cast auctions as games of incomplete information. Thus the structure of information is particularly important. The hallmark of a game of incomplete information is that players are uncertain which branch of the decision tree their opponents are in. This is illustrated in a decision tree by drawing dotted lines among nodes where uncertainty exists as in Figure 5. In the extensive-form game illustrated in Figure 5 player 1 chooses to go Left with some probability and Right with another probability. Thus

[3] To reduce ambiguity, in what follows we shall adopt the convention that the seller is a male while the potential buyers are females.

Figure 5
Decision Tree in a Game of Incomplete Information



player 2 does not know which branch of the decision tree she is in when the choice between Left and Right is to be made. She must weight the payoffs to her decisions in a particular branch of the decision tree by the probability of being in that branch.

Two different informational structures are typically assumed by economic theorists to describe the environments within which potential buyers make decisions. The structures of these two environments represent the polar extremes of information, but by examining these extremes one can understand better how information affects the economic role of an auction. Deciding which informational structure best approximates an economic environment is central to determining the properties of prices determined at auction.

By far the most commonly-assumed informational structure is the independent private-values paradigm (IPVP). Within the IPVP potential buyers are assumed to have individual-specific valuations of the object for sale which they alone know. These valuations are modelled as independent draws of a random variable whose distribution is commonly known. Below, we shall refer generically to this random variable as V.

The simplest way to think about the IPVP is to imagine a large bingo urn with say 100 balls numbered 1 to 100. Each potential buyer gets a draw with replacement from this urn. If potential buyer i gets a ball with the number 29 on it, then the object is worth \$29 to her. In such a case potential buyer i has valuation v_i equal to \$29. None of the other potential buyers knows i's valuation nor does the seller, but everyone (both potential buyers and the seller) knows about the urn and its properties.

Note that within the IPVP a seller can use the auction as a device to find the largest valuation from a group of potential buyers, none of whom has any incentive to tell him her valuation. The rules at any of the four auction formats listed above induce the highest-valuation potential buyer to bid the most and to win the auction. Within the IPVP the auction plays an important allocative role by identifying the highest-valuation potential buyer and allocating the object for sale to that potential buyer. In this way the auction ensures the economically-efficient use of the object. Hence the winning bid at an auction within the IPVP provides important information about the opportunity cost of the object for sale. To wit, under certain circumstances, the winning bid can be useful in estimating the opportunity cost of the object for sale.

The second-most commonly-used informational structure is the common-value paradigm (CVP). Within the CVP the object is assumed to have the same (common) value to all potential buyers, but this value is initially unknown to these potential buyers and the uncertainty concerning the object's value will remain unresolved until after the auction. Potential buyers make different bids at the auction because each has a different *estimate* of the object's *true* value.

To make the discussion concrete consider an example from the oil industry. Oil is typically sold at first-price, sealed-bid auction with considerable heterogeneity in realized bids. But once a tract has been sold and wells have been drilled the volume of oil is known and, to a first approximation, the value of the oil found is the same to all firms as it is sold on the world market. The reason why potential buyers bid differently is because they have different expectations concerning the probability of discovering oil on a particular tract and, should oil be discovered, the volume of oil on that tract.

Within the CVP an auction plays no role in finding the potential buyer with the highest valuation because the object for sale has only one (albeit *ex ante* unknown) value. Hence the auction is just a way of deciding who gets the object and how the value of the object (think of a pie) is divided up once its value has been determined. Within the CVP one could also use a fair $\mathcal{N}$-sided die to choose the winner from $\mathcal{N}$ potential buyers and a 50-50 rule to divide the value of the object, once determined, between the winner and the seller. Such a rule would, on average, lead to the same allocation as that obtained using a first-price, sealed-bid auction except that the value of the object would be divided differently.

3.4. Some Notation

In what follows we shall analyze rational bidding within the IPVP, describing how winning prices are formed under this informational assumption. Before we do this we need to develop a shorthand notation to describe the concepts just reviewed. In what follows we shall assume that a set of $\mathcal{N}$ potential buyers (players) exists, identified generally by the index i where $i = 1, \ldots, \mathcal{N}$. Each potential buyer gets a random draw from V's distribution which we shall hereafter denote $F(v)$. By ordering the $\mathcal{N}$ independent random draws $\{v_1, v_2, \ldots, v_{\mathcal{N}}\}$ from smallest to largest using the following convention:

$$v_{(\mathcal{N}:\mathcal{N})} \leq v_{(\mathcal{N}-1:\mathcal{N})} \leq \cdots \leq v_{(2:\mathcal{N})} \leq v_{(1:\mathcal{N})}$$

we can identify the highest-valuation potential buyer. Here $v_{(1:\mathcal{N})}$ denotes the highest valuation, often referred to as the "first order statistic" of the sample of $\mathcal{N}$ valuations, while $v_{(2:\mathcal{N})}$ denotes the second-highest valuation, often referred to as the "second order statistic" of the sample of $\mathcal{N}$ valuations.

Obviously the four auction formats described above can generate different winning bids from the set of valuations $\{v_1, v_2, \ldots, v_{\mathcal{N}}\}$ because each auction format constitutes a different set of rules for transforming the valuations of potential buyers into the winning prices.

3.5. Rational, Equilibrium Bidding at Auctions

Within the IPVP rational, equilibrium bidding implies that the bids of potential buyers are functions which depend on their valuations.

3.5.1. English Auctions

The optimal bid function for potential buyer i at an English auction with $\mathcal{N}(\geq 2)$ bidders is to stay in the auction until *either* the price reaches i's valuation v_i and then drop out *or* to stop bidding when all others have dropped out. Put formally, for the i^{th} non-winner the dominant-strategy bid function $\alpha(V)$ is to bid b_i her valuation v_i or

$$b_i = \alpha(v_i) = v_i$$

while the potential buyer with the highest valuation should continue to bid until the potential buyer with the second-highest valuation drops out, so the winning bid w will be

$$w = v_{(2:\mathcal{N})}.$$

This is why English auctions are sometimes called "second-price" auctions; the winning bid is the second order statistic of the sample of $\mathcal{N}$ valuations. Note that English auctions ensure the efficient allocation of the object as the potential buyer with the highest valuation wins the object. Note too that the winning bid represents the opportunity cost of the object as this would be the value of the object in its next-best use. Thus winning bids realized at English auctions can be useful in estimating the opportunity cost of the object for sale.

3.5.2. Vickrey Auctions

What should potential bidder i do at a second-price, sealed-bid (Vickrey) auctions? Well, to bid less than v_i would be to reduce her chances of winning without changing the price paid since that is determined by the second-highest bid. This would occur if potential buyer i's bid were lower than some other bid which in turn was less than v_i. To bid more than v_i could change the outcome when some other bidder has submitted a bid higher than i's valuation but lower than her new bid. Raising the bid above v_i causes i to win but to pay more than the object is worth to her. Hence the dominant-strategy bid function for the i^{th} potential buyer is again to bid b_i her valuation v_i or

$$b_i = \alpha(v_i) = v_i$$

while the winner will be the potential buyer with the highest bid $b_{(1:\mathcal{N})}$ or $v_{(1:\mathcal{N})}$, but she will pay what her next nearest opponent was willing to pay $b_{(2:\mathcal{N})}$ or $v_{(2:\mathcal{N})}$, so the winning bid w will also be

$$w = v_{(2:\mathcal{N})}.$$

Again the Vickrey auction ensures the economically-efficient allocation of the object and the winning bid represents the object's opportunity cost.

3.5.3. Dutch and First-Price, Sealed-Bid Auctions

The optimal bid functions which characterize rational, equilibrium behavior at Dutch and first-price, sealed-bid auctions are the same, but they are not dominant-strategy bid functions. Instead these equilibrium bid functions are "Bayes-Nash" bid functions, involving an equilibrium concept that is an extension of Nash's original one. Because considerable mathematics is required to derive these bid function, we shall just state the Bayes-Nash bid function and attempt to provide some intuition for why it is reasonable.

The optimal bid function $\beta(V)$ for potential buyer i who has valuation v_i is

$$\beta(v_i) = v_i - \frac{\int_r^{v_i} F(u)^{\mathcal{N}-1}\, du}{F(v_i)^{\mathcal{N}-1}} \tag{1}$$

where r is the reserve price, the minimum price that must be bid at auction. Basically, when potential buyer i bids, she shaves her valuation v_i by the term to the right of the minus sign in equation (1). How much i shaves depends on the number of potential buyers $\mathcal{N}$, how high the reserve price r is, how high her valuation v_i is, and the shape of the valuation distribution as captured by $F(\cdot)$.

Potential buyer i shaves less when $\mathcal{N}$ is larger than when $\mathcal{N}$ is small: competition matters. In fact, as $\mathcal{N}$ gets very large, this model of an auction converges to the perfectly competitive model of sale. Potential buyer i also shaves less when r is large than when r is small. Note too that potential buyer i shaves more when v_i is large than when v_i is small, but she also bids more when v_i is large than when v_i is small. That is how the auction induces the highest-valuation potential buyer to reveal her willingness to pay for the object.

Suffice it to say that the relationship between β and V is quite complicated. Not surprisingly the relationship between the winning bid at Dutch and first-price, sealed-bid auctions and the opportunity cost $v_{(2:\mathcal{N})}$ is unclear.

3.6. Optimal Mechanism Design

But of the four auction formats described above which is the best selling mechanism? The answer to this question is quite interesting because, under the conditions outlined above, a very remarkable proposition exists: the Revenue Equivalence Proposition.

In order to state the conclusion of this proposition, consider the following thought experiment: instead of having potential buyers sample with replacement from the urn just once, let them do this several times. After each sampling for all $\mathcal{N}$ potential buyers, sell the object "repeatedly" at each of the four auction formats. (Here the thought experiment to imagine is selling the object at each of the four formats given the current sequence of $\mathcal{N}$ valuations.) Under these conditions we have:

> *Revenue Equivalence Proposition:* the average winning bid at English and Vickrey auctions equals the average winning bid at Dutch and first-price, sealed-bid auctions.

But just because the four auction formats garner the same revenue, on average, does not mean they are optimal. In fact, in constructing the optimal selling mechanism for any of the four auction formats, the seller must set the reserve price r, the minimum price which must be tendered, in a particular way. Also, if the potential buyers are risk averse or if the seller is worried about the variance in revenues across different sales, then certain auction formats will be preferred to others. Thus the optimal design of selling mechanisms can be quite a complicated exercise.