Operations Research Test 1

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Question 1

a.

$$\begin{aligned} \max\ & 80x_1 + 100x_2 \\ \text{s.t.}\quad & 20x_1 + 40x_2 \leq 1000 \\ & 60x_1 + 40x_2 \leq 1240 \\ & 12x_1 + 4x_2 \leq 200 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Adding in slack variables, this becomes:

$$\begin{aligned} \max\quad & 80x_1 + 100x_2 \\ \text{s.t.}\quad & 20x_1 + 40x_2 + s_1 = 1000 \\ & 60x_1 + 40x_2 + s_2 = 1240 \\ & 12x_1 + 4x_2 + s_3 = 200 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

Putting in Tableau form:

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & -80 & -100 & 0 & 0 & 0 & 0 \\ 0 & 20 & 40 & 1 & 0 & 0 & 1000 \\ 0 & 60 & 50 & 0 & 1 & 0 & 1240 \\ 0 & 12 & 4 & 0 & 0 & 1 & 200 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & -30 & 0 & \frac{5}{2} & 0 & 0 & 2500 \\ 0 & \frac{1}{2} & 1 & \frac{1}{40} & 0 & 0 & 25 \\ 0 & 40 & 0 & -1 & 1 & 0 & 240 \\ 0 & 10 & 0 & \frac{-1}{10} & 0 & 1 & 100 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & 0 & 2680 \\ 0 & 0 & 1 & \frac{3}{80} & \frac{-1}{80} & 0 & 22 \\ 0 & 1 & 0 & \frac{-1}{40} & \frac{1}{40} & 0 & 6 \\ 0 & 0 & 0 & \frac{3}{20} & \frac{-1}{4} & 1 & 40 \end{array}\right]$$

We can see that the optimal solution is: 2680 obtained where $x_1 = 6, x_2 = 22$.

b.

Note that since $s_3 = 40$, there is slack in the third constraint, and therefore we are not using all the workers. This indicates that there are 40 worker-hours not being utilized, and HR should lay off one worker as only 4 are required.

c.

We would like to relax the costraint on the resource with the highest shadow price, and would therefore like to hire an additional worker in Cutting, as $\frac{7}{4} > \frac{3}{4}$.

d.

Forming the RHS Tableau:

$$\left[\begin{array}{cccccc|c|ccc}
Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS & d_1 & d_2 & d_3 \\
\hline
1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & 0 & 2680 & \frac{7}{4} & \frac{3}{4} & 0 \\
0 & 0 & 1 & \frac{3}{80} & \frac{-1}{80} & 0 & 22 & \frac{3}{80} & \frac{-1}{80} & 0 \\
0 & 1 & 0 & \frac{-1}{40} & \frac{1}{40} & 0 & 6 & \frac{-1}{40} & \frac{1}{40} & 0 \\
0 & 0 & 0 & \frac{3}{20} & \frac{-1}{4} & 1 & 40 & \frac{3}{20} & \frac{-1}{4} & 1
\end{array}\right]$$

Note that since only the workers in the sewing department change: $d_1 = d_3 = 0$
We can see from this that:

$$\begin{aligned}
22 - \frac{1}{80}d_2 &\geq 0 \\
6 + \frac{1}{40}d_2 &\geq 0 \\
40 - \frac{1}{4}d_2 &\geq 0
\end{aligned}$$

This yeilds:

$$\begin{aligned}
d_2 &\leq 1760 \\
d_2 &\geq -240 \\
d_2 &\leq 160
\end{aligned}$$

From this it is plain that: $d_2 \in [-240, 160]$ and that we can remove as many as 6, and add as many as 4 workers without altering the form of the final tableau.

e.

Forming the Objective function tableau:

$$
\begin{array}{c c}
 & \begin{array}{cccccc|c} 1 & d_1 & d_2 & 0 & 0 & 0 & 0 \\ Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \end{array} \\
\begin{array}{c} 1 \\ d_1 \\ d_2 \\ 0 \end{array} & \left[\begin{array}{cccccc|c}
1 & 0 & 0 & \frac{7}{4} & \frac{3}{4} & 0 & 2680 \\
0 & 0 & 1 & \frac{3}{80} & \frac{-1}{80} & 0 & 22 \\
0 & 1 & 0 & \frac{-1}{40} & \frac{1}{40} & 0 & 6 \\
0 & 0 & 0 & \frac{3}{20} & \frac{-1}{4} & 1 & 40
\end{array}\right]
\end{array}
$$

Noting that if only the price of shirts changes, then $d_2 = 0$, our feasibility constraints are:

$$
\begin{aligned}
\frac{7}{4} + \frac{3}{80}d_1 &\geq 0 \\
\frac{3}{4} - \frac{1}{80}d_1 &\geq 0
\end{aligned}
$$

This reduces to:

$$
\begin{aligned}
d_1 &\leq 60 \\
d_1 &\geq \tfrac{-140}{3}
\end{aligned}
$$

We can see that $d_1 = 25$ is within this range, and will therefore not change the optimal solution.
The new maximum will be: $6 * 105 + 22 * 100 = 2830$.

Question 2

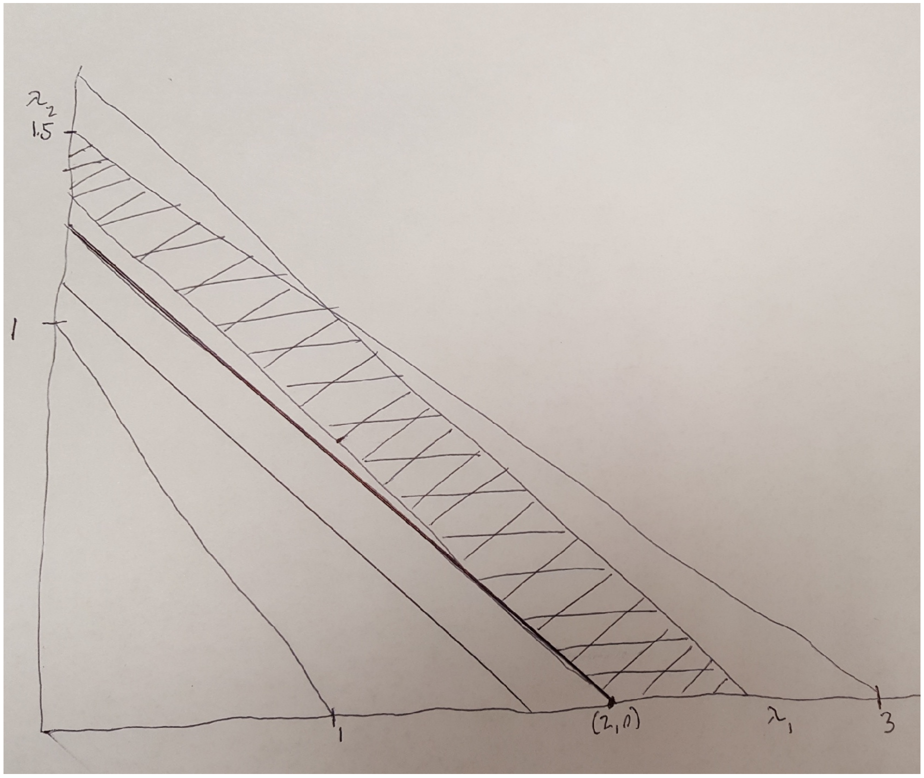
a.

The dual is given by:

$$
\begin{aligned}
\text{min} \quad & 5\lambda_1 + 8\lambda_2 \\
\text{s.t.} \quad & \lambda_1 + 2\lambda_2 \leq 3 \\
& 2\lambda_1 + 3\lambda_2 \geq 4 \\
& \lambda_1 + \lambda_2 \geq 1 \\
& 2\lambda_1 + 3\lambda_2 \leq 5 \\
& \lambda_1, \lambda_2 \geq 0
\end{aligned}
$$

b.

Solving it graphically:



Note that the feasible set is the cross-hatched area, and the objective contour with the minimum is bolded.

We can see that the optimal solution is located at: $\lambda_1 = 2, \lambda_2 = 0$

c.

Via Complementary slackness we arrive at the following equations:

$$\begin{aligned}
(5 - x_1 - 2x_2 - x_3 - 2x_4)\lambda_1 &= 0 \\
(8 - 2x_1 - 3x_2 - x_3 - 3x_4)\lambda_2 &= 0 \\
(3 - \lambda_1 - 2\lambda_2)x_1 &= 0 \\
(4 - 2\lambda_1 - 3\lambda_2)x_2 &= 0 \\
(1 - \lambda_1 - \lambda_2)x_3 &= 0 \\
(5 - 2\lambda_1 - 3\lambda_2)x_4 &= 0
\end{aligned}$$

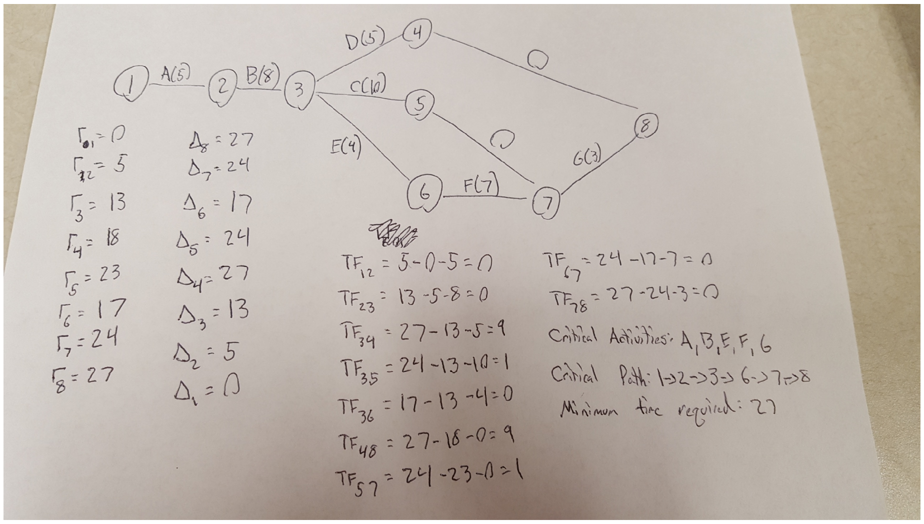
Plugging in: $\lambda_1 = 2, \lambda_2 = 0$.

$$\begin{aligned} 5 - x_1 - 2x_2 - x_3 - 2x_4 &= 0 \\ 0 &= 0 \\ x_1 &= 0 \\ 0 &= 0 \\ -x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

We are left with only x_2 unknown, and can solve for it with the first equation. $x_1 = 0, x_2 = \frac{5}{2}, x_3 = 0, x_4 = 0$.

Question 3.

a.



b.

This can be written as a linear program of the form:

$$
\begin{aligned}
\text{min } & 30d_A + 15d_B + 20d_C + 40d_D + 20d_E + 30d_F + 40d_C \\
\text{s.t.} \quad & t_8 - t_1 \leq 19 \\
& t_8 - t_7 + d_G \geq 3 \\
& t_7 - t_6 + d_F \geq 7 \\
& t_6 - t_3 + d_E \geq 4 \\
& t_5 - t_3 + d_C \geq 10 \\
& t_4 - t_3 + d_D \geq 5 \\
& t_3 - t_2 + d_B \geq 8 \\
& t_2 - t_1 + d_A \geq 5 \\
& d_A \leq 2 \\
& d_B \leq 3 \\
& d_C \leq 1 \\
& d_D \leq 2 \\
& d_E \leq 2 \\
& d_F \leq 3 \\
& d_G \leq 1 \\
& t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8 \geq 0 \\
& d_A, d_B, d_C, d_D, d_E, d_F, d_G \geq 0
\end{aligned}
$$

Via an online simplex calculator: (http://www.phpsimplex.com/en/)
$t_1 = 0, t_2 = 3, t_3 = 8, t_4 = 13, t_5 = 18, t_6 = 10, t_7 = 16, t_8 = 19, d_A = 2, d_B = 3, d_C = 0, d_D = 0, d_E = 2, d_F = 1, d_G = 0$
The Min cost for this speed-up is 175.

Question 4.

a.

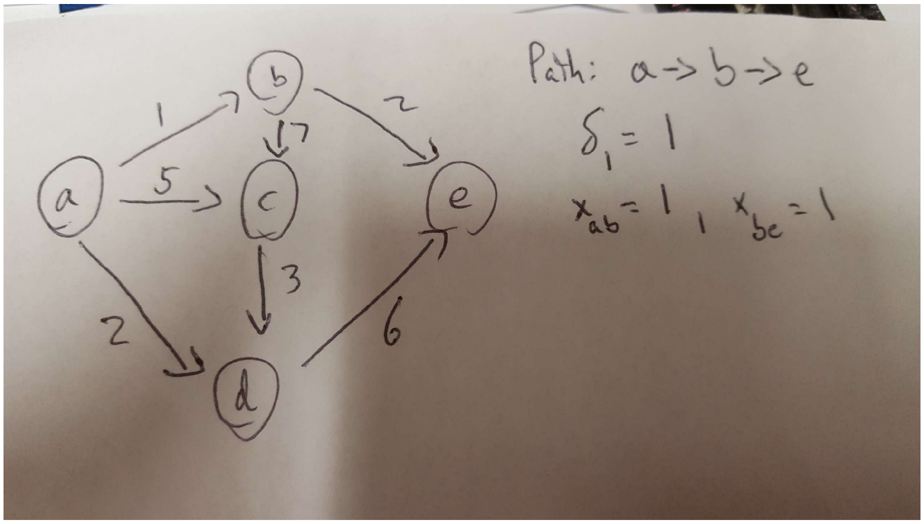


Fig. 1: First Path

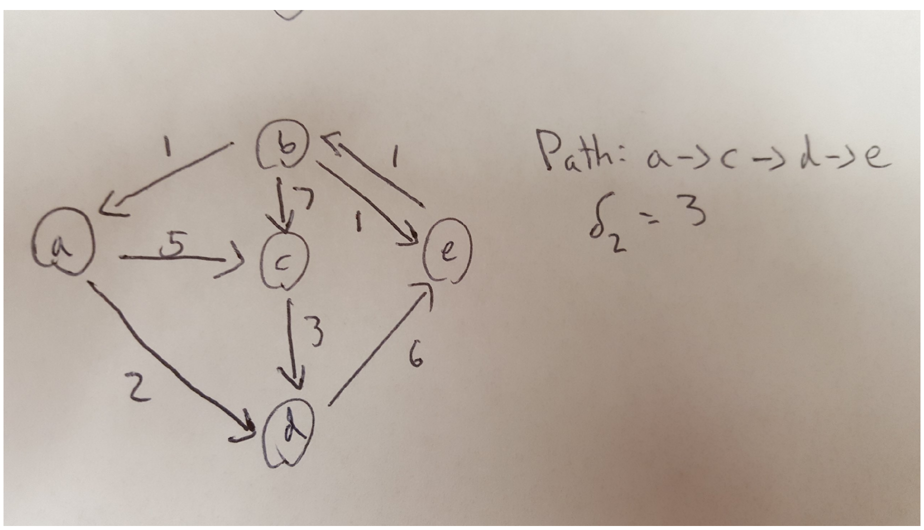


Fig. 2: Second Path

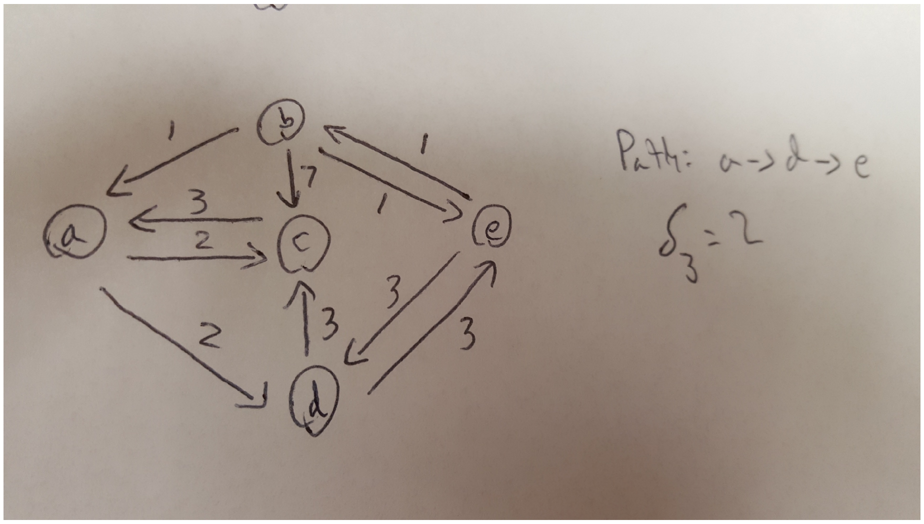


Fig. 3: Third Path

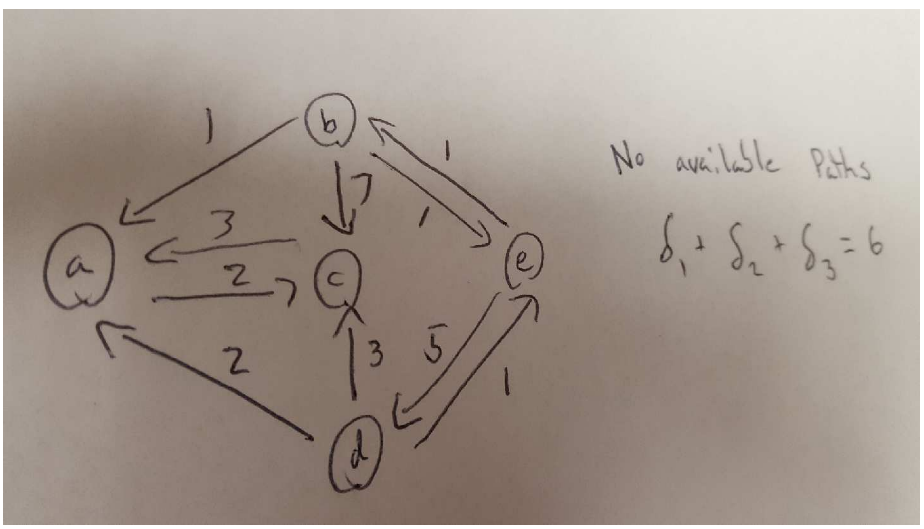


Fig. 4: Fourth Path

This implies that the maximum flow is 6. We can verify this with the dual, and can see that the minimum cut that can be made is 6 as well.

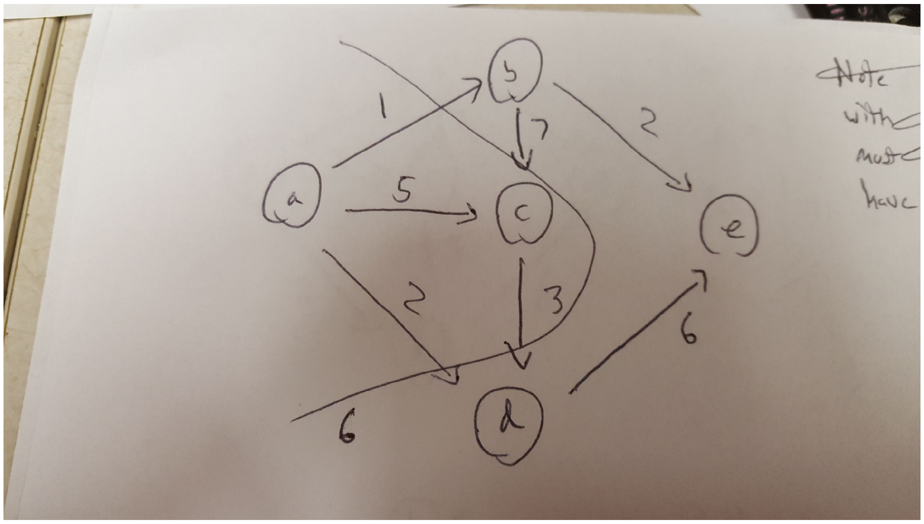


Fig. 5: Minimum Cut

It is clear that no cut can be made smaller while partitioning the graph with the start in one section and the end in the other.

Question 5.

a.

$$
\begin{aligned}
\max \quad & x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 - e_1 = 5 \\
& x_1 - 2x_2 + s_1 = 2 \\
& x_1, x_2, e_1, s_1 \geq 0
\end{aligned}
$$

b.

$$
\begin{aligned}
\max \quad & x_1 + 3x_2 - Ma_1 \\
\text{s.t.} \quad & x_1 + x_2 - e_1 + a_1 = 5 \\
& x_1 - 2x_2 + s_1 = 2 \\
& x_1, x_2, e_1, a_1, s_1 \geq 0
\end{aligned}
$$

c.

$$
\left[\begin{array}{cccccc|c}
Z & x_1 & x_2 & s_1 & e_1 & a_1 & RHS \\
\hline
1 & -1 & -3 & 0 & 0 & M & 0 \\
0 & 1 & 1 & 0 & -1 & 1 & 5 \\
0 & 1 & -2 & 1 & 0 & 0 & 2
\end{array}\right]
$$

$$\left[\begin{array}{ccccccc|c} Z & x_1 & x_2 & s_1 & e_1 & a_1 & & RHS \\ \hline 1 & -1-M & -3-M & 0 & M & 0 & & -5M \\ 0 & 1 & 1 & 0 & -1 & 1 & & 5 \\ 0 & 1 & -2 & 1 & 0 & 0 & & 2 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & e_1 & a_1 & RHS \\ \hline 1 & 2 & 0 & 0 & -3 & -3+M & 15 \\ 0 & 1 & 1 & 0 & -1 & 1 & 5 \\ 0 & 3 & 0 & 1 & -3 & 2 & 12 \end{array}\right]$$

Note that we would like e_1 to enter the basis, but both ratios are negative, leading us to an unbounded problem.

d.

Since the problem has no positive artificial variables, and the final tableau describes a point, $x_2 = 5, x_1 = 0$ that is in the feasible set, the solution is feasible. However the solution is unbounded, and its direction in the x_1, x_2 plane can be determined:

$$\begin{aligned} x_1 + x_2 - e_1 &= 5 \\ 3x_1 - 3e_2 &= 12 \end{aligned}$$

This has a solution of $x_1 = 4 + e_1$ and $x_2 = 1$. Our parametric solution is: $\mathbf{x} = \binom{1}{4} + e_1\binom{1}{0}$.
Thus the direction of unboundedness is: $\binom{1}{0}$.