Behavioral HW5

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1

a

Since we know that $\log V \sim N(\mu, \sigma^2)$ and that since log is a monotonic trasnformation, $\log V_{(2:N)} = (\log V)_{(2:N)}$

$$W_t = V_{(2:N)} = \exp(\log(V_{(2:N)})) = \exp((\log V)_{(2:N)})$$

$$F_{W_t} = P(W_t \leq w) = P(\exp(\log V)_{(2:N)} \leq w) = P((\log V)_{(2:N)} \leq \log w)$$

Since it is known that $\log V$ is distributed normally, we may derive the 2nd order statistic for it. The density of this distribution is given by:

$$f_{W_t} = N(N-1)\Phi(\frac{\log w - \mu}{\sigma})^{N-2}(1 - \Phi(\frac{\log w - \mu}{\sigma}))\phi(\frac{\log w - \mu}{\sigma})\frac{1}{w\sigma}$$

Its distribution is given by the integral from 0 to w

$$F_{W_t} = \int_0^w f_{W_t} = N(N-1)\int \Phi\left(\frac{\log(w) - \mu}{\sigma}\right)^{N-2}\left[1 - \Phi\left(\frac{\log(w) - \mu}{\sigma}\right)\right]\phi\left(\frac{\log(w) - \mu}{\sigma}\right)\frac{1}{w\sigma}dw$$

Applying the substitution of: $u = \Phi\left(\frac{\log(w) - \mu}{\sigma}\right)$

$$F_{W_t} = N(N-1)\int u^{N-2}[1-u]du = Nu^{N-1} - (N-1)u^N$$

$$F_{W_t} = N\left(\Phi\left(\frac{\log(w) - \mu}{\sigma}\right)\right)^{N-1} - (N-1)\left(\Phi\left(\frac{\log(w) - \mu}{\sigma}\right)\right)^N$$

b

$$\mathbb{E}[\log W_t] = \mathbb{E}[(\log V)_{(2:N)}] = \mu + \sigma\mathbb{E}[Z_{(2:N)}]$$

$$\mu + \sigma\int_{-\infty}^{\infty} N(N-1)\Phi(z)^{N-2}[1 - \Phi(z)]\phi(z)dz$$

Using Monte-Carlo Integration, we may numerically estimate this expectation for different values of N.

```julia
using Distributions
N = 2:25
d = Normal()
simSize = 1000000
for i in N
    Guesses = zeros(simSize)
    for j in 1:simSize
        normals = rand(d,i)
        sort!( normals )
        Guesses[j] = normals[i-1]
    end
    average = mean( Guesses )
    println( "For $(i) bidders the expected log is: mu + sigma$(average)")
end
```

```
For 2 bidders the expected log is: mu + sigma-0.5639504557129769
For 3 bidders the expected log is: mu + sigma2.351970563062514e-5
For 4 bidders the expected log is: mu + sigma0.2971588373611547
For 5 bidders the expected log is: mu + sigma0.49521391716804203
For 6 bidders the expected log is: mu + sigma0.641658600596855 2
For 7 bidders the expected log is: mu + sigma0.7565919293131907
For 8 bidders the expected log is: mu + sigma0.8524121727103113
For 9 bidders the expected log is: mu + sigma0.9329847938543535
For 10 bidders the expected log is: mu + sigma1.0024595817655524
For 11 bidders the expected log is: mu + sigma1.0616570276709072
For 12 bidders the expected log is: mu + sigma1.1156927570857202
For 13 bidders the expected log is: mu + sigma1.163894561166882
For 14 bidders the expected log is: mu + sigma1.207704745441353
For 15 bidders the expected log is: mu + sigma1.2480530695785061
For 16 bidders the expected log is: mu + sigma1.2857233343038093
For 17 bidders the expected log is: mu + sigma1.319707380965529
For 18 bidders the expected log is: mu + sigma1.3505116384605473
For 19 bidders the expected log is: mu + sigma1.379540726845045
For 20 bidders the expected log is: mu + sigma1.4085257149350368
For 21 bidders the expected log is: mu + sigma1.432798909304651
For 22 bidders the expected log is: mu + sigma1.4577406641382478
For 23 bidders the expected log is: mu + sigma1.482138982008
For 24 bidders the expected log is: mu + sigma1.5035284386804733
For 25 bidders the expected log is: mu + sigma1.5252116823857267
```

C

$$f(\mu,\sigma^2) = \sum_{t=0}^{T} \log(N_t(N_t-1)) + (N_t-2)\log(\Phi\left(\frac{\log(w_t)-\mu}{\sigma}\right))$$

$$+\log(1-\Phi\left(\frac{\log(w_t)-\mu}{\sigma}\right)) + \log(\phi\left(\frac{\log(w_t)-\mu}{\sigma}\right)) - \log(w_t\sigma)$$

d

Since it is known that $\log(W_t) = \mu + \sigma Z_{(N_t-1):N_t)}$ we may write this as $\log(W_t) = (\mu\sigma)(1 Z_{(N_t-1):N_t)})^T$. We can approximate μ, σ via least squares. We would then fill the y vector with the logarithm of the winning bid, and the X matrix with ones in the first column and the expected values of the order statistics would fill the second column the matrix. From there we could then estimate our values by $(\mu\sigma)^T = (X^TX)^{-1}(X^TY)$ and we would have rough estimates of both μ and σ.

2

a

Since we are in a procurement auction, we obtain our bid in payment, and we pay c in actual costs for doing the task if we win the auction.

$$
\begin{aligned}
&[s-c]\, P(\sigma^{-1}(s) \leq c_j) \forall j \neq i \\
&[s-c]\, \Pi_{n=1}^{N-1} P(\sigma^{-1}(s) \leq c_j) \\
&[s-c]\, \Pi_{n=1}^{N-1} \left[1 - P(c_j \leq \sigma^{-1}(s))\right] \\
&\text{Applying independence} \\
&[s-c] \left(1 - F_c(\sigma^{-1}(s))\right)^{N-1} \\
-[s-c](N-1)[1-F_c(\sigma^{-1}(c_j))^{N-2} f_c(\sigma^{-1}(c_j)) \frac{1}{\sigma'(c)} &+ [1 - F_c(\sigma^{-1}(s))] = 0 \\
&\text{Applying the Bayes-Nash Equilibrium condition of: } s = \sigma(c) \\
\sigma'(c)[1-F_c(c)]^{N-1} &= [\sigma(c) - c](N-1)[1-F-c(c)]^{N-2} f_c(c) \\
\sigma'(c)[1-F_c(c)] + -(N-1) f_c(c)\sigma(c) &= -c(N-1) f_c(c) \\
\sigma'(c) + \frac{-(N-1)f_c(c)}{1-F_c(c)} &= \frac{-c(N-1)f_c(c)}{1-F_c(c)}
\end{aligned}
$$

Using an integrating factor: $\mu = \exp\left(-\int \frac{(N-1)f_c(c)}{1-F_c(C)}\right) = \exp(-\int h_c)^{N-1} = S_c(c)^{N-1} = (1 - F_C(c))^{N-1}$

$$
\begin{aligned}
\left(\sigma(c)(1-F_c(c)^{N-1})\right)' &= -c(N-1) f_c(c)(1-F_c(c))^{N-1} \\
\sigma(c) &= \frac{\int_0^c -\xi(N-1) f_c(\xi)(1-F_c(\xi))^{N-2} d\xi}{(1-F_c(c))^{N-1}}
\end{aligned}
$$

Integrating by parts and applying the boundry condition

$$\sigma(c) = c - \frac{\int_0^c (1 - F_c(\xi))^{N-1} d\xi}{(1 - F_c(c))^{N-1}}$$
$$\sigma(c) = c + \frac{\int_c^\infty (1 - F_c(\xi))^{N-1} d\xi}{(1 - F_c(c))^{N-1}}$$

b

The convergence of this integral relies upon the premise that $\theta_1(N-1) > 1$

$$\sigma(c) = c + \frac{\int_c^\infty \frac{\theta_0}{\xi}^{\theta_1(N-1)}}{\left(\frac{\theta_0}{c}\right)^{\theta_1(N-1)}}$$
$$\sigma(c) = c - \frac{\theta_0^{\theta_1(N-1)} c^{-\theta_1(N-1)+1}}{[-\theta_1(N-1)+1]\theta_0^{\theta_1(N-1)} c^{-\theta_1(N-1)}}$$
$$\sigma(c) = c - \frac{c}{-\theta_1(N-1)+1} = c\left(\frac{\theta_1(N-1)}{\theta_1(N-1)-1}\right)$$

c

To evaluate the winning bid, we must first find the density of the bid function, and we will do so via the method of transformations:

$$\begin{aligned}
\sigma^{-1}(s) = & \quad -s\left(\tfrac{1-\theta_1(N-1)}{\theta_1(N-1)}\right) \\
\frac{\partial \sigma^{-1}(s)}{\partial s} = & \quad -\left(\tfrac{1-\theta_1(N-1)}{\theta_1(N-1)}\right) \\
f_{\sigma(c)} = & \quad \tfrac{\theta_1 \theta_0^{\theta_1}(\theta_1(N-1))^{\theta_1}}{c^{\theta_1+1}(\theta_1(N-1)-1)^{\theta_1}} \\
F_{\sigma(c)}(z) = & \quad P(\sigma(c) \le z) = P(c\tfrac{\theta_1(N-1)}{\theta_1(N-1)-1} \le z) = P(c \le z\tfrac{\theta_1(N-1)-1}{\theta_1(N-1)}) = F_c(z\tfrac{\theta_1(N-1)-1}{\theta_1(N-1)})
\end{aligned}$$

The winning bid is therefore just the last order statistic of this distribution whose density is given by:

$$\begin{aligned}
f_w(w) & = N[1 - F_{\sigma(c)}(w)]^{N-1} f_{\sigma(c)}(w) \\
f_w(w) & = N[1 - F_c(w\tfrac{\theta_1(N-1)-1}{\theta_1(N-1)})]^{N-1} \tfrac{\theta_1 \theta_0^{\theta_1}(\theta_1(N-1))^{\theta_1}}{w^{\theta_1+1}(\theta_1(N-1)-1)^{\theta_1}} \\
f_w(w) & = N\left(\tfrac{\theta_1 \theta_0 (N-1)}{w(\theta_1(N-1)-1)}\right)^{\theta_1(N-1)} \tfrac{\theta_1 \theta_0^{\theta_1}(\theta_1(N-1))^{\theta_1}}{w^{\theta_1+1}(\theta_1(N-1)-1)^{\theta_1}} \\
f_w(w) & = N\tfrac{\theta_1 \theta_0^{\theta_1 N}(\theta_1(N-1))^{\theta_1 N}}{w^{\theta_1 N+1}(\theta_1(N-1)-1)^{\theta_1 N}}
\end{aligned}$$

Since we know that the support of the Pareto is $c > \theta_0$, since σ is a linear trasnformation of c, the support for $\sigma(c)$ is $\sigma(c) > \sigma(\theta_0) = \frac{\theta_0\theta_1(N-1)}{\theta_1(N-1)-1}$ The minimum of these shares the same support.

d

$$f(\theta_0, \theta_1) \quad = \log(N) + \log(\theta_1) + \theta_1 N \log(\theta_0) + \theta_1 N \log(\theta_1(N-1)) - (\theta_1 N + 1)\log(w) - \theta_1 N \log(\theta_1(N-1) - 1)$$

Since the support of our distribution depends upon a parameter, we will not be able to apply conventional maximum liklihood.

e

We seek to optimize the liklihood, with a constraint that all of the datapoints being within the support as determined by our choice of the parameters. This becomes equivalent to

$$\begin{aligned} &\max_{\theta_0,\theta_1} f(\theta_0, \theta_1) \\ \text{s.t. } &\frac{\theta_0\theta_1(N-1)}{\theta_1(N-1)-1} - w_i \leq 0 \\ &\theta_0 > 0 \\ &\theta_1 > 0 \end{aligned}$$

From the start the problem of strict inequalities arises, but we shall ignore that with the traditional hand wave of the economist. They shall become non-strict inequalities. Ignoring this problem, we next we must verify that this is in fact a convex optimization problem. We shall first investigate the constraint function:

$$\frac{\theta_0\theta_1(N-1) + \theta_0 - \theta_0}{\theta_1(N-1)-1} - w_i \leq 0$$
$$\theta_0 + \frac{\theta_0}{\theta_1(N-1)-1} - w_i \leq 0$$

Taking the hessian yields: $H(\theta_0, \theta_1, N) = \begin{bmatrix} 0 & \frac{-(N-1)}{(\theta_1(N-1)-1)^2} \\ \frac{-(N-1)}{(\theta_1(N-1)-1)^2} & \frac{2\theta_0(N-1)^2}{(\theta_1(N-1)-1)^3} \end{bmatrix}$

Clearly the determinant of this matrix is less than zero, since the product of the diagonals is zero, and since it is symmetric the product of the off-diagonals is positive. This implies the matrix is not positive definite, and the problem is not one of convex optimization.

At this point, we lack a convex optimization problem, and while we could try to throw this into a solver, we have no reason to believe that we would reach the global maximum, if we even converged to a local maximum.

Instead of despairing that we are not being able to solve this problem, we will approach it ala Paarsch and Donald (1993). We may estimate this lower bound of the support by $\underline{\hat{w}} = \min_t\{w_t | m_t =$

$m\}$ and by using $\theta_0(\theta_1, \underline{\hat{w}}, N) = \sigma^{-1}(\underline{\hat{w}})$ we may apply this to our liklihood function. Our liklihood function then becomes:

$$f(\theta_1) = \log\theta_1 + \log(N_t) + \theta_1(N_t)\log(\underline{\hat{w}}(N_t - 1)) - (\theta_1 N + 1)\log w_t$$

Taking the derivative of this will yeild us our estimate of θ_1. From this we can use Paarsch's estimate of $\hat{\theta_0}$ given by:

$$\theta_0^{\min} = \min\{\hat{\theta_0^1}, \hat{\theta_0^2}, ..., \theta_0^{\hat{N-1}}\}$$

Where $\theta_0^i = \underline{\hat{w}}\left(\frac{\hat{\theta_1}i-1}{\hat{\theta_1}i}\right)$