Operations Research HW6

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Question 1

a.

At time t, the number of customers $N(t)$ is the counting process of the Poisson Process. This has distribution: $N(t) \sim Poisson(10t)$
$P(N(t) = n) = \frac{e^{-10t}(10t)^n}{n!}$
$\mathbb{E}[N(t)] = 10t$. So during an 8 hour day, 80 Customers are expected.
$P(N(T) - N(T-1) = 3) = P(N(1) = 3) = \frac{e^{-10}10^3}{3!} \approx 0.00756665$

b

The nth customer's arrival time follow an Erlang(n, 10) distribution, which coincides with a gamma distribution for integer counts.
The Density function is given by: $\frac{10^n x^{n-1} e^{-\lambda x}}{(n-1)!}$
Since the mean of an erlang is $\frac{n}{10}$, we would expect it would take $\frac{4}{5}$ of an hour for eight people to arrive.
I am intpreting this question as: what is the probability that the eighth customer has arrived within one hour. $P(S_8 < 1) = \int_0^1 \frac{10^8 x^7 e^{-10x}}{7!} \approx .77978$

c

The inter-arrival time follows an exponential distribution, the parameter is: $\lambda = 10$ so $(S_{n+1} - S_n) \sim exp(10)$
$\mathbb{E}[S_{n+1} - S_n] = .1$ This is equivalent to 6 minutes.

Question 2

We first note that the queue described is an M/M/1 queue with $\lambda = \frac{2}{5}$ and $\mu = \frac{3}{2}$. We calculate $\rho = \frac{4}{15}$.

a

The Probability that the queue is idle is the probability that there is 0 people in the queue: This is $p_0 = (1 - \rho) = \frac{11}{15}$

b

The Expected number of people waiting in the queue is: $L_q = \frac{\rho^2}{1-\rho} = \frac{16}{165}$

c

The Expected Waiting time of people in the queue is: $W_q = \frac{1}{\mu - \lambda} - \frac{1}{\mu} = \frac{10}{15-4} - \frac{2}{3} = \frac{8}{33}$

d

Now the queue has been changed to an M/M/1/5 queue.

$$\begin{aligned}
p_0 &= \frac{1-\rho}{1-\rho^6} = \frac{759375}{1035139} \approx .73360 \\
p_5 &= \rho^5 \frac{1-\rho}{1-\rho^6} = \frac{1024}{1035139} \approx .000989239 \\
\lambda_{eff} &= \lambda(1-p_5) = \frac{413646}{1035139} \approx .39960430 \\
L_s &= \frac{\rho[1-6\rho^5+5\rho^6]}{(1-\rho)(1-\rho^6)} = \frac{374180}{1035139} \approx .36147802 \\
L_q &= L_s - \frac{\lambda_{eff}}{\mu} = \frac{98416}{1035139} \approx .09507515 \\
W_q &= \frac{L_q}{\lambda_{eff}} = \frac{49208}{206823} \approx .237923
\end{aligned}$$

Question 3

The Notation and formulas used in this question are found in "Modeling and Analysis of Stochastic Systems" Third Edition by Vidyadhar G. Kulkarni.
We note that this is an M/M/s queue. Since the arrival time is a Poisson Process with $\lambda = \frac{1}{3}$, but only 80% seek the queue; this is a split Poisson Process with parameter $\frac{4}{15}$. We can also see that: $\lambda = \frac{4}{15}, \mu = \frac{1}{5}, s = 2, r = \frac{\lambda}{\mu}, \rho = \frac{2}{3}$
Calculating the $p_n : p_0 = 1, p_1 = r = \frac{4}{3}, p_i = 2\rho^i, \forall i \geq 2$
$\sum_{n=0}^{\infty} \rho_n = 1 + \frac{4}{3} + \frac{2\rho^2}{1-\rho} = 1 + \frac{4}{3} + \frac{8}{3} = 5$

a

If an arriving customer waits in line, he must seek the service window, and there must be 2 or more already in queue. This is the compliment of there be being 0 or 1 persons in queue. Thus the probability that he waits in line is: $\frac{4}{5}(1 - p_0 - p_1) = \frac{4}{5}(1 - \frac{4}{15} - \frac{1}{5}) = \frac{32}{75}$

b

$p_0 = \frac{1}{\sum_{n=0}^{\infty} \rho_n} = \frac{1}{5}$

c

$L_q = \frac{\rho}{1-\rho} C(s,r)$ Where $C(s,r)$ is the Erlang-C formula $= \frac{\rho}{1-\rho} \frac{r^s \frac{s}{s-r}}{\sum_{j=0}^{s-1} \frac{r^j}{j!} + \frac{r^s s}{s!(s-r)}} = \frac{32}{15}$

d

If there was only one window, people would coninue to seek service with rate $\lambda = \frac{4}{15}$, however they would now be entering an M/M/1 queue with $\mu = \frac{1}{5}$ This would lead to a $\rho = \frac{4}{3} > 1$ Which has no steady-state distribution, as more people are entering than are being served. Thus reasonable service cannot be offered with only one window.