JOURNAL OF APPLIED ECONOMETRICS
J. Appl. Econ. **18**: 179–207 (2003)
Published online 14 January 2003 in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/jae.697

ASYMMETRY IN FIRST-PRICE AUCTIONS WITH AFFILIATED PRIVATE VALUES

SANDRA CAMPO,[a] ISABELLE PERRIGNE[b] AND QUANG VUONG[b]*

[a] *University of North Carolina, Department of Economics, Chapel Hill, NC 27599-3305, USA*
[b] *University of Southern California, Department of Economics, Los Angeles, CA 90089-0253, USA*

SUMMARY

Collusion and heterogeneity across firms may introduce asymmetry in bidding games. A major difficulty in asymmetric auctions is that the Bayesian Nash equilibrium strategies are solutions of an intractable system of differential equations. We propose a simple method for estimating asymmetric first-price auctions with affiliated private values. Considering two types of bidders, we show that these differential equations can be rewritten using the observed bid distribution. We establish the identification of the model, characterize its theoretical restrictions, and propose a two-step non-parametric estimation procedure for estimating the private value distributions. An empirical analysis of joint bidding in OCS auctions is provided.

1. INTRODUCTION

Starting from Paarsch (1992), these past years have seen the development of the structural approach for analysing auction data. Various estimation methods have been proposed mostly for the independent private value (IPV) paradigm. Donald and Paarsch (1993, 1996), and Laffont *et al.* (1995) have developed parametric estimation methods such as maximum likelihood and simulated non-linear least squares, while Elyakime *et al.* (1994) and Guerre *et al.* (2000) have proposed some non-parametric ones. More recently, relying on the latter, the structural approach has been extended to other auction paradigms such as the more general affiliated private value (APV) by Li *et al.* (2000, 2002) and the pure common value (CV) paradigm by Haile *et al.* (2000), Li *et al.* (2000) and Hendricks *et al.* (2001). A common feature of this literature is the consideration of symmetric auction models, where all bidders are *ex ante* identical.

Auction situations, however, provide many examples where the symmetry assumption is not tenable. This can arise because of bidders' differences in size as noted in Laffont *et al.* (1995), in geographic locations in Bajari (1999), Flambard and Perrigne (2001) and Hong and Shum (2002), and in capacity constraints in Jofre-Bonet and Pesendorfer (2000). Other examples include collusion among some bidders known as cartel (see Porter and Zona, 1993; Baldwin *et al.*, 1997; Pesendorfer, 2000; Bajari and Ye, 2002), asymmetrically informed bidders in OCS drainage auctions (see Hendricks and Porter, 1988; Hendricks *et al.*, 1994) or joint bidding in OCS auctions (see Hendricks and Porter, 1992). These examples illustrate the necessity of developing general structural econometric methods in asymmetric auctions. This is the main contribution of the paper for first-price sealed-bid auctions.

* Correspondence to: Quang Vuong, Department of Economics, University of Southern California, Los Angeles, CA 90089-0253, USA. E-mail: qvuong@usc.edu
Contract/grant sponsor: National Science Foundation; Contract/grant number: SBR-9631212.

Received 22 June 2000
Revised 7 November 2002

Simultaneously, the theoretical economic literature has seen a renewal of interest in asymmetric auctions. Hong (1997) characterizes the equilibria in ascending auctions within the pure CV and a particular APV paradigms. Within the IPV paradigm, Lebrun (1999), Maskin and Riley (2000a,b) and Bajari (2001) study the properties of the Bayesian Nash equilibrium in first-price sealed-bid auctions. In particular, these authors derive the system of differential equations characterizing the equilibrium strategies with two types of bidders. As is well known, however, a major difficulty is that no closed-form solutions can be obtained in general unlike in the symmetric case first studied by Vickrey (1961). This represents an important drawback for developing the structural approach since numerical methods are required to solve for the bidding strategies (see Marshall *et al.* , 1994; Bajari, 1999). As a result, only a few researchers (see Bajari, 1999; Crooke *et al.* , 1997; Jofre-Bonet and Pesendorfer, 2000; Flambard and Perrigne, 2001) have tackled the structural empirical analysis of asymmetric first-price sealed-bid auctions within the IPV paradigm.

In this paper, we consider the asymmetric APV model for first-price sealed-bid auctions, which encompasses the IPV model as a special case. It is well known that the IPV model can be restrictive in practice as bidders may have private values related to each other. In contrast, the APV model allows for dependence among bidders' private values while retaining a bidder's private utility for the auctioned object. See Li *et al.* (2000, 2002) for a discussion of the economic justification of the APV model in the symmetric case.[1] For the asymmetric APV model, we derive the differential equations that characterize the Bayesian Nash equilibrium strategies. For simplicity, we consider two types of bidders only but our method can be straightforwardly generalized to many types of bidders. We then establish the non-parametric identification of the model from observed bids and we propose a convenient two-step non-parametric procedure for estimating the underlying distribution of the model, namely the joint distribution of private values. Our method extends Guerre *et al.* (2000) to asymmetric auctions, and is especially convenient computationally as it circumvents the numerical resolution of the differential equations characterizing the equilibrium strategies. Moreover, because of its non-parametric nature, our method does not require *a priori* parametric specifications of the underlying private value distribution.

As an illustration, we analyse the effects of joint bidding in OCS wildcat auctions, specifically whether joint bidding can be a source of asymmetry among bidders. Such a practice allows a firms' consortium to submit one bid and was allowed to some extent by the federal government. The motivation was to encourage the participation of relatively small oil companies because of the large capital requirements for buying and developing OCS leases (see Hendricks and Porter, 1992). Two major issues then need to be addressed. First, it is likely that unobserved tract heterogeneity is present, affecting in particular the number of participants and the relative proportion of bidding consortia on any given tract. We show that our estimation method can deal with such unobserved heterogeneity under some reasonable assumptions. Second, we show that the asymmetric pure CV model is observationally equivalent to some asymmetric APV model while imposing some additional testable restrictions. The latter are not supported by the data.

The paper is organized as follows. In Section 2, we introduce the asymmetric APV model with two types of bidders and derive the system of differential equations characterizing the equilibrium strategies. We then establish its non-parametric identifiability and characterize the restrictions on

[1] Intuitively, affiliation means that when one bidder evaluates the object highly it is likely that others will evaluate it highly too. The APV model is a special case of the general (symmetric) affiliated value model developed by Wilson (1977) and Milgrom and Weber (1982). This general model encompasses both the symmetric IPV and pure CV models as polar cases.

the bid distribution imposed by the model to assess its empirical validity. We also propose a two-step non-parametric procedure to estimate the underlying private values distributions of the model. Section 3 illustrates our estimation procedure and presents our empirical findings. Unobserved tract heterogeneity and the pure CV assumption are discussed there. Section 4 collects some concluding remarks. Proofs are given in an appendix.

2. THE MODEL AND THE STRUCTURAL APPROACH

In this section, we first present the asymmetric APV model. We establish its non-parametric identification and we characterize the restrictions it imposes on the observed bid distribution. We then propose a computationally convenient two-step nonparametric procedure for estimating the latent distribution of the asymmetric APV model.

2.1. The Asymmetric APV Model

A single and indivisible object is auctioned to n bidders who are assumed to be risk neutral. Though our results can be easily generalized to a larger number of types, for simplicity, we assume that there are only two types of bidders as in Maskin and Riley (2000a). For instance, subgroup G1 can be characterized by larger bidders, better-informed bidders, a cartel of bidders or joint bidders as is the case in Section 3. This group is referred as the group of 'strong' bidders. Subgroup G0 gathers the other bidders, i.e. the 'weak' bidders. Type 1 contains n_1 bidders while type 0 contains n_0 bidders, with $n_1 + n_0 = n$ and $n \geq 2$. Let v_{1i}, $i = 1, \ldots, n_1$, denote the strong bidders' private values and v_{0i}, $i = 1, \ldots, n_0$ be the weak bidders' private values.

It is assumed that the vector $(v_{11}, \ldots, v_{1n_1}, v_{01}, \ldots, v_{0n_0})$ is the realization of a random vector whose n-dimensional cumulative distribution function is $F(\cdot)$. The latter is assumed to belong to the set $\mathcal{P}$ of n-dimensional absolutely continuous (with respect to Lebesgue measure) distributions with hypercube support that are affiliated and exchangeable (symmetric) in the first n_1 and last n_0 arguments. Let $[\underline{v}, \overline{v}]^n$ denote the support of $F(\cdot)$, where $\underline{v} \geq 0$.[2] This probabilistic structure can be interpreted as follows. While there is symmetry within each subgroup as bidders in each group jointly draw their private values from an affiliated exchangeable distribution, there is possible asymmetry between the two subgroups since their marginal distributions may differ across subgroups. Moreover, because of affiliation, there is general dependence among all private values. The distribution $F(\cdot)$ is assumed to be common knowledge and defines the asymmetric APV model. Each player i knows the value of his own signal, but does not know other players' private signals. We focus below on the first-price sealed-bid auction.

At the Bayesian Nash equilibrium, each bidder i of type 1 chooses his bid b_{1i} to maximize his expected payoff $E[(v_{1i} - b_{1i})\mathbb{I}(B_{-i} \leq b_{1i})|v_{1i}]$, where $B_{-i} = \max\{s_1(y_{1i}^*), s_0(y_{0i})\}$, $y_{1i}^* = \max_{j \neq i, j \in G1} v_{1j}$ and $y_{0i} = \max_{j \in G0} v_{0j}$, $s_1(\cdot)$ and $s_0(\cdot)$ are the equilibrium strategies of bidders of types 1 and 0, respectively. The term $E[\cdot|v_{1i}]$ denotes the expectation with respect to all random elements conditional upon v_{1i}. As usual, we restrict ourselves to strictly increasing differentiable equilibrium strategies and, because of the symmetry within each group, we assume that bidders

[2] See Milgrom and Weber (1982) for a formal definition of affiliation. As in the theoretical literature on asymmetric auctions, we assume that all private values have the same finite support, namely $[\underline{v}, \overline{v}]$. See Maskin and Riley (2000a).

in the same subgroup adopt the same strategy.[3] Hence the problem for any bidder i of type 1 can be written as

$$\max_{b_{1i}}(v_{1i} - b_{1i}) \Pr(y_{1i}^* \leq s_1^{-1}(b_{1i}) \text{ and } y_{0i} \leq s_0^{-1}(b_{1i})|v_{1i})$$

where $s_j^{-1}(\cdot)$ denotes the inverse of the equilibrium strategy $s_j(\cdot)$, $j = 0, 1$. The above probability can be written as $F_{y_1^*,y_0|v_1}(s_1^{-1}(b_{1i}), s_0^{-1}(b_{1i})|v_{1i})$.

Differentiating with respect to b_{1i}, the equilibrium strategy $s_1(\cdot)$ for any strong bidder $i, i = 1, \ldots, n_1$ satisfies the first-order differential equation

$$\begin{aligned}
&- F_{y_1^*,y_0|v_1}(s_1^{-1}(b_{1i}), s_0^{-1}(b_{1i})|v_{1i}) \\
&+ (v_{1i} - b_{1i}) \left[\frac{\partial F_{y_1^*,y_0|v_1}(s_1^{-1}(b_{1i}), s_0^{-1}(b_{1i})|v_{1i})}{\partial y_1^*} \times \frac{1}{s_1'(s_1^{-1}(b_{1i}))} \right. \\
&\left. + \frac{\partial F_{y_1^*,y_0|v_1}(s_1^{-1}(b_{1i}), s_0^{-1}(b_{1i})|v_{1i})}{\partial y_0} \times \frac{1}{s_0'(s_0^{-1}(b_{1i}))} \right] = 0
\end{aligned} \tag{1}$$

for all $v_{1i} \in [\underline{v}, \overline{v}]$, where $b_{1i} = s_1(v_{1i})$. Similarly, the equilibrium strategy $s_0(\cdot)$ for any weak bidder $i, i = 1, \ldots, n_0$ satisfies the first-order differential equation

$$\begin{aligned}
&- F_{y_1,y_0^*|v_0}(s_1^{-1}(b_{0i}), s_0^{-1}(b_{0i})|v_{0i}) \\
&+ (v_{0i} - b_{0i}) \left[\frac{\partial F_{y1,y_0^*|v_0}(s_1^{-1}(b_{0i}), s_0^{-1}(b_{0i})|v_{0i})}{\partial y_1} \times \frac{1}{s_1'(s_1^{-1}(b_{0i}))} \right. \\
&\left. + \frac{\partial F_{y1,y_0^*|v_0}(s_1^{-1}(b_{0i}), s_0^{-1}(b_{0i})|v_{0i})}{\partial y_0^*} \times \frac{1}{s_0'(s_0^{-1}(b_{0i}))} \right] = 0
\end{aligned} \tag{2}$$

for all $v_{0i} \in [\underline{v}, \overline{v}]$, where $b_{0i} = s_0(v_{0i})$.

The equilibrium strategies $s_1(\cdot)$ and $s_0(\cdot)$ are the solutions of the system of differential equations (1)–(2) subject to the boundary conditions $s_1(\underline{v}) = s_0(\underline{v}) = \underline{v}$ and $s_1(\overline{v}) = s_0(\overline{v})$. This system is quite complex and intractable in general. When private values are mutually independent, it can be verified readily that this system reduces to the system of differential equations characterizing the Bayesian Nash equilibrium in the asymmetric IPV model studied by Maskin and Riley (2000a).[4] Moreover, when $n_1 = 0$, or $n_0 = 0$, or $F(\cdot)$ is exchangeable in all its n arguments, it can be verified that the system (1)–(2) reduces to the single differential equation characterizing the symmetric Bayesian Nash equilibrium in the symmetric APV model studied in Li *et al.* (2002).

[3] In the asymmetric IPV case, Lebrun (1999) shows that the Bayesian Nash equilibrium exists and is unique. Moreover, it is in continuous strictly increasing pure strategies, which are fully characterized by the first-order and boundary conditions only. For the asymmetric APV case, Maskin and Riley (2000a) and Athey (2001) have established the existence of a monotonic pure strategy Nash equilibrium. See also Lizzeri and Persico (2000) who consider a broad class of games with a common value component.

[4] Because of its intractibility even in the IPV case, Marshall *et al.* (1994) propose some numerical algorithms for solving the system when the latent distributions are uniform, while Bajari (1999) proposes some numerical procedures within a Bayesian estimation context.

2.2. Non-Parametric Identification

The structural approach relies upon the hypothesis that observed bids are the equilibrium bids of the auction model under consideration. Depending on their type, bidders will bid according to the equilibrium strategies $s_1(\cdot)$ or $s_0(\cdot)$ defined by the system (1)–(2) of differential equations. Formally, this means that, given an n-dimensional joint distribution $F(\cdot)$ of private values belonging to $\mathcal{P}$, the structural econometric model is

$$\begin{cases} b_{1i} = s_1(v_{1i}, F), i = 1, \ldots, n_1 \\ b_{0i} = s_0(v_{0i}, F), i = 1, \ldots, n_0 \end{cases} \tag{3}$$

where the dependence on $F(\cdot)$ of both equilibrium strategies $s_1(\cdot)$ and $s_0(\cdot)$ is indicated. As private values are random, bids are naturally random with an n-dimensional joint distribution $G(\cdot)$. The equilibrium bid distribution $G(\cdot)$ depends on the underlying distribution $F(\cdot)$ in two ways: (i) through the unobservables $(v_{11}, \ldots, v_{1n_1}, v_{01}, \ldots, v_{0n_0})$ which are jointly drawn from the distribution $F(\cdot)$, and (ii) through the equilibrium strategies $s_1(\cdot)$ and $s_0(\cdot)$ that are both complex functions of $F(\cdot)$. This feature is common to auction models and complicates their identification and structural estimation. See Guerre *et al.* (2000) for a discussion.

The structural element of the asymmetric APV model is the joint distribution $F(\cdot)$ of private values. A question of interest is to know whether this distribution is identified from observables, which are the bids $(b_{11}, \ldots, b_{1n_1}, b_{01}, \ldots, b_{0n_0})$. In particular, the number of bidders of each type as well as the bid and type of each bidder are assumed to be observed. A second issue is whether the structural econometric model (3) imposes some restriction(s) on the joint distribution of observed bids. In other words, can any bid distribution $G(\cdot)$ be rationalized by an asymmetric APV model? Such a question is important for assessing the empirical validity of the asymmetric APV model.

We introduce the conditional distribution $G_{B_1^*,B_0|b_1}$ of (B_1^*, B_0) given b_1, where $B_1^* = \max_{i\neq 1} b_{1i}$ and $B_0 = \max_i b_{0i}$. This gives

$$\begin{aligned} G_{B_1^*,B_0|b_1}(X, X|x) &= \Pr(B_1^* \leq X, B_0 \leq X | b_1 = x) \\ &= \Pr(y_1^* \leq s_1^{-1}(X), y_0 \leq s_0^{-1}(X) | v_1 = s_1^{-1}(x)) \\ &= F_{y_1^*,y_0|v_1}(s_1^{-1}(X), s_0^{-1}(X) | s_1^{-1}(x)) \end{aligned}$$

It follows that

$$\begin{aligned} \frac{dG_{B_1^*,B_0|b_1}(X, X|x)}{dX} &= \frac{\partial F_{y_1^*,y_0|v_1}(s_1^{-1}(X), s_0^{-1}(X)|x)}{\partial y_1^*} \times \frac{1}{s_1'(s_1^{-1}(X))} \\ &\quad + \frac{\partial F_{y_1^*,y_0|v_1}(s_1^{-1}(X), s_0^{-1}(X)|x)}{\partial y_0} \times \frac{1}{s_0'(s_0^{-1}(X))} \end{aligned}$$

Using the last two equations and $v_1 = s_1^{-1}(b_1)$, the first-order differential equation (1) can be written as

$$v_1 = b_1 + \frac{G_{B_1^*,B_0|b_1}(b_1, b_1|b_1)}{dG_{B_1^*,B_0|b_1}(b_1, b_1|b_1)/dX} \equiv \xi_1(b_1, G) \tag{4}$$

A similar treatment applies to the first-order differential equation (2) by introducing the conditional distribution $G_{B_1,B_0^*|b_0}(\cdot,\cdot|\cdot)$ and the derivative $dG_{B_1,B_0^*|b_0}(\cdot,\cdot|\cdot)/dX$. This gives

$$v_0 = b_0 + \frac{G_{B_1,B_0^*|b_0}(b_0, b_0|b_0)}{dG_{B_1,B_0^*|b_0}(b_0, b_0|b_0)/dX} \equiv \xi_0(b_0, G) \tag{5}$$

where $B_1 = \max_i b_{1i}$ and $B_0^* = \max_{i\neq 1} b_{0i}$. Hence, each private value in any of the two subgroups can be expressed as a function of the corresponding bid, an appropriate conditional bid distribution and its *total* derivative without solving the system of differential equations (1)–(2). As in Guerre *et al.* (2000) and Li *et al.* (2002), this is the key result that allows us to identify the asymmetric APV model as well as to estimate it easily without solving the mathematically intractable system of differential equations (1)–(2). In contrast to these previous papers, (4)–(5) involve in general a trivariate distribution and a total derivative in their denominators, which is not a density.

The next proposition, whose proof is given in the Appendix, shows that the asymmetric APV model is identified from observed bids. It also gives a necessary and sufficient condition on the joint distribution $G(\cdot)$ of observed bids for the existence of a latent distribution $F(\cdot)\in\mathcal{P}$ that can rationalize the bid distribution $G(\cdot)$, i.e. for which $G(\cdot)$ is the corresponding equilibrium bid distribution.

Proposition 1: The asymmetric APV model is identified. Moreover, the joint distribution $G(\cdot)$ of observed bids can be rationalized by an asymmetric APV model with $F(\cdot)\in\mathcal{P}$ if and only if (i) $G(\cdot)$ belongs to $\mathcal{P}$, and (ii) the functions $\xi_1(b, G)$ and $\xi_0(b, G)$ are strictly increasing in $b \in [\underline{b}, \overline{b}]$ with $\xi_1(\overline{b}, G) = \xi_0(\overline{b}, G)$, where $[\underline{b}, \overline{b}]^n$ is the support of $G(\cdot)$.[5]

The identification result of Proposition 1 is non-parametric in nature as it does not require any parametric specification of $F(\cdot)$. This contrasts with parametric identification results which can be achieved through misspecified parametric specifications. Second, it complements the few existing identification results for asymmetric models (see Laffont and Vuong, 1996, for the IPV model and a special CV model). Third, our result extends the identification result established by Li *et al.* (2002) for the symmetric APV model. Unlike the symmetric APV model, however, it is unknown whether the asymmetric APV model is the most general auction model identified from observed bids in the class of asymmetric auction models.[6] Fourth, because $\mathcal{P}$ allows $F(\cdot)$ to be exchangeable in all its n arguments, our identification result implies that the asymmetric APV model can be distinguished from the symmetric one in view of observed bids. This is used to assess the presence of asymmetry in Section 3.

Besides identification, Proposition 1 characterizes the game-theoretic restrictions imposed by the asymmetric APV model on the distribution of observed bids. In particular, the fact that bidders adopt the Bayesian Nash equilibrium strategies defined by the system of first-order differential equations (1)–(2) imposes some restrictions on the distribution of observed bids in the form of the monotonicity of the functions $\xi_1(\cdot, G)$ and $\xi_0(\cdot, G)$. This monotonicity can be used as a basis of a formal test of the validity of the theoretical model. Rejection of such restrictions would imply

[5] We assume that the first-order conditions (1)-(2) with boundary conditions are sufficient for characterizing the equilibrium strategies, i.e. the second-order conditions are automatically satisfied. See also footnote 3.

[6] This question is related to the problem of whether two asymmetric auction models can be discriminated from each other from observed bids as discussed in Laffont and Vuong (1996).

that bidders do not adopt a Bayesian Nash equilibrium and/or that one or more of the underlying assumptions of the model are violated. These include situations with some common value, bidders' risk aversion or an unknown number of participants.

It should be noted that the rationalization conditions given in Proposition 1 and the identification of the APV model provide an empirical response to the existence and uniqueness of the Bayesian Nash equilibrium. In particular, if the observed bid distribution satisfies the rationalization conditions, then such a distribution can be viewed as the outcome of a (strictly increasing and differentiable) Bayesian Nash equilibrium of an APV model. Moreover, the identification property shows that such an APV model is unique whenever only strictly increasing and differentiable Bayesian Nash equilibria are considered.

From equations (4) and (5), we note that, provided $G_{B_1^*,B_0|b_1}(\cdot,\cdot|\cdot)$, $G_{B_1,B_0^*|b_0}(\cdot,\cdot|\cdot)$ and their total derivatives are known, one has neither to solve the complex system of differential equations (1)–(2) for determining the equilibrium strategies $s_1(\cdot)$ and $s_0(\cdot)$. Specifically, knowledge of $G_{B_1^*,B_0|b_1}(\cdot,\cdot|\cdot)$ and $G_{B_1,B_0^*|b_0}(\cdot,\cdot|\cdot)$ and hence of $\xi_1(\cdot)$ and $\xi_0(\cdot)$ determines the private values v_1 and v_0 for any given bid b_1 and b_0 through (4) and (5), respectively. This provides a method for circumventing the extreme computational difficulties encountered in the structural analysis of auction data with asymmetric bidders. These equations form the basis upon which our proposed estimation procedure rests.

2.3. Structural Estimation

We focus hereafter on the bivariate densities $f_{1,1}(\cdot,\cdot)$, $f_{0,0}(\cdot,\cdot)$ and $f_{10}(\cdot,\cdot)$ of the pairs (v_{1i}, v_{1j}), (v_{0i}, v_{0j}) and (v_{1i}, v_{0j}), respectively.[7] If one knew $G_{B_1^*,B_0|b_1}(\cdot,\cdot|\cdot)$ and $G_{B_1,B_0^*|b_0}(\cdot,\cdot|\cdot)$, one could use (4) and (5) to compute the private values for all bidders and then estimate $f(\cdot)$ from the latter. This suggests the following two-step estimation procedure. In a first step, the conditional distributions $G_{B_1^*,B_0|b_1}(\cdot,\cdot|\cdot)$ and $G_{B_1,B_0^*|b_0}(\cdot,\cdot|\cdot)$ and their total derivatives are estimated from observed bids. In a second step, private values estimated from (4) and (5) are used to estimate the aforementioned densities of private values. Though parametric methods could be used, in the continuation of our non-parametric identification result, we consider non-parametric techniques in each step. Specifically, the procedure is as follows.

- Step 1. Construct a sample of pseudo private values based on (4) and (5) using non-parametric estimates of $G_{B_1^*,B_0|b_1}(\cdot,\cdot|\cdot)$, $dG_{B_1^*,B_0|b_1}(\cdot,\cdot|\cdot)/dX$, $G_{B_1,B_0^*|b_0}(\cdot,\cdot|\cdot)$, and $dG_{B_1,B_0^*|b_0}(\cdot,\cdot|\cdot)/dX$ from observed bids.
- Step 2. Use the pseudo private values constructed in Step 1 to estimate non-parametrically the bivariate densities of interest.

The analysis must be performed separately for each given pair (n_1, n_0) since the bid distributions and the inverse bidding strategies $\xi_1(\cdot)$ and $\xi_0(\cdot)$ estimated non-parametrically in the first step actually depend on (n_1, n_0). This is so because the bidding strategy of any bidder depends on the number and types of his opponents. If n_1 or n_0 is equal to zero, there is no asymmetry and the estimation procedure reduces to that presented in Li *et al.* (2002). On the other hand, when

[7] Knowing these three bivariate densities, however, is not sufficient for recovering the joint distribution $F(\cdot)$. Our results can be extended to the estimation of other multivariate densities. A vector of observed characteristics Z for the auctioned objects can be included as in Guerre *et al.* (2000).

n_1 and n_0 are both strictly positive, the estimation procedure is in general more involved with a trivariate distribution and a total derivative. In particular, the analysis cannot be done by applying Li *et al.*'s (2002) method separately on each group.

Hereafter, we focus on the case where n_0 and n_1 are both strictly positive. Let L be the number of corresponding auctions and let ℓ index the ℓth auction, $\ell = 1, \ldots, L$. In the first step, we note that the conditioning on b_1 or b_0 disappears from the ratios in (4) and (5) as, for instance, the ratio in (4) can be interpreted as

$$\frac{\Pr(B_1^* \le b_1, B_0 \le b_1, b_1 = b_1)}{\Pr(B_1^* = b_1, B_0 \le b_1, b_1 = b_1) + \Pr(B_1^* \le b_1, B_0 = b_1, b_1 = b_1)} \tag{6}$$

Hence, using the observed bids $\{b_{1i\ell}; i = 1, \ldots, n_1\}$ and $\{b_{0i\ell}; i = 1, \ldots, n_0\}$ for $\ell = 1, \ldots, L$, we can estimate the numerators $G_{B_1^*,B_0,b_1}(b_1, b_1, b_1)$ and $G_{B_1,B_0^*,b_0}(b_0, b_0, b_0)$ non-parametrically by $\hat{G}_{B_1^*,B_0,b_1}(b_1, b_1, b_1)$ and $\hat{G}_{B_1,B_0^*,b_0}(b_0, b_0, b_0)$, where

$$\hat{G}_{B_1^*,B_0,b_1}(x, y, z) = \frac{1}{h_{G1}L} \sum_{\ell=1}^{L} \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbb{I}(B_{1i\ell}^* \le x)\mathbb{I}(B_{0\ell} \le y) K_G\left(\frac{z - b_{1i\ell}}{h_{G1}}\right)$$

$$\hat{G}_{B_1,B_0^*,b_0}(x, y, z) = \frac{1}{h_{G0}L} \sum_{\ell=1}^{L} \frac{1}{n_0} \sum_{i=1}^{n_0} \mathbb{I}(B_{1\ell} \le x)\mathbb{I}(B_{0i\ell}^* \le y) K_G\left(\frac{z - b_{0i\ell}}{h_{G0}}\right)$$

with $\mathbb{I}(\cdot)$ the indicator function, $B_{1i\ell}^* = \max_{j \ne i} b_{1j\ell}$, $B_{0\ell} = \max_i b_{0i\ell}$, $B_{0i\ell}^* = \max_{j \ne i} b_{0j\ell}$, $B_{1\ell} = \max_i b_{1i\ell}$, some bandwidths h_{G1} and h_{G0} and a kernel $K_G(\cdot)$. Note that we have used symmetry within each subgroup by averaging over n_1 and n_0, respectively. Regarding the denominator of (6), we estimate it by the sum of $\hat{D}_{11}(b_1, b_1, b_1)$ and $\hat{D}_{12}(b_1, b_1, b_1)$, where

$$\hat{D}_{11}(x, y, z) = \frac{1}{h_{g1}^2 L} \sum_{\ell=1}^{L} \frac{1}{n_1} \sum_{i=1}^{n_1} K_g\left(\frac{x - B_{1i\ell}^*}{h_{g1}}\right) \mathbb{I}(B_{0\ell} \le y) K_g\left(\frac{z - b_{1i\ell}}{h_{g1}}\right)$$

$$\hat{D}_{12}(x, y, z) = \frac{1}{h_{g1}^2 L} \sum_{\ell=1}^{L} \frac{1}{n_1} \sum_{i=1}^{n_1} \mathbb{I}(B_{1i\ell}^* \le x) K_g\left(\frac{y - B_{0\ell}}{h_{g1}}\right) K_g\left(\frac{z - b_{1i\ell}}{h_{g1}}\right)$$

with a bandwidth h_{g1} and a kernel $K_g(\cdot)$. Similarly, we can estimate the denominator of (6) corresponding to type 0 using

$$\hat{D}_{01}(x, y, z) = \frac{1}{h_{g0}^2 L} \sum_{\ell=1}^{L} \frac{1}{n_0} \sum_{i=1}^{n_0} K_g\left(\frac{x - B_{1\ell}}{h_{g0}}\right) \mathbb{I}(B_{0i\ell}^* \le y) K_g\left(\frac{z - b_{0i\ell}}{h_{g0}}\right)$$

$$\hat{D}_{02}(x, y, z) = \frac{1}{h_{g0}^2 L} \sum_{\ell=1}^{L} \frac{1}{n_0} \sum_{i=1}^{n_0} \mathbb{I}(B_{1\ell} \le x) K_g\left(\frac{y - B_{0i\ell}^*}{h_{g0}}\right) K_g\left(\frac{z - b_{0i\ell}}{h_{g0}}\right)$$

where h_{g0} is a bandwidth. Hence, estimates of the private values $v_{1i\ell}$ and $v_{0i\ell}$ are

$$\hat{v}_{1i\ell} = \hat{\xi}_1(b_{1i\ell}) = b_{1i\ell} + \frac{\hat{G}_{B_1^*,B_0,b_1}(b_{1i\ell}, b_{1i\ell}, b_{1i\ell})}{\hat{D}_{11}(b_{1i\ell}, b_{1i\ell}, b_{1i\ell}) + \hat{D}_{12}(b_{1i\ell}, b_{1i\ell}, b_{1i\ell})} \tag{7}$$

$$\hat{v}_{0i\ell} = \hat{\xi}_0(b_{0i\ell}) = b_{0i\ell} + \frac{\hat{G}_{B_1,B_0^*,b_0}(b_{0i\ell}, b_{0i\ell}, b_{0i\ell})}{\hat{D}_{01}(b_{0i\ell}, b_{0i\ell}, b_{0i\ell}) + \hat{D}_{02}(b_{0i\ell}, b_{0i\ell}, b_{0i\ell})} \tag{8}$$

for $i = 1, \ldots, n_1$ and $i = 1, \ldots, n_0$, respectively. Because of well-known boundary effects in non-parametric estimation, private values as defined in (7) and (8) may not be well estimated near the boundaries. This effect is corrected by introducing a trimming, which is explained in the next subsection.

The second step consists in the estimation of various densities of interest, such as $f_{11}(\cdot, \cdot)$, $f_{00}(\cdot, \cdot)$, and $f_{10}(\cdot, \cdot)$, using the pseudo private values $\{\hat{v}_{1i\ell}, i = 1, \ldots, n_1, \hat{v}_{0i\ell}, i = 1, \ldots, n_0, \ell = 1, \ldots, L\}$. To estimate $f_{11}(\cdot, \cdot)$, let $f_{(i,j)_1}(\cdot, \cdot)$ be the bivariate density of the pair (v_{1i}, v_{1j}). This density can be estimated non-parametrically by

$$\hat{f}_{(i,j)_1}(x, y) = \frac{1}{h_{f1}^2 L} \sum_{\ell=1}^{L} K_f \left(\frac{x - \hat{v}_{1i\ell}}{h_{f1}}, \frac{y - \hat{v}_{1j\ell}}{h_{f1}} \right)$$

where h_{f1} is a bandwidth and $K_f(\cdot)$ is a bivariate kernel. Imposing symmetry, this gives $\hat{f}^S_{(i,j)_1} = [\hat{f}_{(i,j)_1}(x, y) + \hat{f}_{(i,j)_1}(y, x)]/2$. Thus, by considering all possible pairs (i, j), a (symmetric) estimate of the bivariate density of private values for strong bidders is

$$\hat{f}_{11}(x, y) = \frac{2!(n_1 - 2)!}{n_1!} \sum_{i<j} \hat{f}^S_{(i,j)_1}(x, y) \tag{9}$$

Similarly, a (symmetric) estimate of the bivariate density of private values for weak bidders is

$$\hat{f}_{00}(x, y) = \frac{2!(n_0 - 2)!}{n_0!} \sum_{i<j} \hat{f}^S_{(i,j)_0}(x, y) \tag{10}$$

where $\hat{f}^S_{(i,j)_0} = [\hat{f}_{(i,j)_0}(x, y) + \hat{f}_{(i,j)_0}(y, x)]/2$, and

$$\hat{f}_{(i,j)_0}(x, y) = \frac{1}{h_{f0}^2 L} \sum_{\ell=1}^{L} K_f \left(\frac{x - \hat{v}_{0i\ell}}{h_{f0}}, \frac{y - \hat{v}_{0j\ell}}{h_{f0}} \right)$$

with h_{f0} a bandwidth. To estimate the bivariate density $f_{10}(\cdot, \cdot)$, we can proceed as follows. Let $\hat{f}_{i_1, j_0}(\cdot, \cdot)$ be the estimate of the bivariate density of the pair $(v_{1i\ell}, v_{0j\ell})$ for any $i = 1, \ldots, n_1$ and $j = 1, \ldots, n_0$, where

$$\hat{f}_{i_1, j_0}(x, y) = \frac{1}{h'_{f0} h'_{f1} L} \sum_{\ell=1}^{L} K_f \left(\frac{x - \hat{v}_{1i\ell}}{h'_{f1}}, \frac{y - \hat{v}_{0j\ell}}{h'_{f0}} \right)$$

and h'_{f1} and h'_{f0} are two bandwidths. The bivariate density $f_{10}(\cdot, \cdot)$ can be estimated by

$$\hat{f}_{10}(x, y) = \frac{1}{n_1 n_0} \sum_{i \in G1, j \in G0} \hat{f}_{i_1, j_0}(x, y) \tag{11}$$

Using a similar argument as in Li *et al.* (2002), it can be shown that our two-step estimators are uniformly consistent using appropriate bandwidths. Moreover, simulations performed in that paper show the good behaviour of the non-parametric estimator in small samples. The choice of bandwidths and kernel functions is discussed in the next subsection. Each bivariate density provides information on the degree of affiliation among bidders' private values within the same type or across types. In particular, the shape of $\hat{f}_{10}(\cdot,\cdot)$ tells us whether there is some asymmetry between bidders of type 1 and 0. Because our estimation method is fully non-parametric, such information is revealed by the data. A related advantage is that we do not have to parameterize the affiliation among private values, which can be linear or more complex. Lastly, a significant advantage of our method is its computational simplicity. Indeed, our method does not require solving the differential equations (1)-(2) and hence to compute the equilibrium strategies.[8]

2.4. Practical Issues

When working with real data, one frequently observes a highly skewed bid distribution with a large number of observations on the lower end. To minimize skewness effects, the data are transformed using a logarithmic function, which has been frequently used in empirical work. Using the logarithmic transformation, (4) and (5) become

$$v_1 = \exp(d_1)\left(1 + \frac{G_{D_1^*,D_0,d_1}(d_1, d_1|d_1)}{dG_{D_1^*,D_0,d_1}(d_1, d_1|d_1)/dX}\right) - 1 \equiv \tau_1(d_1) \tag{12}$$

$$v_0 = \exp(d_0)\left(1 + \frac{G_{D_1,D_0^*,d_0}(d_0, d_0|d_0)}{dG_{D_1,D_0^*,d_0}(d_0, d_0|d_0)/dX}\right) - 1 \equiv \tau_0(d_0) \tag{13}$$

where $d \equiv \log(1+b)$, $G_{D_1^*,D_0,d_1}(\cdot,\cdot|\cdot)$ is the conditional density of $(D_1^*, D_0) = (\max_{i\neq 1}\log(1+b_{1i}, \max_i \log(1+b_{0i}))$ given $\log(1+b_1)$, b_1 being chosen arbitrarily among n_1 values, and $dG_{D_1^*,D_0,d_1}(\cdot,\cdot|\cdot)/dX$ is the appropriate total derivative. The notation in (13) is similarly defined with $d_0 = \log(1+b_0)$.[9]

Next, we adopt a trimming similar to that of Guerre *et al.* (2000). Let $d_{\max} = \max\{d_{\max 1}, d_{\max 0}\}$, where $d_{\max 1}$ and $d_{\max 0}$ are the maximum values of the log-transformed strong bids and weak bids, respectively. Thus $d_{\max}$ estimates the common upper bound $\overline{d}$ of the supports of d_1 and d_0. For $j = 0, 1$, let $\hat{\tau}_j(\cdot)$ be the estimate of $\tau_j(\cdot)$ using the previously defined non-parametric estimators for bid distributions with $(d_{0i\ell}, d_{1i\ell})$ instead of $(b_{0i\ell}, b_{1i\ell})$. The pseudo private values $\hat{v}_{1i\ell}$ and $\hat{v}_{0j\ell}$ are defined as

$$\hat{v}_{1i\ell} = \begin{cases} \hat{\tau}_1(d_{1i\ell}) & \text{if} \max\{h_{G1}, h_{g1}\} \le d_{1i\ell} \le d_{\max} - \max\{h_{G1}, h_{g1}\} \\ +\infty & \text{otherwise} \end{cases}$$

and

$$\hat{v}_{0i\ell} = \begin{cases} \hat{\tau}_0(d_{0i\ell}) & \text{if} \max\{h_{G0}, h_{g0}\} \le d_{0i\ell} \le d_{\max} - \max\{h_{G0}, h_{g0}\} \\ +\infty & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n_1$ and $i = 1, \ldots, n_0$, respectively, and $\ell = 1, \ldots, L$. Because the kernel $K_f(\cdot)$ in the second step has a compact support, the auctions that are relevant for estimating the densities of interest are those for which the values $\hat{v}_{1i\ell}, i = 1, \ldots, n_1$ and $\hat{v}_{0i\ell}, i = 1, \ldots, n_0$ are all finite, i.e. not trimmed. Let L_T denote the number of remaining auctions.

[8] Under the assumption that the underlying private value distribution is independent of (n_1, n_0), one can pool the pseudo private values (7)–(8) obtained for every pair (n_1, n_0) to improve estimates of the above bivariate densities.

[9] The transformation $\log(1+\cdot)$ ensures that the support $[\underline{d}, \overline{d}]$ is included in $\Re^+$ and compact whenever $\overline{v} < \infty$. In the application, we assume that $\underline{v} = 0$ so that $\underline{b} = 0$ and hence $\underline{d} = 0$.

It remains to discuss the choice of kernels and bandwidths. As is well known, the choice of kernels does not have much effect in practice. We choose the triweight kernel, which satisfies the assumptions in Guerre *et al.* (2000). This kernel is of the form $K(u) = (35/32)(1 - u^2)^3 \mathbb{I}(|u| \leq 1)$. It is used for $K_g(\cdot)$ and $K_G(\cdot)$, while $K_f(\cdot, \cdot)$ is the product of two univariate triweight kernels.

In contrast, the choice of bandwidths requires more attention. We use bandwidths of the form $h_{G1} = c_{G1}(n_1 L)^{-1/5}$, $h_{g1} = c_{g1}(n_1 L)^{-1/6}$, $h_{G0} = c_{G0}(n_0 L)^{-1/5}$, $h_{g0} = c_{g0}(n_0 L)^{-1/6}$ for the first step, and $h_{f1} = c_{f1}(2!(n_1 - 2)!L_T/n_1!)^{-1/15}$, $h_{f0} = c_{f0}(2!(n_0 - 2)!L_T/n_0!)^{-1/15}$, $h'_{f1} = c_{f1}(n_1 n_0 L_T)^{-1/15}$, $h'_{f0} = c_{f0}(n_1 n_0 L_T)^{-1/15}$ for the second step, following Guerre *et al.* (2000) and Li *et al.* (2002). The factors involving n_1 and n_0 are due to the additional averaging of our estimators (see e.g. (9), (10), and (11)). Regarding the constants c_{G1}, c_{g1}, c_{G0}, c_{g0}, c_{f1}, and c_{f0} we note that all the bandwidths except those in the second step correspond to the usual rate so that their constants can be obtained by the so-called rule of thumb. Hence, we use $c_{G1} = c_{g1} = 2.978 \times 1.06\hat{\sigma}_{d_1}$ and $c_{G0} = c_{g0} = 2.978 \times 1.06\hat{\sigma}_{d_0}$, where $\hat{\sigma}_{d_1}$ and $\hat{\sigma}_{d_0}$ are the standard deviations of the log (1 + bids) for each type, respectively. The factor 2.978 follows from the use of the triweight kernel instead of the Gaussian kernel (see Hardle, 1991). For the bandwidths in the second step, we use constants $c_{f1} = 2.978 \times 1.06\hat{\sigma}_{v1}$ and $c_{f0} = 2.978 \times 1.06\hat{\sigma}_{v0}$, where $\hat{\sigma}_{v1}$ and $\hat{\sigma}_{v0}$ are the standard deviations of the trimmed pseudo private values of strong bidders and weak bidders, respectively.

3. AN APPLICATION TO JOINT BIDDING

This section illustrates our method with an empirical study of possible asymmetry arising from joint bidding in OCS wildcat auctions. We first present the data and motivate the problem. We then address two issues, namely unobserved heterogeneity and the adequacy of the pure CV model, that have been frequently raised in the empirical analysis of such data. Lastly, we apply our procedure and discuss our empirical results.

3.1. The Data

The US federal government began auctioning its mineral rights on oil and gas of offshore lands or Outer Continental Shelf (OCS) in 1954. Our study focuses on the wildcat lease sales in the Mexico and Louisiana gulfs. Before each sale, the government announces that an area is available for exploration. This area is divided into a number of tracts, usually a block of 5000 or 5760 acres. Leases of drilling rights on these tracts are sold simultaneously through first-price sealed-bid auctions. The highest bidder wins the tract and pays his bid.[10] The highest bidder wins the tract and pays his bid. The participants to each auction are oil companies.

A characteristic of interest in OCS auctions is the practice of joint bidding, which was allowed to some extent by the federal government.[11] Though joint bidding was allowed since 1954, its practice developed significantly in the 1970s. Before 1970 joint bidding affected fewer than 20% of auctions, while this percentage rose to more than 80% after 1970. On average a joint bid

[10] It can be considered that the reserve price at \$15 per acre does not act as a screening device for participating to the auction, as recognized by many economists. See e.g. McAfee and Vincent (1992).

[11] Until December 1975, any set of firms could organize a consortium so as to submit a so-called joint bid. Because of some concern about competition, the federal government restricted the practice of joint bidding by barring the eight largest firms from bidding jointly with each other.

involves two or three firms. A number of arguments for joint bidding have been given in the literature. For instance, joint bidding can weaken financial constraints, reduce costs by pooling cartel members' information and capital through the joint venture and spread risks among firms. See e.g. DeBrok and Smith (1983), Millsaps and Ott (1985), Gilley *et al.* (1985) and Hendricks and Porter (1992). As noted by many economists, however, joint bidding may have introduced some *ex ante* asymmetry among bidders.

Because joint bidding is negligible in the 1950s–1960s, our study focuses on auctions held between December 1972 and 1979.[12] Because of data requirements explained subsequently, we consider auctions with two bidders who can be either joint or solo. This gives a total of 227 auctions from which 55 auctions have two solo bids, 60 auctions have two joint bids and 112 auctions have one solo bid and one joint bid. Among the latter, 63 auctions are won by the joint bidder. Using a normal approximation, the ratio 63/112 is greater than 1/2 at the 10% significance level in a one-sided test, where 1/2 would be the expected ratio if the two participants have equal chance of winning.[13] Thus joint bidding has increased the probability of winning suggesting some *ex ante* asymmetry among participants.

For each wildcat auction, we know the date, the acreage of the tract, the number of bidders, their bids in constant 1972 dollars and whether the bid is a solo or a joint bid. Table I gives some summary statistics in $ per acre for the 454 bids considered in our empirical study as well as on solo and joint bids separately, whether the opponent's bid is of the same type or of a different type.

A first feature revealed by the means displayed in Table I is that joint bids tend to be higher on average than solo bids, as a number of empirical studies have found. Moreover, joint bidders tend to bid higher when they face a joint bidder than when they face a solo bidder. Likewise, though their bids are lower than those of joint bidders, solo bidders tend to bid on average higher when they face a joint bidder than when they face another solo bidder. This suggests that the bidding strategy of each type of bidder depends on the type of their opponent. This could arise from bidders taking into account some possible asymmetry in their bidding strategies. For instance, a test of the equality of means for solo bids versus joint bids in the 112 auctions with one bidder of each type gives a t-statistic equal to 1.66, which (weakly) rejects their equality. It is also interesting to note that the within variability of solo versus solo bids is much smaller than the within variability

Table I. Summary statistics on bids

Variable	# Obs	Mean	STD	Minimum	Maximum	Within STD
All bids	454	687.30	1,431.31	19.51	20,751.32	1,258.91
Joint bids	232	837.32	1,717.54	21.46	20,751.32	—
Solo bids	222	532.53	1,033.20	19.51	11,019.08	—
Joint vs joint	120	875.13	2,056.12	33.94	20,751.32	2,011.99
Joint vs solo	112	796.83	1,266.32	21.46	6,377.94	—
Solo vs joint	112	603.28	1,226.61	19.51	11,019.08	—
Solo vs solo	110	456.45	788.19	20.80	7,009.10	747.43

[12] We exclude auctions after 1979 since the rules of the auction mechanism have changed somewhat after this date. We also exclude the unique sale held in 1970 and the first sale in 1972 because the water depth of the tracts sold at these sales was much greater than usual.

[13] Hereafter, all tests are conducted at the 10% significance level.

of joint versus joint bids. This may again support the hypothesis of an asymmetry between joint and solo bidders.

A second feature of Table I is the large total variability of bids in all cases. This is confirmed by the wide range of bids. Such a large variability could arise from tract heterogeneity and/or bidders' heterogeneity. Tract heterogeneity may also explain the differences in means noted above. In particular, higher value tracts could attract more likely joint bidders than solo ones, and conversely for lower value tracts. That is, the joint/solo structure could be endogenous in the sense of depending on tract characteristics. Moreover, the within standard deviation is important relative to the total standard deviation ranging from 70% to 98%.[14] For instance, when considering auctions with two joint bidders, the within variability explains about 98% of the total variability of bids. This suggests that a large part of the differences across tracts can be explained by the joint/solo bidders' composition.

To further assess tract heterogeneity, we first regress the log of bids on a set of tract dummies and obtain a weak rejection of tract homogeneity. This may be due to the fact that the 227 tracts are sold through 16 sales spread over the 1972–1979 period. We then consider a regression with sale dummies and, in view of Table I, three bidding structure dummies (whether the bid is the result of a joint consortium, whether the opponent in the auction is a joint bidder, and an interaction dummy between these two dummies). The structure dummies coefficients are jointly significant while the equality of the 16 sale dummies is nearly accepted. Thus, controlling for bidding structure greatly decreases heterogeneity across tracts, while also controlling for changes in bidders' types over time. It remains to discuss whether the oil crisis has increased bids after 1973 as well as whether the 1975 ban on large firms bidding jointly has had some lowering effects on bids. Using dummies, none of these events turns out to be significant.[15]

3.2. Unobserved Heterogeneity and Common Value

In view of the previous analysis of bid data, we first discuss the heterogeneity issue within our econometric model presented in Section 2. Furthermore, the paradigm of the model in Section 2 is that of private values, while the pure CV model (also called the mineral right model) has been widely entertained in the empirical analysis of bidding in OCS auctions. This section derives some testable restrictions imposed by the (asymmetric) pure CV model, which allow us to bring some evidence on the debate private versus common value in such auctions.

Unobserved Heterogeneity

From a broader perspective, heterogeneity in bid data can arise from differences across tracts and/or differences among bidders. Such tracts and bidders' differences can be observed or unobserved by the analyst. Regarding bidders, their unobserved differences/heterogeneity are captured by their unobserved private values or information v_i, while the observed ones lead to an asymmetric game. For *ex ante* differences known to all bidders such as their size or location define the asymmetries in the auction game. As indicated in Section 1, the main contribution of this paper is to deal with observed heterogeneity among bidders despite generally intractable equilibrium strategies. Regarding tracts, their differences can be summarized by two vectors of characteristics, namely Z for the observed and W for the unobserved ones. The observed characteristics Z can be introduced

[14] The within-variability for auctions with solo and joint bids is equal to 877.04.
[15] All the regressions and tests results are available upon request from the authors.

by conditioning the latent distributions in the econometric model by Z as in Guerre *et al.* (2000) among others. As far as we know, the issue of unobserved heterogeneity has not been addressed formally in the previous literature.

We provide here some assumptions, which allow us to deal with unobserved tract heterogeneity. Without such assumptions, major identification issues arise in the structural approach as it is impossible to disentangle the two types of unobserved heterogeneity coming from v_i and W. Specifically, for the asymmetric APV model of Section 2, we make the following assumptions.

Assumptions:
(i) $n_1 = n_1(Z, W)$, $n_0 = n_0(Z, W)$ for some functions $n_0(\cdot)$ and $n_1(\cdot)$.
(ii) The joint distribution of private values conditional upon the characteristics (Z, W) of a tract is equal to the joint distribution of private values conditional upon (Z, n_1, n_0), i.e. $F(\cdot|Z, W) = F(\cdot|Z, n_1, n_0)$.

Assumption (i) requires that the number of bidders from each type is a deterministic function of the tract characteristics, observed and unobserved. This is a natural assumption satisfied by any entry model, which determines n_0 and n_1 endogenously, whenever bidders decide about their participation prior to knowing their private information as in McAfee and McMillan (1987) and Levin and Smith (1994). Intuitively, Assumption (ii) says that, conditionally upon Z, unobserved tract heterogeneity is fully aggregated into (n_1, n_0), i.e. the latter are sufficient statistics for W. It fully justifies our estimation procedure, which is conducted at (n_1, n_0) given.

On empirical grounds, Assumption (i) is compatible with the general belief that higher value tracts are more likely to attract joint bidders as pointed out in Section 3.1. Likewise, Assumption (ii) is consistent with our previous findings that within-variability is very important relative to total bid variability so that tract heterogeneity becomes negligible when conditioning upon the structure (n_1, n_0). Because no Z variable was found to be significant, tract heterogeneity is summarized by the structure (n_1, n_0) in our empirical analysis. Similarly, bidders' observed heterogeneity is reduced to the joint or solo nature of the bidder. This approximation is justified as solo bids pertain mostly to large firms, while joint bids pertain to consortia composed mainly by a large and some fringe firms, as noted by Hendricks and Porter (1992).

An important empirical issue is to assess whether differences in observed bids arise from structural asymmetry due to the solo or joint nature of the bidder or from unobserved tract heterogeneity. This issue can be answered under the preceding assumptions. Our estimation method delivers estimates of the joint densities $f(\cdot, \cdot|n_1, n_0)$ for (n_1, n_0) equal to (2,0), (0,2) and (1,1). A comparison of these densities, however, does not provide a clear answer to this issue as the structure (n_1, n_0) is likely to depend on the unobserved tract characteristics W. On the other hand, under our assumptions, a comparison of the marginal densities for the joint and solo bidders in the (1, 1) case provides a direct test of asymmetry among these two types of bidders. Moreover, the comparison of the marginal densities for the joint bidder in the (1, 1) and (2, 0) cases (for the solo bidder in the (1, 1) and (0, 2) cases) can indicate the presence of unobserved tract heterogeneity. For, if these two marginal densities are different, then tracts attracting bidders of each type differ from tracts attracting two joint (solo) bidders because of unobserved tract heterogeneity.

The Asymmetric Pure Common Value Model
The pure CV model has been largely considered for explaining bidding behaviour in OCS auctions. See Porter (1995) for a survey. Recently, Laffont (1997) has raised serious concerns about the

adequacy of the empirical evidence from reduced-form analyses supporting the CV paradigm in such auctions. This led to the recent debate of private versus common value in OCS auctions, while initiating a search for formal tests of either paradigm.

Using OCS auctions prior to 1970, Hendricks *et al.* (2002) consider a symmetric pure CV model. Their tests rely on *ex post* returns of tracts and a different boundary condition of the bid distribution under the CV and PV paradigms with a binding reserve price. They conclude that bidding behaviour appears to be more consistent with a pure symmetric CV model than a PV one, though recognizing that both components are probably present. Haile *et al.* (2000) also consider a symmetric game and develop an interesting non-parametric test for PV versus CV models. Their test requires that the underlying structure be independent of the number n of bidders, which precludes unobserved tract heterogeneity when n is endogenous. Moreover, their test requires that n varies across auctions, which is then exploited by estimating non-parametrically some functions for each number of bidders. As a result, this test is very demanding in auction data. In contrast, we study here asymmetric situations where unobserved heterogeneity may lead to latent distributions depending on the number of bidders because of unobserved tract heterogeneity (see Assumption (ii)). Moreover, our empirical analysis relies on auctions with two bidders. For, considering auctions with more than 2 bidders leads to a greater number of (n_1, n_0) situations to be entertained with fewer auctions in each case.[16] Thus, both previous tests do not apply to our situation.

In this section, we study the asymmetric pure CV model with two players and derive some restrictions imposed by such a model. The latter are used to propose some simple tests of the pure CV model in the (1,1) case. Following Wilson (1969), the asymmetric pure CV model is defined as follows. Consider 2 bidders bidding for a tract of unknown but common value V with density $f_V(\cdot)$ and support $[\underline{v}, \overline{v}]$. Each bidder $i, i = 0, 1$ receives a signal or estimate σ_i for V. Conditional upon V the signals are independent but not necessarily identically distributed with a joint density $f(\cdot, \cdot|V) = f_0(\cdot|V) \times f_1(\cdot|V)$ on the support $[\underline{\sigma}, \overline{\sigma}]^2$. Because $f_0(\cdot|\cdot)$ and $f_1(\cdot|\cdot)$ need not be equal, the players have disparate information and hence, are asymmetric *ex ante*. Each bidder has a utility function $U(\sigma_i, V) = V$, which defines the pure CV model.

The maximization for bidder 1 (say) is

$$\max_{b_1} \mathrm{E}[(V - b_1)\mathbb{I}(\sigma_0 \le s_0^{-1}(b_1))|\sigma_1]$$
$$= \max \int_{-\infty}^{s_0^{-1}(b_1)} V(\sigma_1, \sigma_0) f_{\sigma_0|\sigma_1}(\tilde{\sigma}_0|\sigma_1) d\tilde{\sigma}_0 - b_1 F_{\sigma_0|\sigma_1}(s_0^{-1}(b_1)|\sigma_1)$$

where $s_0(\cdot)$ is the strictly increasing and differentiable equilibrium strategy of bidder 0, $F_{\sigma_0|\sigma_1}(\cdot|\cdot)$ is the conditional distribution of σ_0 given σ_1 with density $f_{\sigma_0|\sigma_1}(\cdot|\cdot)$, and $V(\sigma_1, \sigma_0) \equiv \mathrm{E}[V|\sigma_1, \sigma_0]$. The first-order condition gives for any $\sigma_1 \in [\underline{\sigma}, \overline{\sigma}]$

$$-F_{\sigma_0|\sigma_1}(s_0^{-1}(b_1)|\sigma_1) + [V(\sigma_1, s_0^{-1}(b_1)) - b_1] \frac{f_{\sigma_0|\sigma_1}(s_0^{-1}(b_1)|\sigma_1)}{s_0'(s_0^{-1}(b_1))} = 0 \tag{14}$$

[16] With three bidders, we have 18 auctions for the (3,0) case, 19 auctions for the (0,3) case, and 48 and 60 auctions for the (2,1) and (1,2) cases, respectively. Moreover, the (2,1) and (1,2) cases involve the non-parametric estimation of trivariate distributions (see (4) and (5)). In contrast, the two bidders cases involve, at most, bivariate distributions. See (18) and (19).

Similarly, the first-order condition for bidder 0 gives for any $\sigma_0 \in [\underline{\sigma}, \overline{\sigma}]$

$$-F_{\sigma_1|\sigma_0}(s_1^{-1}(b_0)|\sigma_0) + [V(s_1^{-1}(b_0), \sigma_0) - b_0]\frac{f_{\sigma_1|\sigma_0}(s_1^{-1}(b_0)|\sigma_0)}{s_1'(s_1^{-1}(b_0))} = 0 \quad (15)$$

As in Section 2.2, we have $G_{b_0|b_1}(b_0|b_1) = F_{\sigma_0|\sigma_1}(s_0^{-1}(b_0)|s_1^{-1}(b_1))$ and $g_{b_0|b_1}(b_0|b_1) = f_{\sigma_0|\sigma_1}(s_0^{-1}(b_0)|s_1^{-1}(b_1))/s_0'(s_0^{-1}(b_0))$. So (14) and (15) become

$$V(\sigma_1, s_0^{-1}(b_1)) = b_1 + \frac{G_{b_0|b_1}(b_1|b_1)}{g_{b_0|b_1}(b_1|b_1)} = \xi_1(b_1, G) \quad (16)$$

$$V(s_1^{-1}(b_0), \sigma_0) = b_0 + \frac{G_{b_1|b_0}(b_0|b_0)}{g_{b_1|b_0}(b_0|b_0)} = \xi_0(b_0, G) \quad (17)$$

for any $\sigma_i \in [\underline{\sigma}, \overline{\sigma}]$, where $b_i = s_i(\sigma_i) \in [\underline{b}, \overline{b}]$, $i = 0, 1$. Equations (16) and (17) are directly comparable to (4) and (5) for the asymmetric APV model with two players. In particular, the right-hand sides of (4) and (16) (or (5) and (17)) are identical. On the other hand, the left-hand sides have a different economic interpretation. In the pure CV model, it is $V(\sigma_1, s_0^{-1}(b_1)) = \mathrm{E}[V|\sigma_1, s_0^{-1}(b_1))]$, while it is bidder's 1 private value v_1 in the APV model.

These equations are the key to the following proposition. We say that a bivariate distribution $G_{b_0,b_1}(\cdot, \cdot)$ with support $[\underline{b}, \overline{b}]^2$ is *quasisymmetric* if $\Pr[b_1 \leq b_0|b_1 \leq b, b_0 \leq b] = \Pr[b_1 \geq b_0|b_1 \leq b, b_0 \leq b]$ for any $b \in [\underline{b}, \overline{b}]$. In other words, a distribution is quasisymmetric if the likelihood for bid 1 being smaller than bid 0 is equal to one half given that both bids are less than any arbitrary value b. In particular, a distribution is quasisymmetric if it is exchangeable or symmetric. The converse is not true.

Proposition 2: Suppose that observed bids are the equilibrium bids of an asymmetric pure CV model with two players. Then their joint distribution $G(\cdot)$ must be quasisymmetric. The latter condition is equivalent to $\xi_1(b, G) = \xi_0(b, G)$ for any $b \in [\underline{b}, \overline{b}]$. Moreover, the asymmetric pure CV model is observationally equivalent to some asymmetric APV model with a quasisymmetric private value distribution and identical equilibrium strategies for both players.

To our knowledge, this result has not appeared in the literature. In particular, by letting $b = \overline{b}$, the quasisymmetry of the joint bid distribution implies that $\Pr(b_1 \leq b_0) = \Pr(b_0 \leq b_1) = 1/2$, i.e. at equilibrium both players have an equal probability of winning despite their disparate information. It is worth noting that a related result holds in the extreme asymmetric pure CV model, where one bidder is perfectly informed about the value V of the object, while the other bidder is completely uninformed. As shown by Wilson (1967) in this case, the equilibrium is such that the uninformed bidder adopts a mixed strategy that is identical to the bid distribution of the informed bidder. Hence, both players have the same probability of winning. See Hendricks and Porter (1988) and Hendricks *et al.* (1994) for empirical applications and extensions.

Such a property provides a very simple test for the adequacy of the pure CV model in the two players case irrespective of symmetry. For, if players were symmetric *ex ante*, then their probabilities of winning will also be equal. Hence, the probability of winning for one bidder should be one half whether or not the game is symmetric in the pure CV paradigm with two bidders. As noted in Section 3.1, among the 112 auctions with one solo and one joint bidders, 63 were won by the joint bidder, suggesting that the joint bidder has a higher probability of winning. To investigate further this issue, it is interesting to split the auctions in half and to test quasisymmetry on each half. Specifically, we consider the 56 auctions for which both bids are smaller than \$460 per acre and the 56 auctions for which at least one bid is larger than \$460. The joint bidder wins 28 times in the first subset and 35 times in the second subset. Quasisymmetry requires that the joint bidder's probability of winning is one half in each subset.[17] In the second subset, such an equality is rejected with a t-test statistic equal to 1.87. Thus, the pure CV model does not appear to be appropriate for these auctions. This does not, however, exclude the existence of a common value in a general model, where the utility of each player is of the form $U(\sigma_i, V)$, as in Wilson (1977). Such a model is known to be unidentified (see Laffont and Vuong, 1996). In our empirical analysis, we choose to consider the asymmetric APV model though we believe that most auction data contain both private and common values. Consideration of private values can be justified by important idiosyncratic differences among firms such as productive inefficiencies, capital constraints and opportunity costs. It should be noted that affiliation among private values may arise from an unknown common component (see Li *et al.* , 2000).

Regarding the last part of Proposition 2, note that the observationally equivalent APV model is a particular asymmetric APV model since its latent distribution must be quasisymmetric and the bidders' equilibrium strategies must be equal. Our result extends the observationally equivalence of the symmetric pure CV model and the symmetric APV model established in Laffont and Vuong (1996) to the asymmetric case. It also extends the observationally equivalence of the extreme asymmetric pure CV model and the symmetric IPV model established in Laffont and Vuong (1996). Proposition 2 states that any asymmetric pure CV model is observationally equivalent to some quasisymmetric APV model. When considering the extreme asymmetric pure CV case, the two players' bids are independent. Such independence combined with quasisymmetry leads to a symmetric bid distribution and hence, a symmetric IPV model.

3.3. Structural Estimation Results

We have 227 auctions of which 60 are with two joint bids, 55 with two solo bids and 112 with one joint and one solo bids corresponding to $(n_1, n_0) = (2, 0)$, $(n_1, n_0) = (0, 2)$ and $(n_1, n_0) = (1, 1)$, respectively. Following Section 2, estimation is performed separately for each pair. Because $n = 2$, the fundamental equations (4)-(5) simplify greatly.

Cases (2,0) and (0,2)

As only one type of bidder is involved, these cases correspond to the symmetric APV model studied by Li *et al.* (2002). We estimate both cases separately. For each case, we have pairs $(b_{1\ell}, b_{2\ell})$ from the same type, dropping the 1/0 index for type. Thus, the ratio in (4) and (5) reduces to $G_{b_2|b_1}(\cdot|\cdot)/g_{b_2|b_1}(\cdot|\cdot)$. Because of the log transformation (see Section 2.4), we estimate the joint distribution $G_{d,d}(\cdot,\cdot)$ from the pairs $(d_{1\ell}, d_{2\ell})$ as well as its corresponding density $g_{d,d}(\cdot,\cdot)$. The former can be estimated by averaging the product of a counting process and a kernel, while the latter can be estimated by averaging the product of two kernels as shown in Li *et al.* (2002). For the (2,0) case, the bandwidths h_G and h_g are equal to 1.41 and 1.65, while they are equal to 1.54 and 1.81 for the (0,2) case. Then, using an equation similar to (12), we obtain the pseudo private values for each case.

[17] It is easy to see that $\Pr[b_1 \le b_0 | b_1 \le b, b_0 \le b] = 1/2$ for any b is equivalent to $\Pr[b_1 \le b_0 | b_1 > b \text{ or } b_0 > b] = 1/2$ for any b. Testing simultaneously that this equality holds for every b is left for future research.

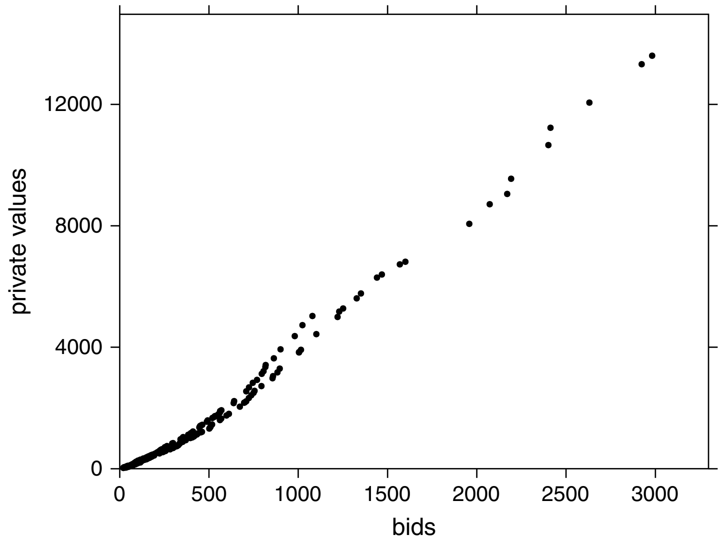

Figure 1. Inverse bidding strategies

Because of boundary effects, three auctions are trimmed out in the (2,0) case, while six auctions are trimmed out in the (0,2) case. Figure 1 displays the pairs $(b_{i\ell}, \hat{v}_{i\ell})$, $i = 1, 2$ $\ell = 1, \ldots, 57$, which trace out the (inverse) equilibrium strategy $\hat{\xi}_{11}(\cdot)$, as well as the pairs $(b_{i\ell}, \hat{v}_{i\ell})$, $i = 1, 2$, $\ell = 1, \ldots, 49$ tracing out $\hat{\xi}_{00}(\cdot)$.[18] The important feature is that both estimated functions are strictly increasing with $\hat{\xi}_{11}(\cdot)$ to the right of $\hat{\xi}_{00}(\cdot)$. Hence, for each case, the data do not reject the symmetric APV model in view of Proposition 2.1 of Li *et al.* (2002). It is interesting to compare the estimated $\xi(\cdot)$ functions. In particular, on the common interval where both functions can be estimated in Figure 1, a same private value leads to a higher bid when a joint bidder faces a joint bidder than when a solo bidder faces a solo bidder. This is in agreement with Table I.

Turning to private values, we estimate the univariate marginal private value density for joint bidders using

$$\hat{f}_1^{(2,0)}(v) = \frac{1}{2h_f L_T} \sum_{\ell=1}^{L_T} \sum_{i=1}^{2} K_f\left(\frac{v - \hat{v}_{1i\ell}}{h_f}\right)$$

where $L_T = 57$ and the bandwidth h_f is equal to $2.978 \times 1.06\hat{\sigma}_{\hat{v}}(2L_T)^{-1/5} = 3702.08$. Similarly, we estimate the univariate marginal private value density for solo bidders $f_0^{(0,2)}(\cdot)$ with $L_T = 49$ and h_f equal to 1476.48. Figure 2 displays the marginal density of joint versus joint by dashed lines and the marginal density of solo versus solo by unbroken lines. The comparison of both marginal private value densities in Figure 2 shows some differences between joint and solo bidders. Specifically, the mean, mode and variance for solo bidders are smaller than the respective quantities for joint bidders. In fact, the estimated cumulative distribution function for joint bidders first-order stochastically dominates that for solo bidders.[19] Hence, joint bidders are likely to draw larger

[18] The first index for the $\xi(\cdot)$ function refers to bidder's type while the second index refers to his opponent's type.

[19] A one-sided Kolmogorov–Smirnov test clearly rejects the equality in favour of stochastic dominance for joint bidders. The graph is available upon request from the authors. For the (2,0) case, the mean of the 114 trimmed private values is

private values than solo bidders with a relatively more important variability for the former. As pointed out in Section 3.2, these differences can be explained by unobserved tract heterogeneity and differences between joint and solo bidders. This issue is further investigated below.

As Figure 2 does not provide information on the affiliation between private values within the same auction whether they are both joint or solo, it is useful to test for their independence. We use the non-parametric test proposed by Blum *et al.* (1961) (BKR hereafter), which is consistent and distribution free. For two variables X and Y, the test statistic is equal to $(1/2)\pi^4 B$, with $B = N^{-4}\sum_{\ell=1}^{N}(N_1(\ell)N_4(\ell) - N_2(\ell)N_3(\ell))^2$, with N the number of observations and $N_1(\ell), N_2(\ell), N_3(\ell), N_4(\ell)$ the numbers of points lying respectively in the regions $\{(x,y)|x \le X_\ell, y \le Y_\ell\}$, $\{(x,y)|x > X_\ell, y \le Y_\ell\}$, $\{(x,y)|x \le X_\ell, y > Y_\ell\}$ and $\{(x,y)|x > X_\ell, y > Y_\ell\}$. To impose symmetry among bidders of the same type, we duplicate the observations so that $N = 2 \times L$. We find a test statistic equal to 6.57 using observed bids and to 4.69 using trimmed private values for joint bidders. For solo bidders, we obtained a test statistic equal to 9.52 using observed bids and equal to 6.92 using trimmed private values. The null hypothesis of independence is clearly rejected in all cases.

Case $(n_1, n_0) = (1,1)$

The potentially asymmetric case is estimated using the 112 auctions with both types. Because there is only one bidder of each type, (4) and (5) simplify as B_1^* and B_0^* are void. In particular, their denominators reduce to the conditional densities $g_{b_0|b_1}(b_1|b_1)$ and $g_{b_1|b_0}(b_0|b_0)$. Hence, (4) and (5) reduce to

$$v_1 = \xi_{10}(b_1) = b_1 + G_{b_0|b_1}(b_1|b_1)/g_{b_0|b_1}(b_1|b_1) \tag{18}$$

$$v_0 = \xi_{01}(b_0) = b_0 + G_{b_1|b_0}(b_0|b_0)/g_{b_1|b_0}(b_0|b_0) \tag{19}$$

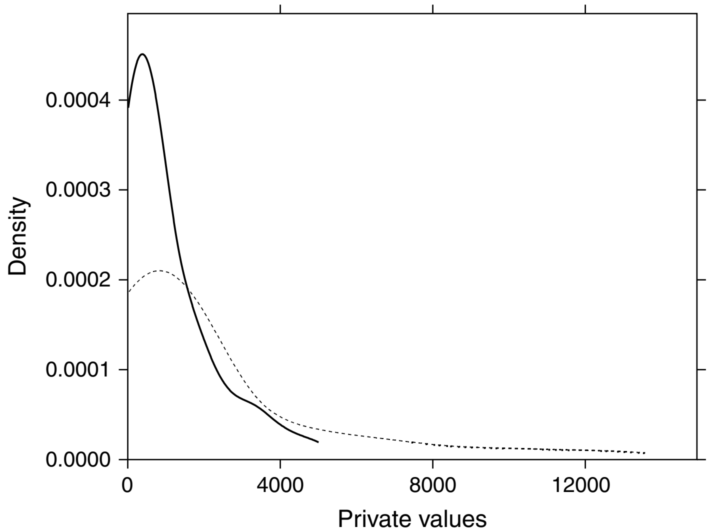

Figure 2. Marginal densities of private values

$2195.14 per acre with a standard deviation equal to $3024.10 and a range of [$33.94;$13 605.86]. For the (0,2) case, these numbers are $1027.90, $1170.15 and [$26.89; $5031.39], respectively from the 98 trimmed private values.

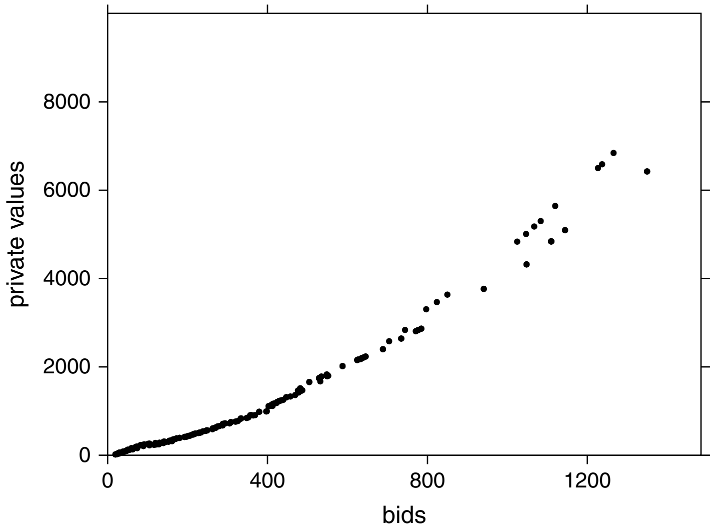

Figure 3. Inverse bidding strategies

respectively. Consequently, (7) and (8) and the first step in our non-parametric estimation procedure simplify accordingly.

For the first step, the bandwidths h_{G1}, h_{g1}, h_{G0} and h_{g0} are equal to 1.73, 2.03, 1.54 and 1.80, respectively. Figure 3 displays the estimated inverse bidding strategies $\hat{\xi}_{10}(\cdot)$ and $\hat{\xi}_{01}(\cdot)$ traced out by the pairs $(b_{1\ell}, \hat{v}_{1\ell})$ and $(b_{0\ell}, \hat{v}_{0\ell})$, $\ell = 1, \ldots, 91$, for the joint and solo bidders, as 21 auctions are trimmed out. In particular, both functions are increasing, indicating that the asymmetric APV model is not rejected by the data. Moreover, the inverse bidding strategy for solo bidders is to the right of that for joint bidders. Given the same valuation, a solo bidder bids more aggressively than a joint bidder. Moreover, from Proposition 2, the comparison of $\xi_{10}(\cdot)$ and $\xi_{01}(\cdot)$ indicates whether the joint bid distribution is quasisymmetric. Figure 3 suggests that this is not the case, especially for bids larger than \$500 when the two $\xi(\cdot)$ curves start to diverge. This corroborates our findings of Section 3.2. Overall, this again agrees with the rejection of the pure CV model.

The difference between the inverse equilibrium strategies constitutes a partial picture as bids also depend on bidders' valuations and their distributions. The joint distribution is estimated using (11) and displays some asymmetry and correlation.[20] Figure 4 displays the marginal density of private values for each type, namely $f_1^{(1,1)}(\cdot)$ by dashed lines and $f_0^{(1,1)}(\cdot)$ by unbroken lines with bandwidths of the form $2.978 \times 1.06\hat{\sigma}_{\hat{v}1}(L_T)^{-1/5} = 2277.61$ and $2.978 \times 1.06\hat{\sigma}_{\hat{v}0}(L_T)^{-1/5} = 1,708.63$ for joint and solo bidders, respectively. It appears that the density of solo private values has a slightly smaller mean, mode and variance than that of joint private values.[21] Hence, solo bidders seem less likely to draw large private values than joint bidders are. This suggests some

[20] The graph is available upon request from the authors. The BKR test strongly rejects independence of both private values and bids with a test statistic equal to 6.45 for private values and 34.23 for bids.

[21] The mean of the trimmed pseudo private values for joint bidders is \$1208.73 per acre with a standard deviation equal to \$1622.34, while the mean of the trimmed pseudo private values for solo bidders is \$1161.62 per acre with a standard deviation equal to \$1334.21.

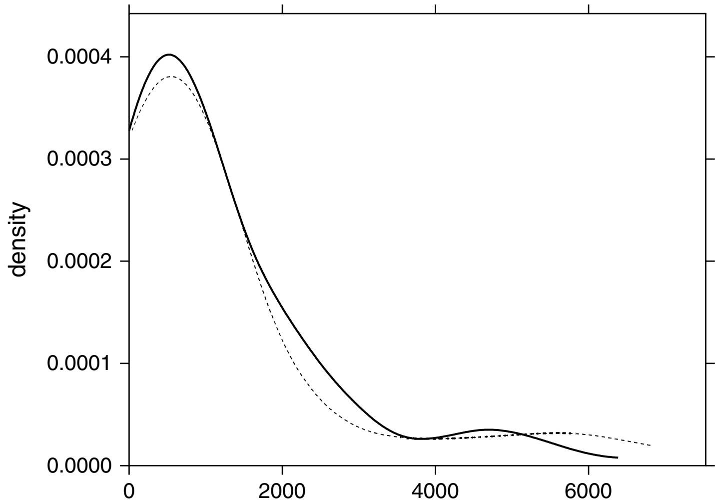

Figure 4. Marginal densities of private values

asymmetry though weak between joint and solo bidders as the empirical cumulative distribution functions slightly differ with a single crossing.[22]

Such an asymmetry leads the solo bidders to shade less their private values than joint bidders, as found in Figure 3, so as to increase their probability of winning the auctions. See also Maskin and Riley (2000a) and Pesendorfer (2000). However, the shading effect does not counterbalance fully the asymmetry in terms of valuation distributions, as indicated by the bid averages for joint versus solo and solo versus joint in Table I and the empirical probability of winning. As is well known, the aggressiveness of the weak bidder relative to the strong bidder may introduce some inefficiency in the auction in the sense that the winner of the auction has the lowest valuation. It turns out that this does not happen in our data set, which can be explained by the relatively weak asymmetry and the important variability of private values within each auction.

It is interesting to compare these results to the first two cases where bidders are of the same type. Figure 5 displays the inverse bidding strategies for a joint bidder when facing a joint bidder ($\hat{\xi}_{11}(\cdot)$) and when facing a solo bidder ($\hat{\xi}_{10}(\cdot)$), the former being to the right of the latter. Given a same tract valuation, a joint bidder will bid more aggressively when facing a joint bidder than when facing a solo bidder. For, the joint bidder faces less 'competition' when facing a solo bidder who is more likely to draw a lower private value. Figure 6 displays the inverse bidding strategies for a solo bidder when facing a solo bidder ($\hat{\xi}_{00}(\cdot)$) and when facing a joint bidder ($\hat{\xi}_{01}(\cdot)$), the former being to the left of the latter. Thus, a solo bidder will bid slightly more aggressively when facing a joint bidder than when facing a solo bidder to compensate for his lower private value. These results confirms the descriptive statistics of Table I. Both figures indicate that bidders have integrated the type of their opponents in their bidding strategies.

[22] The graph is available upon request from the authors. A Kolmogorov–Smirnov test does not reject the equality of the c.d.f.s on either private values or bids. Note, however, that the Kolmogorov–Smirnov test is based on the independence of the two samples. This is not the case as joint and solo private values (or bids) are affiliated, which decreases the power of the test.

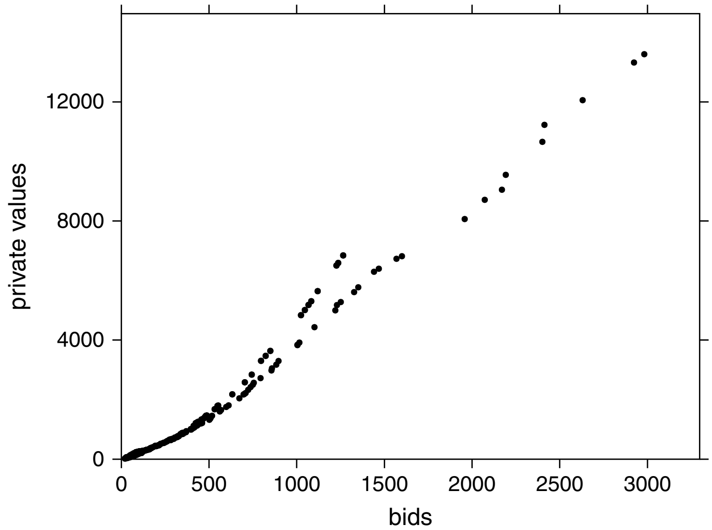

Figure 5. Inverse bidding strategies of joint bidders

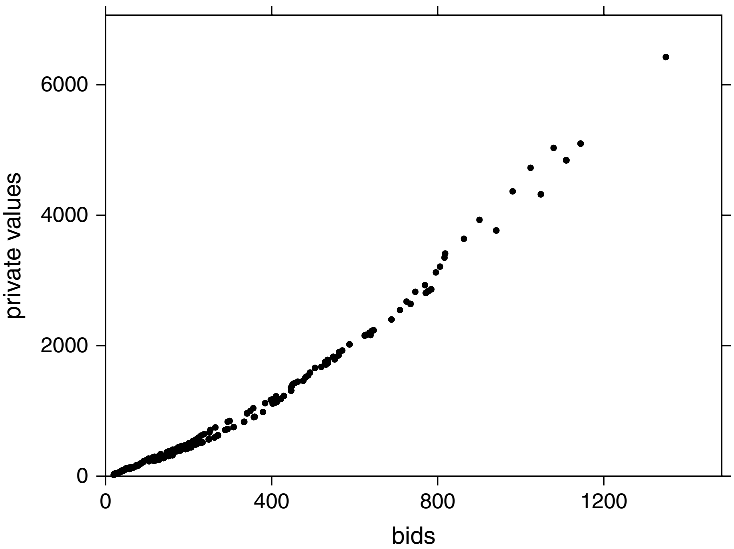

Figure 6. Inverse bidding strategies of solo bidders

Lastly, as indicated in Section 3.2, the comparisons of $\hat{f}_1^{(2,0)}(\cdot)$ and $\hat{f}_1^{(1,1)}(\cdot)$ as well as of $\hat{f}_0^{(0,2)}(\cdot)$ and $\hat{f}_0^{(1,1)}(\cdot)$ provide some information about unobserved tract heterogeneity. A Kolmogorov–Smirnov test gives a test statistic equal to 0.1917 and 0.0965 for the former and latter comparisons, respectively. This represents a clear rejection of $\hat{f}_1^{(2,0)}(\cdot) = \hat{f}_1^{(1,1)}(\cdot)$. Under the assumptions of Section 3.2, this means that there is some unobserved tract heterogeneity as tracts attracting two joint bidders differ significantly from tracts attracting one joint bidder and one solo bidder. This is in agreement with the idea that higher value tracts are more likely to attract joint bidders. It also indicates that differences in bids observed in Table I are due to both asymmetry among solo and joint bidders and unobserved tract heterogeneity though the latter appear to be more important.

Informational Rents

An attractive feature of the structural approach is to estimate the underlying private value distribution as well as bidders' private values. In particular, we can assess the informational rents left to the winners via $(\hat{v}_w - b_w)/\hat{v}_w$, where $\hat{v}_w$ is the estimate of the winner's private value and b_w is his bid. For those auctions that are kept after trimming, Table II gives the summary statistics on the winners' informational rents in percentage decomposed as in Table I. On average, the winner's informational rent is about 65%, indicating that the federal government is capturing 35% of the winner's willingness-to-pay at the auction. Introducing a common value component in the model may lower the informational rents left to winning firms.

Table II shows that the winner's informational rent is the same on average irrespective of whether the winner is a solo or joint bidder and irrespective of the type of his opponent. This suggests that bidding behaviour and bidding structure correspond to an equilibrium as a large firm can choose to be either a solo bidder or constitute a consortium with fringe firms. Such a result indicates that our theoretical model is not rejected by the data. Finding significantly different informational rents across structures would question bidders' use of the Bayesian Nash equilibrium and the adequacy of the model and its assumptions.

Though informational rents in percentage are about the same, they differ in absolute terms. For instance, the average informational rent $\hat{v}_w - b_w$ for a winning consortium is about \$1954.42 per acre compared to \$1151.40 per acre for a solo winner, i.e. 69.7% higher. This arises because the average valuation for a joint winner is \$2639.47 per acre, while it is only \$1608.34 per acre for a solo winner. Following our results, such a difference can be mainly explained by unobserved tract heterogeneity. The remainder is due to bidders' asymmetry as pooling of capital contributes to a higher valuation for a consortium. Because asymmetry is relatively weak between joint and solo bidders, a ban on joint bidding would not have much impact on the federal government's revenue provided participation remains the same. Previous empirical studies indicate that joint bidding occur mostly between a large firm and a number of fringe firms (see Hendricks and Porter, 1992), i.e. a fringe firm seldom bids solo. Some studies find that allowing for joint bidding has increased the level of competition through a larger participation of firms to the auctions. This combined with our results suggests that the federal government has increased its revenue by allowing for joint bidding.

Table II. Summary statistics on winners' informational rents (%)

Variable	# Obs	Mean	STD	Minimum	Maximum
All winners	197	65.07	9.39	31.82	81.51
Joint winners	105	65.50	9.63	42.55	81.51
Solo winners	92	64.59	9.14	31.82	79.00
Joint winners vs joint	57	65.86	9.56	42.55	78.52
Joint winners vs solo	48	65.08	9.80	47.83	81.51
Solo winners vs joint	43	63.85	9.87	31.82	79.00
Solo winners vs solo	49	65.24	8.50	43.17	78.55

4. CONCLUSION

The main contributions of the paper are methodological as it pushes further the frontier of the new structural analysis of auction data (see Perrigne and Vuong, 1999, for a recent survey). While such an approach has been mostly confined to symmetric models, our paper shows that the structural approach can be extended to the asymmetric case known to lead to intractable equilibrium strategies. Moreover, we provide formal assumptions, which can deal with unobserved heterogeneity across auctioned objects. We also develop a simple test of the pure CV paradigm.

Specifically, we consider the asymmetric APV model, which encompasses the IPV model. We establish its non-parametric identification from observed bids and we propose a two-step non-parametric procedure for estimating the underlying bidders' private value distributions. A distinctive feature of our procedure is its computational convenience as it avoids the numerical integration of the system of differential equations characterizing the bidders' equilibrium strategies. Our method is fully nonparametric. This has the main advantage of leaving unspecified the underlying distribution and in particular the affiliation among private values, whose parameterization can be difficult. On the other hand, it requires a large number of data. Thus, parametric estimation methods need to be developed if more than two types of bidders are entertained and if some observed heterogeneity of the auctioned objects needs to be introduced. In this case, there would be as many differential equations as types and one would need to know bidders' identities to follow them across auctions.

Lastly, our paper illustrates the proposed methodology by analyzing joint bidding in wildcat OCS auctions after 1972. In particular, we find that the pure CV model does not seem to be supported by the data. We also find that asymmetry between joint and solo bidders is weak, while unobserved tract heterogeneity is important and well captured by the bidding solo/joint structure. Our findings indicate that the government could have benefited from allowing for joint bidding through mainly increased competition. Our empirical analysis takes joint bidding as given. An important economic issue is the rationale of cartel formation prior to the auction. This constitutes a domain that needs further theoretical development. For a recent contribution, see Hendricks *et al.* (2000).

APPENDIX

Proof of Proposition 1: The proof is in two parts. First, we prove that the asymmetric APV model is identified. Second, we prove the necessary and sufficient conditions under which a distribution $G(\cdot)$ can be rationalized by an asymmetric APV model.

(1) Let $G(\cdot)$ be the joint distribution of the observed (equilibrium) bids with support $[\underline{b}, \overline{b}]^n$ in the asymmetric APV model. Suppose that there are two possible underlying distributions $F(\cdot)$ and $\tilde{F}(\cdot)$ of private values both leading to the same joint bid distribution $G(\cdot)$ in the asymmetric APV model. By assumption, both distributions $F(\cdot)$ and $\tilde{F}(\cdot)$ belong to the set $\mathcal{P}$ of n-dimensional absolutely continuous distributions with hypercube supports that are affiliated and exchangeable in their first n_1 and last n_0 arguments. Let $s_1(\cdot, F)$, $s_0(\cdot, F)$, $\tilde{s}_1(\cdot, \tilde{F})$ and $\tilde{s}_1(\cdot, \tilde{F})$ be the strictly increasing Bayesian Nash equilibrium strategies corresponding to $F(\cdot)$ and $\tilde{F}(\cdot)$, respectively. Thus $s_1(\cdot, F)$, $s_0(\cdot, F)$, $\tilde{s}_1(\cdot, \tilde{F})$ and $\tilde{s}_1(\cdot, \tilde{F})$ satisfy the first-order differential equations (1) and

(2), which can be written as in (4) and (5). Therefore

$$F(\mathbf{v_1}, \mathbf{v_0}) = \Pr(\xi_1(\mathbf{b_1}, G) \leq \mathbf{v_1}, \xi_0(\mathbf{b_0}, G) \leq \mathbf{v_0}) = G(\xi_1^{-1}(\mathbf{v_1}, G), \xi_0^{-1}(\mathbf{v_0}, G))$$

$$\tilde{F}(\mathbf{v_1}, \mathbf{v_0}) = \Pr(\xi_1(\mathbf{b_1}, G) \leq \mathbf{v_1}, \xi_0(\mathbf{b_0}, G) \leq \mathbf{v_0}) = G(\xi_1^{-1}(\mathbf{v_1}, G), \boldsymbol{\xi_0}^{-1}(\mathbf{v_0}, G))$$

It follows that $F(\cdot) = \tilde{F}(\cdot)$ on their common support $[\underline{v}, \overline{v}]^n \equiv [\xi_0(\underline{b}, G), \xi_0(\overline{b}, G)]^n = [\xi_1(\underline{b}, G), \xi_1(\overline{b}, G)]^n$ in the asymmetric APV model. Thus the asymmetric APV model is identified.[23]
(2) We first show necessity of (i)-(ii) in Proposition 1. Let $s_1(\cdot, F)$ and $s_0(\cdot, F)$ be the strictly increasing differentiable Bayesian Nash equilibrium strategies corresponding to $F(\cdot)$, which is affiliated and exchangeable in its first n_1 and last n_0 arguments and $G(\cdot)$ be the joint distribution of observed bids. Thus, $G(\mathbf{b_1}, \mathbf{b_0}) = F(\mathbf{s_1}^{-1}(\mathbf{b_1}, F), \mathbf{s_0}^{-1}(\mathbf{b_0}, F))$ for every $\mathbf{b_1} \in [\underline{b}, \overline{b}]^{n_1} = [\underline{v}, s_1(\overline{v}, F)]^{n_1}$ and $\mathbf{b_0} \in [\underline{b}, \overline{b}]^{n_0} = [\underline{v}, s_0(\overline{v}, F)]^{n_0}$. Because the strategies $s_1(\cdot, F)$ and $s_0(\cdot, F)$ are strictly increasing and $F(\cdot) \in \mathcal{P}$, then $G(\cdot)$ belongs to $\mathcal{P}$. Moreover, these strategies must solve the system of first-order differential equations (1) and (2). Because the system composed by (4) and (5) is equivalent to the system of equations (1) and (2), then $s_1(\cdot, F)$ and $s_0(\cdot, F)$ must satisfy $\xi_1(s_1(v_1, F), G) = v_1$ and $\xi_0(s_0(v_0, F), G) = v_0$ for all $v_1 \in [\underline{v}, \overline{v}]$ and $v_0 \in [\underline{v}, \overline{v}]$. Making the change of variables $b_1 = s_1(v_1, F)$ and $b_0 = s_0(v_0, F)$, we obtain $\xi_1(b_1, G) = s_1^{-1}(b_1, F)$ and $\xi_0(b_0, G) = s_0^{-1}(b_0, F)$ for every b_1 and b_0 in $[\underline{b}, \overline{b}]$. Thus $\xi_1(\cdot, G)$ and $\xi_0(\cdot, G)$ must be strictly increasing on $[\underline{b}, \overline{b}]$ because $s_1^{-1}(\cdot, F)$ and $s_0^{-1}(\cdot, F)$ are strictly increasing on $[\underline{b}, \overline{b}]$.

To prove sufficiency, let the joint distribution $G(\cdot)$ belong to $\mathcal{P}$ with support $[\underline{b}, \overline{b}]^n$. This implies that $G(\cdot)$ is exchangeable in its first n_1 and last n_0 arguments and affiliated. We note that $\lim_{b \to \underline{b}} \xi_1(b, G) = \underline{b}$. This follows from (4) and the fact that (i) $\underline{b}$ is finite, (ii) $\lim_{b \to \underline{b}} \log G_{B_1^*, B_0|b_1}(b, b|\underline{b}) = -\infty$, and (iii) $\partial \log G_{B_1^*, B_0|b_1}(B, b|\underline{b})/\partial B = (\partial G_{B_1^*, B_0|b_1}(B, b|\underline{b})/\partial B)/G_{B_1^*, B_0|b_1}(B, b|\underline{b})$, $\partial \log G_{B_1^*, B_0|b_1}(B, b|\underline{b})/\partial b = (\partial G_{B_1^*, B_0|b_1}(B, b|\underline{b})/\ \partial b)/G_{B_1^*, B_0|b_1}(B, b|\underline{b})$, so that $\lim_{b \to \underline{b}}\ (dG_{B_1^*, B_0|b_1}(b, b|\underline{b})/db)/G_{B_1^*, B_0|b_1}(b, b|\underline{b}) = +\infty$. Using similar arguments in (5), we can show that $\lim_{b \to \underline{b}} \xi_0(b, G) = \underline{b}$.

Next, we define a distribution $F(\cdot)$ as $F(\mathbf{v_1}, \mathbf{v_0}) = G(\xi_1^{-1}(\mathbf{v_1}, G), \xi_0^{-1}(\mathbf{v_0}, G))$, for all $(\mathbf{v_1}, \mathbf{v_0}) \in [\underline{v}, \overline{v}]^n$, where $\underline{v} = \xi_1(\underline{b}, G) = \xi_0(\underline{b}, G) = \underline{b}$ and $\overline{v} = \xi_1(\overline{b}, G) = \xi_0(\overline{b}, G)$. Because $\xi_1(\cdot, G)$ and $\xi_0(\cdot, G)$ are strictly increasing on $[\underline{b}, \overline{b}]$ by assumption, $F(\cdot)$ is a valid distribution. Since $G(\cdot)$ is strictly increasing on $[\underline{b}, \overline{b}]^n$, then $F(\cdot)$ is strictly increasing on $[\underline{v}, \overline{v}]^n$. Hence the support of $F(\cdot)$ is the hypercube $[\underline{v}, \overline{v}]^n$. Moreover, because $\xi_1^{-1}(\cdot, G)$ and $\xi_0^{-1}(\cdot, G)$ are strictly increasing and $G(\cdot) \in \mathcal{P}$, then $F(\cdot)$ belongs to $\mathcal{P}$.

It remains to show that this distribution $F(\cdot)$ can rationalize $G(\cdot)$ in an asymmetric APV model, i.e. that $G(\cdot, \cdot) = F(\mathbf{s_1}^{-1}(\cdot, F), \mathbf{s_0}^{-1}(\cdot, F))$ on $[\underline{b}, \overline{b}]^n$, where $s_1(\cdot, F)$ and $s_0(\cdot, F)$ solve (1) and (2) with the boundary conditions $s_1(\underline{v}, F) = s_0(\underline{v}, F) = \underline{v}$ and $s_1(\overline{v}, F) = s_0(\overline{v}, F)$. By construction of $F(\cdot)$, we have $G(\cdot) = F(\xi_1(\cdot, G), \xi_0(\cdot, G))$. Thus it suffices to show that $\xi_1^{-1}(\cdot, G)$ and $\xi_0^{-1}(\cdot, G)$ solve the system of equations (1) and (2) with the boundary conditions $\xi_1^{-1}(\underline{v}, G) = \xi_0^{-1}(\underline{v}, G) = \underline{v}$ and $\xi_1^{-1}(\overline{v}, G) = \xi_0^{-1}(\overline{v}, G)$. It is easy to see that these boundary conditions are satisfied by construction and assumption, respectively. From the construction of $F(\cdot)$, we note that $F_{y_1^*, y_0|v_1}(\cdot, \cdot|v_1) = G_{B_1^*, B_0|b_1}(\xi_1^{-1}(\cdot, G), \xi_0^{-1}(\cdot, G)|\xi_1^{-1}(v_1, G))$. Differentiating and taking the ratio, this gives $f_{y_1^*, y_0|v_1}(\cdot, \cdot|v_1)/F_{y_1^*, y_0|v_1}(\cdot, \cdot|v_1) = \xi_1^{-1\prime}(\cdot, G)(\partial G_{B_1^*, B_0|b_1}(\xi_1^{-1}(\cdot, G), \xi_0^{-1}(\cdot, G)|\xi_1^{-1}(v_1, G))/\partial B_1^*)/ G_{B_1^*, B_0|b_1}(\xi_1^{-1}(\cdot, G), \xi_0^{-1}(\cdot, G)|\xi_1^{-1}(v_1, G)) + \xi_0^{-1\prime}(\cdot, G)(\partial G_{B_1^*, B_0|b_1}(\xi_1^{-1}(\cdot, G), \xi_0^{-1}(\cdot, G)|\xi_1^{-1}(v_1, G))/ \partial B_0)/G_{B_1^*, B_0|b_1}(\xi_1^{-1}(\cdot, G), \xi_0^{-1}(\cdot, G)|\xi_1^{-1}(v_1, G))$. We can develop similar computations for

[23] To simplify the notation, the vector $\boldsymbol{\xi_1}(\mathbf{b_1}, G)$ denotes the vector $(\xi_1(b_{11}, G), \ldots, \xi_1(b_{1n_1}, G))$, etc.

$F_{y_1,y_0^*|v_1}(\cdot,\cdot|v_0) = G_{B_1,B_0^*|b_1}(\xi_1^{-1}(\cdot,G),\xi_0^{-1}(\cdot,G)|\xi_0^{-1}(v_0,G))$. Thus $\xi_1^{-1}(\cdot,G)$ and $\xi_0^{-1}(\cdot,G)$ solve (1) and (2) if

$$\begin{aligned}
1 &= (v_1 - \xi_1^{-1}(v_1,G))\\
&\left(\frac{\partial G_{B_1^*,B_0|b_1}(\xi_1^{-1}(v_1,G),\xi_0^{-1}(v_1,G)|\xi_1^{-1}(v_1,G))/\partial B_1^*}{G_{B_1^*,B_0|b_1}(\xi_1^{-1}(v_1,G),\xi_0^{-1}(v_1,G)|\xi_1^{-1}(v_1,G))}\right.\\
&\left.+\frac{\partial G_{B_1^*,B_0|b_1}(\xi_1^{-1}(v_1,G),\xi_0^{-1}(v_1,G)|\xi_1^{-1}(v_1,G))/\partial B_0}{G_{B_1^*,B_0|b_1}(\xi_1^{-1}(v_1,G),\xi_0^{-1}(v_1,G)|\xi_1^{-1}(v_1,G))}\right)\\
1 &= (v_0 - \xi_0^{-1}(v_0,G))\\
&\left(\frac{\partial G_{B_1,B_0^*|b_0}(\xi_1^{-1}(v_0,G),\xi_0^{-1}(v_0,G)|\xi_0^{-1}(v_0,G))/\partial B_1}{G_{B_1,B_0^*|b_0}(\xi_1^{-1}(v_0,G),\xi_0^{-1}(v_0,G)|\xi_0^{-1}(v_0,G))}\right.\\
&\left.+\frac{\partial G_{B_1,B_0^*|b_0}(\xi_1^{-1}(v_0,G),\xi_0^{-1}(v_0,G)|\xi_0^{-1}(v_0,G))/\partial B_0^*}{G_{B_1,B_0^*|b_0}(\xi_1^{-1}(v_0,G),\xi_0^{-1}(v_0,G)|\xi_0^{-1}(v_0,G))}\right)
\end{aligned}$$

holds for any $(v_1,v_0) \in [\underline{v},\overline{v}]^2$. This clearly holds by definition of $\xi_1(\cdot,G)$ and $\xi_0(\cdot,G)$.□

Proof of Proposition 2: First, we prove that the joint (equilibrium) bid distribution $G(\cdot)$ in an asymmetric pure CV model with two players must satisfy $\xi_1(\cdot,G) = \xi_0(\cdot,G)$ on $[\underline{b},\overline{b}]$. As noted in the text, such a distribution must satisfy (16) and (17) for any $\sigma_i \in [\underline{\sigma},\overline{\sigma}]$, where $b_i = s_i(\sigma_i)$ for $i = 1, 2$, i.e. for any $b_i \in [\underline{b},\overline{b}]$, where $\sigma_i = s_i^{-1}(b_i)$. Replacing σ_i by $s_i^{-1}(b_i)$ in both left-hand sides of (16) and (17) shows that these two equations are equivalent to

$$V(s_1^{-1}(b), s_0^{-1}(b)) = \xi_1(b,G) \tag{A.1}$$

$$V(s_1^{-1}(b), s_0^{-1}(b)) = \xi_0(b,G) \tag{A.2}$$

for any $b \in [\underline{b},\overline{b}]$. The desired result follows from the equality of the left-hand sides.

Next, we show that the condition $\xi_1(\cdot,G) = \xi_0(\cdot,G)$ on $[\underline{b},\overline{b}]$ is equivalent to the quasisymmetry of $G(\cdot)$. By definition of $\xi_i(\cdot,G)$, the condition $\xi_1(\cdot,G) = \xi_0(\cdot,G)$ on $[\underline{b},\overline{b}]$ is equivalent to $G_{b_0|b_1}(b|b)/g_{b_0|b_1}(b|b) = G_{b_1|b_0}(b|b)/g_{b_1|b_0}(b|b)$ for any $b \in [\underline{b},\overline{b}]$, i.e. to

$$\int_{\underline{b}}^{b} g_{b_1,b_0}(b,\tilde{b})d\tilde{b} = \int_{\underline{b}}^{b} g_{b_1,b_0}(\tilde{b},b)d\tilde{b}$$

for any $b \in [\underline{b},\overline{b}]$, where $g_{b_1,b_0}(\cdot)$ denotes the joint density of (b_1,b_0). This equality can be interpreted as $\Pr[b_0 \le b, b_1 = b] = \Pr[b_0 = b, b_1 \le b]$ for any $b \in [\underline{b},\overline{b}]$. Hence, it is equivalent to the condition $\Pr[b_0 \le b_1 \le b] = \Pr[b_1 \le b_0 \le b]$ for any $b \in [\underline{b},\overline{b}]$, which can be proved formally by integrating the former and differentiating the latter with respect to b. It remains to note that the latter condition is equivalent to $\Pr[b_1 \le b_0 | b_1 \le b, b_0 \le b] = \Pr[b_1 \ge b_0 | b_1 \le b, b_0 \le b]$ for any $b \in [\underline{b},\overline{b}]$. i.e. to the quasisymmetry of $G(\cdot)$.[24]

[24] Note that the equivalences in this paragraph do not depend on the fact that $G(\cdot)$ is the equilibrium distribution. Hence, they apply to any bivariate distribution.

Lastly, to prove the third statement in Proposition 2, consider the bivariate private value distribution $F(\cdot)$ generated by (v_1, v_0), where (v_1, v_0) is given by (4) and (5), while (b_1, b_0) is distributed as $G(\cdot)$, which is the bivariate equilibrium bid distribution in the pure CV model under consideration. In the two players case, it is easily seen that (4) and (5) reduce to (18) and (19), respectively. Now, by Theorem 3 in Milgrom and Weber (1982) note that $G(\cdot)$ is affiliated because $(b_1, b_0) = (s_1(\sigma_1), s_0(\sigma_0))$, σ_1 and σ_0 are affiliated, and the equilibrium strategies $s_1(\cdot)$ and $s_0(\cdot)$ in the (asymmetric) pure CV model are restricted to be strictly increasing. Second, (A.1) and (A.2) imply that $\xi_i(\cdot, G)$, $i = 1, 2$ must be strictly increasing on $[\underline{b}, \overline{b}]$ because $s_i^{-1}(\cdot)$ is strictly increasing, and because $V(\sigma_1, \sigma_0)$ is strictly increasing in (σ_1, σ_0) by Theorem 5 in Milgrom and Weber (1982). Hence, from Proposition 1 it follows that the bid distribution $G(\cdot)$ can be rationalized by an (asymmetric) APV model. Therefore, such an APV model is observationally equivalent to the (asymmetric) pure CV model under consideration.

To complete the proof, it remains to show that the private value distribution $F(\cdot)$ of the observationally equivalent APV model is quasisymmetric. Because $G(\cdot)$ is the equilibrium bid distribution of the pure CV model, it must be quasisymmetric, as shown above. Moreover, $F(v_1, v_2) = G(\xi_1^{-1}(v_1), \xi_0^{-1}(v_0))$ by definition, where $\xi_i(\cdot)$ is strictly increasing, and $\xi_1(\cdot) = \xi_0(\cdot)$, as previously shown. It follows easily that $F(\cdot)$ is quasisymmetric. Moreover, both players adopt the same equilibrium strategy $\xi_1^{-1}(\cdot) = \xi_0^{-1}(\cdot)$. $\square$

ACKNOWLEDGEMENTS

We are especially grateful to K. Hendricks and R. Porter for providing the data analysed in this paper. We thank Steve Stern, the Co-Editor, a referee and participants at the SITE conference, July 1998, North American Econometric Society Meeting, January 1999, European Econometric Society Meeting, August 1999, European Summer School, September 1999 and UC Berkeley, UCLA, Brown, Harvard-MIT and Stanford seminars for their helpful comments. Financial support was provided by the National Science Foundation under Grant SBR-9631212.

REFERENCES

Athey S. 2001. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica* **69**: 861–889.

Bajari P. 1999. Econometrics of the first-price auctions with asymmetric bidders. Stanford University, Working Paper.

Bajari P. 2001. Comparing competition and collusion: a numerical approach. *Economic Theory* **18**: 187–205.

Bajari P, Ye L. 2002. Deciding between competition and collusion, *Review of Economics and Statistics*, forthcoming.

Baldwin L, Marshall R, Richard J-F. 1997. Bidder collusion at Forest Service timber sales. *Journal of Political Economy* **105**: 657–699.

Blum JR, Kiefer J, Rosenblatt M. 1961. Distribution free tests of independence based on the sample distribution function. *Annals of Mathematical Statistics* **32**: 485–498.

Crooke P, Froeb L, Tschantz S. 1997. Mergers in sealed vs oral asymmetric auctions. Vanderbilt University, Working Paper.

DeBrock L, Smith J. 1983. Joint bidding, information pooling, and the performance of petroleum lease auctions. *The Bell Journal of Economics* **14**: 395–404.

Donald S, Paarsch H. 1993. Piecewise maximum likelihood estimation in empirical models of auctions. *International Economic Review* **34**: 121–148.

Donald S, Paarsch H. 1996. Identification, estimation, and testing in parametric empirical models of auctions within the independent private values paradigm. *Econometric Theory* **12**: 517–567.

Elyakime B, Laffont J-J, Loisel P, Vuong Q. 1994. First-price sealed-bid auctions with secret reservation prices. *Annales d'Economie et de Statistique* **34**: 115–141.

Flambard V, Perrigne I. 2001. Asymmetry in procurement auctions: evidence from snow removal contracts. University of Southern California, Working Paper.

Gilley O, Karels G, Lyon R. 1985. Joint ventures and offshore oil lease sales. *Economic Inquiry* **XXIV**: 321–339.

Guerre E, Perrigne I, Vuong Q. 2000. Optimal nonparametric estimation of first-price auctions. *Econometrica* **68**: 525–574.

Haile P, Hong H, Shum M. 2000. Nonparametric tests for common values in first-price auctions. Princeton University, Working Paper.

Hardle W. 1991. *Smoothing Techniques with Implementation in S*. Springer-Verlag: New York.

Hendricks K, Porter R. 1988. An empirical study of an auction with asymmetric information. *American Economic Review* **78**: 865–883.

Hendricks K, Porter R. 1992. Joint bidding in federal OCS auctions. *American Economic Review* **82**: 506–511.

Hendricks K, Pinkse J, Porter R. 2002. Empirical implications of equilibrium bidding in first-price common value auctions. *Review of Economic Studies*, forthcoming.

Hendricks K, Porter R, Tan G. 2000. Joint bidding in federal offshore oil and gas lease auctions. University of British Columbia, Working Paper.

Hendricks K, Porter R, Wilson C. 1994. Auctions for oil and gas leases with an informed bidder and a random reserve price. *Econometrica* **62**: 1415–1444.

Hong H. 1997. Characterization of equilibria in asymmetric ascending auctions. Stanford University, Working Paper.

Hong H, Shum M. 2002. Econometric models of ascending auctions. *Journal of Econometrics*, forthcoming.

Jofre-Bonet M, Pesendorfer M. 2000. Bidding behavior in a repeated procurement auction. Yale University, Working Paper.

Laffont J-J. 1997. Game theory and empirical economics: The case of auction data. *European Economic Review* **41**: 1–35.

Laffont J-J, Ossard H, Vuong Q. 1995. Econometrics of first-price auctions. *Econometrica* **63**: 953–980.

Laffont J-J, Vuong Q. 1996. Structural analysis of auction data. *American Economic Review* **86**: 414–420.

Lebrun B. 1999. First-Price auctions in the asymmetric N bidder case. *International Economic Review* **40**: 125–142.

Levin D, Smith JL. 1994. Equilibrium in auctions with entry. *American Economic Review* **84**: 585–599.

Li T, Perrigne I, Vuong Q. 2000. Conditionally independent private information in OCS wildcat auctions. *Journal of Econometrics* **98**: 129–161.

Li T, Perrigne I, Vuong Q. 2002. Structural estimation of the affiliated private values model. *Rand Journal of Economics* **33**: 171–193.

Lizzeri A, Persico N. 2000. Uniqueness and existence of equilibrium in auctions with a reserve price. *Games and Economic Behavior* **30**: 83–114.

Marshall R, Meurer M, Richard J-F, Stromquist W. 1994. Numerical analysis of asymmetric sealed high bid auctions. *Games and Economic Behavior* **7**: 193–220.

Maskin E, Riley J. 2000a. Asymmetric auctions. *Review of Economic Studies* **67**: 413–438.

Maskin E, Riley J. 2000b. Existence of equilibrium in sealed high bid auctions. *Review of Economic Studies* **67**: 439–454.

McAfee P, McMillan J. 1987. Auctions with entry. *Economics Letters* **23**: 343–347.

McAfee P, Vincent D. 1992. Updating the reserve price in common-value models. *American Economic Review* **82**: 512–518.

Milgrom P, Weber R. 1982. A theory of auctions and competitive bidding. *Econometrica* **50**: 1089–1122.

Millsaps W, Ott M. 1985. Risk aversion, risk sharing and joint bidding: a study of outer continental shelf petroleum auctions. *Land Economics* **61**: 372–386.

Paarsch H. 1992. Deciding between the common and private value paradigms in empirical models of auctions. *Journal of Econometrics* **51**: 191–215.

Perrigne I, Vuong Q. 1999. Structural econometrics of first-price auctions: a survey of methods. *Canadian Journal of Agricultural Economics* **47**: 203–223.
Pesendorfer M. 2000. A study of collusion in first-price auctions. *Review of Economic Studies* **67**: 381–411.
Porter R. 1995. The role of information in U.S. offshore oil and gas lease auctions. *Econometrica* **63**: 1–27.
Porter R, Zona J. 1993. Detection of bid rigging in procurement auctions. *Journal of Political Economy* **101**: 518–538.
Vickrey W. 1961. Counterspeculation, auctions, and sealed tenders. *Journal of Finance* **16**: 8–37.
Wilson R. 1967. Competitive bidding with asymmetric information. *Management Science* **13**: 816–820.
Wilson R. 1969. Competitive bidding with disparate information. *Management Science* **15**: 446–448.
Wilson R. 1977. A bidding model of perfect competition. *Review of Economic Studies* **44**: 511–518.

