Operations Research HW5

Timothy Schwieg

Question 1

If we let the cost vector be denoted c, the time spent vector be denoted P, due date denoted D, we may construct artifical variables s_i which is the delay on job i, and y_{ij} is an indicator if job i precedes job j, as well as Z which is 1 if job 4 precedes job 3.
This problem becomes:

$$\begin{aligned} \min \quad & c^T s \\ \text{s.t.} \quad & x_i - s_i \le D_i - P_i \\ & My_{ji} + x_j - x_i \ge P_i \\ & My_{ij} + x_i - x_j \ge P_j \\ & y_{ij} + y_{ji} = 1 \\ & x_3 - x_4 - P_4 \le M(1-Z) - \epsilon \\ & x_9 + P_9 - x_3 \le ZM \\ & x, s \in \mathbb{R}_+ \quad y_{ij}, Z \in \{0,1\} \end{aligned}$$

Since there are so many constraints, Julia code to solve this will be generated in python.

```
print( "using JuMP\nusing Cbc")

print( "m = Model( solver = CbcSolver() )")
print( "@variable( m, x[0:9] >= 0)" )
print( "@variable( m, s[0:9] >= 0)" )
for i in range(10):
	for j in range(10):
		if( i == j ):
			continue
		print( "@variable( m, y" + str(i) + str(j) + ", Bin)" )
print( "@variable( m, Z, Bin)" )
print( "@objective( m, Min, 1*s[0] + 2*s[1] + 5*s[2] + 1*s[3] + 3*s[4] + 5*s[5] + 2*s[6] + 4*s[7] + 3*s
    ↪ [8] + 7*s[9])" )
P = [10,3,13,15,9,22,17,30,12,16]
D = [20,98,100,34,50,44,32,60,80,150]
for i in range(10):
	print( "@constraint( m, x[" + str(i) + "] - s[" + str(i) + "] <= " + str(D[i]) + " - " + str(P[i
	    ↪ ]) + ")" )
M = "10000"
Epsilon = "0.0001"
for i in range(10):
	for j in range(10):
		if( i <= j):
			continue
		print( "@constraint( m, " + M + "*y" + str(i) + str(j) + " + x[" + str(i) + "] - x[" +
		    ↪ str(j) + "] >= " + str( P[j] ) + ")" )
		print( "@constraint( m, " + M + "*y" + str(j) + str(i) + " + x[" + str(j) + "] - x[" +
		    ↪ str(i) + "] >= " + str( P[i] ) + ")" )
		print( "@constraint( m, y" + str(i) + str(j) + "+ y" + str(j) + str(i) + " == 1)")
print( "@constraint( m, x[2] - x[3] - " + str(P[4-1]) + " <= " + M + "*(1-Z)- " + Epsilon + ")")
print( "@constraint( m, x[8] + " + str(P[9-1]) +" - x[2] <= "+M+"*Z)" )
print( "status = solve(m)" )
print( "println( \"Objective Value: \", getobjectivevalue(m))" )
print( "println( getvalue( x ))")
```

```
print("println( getvalue( s ))")
```

Solving this system using the computer generated code in Julia:

```
julia> println( "Objective Value: ", getobjectivevalue(m))
Objective Value: 318.0

julia> println( getvalue(x))
x: 1 dimensions:
[0] = 0.0
[1] = 82.99999999999999
[2] = 86.0
[3] = 115.99999999999999
[4] = 32.0
[5] = 10.0
[6] = 99.0
[7] = 41.0
[8] = 71.0
[9] = 130.99999999999997


julia> println( getvalue(s))
s: 1 dimensions:
[0] = 0.0
[1] = 0.0
[2] = 0.0
[3] = 96.99999999999999
[4] = 0.0
[5] = 0.0
[6] = 84.0
[7] = 11.0
[8] = 2.9999999999999925
[9] = 0.0
```

Question 2

We set x_1, x_2, x_3 to be production of product 1,2,3 respectively, and let s_1 be a binary predictor of if product 3 is produced. It is clear that $x_3 \leq 100$ so by setting $x_3 \leq 100s_1$ this ensures $x_3 = 0$ if $s_1 = 0$, and x_3 is otherwise unaffected.

$$\begin{aligned}
\max\ & 25x_1 + 30x_2 + 45x_3 \\
\text{s.t.}\quad & 3x_1 + 4x_2 + 5x_3 \leq 100 \\
& 4x_1 + 3x_2 + 6x_3 \leq 100 \\
& x_3 - 5s_1 \geq 0 \\
& x_3 - 100s_1 \leq 0 \\
& x_1, x_2, x_3 \in \mathbb{R}_+, s_1 \in \{0, 1\}
\end{aligned}$$

Solving the Relaxed Linear Program:

$$\begin{aligned}
\max\ & 25x_1 + 30x_2 + 45x_3 \\
\text{s.t.}\quad & 3x_1 + 4x_2 + 5x_3 \leq 100 \\
& 4x_1 + 3x_2 + 6x_3 \leq 100 \\
& x_3 - 5s_1 \geq 0 \\
& x_3 - 100s_1 \leq 0 \\
& s_1 \leq 1 \\
& x_1, x_2, x_3, s_1 \in \mathbb{R}_+
\end{aligned}$$

This has a maximal value of: $\frac{2500}{3}$ at a maximizer of: $x^* = (0, \frac{100}{9}, \frac{100}{9}), s_1 = \frac{1}{9}$

Branching on $s_1 = 0$ and $s_1 = 1$
Case: $s_1 = 1$

$$\begin{aligned}
\max\ & 25x_1 + 30x_2 + 45x_3 \\
\text{s.t.} \quad & 3x_1 + 4x_2 + 5x_3 \le 100 \\
& 4x_1 + 3x_2 + 6x_3 \le 100 \\
& x_3 - 5s_1 \ge 0 \\
& x_3 - 100s_1 \le 0 \\
& s_1 \le 1 \\
& s_1 \ge 1 \\
& x_1, x_2, x_3, s_1 \in \mathbb{R}_+
\end{aligned}$$

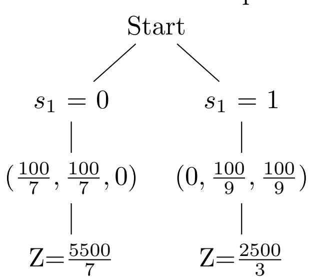
This has a maximal value of: $\frac{2500}{3}$ at a maximizer of: $x^* = (0, \frac{100}{9}, \frac{100}{9}), s_1 = 1$

Case: $s_1 = 0$

$$\begin{aligned}
\max\ & 25x_1 + 30x_2 + 45x_3 \\
\text{s.t.} \quad & 3x_1 + 4x_2 + 5x_3 \le 100 \\
& 4x_1 + 3x_2 + 6x_3 \le 100 \\
& x_3 - 5s_1 \ge 0 \\
& x_3 - 100s_1 \le 0 \\
& s_1 \le 0 \\
& x_1, x_2, x_3, s_1 \in \mathbb{R}_+
\end{aligned}$$

This has maximal value of: $\frac{5500}{7}$ at a maximizer of: $x^* = (\frac{100}{7}, \frac{100}{7}, 0), s_1 = 0$

Since $\frac{2500}{3} > \frac{5500}{7}$ We choose to utilize x_3 and produce: $\frac{2500}{3}$ at a maximizer of: $x^* = (0, \frac{100}{9}, \frac{100}{9})$
The Tree for this process is shown below:



Question 3

Solving the Linear Program with no Integer constraints. We arrive at the optimal solution of $Z = \frac{311}{7}$ where the maximizer is $x = (1, \frac{3}{7}, 0, 0, 0, 1)$
Branching on $x_2 = 0, 1$
Case: $x_2 = 0$: We reach the optimal solution of: 44 located at: $x = (1, 0, \frac{3}{4}, 0, 0, 1)$
Branching upon $x_3 = 0, 1$
Case: $x_3 = 0$: We arrive at: $Z = 43.25$ located at: $x = (1, 0, 0, 0, \frac{3}{4}, 1)$
Branching upon $x_5 = 0, 1$
Case: $x_5 = 0$. Z=43 located at: $x = (1, 0, 0, 1, 0, 1)$ This is our Best Feasible Solution.
Case: $x_5 = 1. Z = 42.8333$ This is below our current best Feasible Solution: 43.
Case: $x_3 = 1 : Z = \frac{263}{6}$ located at: $x = (1, 0, 1, 0, 0, \frac{5}{6})$
Branching upon: $x_6 = 0, 1$
Case: $x_6 = 0. Z = 41.6667$ This is below our current best Feasible Solution: 43
Case: $x_6 = 1. Z = 43.8$ located at: $x = (\frac{4}{5}, 0, 1, 0, 0, 1)$
Branching on $x_1 = 0, 1$
Case: $x_1 = 0. Z = 42$. This is below BFS.
Case: $x_1 = 1$. This is infeasible.
Case: $x_2 = 1$: This has solution: 44.333 located at: $x = (1, 1, 0, 0, 0, \frac{1}{3})$
Branching upon $x_6 = 0, 1$
Case: $x_6 = 0 : Z = 44$ located at $x = (1, 1, \frac{1}{2}, 0, 0, 0)$
Branching on $x_3 = 0, 1$

Case: $x_3 = 0 : Z = 43.5$ located at $x = (1, 1, 0, 0, \frac{1}{2}, 0$
Branching on $x_5 = 0, 1$
Case: $x_5 = 0$. $Z = 43.333$ located at $x = (1, 1, 0, \frac{2}{3}, 0, 0)$
Branching on $x_4 = 0, 1$
Case: $x_4 = 0.Z = 38$. This is below our BFS
Case: $x_4 = 1.Z = 42.8$. This is below our BFS
Case: $x_5 = 1.Z = 42.6$. This is below our BFS
Case: $x_3 = 1.Z = 43.6$ located at: $x = (\frac{3}{5}, 1, 1, 0, 0, 0)$
Branching upon $x_1 = 0, 1$
Case: $x_1 = 0.Z = 42.25$. This is below our BFS
Case: $x_1 = 1$. This is Infeasible.
Case: $x_6 = 1.Z = 44.2$ located at $x = (\frac{1}{5}, 1, 0, 0, 0, 1)$
Branching on $x_1 = 0, 1$
Case: $x_1 = 0.Z = 44$ at $x = (0, 1, \frac{1}{4}, 0, 0, 1)$
Branching on $x_3 = 0, 1$
Case: $x_3 = 0.Z = 43.75$ at $x = (0, 1, 0, 0, \frac{1}{4}, 1)$
Branching on $x_5 = 0, 1$
Case: $x_5 = 0.Z = 43.667$ at $x = (0, 1, 0, \frac{1}{3}, 0, 1)$
Branching on $x_4 = 0, 1$
Case: $x_4 = 0.Z = 41$. This is below our BFS
Case: $x_4 = 1$. Infeasible
Case: $x_5 = 1$ Infeasible
Case: $x_3 = 1$ Infeasible
Case: $x_1 = 1$. Infeasible

Our best Feasible Solution is Z=43 located at: $x = (1, 0, 0, 1, 0, 1)$

Question 4

Let $b_0 = a_0$ mod d, $b_j = a_j$ mod d

a

$31x_1 + 32x_2 + 9x_3 - 29x_4 = 51$, d = 15
$x_1 + 2x_2 + 9x_3 + x_4 \geq 6$

b

$20x_1 + 30x_2 - 5x_3 + 17x_4 + 11x_5 = 168$, d = 17
$3x_1 + 13x_2 + 12x_3 + 11x_5 \geq 15$

Question 5

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 4x_1 + 2x_2 \leq 15 \\ & x_1 + 2x_2 \leq 8 \\ & x_1 + x_2 \leq 5 \\ & x_1, x_2 \in \mathbb{Z}_+ \end{aligned}$$

This has a relaxed linear program of:

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + s_1 = 15 \\ & x_1 + 2x_2 + s_2 = 8 \\ & x_1 + x_2 + s_3 = 5 \\ & x_1, x_2, s_1, s_2, s_3 \in \mathbb{R}_+ \end{aligned}$$

Putting it into Clean Table form

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 2 & 1 & 0 & 0 & 15 \\ 0 & 1 & 2 & 0 & 1 & 0 & 8 \\ 0 & 1 & 1 & 0 & 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & 0 & \frac{-1}{2} & \frac{3}{4} & 0 & 0 & \frac{45}{4} \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{15}{4} \\ 0 & 0 & \frac{3}{2} & \frac{-1}{4} & 1 & 0 & \frac{17}{4} \\ 0 & 0 & \frac{1}{2} & \frac{-1}{4} & 0 & 1 & \frac{5}{4} \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & 0 & \frac{-1}{2} & \frac{3}{4} & 0 & 0 & \frac{45}{4} \\ 0 & 1 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{15}{4} \\ 0 & 0 & \frac{3}{2} & \frac{-1}{4} & 1 & 0 & \frac{17}{4} \\ 0 & 0 & \frac{1}{2} & \frac{-1}{4} & 0 & 1 & \frac{5}{4} \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{25}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -1 & \frac{5}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 1 & -3 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & 2 & \frac{5}{2} \end{array}\right]$$

This is the final tableau for the relaxed linear program. Since it yeilds maximizers of: $x_1 = \frac{5}{2}, x_2 = \frac{5}{2}$, we are not at a feasible integer program solution, and we will take the first constraint from the tableau to make our Gomory cut.
$x_1 + \frac{1}{2}s_1 - s_3 = \frac{5}{2}$
$(1 + \frac{0}{1})x_1 + (0 + \frac{1}{2})s_1 + (-1 + \frac{0}{1})s_3 = (2 + \frac{1}{2})$
So by adding the constraint: $\frac{1}{2}s_1 \geq \frac{1}{2}$ We have a new LP.

$$\begin{aligned} \text{max} \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + s_1 = 15 \\ & x_1 + 2x_2 + s_2 = 8 \\ & x_1 + x_2 + s_3 = 5 \\ & \frac{1}{2}s_1 \geq \frac{1}{2} \\ & x_1, x_2, s_1, s_2, s_3 \in \mathbb{R}_+ \end{aligned}$$

This has solution Z = 12 located at $x = (2, 3, 1, 0, 0)$.

Question 6

a

$\frac{5}{2}y_1 + \frac{7}{3}y_2 + \frac{1}{5}y_3 + 2x_1 \leq \frac{17}{4} + x_2$
We can see that: $f = \frac{1}{4}, f_1 = \frac{1}{2}, f_2 = \frac{1}{3}, f_3 = 15$
So our MIR is: $(2 + \frac{\frac{1}{4}^+}{\frac{3}{4}})y_1 + (2 + \frac{\frac{1}{12}^+}{\frac{3}{4}})y_2 + (0 + \frac{(\frac{1}{5} - \frac{1}{4})^+}{\frac{3}{4}})y_3 \leq 4 + \frac{x_2}{\frac{3}{4}}$
This simplifies to: $\frac{7}{3}y_1 + \frac{19}{9}y_2 \leq 4 + \frac{4x_2}{3}$

b

$\frac{5}{2}y_1 + \frac{5}{2}y_2 + \frac{19}{6}y_3 - x_1 \leq \frac{73}{5} + 2x_2$
$\frac{5}{2}y_1 + \frac{5}{2}y_2 + \frac{19}{6}y_3 \leq \frac{73}{5} + x_1 + 2x_2$
We can see that: $f = \frac{3}{5}, f_1 = \frac{2}{3}, f_2 = \frac{1}{2}, f_3 = 16$
So our MIR is: $(2 + \frac{(\frac{2}{3} - \frac{3}{5})^+}{\frac{2}{5}})y_1 + (2 + \frac{(\frac{1}{2} - \frac{3}{5})^+}{\frac{2}{5}})y_2 + (3 + \frac{(\frac{1}{6} - \frac{3}{5})^+}{\frac{2}{5}})y_3 \leq 14 + \frac{x_1 + 2x_2}{\frac{2}{5}}$
This simplifies to: $\frac{13}{6}y_1 + 2y_2 + 3y_3 \leq 14 + \frac{5}{2}x_1 + 5x_2$