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# The Nonatomic Assignment Model

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## Abstract

We formulate a model with a continuum of individuals to be assigned to a continuum of different positions which is an extension of the finite housing market version due to Shapley and Shubik. We show that optimal solutions to such a model exist and have properties similar to those established for finite models, namely, an equivalence among the following: (i) optimal solutions to the linear programming problem (and its dual) associated with the assignment model; (ii) the core of of the associated market game: (iii) the Walrasian equilibria of the associated market economy.

The assignment model was the subject of the first application, after Edgeworth [1881], of the core to an economic model (Shapley [1955]); and a close relative of the assignment model, the transportation problem, was the subject of one of the earliest developments in linear programming (Kantorovich [1942]). The links between the core of an assignment model and its formulation as a linear programming problem have been established by several authors, notably Shapley and Shubik [1972]. They established one of the first equivalence theorems between the core and Walrasian equilibria, for the assignment model consisting of a *finite* number of individuals exchanging a finite number of commodities.

In this paper we extend the Shapley and Shubik "housing market" version of the assignment model to one with a large number of buyers and a large number of sellers (literally a continuum) each having a distinct house. The analysis is divided into three parts, corresponding to the following alternative formulations of the problem:

(1) As a linear programming problem.

(2) As a market game.

(3) As an exchange economy.

We show that under suitable conditions:

(1') There exist optimal solutions to the linear programming problem and its dual.

(2') The dual solutions are equivalent to the core of the associated market game.

(3') There is an equivalence between the set of solutions to the primal and dual linear programming problems and the Walrasian equilibria of the associated exchange economy.

The conjunction of (1') - (3') extend the Shapley-Shubik result on core equivalence for finite models to the case of a continuum of individuals and a continuum of commodities.

The equivalence of the core and Walrasian equilibria for the assignment model is remarkably different from the equivalence theorems for other economic models demonstrated by, for example, Debreu and Scarf [1963] and Aumann [1964]. In these other models, the core is typically a superset of Walrasian equilibria, and it is only with large numbers of individuals that the core coincides with Walrasian equilibria.[1] Thus, core equivalence typically occurs because the core "shrinks" to the Walrasian

[1] This is not to suggest that large numbers, i.e., a nonatomic continuum of individuals, necessarily implies core equivalence. See Gretsky and Ostroy [1985] and Ostroy and Zame [1988].

equilibria as the number of individuals increases. However, in assignment models the core exhibits no discriminatory power in separating competitive from non-competitive housing markets since no matter what the numbers of individuals and commodities, the core *always* coincides with Walrasian equilibrium. In a follow-up paper, we shall provide an alternative test of competitivity in assignment models based on the no-surplus definition of perfect competition (Ostroy [1980, 1984], Makowski [1980]) and exhibit conditions such that housing markets with a large number of buyers and differentiated commodities (i.e., houses) will or will not be perfectly competitive.

There is a little known but remarkable list of papers related to the nonatomic assignment model. The paper by Kantorovich cited in the first paragraph (translated and reprinted in Kantorovich [1958]) is a nonatomic version of the transportation problem and its dual. Kantorovich and Akilov [1982, p.225-237] summarize several of the contributions of the first-named author and his co-workers to the study of "the translocation of masses".[2] In common with their work, our formulation uses measures on the set of buyer-seller pairs to describe an assignment. Unlike their work, however, we consider a more general class of objective functions as well as the connections among the linear programming, core and market equilibrium problems. In the literature on nonatomic games, the general version of the assignment problem does not appear to have been treated. However, Kaneko and Wooders [1986] have treated a special version. They represent assignments by measure-preserving mappings from buyers to sellers and focus their attention on the non-emptiness of a modification of the core that is particularly suitable for assignment-type models.

In Section 1, the nonatomic version of the assignment problem is formulated and its connection to linear programming, market games and exchange economies is elaborated. We conclude Section 1 with a Portmanteau Theorem summarizing the relationship among the alternative characterizations of the model. In Section 2, we give several examples illustrating the Theorem and the need for its qualifying hypotheses. Finally, more precise statements of the results, along with their proofs, are given in Section 3.

# 1 Alternative Formulations

## 1.1 Preliminaries

For a compact metric space $X$, define

[2] While possibly familiar to those well-acquainted with Russian mathematics, we discovered this work by accident after failing to find any references in standard sources (e.g., Krabs [1979]) and after independently formulating the problem.

- $\mathbf{C}(X)$ to be the Banach space of continuous functions on $X$ equipped with the supremum norm; i.e., $\|f\| = \sup_{x \in X} |f(x)|$

- $\mathbf{M}(X)$ to be the Banach space of countably additive Borel measures on $X$ equipped with the total variation norm; i.e., $\|\nu\| = \sup_\pi \sum_{E_i} |\nu(E_i)|$ over all finite measurable partitions $\pi$ of $X$ and

- $\mathbf{B}(X)$ to be the Banach space of bounded measurable functions on $X$ equipped with the supremum norm.

Recall that $\mathbf{M}(X)$ is the space of continuous linear functionals on $\mathbf{C}(X)$, with the pairing given by integration: $\langle f, \nu \rangle = \int f d\nu$. Of course every element of $\mathbf{C}(X)$ gives rise to a continuous linear functional on $\mathbf{M}(X)$, via the same pairing. More generally, every function in $\mathbf{B}(X)$ gives rise to a continuous linear functional via the pairing $\langle g, \nu \rangle = \int g d\nu$. We shall make particular use of the weak topology $\sigma(\mathbf{M}(X), \mathbf{B}(X))$ on $\mathbf{M}(X)$ that comes from this pairing; convergence of the net $\{\nu_\alpha\}$ to $\nu$ in this topology means that $\int g d\nu_\alpha \to \int g d\nu$ for every $g \in \mathbf{B}(X)$.

If $(T, \mathcal{T}, \tau)$ is a measure space, then a map $h : T \to \mathbf{M}(X)$ is (Gelfand) measurable if the real-valued function $t \mapsto \langle f, h(t) \rangle$ is measurable for each $f \in \mathbf{C}(X)$. Similarly, we say that $h$ is (Gelfand) integrable if is (Gelfand) measurable and for each $T_0 \in \mathcal{T}$, there is a measure $\nu_{T_0} = \int_{T_0} h d\tau \in \mathbf{M}(X)$ such that

$$\langle f, \nu_{T_0} \rangle = \int_{T_0} \langle f, h(t) \rangle d\tau(t),$$

for each $f \in \mathbf{C}(X)$.

## 1.2 Initial Statement of the Problem

There are two classes of individuals, the set $B$ of buyers and the set $S$ of sellers. Denote the Cartesian product of pairs of buyers and sellers as $P = B \times S$ and the disjoint union of individuals as $I = B \cup S$. Each of the classes $B$ and $S$ will be a compact metric space (usually the unit interval $[0,1]$) with a population measure defined on the Borel sigma-algebra of the underlying set; denote these two measures as $\mu_B$ and $\mu_S$ respectively. The combined measure on $I$ will be called simply $\mu$. Each seller $s \in S$ is identified as having a house of type $s$ and a reservation value $\sigma(s)$ indicating the minimum amount of money that $s$ would be willing to accept to sell his house. The list of all seller reservation values is given by the function $\sigma : S \to \mathbf{R}_+$. Each buyer $b \in B$ has a reservation value $\beta(b, s)$ indicating the maximum amount of money he would pay to obtain the house $s$. The list of all buyer reservation values is given by the function $\beta : P \to \mathbf{R}_+$.

In summary, the data for the problem are:

- agents—the set of buyers $B$ and the set of the sellers $S$
- population measure—$\mu$ on $B \cup S$
- agents' characteristics—$(\beta, \sigma)$ where $\beta : B \times S \to \mathbf{R}$ and $\sigma : S \to \mathbf{R}$ are reservation values.

Since $B$ and $S$ are tacitly understood to be fixed, we may describe the problem as given by $(\mu, \beta, \sigma)$.

If buyer $b$ and seller $s$ were to transfer ownership of house $s$, the monetary value of this transfer between the pair $(b, s)$ would be [3]

$$V(b, s) = \beta(b, s) - \sigma(s).$$

Thus, if $b$ and $s$ were paired they would not voluntarily make the exchange if the seller's reservation price exceeded that of the buyer. Alternatively, $V(b, s)$ is the profit available to the pair $(b, s)$. In most instances, the assignment problem can be defined in summary form by the pair $(\mu, V)$.

The goal of the assignment problem is to match buyers and sellers so as to maximize the total available profit. We define an *assignment* of buyers to sellers (i.e., houses) as a measure on $P = B \times S$. To this end, let $\mathbf{M}(P)$ be the countably additive Borel measures on $P$. Interpret the measure $x$ in the assignment model as a statistical summary of activities in the housing market, i.e., $x(E \times F)$ is the distribution of buyers in $E$ purchasing from sellers of houses in $F$. The question arises whether this statistical distribution has a representation in terms of pairs of individual buyers and sellers and whether this individualistic representation preserves the implicit feature of the assignment model that each buyer is assigned to at most one seller and conversely, so that fractional assignments are ignored. This issue will be addressed in Sections 1.2.1 and 1.5.1, below.

**Definition:** The assignment $x \in \mathbf{M}(P)$ is *feasible* for the population $\mu$ if $x \geq 0$ and $x(E \times S) \leq \mu_B(E)$ and $x(B \times F) \leq \mu_S(F)$ for all Borel sets $E$ in $B$ and $F$ in $S$.

Given the definition of feasible assignments, we can state:

[3]The definition given in Shapley and Shubik [1972] makes $V$ non-negative for all $(b, s)$. This has the unfortunate feature of allowing players to appear to trade when they should not. Our definition of $V$ does not change the values computed below but merely rules out the need to extract non-trading players from the solutions.

**Definition:** The *assignment problem* given by the pair $(\mu, V)$, where $V(b,s) = \beta(b,s) - \sigma(s)$, is: find $x \in \mathbf{M}(P)$ to attain

$$g(\mu) = \sup\{\int_P V(b,s)dx(b,s) | x \text{ is feasible for } \mu\}.$$

Call $g(\mu)$ the *optimal value* of the assignment model as a function of the population measure of buyers and sellers, holding $V$ fixed. The function $g$ is readily shown to be both positively homogeneous ($g(t\mu) = tg(\mu)$, $t > 0$) and concave on the set of positive measures in $\mathbf{M}(I)$. The subdifferential of the value function plays a key role in uniting the various solution concepts, below.

**Definition:** The *subdifferential* $\partial g(\mu)$ of the value function $g$ at $\mu$ is the collection of $q \in \mathbf{B}(I)$ such that $\int q d\mu \equiv \langle q, \mu \rangle = g(\mu)$ and $\langle q, \mu' \rangle \geq g(\mu')$ for all $\mu' \geq 0$.

### 1.2.1 Finite Assignment Models and Integral Assignments

In the finite assignment model, the statistical interpretation of an optimal assignment $x$ also yields an individualistic interpretation as a matching of buyers and sellers. In the finite model, $B = \{1, \ldots, m\}$ and $S = \{1, \ldots, n\}$ and $\mu(C)$ equals the cardinality of $C$, a subset of $B \cup S$. Therefore, a feasible assignment can be written as

$$\sum_s x_{bs} \leq 1, \text{ for all } b,$$

$$\sum_b x_{bs} \leq 1, \text{ for all } s,$$

where $x_{bs} = x(\{b\} \times \{s\})$. It is known that optimal assignments include those in which $\mu(B_0) = \mu(S_0) = k \leq \min\{m,n\}$. When the inequalities above are replaced by equalities involving $k$ buyers and $k$ sellers, the constraint set is the set of doubly-stochastic matrices (of order $k$). Since the objective function of the assignment model (i.e., $\sum_s \sum_b V_{bs} x_{bs}$) is linear and the constraint set is convex, an optimal solution will occur at an extreme point of the set of doubly-stochastic matrices. These extreme points are known to be precisely those $x$ in the constraint set such that $x_{bs} \in \{0,1\}$, the set of permutation matrices of order $k$. Thus, there exists a permutation $\pi : B_0 \to S_0$ such that $x_{bs} = 1$ if and only if $\pi(b) = s$, i.e., there is always an integral optimal assignment.

In the nonatomic extension of the assignment model, let $(B, \mu_B)$ and $(S, \mu_S)$ be measure spaces. Recall that a measure $\nu$ on $B \times S$ is called *doubly stochastic* if $\nu(E \times S) = \mu_B(E)$ for all measurable $E \subset B$ and $\nu(B \times F) = \mu_S(F)$ for all measurable

$F \subset S$. A feasible measure for the linear programming formulation of the assignment problem is doubly stochastic precisely when there is equality in the constraints and every doubly stochastic measure (with respect to $(\mu_B, \mu_S)$) is feasible.

Define $\pi : B \to S$ to be a *measure-preserving isomorphism* if $\pi$ a one-to-one measurable map of $B$ onto $S$ and $\mu_B(E) = \mu_S(\pi[E])$. Then we can construct a doubly stochastic measure $\nu$ with $\nu(E \times F) = \nu(E, \pi(E) \cap F)$ whose support is $\{(b, s)|s = \pi(b)\}$. Such a measure $\nu$ is singular with respect to $\mu_B \times \mu_S$. A theorem of Lindenstrauss [1965] asserts that every extreme doubly stochastic measure on $B \times S$ is singular with respect to $\mu_B \times \mu_S$. Unlike the situation in the finite case however, not all such extreme measures arise from a measure-preserving isomorphism. (See Example 5, below, for an extreme point which cannot be so represented.) Therefore, there is no guarantee that there is an integral optimal assignment when the set of individual buyers and sellers is taken to be $B$ and $S$. This conclusion agrees with previous results on the difficulties of providing an individualistic representation of economies in distribution form. In the analysis of the assignment problem as a Walrasian equilibrium in Section 1.5.1, we shall use a device from the theory of economies in distribution form (Hildenbrand [1974]) to provide an integral representation of an assignment based on an alternative description of the set of buyers and sellers.

## 1.3 The Assignment Problem as a Linear Program

In this section we treat the assignment problem as a linear program. The dual of this problem will be of primary interest.

The condition on the feasibility of an assignment can also be described in an operator-theoretic way useful for a linear programming description of the assignment problem. Define the coordinate projections $(\pi_1 x)(U) = x(U \times S)$ for all Borel sets $U$ in $B$ and $(\pi_2 x)(V) = x(B \times V)$ for all Borel sets $V$ in $S$. Define the bounded linear operator $A : \mathbf{M}(P) \to \mathbf{M}(I)$ by $Ax = (\pi_1 x, \pi_2 x)$. Feasibility becomes the condition that $Ax \leq \mu$. In keeping with the finite agent version of the assignment problem, it's appropriate that $A : \delta_{(b,s)} \mapsto \delta_b + \delta_s$ where $\delta_x$ denotes the unit point mass at the point $x$. Moreover, $A$ is the dual of the operator $A^* : \mathbf{C}(I) \to \mathbf{C}(P)$ given by $(A^* q)(b, s) = q(b) + q(s)$. Consequently, the operator $A$ is weak*- to weak*-continuous. Since the unit point masses are the extreme points of the unit ball of $\mathbf{M}(P)$ and the unit ball is weak*-compact, it follows that every element in the unit ball is in the weak*-closed convex hull of the collection of the unit point masses. Thus, from the values of $A$ on the unit point masses $\mathbf{M}(P)$ we can extend $A$ by linearity and weak*-continuity to the unit ball and thence to the whole space, and the values of this extension agree with the values given by the definition of $A$ via the projections.

The problem of finding a solution to the assignment game can be recast in linear programming terms.

The *primal problem* is: find $x \in \mathbf{M}(P)$ so as to achieve

$$g(\mu) = \sup\{\langle V, x\rangle = \int_P V(b,s)dx(b,s) \mid Ax \leq \mu,\ x \geq 0\}.$$

The *dual problem* is to find a function $q \in \mathbf{B}(I)$ so as to achieve

$$h(V) = \inf\{\langle q, \mu\rangle = \int_I q d\mu = \int_B q(b)d\mu_B(b) + \int_S q(s)d\mu_S(s) | A^*q \geq V,\ q \geq 0\}.$$

As in any (infinite-dimensional) linear programming problem, the basic issues are:

- existence of primal solutions, i.e., feasible $x$ such that $\langle V, x\rangle = g(\mu)$,
- existence of dual solutions, i.e., feasible $q$ such that $\langle q, \mu\rangle = h(V)$,
- no gap between the primal and dual values, i.e., $g(\mu) = h(V)$.

## 1.4 The Assignment Problem as a Market Game

A game which is closely related to the assignment problem can be defined by specifying its (game-theoretic) characteristic function. The core of this game has an intimate connection via the Radon-Nikodym theorem to the (dual) solutions of the linear programming problem. In the finite player case a result of Shapley and Shubik [1972] tells us that the core is precisely the set of solutions to the dual problem. In the present context, there are two possible definitions of the core corresponding to two possible interpretations of the set of coalitions. We shall see that under the restriction that the valuation function $V$ is continuous, the two different sets of coalitions yield the same conclusions with respect to assignments in the core. However, when $V$ is not continuous, to capture the desired equivalence between the dual and the core, "distributional" coalitions are essential.

We begin with the standard formulation of a nonatomic game (see Aumann and Shapley [1974] based on the assignment model in which the coalitions are the Borel subsets $\mathcal{B}$ of $I = B \cup S$. The *characteristic function* $w$ of the assignment game is defined on $\mathcal{B}$ as the maximum profit available to the coalition $C$ for each Borel set $C \in \mathcal{B}$, viz.

$$w(C) = \sup\{\int V dx | x \geq 0,\ Ax \leq \mu_C\} = g(\mu_C),$$

where $\mu_C = (\mu_{C\cap B}, \mu_{C\cap S})$.

**Definition:** The *core* of the game given by characteristic function $w$ is defined to be $\mathcal{C}(w) = \{\nu : \mathcal{B} \to \mathbf{R} | \nu$ is a finitely additive set function with $\nu(C) \geq w(C)$ for all $C \in \mathcal{B}$ and $\nu(I) = w(I)\}$.

To enlarge the family of coalitions, first identify the elements $C$ of $\mathcal{B}$ with their indicator functions $\mathbf{1}_C \in \mathbf{B}(I)$. Then, denote by $\mathcal{I}$ the set of those $\tilde{C} \in \mathbf{B}(I)$ such that $0 \leq \tilde{C} \leq \mathbf{1}_I$, i.e., $\tilde{C}$ represents a "fractional group of buyers and sellers" in $C$. In Aumann and Shapley [1974], these coalitions are called *ideal sets.* Each $\tilde{C}$ yields a population measure $\tilde{\mu}$ defined by $\tilde{\mu}(\cdot) = \int_{(\cdot)} \tilde{C} d\mu$

Since we have placed no restrictions on the measure $\mu$ in the definition of an assignment problem, it is indistinguishable from a $\tilde{\mu}$. Therefore, the optimal value associated with the population measure $\tilde{\mu}$ is $g(\tilde{\mu})$. This permits the extension of the game-theoretic characteristic function $w$ from elements of $\mathcal{B}$ to $\mathcal{I}$ via the function $\tilde{w}(\tilde{C}) = g(\tilde{\mu})$. The distributional coalitions associated with the nonatomic game $\tilde{w}$ are the functions $\tilde{C}$ which include not only those ideal sets taking values in $\{0, 1\}$ but also those $\tilde{C}$ taking values in $[0, 1]$ and identified with all the measures $\tilde{\mu} \leq \mu$.

Call $\tilde{\nu} : \mathcal{I} \to \mathbf{R}$ an *ideal set function* if it is finitely additive in the sense that $\tilde{\nu}(\tilde{C}_1 + \tilde{C}_2) = \tilde{\nu}(\tilde{C}_1) + \tilde{\nu}(\tilde{C}_2)$ whenever $\tilde{C}_1 + \tilde{C}_2 \leq \mathbf{1}_I$.

**Definition:** The *distributional core* $\mathcal{D}(\tilde{w})$ of the market game determined by the characteristic function $w$, or equivalently by the assignment model $(\mu, V)$, is defined to consist of all ideal set functions $\tilde{\nu}$ such that $\tilde{\nu}(\tilde{C}) \geq \tilde{w}(\tilde{C})$ for all $\tilde{C} \in \mathcal{I}$ and $\tilde{\nu}(\mathbf{1}_I) = \tilde{w}(\mathbf{1}_I)$.

## 1.5 The Assignment Problem as a Market Economy

The assignment problem can also be formulated as taking place in a market economy. To place the problem in this context, it is necessary to explicitly give the endowments and preferences of the agents and to emphasize the description of the problem in terms of the individual components $\beta$ and $\sigma$ rather than the summary term $V$ used in the linear programming and core approaches.

There are two classes of goods: houses and money. Each seller $s$ has an initial endowment of one house identified with his name. In describing allocations, we face a difficulty here parallel to the discussion of integral assignments in Section 1.2.1, namely the possibility that allocations may not be integral. There are at least two ways of treating this problem: to allow individuals to buy and sell fractional houses or to permit only integral allocations, but to view elements of $B$ and $S$ as *types* rather than individuals. The second of these is more consistent with the spirit of the assignment problem and we shall describe it in Section 1.5.1, below. Here, however, it is convenient to adopt the first interpretation.

The preferences of buyers and sellers are given by utility functions. Buyer $b$ has a feasible trading set $D_b$ consisting of all non-negative measures on $S$ with variation norm less than or equal to 1. The utility of buyer $b$ for $\gamma \in D_b$ is given by $u_b(\gamma) = \langle \beta_b, \gamma \rangle$ where $\beta_b(s) = \beta(b, s)$. Seller $s$ has a feasible trading set $D_s$ consisting of all non-positive measures on $S$ of the form $\alpha\delta_s$ where $-1 \leq \alpha \leq 0$. The utility of seller $s$ for $\alpha\delta_s \in D_s$ is given by $u_s(\alpha\delta_s) = \alpha\sigma(s)$.

An equilibrium will be described by a price function for houses such that when buyer and seller types make purchase and sale decisions by maximizing their utility taking those prices as given, there is an allocation of buyers to houses which is market-clearing.

An *allocation* for a market economy is a Gelfand measurable map $y : B \cup S \to \mathbf{M}(S)$ where $y(b)$ indicates what houses buyer $b$ buys and $y(s)$ indicates what (portion of) the house $\delta_s$ seller $s$ sells. A *feasible allocation* for the market economy with population measure $\mu$ is an allocation such that

$$\int_I y d\mu = 0.$$

(The integral is taken in the Gelfand sense.) A *price system* is a measurable function $p \in \mathbf{B}(S)$ where $p(s)$ indicates the price of the house $s$.

**Definition:** An allocation and price system $(y, p)$ is a *Walrasian equilibrium* for the market economy $(\beta, \sigma, \mu)$ if $y$ is a feasible allocation for $\mu$ and

$$u_i(y(i)) - \langle p, y(i) \rangle = \max\{u_i(\gamma) - \langle p, \gamma \rangle | \gamma \in D_i\}, \ \mu - a.e.$$

The above condition is the requirement that $y(i)$ be a utility maximizing choice when prices are given by $p$. If we define $u_i$ on all of $\mathbf{M}(S)$ by saying that it agrees with the above definition on the feasible set $D_i$ and is $-\infty$ elsewhere, the condition that $y(i)$ is utility-maximizing for $i$ at prices $p$ is evidently equivalent to the condition that $p \in \partial u_i(y(i))$.

### 1.5.1 Representations of Integral Allocations

Let $\Delta = \{\delta_s : s \in S\} \cup \{0\}$. Define $y$ to be an *integral allocation* if, $\mu$-a.e. on $I$, $y(b) \in \Delta$ and $y(s) \in -\Delta$. Note that if $y$ is a Walrasian allocation for prices $p$, and $s$ and $s'$ are in the support of $y(b)$, then $u_b(y(b)) - \langle p, y(b) \rangle = \beta(b, s) - p(s) = \beta(b, s') - p(s')$, i.e., an houses in the support of $y(b)$. Similarly, if $y(s) = \alpha\delta_s$, where $0 < \alpha < 1$, then $\sigma(s) - p(s) = 0$ and any seller with utility function $u_s$ is indifferent between selling and not selling. Thus, the Walrasian equilibrium $(y, p)$ would itself be an integral assignment if $\mu$-a.e. the utility-maximizating choice at $p$ were unique.

In the absence of uniqueness, we face a problem related to that raised in Section 1.2.1 on the inability to guarantee an integral assignment among the optimal solutions of the nonatomic assignment model. *When the set of individuals is constrained to be* $I$, there is no guarantee of an integral Walrasian allocation in the above description of a market economy. (See Example 5, below.) However, there is a representation of a Walrasian allocation as an integral assignment if the space of individuals is expanded to $I \times [0, 1]$. (Note the similarity between this expansion and the construction of ideal sets in Section 1.4.)

Consider an allocation $y$. Regard $y(i) \in \mathbf{M}(S)$ as a summary statistical description of the average behavior of all the individuals of *type* $i$, of whom there are nonatomic continuum represented by Lebesgue measure on $[0,1]$. In effect, the allocation $y$ is an intermediate level description of economic activity which lies between the purely statistical description of $x$ as a measure on types of buyers and types of houses and the following description in terms of individual members of each type.

Define $z = (z_B, z_S)$, where $z_B : B \times [0,1] \to \Delta$ and $z_S : S \times [0,1] \to -\Delta$ with the additional stipulation that for each $s$, $z_S(s,t) \in \{0, -\delta_s\}$ for all $t$. By construction, the mapping $z$ represents an integral allocation among the individuals in $I \times [0,1]$ since it requires each buyer to purchase either no house or exactly one integral house and each seller to sell either all or nothing of his/her house.

If $z$ is $\mu \times \lambda$ measurable, then it is integrable and we may define $y : I \to \mathbf{M}(S)$ by

$$y(i) = \int z(i,t) d\lambda(t). \tag{$*$}$$

Conversely, if $y : I \to \mathbf{M}(S)$ is a $\mu$-integrable function such that for each $i$, $y(i) \in D_i$, there is a $z$ satisfying $(*)$. (This is a result on the disintegration of measures. See Bourbaki [1959].)

## 1.6 Solutions and Equivalences

The point of this paper and its main theorem is that under suitable hypotheses all these ways of looking at the assignment problem are soluble and that within reasonable equivalences the solutions are the same. In terms of the data of the problem, no restrictions are made on $\mu$, but we do assume $\beta$ is upper semi-continuous on $P$ and $\sigma$ is lower semi-continuous on $S$, making $V = \beta - \sigma$ upper semi-continuous.

**PORTMANTEAU THEOREM** *Assume that the population measure $\mu \in \mathbf{M}(I)$ is given and that the valuation function $V \in \mathbf{B}(B \times S)$ is upper semi-continuous. Then*

- *The assignment problem has a solution (Theorem 2).*
- *The linear programming problem has both primal and dual solutions and there is no gap in their values (Theorems 1 and 2).*
- *The distributional core (and hence the core) is non-empty (Theorem 3).*
- *Walrasian equilibria exist (Theorem 4).*
- *The subdifferential of the value function is non-empty (Theorem 5).*
- *All the solution concepts are effectively equivalent (Theorem 1 - 5).*

In addition, we shall establish some stronger conclusions when $V \in \mathbf{C}(B \times S)$ (Theorems 6 - 8).

# 2 Examples

In the following examples, $B$ and $S$ are each taken to be the unit interval $[0, 1]$. The population measures $\mu_B$ and $\mu_S$ are each taken to be Lebesgue measure on the Borel sets of $[0, 1]$, i.e. $\mu$ on $I$ is taken to be $\mu = (\mu_B, \mu_S) = (\lambda, \lambda)$. Also we take $\sigma(s) = 0$ for all $s \in S$. Thus, in these examples, the assignment model is defined by $V(b, s) = \beta(b, s) - \sigma(s) = \beta(b, s)$.

Let us recall the distinctions among the following: A solution to the dual problem is a $q \in \mathbf{B}(I)$, whereas a Walrasian price system is a $p \in \mathbf{B}(S)$. Define the conjugate of the utility function $u_i$, or the indirect utility function, as $u_i^*(p) = \sup\{u_i(\gamma) - \langle p, \gamma \rangle \mid \gamma \in D_i\}$. Taking advantage of the linearity of utility in the assignment model as well as the additional restrictions imposed above, we have

$$u_i^*(p) = \begin{cases} \sup_s \{\beta(b, s) - p(s)\} & \text{if } i = b \\ p(s) & \text{if } i = s, \end{cases}$$

as long as $p \geq 0$. The relation between (the canonical representative of the equivalence class of) dual solutions and Walrasian prices is given by the one-to-one mapping $q(i) = u_i^*(p)$.

The first example illustrates a "typical" case when $V$ is continuous.

**Example 1.** *Bigger is better.* Let $V(b,s) = bs$. In this example, the primal and dual linear programs each have a unique solution, and the dual solution lies in $\mathbf{C}(S)$.

Let $x$ be an optimal primal solution. In this case, any optimal solution must achieve equality in the constraint conditions. Let $q$ be any optimal dual solution in $C(I)$. Notice that $x$ has support on the main diagonal. To see this, note that $q(b) + q(s) \geq V(b,s)$ for all $(b,s)$ and that $q(b) + q(s) = V(b,s)$ for $(b,s)$ in the support of $x$. Now, take any two points $(b_0, s_0)$ and $(b_1, s_1)$ in the support of $x$. Then,

$$\begin{aligned} q(b_0) + q(s_0) &= b_0 s_0 \\ q(b_1) + q(s_1) &= b_1 s_1 \\ q(b_0) + q(s_1) &\geq b_0 s_1 \\ q(b_1) + q(s_0) &\geq b_1 s_0 \end{aligned}$$

Adding the two inequalities and then subtracting $b_0 s_0 + b_1 s_1$ yields

$$b_0 s_1 + b_1 s_0 - b_0 s_0 - b_1 s_1 \leq q(b_0) + q(s_1) + q(b_1) + q(s_0) - b_0 s_0 - b_1 s_1$$

Substituting in the two equalities the expression becomes $0 \leq (b_1 - b_0)(s_1 - s_0)$. Thus either $(b_0, s_0) \leq (b_1, s_1)$ or $(b_0, s_0) \geq (b_1, s_1)$.

Now suppose that $(b_0, s_0)$ is in the support of $x$. Then $b_0 s_0 = q(b_0) + q(s_0)$ and the support of $x$ lies in the set $([0, b_0] \times [0, s_0]) \cup ([b_0, 1] \times [s_0, 1])$. Since, however, there is equality in the constraint condition, it follows that

$$\begin{aligned} x([0, b_0] \times [0, s_0]) &= x([0, b_0] \times S) = \lambda[0, b_0] = b_0 \\ x([0, b_0] \times [0, s_0]) &= x(B \times [0, s_0]) = \lambda[0, s_0] = s_0 \end{aligned}$$

i.e. $b_0 = s_0$ so that $(b_0, s_0)$ does indeed lie on the main diagonal of the square. There is only one measure on $B \times S$ with support on the main diagonal and Lebesgue measure as its two co-ordinate functions, i.e. $x$ is unique.

For a continuous optimal dual solution $q$ on $I$, as above, $q(b) + q(s) \geq V(b,s)$ for all $(b,s)$, $q(b) + q(s) = bs$ for $b = s$, and $q \geq 0$. (Recall that the function $q$ on $B$ is distinct from the function $q$ on $S$.) The obvious candidates for solutions are $q(b) = b^2/2$ and $q(s) = s^2/2$. A simple application of the arithmetic-geometric mean inequality gives

$$q(b) + q(s) = b^2/2 + s^2/2 \geq (b^2 s^2)^{1/2} = bs$$

A moment's reflection with small perturbations yields the conclusion that this solution is unique.

The Walrasian allocation corresponding to the optimal solution of the linear programming problem is the integral allocation $y(b) = \delta_b$ and $y(s) = -\delta_s$. Walrasian prices are obtained from the dual solution simply by setting $p(s) = q(s)$.

The second example illustrates why uniqueness of dual solutions for a continuous valuation function can be only a generic, rather than a global, result. This example is equivalent to the well-known "glove market" which, in turn, is equivalent to the "master-servant" example Edgeworth [1881] used to illustrate how large numbers by itself does not suffice to guarantee what he called *determinacy*, i.e., unicity of the core.

**Example 2.** *Glove Market.* Let $V(b,s) = 1$ for all $(b,s)$ so that $V$ is certainly a continuous function on $B \times S$. There are many primal solutions; any measurable automorphism on $[0,1]$ describes a matching of buyers and sellers that is optimal. There is also a multiplicity of dual solutions; fix $0 \leq \alpha \leq 1$ and let $q(b) \equiv \alpha$ and $q(s) \equiv 1 - \alpha$.

Note that this example has a unstable core in the following sense. If a small coalition of buyers (sellers) reduces (increases) their reservation prices, there will be a reduction in the highest equilibrium price (increase in the lowest equilibrium price). In particular, let the buyers in $[0, 1/n]$ announce the reservation values $\beta(b,s) = \epsilon$ so that $V(b,s) = \epsilon$ if $b \leq 1/n$. Then the set of dual solution becomes $q(b) = \alpha$ and $q(S) = 1 - \alpha$ for $\alpha \in [1-\epsilon, 1]$. This illustrates the instability/manipulability of the core when it is not a singleton.

**Example 3.** *Diagonal.* Let $V(b,s) = 1$ if $b = s$ and 0 otherwise. Evidently, $V$ is not continuous, but it is upper semi-continuous. The primal solution is unique: each buyer $b$ is matched with the seller $s$ of the same name. The multiplicity of dual solution is, however, enormous. Let $q$ be defined on $B$ as any measurable function with values between 0 and 1. Then define $q(s) = 1 - q(b)$ for $b = s$. These are all dual solutions. Notice that unlike the previous example, where the price of any house could be any number between 0 and 1 but *the prices of all houses had to be the same*, here the price of any house is between 0 and 1 but there is no relation between the price of one house and another. This is because the example essentially depicts a continuum of isolated "Edgeworth boxes".

The next example shows the need for the valuation function to be upper semi-continuous in order for there to be an optimal solution to the primal problem.

**Example 4.** *Snob.* In this example every buyer likes equally well any house *strictly* above his own name. Let $\beta(b,s) = 1$ if $s > b$ and $= 0$ otherwise. By reasoning similar to the previous case, one sees that the primal solution "wants" to be the uniform measure of mass 1 supported on the diagonal of the square (or at least arbitrarily close to that from above); unfortunately, this is not a possible solution since the valuation function vanishes on the main diagonal. The dual problem has multiple solutions,viz. for each $\alpha \in [0,1]$ the distribution of $\alpha$ to each buyer and $1-\alpha$ to each seller provides the collection of dual solutions. Note that there is no gap in this problem since both values are 1.

If the valuation function is changed to become upper semi-continuous by setting $\beta(b,s) = 1$ if $s \geq b$ and $= 0$ otherwise, then the measure mentioned in the above paragraph is indeed the unique primal solution.

The next example illustrates two related points: (i) there need not exist an integral Walrasian allocation $y$, i.e., the range of which lies in $\Delta$, and (ii) the core and the distributional core need not coincide.

**Example 5.** *Double Your Pleasure.* Let $V(b,s) = 1$ if $s \geq 2b$ and be 0 otherwise. This function is upper semi-continuous (it is the characteristic function of a closed set). The primal solution is unique: the measure $x$ is supported on the line $s = 2b$ and carries mass 1/2. The dual solution is also unique: for $b \in [0,1/2]$, $q(b) = 1$, whereas for the remaining $b$ and all $s$, $q(b) = q(s) = 0$.

The price system associated with this $q$ is $p = 0$. At zero prices, the buyers in $[0,1/2]$ are each able to purchase one house for free and the remaining buyers are content not to purchase since there is nothing in the market they like. The sellers, having zero reservation values, are indifferent between supplying and not supplying.

The associated Walrasian allocation $y$ has $y(b) = \delta_{2b}$ for $b \in [0,1/2]$, $y(b) = 0$ for $b \in (1/2,1]$ and $y(s) = (1/2)\delta_s$ for all $s$. If we interpret the individuals in the model as consisting of $I \times [0,1]$, so that there is a continuum of each *type* $b \in B$ and $s \in S$, then there is an integral $z = (z_B, z_S)$ given by $z(b,t)$ equal to $y(b)$ in the previous sentence and, for example,

$$z(s,t) = \begin{cases} \delta_s & \text{if } t \in [0,1/2] \\ 0 & \text{if } t \in (1/2,1]. \end{cases}$$

With this $z$ there is an integral allocation over the individuals in $I \times [0,1]$ leading to the summary statistical description $y$ of the allocation over types, i.e., $y(i) = \int z(i,t)d\lambda(t)$.

If, however, we regard $I$ as itself the set of (indivisible) individuals, then there is no integral Walrasian allocation. Further, if we similarly regard the set of coalitions in the definition of the core as the Borel sets of $I$, the core is much bigger than the unique $q$ given above. In fact it is readily verified that the characteristic function $w$ for the market game associated with this problem is $w(C) = \min\{\lambda(C \cap [0,1/2] \cap B), \lambda(C \cap S)\}$. The set of core solutions $\nu$ are given by $\nu(C) = \int_C q_\alpha(i)d\lambda(i)$ where for each $\alpha \in [0,1]$ set $q_\alpha(b) = \alpha$ if $0 \leq b \leq 1/2$ and $= 0$ elsewhere, and set $q_\alpha(s) = 1-\alpha$

for all $s$.

There are neither Walrasian allocations of the market economy nor dual solutions of the linear programming problem that correspond to these core solutions. Thus, the example shows that if we are to obtain core equivalence as one of the alternative characterizations of the assignment problem, it will be necessary to consider the distributional core. However, as shown in Section 3.5, if $V$ is restricted to be continuous, the complications found in this example disappear in the sense that the core and the distributional core coincide.

As we have noted, optimal assignments need not be matchings. When the valuation $V$ is continuous, it can be shown that every optimal assignment can be approximated by matchings which come arbitrarily close to realizing the optimal value. The last example shows, however, that assignments which are the limits of matchings need not be matchings, and that — even when $V$ is continuous — it may not be possible to realize the value via a matching.

**Example 6.** *Almost a matching.* Let $D$ denote the main diagonal $\{(b,s) \in B \times S : b = s\}$ and denote the distance from a point $(b,s)$ to $D$ by $d((b,s),D)$. Define

$$V(b,s) = \begin{cases} 1 & \text{if } s = 2b \\ 1 - d((b,s),D) & \text{if } s > 2b \\ 0 & \text{otherwise} \end{cases}$$

Fix a positive integer $n$. Define $f_n(b)$ to be the piecewise linear function taking the intervals $[(k-1)/n, k/n]$ to $[(2k-1)/n, 2k/n]$ for $k = 1, \ldots, n$. Note that each $f_n$ is indeed a matching and is uniformly within $1/n$ of the function $f(x) = 2x$ which is not a matching. In fact, $f_n$ converges uniformly to $f$ which is not a matching. The (unique) optimal solution here is normalized arc length on the line $s = 2b$.

# 3 Statements and Proofs of the Theorems

## 3.1 The Linear Program and Its Dual

We must be more explicit about the spaces to be paired in duality with $\mathbf{M}(P)$ and $\mathbf{M}(I)$. The simplest choice of pairings would use the duality pairing $(\mathbf{M}(X), \mathbf{C}(X))$ given by $\mathbf{M}(X)$ being the space of continuous linear functionals on $\mathbf{C}(X)$. A more comprehensive choice is the pairing $(\mathbf{M}(X), \mathbf{B}(X))$ given by $\langle \nu, f \rangle = \int_X f d\nu$. In this Section, we shall proceed with an analysis in the more comprehensive setting and specialize to examine the special properties of the simpler case in Section 3.5.

A minor difficulty in adopting the $(\mathbf{M}, \mathbf{B})$ pairing is that dual solutions which are really the same (i.e., which are equal to each other $\mu$-almost everywhere) are regarded as distinct. To adopt a suitable LP framework we will make appropriate equivalence class identifications. Denote by $\mathbf{M}_\mu(I)$ the set of $\mu$-absolutely continuous measures on $I$ and (in a slight mangling of notation) by $\mathbf{M}_\mu(P)$ the pre-image of $\mathbf{M}_\mu(I)$ under the map $A$. By the Radon-Nikodym Theorem, for every $\nu \in \mathbf{M}_\mu(I)$, there is an integrable function $f \in \mathbf{B}(I)$ (its Radon-Nikodym derivative) such that $\nu(C) = \int_C f d\nu$. As usual, we equate integrable functions on $I$ which are equal $\mu$-almost everywhere. Write $L^1(\mu)$ for the space of equivalence classes of integrable functions. Let $\iota : \mathbf{M}_\mu(I) \rightarrow L^1(\mu)$ be the map which assigns to $\nu$ (the class of) its Radon-Nikodym derivative, $\frac{d\nu}{d\mu}$.

Writing $L^\infty(\mu)$ for the space of equivalences classes of bounded measurable functions on $I$, we have:

$$\mathbf{M}_\mu(P) \xrightarrow{A} \mathbf{M}_\mu(I) \overset{\iota}{\cong} L^1(\mu)$$

with dual

$$\mathbf{M}_\mu(P)^* \xleftarrow{A^*} \mathbf{M}_\mu(I)^* \overset{\iota^*}{\cong} L^\infty(\mu)$$

and second dual [4]

$$\mathbf{M}_\mu(P)^{**} \xrightarrow{A^{**}} \mathbf{M}_\mu(I)^{**} \overset{\iota^{**}}{\cong} ba(\mu)$$

Now the primal constraint $Ax \le \mu$ which holds in $\mathbf{M}_\mu(I)$ can be written as $\frac{d}{d\mu_B}x(\cdot \times S) \le 1$ $\mu_B$-a.e. and $\frac{d}{d\mu_S}x(B \times \cdot) \le 1$ $\mu_S$-a.e. The dual constraint is a little less obvious. Let $F_V$ be the linear functional on $\mathbf{M}_\mu(P)$ defined by integration against the valuation $V$. Define a set $U$ in $B \times S$ as $\mathbf{M}_\mu(P)$-*null* if $x(U) = 0$ for all $x \in \mathbf{M}_\mu(P)$. The condition $A^*q \ge F_V$ in $\mathbf{M}_\mu(P)^*$ means that $q(b) + q(s) \ge V(b, s)$ for all $(b, s) \in B \times S$ except for a $\mathbf{M}_\mu(P)$-null set.

**Lemma 1** *A Borel set $U$ in $B \times S$ is $\mathbf{M}_\mu(P)$-null if and only if $U$ may be written as $U = U_1 \cup U_2$ with $\mu_B(\pi_B(U_1)) = 0$ and $\mu_S(\pi_S(U_2)) = 0$.*

*Proof:* Let $U \subseteq B \times S$ be a $\mathbf{M}_\mu(P)$-null Borel set. Define $\mathcal{U} = \{T \subseteq U \mid \mu_B(\pi_B(T)) = 0\}$ and, for $T \in \mathcal{U}$, let $\sigma(T) = \mu_S(\pi_S(T))$. Note that $\mathcal{U}$ is a monotone class in $U$ and that $\sigma$ is an increasing function on $\mathcal{U}$. Thus $\sigma$ attains its maximum, say at $T_1$, on $\mathcal{U}$. Now repeat the process with the roles of $B$ and $S$ reversed to get the set $T_2$. Set $T_0 = T - T_1 - T_2$. The claim is that both coordinate projections of $T_0$ have measure 0.

To see this, suppose that $\mu_B(\pi_B(U)) > 0$. (The argument for $\mu_S$ is entirely similar.) Let $f : \pi_B(U) \rightarrow U$ be a (Borel) measurable selection of ${\pi_B}^{-1}$, i.e., a

[4] $ba(\mu)$, the set of bounded additive set functions which vanish on $\mu$-null sets, is the dual of $L^\infty(\mu)$ and therefore $ba(\mu) \cong L^1(\mu)^{**}$.

measurable map such that $\pi_B \circ f = \iota_{\pi_B(E)}$, $\mu$-almost everywhere. Then, on the one hand, $f^*(\mu|_{\pi_B(U)})$ is a measure on $B \times S$ whose $B$-marginal measure is dominated by $\mu_B$ and whose $S$-marginal measure is dominated by $\mu_S$, i.e., $f^*(\mu|_{\pi_B(U)})$ is a member of $\mathbf{M}_\mu(P)$. On the other hand, $f^*(\mu|_{\pi_B(U)})$ assigns positive measure to the set $U$. This contradicts the hypothesis that $U$ is a $\mathbf{M}_\mu(P)$-null set. □

**Theorem 1** *The linear programming problem has a dual solution and there is no gap between the primal and dual values.*

*Proof:* We begin by viewing the primal LP problem in the space $\mathbf{M}_\mu(P)$, as above. The constraint subset for the dual LP problem is a subset of $L^\infty(\mu)$ and may be taken, without loss of generality to be bounded. Since the Banach space $L^\infty(\mu)$ is the norm dual of the space $L^1(\mu)$ the constraint set is weak*-compact by the Banach-Alaoglu theorem. Thus the dual objective function, which is weak*-continuous, attains its minimum on this set, i.e. the dual problem has a solution.

Now let $q : I \to \mathbf{R}$ be any Borel function whose $L^1(\mu)$ class is a dual solution in the sense above. By construction, $q(b) + q(s) \geq V(b,s)$, except on a set $U \subseteq B \times S$ which is $\mathbf{M}_\mu$-null. By Lemma 1, we can write $U = U_1 \cup U_2$ where $\mu_B(\pi_B(U_1)) = \mu_S(\pi_S(U_2)) = 0$. Set

$$\bar{q}(i) = \begin{cases} q(i) & \text{if } i \notin U_1 \cup U_2 \\ \sup V(b,s) & \text{otherwise.} \end{cases}$$

Then $\bar{q} = q$ $\mu$-almost everywhere and $q(b) + q(s) \geq V(b,s)$ everywhere, so $\bar{q}$ is a dual solution to the original LP problem. We shall return to extraction of a the point function solution from its equivalence class in Section 3.5.

In order to see that there is no duality gap, we will apply to the current dual problem a fundamental theorem of infinite-dimensional linear programming which is due to R.J. Duffin, L.A. Karlovitz, and K.S. Kretschmer and may be found in Krabs [1979]. To apply this theorem, we must first ensure that the positive cone of $\mathbf{M}_\mu(P)^*$ has a non-empty interior. Note that $\mathbf{M}_\mu(P)$ is a Banach space in its own right and that the lattice operations defined in $\mathbf{M}(P)$ are still lattice operations when restricted to $\mathbf{M}_\mu(P)$. Thus $\mathbf{M}_\mu(P)$ is a Banach lattice. Moreover, $\|x+y\| = \|x\| + \|y\|$ for any $0 \geq x, y \in \mathbf{M}_\mu(P)$ so that $\mathbf{M}_\mu(P)$ is an AL-space in the sense of G. Birkhoff and S. Kakutani (see Dunford and Schwartz [1958]). Hence, by a theorem of Kakutani, $\mathbf{M}_\mu(P)^*$ is an AM-space whose positive cone has interior. Consequently, Duffin's theorem applies and this problem (i.e. the current dual) and *its* dual (i.e. the current double dual) have no duality gap between their values and, moreover, the dual (i.e. the current double dual) has a solution. The second dual $\mathbf{M}_\mu(P)^{**}$ is again an AL-space and has a norm 1 positive projection onto $\mathbf{M}_\mu(P)$ which undoes the canonical injection of $\mathbf{M}_\mu(P)$ into its second dual. Consequently, the value for the double dual is equal to the value for the primal LP problem and there is no gap for the primal and dual values. □

A stronger hypothesis is needed for the existence of primal solutions as is indicated by Example 5 which has no primal solutions.

**Theorem 2** *If the valuation function $V$ is upper semi-continuous then the LP assignment problem has a primal solution.*

*Proof:* Define the linear functional $f(x) = \int_P V dx$ on $\mathbf{M}(P)$. By a theorem of Simon and Zame [1990] f is a weak* upper semi-continuous linear functional on the positive cone $\mathbf{M}_+(P)$. Since the feasible set $\mathbf{M}_\mu(P)$ is bounded in $\mathbf{M}_+(P)$ and is evidently weak* closed, it must be a weak* compact subset. Thus, the functional $f$ attains its maximum on this set. □

## 3.2 The Market Game

As described in Section 1.3, $w : \mathcal{C} \to \mathbf{R}$ is the characteristic function of the market game induced by the LP data $(\mu, V)$, and $\tilde{w} : \mathcal{I} \to \mathbf{R}$ is the characteristic function of that game extended to its ideal sets.

**Lemma 2** *The distributional core $\mathcal{D}(\tilde{w})$ is a subset of the core $\mathcal{C}(w)$.*

*Proof:* Recall that the core $\mathcal{C}(w)$ of the game given by characteristic function $w$ consists of all finitely additive set-functions $\nu$ (on the Borel coalitions $\mathcal{B}$ of $I$) with the properties that $\nu(C) \geq w(C)$ for all $C \in \mathcal{B}$ and $\nu(I) = w(I)$.

Moreover, every member of the core is countably additive and $\mu$-absolutely continuous. To see this, compute

$$\begin{aligned} w(C) &= \sup\{\int_{(C\cap B)\times(C\cap S)} V dx \mid x \in M_\mu\} \\ &\leq \|V\|_\infty \sup\{x(C\cap B, C\cap S) \mid x \in M_\mu\} \end{aligned}$$

Note, however, that $x(C \cap B \times C \cap S) \leq x(B \times C \cap S) \leq \mu_S(C \cap S)$ and, similarly, $x(C\cap B \times C\cap S) \leq \mu_B(C\cap B)$; thus, $x(C\cap B\times C\cap S) \leq \mu_S(C\cap S)+\mu_B(C\cap B) = \mu(C)$. In total, $w(C) \leq \|V\|_\infty \mu(C)$. Consequently, $w$ is a $\mu$-continuous set function. By a standard result (Aumann and Shapley [1974]) it follows that every member of $\mathcal{C}(w)$ is countably additive and $\mu$-absolutely continuous.

Now, recall that a member of the distributional core $\mathcal{D}(\tilde{w})$ is an additive function $\tilde{\nu} : \mathcal{I} \to \mathbf{R}$ with the property that $\tilde{\nu}(\mathbf{1}_I) = \tilde{w}(\mathbf{1}_I)$ and $\tilde{\nu}(\tilde{C}) \geq \tilde{w}(\tilde{C})$ for all $\tilde{C} \leq \mathbf{1}_I$. Since $\tilde{w}(\mathbf{1}_I) = w(I)$ and $\tilde{w}(\mathbf{1}_C) = w(C)$ for every Borel set of $I$, the set of inequalities defining the distributional core is strictly larger than those defining the core. Therefore, since we can uniquely extend any $\mu$-continuous measure $\nu$ on the Borel sets of $I$ in $\mathcal{C}(w)$ to a set function $\tilde{\nu}$ on the ideal sets of $I$ via $\tilde{\nu}(\tilde{C}) = \int \tilde{C} q d\mu$ (where $q \in L^1(\mu)$ is the Radon-Nikodym derivative of $\nu$), and any member of the core of $w$ is $\mu$-continuous, the distributional core can be no larger than the core. $\square$

Note that Example 5 illustrates a case in which the distributional core is a proper subset of the core. In Section 3.5 we show that when the valuation function is continuous the two coincide.

**Theorem 3** *Up to equivalence almost everywhere, the collection of dual LP solutions coincides (via Radon-Nikodym identification) with the distributional core.*

*Proof:* First, we show that every LP dual solution induces (via identification) a member of the distributional core. Let $q$ be a dual solution for the linear programming problem; set $\tilde{\nu}(\tilde{C}) = \int \tilde{C} q d\mu$ for each ideal set $\tilde{C}$. Then $\tilde{\nu}$ is an ideal set function and $\tilde{\nu}(\mathbf{1}_I) = \int_I q d\mu = g(\mu) = w(I)$. For any other ideal coalition $\tilde{C}$ ($\leq \mathbf{1}_I$) representing the measure $\mu' = (\mu'_B, \mu'_S)$ ($\leq \mu$), given $\epsilon > 0$ there is a $x_1 \in M_{\mu'}$ such that

$$\begin{aligned}
w(\tilde{C}) &= \sup\{\int \tilde{C}(b)\tilde{C}(s)V(b,s)dx(b,s) | x \in M_\mu\} \\
&= \sup\{\int V dx | x \in M_{\mu'}\} \\
&\leq \int V dx_1 + \epsilon \\
&\leq \int (q(b) + q(s)) dx_1(b,s) + \epsilon \\
&= \int q(b) x_1(db, S) + \int q(s) x_1(B, ds) + \epsilon \\
&\leq \int q(b)\mu'_B(db) + \int q(s)\mu'_S(ds) + \epsilon \\
&= \int q d\mu' + \epsilon \\
&= \int \tilde{C} q d\mu + \epsilon \\
&= \tilde{\nu}(\tilde{C}) + \epsilon
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, $\tilde{w}(\tilde{C}) \leq \tilde{\nu}(\tilde{C})$ for all ideal coalitions $\tilde{C}$; thus, $\tilde{\nu}$ (the correspondent of $q$) is in the distributional core of $w$.

Now, let $\tilde{\nu} \in \mathcal{D}(w)$. Define $\nu(C) = \tilde{\nu}(\mathbf{1}_C)$. Since an element of the distributional core is an element of the core and the latter was shown in the first lemma of this Section to contain only $\mu$-continuous set functions, we may define $\hat{q} \in L^1(\mu)$ as the Radon-Nikodym derivative of $\nu$. In fact, since $\hat{q}$ is bounded $\mu$- a.e. by $\|V\|$, $\hat{q}$ is in $L^\infty(\mu)$. To see that $\hat{q}$ is an optimal LP-dual solution, we need only translate the conditions on $\tilde{\nu}$ to ones on $\hat{q}$.

The condition that $\tilde{\nu}(I) = w(I)$ is just $\int \hat{q} d\mu = g(\mu)$,i.e., that $\hat{q}$ attains the optimal value. The other condition is that $\tilde{\nu}(\tilde{C}) \geq \tilde{w}(\tilde{C})$ for all ideal sets $\tilde{C}$. Suppose that $\hat{q}$ is not feasible, i.e., suppose that $\hat{q}(b) + \hat{q}(s) < V(b,s)$ for all $(b,s) \in A$ where $A$ is a $\mathbf{M}_{\mu}$- non-null set. Then there is a non-negative feasible Borel measure $x'$ on $B \times S$ such that the carrier of $x'$ is $A$. The corresponding ideal set $\tilde{C}$ is defined on $B$ as the Radon-Nikodym derivative of $\pi_B x'$ and, similarly on $S$ as the Radon-Nikodym derivative of $\pi_S x'$. (Recall that $\pi_B$ and $\pi_S$ are defined in Section 1.3 as projection mappings.) Thus,

$$\begin{aligned}
\tilde{w}(\tilde{C}) &\geq \int V dx' \\
&> \int (\hat{q}(b) + \hat{q}(s)) dx' \\
&= \int \hat{q} dx'_B + \int \hat{q} dx'_S \\
&= \int \tilde{C} \hat{q} d\mu \\
&= \tilde{\nu}(\tilde{C})
\end{aligned}$$

which is a contradiction.

Finally, as in the proof of Theorem 1, we obtain a representative $\bar{q}$ of $\hat{q}$ which is the dual solution of our original LP problem. □

## 3.3 Walrasian Equilibria of an Exchange Economy

The existence of Walrasian equilibrium will follow from its relation to the linear programming problem. In order to address the question of core equivalence one must first establish a canonical correspondence between the solution concepts for the market game and the market economy. Since the solution concepts for the market economy correspond to those for the linear programming problem, it will be enough to establish an identification between pairs $(x, q)$ of LP solutions, where $q$ is the canonical representative of the equivalence class, and Walrasian equilibria $(y, p)$. Here $x$ is an optimal (LP primal) measure of correspondence between buyers and sellers, $q$ is a optimal (LP dual) distribution of profit (i.e. gains from trade) to the agents, $y$ is a Walrasian allocation and $p$ is a Walrasian price system.

We have given above the agents, endowments, and preferences (via utility functions) for the formulation of the assignment problem as a market economy. Our goal is to establish the canonical correspondence between the solution concepts for the LP problem and the market economy. First, given feasible $x$ and $q$, construct the allocation $y$ and the price system $p$. For any Borel subset $E$ of $B$ and Borel subset $F$ of $S$ define the set function $(Y_B(E))(F) = x(E \times F)$. Then $Y_B : \mathcal{B}(B) \to \mathbf{M}(S)$ since it is easily verified that $(Y_B(E))(\cdot)$ is countably additive for each fixed $E$. Moreover, $Y_B$ is itself also countably additive in the norm topology of $\mathbf{M}(S)$ , i.e. $Y_B$ is a $\mathbf{M}(S)$-valued vector measure. Note that if $\mu_B(E) = 0$ then so is $x(E \times S) = 0$ and ,hence, so is $x(E \times F) = 0$ for all Borel $F \subseteq S$. Consequently, $Y_B$ is $\mu_B$- continuous. There is a Gelfand density (i.e. weak*-measurable and weak*-bounded) $y_B : B \to \mathbf{M}(S)$ such that $\int_B f(b)Y(db) = \int_B f(b)y_B(b)d\mu_B(b)$ for all $f \in \mathbf{B}(B)$ (see Diestel and Uhl [1977]). The allocation to buyer $b$ is then $y_B(b)$. Similarly, let $Y_S(F)(E) = x(E \times F)$ define a $\mathbf{M}(B)$-valued set function which is $\mu_S$-continuous on the Borel sets of $S$. Call the Gelfand derivative $y_S$. Since the endowment of a seller $s$ is $\delta_s$, the trade by sellers of type $s$ is $y_S(s) = -z(s)$ for $z(s) \in [0,1]$. The supporting price $p$ is determined by noting that, since $q$ on $B \cup S$ is the profit that each player receives, the price of the $s$-th house must be $p(s) = q(s) + \sigma(s)$.

Secondly, given an allocation $y = (y_B, y_S)$ and a price system $p$, construct a feasible $x$ and $q$. The profit distribution is given by $q(s) = p(s) - \sigma(s)$. The buyer-seller measure is given by $x(E \times F) = \int_E \langle y_B(b), \chi_F \rangle d\mu_B(b)$.

The following theorem (after the lemma) establishes distributional core equivalence since it states that the above canonical correspondence matches pairs of *optimal* solutions to the linear program and *Walrasian* equilibria of the market economy. It has already been established that the dual solutions of the program are exactly the distributional core allocations of the market game.

**Lemma 3** *Complementary slackness holds: Let $x$ be an optimal primal solution and $q$ an optimal dual solution so that, in particular, $V(b,s) \leq q(b) + q(s)$ for all $b$ and $s$ and $V(b,s) = q(b) + q(s)$ for all $b$ and $s$ in the support of $x$.*

*Proof:* By the optimality of the solutions, all of the available profit must be distributed between buyers and sellers, i.e.,

$$\int q(b)d\mu_B + \int q(s)d\mu_S = g(\mu) = \int V dx.$$

Since the marginals of $x$ for the buyers and sellers are at most $\mu_B$ and $\mu_S$, respectively, we have

$$\begin{aligned} \int q(b)d\mu_B &\geq \int q(b)dx(b,s) \\ \int q(s)d\mu_S &\geq \int q(s)dx(b,s). \end{aligned}$$

Combining this with the fact that $q(b) + q(s) \geq V(b,s)$ for all $b$ and $s$ yields the desired conclusion that $q(b) + q(s) = V(b,s)$ except on an $x$-null set. □

**Theorem 4** *There is a a one-to-one correspondence between pairs of optimal linear programming solutions and Walrasian equilibria.*

*Proof:* Let $x$ and $q$ be optimal solutions to the linear program. Construct $y$ and $p$ as above. That markets clear will be a consequence of the definition of a feasible allocation after computing $\int_I y(i)d\mu(i) = \int_B y(b)d\mu_B(b) + \int_S y(s)d\mu_S(s)$. These latter two Gelfand integrals will be evaluated separately; it suffices to do so on subsets of $S$. Let $F$ be a Borel subset of $S$. Then for the buyers

$$\begin{aligned}
\langle \int_B y_B(b)d\mu_B(b), \chi_F\rangle &= \int_B \langle y(b), \chi_F\rangle d\mu_B(b) \\
&= \langle Y(B), \chi_F\rangle \\
&= Y(B)(F) \\
&= x(B \times F)
\end{aligned}$$

On the other hand, for the sellers

$$\begin{aligned}
\langle \int y_S(s)d\mu_S(s), \chi_F\rangle &= \langle \int_S z(s)\, \delta_s d\mu_S(s), \chi_F\rangle \\
&= -\langle \int_S z(s)(B)\delta_s d\mu_S(s), \chi_F\rangle \\
&= -\int_S z(s)(B)\langle \delta_s, \chi_F\rangle d\mu_S(s) \\
&= -\int_S z(s)(B)\chi_F(s)d\mu_S(s) \\
&= -\int_F z(s)(B)d\mu_S(s) \\
&= -\langle Z(F), \chi_B\rangle \\
&= -x(B \times F)
\end{aligned}$$

Thus, $\int_I y(i)d\mu(i) = 0$ as desired.

The individual support property requires that $p \in \partial u_i(y(i))$ for $\mu$-almost all $i$. By conjugate duality this is the same as requiring that $y(i) \in \partial u_i^*(p)$ where $u_i^*(p) = \sup\{u_i(\gamma) - \langle p, \gamma\rangle | \gamma \in D_i\}$. Consider the buyers and sellers separately. Let $b$ be a fixed buyer. Then $u_b(\gamma) = \langle \beta_b, \gamma\rangle$ for any $0 \leq \gamma \in \mathbf{M}(S)$ with $\gamma \leq 1$. The conjugate function is $u_b^*(p) = \sup\{\beta_b - p, \gamma\rangle : \gamma \in D_b\}$. To find a member of the subdifferential of $u_b^*$ it thus suffices to find $z \in D_b$ which maximizes $\langle \beta_b - p, z\rangle$. The claim is that any $\gamma \in D_b$ with the property that the support of $\gamma$ is contained among those $s$ for which $(b, s)$ is in the support of $x$. Note that $y(b)$ is such a measure by construction.

The proof of the claim is established from the following use of complementary slackness. For any $s' \in S$ duality gives $q(b) + q(s') \geq V(b, s') = \beta_b(s') - \sigma(s')$ so that $q(b) \geq \beta_b(s') - p(s')$. If, moreover, $(b, s)$ is in the support of the optimal $x$ then $q(b) + q(s) = V(b, s) = \beta_b(s) - \sigma(s)$ so that $q(b) = \beta_b(s) - p(s)$. Thus, $\beta_b(s) - p(s) \geq \beta_b(s') - p(s')$ for $s$ such that $(b, s)$ is in the support of $x$ and any $s' \in S$. Consequently, the function $\beta_b - p$ is maximized at such $s$.

Finally to see that $p \in \partial u_s(y(s))$ recall that $u_s(\alpha\delta_s) = \alpha\sigma(s)$ so that the condition becomes $p(s) \geq \sigma(s)$ for almost all $s$ which is true for this $p$. □

## 3.4 The Subdifferential of the Value Function

The relationships between the set of optimal solutions for the linear programming formulation of the assignment problem, Walrasian equilibria, and the distributional core (as well as the set of core allocations) for the corresponding market game has been established in previous sections. In order to complete this circle of ideas we will relate these concepts to the notion of subdifferential of the function which gives the value of the linear program.

The value function for the assignment problem $(\mu, V)$ is defined as the optimal value $g(\mu) = \sup\{\int V dx | Ax \leq \mu,\ x \geq 0\}$ as $\mu$ varies holding $V$ fixed. It was observed in Section 1.1 that $g$ is positively homogeneous and concave. Therefore, the subdifferential $\partial g(\mu) = \{q \in \mathbf{B}(I) | \langle q, \mu\rangle = g(\mu),\ \langle q, \mu'\rangle \geq g(\mu'),\ \mu' \geq 0\}$.

Let $Q(\mu)$ be the set of dual solutions to the assignment problem $(\mu, V)$, where again we suppress the dependence on $V$.

**Lemma 4** $Q(\mu) \subset \partial g(\mu)$.

*Proof:* Let $q$ be an optimal dual solution, i.e., $0 \leq q \in \mathbf{B}(I)$ with $q(b) + q(s) \geq V(b, s)$ such that $\int_I q d\mu$ is a minimum among all feasible solutions to the dual. By duality and the no-gap theorem $\int_I q d\mu = \langle q, \mu\rangle = g(\mu)$.

Consider any other $\mu' \in \mathbf{M}_+(I)$. Then $q$ is still feasible for the problem $(\mu', V)$. Hence $g(\mu') \leq \langle q, \mu'\rangle$. Consequently, $g(\mu') - g(\mu) \leq \langle q, \mu' - \mu\rangle$ for all $\mu'$. Therefore, $q \in \partial g(\mu)$. □

The distributional core $\mathcal{D}(\tilde{w})$ of the game derived from $(\mu, V)$ consists of ideal set functions $\tilde{\nu} : \mathcal{I} \to \mathbf{R}$. For any $q \in \mathbf{B}(I)$, an ideal set function can be obtained through the Radon-Nikodym correspondence $\tilde{\nu}(\tilde{C}) = \int \tilde{C} d\mu$.

**Lemma 5** $\partial g(\mu) \subset \mathcal{D}(\tilde{w})$.

*Proof:* Let $q \in \partial g(\mu)$, i.e. $\langle q, \mu\rangle = g(\mu)$ and $\langle q, \mu'\rangle \geq g(\mu')$ for all $\mu' \in \mathbf{M}_+(I)$. Define $\tilde{\nu}(\tilde{C}) = \int \tilde{C} q d\mu$ for Borel ideal sets $\tilde{C}$. Then $\langle q, \mu = g(\mu)$ implies that $\tilde{\nu}(\mathbf{1}_I) = \langle q, \mu\rangle = g(\mu) = w(I)$. Furthermore, letting $\mu'(\cdot) = \int_{(\cdot)} \tilde{C} d\mu$, then $\tilde{\nu}(\tilde{C}) = \int \tilde{C} q d\mu = \int_I q d\mu' \geq g(\mu') = \sup\{\int V dx | Ax \leq \mu',\ x \geq 0\} = w(\tilde{C})$. Consequently, $\tilde{\nu}$ is in the distributional core of $w$ as desired. □

**Theorem 5** *For the assignment model defined by $(\mu, V)$, there is equality between the distributional core of the market game, the set of optimal dual solutions, and the* *subdifferential of the value function, i.e.,* $Q(\mu) = \partial g(\mu) = \mathcal{D}(\tilde{w})$.

*Proof:* It has just been established that $Q(\mu) \subseteq \partial g(\mu) \subseteq \mathcal{D}(\tilde{w})$. It is known, however, from Theorem 4 that the dual coincides with the distributional core. Hence, there is equality throughout. □

## 3.5 Continuous Valuations

In the event that the reservation prices $\beta$ and $\sigma$ are continuous functions or, in particular, that the valuation function $V(b,s) = \beta(b,s) - \sigma(s)$ is continuous, then there are many stronger conclusions that can be made: the dual solutions are also continuous, the set of such solutions is compact, and the distributional core coincides with the core of the associated market game.

**Theorem 6** *If the valuation function* $V : B \times S \to \mathbf{R}$ *is continuous, then the LP dual solutions are also continuous (more precisely, each of the* $L^\infty(\mu)$ *equivalence classes which are LP dual solutions contains a continuous representative).*

*Proof:* The proof of Theorem 1 shows how to extract from a dual solution $q \in L^\infty(\mu)$ which minimizes $\int q d\mu$ subject to $q \geq 0$ $\mu$-a.e. and $q(b) + q(s) \geq V(b,s)$ for all $(b,s)$ except in a $\mathbf{M}_\mu$-null set, a $\bar{q} \in \mathbf{B}(I)$ such that $\int \bar{q} d\mu = \int q d\mu$ and $\bar{q}(b) + \bar{q}(s) \geq V(b,s)$ for *all* $b,s$. We need to show that $\bar{q}$ can be chosen to be continuous. This will be done by the following "shrink-wrap" argument on $\bar{q}$.

The construction proceeds as follows: Let $\bar{q}$ be a bounded measurable function from the equivalence class of the LP dual $q$, as above.

Define for each $b \in B$

$$q_1(b) = \sup\{V(b,s) - \bar{q}(s) | s \in S\}$$

Observe that $0 \leq q_1(b) \leq \bar{q}(b)$ for $\mu_B$-almost all $b$ and that $q_1(b) + \bar{q}(s) \geq V(b,s)$ for every $b$ and $s$. Consequently, $q_1$ is the B-component of a solution when paired with the S-component of $\bar{q}$. Repeat the process by setting for each $s \in S$

$$q_1(s) = \sup\{V(b,s) - q_1(b)\}$$

The result is a function $q_1 \in \mathbf{B}(I)$ which satisfies $q_1(b) + q_1(s) \geq V(b,s)$ for every $b$ and $s$ and $\int q_1 d\mu = \int \bar{q} d\mu$. Note that up to this point, no assumption on continuity of $V$ has been used.

We now show that $q_1 \in \mathbf{C}(I)$. In fact, the proof will be shown for $q_1$ restricted to $B$; the proof for $S$ is similar. Fix $\epsilon > 0$ and let $\delta > 0$ be such that $|V(b_1,s) - V(b_2,s)| < \epsilon$ for all $s$ whenever $d(b_1,b_2) < \delta$. Let $b_1, b_2$ be arbitrary such that $d(b_1,b_2) < \delta$. For

$b_2$ observe that $q_1(b_2) \geq V(b_2, s) - \bar{q}(s)$ for almost all $s$. For $b_1$ there is $s_\epsilon$ such that $q_1(b_1) \leq V(b_1, s_\epsilon) - \bar{q}(s_\epsilon) + \epsilon$. So compute

$$\begin{aligned} q_1(b_1) - q_1(b_2) &= \sup\{V(b_1, s) - \bar{q}(s) | s \in S\} - \sup\{V(b_2, s) - \bar{q}(s) | s \in S\} \\ &\leq (V(b_1, s_\epsilon) - \bar{q}(s_\epsilon) + \epsilon) - (V(b_2, s_\epsilon) - \bar{q}(s_\epsilon)) \\ &= V(b_1, s_\epsilon) - V(b_2, s_\epsilon) + \epsilon \\ &< 2\epsilon \end{aligned}$$

Changing the roles of $b_1$ and $b_2$ establishes the inequality that $|q_1(b_1) - q_1(b_2)| < 2\epsilon$ whenever $d(b_1, b_2) < \delta$. Thus $q_1$ is continuous on $B$. A similar computation establishes continuity of $q_1$ on $S$. □

**Theorem 7** *If the valuation function $V$ is continuous, then the set of LP dual solutions is a norm-compact set in $\mathbf{C}(I)$.*

*Proof:* Since the set of LP dual solutions is clearly closed, by the Arzela-Ascoli theorem, it will suffice to show that the set is an equicontinuous family in $\mathbf{C}(I)$. This is easily seen from the observation that in the proof of the above theorem, the choice of $\delta$ for a given $\epsilon$ in the continuity calculation for $q_1$ depended only $V$ and not on the particular solution. □

We remarked above that the finite assignment model has the remarkable property that the core coincides with its Walrasian equilibria. Another perspective on this property is that replication of the model does nothing to shrink the core. Note that in the finite assignment model, the coalitions present in the distributional core may be interpreted as those which create an infinite replica of the assignment model. Thus, the Shapley-Shubik result on the coincidence of the core and Walrasian equilibria for finite assignment models can be reinterpreted in our terms to say that if the measure $\mu$ describing the population of buyers and sellers in the assignment model consists of a finite number of atoms, each with a mass of unity, the core of this model coincides with the distributional core (consisting of the infinite replication of the model). The following shows that this result can be extended to the nonatomic version of the assignment model provided the valuation function is continuous.

**Theorem 8** *If $\mu$ is nonatomic and the valuation function $V$ is continuous, then the distributional core of the associated market game is the same as the core of the market game.*

*Proof:* As observed in the proof of Theorem 2, $\mathbf{M}_\mu(P) = \{x | Ax \leq \mu,\ x \geq 0\}$ is weak* compact in $\mathbf{M}(P)$. Since $V \in \mathbf{C}(P)$, then $\langle V, x\rangle$ is weak* continuous on $\mathbf{M}_\mu(P)$. Therefore, $g(\mu') = \sup\{\langle V, x\rangle | Ax \leq \mu',\ x \geq 0\}$ is weak* continuous on $\{\mu' | 0 \leq \mu' \leq \mu\}$.

Define $\mu_C$ as the measure $\mu_C(\cdot) = \int_{(\cdot)} \mathbf{1}_C d\mu$ and $\tilde{\mu}_{\tilde{C}}$ as the measure $\tilde{\mu}_{\tilde{C}}(\cdot) = \int_{(\cdot)} \tilde{C} d\mu$. The set $\{\mu_C | C \in \mathcal{B}\}$ is the range of a nonatomic vector measure. Therefore its weak* closure is convex (Kluvanek [1973]); so the closure contains $\{\tilde{\mu}_{\tilde{C}} | \tilde{C} \in \mathcal{I}\}$.

From Lemma 2, $\mathcal{D}(\tilde{w}) \subset \mathcal{C}(w)$, where $\tilde{w}(\tilde{C}) = g(\tilde{\mu}_{\tilde{C}})$ and $w(C) = g(\mu_C)$. To show that the two cores coincide, let $\nu \in \mathcal{C}(w)$. From the proof of Lemma 2, there is a $q \in L^1(\mu)$ such that $\nu(C) = \langle q, \mu_C \rangle$. Therefore, it suffices to show that if we set $\tilde{\nu}(\tilde{C}) = \langle q, \tilde{\mu}_{\tilde{C}} \rangle$, then $\tilde{\nu} \in \mathcal{D}(\tilde{w})$.

Suppose the contrary. Then there exists a $\tilde{C}$ and an $\epsilon > 0$ such that

$$(\dagger) \qquad (\langle q, \tilde{\mu}_{\tilde{C}} \rangle) + \epsilon = \tilde{\nu}(\tilde{C}) < \tilde{w}(\tilde{C}) = g(\tilde{\mu}_{\tilde{C}}).$$

But, by the hypothesis that $\nu \in \mathcal{C}(w)$, $\langle q, \mu_C \rangle \geq g(\mu_C)$ for all Borel sets $C$ in $I$. Adding this to the facts that $g$ is weak* continuous and its weak* closure contains $\{\tilde{\mu}_{\tilde{C}} | \tilde{C} \in \mathcal{I}\}$ leads to a contradiction. □

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