$$f_Y(y|\mu,\sigma^2) = \frac{1}{y}\frac{1}{\sqrt{2\pi\sigma^2}}exp[\frac{-(\log y-\mu)^2}{2\sigma^2}]$$

a.

$$L(\mu,\sigma^2) = \prod_{n=1}^{N}\frac{1}{y_n}\frac{1}{\sqrt{2\pi\sigma^2}}exp[\frac{-(\log y_n-\mu)^2}{2\sigma^2}]$$

$$f(\mu,\sigma^2) = \log L(\mu,\sigma^2) = -\sum_{n=1}^{N}\log y_n - \frac{N}{2}\log 2\pi\sigma^2 - \sum_{n=1}^{N}\frac{(\log y_n-\mu)^2}{2\sigma^2}$$

$\mathbf{g}(\mu,\sigma^2) =$

$$\begin{bmatrix} \sum_{n=1}^{N}\log y_n - \hat{\mu} \\ -\frac{N}{2\hat{\sigma}^2} + \sum_{n=1}^{N}\frac{(\log y_n-\hat{\mu})^2}{2\hat{\sigma^2}^2} \end{bmatrix}$$

By solving the first component of the vector for $\hat{\mu}$ we arrive at: $\hat{\mu} = \frac{1}{N}\sum_{n=1}^{N}\log y_n$. Plugging this into the second component of the gradient vector, we arrive at: $\hat{\sigma}^2 = \frac{1}{N}\sum_{n=1}^{N}(\log y_n - \hat{\mu})^2$

b.

Note that: $\mathbb{E}[\log y] = \int_0^\infty \frac{\log y}{y\sqrt{2\pi\sigma^2}}exp[\frac{-(\log y-\mu)^2}{2\sigma^2}]$ Applying the substitution: $u = \log y$

We arrive at: $\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}uexp[\frac{-(u-\mu)^2}{2\sigma^2}] = \mu$ as this is the expected value of a normal distribution.

Also Note: $\mathbb{E}[\log y^2] = \int_0^\infty \frac{\log y^2}{y\sqrt{2\pi\sigma^2}}exp[\frac{-(\log y-\mu)^2}{2\sigma^2}]$ Substituting: $u = \log y$

$\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}u^2exp[\frac{-(u-\mu)^2}{2\sigma^2}]$ This is the second moment of a normal distribution and is: $\mu^2+\sigma^2$

$$\mathbb{E}[\hat{\mu}] = \mathbb{E}[\frac{1}{N}\sum_{n=1}^{N}\log y_n] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[\log y_n] = \mu$$

$$\mathbb{V}[\hat{\mu}] = \frac{1}{N^2}\sum_{n=1}^{N}\mathbb{V}(\log y_n) = \frac{1}{N^2}\sum_{n=1}^{N}(\mathbb{E}[\log {y_n}^2] - \mathbb{E}[\log y_n]^2) = \frac{1}{N^2}N(\mu^2+\sigma^2-\mu^2) = \frac{\sigma^2}{N}$$

$$\mathbb{E}[\hat{\sigma}^2] = \mathbb{E}[\frac{1}{N}\sum_{n=1}^{N}(\log y_n-\hat{\mu})^2] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[\log {y_n}^2 - 2\hat{\mu}\log y_n + \hat{\mu}^2] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[\log {y_n}^2] - \mathbb{E}[\hat{\mu}^2]$$

$$= \mu^2+\sigma^2 - (\mathbb{V}(\hat{\mu}) + \mathbb{E}[\hat{\mu}]^2) = \mu^2+\sigma^2-(\frac{\sigma^2}{N}+\mu^2) = \frac{N-1}{N}\sigma^2$$

Thus we can see that $\hat{\mu}$ is unbiased and $\hat{\sigma}^2$ is biased. However we can see that $\frac{N}{N-1}\hat{\sigma}^2$ is unbiased.

Let us examine the moment-generating function for $\log y$. This takes the form: $\mathbb{E}[exp[t \log y]] = \mathbb{E}[y^t] = exp[t\mu + \frac{t^2}{2}\sigma^2]$. This is exactly the moment generating function of a normal distribution, so $\log y \sim N(\mu, \sigma^2)$. As a linear combination of normals, $\hat{\mu} \sim N(\mu, \frac{\sigma^2}{N})$

c.

Applying the Criterion for Method of Moments Estimators: $\frac{1}{N}\sum_{n=1}^{N} y_i^k = \mathbb{E}[Y^k]$

$$\frac{1}{N}\sum_{n=1}^{N} y_i = exp[\mu + \frac{\sigma^2}{2}]$$

$$\frac{1}{N}\sum_{n=1}^{N} y_i^2 = exp[2\mu + 2\sigma^2]$$

Solving for μ, σ^2 We arrive at:

$$\widetilde{\mu} + \widetilde{\sigma^2} = \frac{1}{2}\log\frac{1}{N}\sum_{n=1}^{N} y_i^2$$

$$\widetilde{\mu} + \frac{1}{2}\widetilde{\sigma^2} = \log\frac{1}{N}\sum_{n=1}^{N} y_i$$

$$\widetilde{\mu} = 2\log\frac{1}{N}\sum_{n=1}^{N} y_i - \frac{1}{2}\log\frac{1}{N}\sum_{n=1}^{N} y_i^2$$

$$\widetilde{\sigma^2} = \log\frac{1}{N}\sum_{n=1}^{N} y_i^2 - 2\log\frac{1}{N}\sum_{n=1}^{N} y_i$$

It is clear from the obviously nonlinear nature of the functions that characterize $\widetilde{\mu}$ and $\widetilde{\sigma^2}$ that they not unbiased.

d.

We can see that since $\log x \in C \quad \forall x > 0$ We may invoke the continous mapping theorem and pass the plim inside the function, and then invoking the strong law of large numbers:

$$plim\,\widetilde{\mu} = 2\log(plim\,\frac{1}{N}\sum_{n=1}^{N} y_i) - \frac{1}{2}\log(plim\,\frac{1}{N}\sum_{n=1}^{N} y_i) = 2\log exp[\mu + \frac{1}{2}\sigma^2] - \frac{1}{2}\log exp[2\mu + 2\sigma^2] = \mu$$

$$plim\,\widetilde{\sigma^2} = \log(plim\,\frac{1}{N}\sum_{n=1}^{N} y_i^2) - 2\log(plim\,\frac{1}{N}\sum_{n=1}^{N} y_i) = \log exp[2\mu + 2\sigma^2] - 2\log exp[\mu + \frac{1}{2}\sigma^2] = \sigma^2$$

e.

Note that $\widetilde{\mu}$ is a function of two statistics. If we would like knowledge of σ^2 to be reflected in our MoM estimator, we would need to create a new estimator. Let us call this new estimator $\bar{\mu}$. Applying the method of moments:

$$\frac{1}{N}\sum_{n=1}^{N} y_i = E[Y] = exp[\bar{\mu} + \frac{1}{2}]$$

$$\bar{\mu} = \log(\frac{1}{N}\sum_{n=1}^{N} y_i) - \frac{1}{2}$$

Let $g(\bar{y}) = \bar{\mu} = \log \bar{y} - \frac{1}{2}$ Via the delta method:

$$\sqrt{N}(g(\bar{y}) - g(\mu)) \sim N(0, g'(\mu)^2 \mathbb{V}(\bar{y}))$$

Note that: $g'(\bar{y}) = \frac{1}{\bar{y}}$ and $\mathbb{V}(\bar{y}) = \frac{1}{N}\mathbb{V}(Y) = \frac{exp[2\mu + 2] - exp[2\mu + 1]}{N}$

So the Variance of $\bar{\mu} = \frac{exp[2\mu + 2] - exp[2\mu + 1]}{Nexp[2\mu + 1]} = \frac{e - 1}{N} \approx \frac{1.71828}{N}$

This Estimator has higher variance than the Maximum Liklihood Estimator and is relatively less efficient.