# 3 Questions

## *Timothy Schwieg*

### 1 How much does a Volvo V60 Cost?

Consider an agent who is trying to determine how much a Volvo v60 costs. He must take into account the cost of the car, whether or not he is taking out a loan, the maintenance that the car requires, as well as repairs, insurance, fuel and registration. Since many of these payments are made in the future, they will time discounted. Besides these explicit costs, there is also the implicit cost of not investing the money at some prevailing interest rate.

The Buyer faces some upfront prices: Down payment size, insurance level, warranty.

In each time period the buyer must pay a loan payment, insurance, fuel costs, registration, maintenance and repairs. Since insurance is mandatory, we will claim insurance is actuarilly fair, so the choice of insurance is irrelevant. We will take the minimum insurance package as a result as it is easiest computationally.

The buyer wishes to minimize the present value of these effects, using geometric time discounting to figure out the cost of the Volvo V60.

We will consider loan payments to be monthly, while maintenance, repairs, registration, insurance, and fuel costs to be yearly payments. Time will be discounted by $\beta$ which is the discount per month.

The amount of time the buyer spends driving will be taken as exogenous, as the driver is purchasing the car for some task, hopefully driving.

### Setup

$$\min_{DP} \mathbb{E}DP + \sum_{t=0}^{T} \beta^t (P) + \sum_{y=0}^{Y} \beta^{12y}(I + F_y(M_y + Rep_y) + Reg_y)$$

$$P = \frac{r(MSRP - DP)}{1 - (1+r)^{-T}}$$

$$\beta = \frac{1}{1+r}$$

$M_y, Rep_y$ are a sequence of maintence and repair costs. $Reg_y$ is the cost of registering the vehicle yearly. $F_y$ is the cost of gasoline in time period y, MSRP is the cost of the Car, r is the prevailing interest rate.

$$
\begin{aligned}
MSRP &= 36150 \\
M_Y + Rep_y &= 229 \\
F_t = \frac{15000}{12} &* 25 * C_g(t) \\
C_g &= 2.607
\end{aligned}
$$

## Code

```
using JuMP
using SCIP
m = Model(solver=SCIPSolver())

MSRP = 36150
Main = 229
Fuel = 15000*25*2.607/12
r = 0.07/12.0
beta = 1/(1+r)
T = 36
Y = 10
I = 227
Reg_initial = 225
Reg = 72.40

@variable( m, 0<=D<=MSRP)
@expression( m, P, (r*(MSRP-D))/ (1-(1+r)^(-T) ) )
@objective(m, Min, Reg_initial +  sum( beta^i *(P+I) for i = 1:T ) + sum( beta^(12*i) * ( Fuel + Main + Reg )
status = solve(m)
```

```
feasible solution found by trivial heuristic after 0.0 seconds, objective v
alue -3.615000e+04
presolving:
presolving (1 rounds: 1 fast, 0 medium, 0 exhaustive):
 1 deleted vars, 0 deleted constraints, 0 added constraints, 0 tightened bo
unds, 0 added holes, 0 changed sides, 0 changed coefficients
 0 implications, 0 cliques
transformed 1/3 original solutions to the transformed problem space
Presolving Time: 0.00

SCIP Status        : problem is solved [optimal solution found]
Solving Time (sec) : 0.00
Solving Nodes      : 0
Primal Bound       : -3.61499999999998e+04 (3 solutions)
Dual Bound         : -3.61499999999998e+04
Gap                : 0.00 %
```

```
println("Objective value: ", getobjectivevalue(m))
```

```
Objective value: 575865.3235946147
```

## 2 How Much Should I save?

We will consider this problem in the context of an LQ-control problem of finding the optimal consumption path when facing some income.

We are considering a household budget of:

$$a_{t+1} + c_t = (1+r)a_t + y_t$$

We consider $a_t$ to be savings at time t, and $c_t$ to be consumption, and $y_t$ to be non-financial income, or income earned by working. We would like $y_t$ to be random, following some linear law of motion. For simplicity let: $y_t \sim N(\mu, \sigma^2)$. We shall also represent consumption as deviation from some ideal level of consumption. That is we choose $u_t := c_t - \bar{c}$. Our budget equation now becomes:

$$a_{t+1} = (1+r)a_t - u_t - \bar{c} + \sigma w_{t+1} + \mu$$

where $w_{t+1} = N(0,1)$ We wish to represent this in a linear State space model, however we currently have an affline function, so we must add a fake variable to allow for the model to be written as:

$$x_{t+1} = Ax_t + Bu_t + Cw_{t+1}$$

We shall define

$$x_t := \begin{bmatrix} a_t \\ 1 \end{bmatrix} A := \begin{bmatrix} 1+r & -\bar{c}+\mu \\ 0 & 1 \end{bmatrix} B := \begin{bmatrix} -1 \\ 0 \end{bmatrix} C := \begin{bmatrix} \sigma \\ 0 \end{bmatrix}$$

### Preferences

We wish to represent preferences as trying to minimize deviations of the consumer from the optimal level of consumption. In fitting with the LQ control model, we wish to minimize $u_t 1 u_t = u_t^2 = (c_t - \bar{c})^2$ Since the LQ objective function is $x_t^T R x_t + u_t^T Q u_t$ we shall set $R = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = 1$.

### The Objective

We wish to maximize time discounted consumption in each period with an incentive to be debt-free at the end of life, as otherwise the consumer would simply rack up endless debt to pay for optimal consumption. We shall achieve this by defining the objective in the final time period as: $R_f = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}$. Where q is suitably large enough. Our objective function then is:

$$\min_{\{c_t\}_{t=0}^{T}} \mathbb{E} \sum_{t=0}^{T-1} \beta^t (c_t - \bar{c})^2 + \beta^T q a_t^2$$

### Further Investigation

It is pretty unrealistic to assume income doesn't change over time. We would like to consider parabolic income where $y_t = m_0 + m_1 t + m_2 t^2 + \sigma w_{t+1}$ We will choose the m values to determine the peek of income. Income begins at 0, rises to $\mu$ at some point in his life, and falls to 0 again somewhere later outside the "end" of his life. We will believe that the peak of the agent's career occurs two thirds of the way through his life. That is: $\frac{m_1}{2m_0} = \frac{2}{3}T$ and the roots of the parabola occur at: 0, $\frac{4}{3}T$. This yields: $m_0 = 0, m_1 = 3\mu, m_2 = \frac{-9\mu}{4}$.

Our income model has become slightly less fun:

$$a_{t+1} = (1+r)a_t - u_t - \bar{c} + \frac{3\mu t}{K} - \frac{9\mu}{4K^2}t^2 + \sigma w_{t+1}$$

Our states have become:

$$x_t = \begin{bmatrix} a_t \\ 1 \\ t \\ t^2 \end{bmatrix} A = \begin{bmatrix} 1+r & -\bar{c} & \frac{3\mu}{K} & \frac{9\mu}{4K^2} \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} C = \begin{bmatrix} \sigma \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Q,R,$R_f$ remain unchanged except for the increase in size.

## Retirement

Naturally, we only care about savings in order to figure out how much we should save for retirement. This means our model cannot have the lifetime of our agent ending before retirement. Our model simply creates an incentive of being debt-free at the time of retirement, which doesn't really make a lot of sense in the context of saving for retirement. We wish to extend the lifetime infinitely in retirement where his income is fixed but at a smaller amount than he could earn by working. Hopefully this will incentives the agent to save money so that he can still consume at a level he is used to in retirement.

This introduces a kink into our income formula.

$$y = \begin{cases} m_0 + m_1 t + m_2 t^2 + \sigma w_{t+1} & \text{if } t \leq K \\ s & \text{otherwise} \end{cases}$$

To solve this problem, we will solve the infinite dynamic program, finding the value function at the start of this program, and this will become the value function at the end of the finite LQ program. This means that we will be solving an infinite LQ Control problem in order to figure out $R_f$ to determine the savings path.

## Solving the Model

### Infinite Horizon

We seek the solution to the LQ-Bellman equation given by:

$$P = R(B^T PA)^T (Q + B^T PB)^{-1}(B^T PA) + A^T PA$$

Our optimal policy will be given by:

$$u = -Fx \text{ where } F = (Q + \beta B^T PB)^{-1}(\beta B^T PA)$$

Our optimal savings policy is then given by:

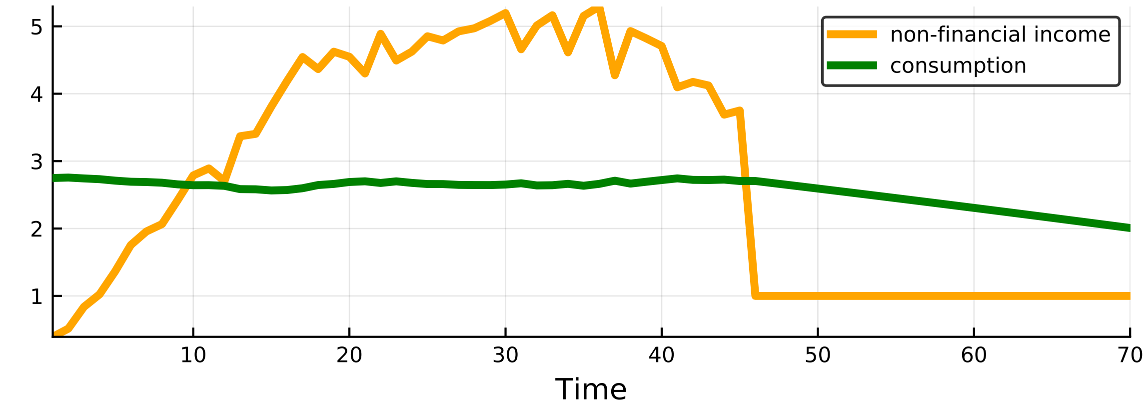
$$x_{t+1} = (A - BF)x_t + Cw_{t+1}$$

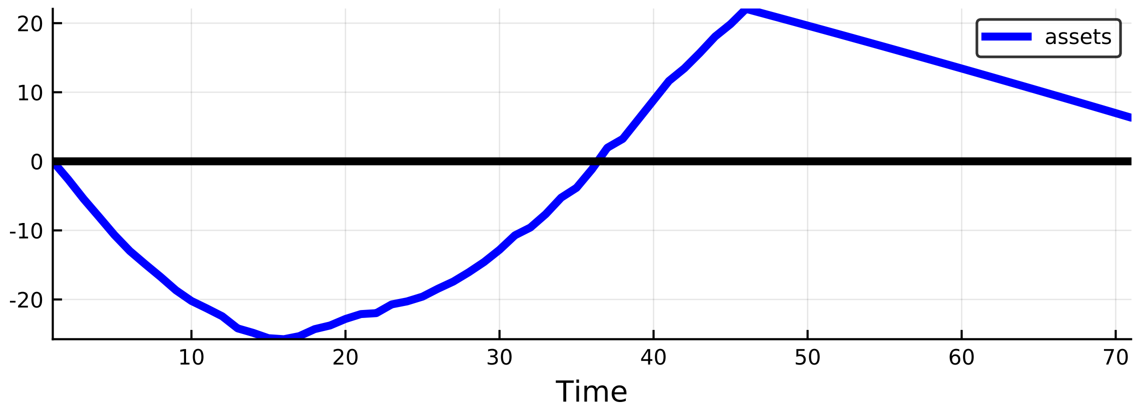
This P, the stationary distribution will be passed to the Discrete interval question as the final state, allowing us to account for the discontinuity in the income process.

## Code

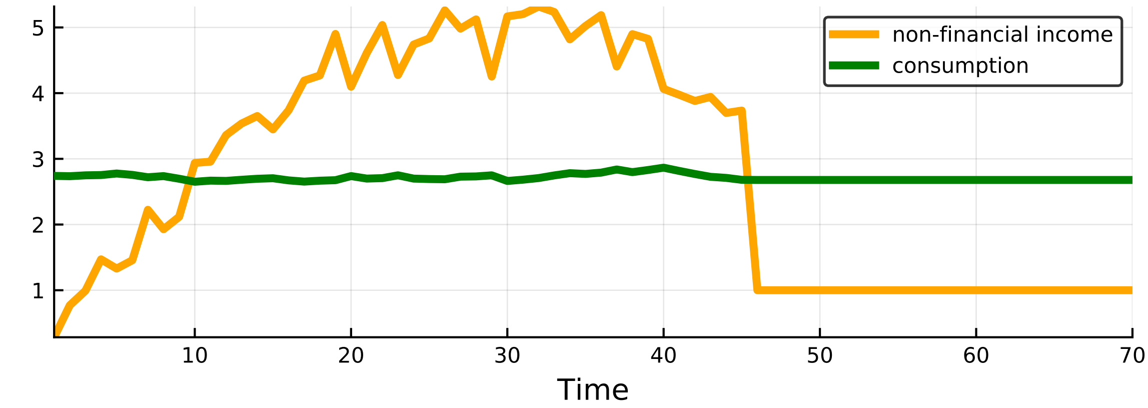
```
using QuantEcon
using Plots
using Plots.PlotMeasures
pyplot()

r = 0.05
#This had to be a lot lower than expected to not get massive debt.
#I think this is because there is no incentive to be debt-free at the end of life.
#The current Linear State space cannot accomodate Only punishing negative a_t so
#I am not sure how this could be worked around without provididing a completely different model
deathParam = 0.004
beta = 1/(1+r+deathParam)
T = 70
c_bar = 10.0
sigma = .5
mu = 5.0
s = 1
K = 45
q = 1e6
m1 = 3*mu/(K)
m2 = -9*mu/(4*(K)^2)

# == Formulate as an LQ problem == #
Q = 1.0
R = zeros(4, 4)



Rf = zeros(4, 4);
Rf[1, 1] = q
A = [1+r  s-c_bar 0 0;
     0     1      0 0;
     0     1      1 0;
     0     1      2 1]
B = [-1.0; 0.0; 0.0; 0.0]
C = [0.0; 0.0; 0.0; 0.0]

#Seek out the stationary distribution of the infinite LQ
lq_retired = LQ( Q, R, A, B, C, bet=beta)
stationary_values!( lq_retired )


Rf2 = lq_retired.P
beta = 1/(1+r)
Q = 1.0
```

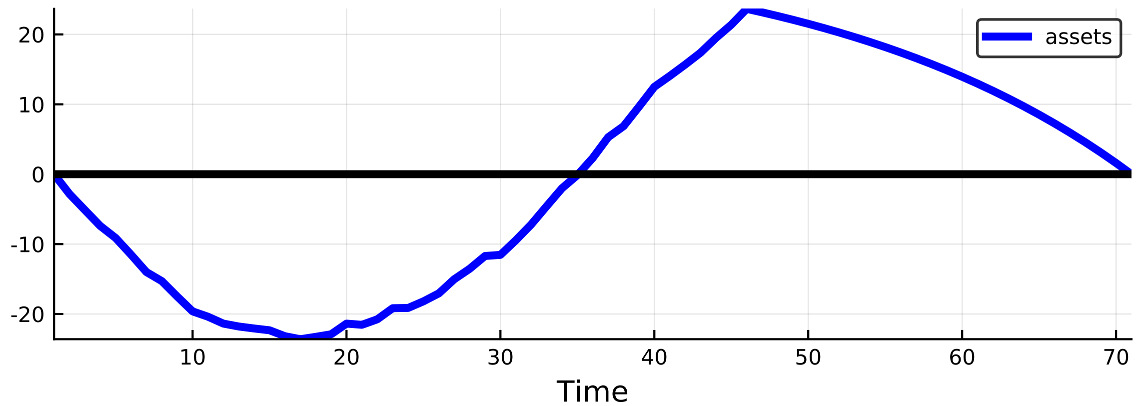
```
R = zeros(4, 4)
A = [1 + r -c_bar m1 m2;
     0     1      0  0;
     0     1      1  0;
     0     1      2  1]
B = [-1.0; 0.0; 0.0; 0.0]
C = [sigma; 0.0; 0.0; 0.0]

# == Set up working life LQ instance with terminal Rf from lq_retired == #
lq_working = LQ(Q, R, A, B, C; bet = beta, capT = K, rf = Rf2)

# == Simulate working state / control paths == #
x0 = [0.0; 1.0; 0.0; 0.0]
xp_w, up_w, wp_w = compute_sequence(lq_working, x0)
# == Simulate retirement paths (note the initial condition) == #
xp_r, up_r, wp_r = compute_sequence(lq_retired, xp_w[:, end],T-K )

# == Convert results back to assets, consumption and income == #
xp = [xp_w xp_r[:, 2:end]]
assets = vec(xp[1, :])               # Assets

up = [up_w up_r]
c = vec(up + c_bar)                  # Consumption

time = 1:K
income_w = sigma * vec(wp_w[1, 2:K+1]) + m1 .* time + m2 .* time.^2   # Income
income_r = ones(T-K) * s
income = [income_w; income_r]

# == Plot results == #
p1 = plot(Vector[income, assets, c, zeros(T + 1)], lab = ["non-financial income" "assets" "consumption" ""],
     color = [:orange :blue :green :black], width = 3, xaxis = ("Time"), layout = (2,1),
          bottom_margin = 20mm, size = (600, 600), show = true)
```




This can be contrasted against the consumer who knows when the date he will pass is. He is incentivised to be debt-free at the end of his life, and knows the time when he will pass, he is able to leverage this into a constant consumption level along his entire lifetime, while the infinitely lived, but death discounting agent decreases his consumption throughout retirement as he expects to die in the future, and more highly values present consumption.




We can see that the myopic consumer with the death wish consumes higher in his lifetime, possibly leading to the death that he is so certain of. The consumer who is uncertain of his death maintains his savings so that he would be able to live off it indefiniety. One explanation is he plans to pass on his wealth to his family, so considering an infinite time horizen is legitimate.

## 3 Can I afford a House?

### The Problem

I have chosen to consider that this question is asking whether or not you should rent a house or buy one. I am taking the value of the house as exogenous, as otherwise you could simply find a cheaper one and this would be trivial.

By applying the Law of renting costs equals owning costs popularized by the famous Economist Harry Paarsch in 2017, we can understand that the costs of owning a house and renting should be relatively close to each other. I will consider this arbitrage arguement true for no down payment made on a house.

This condition leads us to a choice of down payment on a house, and if we apply an income process that is increasing over the time of the loan, we may choose optimal consumption over the lifetime of the agent who pays payments on a loan for some fixed period of time, and always pays maintenence fees on his house. The consumption under this model may be compared to consumption under a model where the agent only pays rent costs for his entire life. The question is whether or not there is a down payment that is strictly better off in terms of consumption over the renting option.

The arbitrage condition that this model rests upon is:

$$\mathbb{E}[\sum_{t=0}^{\infty} \beta^t C_R(t)] = \sum_{n=0}^{N} \beta^n P + \mathbb{E}[\sum_{t=0}^{\infty} \beta^t C_m(t)$$

Where $C_R$ is the cost of renting, $C_m$ is the cost of maintaining the house, P is the payments made on a house and $\beta$ is some time discounting factor.

We can simplify this by setting $C_R(t) = C_R$ and $C_m(t) = C_m$ Our arbitrage condition then simplifies to:

$$\begin{aligned} \frac{C_R}{1-\beta} &= P\frac{1-\beta^{N+1}}{1-\beta} + \frac{C_m}{1-\beta} \\ C_R &= P(1-\beta^{N+1}) + C_m \\ C_R - C_m &= P(1-\beta^{N+1}) \end{aligned}$$

## Law of Motion

We would like to follow the same law of motion as the savings problem, where assets and consumption are equal to working and capital income. We will seperate consumption from housing payments, and consider the deviation of consumption from some ideal level as the utility function.

### Renter

$$a_{t+1} = (1+r)a_t - u_t - \bar{c} + \frac{3\mu t}{K} - \frac{9\mu}{4K^2}t^2 + \sigma w_{t+1} - C_R$$

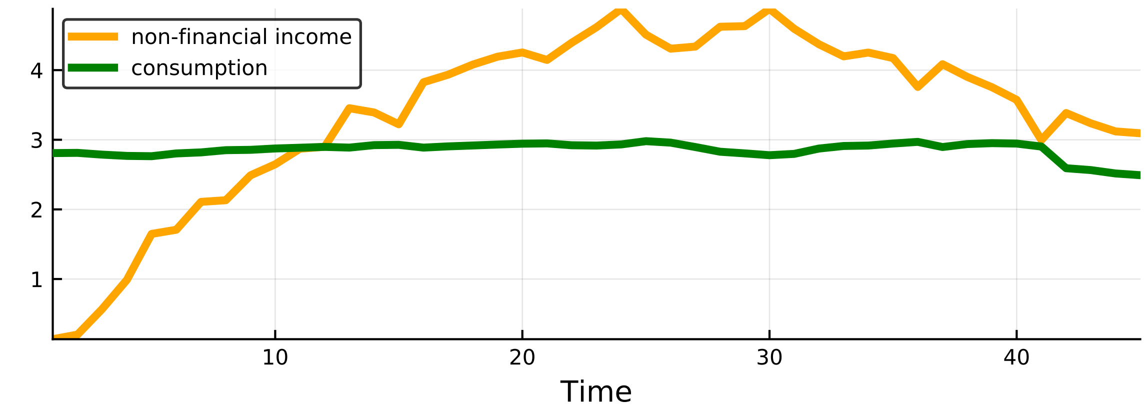
$$x_t = \begin{bmatrix} a_t \\ 1 \\ t \\ t^2 \end{bmatrix} A = \begin{bmatrix} 1+r & -\bar{c} - C_R & \frac{3\mu}{K} & \frac{9\mu}{4K^2} \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} C = \begin{bmatrix} \sigma \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

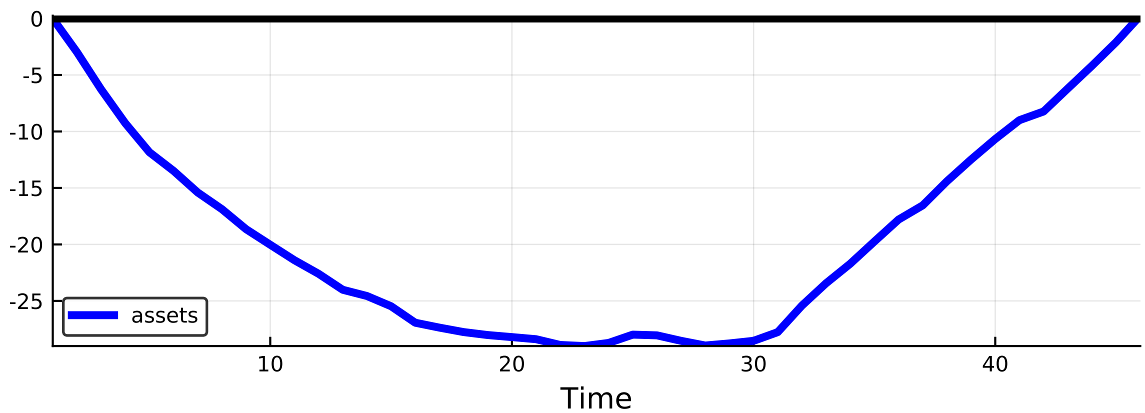
This is effectively the same problem as before, as the renter simply has a lower level of ideal consumption, or can be treated as having the same amount of ideal consumption, but measurely less wealth in each period.

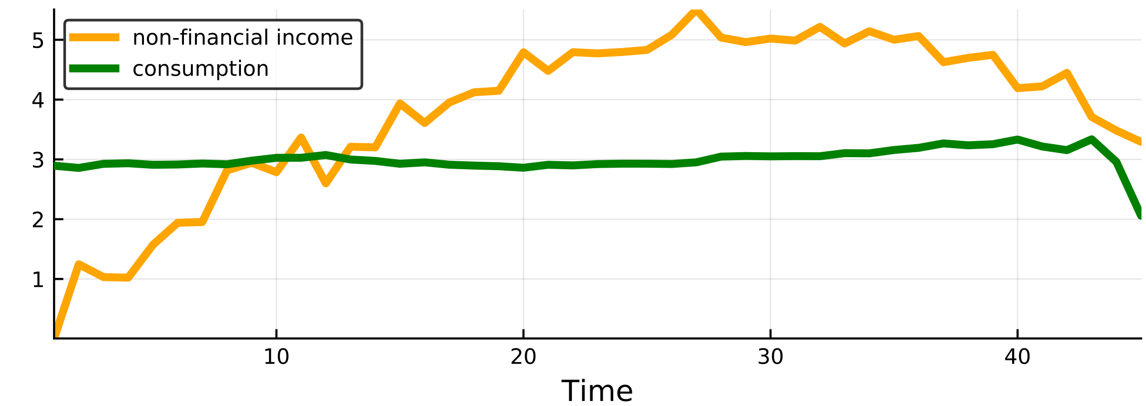
### Buyer

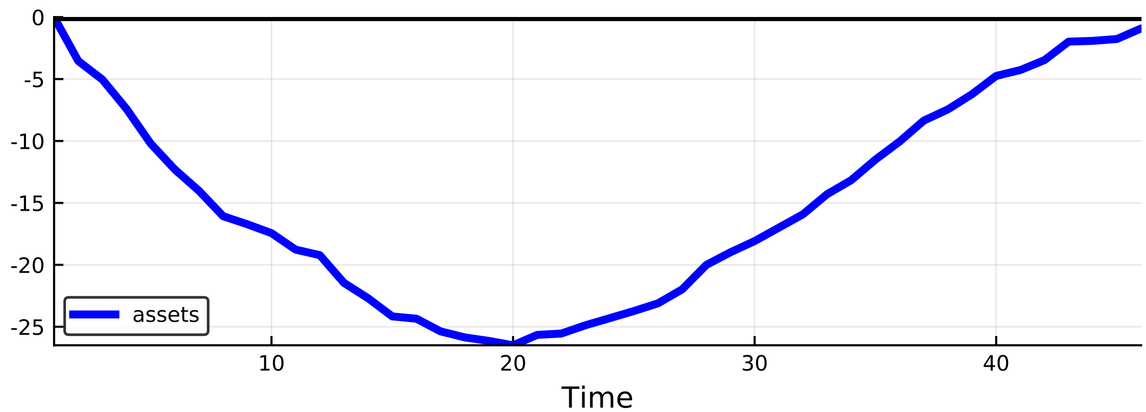
The buyer however has a slightly different story: The buyer faces a lower income for the periods in which he is paying off the hoouse, but then has more income later in his life to enjoy. For the period in which he is paying back his loans: $Y_R + P(1 - \beta^{N+1}) - P = Y_R - P\beta^{N+1}$ then changes after N periods to $Y_R + P(1 - \beta^{N+1})$.

## Code






We can see that under this arbitrage arguement, there is no reason to buy a house over renting one of equal value. The Renter enjoys higher consumption, and goes into less debt compared to the house buyer.

I am actually uncertain of how to properly cite my source here. But almost everything I did with LQ control borrows very heavily from the QuantEcon website by Thomas J. Sargent and John Stachurski. Their section on LQ control from which all of my work is basically stolen and then modified can be found at: https://lectures.quantecon.org/jl/lqcontrol.html.

I attempted to keep their code as intact as possible when I made my additions so that it could be followed easily, but if it would have been better to not follow that please tell me so I don't plaigerize on something much more important.