Operations Research HW3

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Question 1

$$
\begin{aligned}
\max \quad & 3x_1 + x_2 \\
\text{s.t.} \quad & 4x_1 + x_2 \geq 4 \\
& 2x_1 + x_2 \leq 4 \\
& x_1 + x_2 = 3 \\
& x_1, x_2, x_3, s_1, s_2, a_1 \geq 0
\end{aligned}
$$

a.

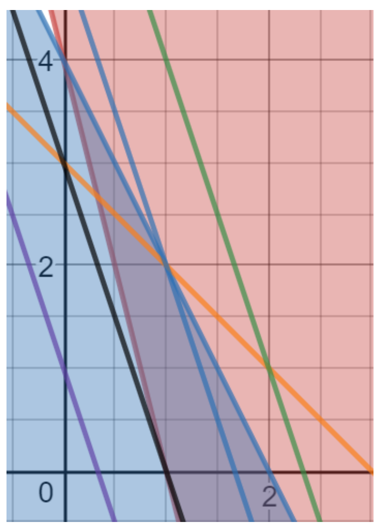

Fig. 1: Feasible set and level sets

Note that the orange line is a constraint.

b.

$$
\begin{aligned}
Z - 3x_1 - x_2 \qquad\qquad & - Ma_1 - Ma_3 = 0 \\
4x_1 + x_2 - s_1 \qquad\qquad & + a_1 \qquad\quad = 4 \\
2x_1 + x_2 \quad + s_2 \qquad\qquad & \qquad\qquad\quad = 4 \\
x_1 + x_2 \qquad\qquad & \qquad + a_3 \quad = 3
\end{aligned}
$$

Putting it into Clean Table form

$$
\left[\begin{array}{ccccccc|c}
Z & x_1 & x_2 & s_1 & s_2 & a_1 & a_2 & RHS \\
\hline
1 & -3-5M & -1-2M & M & M & 0 & 0 & -7M \\
0 & 4 & 1 & -1 & 0 & 1 & 0 & 4 \\
0 & 2 & 1 & 0 & 1 & 0 & 0 & 4 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 3
\end{array}\right]
$$

$$
\left[\begin{array}{ccccccc|c}
Z & x_1 & x_2 & s_1 & s_2 & a_1 & a_2 & RHS \\
\hline
1 & 0 & \frac{-1-3m}{4} & \frac{-3-M}{4} & M & \frac{3+5M}{4} & 0 & 3-2M \\
0 & 1 & \frac{1}{4} & \frac{-1}{4} & 0 & \frac{1}{4} & 0 & 1 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 1 & \frac{-1}{2} & 0 & 2 \\
0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 & \frac{-1}{4} & 1 & 2
\end{array}\right]
$$

$$
\left[\begin{array}{ccccccc|c}
Z & x_1 & x_2 & s_1 & s_2 & a_1 & a_2 & RHS \\
\hline
1 & 0 & 0 & \frac{-2}{3} & M & \frac{2+3M}{3} & \frac{1+3M}{3} & \frac{11}{3} \\
0 & 1 & 0 & \frac{-1}{3} & 0 & \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} \\
0 & 0 & 0 & \frac{1}{3} & 1 & \frac{-1}{3} & \frac{-2}{3} & \frac{2}{3} \\
0 & 0 & 1 & \frac{1}{3} & 0 & \frac{-1}{3} & \frac{4}{3} & \frac{8}{3}
\end{array}\right]
$$

$$
\left[\begin{array}{ccccccc|c}
Z & x_1 & x_2 & s_1 & s_2 & a_1 & a_2 & RHS \\
\hline
1 & 0 & 0 & 0 & M+2 & M & M-1 & 5 \\
0 & 1 & 0 & 0 & 1 & 0 & -1 & 1 \\
0 & 0 & 0 & 1 & 3 & -1 & -2 & 2 \\
0 & 0 & 1 & 0 & -1 & 0 & 2 & 2
\end{array}\right]
$$

From here we can see that $x_1 = 1$, $x_2 = 2$, $s_1 = 2$, and the maximum obtained is 5.

c.

At the start of the algorithm, s_1, a_2, a_2 are in the basis, and our solution is not feasible, with objective value 0. This corresponds to the origin on the graph.

After One step, x_1, s_2, a_2 are in the basis. Since $x_1 = 1$, our solution is still not feasible, it corresponds to being along the x-axis on the graph, and the objective value is -3.

After the second step, x_1, x_2, s_2 are in the basis, but we are still not at a feasible solution, since $x_1 + x_2 \neq 3$. The objective value is $\frac{11}{3}$

At the final step, x_1, x_2, s_1 are in the basis, and we are finally at a feasible solution. The objective function value is 5, and on the graph we correspond to the maximum shown.

d.

We can see that the problem is not unbounded, as there are no negative elements on the Z-row with a DNE or negative minimum ratio-test. The problem is feasible because there is no artificial variables present in the basis at the final solution.

Question 2.

a.

$$\begin{aligned} \min\ & 4x_1 + 4x_2 + x_3 \\ \text{s.t.}\quad & x_1 + x_2 + x_3 \leq 2 \\ & 2x_1 + x_2 \leq 3 \\ & 2x_1 + x_2 + 3x_3 \geq 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

This statement is equivalent to:

$$\begin{aligned} \max\ & -4x_1 - 4x_2 - x_3 - Ma_1 \\ \text{s.t.}\quad & x_1 + x_2 + x_3 + s_1 = 2 \\ & 2x_1 + x_2 + s_2 = 3 \\ & 2x_1 + x_2 + 3x_3 - s_3 + a_1 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$$\left[\begin{array}{cccccccc|c} Z & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & a_1 & RHS \\ \hline 1 & 4-2M & 4-M & 1-3M & 0 & 0 & -M & 0 & -3M \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 2 & 1 & 3 & 0 & 0 & -1 & 1 & 3 \end{array}\right]$$

$$\left[\begin{array}{cccccccc|c} Z & x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & a_1 & RHS \\ \hline 1 & \frac{10}{3} & \frac{11}{3} & 0 & 0 & 0 & \frac{1}{3} & \frac{-1}{3} + M & -1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 & 1 & 0 & \frac{1}{3} & \frac{-1}{3} & 1 \\ 0 & 2 & 1 & 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & \frac{2}{3} & \frac{1}{3} & 1 & 0 & 0 & \frac{-1}{3} & \frac{1}{3} & 1 \end{array}\right]$$

Thus the maximum of -Z is -1, and the minimum of Z is 1. Occurring where $x_1 = 0, x_2 = 0, x_3 = 1$.

b.

The problem is not infeasible as there is are no artificial variables that are still in the basis, only the slack variables.

Question 3.

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & -x_1 + x_2 \leq 2 \\ & -2x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

a.

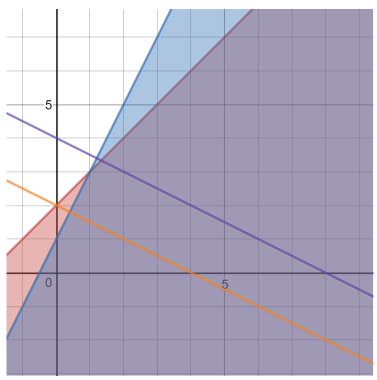

Fig. 2: Feasible set with Level Sets

It is clear from the graph that the linear program is unbounded, as the feasible region is not bounded, and level sets can be continued on infinitely increasing the maximal value.

b.

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & -x_1 + x_2 + s_1 = 2 \\ & -2x_1 + x_2 + s_2 = 1 \\ & x_1, x_2, s_1, s_2 \geq 0 \end{aligned}$$

$$\left[\begin{array}{ccccc|c} Z & x_1 & x_2 & s_1 & s_2 & RHS \\ \hline 1 & -1 & -2 & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} Z & x_1 & x_2 & s_1 & s_2 & RHS \\ \hline 1 & -5 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & -2 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} Z & x_1 & x_2 & s_1 & s_2 & RHS \\ \hline 1 & 0 & 0 & 5 & -3 & 7 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 2 & -1 & 3 \end{array}\right]$$

c.

In the final tableau we can see that we would like s_2 to enter the basis, but the ratio for both rows is negative. This indicates that the program is unbounded, and the objective function can be made as large as possible.

d.

Examining the two constraints:
$x_1 + s_1 - s_2 = 1$
$x_2 + 2s_1 - s_2 = 3$
Note that s_2 is what we want to enter the basis, so s_1 will remain 0.
$x_1 = 1 + s_2$ and $x_2 = 3 + s_2$
Thus $\mathbf{x} = \binom{1}{3} + s_2\binom{1}{1}$ and the direction of unboundedness is: $\binom{1}{1}$

Question 4

a.

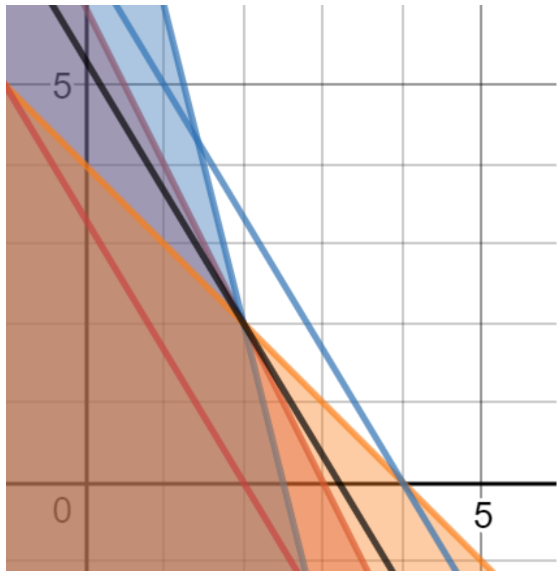

Fig. 3: Feasible set and level sets

b.

$$
\begin{aligned}
\max \quad & 5x_1 + 3x_2 \\
\text{s.t.} \quad & 4x_1 + 2x_2 + s_1 = 12 \\
& 4x_1 + x_2 + s_2 = 10 \\
& x_1 + x_2 + s_3 = 4 \;\; x_1, x_2, s_1, s_2, s_3 \geq 0
\end{aligned}
$$

$$
\left[\begin{array}{cccccc|c}
Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\
\hline
1 & -5 & -3 & 0 & 0 & 0 & 0 \\
0 & 4 & 2 & 1 & 0 & 0 & 12 \\
0 & 4 & 1 & 0 & 1 & 0 & 10 \\
0 & 1 & 1 & 0 & 0 & 1 & 4
\end{array}\right]
$$

$$
\left[\begin{array}{cccccc|c}
Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\
\hline
1 & 0 & \frac{-7}{4} & 0 & \frac{5}{4} & 0 & \frac{25}{2} \\
0 & 0 & 1 & 1 & -1 & 0 & 2 \\
0 & 1 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{5}{2} \\
0 & 0 & \frac{3}{4} & 0 & \frac{-1}{4} & 1 & \frac{3}{2}
\end{array}\right]
$$

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & 0 & 0 & \frac{7}{4} & \frac{-1}{2} & 0 & 16 \\ 0 & 0 & 1 & 1 & -1 & 0 & 2 \\ 0 & 1 & 0 & \frac{-1}{4} & \frac{1}{2} & 0 & 2 \\ 0 & 0 & 0 & \frac{-3}{4} & \frac{1}{2} & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccccc|c} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & 0 & 0 & \frac{1}{4} & 0 & 2 & 16 \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & 2 & 2 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -1 & 2 \\ 0 & 0 & 0 & \frac{-3}{2} & 1 & 2 & 0 \end{array}\right]$$

We can see that a maximum of 16 is obtained at: $x_1 = 2, x_2 = 2, s_3 = 2, x_3 = s_1 = s_2 = 0$

c.

The Algorithm begins at the origin with the point: (0,0,12,10,4), it then moves to $(\frac{5}{2}, 0, 0, 2, 0, \frac{3}{2})$ found along the x-axis. After this x_2 enters the basis and the algorithm moves to: $(2, 2, 0, 0, 0)$ This corresponds to the maximum on the graph. The final step remains here.

d.

Yes the LP is degenerate. Notice in the second tableau there is two options of equal ratio, meaning that one constraint is redundant, and the linear program is degenerate.

e.

The linear program is not unbounded since we can see that there are no negative elements in the Z row. Since we do not see a nonbasic variable become zero in the Z row, we can conclude that there are not multiple solutions.

Question 5.

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 + s_1 = 100 \\ & 4x_1 + x_2 + s_2 = 80 \\ & x_1 + s_3 = 40 \\ & x_1, x_2, s_1, s_2, s_3 \geq 0 \end{aligned}$$

a.

The new Tableau is:

$$\left[\begin{array}{ccccccc|c} & 1 & d_1 & d_2 & 0 & 0 & 0 & 0 \\ & Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS \\ \hline 1 & 1 & 0 & 0 & 1 & 1 & 0 & 180 \\ d_1 & 0 & 1 & 0 & 1 & -1 & 0 & 20 \\ d_2 & 0 & 0 & 1 & -1 & 2 & 0 & 60 \\ 0 & 0 & 0 & 0 & -1 & 1 & 1 & 20 \end{array}\right]$$

The reduced cost for s_1 is: $1 + d_1 - d_2$ The reduced cost for s_2 is $1 - d_1 + 2d_2$

If only d_1 changes, d_2 is 0. So for s_1, s_2 to be feasible. $1 + d_1 \geq 0$ and $1 - d_1 \geq 0$. This implies that: $d_1 \in [-1, 1]$. This means that the basis will remain optimal as long as the coefficient of $x_1 \in [2, 4]$. If the objective function coefficient becomes 3.5, the basis remains optimal, and since x_1, x_2 are in the basis and completely determine the maximal value, it will remain unchanged at 180.

b.

If only d_2 changes, d_1 is 0. We instead arrive at: $1 - d_2 \geq 0$ and $1 + 2d_2 \geq 0$. This implies that $d_2 \in [\frac{-1}{2}, 1]$ and the basis will remain optimal as long as the coefficient of $x_2 \in [\frac{3}{2}, 3]$.

c.

The RHS Tableau will be:

$$\left[\begin{array}{cccccc|cccc} Z & x_1 & x_2 & s_1 & s_2 & s_3 & RHS & d_1 & d_2 & d_3 \\ \hline 1 & 0 & 0 & 1 & 1 & 0 & 180 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 & 20 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 & 60 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & 20 & -1 & 1 & 1 \end{array}\right]$$

If only d_1 changes, $d_2, d_3 = 0$. Feasibility conditions are:

$$\begin{aligned} 20 + d_1 &\geq 0 \\ 60 - d_1 &\geq 0 \\ 20 - d_1 &\geq 0 \end{aligned}$$

We can see that $d_1 \in [-20, 20]$. This allows for the RHS to be $\in [80, 120]$. If the RHS for the first constraint changes to 90, that means $d_1 = 10$. This changes the optimal value of the objective function to: $180 + d_1 + d_2 = 190$

Question 6.

It is equivalent to state the problem in this manner:

$$\begin{aligned} \max\ & 4x_1 - e_1 + e_2 + e_3 - e_3 \\ \text{s.t.}\ & x_1 + e_1 - e_2 \leq 5 \\ & 2x_1 + e_1 - e_2 \leq 7 \\ & -2e_1 + 2e_2 - e_3 + e_4 \leq -6 \\ & x_1 + e_3 - e_4 \leq 4 \\ & -x_1 - e_3 + e_4 \leq -4 \\ & x_1, e_1, e_2, e_3, e_4 \geq 0 \end{aligned}$$

The dual of such a problem is:

$$\begin{aligned} \min\ & 5\lambda_1 + 7\lambda_2 - 6\lambda_3 + 4\lambda_4 - 4\lambda_5 \\ \text{s.t.}\ & \lambda_1 + 2\lambda_2 + \lambda_4 - \lambda_5 \geq 4 \\ & -\lambda_1 + \lambda_2 + \lambda_3 - 2\lambda_4 \geq -1 \\ & \lambda_1 - \lambda_2 - \lambda_3 + 2\lambda_4 \geq 1 \\ & -\lambda_3 + \lambda_4 - \lambda_5 \geq 1 \\ & \lambda_3 - \lambda_4 + \lambda_5 \geq -1 \\ & \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0 \end{aligned}$$

Note that I elected to change the equality constraint to two constraints instead of allowing a dual variable to become unrestricted. I also forced all constraints in the primal to be $\leq$ so that all the constraints in the dual would be $\geq$. It is possible to form the dual without these changes, however my answer is equivalent, and follows the procedure burned into me by the math department.

Question 7.

$$\begin{aligned} \max\ & 3x_1 + 4x_2 + x_3 + 5x_4 \\ \text{s.t.}\ & x_1 + 2x_2 + x_3 + 2x_4 \leq 5 \\ & 2x_1 + 3x_2 + x_3 + 3x_4 \leq 8 \\ & x_1, x_2, x_3, x_4, \geq 0 \end{aligned}$$

This problem has a dual of:

$$\begin{aligned} \min\ & 5\lambda_1 + 8\lambda_2 \\ \text{s.t.}\ & \lambda_1 + 2\lambda_2 \geq 3 \\ & 2\lambda_1 + 3\lambda_2 \geq 4 \\ & \lambda_1 + \lambda_2 \geq 1 \\ & 2\lambda_1 + 3\lambda_2 \geq 5 \\ & \lambda_1, \lambda_2 \geq 0 \end{aligned}$$

b.

By inspection we may note that constraint 2 is irrelevant, and that $\lambda_1 = 1, \lambda_2 = 1$ is the smallest values we may achieve. This gives us a minimum of: 13

c.

Via Complementarity Slackness: $(c - A^T\lambda)^T x = 0$ This amounts to:

$$\left(\begin{bmatrix} 3 \\ 4 \\ 1 \\ 5 \end{bmatrix} - \begin{bmatrix} 3 \\ 5 \\ 2 \\ 5 \end{bmatrix}\right)^T \mathbf{x} = \begin{bmatrix} 0 \\ -1 \\ -1 \\ 0 \end{bmatrix}^T \mathbf{x} = 0$$

Thus we can see that: $x_2, x_3 = 0$ Using $(b - Ax)^T\lambda = 0$ tells us that the first two constraints must bind, as both λ_1 and λ_2 are positive. Using the constraints to form a linear system we arrive at:

$$\begin{aligned} x_1 + 2x_4 &= 5 \\ 2x_1 + 3x_4 &= 8 \end{aligned}$$

This has a solution of $x_1 = 1, x_4 = 2$. Thus the maximum is: 13 at: $(1, 0, 0, 2)$

Question 8.

a.

$$\begin{aligned} \max\ & -3x_1 + x_2 + 2x_3 \\ \text{s.t.}\ & x_2 + 2x_3 \leq 3 \\ & -x_1 + 3x_3 \leq -1 \\ & -2x_1 - 3x_2 \leq -2 \\ & x_1, x_2, x_3, \geq 0 \end{aligned}$$

We can see that the dual of this problem is:

$$\begin{aligned} \min\ & 3\lambda_1 - \lambda_2 - 2\lambda_3 \\ \text{s.t.}\ & -\lambda_2 - 2\lambda_3 \geq -3 \\ & \lambda_1 - 3\lambda_2 \geq 1 \\ & 2\lambda_1 + 3\lambda_2 \geq 2 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

b.

By Multiplying each of the constraints in the dual by -1 we arrive at this equivalent statement for the dual:

$$\begin{aligned} \min\ & 3\lambda_1 - \lambda_2 - 2\lambda_3 \\ \text{s.t.}\quad & \lambda_2 + 2\lambda_3 \leq 3 \\ & -\lambda_1 + 3\lambda_2 \leq -1 \\ & -2\lambda_1 - 3\lambda_2 \leq -2 \\ & \lambda_1, \lambda_2, \lambda_3 \geq 0 \end{aligned}$$

It is clear that this is the same feasible set as the primal.

c.

Let F be the feasible region for both the dual and the primal. Via weak duality:
$-3x_1 + x_2 + 2x_3 \leq 3\lambda_1 - \lambda_2 - 2\lambda_3 \quad \forall x, \lambda \in F$
Fix x, λ at their optimal values $x^* and \lambda^*$.
$-3x_1^* + x_2^* + 2x_3^* \leq 3\lambda_1^* - \lambda_2^* - 2\lambda_3^*$
However since $x^* \in F$ for the dual and $\lambda^* \in F$ for the primal. We may apply weak duality for $x = \lambda^*$ and $\lambda = x^*$.
$-3\lambda_1^* + \lambda_2^* + 2\lambda_3^* \leq 3x_1^* - x_2^* - 2x_3^*$
Thus $-3\lambda_1^* + \lambda_2^* + 2\lambda_3^* = 3x_1^* - x_2^* - 2x_3^*$ and both objective functions are equal.
Note also that since $(0, 0, 0) \in F$, $-3x_1^* + x_2^* + 2x_3^* \leq 0$ and $-3\lambda_1^* + \lambda_2^* + 2\lambda_3^* \geq 0$
Thus: $-3\lambda_1^* + \lambda_2^* + 2\lambda_3^* = 3x_1^* - x_2^* - 2x_3^* = 0$