Problem Set 1

1. Irreps of $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$:

   (a) Beginning from the $|++\rangle$ state apply the the lowering operator defined in class until you run out of states (i.e. "start at the top of the ladder and lower to the bottom"). You should find the three states defined in class.

   (b) Redefine these states so they have normalization 1, i.e. $\langle\text{state}|\text{state}\rangle = 1$.

   (c) Since we know there are four states in $\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$, we have missed one state. Use that this state must be orthogonal to the others to derive the $|00\rangle$ state and normalize it as above.

   (d) Apply $\mathcal{S}^2$ and $\mathcal{S}_z$ (i.e. the operators in product space) to each state and confirm that they do not result in states outside the irreducible representations you derived.

2. Gaussian Integrals – there are few integrals we know how to do, one of the most useful is the Gaussian:

   (a) Evaluate the one-dimensional Gaussian integral,

   $$I = \int_{-\infty}^{\infty} dp \, e^{-\frac{1}{2}ap^2 + Jp} \tag{1}$$

   hints: complete the square, make a substitution to obtain a simple square in the exponential, use polar coordinates to evaluate the square of I which will allow you to obtain the correct normalization.
   You should find:

   $$I = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \tag{2}$$

   (b) Using the same approach evaluate the multidimensional integral:

   $$I = \int_{-\infty}^{\infty} d\vec{p} e^{-\frac{1}{2}p_i^* A_{ij} p_j + J_i^* p_i} \tag{3}$$

   hints: follow the above, also diagonalize A_{ij}, recall that $\det A_{ij} = \Pi_i \lambda_i = \mathrm{Tr} D_{ij}$ where λ_i are the eigenvalues of A_{ij} and D_{ij} is the matrix resulting from diagonalizing A_{ij}.
   You should find:

   $$I = \sqrt{\frac{(2\pi)^n}{\det A_{ij}}} e^{\frac{1}{2} J_i^* A_{ij}^{-1} J_j} \tag{4}$$

3. Lie Algebra of the Lorentz group:

(a) Derive the Lie Algebra of the Lorentz group, i.e. show:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\,[g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma) \tag{5}$$

(b) Using the definitions of the angular momentum operator $J_i = \epsilon_{ijk}M^{jk}$ and boost operator $K_i = M^{i0}$, show:

$$[J_i, J_j] = i\epsilon_{ijk}J_k \tag{6}$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k \tag{7}$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k \tag{8}$$

(c) finally define $N_i = \frac{1}{2}(J_i - iK_i)$ to obtain the following:

$$[N_i, N_j] = i\epsilon_{ijk}N_k \tag{9}$$
$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk}N_k^\dagger \tag{10}$$
$$[N_i, N_j^\dagger] = 0 \tag{11}$$