Problem Set 6

1. Transverse and Longitudinal projection operators
   We define the Transverse and Longitudinal projection operators as:

$$\Pi_T^{\mu\nu} = \eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \quad (1)$$

$$\Pi_L^{\mu\nu} = \frac{p^\mu p^\nu}{p^2} \quad (2)$$

   Using these definitions show that these are indeed projection operators, i.e:

   (a) $\Pi_T^{\mu\nu}\Pi_{T,\nu\rho} = \eta_{\rho\sigma}\Pi_T^{\mu\sigma}$ (analogous to $P^2 = 1$)
   (b) $\Pi_L^{\mu\nu}\Pi_{L,\nu\rho} = \eta_{\rho\sigma}\Pi_L^{\mu\sigma}$ (analogous to $P^2 = 1$)
   (c) $\Pi_T^{\mu\nu}\Pi_{L,\nu\rho} = 0$ (analogous to $P_i P_{j\neq i} = 0$)
   (d) $\Pi_T^{\mu\nu} + \Pi_L^{\mu\nu} = \eta^{\mu\nu}$ (analagous to $\sum_i P_i = 1$)

2. Magnetic Dipole Moment operator
   We found the magnetic dipole moment of the electron was given by the term:

$$\frac{e}{2m}\bar{\psi} i\sigma^{\mu\nu}\psi p_\mu^\gamma \quad (3)$$

   (a) Show that the effective operator:

$$\mathcal{L}_{\text{eff}} = \bar{\psi}\sigma_{\mu\nu}\psi F^{\mu\nu} \quad (4)$$

   gives a Feynman Rule proportional to this. Note the Feynman rule won't contain the external fields $\bar{\psi}$ and ψ. (This operator is *not* hermitian, you can ignore this for simplicity)

   (b) What is the dimension of this operator in $d = 4$ dimensions?

3. RGE for onshell scheme in QED
   In class we found:

$$\delta_\psi = -\frac{e^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + 4 + \ln\frac{\mu^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right] \quad (5)$$

$$\delta_m = -\frac{4e^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2 + \ln\frac{\mu^2}{m^2} - \frac{1}{2}\ln\frac{m^2}{m_\gamma^2}\right] \quad (6)$$

$$\delta_e = -\frac{e^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + 4 + \ln\frac{\mu^2}{m^2} - 2\ln\frac{m^2}{m_\gamma^2}\right] \quad (7)$$

$$\delta_A = -\frac{e^2}{(4\pi)^2}\frac{4}{3}\left[\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2}\right] \quad (8)$$

   where, for simplicity, I have set $\tilde{\mu} \to \mu$ and dropped the subscript $e_R \to e$ and $m_R \to m$. We also had the relations between the bare and renormalized quantities:

$$e_0 = e\mu^\epsilon Z_e Z_\psi^{-1} Z_A^{-1/2} \quad (9)$$

$$m_0 = m Z_m Z_\psi^{-1} \quad (10)$$

   From this we saw,

$$0 = \mu\frac{d}{d\mu}\ln e + \mu\frac{d}{d\mu}\ln Z_e Z_\psi^{-1} Z_A^{-1/2} + \epsilon \quad (11)$$

$$0 = \mu\frac{d}{d\mu}\ln m + \mu\frac{d}{d\mu}\ln Z_m Z_\psi^{-1} \quad (12)$$

Using the above, derive the beta function for QED and the anomalous dimension of the electron mass:

$$\begin{aligned} \mu \frac{d}{d\mu} \ln e &= \frac{4}{3} \frac{e^2}{(4\pi)^2} && (13) \\ \mu \frac{d}{d\mu} \ln m &= 0 && (14) \end{aligned}$$

Use the following assumption:

$$\frac{dm_\gamma}{d\mu} = 0 \tag{15}$$

Be careful about, and remember to, drop terms of two-loop and higher order. If you get a non-zero anomalous dimension check if it is order e^4 in which case you accidentally kept a two-loop contribution