Path Integrals for Vector Fields, Quantum Electrodynamics (QED)

1) Vector Lorentz Invariants

→ Schwartz 8.2

→ Srednicki just draws from the classical field thry of E&M Sec. 54

Consider the free field $\mathcal{L}$:

$$\mathcal{L}_1 = -\frac{1}{2}(\partial^{\nu}A_{\mu})(\partial_{\nu}A^{\mu}) + \frac{1}{2}m^2 A_{\mu}A^{\mu}$$

Recall the Euler-Lagrange Equations of motion (EOM):

$$\frac{\partial \mathcal{L}}{\partial \varphi} = \partial_{\mu}\frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}$$

For $\mathcal{L}_1$ we have:

$$\frac{\partial \mathcal{L}_1}{\partial A_{\alpha}} = m^2 A^{\alpha}$$

$$\frac{\partial \mathcal{L}_1}{\partial(\partial_{\beta}A_{\alpha})} = \frac{\partial}{\partial(\partial_{\beta}A_{\alpha})}\left[-\frac{1}{2}\eta_{\mu\rho}(\partial^{\nu}A^{\rho})\eta_{\nu\sigma}(\partial^{\sigma}A^{\mu})\right]$$

$$= -\frac{1}{2}\left[(\partial^{\nu}A_{\mu})\eta_{\nu\sigma}\eta^{\sigma\beta}\eta^{\mu\alpha} + \eta_{\mu\rho}\eta^{\nu\beta}\eta^{\rho\alpha}(\partial_{\nu}A^{\mu})\right]$$

$$= -(\partial^{\beta}A^{\alpha})$$

We conclude:

$$\partial_{\beta}\frac{\partial \mathcal{L}_1}{\partial(\partial_{\beta}A_{\alpha})} = m^2 \rightarrow (\Box + m^2)A^{\alpha}$$

But this is just the EOM for 4 scalar fields:

$$(\Box + m^2)\varphi^i = 0, \quad i = 0 \ldots 4$$

Instead consider

$$\mathcal{L}_2 = (\partial^\mu A_\mu)(\partial^\nu A_\nu)$$

The EOM is:

$$\partial_\beta \frac{\partial \mathcal{L}_2}{\partial(\partial_\beta A_\alpha)} = \partial_\beta \frac{\partial}{\partial(\partial_\beta A_\alpha)}\left[\eta_{\mu\rho}(\partial^\mu A^\rho)\eta_{\nu\sigma}(\partial^\nu A^\sigma)\right]$$

$$= \partial_\beta\left[\eta_{\mu\rho}\eta^{\beta\mu}\eta^{\alpha\rho}(\partial^\nu A_\nu) + (\partial^\mu A_\mu)\eta_{\nu\sigma}\eta^{\beta\nu}\eta^{\alpha\sigma}\right]$$

$$= 2\,\partial^\alpha(\partial^\nu A_\nu)$$

$$\rightarrow \partial^\alpha \partial^\nu A_\nu = 0$$

Notice if $\partial^\nu A_\nu$ xformed as 4 scalars, $\partial^\nu A_\nu$ wouldn't be LI, so now $(\partial^\nu A_\nu)=0$ is a condition that removes 1 dof from A_ν, since it is a LI condition it must remove the 1 in $2\otimes 2 = 3\oplus 1$

Since we expect $\mathcal{L}_2$ to restrict us to a Lorentz vector our most general

$$\mathcal{L}_{free} = \frac{a}{2}(\partial^\nu A_\mu)(\partial_\nu A^\mu) + \frac{b}{2}(\partial^\mu A_\mu)(\partial^\nu A_\nu) + \frac{m^2}{2}A^\mu A_\mu$$

w/ EOM:

$$(a\Box\eta^{\alpha\nu} + b\partial^\alpha\partial^\nu + m^2\eta^{\alpha\nu})A_\nu$$

Taking the ∂_α derivative of this:

$$[(a+b)\Box + m^2](\partial^\nu A_\nu) = 0$$

If $a=-b$ we retain $(\partial^\nu A_\nu)=0$ (for $m\neq 0$) and we remove the A_0 component (or at least 1 degree of freedom)

Taking $a = 1 = -b$:

$$\mathcal{L}_{free} = \frac{1}{2}(\partial^\nu A_\mu)(\partial_\nu A^\mu) - \frac{1}{2}(\partial^\mu A_\mu)(\partial^\nu A_\nu) + \frac{m^2}{2} A^\mu A_\mu$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A^\mu A_\mu$$

where we have defined:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

The EOM is then:

$$\partial_\nu \partial^\nu A^\alpha - \partial^\alpha \partial_\nu A^\nu + m^2 A^\alpha = 0$$

$$\partial_\nu F^{\nu\alpha} + m^2 A^\alpha = 0$$

Unfortunately if we take $m^2 \to 0$ we lose $(\partial^\nu A_\nu) = 0$:

$$\partial_\nu F^{\nu\alpha} = 0 \to (\Box \eta^{\alpha\nu} - \partial^\alpha \partial^\nu) A_\nu = 0$$

$$\partial_\alpha (\quad) A_\nu = \underbrace{(\Box \partial^\nu - \Box \partial^\nu)}_{=0,\ \text{so}\ \partial^\nu A_\nu\ \text{need not}} A_\nu = 0$$

But we can use $\mathcal{L}_{free}(m \to 0)$ as a starting point

↳ You may recognize $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ as Maxwell's Eqns in covariant form

→ I don't know any text that shows this is the most general $\mathcal{L}_{free}$ w/o invoking gauge symmetry/ Maxwell's Eqns

2) U(1) gauge symmetry & massless vectors - Schwartz 14.5

Starting from $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ & $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

We can show this $\mathcal{L}$ is invariant under the xformation:

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

dependence on x implicit

We want to impose $\partial_\mu A^\mu = 0$ to obtain physical degrees of freedom for A_μ

We can do this w/ Lagrange moltipliers:

$$\mathcal{L} \to -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \underbrace{\frac{1}{2\xi}}_{\text{Lagrange multiplier enforces } \partial_\mu A^\mu = 0} (\partial_\mu A^\mu)^2$$

To achieve this we begin w/

$$\underbrace{f(\xi) = \int D\pi \, e^{-i \int d^4x \frac{1}{2\xi} (\Box \pi)^2}}_{\substack{\text{A scalar free field thry} \to \text{infinite constant} \\ \text{(w/ too many } \partial\text{'s)}}}$$

However, this integral doesn't change if we allow

$$\pi \to \pi + \alpha(x)$$

Further, since $A_\mu \to A_\mu + \partial_\mu \alpha(x)$

Our constraint $\partial_\mu A^\mu$ looks like:

$$\partial_\mu A^\mu + \Box \alpha = 0 \to \alpha = \frac{-1}{\Box} \partial_\mu A^\mu$$

Applying both of these we have:

$$f(\xi) = \int D\pi \, e^{-i \int d^4x \frac{1}{2\xi} (\Box \pi - \partial_\mu A^\mu)^2}$$

So our free field thry can be manipulated as:

$$\langle 0|0\rangle = \int DA_\mu \, e^{i\int d^4x\,(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu})}$$

$$= \frac{\xi(\xi)}{\xi(\xi)} \int DA_\mu \, e^{i\int d^4x\,(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu})}$$

$$= \frac{1}{\xi(\xi)} \int DA_\mu \, D\pi \, e^{i\int d^4x\,(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\Box\pi - \partial_\mu A^\mu)^2)}$$

Making the gauge xformation:

$$A_\mu \to A_\mu + \partial_\mu \pi$$

leaves $\underset{\text{gauge invar.}}{\underset{\uparrow}{F^{\mu\nu}F_{\mu\nu}}}$ $\dot{\varepsilon}$, $\underset{\text{shift invar}}{\underset{\uparrow}{D\pi\, DA_\mu}}$ invariant

So we find the Path Integral is:

$$\langle 0|0\rangle = \underbrace{\left[\frac{1}{\xi(\xi)}\int D\pi\right]}_{\text{Some infinite constant}} \int DA_\mu \, e^{i\int d^4x\,[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2]}$$

And we have a $\mathcal{L}$ multiplier to enforce the gauge condition $(\partial_\mu A^\mu)=0$ which removes 1 unphysical degree of freedom

Our new unphysical constant will cancel in any observable guaranteeing gauge invariance and gauge parameter (ξ) independence.

3) Free vectors

We have:

$$\mathcal{L}_{free} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

$$= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2\xi}(\partial_\mu A^\mu)(\partial_\nu A^\nu)$$

$$= -\frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial^\mu A^\nu + \frac{1}{2\xi}A^\mu(\partial_\mu \partial_\nu A^\nu) + \partial_\mu(\cdots)$$

$$= \frac{1}{2}A_\nu \partial^\mu \partial_\mu A^\nu - \frac{1}{2}A_\mu \partial_\nu \partial^\mu A^\nu + \frac{1}{2\xi}A^\mu(\partial_\mu \partial_\nu A^\nu) + \cdots$$

$$= \frac{1}{2}A^\rho\left(\partial^\mu\partial_\mu \eta_{\rho\nu}\delta^\nu_\sigma - \partial_\nu\partial^\mu \eta_{\mu\rho}\delta^\nu_\sigma + \frac{1}{\xi}\partial_\nu\partial^\mu\eta_{\mu\rho}\delta^\nu_\sigma\right)A^\sigma + \cdots$$

$$= \frac{1}{2}A^\rho\left[\partial^\mu\partial_\mu\eta_{\rho\sigma} - (1-\tfrac{1}{\xi})\partial_\sigma\partial_\rho\right]A^\sigma$$

From our discussion of the effective action we have:

$$\left.\frac{\partial^2\Gamma}{\partial A^\alpha \partial A^\beta}\right|_{A\to 0} = \frac{1}{(2\pi)^4}\delta^4(k_1+k_2)\, i\Gamma^{(2)\alpha\beta}(k_1,k_2)$$

Associating $\Gamma = iS = i\int d^4x\, \mathcal{L}_{free}$ we have:

$$i\int d^4x\, \mathcal{L}_{free} = i\int d^4x\, \frac{1}{2}A^\rho\left[\partial^\mu\partial_\mu\eta_{\rho\sigma} - (1-\tfrac{1}{\xi})\partial_\sigma\partial_\rho\right]A^\sigma$$

$$= i\int d^4x\, d^4y\, \frac{1}{2}A^\rho(y)\left[\partial^\mu_x\partial_{x\mu}\eta_{\rho\sigma} - (1-\tfrac{1}{\xi})\partial_{x\sigma}\partial_{x\rho}\right]A^\sigma\, \delta^4(x-y)$$

$$= i\int d^4x\, d^4y\, \delta^4(x-y)\int\frac{d^4p_1 d^4p_2}{(2\pi)^{2\cdot 4}}\,\frac{1}{2}e^{-ip_1\cdot x}\left[\partial^\mu_x\partial_{x\mu}\eta_{\rho\sigma} - (1-\tfrac{1}{\xi})\partial_{x\sigma}\partial_{x\rho}\right]e^{-ip_2\cdot x}A^\rho(p_1)A^\sigma(p_2)$$

$$= i\int d^4x\, d^4y\, \delta^4(x-y)\int\frac{d^4p_1 d^4p_2}{(2\pi)^{2\cdot 4}}\,\frac{1}{2}e^{-ip_1\cdot x}\left[-p_2^2\eta_{\rho\sigma} + (1-\tfrac{1}{\xi})p_{2\sigma}p_{2\rho}\right]e^{-ip_2\cdot x}A^\rho(p_1)A^\sigma(p_2)$$

$$= \frac{-i}{2}\int\frac{d^4p_1 d^4p_2}{(2\pi)^{2\cdot 4}}(2\pi)^4\delta^4(p_1+p_2)\left[p_2^2\eta_{\rho\sigma} - (1-\frac{1}{\xi})p_{2\sigma}p_{2\rho}\right]A^\rho(p_1)A^\sigma(p_2)$$

taking variations we have:

$$\frac{\partial^2 i S_{free}}{\partial A^\alpha(k_1)\,\partial A^\beta(k_2)} = \frac{-i}{2}\int \frac{d^4p_1 d^4p_2}{(2\pi)^4}\, \underbrace{\delta^4(p_1+p_2)}_{\text{Symm in } p_1 \leftrightarrow p_2} \underbrace{\left[p_2^2\eta_{\rho\sigma} - (1-\tfrac{1}{\xi})p_{2\sigma}p_{2\rho}\right]}_{\text{symm in } \rho\sigma} \Big(\eta^{\rho\alpha}\eta^{\sigma\beta}\delta^4(p_1-k_1)\delta^4(p_2-k_2)$$

$$+ \underbrace{\eta^{\rho\beta}\eta^{\sigma\alpha}\delta^4(p_1-k_2)\delta^4(p_2-k_2)}_{\text{so we can sum these two terms}}\Big)$$

$$= -i\left[k_1^2\eta^{\alpha\beta} - (1-\tfrac{1}{\xi})k_1^\alpha k_1^\beta\right]\delta^4(k_1+k_2)\frac{1}{(2\pi)^4}$$

So we conclude:

$$i\Gamma^{(2)\alpha\beta}(k_1) = -i\left[k_1^2\eta^{\alpha\beta} - (1-\tfrac{1}{\xi})k_1^\alpha k_1^\beta\right]$$

So our propagator is given by:

$$\underbrace{-i\left[k_1^2\eta^{\alpha\beta} - (1-\tfrac{1}{\xi})k_1^\alpha k_1^\beta\right]}_{\underbrace{-ik_1^2}_{A}\underbrace{\left[\eta^{\alpha\beta} - (1-\tfrac{1}{\xi})k_1^\alpha k_1^\beta/k^2\right]}_{B^{\alpha\beta}}} iG^{\alpha\beta}(k_1) = -1 \qquad *$$

$$A^{-1} = \frac{i}{k_1^2}$$

$$(B^{\mu\alpha})^{-1} = \left[\eta^{\mu\alpha} - (1-\xi)k_1^\mu k_1^\alpha/k^2\right]$$

$$(B^{\mu\nu})^{-1}\eta_{\nu\alpha}B^{\alpha\beta} = \left[\delta^\mu_\alpha - (1-\xi)k_1^\mu k_{1\alpha}/k^2\right]\left[\eta^{\alpha\beta} - (1-\tfrac{1}{\xi})k_1^\alpha k_1^\beta/k^2\right]$$

$$= \eta^{\mu\beta} \underbrace{- (1-\xi)\frac{k_1^\mu k_1^\beta}{k^2} - (1-\tfrac{1}{\xi})\frac{k_1^\mu k_1^\beta}{k^2} + (1-\xi)(1-\tfrac{1}{\xi})k_1^\mu k_1^\beta/k^2}_{-1-1+\xi+\frac{1}{\xi}+(1-\frac{1}{\xi}-\xi+1)=0}$$

$$= \eta^{\mu\beta}$$

this (-1) comes from RHS of *

We conclude $\quad iG^{\alpha\beta} = \frac{-i}{k^2}\left(\eta^{\alpha\beta} - (1-\xi)k^\alpha k^\beta/k^2\right)$

4) Quantum Electrodynamics as a gauge theory

Recall our free vector theory had a gauge symmetry:

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \to -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \text{for } A_\mu \to A_\mu + \partial_\mu \alpha(x)$$
$$\text{\& } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Consider a Dirac Field ψ that transforms under this same symmetry as:

$$\psi \to e^{-i\alpha(x)}$$

$$\bar{\psi} = \psi^\dagger \gamma_0 \to \bar{\psi} e^{+i\alpha(x)}$$

Then the mass term is invariant:

$$m\bar{\psi}\psi \to m\bar{\psi}\psi$$

But the derivative term isn't:

$$i\bar{\psi}\partial\!\!\!/\psi \to i\bar{\psi}e^{+i\alpha(x)}\gamma^\mu\partial_\mu\left[e^{-i\alpha(x)}\psi\right]$$
$$= i\bar{\psi}e^{+i\alpha(x)}\gamma^\mu\left[e^{-i\alpha(x)}\partial_\mu\psi - ie^{i\alpha(x)}(\partial_\mu\alpha(x))\psi\right]$$

But the $\partial_\mu\alpha(x)$ term is exactly how the vector transforms

So let's make the replacement: $\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu$

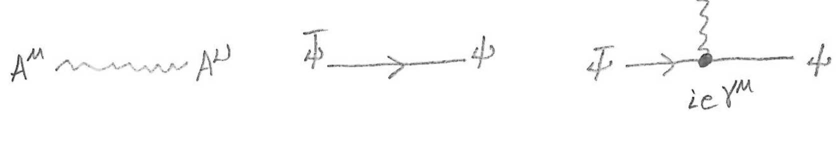
these cancel

$$i\bar{\psi}D\!\!\!\!/\,\psi \to i\bar{\psi}e^{+i\alpha(x)}\gamma^\mu\left[e^{-i\alpha(x)}\partial_\mu\psi - ie^{i\alpha(x)}(\partial_\mu\alpha(x))\psi + ie\underbrace{(A_\mu + \partial_\mu\alpha(x))}_{A_\mu \to A_\mu + \partial_\mu\alpha}\psi\right]$$
$$= i\bar{\psi}D\!\!\!\!/\,\psi$$

So we arrive at the $\mathcal{L}$ for QED:

$$\mathcal{L}_{QED} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(iD\!\!\!\!/ - m)\psi$$

In addition to the photon (A^μ) and electron (ψ) propagators we have an interaction between $A\bar{\psi}\psi$:



5) Making predictions in QED

To calculate amplitudes we need the LSZ reduction for spinors & vectors

Spinors:

$$\langle \beta\ \mathrm{out} | \alpha\ \mathrm{in} \rangle = \int \prod_i d^4x_i \left(i e^{-ip_i x_i}\right) \Big[(i\not{\partial} + m) \underset{\substack{\uparrow \\ \text{onshell spinor} \\ \text{for in particles}}}{u_i} \Big]_{\alpha_i} \prod_l d^4y_l \, i e^{ip_l x_l} \Big[\underset{\substack{\uparrow \\ \text{onshell spinor} \\ \text{for out particles}}}{\bar{u}_l} (-i\not{\partial} + m) \Big]_\beta \langle 0 | T\{ \psi_\beta \cdots \bar{\psi}_\alpha \cdots \} | 0 \rangle$$

this is the same as for scalars w/ the addition of onshell spinors (1 for scalars)

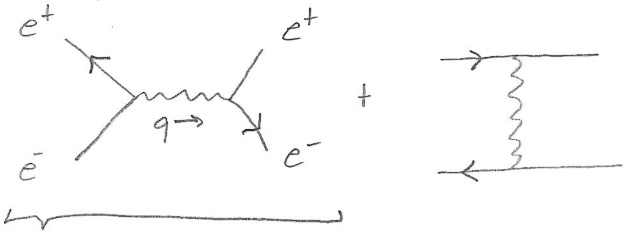
$u \to$ incoming electron

$\bar{u} \to$ outgoing electron

analogously we also have

$v \to$ outgoing positron
$\bar{v} \to$ incoming positron
} notice the role of the bar is swapped

So the amplitude for $e^+e^- \to e^+e^-$



$$i\mathcal{M} = \bar{v}_a (ie\gamma^\mu_{ab}) u_b \underbrace{iG^A_{\mu\nu}}_{\frac{-i}{q^2+i\epsilon}\left(\eta_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right)} \bar{u}_c (ie\gamma^\nu_{cd}) v_d$$

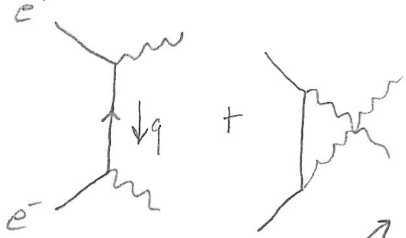
For vectors (photons) LSZ reads:

$$\langle \beta\, out | \alpha\, in \rangle = \int \prod_j d^4x_j \left(i e^{-ip_j x_j}\right)\left(\epsilon_j^{\mu_j}\right)^* (-\partial^2)_{j\beta} \prod_\ell d^4 y_\ell \left(+i e^{i p_\ell x_\ell}\right)\left(\epsilon_\ell^{\mu_\ell}\right)(-\partial^2)_{\ell\alpha} \langle 0 | T\{A \cdots A\} | 0 \rangle$$

onshell pol. vectors

Now we've introduced onshell polarization vectors for external photon lines

So the amplitude for $e^+e^- \to \gamma\gamma$:



$$\frac{i(-\not{q}-m)}{q^2-m^2}$$

$$i\mathcal{M} = \bar{v}_a (ie\gamma^\mu_{ab})\, i G^F_{bc} (ie\gamma^\nu_{cd}) u_d\, \epsilon^*_\mu \epsilon^*_\nu$$

Some important points:

incoming $\bar{v} \times \leftarrow$
$u \times \rightarrow$

outgoing $v \times \rightarrow$
$\bar{u} \times \leftarrow$

} arrow points toward barred, away from unbarred

mtm for fermionic propagator goes w/ the arrow

→ my choice of momentum assignment is opposite that → $-\not{q}$

Look at charge assignment:

← is this e^+ or e^-? neither, the prop connects +−: $\langle 0 | T\{\psi \bar{\psi}\} | 0 \rangle$

Explicit forms for the spinors can be derived from the classical EOM:

$$\mathcal{L}_{free} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})}$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \underbrace{(i\partial\!\!\!/ - m)\psi}_{\text{Dirac Eqn}} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = \partial_\mu(0) = 0$$

And similarly for vectors:

$$\Box A_\mu - \partial_\mu \partial_\nu A_\nu = 0$$

We'll skip this to save time (See Schwartz 8.2.3 for Vectors
11.2 for Spinors)

We'll instead make use of the following identities:

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = p\!\!\!/ + m$$

$$\sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = p\!\!\!/ - m$$

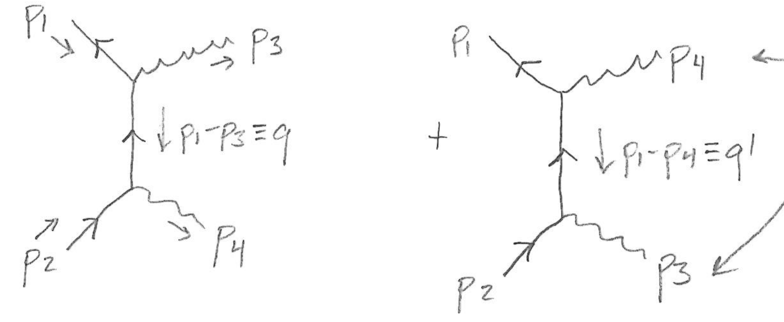
$$\sum_{\lambda} (\epsilon_\lambda^\mu)^* \epsilon_\lambda^\nu = -g_{\mu\nu}$$ (techically there are gauge dep parts to this but they vanish if we sum all diagrams for a given process)

The Dirac Eqn also tells us:

$$(p\!\!\!/ - m)u_s(p) = \bar{u}_s(p)(p\!\!\!/ - m) = 0$$

$$(p\!\!\!/ + m)v_s(p) = \bar{v}_s(p)(p\!\!\!/ + m) = 0$$

So let's calculate $e^+e^- \to \gamma\gamma$ at tree level:



(1) (2)

We don't know which vertex the γ come from as they are indistinguishable

$$i\mathcal{M}_1 = \bar{V}_a(p_1)\left(ie\gamma^\mu_{ab}\right)\frac{i(-\not q+m)_{bc}}{q^2-m^2}\left(ie\gamma^\nu_{cd}\right)U_d(p_2)\epsilon^*_\mu(p_3)\epsilon^*_\nu(p_4)$$

↑ spinor index

$$= -ie^2\left[\bar{V}_1\gamma^\mu(-\not q+m)\gamma^\nu U_2\right]\epsilon^*_{\mu 3}\epsilon^*_{\nu 4}\frac{1}{q^2-m^2}$$ ← shorthand

$$(i\mathcal{M}_1)^\dagger = +ie^2\left[U_2^\dagger\gamma^{\beta\dagger}(-\not q^\dagger+m)\gamma^{\alpha\dagger}\underset{\gamma_0^\dagger=\gamma_0}{\gamma_0^\dagger}V_1\right]\epsilon_{\alpha 3}\epsilon_{\beta 4}\frac{1}{q^2-m^2}$$

$$= ie^2\left[U_2^\dagger\gamma_0^2\gamma^{\beta\dagger}\gamma_0^2(-\not q^\dagger\gamma_0\gamma_0+m)\gamma_0\underbrace{\gamma_0\gamma^{\alpha\dagger}\gamma_0}_{\gamma_0\gamma^{\mu\dagger}\gamma_0=\gamma^\mu}V_1\right]\epsilon_{\alpha 3}\epsilon_{\beta 4}\frac{1}{q^2-m^2}$$

$$= ie^2\left[\bar{U}_2\gamma^\beta(-\not q+m)\gamma^\alpha V_1\right]\epsilon_{\alpha 3}\epsilon_{\beta 4}\frac{1}{q^2-m^2}$$

Generally we don't know the incoming chiralities → average over spins
outgoing polarization → sum over pol.

$$\left(\frac{1}{2}\right)^2\sum_s\sum_\lambda(i\mathcal{M}_1)(i\mathcal{M}_1)^\dagger = \frac{1}{4}\sum_s\sum_\lambda e^4\left[\bar{V}_{1a}\gamma^\mu_{ab}(-\not q+m)_{bc}\gamma^\nu_{cd}U_{2d}\right]\left[\bar{U}_{2e}\gamma^\beta_{ef}(-\not q+m)_{fg}\gamma^\alpha_{gh}V_{1h}\right]\epsilon^*_{\mu 3}\epsilon^*_{\nu 4}\epsilon_{\alpha 3}\epsilon_{\beta 4}\left(\frac{1}{q^2-m^2}\right)^2$$

↑ average over 2 chiralities x 2 spinors

$$= \frac{e^4}{4}\underbrace{(\not p_1-m)_{ha}\gamma^\mu_{ab}(-\not q+m)_{bc}\gamma^\nu_{cd}(\not p_2+m)_{de}\gamma^\beta_{ef}(-\not q+m)_{fg}\gamma^\alpha_{gh}}_{\text{this is a trace!}}g_{\mu\alpha}g_{\nu\beta}\left(\frac{1}{q^2-m^2}\right)^2$$

$$= \frac{e^4}{4}\mathrm{Tr}\left[(\not p_1-m)\gamma^\mu(-\not q+m)\gamma^\nu(\not p_2+m)\gamma_\nu(-\not q+m)\gamma_\mu\right]\left(\frac{1}{q^2-m^2}\right)^2$$

Recall $\gamma_\mu \gamma^\nu \gamma^\mu = -2\gamma^\nu$ & $\gamma^\mu \gamma_\mu = 4$

Then $\gamma_\mu (\not p \pm m)\gamma^\mu = -(2\not p \mp 4m)$

And that Traces are cyclic: $Tr[ab \cdots de] = Tr[eab \cdots d]$

Then we have:

$$\left(\frac{1}{2}\right)^2 \sum_{s,\lambda} |M_1|^2 = \frac{e^4}{4} Tr[\underbrace{(2\not p_1 + 4m)(-\not q + m)(2\not p_2 - 4m)(-\not q + m)}_{A}]\left(\frac{1}{q^2 - m^2}\right)^2$$

$$A = (-2\not p_1 \not q + 2\not p_1 m - 4m\not q + 4m^2)(-2\not p_2 \not q + 4m\not q - 4m^2 + 2\not p_2 m)$$

$$= 4\not p_1 \not q \not p_2 \not q - \underline{8m\not p_1 \not q \not q} + 8m^2 \not p_1 \not q - \underline{4m \not p_1 \not q \not p_2}$$

$$-\underline{4m\not p_1 \not p_2 \not q} + 8m^2 \not p_1 \not q - \underline{8m^3 \not p_1} + 4m^2 \not p_1 \not p_2$$

$$+\underline{8m\not q \not p_2 \not q} - 16m^2 \not q \not q + \underline{16m^3 \not q} - 8m^2 \not q \not p_2$$

$$-8m^2 \not p_2 \not q + \underline{16m^3 \not q} - 16m^4 + \underline{8m^3 \not p_2}$$

$\underline{\quad}$: $Tr[\text{odd \# } \gamma\text{'s}] = 0$

$$\rightarrow 4\not p_1 \not q \not p_2 \not q + m^2(16 \not p_1 \not q + 4\not p_1 \not p_2 - 16\not q \not q - 8\not q \not p_2 - 8\not p_2 \not q) - 16m^4 Tr[\mathbb{1}]$$

$$Tr[A] = 16(p_1\cdot q\, p_2\cdot q - p_1\cdot p_2\, q\cdot q + p_1\cdot q\, p_2\cdot q) + 4m^2(8\, p_1\cdot q + 4p_1\cdot p_2 - 16 q\cdot p_2) - 64m^4$$

$$q = p_1 - p_3$$

$$= 16(p_1^2\, p_1\cdot p_2 - 2p_1^2\, p_2\cdot p_3 + 2\, p_1\cdot p_3\, p_2\cdot p_3 - p_1\cdot p_2\, p_3^2)$$

$$+ 4m^2(-12 p_1\cdot p_2 + 16 p_1\cdot p_3 + 16\, p_2\cdot p_3 - 16 p_3^2)$$

$$- 64m^4$$

but $p_1^2 = p_2^2 = m^2$ $\quad p_3^2 = 0$ (photons are massless)

$$= 32(p_1\cdot p_3\, p_2\cdot p_3 + m^2 p_2\cdot p_3 + 2m^2 p_1\cdot p_3 - p_1\cdot p_2 - 2m^2)$$

We can make use of Mandelstam variables

$$p_1 \cdot p_2 = \frac{1}{2}(S - m_1^2 - m_2^2) = \frac{1}{2}(S - 2m^2) \qquad p_3 \cdot p_4 = \frac{1}{2}S$$

$$p_1 \cdot p_3 = \frac{-1}{2}(T - m_1^2 - m_3^2) = -\frac{1}{2}(T - m^2) \qquad p_2 \cdot p_4 = -\frac{1}{2}(T - m^2)$$

$$p_1 \cdot p_4 = -\frac{1}{2}(U - m_1^2 - m_4^2) = -\frac{1}{2}(U - m^2) \qquad p_2 \cdot p_3 = -\frac{1}{2}(U - m^2)$$

$$\mathrm{Tr}[A] = 8TU - 8(2S + 5T + 3U)m^2 + 24m^4$$
$$= 8\left[TU - (3T+U)m^2 - m^4\right] \qquad (S = 2m^2 - T - U)$$

$$\frac{1}{4}\sum_{s,\lambda} |m_1|^2 = 2e^4\left[TU - (3T+U)m^2 - m^4\right]\left(\frac{1}{T - m^2}\right)^2$$

For the 2^{nd} diagram:

$$i\mathcal{M}_2 = -ie^2[\bar{v}_1\gamma^\mu(-\not{q}'+m)\gamma^\nu u_2]\epsilon^*_{\mu 4}\epsilon^*_{\nu 3}\frac{1}{(q')^2-m^2}$$

$$(i\mathcal{M}_2)^\dagger = ie^2[\bar{u}_2\gamma^\beta(-\not{q}'+m)\gamma^\alpha v_1]\epsilon_{\alpha 4}\epsilon_{\beta 3}\frac{1}{(q')^2-m^2}$$

This just trades $q \leftrightarrow q'$, or $p_3 \leftrightarrow p_4$, or $U \leftrightarrow T$

So $$\frac{1}{4}\sum_{s,\lambda}|\mathcal{M}_2|^2 = 2e^4[TU-(3U+T)m^2-m^4]\left(\frac{1}{U-m^2}\right)^2$$

But nature is Quantum so our diagrams interfere:

$$(i\mathcal{M}_1)^\dagger(i\mathcal{M}_2)+(i\mathcal{M}_2)^\dagger(i\mathcal{M}_1)$$
$$= 2\,\mathrm{Re}[\mathcal{M}_1^\dagger\mathcal{M}_2]$$

$$\frac{1}{4}\sum_{s,\lambda}2\,\mathrm{Re}[\mathcal{M}_1^\dagger\mathcal{M}_2] = \frac{1}{4}\sum_{s,\lambda}2e^4[\bar{u}_2\gamma^\beta(-\not{q}+m)\gamma^\alpha v_1][\bar{v}_1\gamma^\mu(-\not{q}'+m)\gamma^\nu u_2]\epsilon_{\alpha 3}\epsilon_{\beta 4}\epsilon^*_{\mu 4}\epsilon^*_{\nu 3}$$
$$\times\left(\frac{1}{q^2-m^2}\right)\left(\frac{1}{(q')^2-m^2}\right)$$

$$= \frac{1}{2}e^4\,\mathrm{Tr}[(\not{p}_1-m)\gamma^\mu(-\not{q}'+m)\gamma^\nu(\not{p}_2+m)\gamma_\mu(-\not{q}+m)\gamma_\nu]\left(\frac{1}{q^2-m^2}\right)\left(\frac{1}{(q')^2-m^2}\right)$$

$$= 32\,p_1\cdot p_2(p_1\cdot p_3+p_1\cdot p_4-p_3\cdot p_4)$$
$$+16(p_1\cdot p_2+2p_1\cdot p_3+2p_1\cdot p_4-p_2\cdot p_3-p_2\cdot p_4-p_3\cdot p_4)m^2$$
$$-16m^4$$
$$= 8m^2(s-4m^2)$$

So we have

$$|\mathcal{M}_1+\mathcal{M}_2|^2 = 2e^4\left[\frac{TU-(3T+U)m^2-m^4}{(T-m^2)^2}+\frac{TU-(3U+T)m^2-m^4}{(U-m^2)^2}+\frac{4m^2(s-4m^2)}{(T-m^2)(U-m^2)}\right]$$

For massless e^-, e^+ we have

$$|\mathcal{M}_1 + \mathcal{M}_2|^2 = 2e^2\left(\frac{T}{U} + \frac{U}{T}\right)$$

Taking $\quad T = -\frac{s}{2}(1-\cos\theta)$

$\quad\quad\quad\;\; U = -\frac{s}{2}(1+\cos\theta)$

We have:

$$|\mathcal{M}_1 + \mathcal{M}_2|^2 = \frac{4e^2(1+\cos^2\theta)}{1-\cos^2\theta}$$

The cross section is given by:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{1}{2}|\mathcal{M}_1+\mathcal{M}_2|^2 \quad \leftarrow \tfrac{1}{2} \text{ for identical particles}$$

$$= \frac{e^2}{32\pi^2 s}\frac{(1+\cos^2\theta)}{(1-\cos^2\theta)}$$

$$= \frac{e^2}{32\pi^2 s}(2\pi)\underbrace{\int_{-1}^{1} d\cos\theta \frac{(1+\cos^2\theta)}{(1-\cos^2\theta)}}_{\text{This is ill defined} \to \text{collinear divergence}}$$

this is regulated by m_e

but hints at a problem w/ defining single particle states in QFT

e.g. what's the difference between

1 photon & 2 collinear photons?

or 1 photon and 1 soft ($E_\gamma \to 0$) photon?

Restoring the mass gives:

$$\frac{1}{[s(1-\cos\theta)+m^2(1+\cos\theta)]^2} \times \frac{1}{[s(1+\cos\theta)+m^2(1-\cos\theta)]^2}$$

In the denominator and we see the problem goes away.

Integrating the full expression:

$$\sigma(s) = \frac{e^4}{4\pi}\left[2(s^2+6m^2s-16m^4)\operatorname{arctanh}\left(\sqrt{1-\frac{4m^2}{s}}\right)-(s+4m^2)\sqrt{s(s-4m^2)}\right]\frac{1}{s^2\sqrt{s(s-4m^2)}}$$