Path Integrals

1) In QM: (Schwartz 14.2)

consider the Hamiltonian:

$$\hat{H}(t) = \frac{\hat{p}^2}{2m} + V(\hat{x},t)$$

w/ initial and final states:

$$|i\rangle = |x_i\rangle$$
$$|f\rangle = |x_f\rangle$$

For $\hat{H}$ indep of t, the projection of $|i\rangle$ onto $|f\rangle$ is just:

$$\langle f|i\rangle = \langle x_f|e^{-i(t_f - t_i)\hat{H}}|x_i\rangle$$

So to deal w/ $\hat{H}(t)$ we break t into n intervals approx. constant in t:

$$t_j = t_i + j\delta t$$
$$\text{w/ } t_n = t_f$$

Then we evolve $|i\rangle$ step-by-step to $|f\rangle$:

$$\langle f|i\rangle = \int dx_n \cdots dx_1 \, \langle x_f|e^{-iH(t_f)}|x_n\rangle\langle x_n|\cdots|x_2\rangle\langle x_2|e^{-iH(t_2)\delta t}|x_1\rangle\langle x_1|e^{-iH(t_1)\delta t}|x_i\rangle$$

Each step can be simplified by inserting a complete set of mta eigenstates:

$$\langle x_{j+1}|e^{-iH\delta t}|x_j\rangle = \int \frac{dp}{2\pi} \underbrace{\langle x_{j+1}|p\rangle}_{=e^{ipx_{j+1}}}\langle p|e^{-i[\frac{\hat{p}^2}{2m} + V(\hat{x}_j,t_j)]\delta t}|x_j\rangle \qquad \hat{p}|p\rangle = p \qquad \hat{x}|x\rangle = x$$

$$= \int \frac{dp}{2\pi} e^{ipx_{j+1}} e^{-i\frac{p^2}{2m}\delta t} e^{-iV(x_j,t_j)\delta t} \langle p|x_j\rangle$$

$$= e^{-iV(x_j,t_j)\delta t} \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\delta t} e^{ip(x_{j+1} - x_j)}$$

Using the Gaussian integral result:

$$\int dp\, e^{-\frac{1}{2}ap^2 + Jp} = \sqrt{\frac{2\pi}{a}}\, e^{J^2/2a}$$

We identify: $a = i\frac{\delta t}{m}$

$J = i(x_{j+1} - x_j)$

$$\Rightarrow \langle x_{j+1}| e^{-iH\delta t} |x_j\rangle = e^{-iV(x_j,t_j)\delta t} \sqrt{\frac{m}{2\pi i \delta t}}\, e^{\frac{-(x_{j+1}-x_j)^2 m}{2i\delta t}}$$

$$\equiv \underset{\substack{\uparrow \\ x \,\&\, t\text{-indep}}}{N}\, e^{-iV(x_j,t_j)\delta t}\, e^{\frac{im}{2}\delta t \underbrace{\frac{(x_{j+1}-x_j)^2}{(\delta t)^2}}_{=\dot{x}^2}}$$

$$= N e^{iL(x,\dot{x})\delta t}$$

w/ $L(x,\dot{x}) = \frac{m}{2}\dot{x}^2 - V(x,t)$

Then returning to the original calculation:

$$\langle f|i\rangle = N^n \int dx_n \cdots dx_1\, e^{iL(x_n,\dot{x}_n)\delta t} \cdots e^{iL(x_1,\dot{x}_1)\delta t}$$

In the limit $\delta t \to 0$ we obtain an itegral over $\delta t \to dt$

$$\langle f|i\rangle = \bar{N} \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\, e^{i\int dt\, L(x(t),\dot{x}(t))}$$

formally ∞, but drops out of physical quantities

integrate over <u>all paths</u> $x(t)$

Srednicki 6 & 7 goes into more detail and discusses some nuances we've skipped over.

2) In QFT (Schwartz 14.2)

In analogy w/ QM we start w/

$$\hat{x},\ |x\rangle \leftrightarrow \hat{\varphi},\ |\varphi\rangle$$

$$\hat{p},\ |p\rangle \leftrightarrow \hat{\pi},\ |\pi\rangle$$

$$[\hat{x},\hat{p}]=i \leftrightarrow [\hat{\varphi}(x),\hat{\pi}(y)]=i\delta^3(\vec{x}-\vec{y})$$

continuum analogue, exists also in QM

We have associated eigenvalues:

$$\hat{\varphi}(\vec{x})|\varphi\rangle=\varphi(\vec{x})|\varphi\rangle$$
$$\hat{\pi}(\vec{x})|\pi\rangle=\pi(\vec{x})|\pi\rangle$$

Similar to $\langle\vec{p}|\hat{\vec{x}}\rangle=e^{-i\vec{p}\cdot\vec{x}}$,

$$\langle\pi|\varphi\rangle=e^{-i\int d^3x\,\pi(\vec{x})\varphi(\vec{x})}$$

And analogous w/ $\langle\vec{x}'|x\rangle=\delta(\vec{x}-\vec{x}')=\int\frac{dp}{2\pi}e^{-i\vec{p}\cdot(\vec{x}-\vec{x}')}$ we have:

$$\langle\varphi'|\varphi\rangle=\int D\pi\langle\varphi'|\pi\rangle\langle\pi|\varphi\rangle=\int D\pi\, e^{-i\int d^3x\,\pi(\vec{x})[\varphi(\vec{x})-\varphi'(\vec{x})]}$$

This is quite abstract, Schwartz problem 14.4 can be followed to construct these states explicitly.

In advanced particles it was shown:

$$\hat{H} = \frac{1}{2}\hat{\pi}^2 + \frac{1}{2}(\nabla\hat{\varphi})^2 + \frac{1}{2}m^2\hat{\varphi}^2$$

We'll take the more general:

$$\hat{H} = \frac{1}{2}\hat{\pi}^2 + V(\hat{\varphi})$$

And calculate:

$$\langle \Omega; t_f | \Omega; t_i \rangle = \langle \Omega | \Omega \rangle$$

$$= \int D\varphi_1(x) \cdots D\varphi_n(x) \langle \Omega | e^{-i\delta t \hat{H}(t_n)} | \varphi_n \rangle \langle \varphi_n | \cdots | \varphi_1 \rangle \langle \varphi_1 | e^{-i\delta t \hat{H}(t_0)} | \Omega \rangle$$

Inspecting a particular piece:

$$\langle \varphi_{j+1} | e^{-i\delta t \hat{H}(t_j)} | \varphi_j \rangle = \int D\pi_j \langle \varphi_{j+1} | \pi_j \rangle \langle \pi_j | e^{-i\delta t \int d^3x \left(\frac{1}{2}\hat{\pi}^2 + V(\hat{\varphi})\right)} | \varphi_j \rangle$$

$$\langle \pi_j | \hat{\pi} = \langle \pi_j | \pi_j \qquad \hat{\varphi} | \varphi_j \rangle = \varphi_j | \varphi_j \rangle$$

$$= \int D\pi_j \langle \varphi_{j+1} | \pi_j \rangle \langle \pi_j | \varphi_j \rangle e^{-i\delta t \int d^3x \left(\frac{1}{2}\pi_j^2 + V(\varphi_j)\right)}$$

$$= \int D\pi_j \, e^{-i\int d^3x (\pi_j \varphi_j - \pi_j \varphi_{j+1})} e^{-i\delta t \int d^3x \left(\frac{1}{2}\pi_j^2 + V(\varphi_j)\right)}$$

$$= e^{-i\delta t \int d^3x V(\varphi_j)} \int D\pi_j \exp\left[\int d^3x \left(-i\underbrace{\frac{\delta t}{2}}_{a = i\delta t}\pi_j^2 + \underbrace{i\pi_j(\varphi_{j+1} - \varphi_j)}_{J = i(\varphi_{j+1} - \varphi_j)}\right)\right]$$

$$= e^{-i\delta t \int d^3x V(\varphi_j)} \sqrt{\frac{2\pi}{i\delta t}} \exp\left[\int d^3x \underbrace{\frac{-(\varphi_{j+1} - \varphi_j)^2}{2i\delta t}}_{\frac{i}{2}\delta t (\partial_t \varphi_j)^2}\right]$$

$$= N \exp\left[i\delta t \int d^3x \underbrace{\left(\frac{1}{2}(\partial_t \varphi_j)^2 - V(\varphi_j)\right)}_{\equiv \mathcal{L}[\varphi_j, \partial_t \varphi_j]}\right]$$

Then we have:

$$\langle\Omega|\Omega\rangle = \int D\varphi_1 \cdots D\varphi_n \prod_{j=1}^{n} N \exp\left[i\delta t \int d^3x \left(\tfrac{1}{2}(\partial_t \varphi_j)^2 - V(\varphi_j)\right)\right]$$

$$= \bar{N} \int D\varphi(\vec{x},t)\, e^{iS[\varphi]} \qquad \text{in the continuum limit}$$

↑ integral over all classical field configurations

Now consider a similar integral:

$$I = N \int D\varphi\, e^{iS} \varphi(\vec{x}_i, t_i)$$

$$= N \int D\varphi_1 \cdots D\varphi_n \langle\Omega| e^{-iH(t_n)\delta t} |\varphi_n\rangle \cdots \langle\varphi_2| e^{-iH(t_2)\delta t} |\varphi_1\rangle\langle\varphi_1| e^{-iH(t_1)\delta t} |\Omega\rangle \varphi_i(\vec{x}_i)$$

↑ i implicit @ t_i

φ_i is just a classical field, we want to replace it w/ an operator $\hat{\varphi}_i$

recalling $\hat{\varphi}_i |\varphi_i\rangle = \varphi_i |\varphi_i\rangle$

Then $\int D\varphi_i(\vec{x}) [e^{-iH(t_i)\delta t} |\varphi_i\rangle \varphi_i \langle\varphi_i| = \int D\varphi_i\, \hat{\varphi}_i\, e^{-iH(t_i)\delta t} |\varphi_i\rangle\langle\varphi_i|$

So we can essentially pair φ_i w/ $|\varphi_i\rangle$, rewrite it as $\hat{\varphi}_i$, and then regroup our integrals:

$$I = \int D\varphi_1 \cdots D\varphi_n \langle\Omega| e^{-iH(t_n)\delta t} |\varphi_n\rangle\langle\varphi_n| \cdots \hat{\varphi}_i |\varphi_i\rangle\langle\varphi_i| \cdots |\varphi_1\rangle\langle\varphi_1| e^{-iH(t_1)\delta t} |\Omega\rangle$$

$$= \langle\Omega|\hat{\varphi}(\vec{x}_i, t_i)|\Omega\rangle$$

If we consider two fields:

$$\int D\varphi\, e^{iS} \varphi(\vec{x}_1, t_1)\varphi(\vec{x}_2, t_2)$$

The same process happens, but since our complete set of states goes $t_1 \to t_2 \to \cdots \to t_n$, φ_1 & φ_2 will end up time ordered:

$$N \int D\varphi\, e^{iS} \varphi(\vec{x}_1, t_1)\varphi(\vec{x}_2, t_2) = \langle\Omega| T\{\hat{\varphi}(x_1)\hat{\varphi}(x_2)\}|\Omega\rangle$$

Recall from Particle Physics (or see Schwartz 6.1) the LSZ reduction formula relates S-matrix (scattering) amplitudes to t-ordered products:

$$\langle p_3 \cdots p_n | S | p_1 p_2\rangle = \left[\prod_{i=1}^{n} i\int d^4x_i\, e^{-ip_i x_i}(\Box_i + m^2)\right] \langle\Omega| T\{\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)\}|\Omega\rangle$$

So we want to obtain these t-ordered products from the Path Integral, w/ some goals:

1) manifest Lorentz-invariance
2) gauge invariance (e.g. Quantum Electrodynamics, but soon Yang Mills)

3) Generating Functionals (Schwartz 14.3)

consider:

$$Z[J] \equiv \int D\varphi \exp\left[iS[\varphi] + i\int d^4x\, J(x)\varphi(x)\right]$$

$$Z[0] = \langle\Omega|\Omega\rangle = \int D\varphi\, e^{iS[\varphi]}$$

We introduce the variational derivative:

$$\frac{\partial J(x)}{\partial J(y)} = \delta^4(x-y)$$

So:

$$\frac{\partial}{\partial J(x_1)}\int d^4x\, J(x)\varphi(x) = \int d^4x\, \delta^4(x-x_1)\varphi(x) = \varphi(x_1)$$

Then

$$\left.\frac{-i\partial Z}{\partial J(x_1)}\right|_{J=0} = \int D\varphi \left(-i\frac{\partial}{\partial J(x_1)}\right)\exp\left[iS[\varphi] + i\int d^4x\, J(x)\varphi(x)\right]\Big|_{J=0}$$

$$= \int D\varphi \exp\left[iS[\varphi] + i\int d^4x\, J(x)\varphi(x)\right]\left[(-i)i\int d^4x\, \delta^4(x-x_1)\varphi(x)\right]\Big|_{J=0}$$

$$= \int D\varphi \exp\left[iS[\varphi] + i\int d^4x\, J(x)\varphi(x)\right]\varphi(x_1)\Big|_{J=0}$$

$$= \langle\Omega|\hat{\varphi}(x_1)|\Omega\rangle$$

In this way we obtain t-ordered products:

$$\left[\prod_{i=1}^{n}\left(-i\frac{\partial}{\partial J(x_i)}\right)\right]Z\Big|_{J=0} = \langle\Omega|T\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\}|\Omega\rangle$$

Which we need for scattering amplitudes

3.1) Free fields

consider a free scalar:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{m^2}{2}\varphi^2$$

$$= -\frac{1}{2}\varphi(\Box + m^2)\varphi + \underbrace{\partial_\mu(\text{stuff})}$$

we assume fields drop off sufficiently fast at ∞ so surface terms can be ignored
→ overall ∂'s don't contribute

Then:

$$Z[J] = \int D\varphi \exp\left[i\int d^4x\left(-\frac{1}{2}\varphi(\Box + m^2)\varphi + J(x)\varphi(x)\right)\right]$$

This is exactly solvable as it is quadratic in fields, recall:

$$\int_{-\infty}^{\infty} d\vec{p}\, e^{-\frac{1}{2}p_i A_{ij} p_j + J_i p_i} = \sqrt{\frac{(2\pi)^n}{\det A}}\, e^{\frac{1}{2}J_i A^{-1}_{ij} J_j}$$

Here: $A = i(\Box + m^2)$

A^{-1} is then the solution to

$$\underbrace{i(\Box + m^2)}_{A}\underbrace{iG(x-y)}_{A^{-1}} = -\delta(x-y)$$

From advanced particles (also inspection) the solution can be seen to be the scalar propagator:

$$iG(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m^2} e^{ip\cdot(x-y)}$$

← notice $p^2 \neq m^2$ generally so propagators can be "off-shell" or off the mass shell

note: this is symmetric in $x \leftrightarrow y$, though it's not obvious

$(x-y) \to -(y-x)$, but $p \in (-\infty, \infty)$ so we can simultaneously take $p \leftrightarrow -p$

So we arrive at:

$$Z[J] = N\exp\left[-i\int d^4x\, d^4y\, \tfrac{1}{2} J(x) G(x-y) J(y)\right]$$

And therefore:

$$\langle 0|T\{\varphi(x_1)\varphi(x_2)\}|0\rangle \sim (-i)^2 \frac{\partial^2 Z[J]}{\partial J(x_1)\,\partial J(x_2)}\bigg|_{J=0}$$

For free fields we label our vacuum $|0\rangle$

$$= (-i)^2 \frac{\partial}{\partial J(x_1)} Z[J] \frac{-i}{2} \underbrace{\int d^4x\, d^4y\, G(x-y)\big(\delta(x-x_2)J(y) + J(x)\delta(y-x_2)\big)}_{\text{use symmetry in } x,y \text{ to combine}}\bigg|_{J=0}$$

$$= (-i)^2 \frac{\partial}{\partial J(x_1)} Z[J](-i)\int d^4x\, d^4y\, G(x-y) J(y)\delta(x-x_2)\Big|_{J=0}$$

$$= (-i)^2 Z[J]\Big((-i)^2\int d^4x\, d^4y\, d^4x'\, d^4y'\, G(x-y)J(y)\delta(x-x_2)G(x'-y')J(y)\delta(x'-x_1)$$

$$-i\int d^4x\, d^4y\, G(x-y)\,\delta(x-x_2)\delta(y-x_1)\Big)\Big|_{J=0}$$

$$= iG(x_1-x_2)\, Z[0]$$

this is suspect

So our t-ordered product is more like:

$$\frac{1}{Z[0]}\left(\prod_{i=1}^{n} \frac{1}{i}\frac{\partial}{\partial J(x_i)}\right) Z[J]\bigg|_{J=0} = \langle 0|T\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\}|0\rangle$$

This conveniently cancels the infinite constant "N"

For 4-pt let's introduce

$$\frac{\partial}{\partial J_i} \equiv \frac{\partial}{\partial J(x_i)}$$

$$G_{ij} = G(x_i - x_j) = G(x_j - x_i)$$

Then, letting the integrals be implied:

$$\frac{\partial^4}{\partial J_1 \partial J_2 \partial J_3 \partial J_4} e^{-\frac{1}{2} J_x G_{xy} J_y}\Big|_{J=0}$$

$$= \frac{\partial^3}{\partial J_1 \partial J_2 \partial J_3} (-J_z G_{z4}) e^{-\frac{1}{2} J_x G_{xy} J_y}\Big|_{J=0}$$

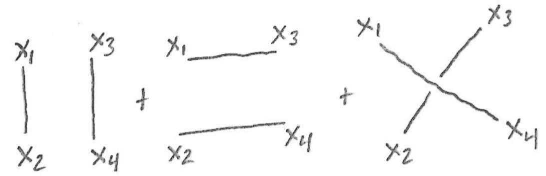
$$= \frac{\partial^2}{\partial J_1 \partial J_2} (-G_{34} + J_z G_{z4} J_w G_{w3}) e^{-\frac{1}{2} J_x G_{xy} J_y}\Big|_{J=0}$$

$$= \frac{\partial}{\partial J_1} \Big(G_{34} J_z G_{z2} + G_{24} J_w G_{w3} + J_z G_{z4} G_{23} \underbrace{- J_z G_{z4} J_w G_{w3} J_r G_{r2}}_{\substack{\text{not enough } \partial_J\text{'s to remove} \\ \text{all } J\text{'s so will vanish when} \\ J \to 0}} \Big) e^{-\frac{1}{2} J_x G_{xy} J_y}\Big|_{J=0}$$

$$= (G_{34} G_{12} + G_{24} G_{13} + G_{14} G_{23})$$

Diagramatically we have:

$\langle 0 | T\{\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3, \hat{\varphi}_4\} | 0 \rangle =$



3.2) Interacting fields (Sterman 3.4)

Consider φ^3 theory:

$$\mathcal{L} = -\frac{1}{2}\varphi(\Box + m^2)\varphi + \frac{g}{3!}\varphi^3$$

Then:

$$Z[J] = \int D\varphi \exp\left[i\int d^4x \left(-\frac{1}{2}\varphi(\Box+m^2)\varphi + J\varphi + \frac{g}{3!}\varphi^3\right)\right]$$

$$= N\int D\varphi \exp\left[i\int d^4x \frac{g}{3!}\varphi^3\right] \exp\left[-i\int d^4x d^4y \frac{1}{2} J(x) G(x-y) J(y)\right]$$

We will consider the interactions as perturbations about the free thry:

$$\frac{g}{3!}\varphi(x)^3 \rightarrow \frac{g}{3!}\left(\frac{\delta}{\delta J(x)}\right)^3 \equiv \frac{g}{3!}(\delta_{Jx})^3$$

↑ at the same space-time pt

$$\exp\left[i\int d^4x \frac{g}{3!}(\delta_{Jx})^3\right] \sim 1 + i\int d^4x \frac{g}{3!}(\delta_{Jx})^3 + \frac{i^2}{2!}\left[\int d^4x \frac{g}{3!}(\delta_{Jx})^3\right]\left[\int d^4y \frac{g}{3!}(\delta_{Jy})^3\right] + \dots$$

* Up until this point we weren't using Perturbation thry, our results were <u>exact</u>!!
Now we need to distinguish interacting fields, φ,
from the free fields about which we're perturbing our theory, φ_0
& the interacting vacuum $|\Omega\rangle$ from free $|0\rangle$

Our leading order 2-pt function is:

$$\langle\Omega| T\{\varphi(x_1), \varphi(x_2)\}|0\rangle = \frac{1}{Z[0]}\langle 0| T\{\varphi_0(x_1), \varphi_0(x_2)\}|0\rangle$$

$$= iG(x_1 - x_2)$$

So to leading order, the two pt function/propagator is identical to that of the free theory

For the next order, using shorthand notation, we obtain:

$$\delta_{x_1}\delta_{x_2}\delta_z^3\delta_y^3\, e^{-\frac{1}{2}J_x G_{xy} J_y}$$

$$= \delta_{x_1}\delta_{x_2}\delta_z^3\delta_y^2\left(-J_a G_{ay}\right)e$$

$$= \delta_{x_1}\delta_{x_2}\delta_z^3\delta_y\left(+J_a G_{ay} J_b G_{by} - G_{yy}\right)e$$

$$= \delta_{x_1}\delta_{x_2}\delta_z^3\left(-J_a G_{ay} J_b G_{by} J_c G_{cy} + 3G_{yy} J_b G_{by} + \underline{\cancel{J_a G_{ay} G_{yy}}}\right.$$
$$\left. + \cancel{G_{yy} J_a G_{ay}}\right)e$$

$$= \delta_{x_1}\delta_{x_2}\delta_z^2\left(+J_a G_{ay} J_b G_{by} J_c G_{cy} J_d G_{dz} \underline{-3}G_{yz} J_b G_{by} J_c G_{cy}\right.$$
$$-J_a G_{ay} \underline{\cancel{G_{yz} J_c G_{cy}}} - J_a G_{ay} \underline{\cancel{J_b G_{by} G_{zy}}}$$
$$\left.-3 G_{yy} J_b G_{by} J_c G_{cz} + 3G_{yy} G_{yz}\right)e$$

$$= \delta_{x_1}\delta_{x_2}\delta_z\left(\underline{6}G_{zy} J_b G_{by} J_c G_{cy} J_d G_{dz} + \underline{\cancel{J_a G_{ay} G_{zy} J_c G_{cy} J_d G_{dz}}}\right.$$
$$+ \underline{\cancel{J_a G_{ay} J_b G_{by} G_{zy} J_d G_{zd}}} + J_a G_{ay} J_b G_{by} J_c G_{cy} G_{zz}$$
$$+ 3\underline{\cancel{G_{yz} J_b G_{by} J_c G_{cy} J_d G_{dz}}} - \overset{6}{\underline{\cancel{3}}} G_{yz} G_{zy} J_c G_{cy}$$
$$-3\, \underline{\cancel{G_{yz} J_b G_{by} G_{zy}}} + 3G_{yy} J_b G_{by} J_c G_{cz} J_d G_{dz}$$
$$- \overset{6}{\underline{\underline{\cancel{3}}}} G_{yy} G_{zy} J_c G_{cz} - 3G_{yy} J_b G_{by} G_{zz} - \underline{\underline{\cancel{3 G_{yy} G_{yz} J_a G_{az}}}}$$
$$\left.+(>4\ J\text{'s})\right)e$$

$$= \delta_{x_1}\delta_{x_2}\Big(\overset{18}{\cancel{6}}\, G_{zy}G_{zy}J_cG_{cy}J_dG_{dz} + \cancel{6\, G_{zy}J_bG_{by}G_{zy}J_dG_{dz}}$$

$$+ \overset{18}{\cancel{6}}\, G_{zy}J_bG_{by}J_cG_{cy}G_{zz} + \cancel{G_{zy}J_bG_{by}J_cG_{cy}G_{zz}}$$

$$+ \cancel{J_aG_{ay}G_{zy}J_cG_{cy}G_{zz}} + \cancel{J_aG_{ay}J_bG_{by}G_{zy}G_{zz}}$$

$$+ \cancel{6\, G_{yz}G_{zy}J_cG_{cy}J_dG_{dz}} - 6\, G_{yz}G_{zy}G_{zy}$$

$$+ 3\, G_{yy}G_{zy}J_cG_{cz}J_dG_{dz} + \overset{9}{\cancel{3}}\, G_{yy}J_bG_{by}G_{zz}J_dG_{dz}$$

$$+ \cancel{3\, G_{yy}J_bG_{by}J_cG_{cz}G_{zz}} + \cancel{6\, G_{yy}G_{zy}J_cG_{cz}J_dG_{dz}}$$

$$- \overset{9}{\cancel{6}}\, G_{yy}G_{zy}G_{zz} + \cancel{3\, G_{yy}J_bG_{by}G_{zz}J_cG_{cz}} - \cancel{3\, G_{yy}G_{zy}G_{zz}} + (>2J\text{'s})\Big)e$$

$$= \delta_{x_1}\Big(\overset{36}{\cancel{18}}\, G_{zy}G_{zy}G_{x_2y}J_dG_{dz} + \cancel{18\, G_{zy}G_{zy}J_cG_{cy}G_{x_2z}} + \overset{36}{\cancel{18}}\, G_{zy}G_{x_2y}J_cG_{cy}G_{zz}$$

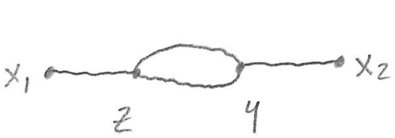
$$+ \cancel{18\, G_{zy}J_bG_{by}G_{x_2y}G_{zz}} + 6\, G_{yz}G_{zy}G_{zy}J_aG_{ax_2}$$

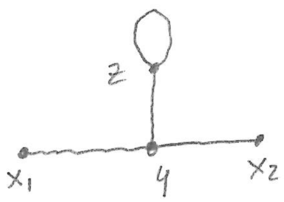
$$+ \overset{18}{\cancel{9}}\, G_{yy}G_{x_2y}G_{zz}J_dG_{dz} + \cancel{9\, G_{yy}J_bG_{by}G_{zz}G_{x_2z}}$$

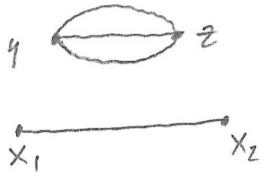
$$+ 9\, G_{yy}G_{zy}G_{zz}J_aG_{ax_2} + (>1J)\Big)e$$

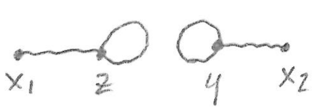
$$= 36\, G_{zy}G_{zy}G_{x_2y}G_{x_1z} + 36\, G_{zy}G_{x_2y}G_{x_1y}G_{zz} + 6\, G_{yz}G_{zy}G_{zy}G_{x_1x_2}$$

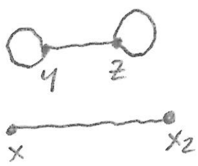
$$+ 18\, G_{yy}G_{x_2y}G_{zz}G_{x_1z} + 9\, G_{yy}G_{zy}G_{zz}G_{x_1x_2}$$

= 36 

+ 36 

+ 6 

+ 18 

+ 9 

restoring the prefactors from expanding the potential, etc:

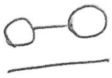
$$\frac{i^2}{2!}\left(\frac{g}{3!}\right)^2 = \frac{-g^2}{2\times 36} \qquad \text{from potential}$$

$\left(\frac{1}{i}\right)^8 = +1$ for $2\times$ external variations + $2\times 3\times$ variations from the potential

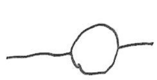
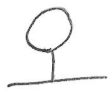
$(+i)^4 = 1$ for each propagator

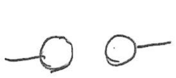
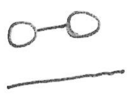
$$\Rightarrow -\frac{g^2}{72}$$

$\Rightarrow -1\Big(\frac{g^2}{2}$  $+\ \frac{g^2}{2}$  $+\ \frac{g^2}{72}$ 

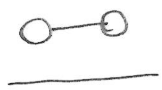
$+\ \frac{g^2}{4}$  $+\ \frac{g^2}{8}$  $\Big)$

Terminology:

 &  = connected

 ,  ,  = disconnected

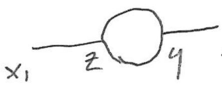
 &  = tadpoles

 &  = vaccuum bubbles

 = one-particle irreducible = 1PI

Tadpoles like  indicate the vacuum in the interacting thry is a mix of the vaccuum & 1 particle states of the free thry

Bubbles like  &  indicate the vacuum in the interacting thry also mixes the free vacuum w/ multiparticle states let's consider  in more detail:



$$x_2 = -\frac{g^2}{2}\int d^4y\, d^4z \underbrace{\int \frac{d^4p_1}{(2\pi)^4}\frac{e^{ip_1\cdot(x_1-z)}}{p_1^2-m^2}}_{G(x_1-z)} \int\frac{d^4p_2}{(2\pi)^4}\frac{e^{ip_2\cdot(z-y)}}{p_2^2-m^2}\int\frac{d^4p_3}{(2\pi)^4}\frac{e^{ip_3\cdot(y-z)}}{p_3^2-m^2}\int\frac{d^4p_4}{(2\pi)^4}\frac{e^{ip_4\cdot(y-x_2)}}{p_4^2-m^2}$$

recall:

$$\int d^4y\, e^{iy\cdot(p_3-p_2+p_4)} = (2\pi)^4\delta^4(p_3-p_2+p_4)$$

$$\int d^4z\, e^{iz\cdot(p_2-p_3-p_1)} = (2\pi)^4\delta^4(p_2-p_3-p_1)$$

$$= \frac{-g^2}{2}\int\frac{d^4p_1 d^4p_2 d^4p_3 d^4p_4}{(2\pi)^{4\cdot 4}}\frac{e^{i(p_1\cdot x_1 - p_4\cdot x_2)}}{(p_1^2-m^2)(p_2^2-m^2)(p_3^2-m^2)(p_4^2-m^2)}(2\pi)^{4\cdot 2}\delta^4(p_3+p_4-p_2)\,\delta^4(p_2-p_3-p_1)$$

rename $p_3 = l$ & use 2^{nd} δ

$$= \int\frac{d^4p_1 d^4p_4}{(2\pi)^{2\cdot 4}} e^{i(p_1\cdot x_1 - p_4\cdot x_2)}\left(\frac{-g^2}{2}\right)\int\frac{d^4l}{(2\pi)^4}\frac{(2\pi)^4\delta^4(p_4-p_1)}{(p_1^2-m^2)[(p_1+l)^2-m^2](l^2-m^2)(p_4-m^2)}$$

LSZ then says:

$$\langle f|S|i\rangle = \left(-i\int d^4x_1 \underbrace{e^{-ip_i\cdot x_1}}_{\text{incoming}}\underbrace{(p_i^2-m^2)}_{\text{cancels external prop}}\right)\left(-i\int d^4x_2 \underbrace{e^{ip_f\cdot x_2}}_{\text{outgoing}}\underbrace{(p_f^2-m^2)}_{\text{cancels external prop}}\right)\langle\Omega|T\{\phi_1(x_1)\phi_2(x_2)\}|\Omega\rangle$$

For  this gives (an incomplete result as we must sum all diagrams):

$$\langle f|S|i\rangle = +\frac{g^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{[(p_i+l)^2-m^2](l^2-m^2)}(2\pi)^4\delta^4(p_i-p_f)$$

To find S matrix elements, recall:

$$S = \mathbb{1} + (2\pi)^4\delta^4\left(\sum_i p_i\right) i\mathcal{M}$$

$$\Rightarrow i\mathcal{M} = +\frac{g^2}{2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{[(p_i+l)^2-m^2](l^2-m^2)}$$

Some notes:

1) the exponentials in the propagators enforce mtm conservation at each vertex, we finish w/ a δ-function enforcing mtm cons. of the external legs

2) LSZ removes external propogators

→ for fermions & vectors we will see it appends on-shell states, for scalars this is just 1 as they have a single degree of freedom

3) We need to integrate over all possible mta for a closed loop, we will learn more about this later

Consider instead the Feynman rules:

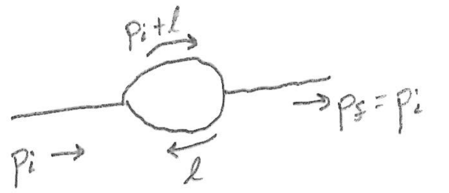
1) propagator —— $= \frac{i}{p^2 - m^2}$

2) vertex  $= ig$

3) loops $\Rightarrow \int \frac{d^4 l}{(2\pi)^4}$

I claim these combined give $i\mathcal{M}$:

$$i\mathcal{M} = (ig)^2 \int \frac{d^4 l}{(2\pi)^4} \frac{i^2}{[(p_i + l)^2 - m^2](l^2 - m^2)}$$



Notice:

1) we're off by a factor of $\frac{1}{2}$!

→ extra rule, symmetry factors, not completely obvious how to implement, see Sterman 3.4

Generally we can cheat and get these w/ a computer algebra system that does Feynman diagram calculations, these use the Wick expansion to guarantee correct results

2) different mtm assignment → different loop integrals, but the integral is unchanged bc $\ell \to \ell + K \Rightarrow d^4\ell \to d^4\ell$

3) From the PI we arrived at "configuration space" Feynman rules like $G(x-y)$, but the exponentials in these made it trivial to transform to mtm space.

The FRs I wrote above are in mtm space and allow us to directly obtain the amplitude, $i\mathcal{M}$

3.2) Connected & disconnected diagrams

We have largely been sweeping the $\frac{1}{Z[0]}$ under the rug, let's be more careful:

$$\langle \Omega | T\{\varphi(x_1)\cdots\varphi(x_n)\}|\Omega\rangle = \frac{1}{Z[0]}\left(\prod_{i=1}^{n}\frac{1}{i}\frac{\partial}{\partial J(x_i)}\right) Z[J]\Big|_{J=0}$$

$$= \frac{1}{Z[0]}\left(\prod_{i=1}^{n}\frac{1}{i}\frac{\partial}{\partial J(x_i)}\right)\exp\left(i\int d^4z\, V\left(\frac{1}{i}\frac{\partial}{\partial J(z)}\right)\right) Z[J]\Big|_{J=0}$$

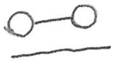
For our example:

$$\langle \Omega | T\{\varphi_1,\varphi_2\}|\Omega\rangle = \frac{\langle 0|T\{\varphi_1,\varphi_2\}|0\rangle + \left(\frac{ig}{3!}\right)^2\langle 0|T\{\varphi_1,\varphi_2,\varphi(z)^3,\varphi(y)^3\}|0\rangle + \cdots}{\underbrace{\langle 0|0\rangle}_{1} + \left(\frac{ig}{3!}\right)^2\langle 0|T\{\varphi(z)^3,\varphi(y)^3\}|0\rangle + \cdots}\Bigg|_{J=0}$$

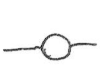
$$\sim \langle 0|T\{\varphi_1,\varphi_2\}|0\rangle + \left(\frac{ig}{3!}\right)^2\langle 0|T\{\varphi_1,\varphi_2,\varphi(z)^3,\varphi(y)^3\}|0\rangle$$
$$-\left(\frac{ig}{3!}\right)^3\langle 0|T\{\varphi_1,\varphi_2\}|0\rangle\underbrace{\langle 0|T\{\varphi(z)^3,\varphi(y)^3\}|0\rangle}_{\text{something vaccuumy}} + \cdots$$

$$-\left[\frac{1}{2}\left(\frac{ig}{3!}\right)^2\left(\frac{1}{i}\right)^6(i)^3\,\delta_z^3\delta_y^3\, e^{-\frac{i}{2}J_x G_{xy} J_y}\right]\times iG_{x_1x_2}$$

$$= -\frac{g^2}{72}\left(-6G_{yz}G_{zy}G_{zy}G_{x_1x_2} - 9G_{yy}G_{zy}G_{zz}G_{x_1x_2}\right)G_{x_1x_2}$$

$$= +g^2\Big(\frac{1}{12}$$  $$+\frac{1}{8}$$  $$\Big)$$

All together w/ our previous result this gives:

$$\langle\Omega|T\{\varphi_1,\varphi_2\}|\Omega\rangle = -\frac{g^2}{2}$$  $$-\frac{g^2}{2}$$  $$-\frac{g^2}{4}$$ 

$$-\frac{g^2}{12}$$  $$-\frac{g^2}{8}$$ 

$$+\frac{g^2}{12}$$  $$+\frac{g^2}{8}$$  } these cancel!

I claim that  doesn't contribute to the process

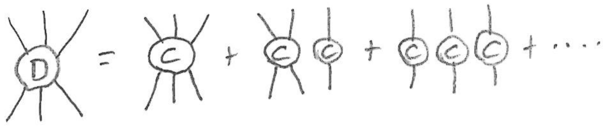
→ easy/handwavey argument: it's included at lower order in the S-matrix

→ more detailed discussion see: Schwartz: 7.3.2
Srednicki: 9
Sterman: 3.4

3.3) Generating functionals for connected & 1PI diagrams:
(Abbott, literature on moodle)

we had $Z[J] = \int D\varphi \exp\left(i S[\varphi] + i \int d^4x\, J\varphi \right)$

and saw it generates Disconnected "D" and connected "C" diagrams, schematically we can represent a disconnected graph as a sum of various combinations of connected graphs:



We argued disconnected pieces of Green's functions don't contribute to the S-matrix

We can, surprisingly, generate all connected Green's functions with the the functional:

$$W[J] = -i \ln Z[J]$$

to see this, let's identify some time ordered products

$\langle \Omega | \varphi | \Omega \rangle =$  (the tadpole/1-pt function is implicitly connected as there is only one external line)

$\langle \Omega | T\{\varphi, \varphi\} | \Omega \rangle =$ 

$\langle \Omega | T\{\varphi, \varphi, \varphi\} | \Omega \rangle =$

... and so on

Taking variations of W we find:

$$\frac{1}{i}\frac{\partial W}{\partial J} = \frac{\frac{\partial Z[J]}{\partial J}}{Z[J]} = \frac{\langle \Omega | \varphi | \Omega \rangle}{\langle \Omega | \Omega \rangle} =$$ 

$$\frac{1}{i}\frac{\partial^2 W}{\partial J^2} = \frac{\frac{\partial^2 Z[J]}{\partial J^2}}{Z[J]} - \frac{\left(\frac{\partial Z[J]}{\partial J}\right)^2}{Z[J]^2} = \frac{\langle \Omega | T\{\varphi,\varphi\} | \Omega \rangle}{\langle \Omega | \Omega \rangle} - \left(\frac{\langle \Omega | \varphi | \Omega \rangle}{\langle \Omega | \Omega \rangle}\right)^2$$

=  -  = 

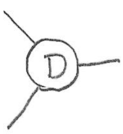
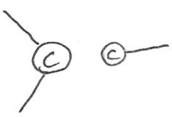
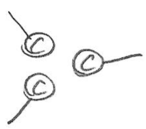
$$\frac{1}{i}\frac{\partial^3 W}{\partial J^3} = \frac{\partial^3 Z[J]/\partial J^3}{Z[J]} - 3\,\frac{\partial^2 Z[J]/\partial J^2 \;\; \partial Z[J]/\partial J}{Z[J]^2} + 2\left(\frac{\partial Z[J]/\partial J}{Z[J]}\right)^3$$

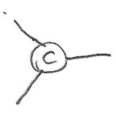
$$= \frac{\langle \Omega | T\{\varphi,\varphi,\varphi\} | \Omega \rangle}{\langle \Omega | \Omega \rangle} - 3\,\frac{\langle \Omega | T\{\varphi,\varphi\} | \Omega \rangle \langle \Omega | \varphi | \Omega \rangle}{(\langle \Omega | \Omega \rangle)^2} + 2\left(\frac{\langle \Omega | \varphi | \Omega \rangle}{\langle \Omega | \Omega \rangle}\right)^3$$

recall: $\frac{1}{i}\frac{\partial^2 W}{\partial J^2} =$  $= \frac{\langle \Omega | T\{\varphi,\varphi\} | \Omega \rangle}{\langle \Omega | \Omega \rangle} - \left(\frac{\langle \Omega | \varphi | \Omega \rangle}{\langle \Omega | \Omega \rangle}\right)^2$

$$= \frac{\langle \Omega | T\{\varphi,\varphi,\varphi\} | \Omega \rangle}{\langle \Omega | \Omega \rangle} - 3\left[\frac{1}{i}\frac{\partial^2 W}{\partial J^2} + \left(\frac{\langle \Omega | \varphi | \Omega \rangle}{\langle \Omega | \Omega \rangle}\right)^2\right]\frac{\langle \Omega | \varphi | \Omega \rangle}{\langle \Omega | \Omega \rangle} + 2\left(\frac{\langle \Omega | \varphi | \Omega \rangle}{\langle \Omega | \Omega \rangle}\right)^3$$

$-3+2=-1$

=  -3  $-$ 

= 

So we conclude W is the generating functional of <u>connected</u> diagrams

Next we want the generating functional of "1PI" or one-particle irreducible diagrams.

1PI = cannot be split into two disjointed pieces by cutting an internal line

We define the "effective action:"

$$\Gamma(\bar{Q}) = W[J] - \int d^4x\, J\bar{\varphi} \qquad w/\ \bar{\varphi} \equiv \frac{\partial W}{\partial J}$$

Note: $\bar{\varphi} \neq \varphi$!!!

Consider:

$$\frac{\partial \Gamma}{\partial \bar{\varphi}} = -J \qquad \text{this replaces } \frac{\partial S}{\partial \varphi} = -J \text{ w/ } \bar{\varphi}$$

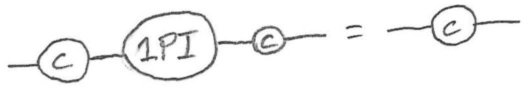
But notice:

$$-\frac{\partial J}{\partial \bar{\varphi}} = \left[-\frac{\partial \bar{\varphi}}{\partial J}\right]^{-1} = \left[-\frac{\partial^2 W}{\partial J^2}\right]^{-1} = i\, \underset{\substack{\uparrow \\ \text{the propagator}}}{G^{-1}} \qquad *$$

Since Γ has J dep in W & linear J dep in the $\bar{\varphi}$ term we conclude:

$$\frac{1}{i}\frac{\partial^2 \Gamma}{\partial \bar{\varphi}^2} = G^{-1}$$

$$\rightarrow \frac{1}{i} G \frac{\partial^2 \Gamma}{\partial \bar{\varphi}^2} G = G \quad \Rightarrow$$



**

i.e. the full propagator is the 1PI propagator dressed w/ external propagators

$*$ also allows us to derive:

$$\frac{\partial}{\partial\varphi} = \underbrace{\frac{\partial J}{\partial\varphi}\frac{\partial}{\partial J}}_{\text{chain rule}} = \frac{1}{i}G^{-1}\frac{\partial}{\partial J}$$

This elucidates how Γ generates 1PI diagrams:

$\frac{\partial}{\partial J}$ acting on $W \to$ an external line to a Green's function

$\frac{\partial}{\partial\varphi}$ acting on $\Gamma \to$ an external line and removes the propagator

Consider:

$$\frac{\partial^3\Gamma}{\partial\varphi^3} = G^{-1}\frac{1}{i}\frac{\partial}{\partial J}\Big[\underbrace{\frac{\partial^2\Gamma}{\partial\varphi^2}}_{\substack{= iG^{-1} \text{ by } ** \\ = \left[-\frac{\partial^2 W}{\partial J^2}\right]^{-1} \text{ by } *}}\Big]$$

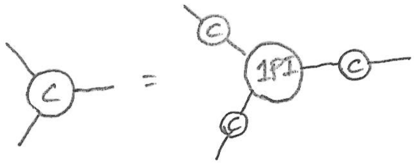
$$= G^{-1}\frac{1}{i}\frac{\partial}{\partial J}\left[-\frac{\partial^2 W}{\partial J^2}\right]^{-1}$$

$$= G^{-1}\frac{1}{i}\left(\frac{\partial^3 W}{\partial J^3}\right)\Big/\left(\frac{\partial^2 W}{\partial J^2}\right)^2$$

$$= G^{-1}\frac{1}{i}(-iG^{-1})^2\frac{\partial^3 W}{\partial J^3}$$

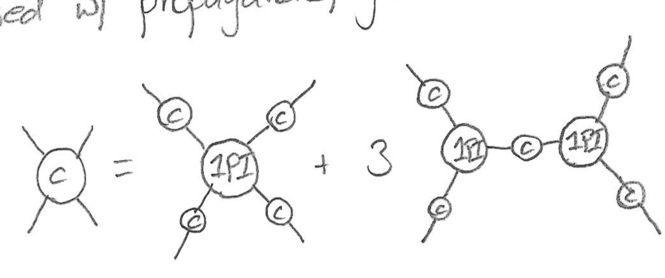
$$= iG^{-3}\frac{\partial^3 W}{\partial J^3}$$

Or: $\left(\frac{1}{i}\right)^3\frac{\partial^3 W}{\partial J^3} = G^3\frac{\partial^3\Gamma}{\partial\varphi^3}$



Again, the 1PI diagrams, dressed w/ propagators, gives the connected diagram

For 4 external lines you will show:



4) Feynman Rules (FRs)

For deriving the FRs for vertices, it is sufficient to take:

$$iS = i\int d^4x\, \mathcal{L} = \Gamma \qquad (iS \text{ is just convenient})$$

ie the FRs are tree level 1PI diagrams, and we can construct all connected (including loops) diagrams from them.

First let's parameterize Γ:

$$\Gamma = i\int d^4x\, \mathcal{L}(\varphi(x))$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4x_1 \cdots d^4x_n \; i\Gamma^{(n)}(x_1, x_2, \ldots)\, \delta^4(x_2 - x_1)\,\delta^4(x_3 - x_1) \cdots \delta^4(x_n - x_1)\, \varphi(x_1) \cdots \varphi(x_n)$$

0 = vacuum E
1 = tadpoles

symbolizes we need to be careful counting identical particles

We want to work in mtm space so we xform:

$$\varphi(x) = \int \frac{d^4p}{(2\pi)^4}\, \varphi(p)\, e^{-ip\cdot x}$$

the sign is convention, we treat all particles as incoming $\rightarrow \sum_i p_i = 0$

Then, for example, the $n=3$ term looks like:

$$\frac{1}{3!} \int d^4x_1\, d^4x_2\, d^4x_3\, \delta^4(x_2 - x_1)\, \delta^4(x_3 - x_1) \int \frac{d^4p_1\, d^4p_2\, d^4p_3}{(2\pi)^{3\cdot 4}}\, i\Gamma^{(n)}(x_1, \ldots, x_n)$$

$$\times e^{-ip_1\cdot x_1}\, e^{-ip_2\cdot x_2}\, e^{-ip_3\cdot x_3}\, \varphi(p_1)\varphi(p_2)\varphi(p_3)$$

$$= \frac{1}{3!} \int \frac{d^4p_1\, d^4p_2\, d^4p_3}{(2\pi)^{3\cdot 4}}\, (2\pi)^4 \delta^4(p_1 + p_2 + p_3)\, i\Gamma^{(n)}(p_1, \ldots, p_n)\, \varphi(p_1) \cdots \varphi(p_3)$$

Extrapolating we have:

$$\Gamma = \sum_{n=0}^{\infty} \frac{1}{n!} \int \frac{d^4p_1}{(2\pi)^4} \cdots \frac{d^4p_n}{(2\pi)^4}\, \underbrace{(2\pi)^4 \delta^4(p_1 + \cdots + p_n)}_{\text{4mta conservation}}\, i\Gamma^{(n)}(p_1, \ldots, p_n)\, \varphi(p_1) \cdots \varphi(p_n)$$

So our n-point FR is the variation of Γ w/r to Φ, which for us is just φ:

$$\left.\frac{\partial^2\Gamma}{\partial\varphi^2}\right|_{\varphi=0} = \frac{\partial^2\Gamma}{\partial\varphi(k_1)\partial\varphi(k_2)} = \frac{1}{2!}\int\frac{d^4p_1\,d^4p_2}{(2\pi)^{2\cdot4}}(2\pi)^4\delta^4(p_1+p_2)\,i\Gamma^{(n)}(p_1,p_2)\left[\delta^4(p_1-k_1)\delta^4(p_2-k_2)+(p_1\leftrightarrow p_2)\right]$$

$$= \frac{1}{2!}\,\frac{1}{(2\pi)^4}\delta^4(k_1+k_2)\,i\Gamma^{(n)}(k_1,k_2)\times 2$$

We see the origin of the counting factor

Associating Γ w/ $iS = i\int d^4x\,\mathcal{L}_{free}$:

$$\int d^4x\,\mathcal{L}_{free} = \int d^4x\left(\frac{-1}{2}\right)\varphi(x)(\Box+m^2)\varphi(x)$$

$$= \int d^4x\,d^4y\,\delta^4(x-y)\left(\frac{-1}{2}\right)\varphi(x)(\Box_y+m^2)\varphi(y)$$

$$= \int d^4x\,d^4y\,\delta^4(x-y)\int\frac{d^4p_1\,d^4p_2}{(2\pi)^{4\cdot2}}\left(-\frac{1}{2}\right)e^{-ip_1\cdot x}(\Box_y+m^2)e^{-ip_2\cdot y}\varphi(p_2)\varphi(p_1)$$

$$\Box_y \to (-ip_2)^2$$

$$= \int\frac{d^4p_1\,d^4p_2}{(2\pi)^{4\cdot2}}(2\pi)^4\delta^4(p_1+p_2)\,\frac{1}{2}\,(p_2^2-m^2)\varphi(p_1)\varphi(p_2)$$

$$\left.\frac{\partial^2 iS}{\partial\varphi(k_1)\partial\varphi(k_2)}\right|_{\varphi=0} = \frac{1}{2}\int\frac{d^4p_1\,d^4p_2}{(2\pi)^{4\cdot2}}(2\pi)^4\delta^4(p_1+p_2)\,i(p_2^2-m^2)\left[\delta^4(p_1-k_1)\delta^4(p_2-k_2)+(p_1\leftrightarrow p_2)\right]$$

$$= \frac{1}{(2\pi)^4}\delta^4(k_1+k_2)\,i(k_1^2-m^2)$$

Requiring $\left.\frac{\partial^2\Gamma}{\partial\varphi^2}\right|_{\varphi=0} = \left.\frac{\partial^2 iS}{\partial\varphi^2}\right|_{\varphi=0} \rightarrow i\Gamma^{(2)} = i(k_1^2-m^2)$

We had before (in config space): $i(\Box+m^2)\,iG(x-y) = -\delta(x-y)$

In mtm space we've found $\quad i(k_1^2-m^2)\,iG(k_1) = -1$

$$\rightarrow iG(k_1^2) = \frac{i}{k_1^2-m^2}$$

For our interaction, $\mathcal{L}_{Int} = \frac{g}{3!}\varphi(x)^3$:

$$\left.\frac{\partial^3 \Gamma}{\partial \varphi^3}\right|_{\varphi=0} = \frac{1}{3!}\int \frac{d^4p_1 d^4p_2 d^4p_3}{(2\pi)^{3\cdot 4}} (2\pi)^4 \delta^4(p_1+p_2+p_3)\, i\Gamma^{(n)}(p_1,p_2,p_3)$$

$$\times \underbrace{\Big[\delta^4(p_1-k_1)\delta^4(p_2-k_2)\delta^4(p_3-k_3) + \delta^4(p_1-k_1)\delta^4(p_2-k_3)\delta^4(p_3-k_2) + \delta^4(p_1-k_2)\delta^4(p_2-k_1)\delta^4(p_3 \\ + \delta^4(p_1-k_2)\delta^4(p_2-k_3)\delta^4(p_3-k_1) + \delta^4(p_1-k_3)\delta^4(p_2-k_2)\delta^4(p_3-k_1) + \delta^4(p_1-k_3)\delta^4(p_2-k_1)\delta^4(p_3-k}$$

these 6 terms add trivially to 3!
let's call this N and return below

$$= \frac{1}{(2\pi)^{2\cdot 4}} \delta^4(k_1+k_2+k_3)\, i\Gamma^{(3)}(k_1,k_2,k_3)\frac{N}{3!}$$

$$\left.\frac{\partial^3 iS}{\partial \varphi^3}\right|_{\varphi=0} = i\int \frac{d^4p_1 d^4p_2 d^4p_3}{(2\pi)^{3\cdot 4}} (2\pi)^4 \delta^4(p_1+p_2+p_3)\left(\frac{g}{3!}\right) \times N$$

if our interaction were $\varphi^2\Box\varphi$ this would have mta dep

our choice of normalization here conveniently cancels against $N = 3!$

In the exercises we will derive the FR for $\mathcal{L}_{int} = c\,\varphi^2\Box\varphi$

a coupling constant

In this case what we called N above will not be so simple.