$$\Pi_T^{\mu\nu} = +\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}$$

$$\Pi_T^{\mu\nu}\Pi_{T\nu\rho} = \left(+\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(+\eta_{\nu\rho} - \frac{p_\nu p_\rho}{p^2}\right)$$

$$= \left(\delta^\mu{}_\rho - \frac{p^\mu p_\rho}{p^2} - \frac{p^\mu p_\rho}{p^2} + \frac{p^\mu p_\rho}{p^2}\right)$$

$$= \eta_{\rho\sigma}\left(\eta^{\mu\sigma} - \frac{p^\mu p^\sigma}{p^2}\right) = \eta_{\rho\sigma}\Pi_T^{\mu\sigma}$$

$$\Pi_L^{\mu\nu} = \frac{p^\mu p^\nu}{p^2}$$

$$\Pi_L^{\mu\nu}\Pi_{L\nu\rho} = \left(\frac{p^\mu p^\nu}{p^2}\right)\left(\frac{p_\nu p_\rho}{p^2}\right) = \frac{p^\mu p_\rho}{p^2} = \eta_{\rho\sigma}\frac{p^\mu p^\sigma}{p^2} = \eta_{\rho\sigma}\Pi_L^{\mu\sigma}$$

$$\Pi_T^{\mu\nu}\Pi_{L\nu\rho} = \left(+\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)\left(\frac{p_\nu p_\rho}{p^2}\right) = +\frac{p^\mu p_\rho}{p^2} - \frac{p^\mu p_\rho}{p^2} = 0$$

$$\Pi_T^{\mu\nu} + \Pi_L^{\mu\nu} = \left(+\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) + \frac{p^\mu p^\nu}{p^2} = +\eta^{\mu\nu}$$

$$iS = i\int d^4x\, \bar{\psi}_a (\sigma_{\mu\nu})_{ab} \psi_b (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

Dirac Indices

$\bar{\psi}$, ψ, & A are distinguishable, so from Week 2s notes we have:

$$i\Gamma = \int \frac{d^4p_1 d^4p_2 d^4p_3}{(2\pi)^{3\cdot 4}} \underset{(2\pi)^4}{\delta^4}(p_1+p_2+p_3)\, i\Gamma^{(3)}_{cd\rho}(p_1,p_2,p_3)\, \bar{\psi}_c(p_1)\psi_d(p_2)A^\rho(p_3)$$

$$\frac{\partial^3 i\Gamma}{\partial\bar{\psi}_\xi(k_1)\,\partial\psi_g(k_2)\,\partial A^\sigma(k_3)} = \int \frac{d^4p_1 d^4p_2 d^4p_3}{(2\pi)^{3\cdot 4}} \overset{(2\pi)^4}{\delta^4}(p_1+p_2+p_3)\, i\Gamma^{(3)}_{cd\rho}(p_1,p_2,p_3) \frac{\partial^2}{\partial\bar{\psi}_\xi(k_1)\partial\psi_g(k_2)} \bar{\psi}_c(p_1)\psi_d(p_2)\delta^\rho_\sigma \delta(p_3-$$

$$=(-1)\int \frac{d^4p_1 d^4p_2}{(2\pi)^{3\cdot 4}}(2\pi)^4\delta^4(p_1+p_2+k_3)\, i\Gamma^{(3)}_{cd\sigma}(p_1,p_2,k_3)\frac{\partial}{\partial\bar{\psi}_\xi(k_1)}\bar{\psi}_c(p_1)\,\delta_{dg}\,\delta(p_2-k_2)$$

$\psi, \bar{\psi}$ anticommute

$$=(-1)\int \frac{d^4p_1}{(2\pi)^{3\cdot 4}}(2\pi)^4\delta^4(p_1+k_2+k_3)\, i\Gamma^{(3)}_{cg\sigma}(p_1,k_2,k_3)\,\delta_{\xi c}\,\delta(p_1-k_1)$$

$$=(-1)\frac{1}{(2\pi)^{3\cdot 4}}(2\pi)^4\delta^4(k_1+k_2+k_3)\, i\Gamma^{(3)}_{\xi g\sigma}(k_1,k_2,k_3)$$

This is equal to (at treelevel) the variations acting on iS:

$$\frac{\partial^3 i\Gamma}{\partial\bar{\psi}_\xi(k_1)\,\partial\psi_g(k_2)\,\partial A^\sigma(k_3)} = \frac{\partial^3 iS}{\partial\bar{\psi}_\xi(k_1)\,\partial\psi_g(k_2)\,\partial A^\sigma(k_3)}$$

$$iS = i\int d^4x\, \bar{\psi}_a (\sigma_{\mu\nu})_{ab}\, \psi_b\, (\partial^\mu A^\nu - \partial^\nu A^\mu)$$

Transforming to mtm space for a field $F_a(x)$
some index

$$F_a(x) = \int \frac{d^4p}{(2\pi)^4}\, F_a(p)\, e^{-ip\cdot x}$$

$$= i\int d^4x\, d^4y\, d^4z\, \delta^4(x-y)\,\delta^4(x-z) \int \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4p_3}{(2\pi)^4}\, \bar{\psi}_a(p_1)(\sigma_{\mu\nu})_{ab}\,\psi_b(p_2)\, e^{-ip_1\cdot z - ip_2\cdot y}$$

$$\times\left(\partial_x^\mu A^\nu(p_3) e^{-ip_3\cdot x} - \underbrace{\partial_x^\nu A^\mu(p_3) e^{-ip_3\cdot x}}_{\to (-ip_3^\nu) e^{-ip_3\cdot x}}\right)$$

$$= i\int d^4x \underbrace{\int \frac{d^4p_1\, d^4p_2\, d^4p_3}{(2\pi)^{4\cdot 3}}\,(-i)\, e^{-i(p_1+p_2+p_3)\cdot x}}_{(2\pi)^4\delta^4(p_1+p_2+p_3)}\, \bar{\psi}_a(p_1)(\sigma_{\mu\nu})_{ab}\,\psi_b(p_2)\left(p_3^\mu A^\nu(p_3) - p_3^\nu A^\mu(p_3)\right)$$

Antisymmetric; so we can add the RH terms

$$= i\int \frac{d^4p_1\, d^4p_2\, d^4p_3}{(2\pi)^{3\cdot 4}}\,(2\pi)^4\delta^4(p_1+p_2+p_3)\, \bar{\psi}_a(p_1)(\sigma_{\mu\nu})_{ab}\,\psi_b(p_2)\; 2\, p_3^\mu A^\nu(p_3)$$

Taking the variations:

$$\frac{\partial^3 iS}{\partial\bar{\psi}_\xi(k_1)\,\partial\psi_g(k_2)\,\partial A^\sigma(k_3)} = \frac{\partial^2}{\partial\bar{\psi}_\xi(k_1)\,\partial\psi_g(k_2)}\, i\int \frac{d^4p_1\, d^4p_2\, d^4p_3}{(2\pi)^{3\cdot 4}}\,(2\pi)^4\delta^4(p_1+p_2+p_3)\,\bar{\psi}_a(p_1)(\sigma_{\mu\nu})_{ab}\,\psi_b(p_2)\; 2\, p_3^\mu\, \delta^\nu_\sigma\, \delta(p_3-k_3)$$

$$= (-1)\frac{\partial}{\partial\bar{\psi}_\xi(k_1)}\, i\int \frac{d^4p_1\, d^4p_2}{(2\pi)^{3\cdot 4}}\,(2\pi)^4\delta^4(p_1+p_2+k_3)\,\bar{\psi}_a(p_1)(\sigma_{\mu\sigma})_{ab}\,\delta_{bg}\,\delta(p_2-k_2)\; 2\, p_3^\mu$$

$$= (-1)\, i\int \frac{d^4p_1}{(2\pi)^{3\cdot 4}}\,(2\pi)^4\delta^4(p_1+k_2+k_3)\,\delta_{a\xi}\,\delta(p_1-k_1)(\sigma_{\mu\sigma})_{ag}\; 2\, p_3^\mu$$

$$= -i\,\frac{1}{(2\pi)^{3\cdot 4}}\,(2\pi)^4\delta^4(k_1+k_2+k_3)\,(\sigma_{\mu\sigma})_{\xi g}\; 2\, p_3^\mu$$

Equating the two gives

$$i\Gamma^{(3)}_{\xi g\sigma} = i(\sigma_{\mu\sigma})_{\xi g}\, 2\, p_3^\mu$$

We also had

$$0 = \mu \frac{d}{d\mu} m Z_m Z_\psi^{-1}$$

$$= Z_m Z_\psi^{-1} \mu \frac{dm}{d\mu} + m\mu \frac{d}{d\mu} Z_m Z_\psi^{-1}$$

$$\Rightarrow 0 = \mu \frac{d\ln m}{d\mu} + \mu \frac{d}{d\mu} \ln Z_m Z_\psi^{-1}$$

$$= \mu \frac{d \ln m}{d\mu} + \mu \frac{d}{d\mu} (\delta_m - \delta_\psi)$$

$$\delta_m - \delta_\psi = -\frac{e^2}{(4\pi)^2}\left[4 + \frac{3}{\epsilon} + 3\ln\frac{\mu^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]$$

$$\mu\frac{d}{d\mu}(\delta_m - \delta_\psi) = -2\frac{e^2}{(4\pi)^2}\mu\frac{d}{d\mu}\ln e\left[4 + \frac{3}{\epsilon} + 3\ln\frac{\mu^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]$$

$$-2\frac{e^2}{(4\pi)^2}3 \cancel{- 2\frac{e^2}{(4\pi)^2}\mu\frac{d}{d\mu}\ln m^2} \cancel{+ 2\frac{e^2}{(4\pi)^2}\mu\frac{d}{d\mu}\ln m^2}$$

$$\mu\frac{d\ln m}{d\mu} = 2\frac{e^2}{(4\pi)^2}\left[4 + \frac{3}{\epsilon} + 3\ln\frac{\mu^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]\left(\left[-2 + \underset{\substack{\uparrow \\ \text{2 loop}}}{\frac{e^2}{(4\pi)^2}\frac{4}{3}}\right]\mu\frac{d}{d\mu}\ln m + \epsilon\right) - 6\frac{e^2}{(4\pi)^2}$$

$$\left[1 + 4\frac{e^2}{(4\pi)^2}\left[4 + \frac{3}{\epsilon} + 3\ln\frac{\mu^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]\right]\mu\frac{d\ln m}{d\mu} = 6\frac{e^2}{(4\pi)^2} - 6\frac{e^2}{(4\pi)^2} + \epsilon\, 2\frac{e^2}{(4\pi)^2}\left[4 + 3\ln\frac{\mu^2}{m^2} - \ln\frac{m}{m}\right.$$

$$\mu\frac{d\ln m}{d\mu} = 0 + \epsilon\, 2\frac{e^2}{(4\pi)^2}\left[4 + 3\ln\frac{\mu^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]\left[1 + 4\frac{e^2}{(4\pi)^2}\left[4 + \frac{3}{\epsilon} + 3\ln\frac{\mu^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]\right]^{-1}$$

$$= 0 + \epsilon\, 2\frac{e^2}{(4\pi)^2}\left[4 + 3\ln\frac{\mu^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right] + \text{2 loop}$$

$$= 0 + O(\epsilon)$$

In class we found:

$$0 = \mu\frac{d}{d\mu}\ln e + \mu\frac{d}{d\mu}\ln Z_e Z_\psi^{-1} Z_A^{-1/2} + \epsilon$$

$$= \mu\frac{d}{d\mu}\ln e + \mu\frac{d}{d\mu}(\delta_e - \delta_\psi - \tfrac{1}{2}\delta A) + \epsilon$$

$$\delta_e - \delta_\psi = \ln\frac{m^2}{m_\gamma^2}$$

$$\mu\frac{d}{d\mu}(\delta_e - \delta_\psi) = 2\mu\frac{d}{d\mu}\ln m$$

$$\mu\frac{d}{d\mu}\delta_A = -2\frac{e^2}{(4\pi)^2}\frac{4}{3}\left[\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2}\right]\mu\frac{d}{d\mu}\ln e \;\overbrace{- \frac{e^2}{(4\pi)^2}\frac{4}{3}\left[2 - 2\mu\frac{d}{d\mu}\ln m\right]}^{\text{explicit } \mu \text{ dep}}$$

So we have:

$$0 = \mu\frac{d}{d\mu}\ln e + 2\mu\frac{d}{d\mu}\ln m + \frac{e^2}{(4\pi)^2}\frac{4}{3}\left[\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2}\right]\mu\frac{d}{d\mu}\ln e + \frac{1}{2}\frac{e^2}{(4\pi)^2}\frac{4}{3}\left[2 - 2\mu\frac{d}{d\mu}\ln m\right] + \epsilon$$

$$\left[1 + \frac{e^2}{(4\pi)^2}\frac{4}{3}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2}\right)\right]\mu\frac{d}{d\mu}\ln e = -2\mu\frac{d}{d\mu}\ln m - \frac{1}{2}\frac{e^2}{(4\pi)^2}\frac{4}{3}\left[2 - 2\mu\frac{d}{d\mu}\ln m\right] - \epsilon$$

$$\mu\frac{d}{d\mu}\ln e = -2\mu\frac{d}{d\mu}\ln m - \frac{1}{2}\frac{e^2}{(4\pi)^2}\frac{4}{3}\left[2 - 2\mu\frac{d}{d\mu}\ln m\right] - \epsilon\left[1 + \frac{e^2}{(4\pi)^2}\frac{4}{3}\left(\frac{1}{\epsilon} + \ln\frac{\mu^2}{m^2}\right)\right]^{-1} + \text{2-loop}$$

↑ this term $\times \frac{d}{d\mu}\ln m$ is 2 loop

$$= \left[-2 + \frac{e^2}{(4\pi)^2}\frac{4}{3}\right]\mu\frac{d}{d\mu}\ln m - \epsilon + \frac{4}{3}\frac{e^2}{(4\pi)^2} + O(\epsilon) + \text{2-loop}$$

Plugging this in to $\mu \frac{d}{d\mu} \ln e$ gives:

$$\mu \frac{d}{d\mu} \ln e = + \frac{4}{3} \frac{e^2}{(4\pi)^2}$$

$$\mu \frac{d}{d\mu} \ln m = 0$$