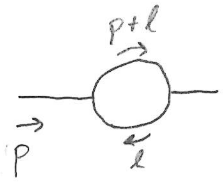
Loop integrals & Renormalization in Scalar theory

1) Loop integrals

For φ^3 theory we found



$$= \frac{g^2}{2}\int\frac{d^4l}{(2\pi)^4}\,\frac{1}{(p+l)^2-m^2+i\epsilon}\,\frac{1}{l^2-m^2+i\epsilon}$$

Writing $\int d^4l = \underbrace{\int d\Omega_4}_{\text{solid angle in 4d}}\int l^3 dl$ (ie spherical coordinates

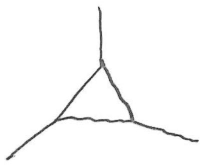
and taking $m\to 0$, $\epsilon\to 0$, we have:

$$\int d^4l\,\frac{1}{l^4} = \int d\Omega_4\int_0^\Lambda \frac{l^3 dl}{l^4}$$

$$= \pi^2 \log\Lambda$$

which is infinite for $\Lambda\to\infty$, "Ultraviolet divergen"/ UV divergent

Consider:



$$\sim ig^3\int\frac{d^4l}{(2\pi)^4}\,\frac{1}{l^6} = ig^3\pi^2\int_0^\Lambda\frac{l^3dl}{l^6}$$

$$\sim ig^3\frac{1}{\Lambda^2}$$

This is the "superficial degree of divergence"

It can be used to estimate/guess if a diagram is divergent

To handle these divergences we'll use "dimensional regularization"

→ "regulates," or makes our integrals finite, by performing them in $d<4$ spacetime dim where they're finite

We'll take $d=4-2\epsilon$ (sometimes other choices are used)

First we'll clean up our integrals using "Feynman Parameterization":

$$\frac{1}{A_1 \cdots A_n} = \int_0^1 d\alpha_1 \cdots d\alpha_n \frac{\Gamma(n)\,\delta(1-\alpha_1 \cdots \alpha_n)}{[\alpha_1 A_1 + \cdots + \alpha_n A_n]^n} \qquad \text{(Homework)}$$

(0,1) is $\alpha_1 + \cdots \alpha_n$, after $\delta(\ )$ integral the limits need to be adjusted

For example:

$$\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(p+l)^2 - m^2 + i\epsilon} \frac{1}{l^2 - m^2 + i\epsilon} = \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{1}{(x[(p+l)^2 - m^2 + i\epsilon] + (1-x)[l^2 - m^2 + i\epsilon])^2}$$

$$= \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{1}{(x[p^2 + l^2 + 2p\cdot l - m^2 + i\epsilon] + (1-x)[l^2 - m^2 + i\epsilon])^2}$$

mix term, we prefer l^2, so we want to rewrite $q = l + xp \rightarrow q^2 = l^2 + 2xp\cdot l + p^2$

$$= \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx \frac{1}{(\underbrace{[l+xp]^2}_{\equiv q^2} \underbrace{- x^2 p^2 + xp^2 - m^2 + i\epsilon}_{\equiv -\Delta + i\epsilon})^2}$$

$$= \int \frac{d^4 q}{(2\pi)^4} \int_0^1 dx \frac{1}{(q^2 - \Delta + i\epsilon)^2}$$

something playing the role of a mass

This gives a cleaner integral which is also <u>even in q</u> → if we had terms like $p\cdot q$ in the numerator they vanish

From here we extend our integral to d-spacetime dimensions, we'll need:

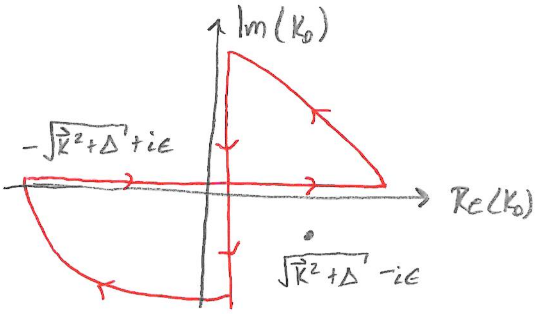
$$\int d^dk = \int d\Omega_d \int k^{d-1} dk$$

$$\text{w/ } \Omega_d = \int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)} \qquad \text{(Homework)}$$

With a generic 1-loop integral, we can rotate to Euclidean space:

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\epsilon)^n}$$

poles at $k_0 = \pm\sqrt{\vec{k}^2 + \Delta} \mp i\epsilon$



Integral over red contour vanishes ⇒ integral over $\mathbb{R}$ & Imaginary axis are equal & opposite → we can write $k_0 \to ik_0$

$\to k^2 \to -k_0^2 - \vec{k}^2 = -k_E^2$ } Wick rotation

$$\int \frac{d^4k}{(2\pi)^4} \frac{(k^2)^a}{[k^2-\Delta]^b} \to \frac{i(-1)^{b-a}}{(2\pi)^4} \int d^4k_E \frac{(k_E^2)^a}{[k_E^2+\Delta]^b}$$

$dk_0 \to i\,dk_0$

$$= \frac{i(-1)^{b-a}}{(2\pi)^4} \int d\Omega_4 \int_0^\infty k_E^3 dk_E \frac{(k_E^2)^a}{[k_E^2+\Delta]^b}$$

Changing to d-dimensions:

$$\int \frac{d^dk}{(2\pi)^d} \frac{(k^2)^a}{[k^2-\Delta]^b} \longrightarrow \frac{i(-1)^{b-a}}{(2\pi)^d} \int d\Omega_d \int_0^\infty k_E^{d-1} dk_E \frac{(k_E^2)^a}{[k_E^2+\Delta]^b}$$

Next we will rewrite these integrals as Γ functions:

$$\Gamma(a)\Gamma(b) = \int_0^\infty e^{-x} x^{a-1} dx \int_0^\infty e^{-y} y^{b-1} dy$$

$$= \int dx dy \, e^{-(x+y)} x^{a-1} y^{b-1}$$

let $x = zt$, $y = z(1-t)$ $\qquad J = \begin{vmatrix} t & z \\ (1-t) & -z \end{vmatrix} = |-zt - z(1-t)| = z$

$$= \int_0^\infty z dz \int_0^1 dt \, e^{-z} (zt)^{a-1} (z(1-t))^{b-1}$$

$z = x+y \rightarrow z \in (0,\infty)$

$t = \frac{x}{x+y} \rightarrow z \in (0,1)$

$$= \underbrace{\int_0^\infty dz \, z^{a+b-1} e^{-z}}_{\Gamma(a+b)} \underbrace{\int_0^1 dt \, t^{a-1}(1-t)^{b-1}}_{\equiv B(a,b)}$$

We have:

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

We just need to make $B(a,b)$ look like our loop integrals in Euclidean space:

$$\int_0^1 dt \, t^{a-1}(1-t)^{b-1} = \int_0^\infty dx \frac{1}{(1+x)^2} \frac{x^{a-1}}{(1+x)^{a-1}} \left(1 - \frac{x}{1+x}\right)^{b-1}$$

for $t = \frac{x}{(1+x)}$, $dt = \frac{dx}{(1+x)^2}$

$$= \int_0^\infty dx \frac{x^{a-1}}{(1+x)^{a+b}}$$

Then our integral over Euclidean mta looks like:

$$\int dk_E \frac{k_E^a}{(k_E^2+\Delta)^b} = \Delta^{-b}\int dk_E \frac{k_E^a}{(k_E^2/\Delta+1)^b}$$

$$= \Delta^{\frac{a+1}{2}-b}\int \frac{dk_E}{\sqrt{\Delta}} \frac{(k_E^2/\Delta)^{a/2}}{(k_E^2/\Delta+1)^b}$$

let $z = \frac{k_E^2}{\Delta}$ $\quad dz^2 = 2z\,dz = 2\,\frac{k_E^2}{\Delta}\,\frac{dk_E}{\sqrt{\Delta}}$

$$= \frac{1}{2}\,\Delta^{\frac{a+1}{2}-b}\int dz^2\, \frac{(z^2)^{(a-1)/2}}{(z^2+1)^b}$$

$$= \frac{1}{2}\,\Delta^{\frac{a+1}{2}-b}\underbrace{\int dz^2\, \frac{(z^2)^{\frac{a+1}{2}-1}}{(z^2+1)^{b-(a+1)/2+(a+1)/2}}}_{B\left(\frac{a+1}{2},\, b-\frac{(a+1)}{2}\right)}$$

$$= \frac{\Delta^{\frac{a+1}{2}-b}}{2}\,\frac{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(b-\frac{a+1}{2}\right)}{\Gamma(b)}$$

Combining w/ the radial part & Wick rotation:

$$\int \frac{d^dk}{(2\pi)^d}\frac{(k^2)^a}{(k^2-\Delta)^b} = \frac{i(-1)^{b-a}}{(2\pi)^d}\int d^dk_E\, \frac{k_E^{2a}}{(k_E^2+\Delta)^b} \qquad \text{Wick}$$

$$= \frac{i(-1)^{b-a}}{(2\pi)^d}\int d\Omega_d \int k_E^{d-1}\,dk_E\, \frac{k_E^{2a}}{(k_E^2+\Delta)^b}$$

$$= \frac{2i(-1)^{b-a}}{(4\pi)^{d/2}\Gamma(d/2)}\int dk_E\, \frac{k_E^{2a+d-1}}{(k_E^2+\Delta)^b}$$

$$= \frac{2i(-1)^{b-a}}{(4\pi)^{d/2}\Gamma(d/2)}\,\Delta^{a-b+d/2}\,\frac{\Gamma(a+d/2)\Gamma(b-a-d/2)}{2\Gamma(b)}$$

$$= \frac{i(-1)^{a-b}}{(4\pi)^{d/2}}\,\frac{1}{\Delta^{b-a-d/2}}\,\frac{\Gamma(a+\frac{d}{2})\Gamma(b-a-d/2)}{\Gamma(b)\Gamma(d/2)}$$

Returning to our example:

$$I \equiv \int \frac{d^dq}{(2\pi)^4} \int_0^1 dx \frac{1}{(q^2-\Delta+i\epsilon)^2} = \frac{i\cancel{(-1)^{-2}}^{\,1}}{(4\pi)^{d/2}} \frac{1}{\Delta^{2-d/2}} \frac{\Gamma(d/2)\Gamma(2-d/2)}{\Gamma(2)\Gamma(d/2)}$$

Taking $d=4-2\epsilon$ (some conventions use $d=4-\epsilon$)

$$I = \int_0^1 dx \frac{i}{(4\pi)^{2-\epsilon}} \frac{1}{\Delta^\epsilon} \frac{\cancel{\Gamma(2-\epsilon)}\Gamma(\epsilon)}{\underbrace{\Gamma(2)}_{1!=1}\cancel{\Gamma(2-\epsilon)}}$$

$$= \frac{i}{(4\pi)^2}\Gamma(\epsilon)\int_0^1 dx \left(\frac{4\pi}{\Delta}\right)^\epsilon$$

We need some more results:

$$A^\epsilon = e^{\ln A^\epsilon}$$
$$= e^{\epsilon \ln A}$$
$$= 1+\epsilon \ln A + O(\epsilon^2)$$

$$\Gamma(1+\epsilon) = \int_0^\infty x^\epsilon e^{-x} dx$$
$$= \int_0^\infty (1+\epsilon\ln x + \dots) e^{-x} dx$$
$$= 1 - \epsilon \gamma_E + O(\epsilon^2)$$

↑ Euler-Mascheroni constant ~ 0.5772

$$\Gamma(1+\epsilon) = \epsilon\Gamma(\epsilon)$$

So:

$$I = \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E\right)\int_0^1 dx\left(1+\epsilon\ln\frac{4\pi}{\Delta}\right) + \dots$$

$$= \frac{i}{(4\pi)^2}\left(\frac{1}{\epsilon} - \gamma_E + \int_0^1 dx \ln\frac{4\pi}{x(x-1)p^2+m^2}\right) + O(\epsilon)$$

↑ unitless → come back to this

$$= \frac{i}{(4\pi)^2}\left[2 + \frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi}{m^2} + \frac{\sqrt{p^2(p^2-4m^2)}}{p^2}\ln\frac{2m^2-p^2+\sqrt{p^2(p^2-4m^2)}}{2m^2}\right]$$

2) φ^4 theory

(φ^3 thry is really pretty in $d=6$, but φ^4 will be more instructive for us)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda\varphi^4}{4!}$$

Notice that S is dimensionless:

$$S = \int d^4x\, \mathcal{L}$$

So $\mathcal{L}$ has units of (inverse length)^4

$$[\mathcal{L}] = \frac{1}{L^4} = M^4 \equiv 4$$

Since we know $[\partial_\mu] = \frac{1}{L} = 1$, we conclude

$$[\varphi] = 1$$

This gives $[\lambda] = 0$

But in dim reg in $d = 4-2\epsilon$:

$$[\partial_\mu] = 1$$

$$[\varphi^2] = d-2 = 2-2\epsilon$$

$$[\varphi] = 1-\epsilon$$

$$[\lambda] = d - 4(1-\epsilon) = 4-2\epsilon-4+4\epsilon$$

$$= 2\epsilon$$

To retain $[\lambda]=0$, we introduce μ, w/ $[\mu]=1$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{m^2}{2}\varphi^2 - \frac{\lambda}{4!}\mu^{2\epsilon}\varphi^4$$

Our Feynman Rules are

 $= \dfrac{i}{p^2-m^2+i\epsilon}$

 $= -i\lambda\mu^{2\epsilon}$

The one-loop correction to the propagator (2pt function) is:

$$= -\frac{i\lambda\mu^{2\epsilon}}{\underset{\leftarrow \text{symm factor}}{2}}\int\frac{d^d\ell}{(2\pi)^d}\frac{i}{\ell^2-m^2+i\epsilon}$$

$$= +\frac{\lambda\mu^{2\epsilon}}{2}\left[\frac{i(-1)^{-1}}{(4\pi)^{2-\epsilon}}\frac{1}{(m^2)^{1-2+\epsilon}}\frac{\Gamma(\cancel{2-\epsilon})\Gamma(1-2+\epsilon)}{\Gamma(1)\cancel{\Gamma(2-\epsilon)}}\right]$$

note $\Gamma(\epsilon) = (\epsilon-1)\Gamma(\epsilon-1)$

$$\Gamma(\epsilon-1) = \frac{1}{\epsilon-1}\Gamma(\epsilon)$$

$$= \frac{1}{\epsilon-1}\left(\frac{1}{\epsilon} - \gamma_E + O(\epsilon)\right)$$

$$= -\frac{1}{\epsilon} + (\gamma_E - 1) + O(\epsilon)$$

$$= \frac{-i\lambda}{2(4\pi)^2} m^2 \left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon}\left(-\frac{1}{\epsilon} + (\gamma_E - 1) + O(\epsilon)\right)$$

$$= \frac{+i\lambda}{2(4\pi)^2} m^2 \left(1 + \epsilon\ln\frac{4\pi\mu^2}{m^2}\right)\left(\frac{1}{\epsilon} + (1-\gamma_E) + O(\epsilon)\right)$$

$$= \frac{+i\lambda}{2(4\pi)^2} m^2 \left[\frac{1}{\epsilon} + \ln\frac{4\pi\mu^2}{m^2} + (1-\gamma_E)\right] + O(\epsilon)$$

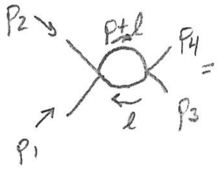
$\uparrow$ take $\gamma_E = \ln e^{-\gamma_E}$

$$= \frac{+i\lambda}{2(4\pi)^2} m^2\left[\frac{1}{\epsilon} + 1 + \ln\frac{4\pi\mu^2}{e^{\gamma} m^2}\right]$$

$\uparrow$ UV divergence appears as $\epsilon \to 0$

argument of ln has correct dim $\left[\frac{\mu^2}{m^2}\right] = 1$

The one-loop correction to the 4pt function is:



$$= +\frac{\lambda^2}{2}\mu^{4\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{1}{[(p+l)^2-m^2][l^2-m^2]} + \text{T-channel} + \text{U-channel}$$

$p \equiv p_1+p_2$

$$= \frac{i\lambda^2\mu^{4\epsilon}}{2(4\pi)^{2-\epsilon}}\int_0^1 dx\,\frac{1}{\Delta^\epsilon}\Gamma(\epsilon)$$

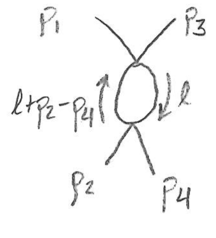
$$\Delta = x(x-1)p^2+m^2$$

$$= \frac{i\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\int_0^1 dx\left(\frac{4\pi\mu^2}{\Delta}\right)^\epsilon\Gamma(\epsilon)$$

$$= \frac{i\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\int_0^1 dx\left(1+\epsilon\ln\frac{4\pi\mu^2}{\Delta}\right)\left(\frac{1}{\epsilon}-\ln e^\gamma\right)+O(\epsilon)$$

$$= \frac{i\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\left(\frac{1}{\epsilon}+\int_0^1 dx\ln\frac{4\pi\mu^2}{e^\gamma\Delta}\right)+O(\epsilon)$$

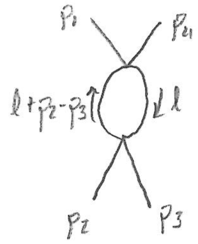
The T-channel contribution is:



$$= \frac{\lambda^2}{2}\mu^{4\epsilon}\int\frac{d^dl}{(2\pi)^d}\frac{1}{[(\underbrace{p_2-p_4}_{p\leftrightarrow p_2-p_4\equiv p_T}+l)^2-m^2+i\epsilon][l^2-m^2+i\epsilon]}$$

$$= \frac{i\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\left(\frac{1}{\epsilon}+\int_0^1 dx\ln\frac{4\pi\mu^2}{e^\gamma\Delta_T}\right)+O(\epsilon)$$

And the U-channel gives:



$$= \frac{i\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\left(\frac{1}{\epsilon}+\int_0^1 dx\ln\frac{4\pi\mu^2}{e^\gamma\Delta_U}\right)+O(\epsilon)$$

The one-loop $2 \to 2$ process is then described by:

$$i\mathcal{M}_{2\to 2} = \frac{i\lambda^2 \mu^{2\epsilon}}{2(4\pi)^2}\left[\frac{3}{\epsilon} + \int_0^1 dx \left(\ln\frac{\tilde{\mu}^2}{\Delta_S} + \ln\frac{\tilde{\mu}^2}{\Delta_T} + \ln\frac{\tilde{\mu}^2}{\Delta_U}\right)\right]$$

Next we will learn to remove these UV divergences

3) Renormalization in φ^4 theory

Now we differentiate between "bare" and "renormalized" fields, masses, couplings

$$\varphi_0 = Z_\varphi^{1/2} \varphi_R$$

$$\mathcal{L}_0 = \frac{1}{2} Z_\varphi (\partial^\mu \varphi_R)(\partial_\mu \varphi_R) - \frac{1}{2} Z_\varphi m_0^2 \varphi_R^2 - \frac{\lambda_0}{4!} \varphi_R^4$$

let $\quad Z_\varphi \equiv 1 + \delta_\varphi$

$\quad Z_\varphi m_0^2 \equiv m_R^2 (1 + \delta m)$

$\quad Z_\varphi^2 \lambda_0 \equiv \lambda_R (1 + \delta \lambda)$

This gives back what looks like two copies of $\mathcal{L}$:

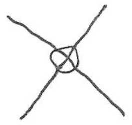
$$\mathcal{L}_0 = \frac{1}{2} (\partial^\mu \varphi_R)(\partial_\mu \varphi_R) - \frac{1}{2} m_R^2 \varphi_R^2 - \frac{\lambda_R}{4!} \varphi_R^4 \quad \leftarrow \text{renormalized } \mathcal{L}_R$$

$$+ \frac{1}{2} \delta_\varphi (\partial^\mu \varphi_R)(\partial_\mu \varphi_R) - \frac{1}{2} m_R^2 \delta m \varphi_R^2 - \frac{\lambda_R}{4!} \delta\lambda \varphi_R^4 \quad \leftarrow \text{counterterm } \mathcal{L}_{ct}$$

So $\mathcal{L}_R$ gives us the usual FRs in terms of Renormalized quantities

$\mathcal{L}_{ct}$ gives new "counter terms" (Feynman Rules)

 $= i(p^2 \delta_\varphi - m_R^2 \delta m)$ $\quad \leftarrow$ treated as a perturbation!

 $= -i \lambda_R \delta\lambda$

These counterterms are treated as formally higher order, and formally infinite

Consider the 2pt function in the presence of the ct:

$$= \underbrace{\frac{+i\lambda_R}{2(4\pi)^2} m_R^2 \left[\frac{1}{\epsilon} + 1 + \ln\frac{4\pi\mu^2}{e^{\gamma} m_R^2}\right]}_{p \text{ indep, notice } m \to m_R,\ \lambda \to \lambda_R} + i(p^2\delta_\varphi - m_R^2 \delta m)$$

Since the loop is p-indep, we can't cancel the divergence w/ δ_φ $\to$ $\delta\varphi = 0$ @ 1 loop

But we can w/ δm:

1) cancel only $\frac{1}{\epsilon}$ $\to$ "minimal subtraction" (MS) scheme $\to$ $\delta m = \frac{+\lambda_R}{2(4\pi)^2 \epsilon}$

2) cancel $\frac{1}{\epsilon}$ & $\ln\frac{4\pi}{e^{\gamma}}$ $\to$ $\overline{MS}$ or modified MS $\to$ $\delta m = \frac{+\lambda_R}{2(4\pi)^2}\left(\frac{1}{\epsilon} + \ln\frac{4\pi}{e^{\gamma}}\right)$

3) require $m_R = m_{pole}$ $\to$ notice 1 & 2 don't necessarily achieve this!
we'll come back to this

Next consider $2 \to 2$ scattering:

$$= \frac{i\lambda_R^2 \mu^{2\epsilon}}{2(4\pi)^2}\left[\frac{3}{\epsilon} + \int_0^1 dx \left(\ln\frac{\tilde{\mu}^2}{\Delta_S} + \ln\frac{\tilde{\mu}^2}{\Delta_T} + \ln\frac{\tilde{\mu}^2}{\Delta_U}\right)\right] - i\lambda_R \delta\lambda\, \mu^{2\epsilon}$$

we can drop these $\mu^{2\epsilon}$ to make life a little easier, only intro $\mu^{2\epsilon}$ for terms beyond Leading order (otherwise we cancel all at end and get same result)

In the MS scheme the divergence is cancelled for

$$\delta\lambda = \frac{+\lambda_R}{2(4\pi)^2}\frac{3}{\epsilon}$$

And for $\overline{MS}$

$$\delta\lambda = +\frac{\lambda_R}{2(4\pi)^2}\left[\frac{3}{\epsilon} + \ln\frac{4\pi}{e^{\gamma}}\right]$$

It is sometimes possible to define an on-shell-like scheme for couplings

eg. In QED $e_R \Gamma^{\mu}_{\bar{e}eA}(0) = e_R \gamma^{\mu}$

$\to$ enforces e_R is the electric charge as measured by coulombs law at large distances

4) On-shell renormalization in φ^4

Consider the 2-point function at 1 loop:

$iG =$ 

We can, now that we know these 1-loop fuctions, include a subset of n-loop contributions:

$iG_R =$ 

$$= \frac{i}{p^2 - m_R^2} + \frac{i}{p^2 - m_R^2}(i\Sigma)\frac{i}{p^2 - m_R^2} + \frac{i}{p^2 - m_R^2}(ip^2\delta_\varphi - im_R^2\delta_m)\frac{i}{p^2-m_R^2} + \dots$$

$$= \frac{i}{p^2 - m_R^2 + \underbrace{\Sigma(p^2) + p^2\delta_\varphi - m_R^2\delta_m}_{\equiv \Sigma_R(p^2)}}$$

If we want a physical pole in iG we need:

$$\Sigma_R(m_p^2)\big|_{p^2 \to m_R^2} = 0 = m_R^2 - m_p^2$$

Which gives:

$$iG_R = \frac{i}{p^2 - m_R^2 + \Sigma_R(p^2)|_{p^2\to m_R^2}} = \frac{i}{p^2 - m_R^2 + m_R^2 - m_p^2} = \frac{i}{p^2 - m_p^2}$$

We also need the pole to have residue i:

$$i = \lim_{p^2 \to m_p^2} (p^2 - m_p^2)\frac{i}{p^2 - m_R^2 + \Sigma_R(p^2)}$$

$$= \lim_{p^2\to m_p^2} \frac{i\frac{d}{dp^2}(p^2 - m_p^2)}{\frac{d}{dp^2}(p^2 - m_R^2 + \Sigma_R(p^2))}$$

$$= \lim_{p^2 \to m_p^2} \frac{i}{1 + \frac{d}{dp^2}\Sigma_R(p^2)} \implies \frac{d}{dp^2}\Sigma_R(p^2)\Big|_{p^2\to m_p^2} = 0$$

Recalling $\Sigma_R(p^2) =$  $+$ 

We have:

$$\Sigma_R(p^2) = \frac{-i\lambda_R}{2(4\pi)^2} m_R^2 \left[\frac{1}{\epsilon} + 1 + \ln\frac{4\pi\mu^2}{e^\gamma m_R^2}\right] + i(p^2\delta_\varphi - m_R^2\delta m)$$

Since the loop has no p-dep we needed $\delta\varphi = 0$

So to satisfy the onshell conditions:

$$\Sigma_R(p^2)\big|_{p^2\to m_R^2} = 0$$

$$\frac{d}{dp^2}\Sigma_R(p^2)\big|_{p^2\to m_p^2} = 0$$

We need δm to cancel the full 1-loop contribution.

The second condition sets $\delta_\varphi = 0$

So $\quad \delta m = \frac{\lambda_R}{2(4\pi)^2}\left[\frac{1}{\epsilon} + 1 + \ln\frac{4\pi\mu^2}{e^\gamma m_R^2}\right]$

5) Renormalization group Equations (RGE) and φ^4 thry

To work w/ dimensional regularization we introduced the dimensionful "μ"

Physics <u>should not</u> depend on this made up quantity → consequences

Recall: $Z_\varphi^2 \lambda_0 = \lambda_R \mu^{2\epsilon}(1+\delta\lambda)$

$$\underbrace{\lambda_0}_{\text{doesn't depend on } \mu!} = \lambda_R \mu^{2\epsilon} Z_\lambda Z_\varphi^2 \qquad \text{w/ } Z_\lambda \equiv 1+\delta\lambda$$

$$\mu \frac{d}{d\mu}\lambda_0 = 0 = \mu\frac{d}{d\mu}(\lambda_R Z_\lambda \underbrace{Z_\varphi^2}_{\delta_\varphi = 0 \Rightarrow Z_\varphi = 1} \mu^{2\epsilon})$$

← all terms in ( ) must change w/ renormalization scale "μ" to compensate eachother → λ_0 is μ-indep

$$\mu\frac{d}{d\mu}\lambda_0 = 0 = \mu\frac{d\lambda_R}{d\mu}(Z_\lambda \mu^{2\epsilon}) + \mu \underbrace{\frac{dZ_\lambda}{d\mu}}_{Z_\lambda = 1 + \frac{\lambda_R}{32\pi^2}\frac{3}{\epsilon}}(\lambda_R \mu^{2\epsilon}) + 2\epsilon \lambda_R Z_\lambda \mu^{2\epsilon}$$

only implicit dep on μ in λ_R

$$= \mu\frac{d\lambda_R}{d\mu}(Z_\lambda \mu^{2\epsilon}) + \mu\frac{d\lambda_R}{d\mu}\frac{\partial Z_\lambda}{\partial\lambda_R}(\lambda_R\mu^{2\epsilon}) + 2\epsilon\lambda_R Z_\lambda \mu^{2\epsilon}$$

Divide through by $\lambda_R Z_\lambda \mu^{2\epsilon}$:

$$0 = \mu\frac{d}{d\mu}\ln\lambda_R + \mu\frac{d}{d\mu}\ln\lambda_R \underbrace{\left(\frac{3\lambda_R}{32\pi^2\epsilon}\right)/Z_\lambda}_{\frac{1}{\epsilon}/(1+1/\epsilon)\, \sim\, \frac{1}{\epsilon} + \text{higher loops}} + 2\epsilon$$

so we can take $Z_\lambda \to 1$ here

$$= \mu\frac{d}{d\mu}\ln\lambda_R\left(1+\frac{3\lambda_R}{32\pi^2\epsilon}\right) + 2\epsilon$$

Solving this gives:

$$\mu \frac{d}{d\mu} \ln \lambda_R = -2\epsilon \left(1 + \frac{3\lambda_R}{32\pi^2\epsilon}\right)^{-1}$$

$$\sim -2\epsilon \left(1 - \frac{3\lambda_R}{32\pi^2\epsilon} + \text{higher order}\right)$$

$$\rightarrow \frac{3\lambda_R}{16\pi^2} + O(\epsilon) \qquad \text{"}\beta \text{ function for } \lambda\text{"}$$

Consider the mass m_R:

$$m_0^2 = m_R^2 Z_m \cancel{Z_\varphi^{-1}}^{\,1}$$

$$\mu \frac{d}{d\mu} m_0^2 = 0 = \left(\mu \frac{d m_R^2}{d\mu}\right) Z_m + m_R^2 \, \mu \frac{d Z_m}{d\mu} \qquad *$$

For MS we had

$$Z_m = 1 + \frac{\lambda_R}{32\pi^2\epsilon}$$

$$\mu \frac{d Z_m}{d\mu} = \frac{1}{32\pi^2\epsilon} \mu \frac{d\lambda_R}{d\mu} = \frac{\lambda_R}{32\pi^2\epsilon} \mu \frac{d \ln \lambda_R}{d\mu}$$

Dividing $*$ through by $m_R^2 Z_m$:

$$0 = \mu \frac{d}{d\mu} \ln m_R^2 + \frac{\lambda_R}{32\pi^2\epsilon} \left(\mu \frac{d \ln \lambda_R}{d\mu}\right) / Z_m$$

← again this is 2-loop, so take $Z_m \to 1$

$$\mu \frac{d}{d\mu} \ln m_R^2 = -\frac{\lambda_R}{32\pi^2\epsilon} (-2\epsilon)\left(1 - \underbrace{\frac{3\lambda_R}{32\pi^2\epsilon}}_{\text{2-loop}} + \cdots\right)$$

$$= + \frac{\lambda_R}{16\pi^2\epsilon} \qquad \text{"Anomalous dimension of the mass"}$$

6) Renormalizability

consider the following $\mathcal{L}$:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{\lambda}{4!}\varphi^4 + \frac{c}{6!}\varphi^6$$

(notice $m=0$ for simplicity)

In $d=4-2\epsilon$ dimensions:

$$[c\varphi^6] = 0$$

$$[\varphi] = 1-\epsilon$$

$$[c] = d - 6[\varphi] = 4-2\epsilon - 6(1-\epsilon) = -2+4\epsilon$$

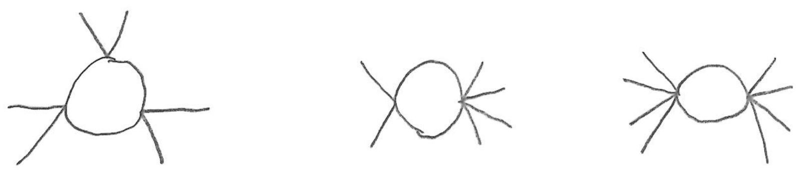
In $d=4$ c has inverse-mass-squared dimension

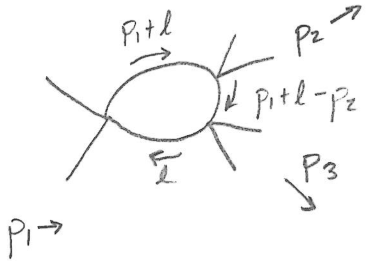
In $d=4-2\epsilon$ we take

$$c \to c\mu^{4\epsilon}$$

In order to retain this.

Let's consider the following loop diagrams:





$p_1 = p_2 + p_3$

$$i\mathcal{M} = (-i\lambda)^3 \int \frac{d^d l}{(2\pi)^d} \frac{i}{(p_1+l)^2+i\epsilon} \frac{i}{(p_1+l-p_2)^2+i\epsilon} \frac{i}{(l^2+i\epsilon)}$$

$$= \lambda^3 \int \frac{d^d l}{(2\pi)^d} \int_0^1 dx\,dy\,dz \frac{\Gamma(3)\,\delta(1-x-y-z)}{(x[(p_1+l)^2+i\epsilon] + y[(p_1+l-p_2)^2+i\epsilon] + z[l^2+i\epsilon])^3}$$

$$= 2\lambda^3 \int_0^1 dx \int_0^{1-x} dy \frac{1}{(x[l^2+2p_1\cdot l+p_1^2+i\epsilon] + y[p_1^2+l^2+p_2^2+2p_1\cdot l-2p_1\cdot p_2-2p_2\cdot l+i\epsilon] + (1-x-y)[l^2+i\epsilon])^3}$$

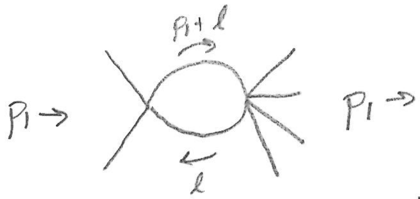
$$l^2+2xp_1\cdot l+2yp_1\cdot l-2yp_2\cdot l = \underbrace{[l+(x+y)p_1-yp_2]^2}_{\equiv q^2} - (x+y)^2p_1^2 - y^2p_2^2 + 2y(x+y)p_1\cdot p_2$$

$$= 2\lambda^3 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d q}{[q^2 \underbrace{- (x+y)(1+x+y)p_1^2 + y(1+y)p_2^2 - 2y(1+x+y)p_1\cdot p_2 + i\epsilon}_{\equiv\Delta}]^3}$$

$$= 2\lambda^3 \int_0^1 dx \int_0^{1-x} dy \frac{i(-1)^{-3}}{(4\pi)^{2-\epsilon}} \frac{1}{\Delta^{3-2+\epsilon}} \frac{\cancel{\Gamma(d/2)}\,\Gamma(3-2+\epsilon)}{\Gamma(3)\,\cancel{\Gamma(d/2)}}$$

$$\Gamma(1+\epsilon) = \epsilon\,\Gamma(\epsilon)$$
$$= \epsilon\left(\frac{1}{\epsilon} + \text{finite}\right)$$
$$= \text{finite!}$$

$= \text{finite}$



leading correction

$$i\mathcal{M} = (-i\lambda\mu^{2\epsilon})(ic)\int \frac{d^d\ell}{(2\pi)^d}\,\frac{i}{(p_1-\ell)^2+i\epsilon}\,\frac{i}{(\ell^2+i\epsilon)}$$

$$= -\lambda c\mu^{2\epsilon}\int \frac{d^d\ell}{(2\pi)^d}\int_0^1 dx\,\frac{1}{\left[x\left[(p_1+\ell)^2+i\epsilon\right]+(1-x)(\ell^2+i\epsilon)\right]^2}$$

$$\ell^2+2x\,p_1\cdot\ell = \underbrace{(\ell+xp_1)^2}_{\equiv q^2} - x^2p_1^2$$

$$= -\lambda c\mu^{2\epsilon}\int \frac{d^d\ell}{(2\pi)^d}\int_0^1 dx\,\frac{1}{[q^2\underbrace{+x(1-x)p_1^2}_{\equiv-\Delta}+i\epsilon]^2}$$

$$= -\lambda c\int_0^1 dx\,\frac{i(-1)^{-2}}{(4\pi)^{2-\epsilon}}\left(\frac{\mu^2}{\Delta}\right)^{\epsilon}\frac{\cancel{\Gamma(d/2)}\,\Gamma(2-2+\epsilon)}{\Gamma(2)\,\cancel{\Gamma(d/2)}}$$

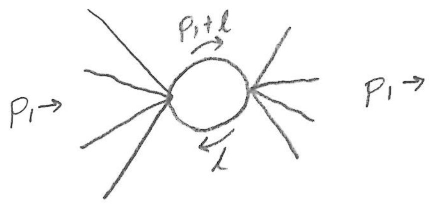
$$\Gamma(\epsilon) = \frac{1}{\epsilon} + \text{finite}$$

$$= \frac{-i\lambda c}{(4\pi)^2}\,\frac{1}{\epsilon}$$

This divergence is removed by

$$Z_\phi^6\,c \equiv c_R(1+\delta c)$$

 $= i\delta c$

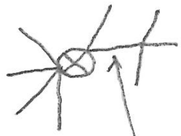


$$i\mathcal{M} = (ic)^2 \int \frac{d^d l}{(2\pi)^d} \frac{i}{(p_1-l)^2+i\epsilon} \frac{i}{l^2+i\epsilon}$$

$$= \frac{ic^2}{(4\pi)^2} \frac{1}{\epsilon}$$

This divergence cannot be removed w/ δc!

We also can't use



$\delta c \lambda$

mtm dep in prop!

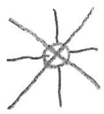
We say this theory is "nonrenormalizable"

This doesn't mean NOT renormalizable

It means an ∞ number of counter terms are needed to render the thry finite

eg $\frac{c_8}{8!}\phi^8 \to$

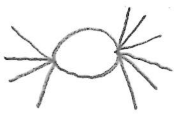
$+$



ic_8 $\quad i\delta c_8$

this cancels the divergence!

But then



$\sim \frac{i c_6 c_8}{(4\pi)^2} \frac{1}{\epsilon}$

This requires c_{10} and so on...

Rewriting our $\mathcal{L}$:

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu \varphi)^2 - \tfrac{\lambda}{4!}\varphi^4}_{\mathcal{L}_4} + \sum_{i=1}^{\infty} \frac{C_{4+2i}}{\Lambda^{2i}} \varphi^{4+2i} \qquad *$$

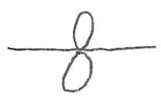
1) a theory is renormalizable if all operators $\hat{O}$ are of dimension:

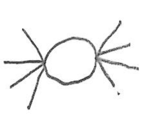
$$0 \leq [\hat{O}] \leq d$$

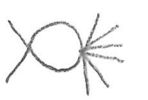
and requires only the number of counter terms present in $\mathcal{L}$

2) a nonrenormalizable theory <u>is</u> renormalizable order-by-order in $\frac{1}{\Lambda}$ of $*$

ie  $\sim \frac{\lambda c_6}{\Lambda^2}$ is renormalizable w/ just δc_6 and the counterterms in $\mathcal{L}_4$

 $\sim \frac{c_6}{\Lambda^2}$ is as well, the # of loops doesn't matter, just the order in $\frac{1}{\Lambda}$

 $\sim \frac{c_6^2}{\Lambda^4}$ requires all counter terms at $\frac{1}{\Lambda^4}$, eg δc_8

 $\sim \frac{\lambda c_8}{\Lambda^4}$ is as above

So nonrenormalizable theories are predictive if $\frac{1}{\Lambda^2}$ is a small number (essentially $p^2 \ll \Lambda^2$)

We'll discuss this more when we discuss <u>Effective field theories</u>