Fundamentals:

what is a vector?

In intro physics - a quantity w/ magnitude & direction

For advanced physics - an object v_i transforming according to $R_{ij} v_j$

e.g. Normal rotations $w_i' = R_{ij} v_j$

$$w_i^T w_i = v_j^T \underbrace{R_{ji}^T R_{ik}}_{} v_k = v_j^T v_j$$

Orthogonal $\rightarrow R_{ji}^T R_{ik} = \delta_{jk}$

Notice the norm is preserved $w^2 = v^2$

This isn't limited to squares, if p_i & x_i are vectors under rotations

$$p_i' x_i' = p_i R_{ji}^T R_{jk} x_k = p_i x_i$$

For an N-dimensional space this is described by the rotation group $O(N)$

This is abstracted by defining a Rank n Tensor:

$$R_{i_1 j_1} R_{i_2 j_2} \cdots R_{i_n j_n} T_{j_1 j_2 \cdots j_n}$$

$$T'_{i_1 i_2 \cdots i_n} T'_{i_1 i_2 \cdots i_n} = T_{i_1 \cdots i_n} T_{i_1 \cdots i_n}$$

ie a rank n tensor transforms under the application of n rotations

In classical mecanics we have the moment of Inertia:

$$L_i = I_{ij} \omega_j \rightarrow R_{ij} L_j = R_{ij} I_{jk} R_{kl}^T R_{lm} \omega_m$$

angular momentum

angular velocity

1) Representation thry, spin example

Recall from QM when we discuss spin,
The spin op, $\vec{S}$ can be defined as:

$$\vec{S} = \frac{1}{2}\vec{\sigma} \qquad (\hbar = 1)$$

w/ $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) = \sigma_i$ w/ σ_i the Pauli Matrices

Further recall:

$$[S_i, S_j] = i\epsilon_{ijk} S_k$$

And if two ops don't commute, a quantum state can't have simultaneous eigenvalues for both ops

So it's convenient to use $S^2 = \vec{S}\cdot\vec{S}$ which commutes w/ all S_i

$$[S^2, S_i] = 0$$

So we can have simultaneous eigenvalues of, e.g. S^2 & S_z

let $|\pm\rangle$ be an eigenket w/ $S_z = \pm 1/2$:

$$S_z|\pm\rangle = \pm 1/2 |\pm\rangle$$
$$S_x|\pm\rangle = 1/2 |\mp\rangle$$
$$S_y|\pm\rangle = \pm i/2 |\mp\rangle$$
$$S^2|\pm\rangle = \frac{1}{2}(1 + 1/2)|\pm\rangle$$

And then we label a spin state by its total spin & z-projection:

$$|s, m\rangle \xrightarrow[\text{spin } 1/2]{} |1/2, \pm 1/2\rangle$$

If we want to add two spin ½ particles we take the tensor product of two spin ½ Hilbert spaces: $\mathcal{H}_{1/2}$

$$\mathcal{H}_{\frac{1}{2}\oplus\frac{1}{2}} = \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$$

For example:

$$\left.\begin{array}{l} |+\rangle \otimes |+\rangle \equiv |++\rangle \\ |+\rangle \otimes |-\rangle = |+-\rangle \\ |-\rangle \otimes |+\rangle = |-+\rangle \\ |-\rangle \otimes |-\rangle = |--\rangle \end{array}\right. \quad \text{(order matters! for the middle two)} \qquad (A)$$

We can define spin operators in this space:

$$\vec{S} = S_i = \frac{1}{2}\sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \frac{1}{2}\sigma_i \equiv S_i \oplus S_i$$

$$S_z|++\rangle = (S_z \oplus S_z)|++\rangle$$

$$= (S_z|+\rangle) \otimes |+\rangle + |+\rangle \otimes (S_z|+\rangle)$$

$$= \left(\frac{1}{2} + \frac{1}{2}\right)|++\rangle$$

$$S_z|--\rangle = -|--\rangle$$

$$S_z|+-\rangle = S_z|-+\rangle = 0$$

Now consider

$$S^2 = (S_i \otimes \mathbb{1} + \mathbb{1} \otimes S_i)^2$$

$$= \frac{1}{4}\left[6\,\mathbb{1} \otimes \mathbb{1} + 2(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z)\right]$$

$$S^2|\pm\pm\rangle = 2|\pm\pm\rangle = 1(1+1)|\pm\pm\rangle$$

$$S^2|\pm\mp\rangle = \quad (|+-\rangle + |-+\rangle)$$

The states we chose in (A) are not simultaneous eigenvalues of S_z & S^2!

How do we form eigenkets w/ simultaneous eigenvalues?

recall ladder ops for $\mathcal{H}_{1/2}$:

$$S_\pm = S_x \pm i S_y$$

$$S_- |+\rangle = |-\rangle$$

$$S_+ |-\rangle = |+\rangle$$

$$S_- |-\rangle = S_+ |+\rangle = 0$$

So for the addition of two spin ½ particles we have:

$$\mathcal{S}'_\pm = S_\pm \otimes \mathbb{1} + \mathbb{1} \otimes S_\pm$$

Since we saw $|++\rangle$ ($|--\rangle$) has simultaneous eigenvalues of S^2 & S_z we can start "at the top (bottom) of the ladder and lower (raise) to the bottom (top)"

You will show this results in the following (normalized) states:

$$\left.\begin{array}{c} |++\rangle \\ \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) \\ |--\rangle \end{array}\right\} \text{Spin 1 combo} \left\{\begin{array}{l} |11\rangle \\ |10\rangle \\ |1-1\rangle \end{array}\right.$$

$$\frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \rightarrow \text{spin } 0 \rightarrow |00\rangle$$

So adding two spin ½ particles yields a spin one eigenket w/ S_z projections $\pm 1, 0$ and a spin zero eigenket

2) Spin Algebra:

recall: $[S_i, S_j] = i\epsilon_{ijk}S_k$

this is the Lie algebra, $su(2)$

Lie algebras correspond to Lie groups. So $su(2) \rightarrow SU(2)$

recall: a group is a set of elements w/ an operation satisfying:

1) closure: if $A, B \in G$ $(A * B) \in G$
2) associativity: $(A*B)*C = A*(B*C)$
3) $\exists$ an identity such that: $A*\mathbb{1} = \mathbb{1}*A = A \quad \forall A \in G$
4) an inverse: $\forall A \in G \; \exists A^{-1}$ such that $A^{-1}A = AA^{-1} = \mathbb{1}$

A group is an abstract mathematical object, they are easier to visualize by choosing a "representation" of the group

e.g. $SU(2)$ has a 2D representation

any $SU(2)$ matrix in the 2d rep can be written as:

$$U(\theta) = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} \quad \text{for arbitrary } \vec{\theta} = (\theta_1, \theta_2, \theta_3)$$

It is possible that a representation is formed of more than one representations.

"representation" refers to the group, but physicists frequently use it to refer to a "vector/tensor transforming according to the representation"

For example, the spin ½ representation is irreducible,

$$|s, m\rangle = |\tfrac{1}{2}, \pm\tfrac{1}{2}\rangle,$$

can't be further decomposed (eg there is no way to add spin $\emptyset$ to make ½, and the only spin less than ½ is $\emptyset$)

But we added $\frac{1}{2} \otimes \frac{1}{2}$ to obtain:

$$|s, m\rangle = |1, 1\rangle, \ |1, 0\rangle, \ |1, -1\rangle$$

$$|0, 0\rangle$$

The $s=1$ states are a 3D irreducible representation of $SU(2)$
$s=0$ state is a 1D irreducible representation of $SU(2)$

We can say: $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0_A$

the $s=0$ rep was Antisymmetric

A mathematician wouldn't use the total spin as a label, but would instead label the representations by dimension:

$$2 \otimes 2 = 3 \oplus 1_A$$

A mathematician would also take issue w/ this vague explanation of a representation.

A <u>representation</u> of a group G on a vector space V is a group homomorphism from G to $GL(V)$ (the general linear group on V):

$$\rho : G \to GL(V) \quad \text{such that} \quad \underbrace{\rho(g_1 g_2) = \rho(g_1)\rho(g_2)}_{\text{group homomorphism}} \quad \forall g_1, g_2 \in G$$

LH: product in G
RH: product in $GL(V)$

A subspace W of V that is invariant under the group action is called a subrepresentation.

If V has two subrepresentations which are the zero-dim subspace and V itself, the representation is irreducible.

$$\left.\begin{array}{l} |++\rangle \\ |+-\rangle \\ |-+\rangle \\ |--\rangle \end{array}\right\} \text{4D representation}$$

$$\left.\begin{array}{l} |11\rangle \\ |10\rangle \\ |1\,-1\rangle \end{array}\right\} \text{3D irrep}$$

$$|00\rangle \quad \text{1D irrep}$$

3) The Lorentz Group

In constructing a QFT we want the theory to respect the tenets of Special Relativity. To do this we want to write the theory in terms of Lorentz invariants.

Analogous to Rotational invariants we have:

$$\bar{x}^\mu = \underset{\uparrow}{\Lambda^\mu{}_\nu} x^\nu$$

Lorentz xform

$$\bar{x}^2 = x^2 = g_{\mu\nu}\bar{x}^\mu \bar{x}^\nu$$
$$= g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \bar{x}^\mu \bar{x}^\nu$$

$g_{\mu\nu}$ = Minkowski Metric

This gives the condition:

$$g_{\mu\nu}\Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma}$$
$$= \Lambda_{\nu\rho}\Lambda^\nu{}_\sigma$$
$$g^{\rho\alpha} g_{\alpha\sigma} = g^{\rho\alpha}\Lambda_{\nu\alpha}\Lambda^\nu{}_\sigma$$
$$\delta^\rho_\sigma = \Lambda_\nu{}^\rho \Lambda^\nu{}_\sigma \Rightarrow (\Lambda^{-1})^\rho{}_\nu = \Lambda_\nu{}^\rho$$

So we have the inverse L xform

L xforms form a group:

1) The product of 2 LT → LT
2) The product is associative
3) There is an identity $\delta^\mu{}_\nu$
4) ∃ an inverse ∀ $\Lambda^\mu{}_\nu$

Consider the infinitesimal LT:

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \delta\omega^{\mu}{}_{\nu}$$

Then we have:

$$g_{\rho\sigma} = g_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = g_{\mu\nu}\left(\delta^{\mu}{}_{\rho} + \delta\omega^{\mu}{}_{\rho}\right)\left(\delta^{\nu}{}_{\sigma} + \delta\omega^{\nu}{}_{\sigma}\right)$$

$$= g_{\rho\sigma} + \underbrace{\delta\omega_{\sigma\rho} + \delta\omega_{\rho\sigma}}_{=0} \rightarrow \delta\omega_{\sigma\rho} = -\delta\omega_{\rho\sigma}$$

$\delta\omega_{\rho\sigma}$ is a 4x4 $\mathbb{C}$ antisymmetrix matrix

→ 3 $\mathbb{C}$ independent LT

or 6 independent LT

→ 3 rotations $\delta\omega_{ij} = -\epsilon_{ijk}\hat{n}_k\delta\theta$ ($\hat{n}$ is a unit vector)

3 boosts $\delta\omega_{i0} = \hat{n}_i\,\delta\eta$

Compounding these infinitesimal LT reaches all "Proper" LTs

ie $\det\Lambda = 1$

So we have identified the Proper subgroup

We will further restrict ourselves to the Proper orthochronous subgroup

$$\Lambda^{0}{}_{0} \geq +1$$

This is essentially removing elements reached by Parity and time reversal operations.

In quantum theories symmetries are represented by unitary operators. So associating $U(\Lambda)$ w/ the LT Λ we have:

$$U(\Lambda'\Lambda) = U(\Lambda')U(\Lambda)$$

For an infinitesimal LT we have:

$$U(1+\delta\omega) = \mathbb{1} + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}$$

"generators" of the L group

antisymmetric, so M must be too

In the exercises we will show:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\mu\rho}M^{\nu\sigma} - (\mu \leftrightarrow \nu)) - (\rho \leftrightarrow \sigma)$$

And associating:

angular momentum $J_i = \frac{1}{2}\epsilon_{ijk}M^{jk}$

boosts $K_i = M^{i0}$

$$[J_i, J_j] = i\epsilon_{ijk}J_k$$
$$[J_i, K_j] = i\epsilon_{ijk}K_k$$
$$[K_i, K_j] = -i\epsilon_{ijk}J_k$$

Generators, symmetries, & conservation laws are closely related e.g:

symmetry	generator	conserved qty
translations	$\hat{p}_i$	momentum
rotational invariance	$\hat{L}_i$	ang. momentum
t-translations	$\hat{H}$	Energy

We need to develop the idea of representations of groups to help make Lorentz covariance manifest in our QFT

Fields as irreps of the Lorentz Group

recall the 4D rep we obtained from adding 2 spin ½ particles

mixed $s=1$ and $s=0$ states:

$$|{\pm}{\mp}\rangle = \frac{1}{\sqrt{2}}\left(|10\rangle \pm |00\rangle\right)$$

so under a general rotation

$$U(\theta_1,\theta_2) = e^{i\vec{\theta}_1\cdot\vec{S}_1}\, e^{i\vec{\theta}_2\cdot\vec{S}_2}$$

this state mixes between all 4 states.

However, the 3D and 1D irreps do not!

For a Special Relativistic theory it is desireable (though not necessary) to form our theory of fields in Irreducible representations of the Lorentz Group.

We also generally restrict ourselves to the

Proper: $\det\Lambda = +1$

Orthochronous: $\Lambda^0{}_0 \geq +1$

subgroup

This is essentially a subgroup of the L group from which the rest of the L group can be reached by Parity and Time reversal transformations.

The Lorentz group is rotations and boosts,

rotations are generated by the angular momentum op: $\vec{J}$

boost are generated by $\vec{K}$

$$J_i \equiv \frac{1}{2} \epsilon_{ijk} M^{jk}$$

$$K_i \equiv M^{i0}$$

where $M^{\mu\nu}$ are the generators of the Lorentz group, which specify the Lie Algebra of the Lorentz group:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho} M^{\nu\sigma} - (\mu \leftrightarrow \nu)\right) - (\rho \leftrightarrow \sigma)$$

Using this one can show:

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

If we instead define N_i & $N_i^\dagger$ which are mixtures of boosts and rotations:

$$N_i = \frac{1}{2}(J_i - iK_i)$$

$$N_i^\dagger = \frac{1}{2}(J_i + iK_i)$$

Then we obtain:

$$\left.\begin{aligned} [N_i, N_j] &= i\epsilon_{ijk} N_k \\ [N_i^\dagger, N_j^\dagger] &= i\epsilon_{ijk} N_k^\dagger \end{aligned}\right\} \text{2 SU(2) algebras exchanged by Hermitian conjugation}$$

$$[N_i, N_j^\dagger] = 0$$

So we have two $SU(2)$ algebras, reps of the L group are just the addition of spin

We can label our reps by:

$$(2n+1, 2n'+1)$$

Which has $(2n+1)\times(2n'+1)$ components

w/ n an n' integers or half integers (as with spin)

Ex: $n=0, n'=0 \rightarrow (1,1)$ w/ 1 component $\rightarrow$ Scalar (Singlet)

$n=\frac{1}{2}, n'=0 \rightarrow (2,1)$ w/ 2 components $\rightarrow$ left handed spinor

$n=0, n'=\frac{1}{2} \rightarrow (1,2)$ w/ 2 components $\rightarrow$ right handed spinor

$n=\frac{1}{2}, n'=\frac{1}{2} \rightarrow (2,2)$ w/ 4 components $\rightarrow$ vector

Notice a vector w/ 4 components, V^{μ}; is a 4D rep analogous to adding two spin ½ particles

$\rightarrow$ one 3D irrep + one 1D irrep

Given $J_i = N_i + N_i^{\dagger}$ we deduce these components are (intrinsic) angular momentum.