Final Exam for "Modern Methods in Particle Physics" – Tyler Corbett

You may use any expressions derived in class or in the assignments without proof/re-deriving them.

1. IBP with covariant derivatives

   Consider a field F with covariant derivative:

$$D_\mu F \equiv (\partial_\mu + igA_\mu) F \tag{1}$$

   Assume A is a field corresponding to a nonabelian gauge symmetry and that it does not commute with itself (e.g. $A_\mu = A_\mu^I \sigma^I$ where the σ^I are generators of some nonabelian gauge symmetry). Show:

$$(D_\mu F)^\dagger (D_\mu F) = -F^\dagger D^2 F \tag{2}$$

   Convince yourself (no need to write anything) that covariant derivatives can be manipulated by integration by parts in the same manner as normal derivatives (∂_μ) and use this result if necessary later in the exam.

2. Equations of motion in the SM

   Derive the equations of motion for the SM fields $\{B_\mu, W_\mu^I, G_\mu^A, Q, L, u, d, e, H\}$ in the SM:

$$\begin{aligned}
\mathcal{L}_{\text{SM}} \;=\; & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W_{\mu\nu}^I W_{\mu\nu}^I - \frac{1}{4}G_{\mu\nu}^A G_{\mu\nu}^A \\
& + i\bar{L}\not{D}L + i\bar{Q}\not{D}Q + i\bar{e}\not{D}e + i\bar{u}\not{D}u + i\bar{d}\not{D}d \\
& + (D^\mu H)^\dagger (D_\mu H) + \mu^2 (H^\dagger H) - \lambda (H^\dagger H)^2 \\
& + \left(Y_e \bar{L} H e + Y_d \bar{Q} H d + Y_u \bar{Q} \tilde{H} u + h.c. \right)
\end{aligned} \tag{3}$$

   Recall $\tilde{H} = i\sigma_2 H^*$. Use the following definitions of the Covariant derivative for the W and G to simplify your results:

$$D_\alpha W_{\alpha\beta}^I = \partial_\alpha W_{\alpha\beta}^I - g_2 \epsilon^{IJK} W_\alpha^J W_{\alpha\beta}^K \tag{4}$$

$$D_\alpha G_{\alpha\beta}^A = \partial_\alpha G_{\alpha\beta}^A - g_3 f^{ABC} G_\alpha^B G_{\alpha\beta}^C \tag{5}$$

   (this is simply the covariant derivative of a field transforming in the adjoint).

3. Tree level Z-pole physics in the SMEFT

   (a) Considering the attached table of the dimension-six operators of the SMEFT, which operators generate $Z\bar{\psi}\psi$ interactions? (There should be five operator forms: accounting for fermion chirality, but otherwise allowing the fermions to be generic.)

   (b) Identify which of the above operators *do not* induce a chiral flip (i.e. don't mix two different chiral fermions). Add these operators to the fermion gauge-kinetic term $i\bar{\psi}\not{D}\psi$. The Feynman rule for this interaction can be written as:

$$\{\bar{\psi}, \psi, Z\} = i\frac{\bar{g}_Z}{2}\gamma_\mu \left(c_L^\psi P_L + c_R^\psi P_R\right) \tag{6}$$

   Where $\{\bar{\psi}, \psi, Z\}$ represents the Feynman rule coupling the Z to two fermions. What are c_L^ψ and c_R^ψ?

   (c) Derive the partial widths of $\Gamma(Z \to e^+e^-)$, $\Gamma(Z \to \bar{\nu}\nu)$ and the leptonic forward-backward asymmetry A_{FB}. Leave them in terms of c_L^ψ and c_R^ψ for convenience. You will expand them later, and this is best done on the computer.

4. Matching a model of a heavy $U(1)$ gauge field, X, mixing with the SM B:

   (a) Derive the equation of motion of the X field from the following UV Lagrangian:

$$\Delta\mathcal{L} = -\frac{1}{4}X_{\mu\nu}X^{\mu\nu} + \frac{M^2}{2}X_\mu X^\mu - \frac{\kappa}{2}B^{\mu\nu}X_{\mu\nu} \tag{7}$$

   (b) Your derived equation of motion should include a term $\partial_\mu B_{\mu\nu}$. Apply the $B_{\mu\nu}$ equation of motion derived in problem 1 to rewrite this in terms of fermionic and Higgs currents. Integrate the part proportional to $B^{\mu\nu}\partial_\mu J_\nu$ by parts so you can again use the EOM of the B to simplify the problem.

   (c) Solve the EOM for X to order $1/M^2$ and use this result in the Lagrangian of Eq. 7 to obtain the mapping of this UV model onto the SMEFT. You should find:

$$\Delta\mathcal{L} = -\frac{\kappa^2}{2M^2}\frac{g_1^2}{4}\left(Q_{H\Box} + 4Q_{HD}\right) - \frac{\kappa^2 g_1^2}{2M^2}\left(\frac{1}{4}Q_{LL} + Q_{Le} + Q_{ee} - \frac{1}{2}Q_{HL}^{(1)} - Q_{He}^{(1)}\right) \tag{8}$$

   Note: You need to use integration by parts to write the H^4D^2 terms in terms of $Q_{H\Box}$ and Q_{HD}.

5. The Real scalar triplet model: A real scalar field with hypercharge 0, which is a singlet under $SU(3)_c$ and a triplet under $SU(2)_L$ has the following transformation properties under the SM gauge group:

$$\Phi \to U_L \Phi U_L^\dagger \qquad \text{w/ } \; U \equiv \exp(i g_2 \theta_L \cdot \sigma/2), \qquad \Phi \equiv \Phi^I \sigma^I/2 \tag{9}$$

   (a) Write down all terms in the renormalizable (dimension ≤ 4) Lagrangian resulting in interactions of Φ with itself and/or the Higgs boson.

   (b) Why doesn't the interaction $\bar{L}\sigma^I L \Phi^I$ exist? (hint: rewrite this term with chiral projection operators explicit).

   (c) Derive the EOM for Φ^A (You don't have to derive the resulting EFT, the result is in the table below)

6. SMEFT fit to Z-pole physics – this problem is best done using a CAS such as Mathematica. Please attach the output from your CAS to the assignment as a pdf.

   (a) Take your expressions from Problem 1, as well as

$$\bar{m}_W^2 = \frac{\bar{g}_2^2 v^2}{4}\,, \tag{10}$$

   and expand them in terms of Wilson Coefficients (i.e. the c_i) and in terms of the input parameter scheme $\{\hat{\alpha}, \hat{m}_Z, \hat{G}_F\}$. We will only consider Wilson coefficients which are generated by the models we have discussed which will slightly reduce the number of contributing operators. (i.e. set any Wilson coefficients not appearing in the table for problem 4 to zero, in the following equations I have also set them to zero.) You should use the following expressions:

$$\begin{aligned}
\bar{m}_Z^2 &= \hat{m}_Z^2 && (11)\\
\bar{g}_Z^2 &= \frac{4\hat{m}_Z^2}{v^2}\left(1 + \frac{v^2}{2} c_{HD}\right) && (12)\\
v^2 &= \hat{v}^2 + \frac{\delta G_F}{\hat{G}_F} && (13)\\
\bar{g}_1 &= \hat{g}_1 + \frac{\hat{g}_1}{2\hat{c}_{2W}} \hat{s}_W^2 \left(\sqrt{2}\delta G_F + \frac{\delta M_Z^2}{\hat{m}_Z^2}\right) = \hat{g}_1 + \delta g_1 && (14)\\
\bar{g}_2 &= \hat{g}_2 - \frac{\hat{g}_2}{2\hat{c}_{2W}} \hat{c}_W^2 \left(\sqrt{2}\delta G_F + \frac{\delta M_Z^2}{\hat{m}_Z^2}\right) = \hat{g}_2 + \delta g_2 && (15)\\
\bar{s}_W^2 &= \hat{s}_W^2 + 2\hat{c}_W^2 \hat{s}_W^2 \left(\frac{\delta g_1}{\hat{g}_1} - \frac{\delta g_2}{\hat{g}_2}\right) = \hat{s}_W^2 + \delta s_W^2 && (16)\\
\delta M_Z^2 &= \frac{1}{2\sqrt{2}} \frac{\hat{m}_Z^2}{\hat{G}_F} c_{HD} && (17)\\
\delta G_F &= -\frac{c_{LL}}{4\hat{G}_F} && (18)
\end{aligned}$$

(Note: We are, as usual, really oversimplifying by ignoring flavor considerations.) Note $\hat{c}_{2W} = \cos\left(2\hat{\theta}_W\right)$. For the input parameter values you should use:

$$\begin{aligned}
\hat{e} &= \sqrt{4\pi\hat{\alpha}} & \hat{g}_1 &= \tfrac{\hat{e}}{\hat{c}_W} & \hat{g}_2 &= \tfrac{\hat{e}}{\hat{s}_W} \\
\hat{s}_W^2 &= \tfrac{1}{2}\left[1 - \sqrt{1 - \tfrac{4\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F\hat{m}_Z^2}}\right] & \hat{v}^2 &= \tfrac{1}{\sqrt{2}\hat{G}_F} & & \\
\hat{\alpha} &= 0.00775489 & \hat{G}_F &= \tfrac{1.16638\cdot 10^{-5}}{\text{GeV}^2} & \hat{m}_Z &= 91.1876 \text{ GeV}
\end{aligned} \tag{19}$$

(b) From your expressions, subtract off the tree-level SM contribution and add on the two-loop SM theory predictions:

$$\Gamma_{\ell\ell}^{\text{th}} = (0.083978 \pm 0.00013) \text{ GeV} \tag{20}$$

$$\Gamma_{\nu\nu}^{\text{th}} = (0.167166 \pm 0.000015) \text{ GeV} \tag{21}$$

$$A_{FB}^{\text{th}} = 0.01632 \pm 0.00022 \tag{22}$$

$$m_W^{\text{th}} = (80.36 \pm 0.01) \text{ GeV} \tag{23}$$

Create a χ^2 for these three predictions as a function of c_{HD}, c_{LL}, $c_{Hl}^{(1)}$, and c_{He}. Use these experimentally measured values:

$$\Gamma_{\ell\ell}^{\text{exp}} = (0.08392 \pm 0.00012) \text{ GeV} \tag{24}$$

$$\Gamma_{\nu\nu}^{\text{exp}} = (0.166333 \pm 0.0005) \text{ GeV} \tag{25}$$

$$A_{FB}^{\text{exp}} = 0.0171 \pm 0.0010 \tag{26}$$

$$m_W^{\text{exp}} = (80.387 \pm 0.016) \text{ GeV} \tag{27}$$

Minimize this χ^2 to find the best fit points for the four Wilson coefficients. Then separately vary each Wilson coefficient from its best fit value while holding the others at their best fit value in order to find the 99% confidence level region for each of these ($\chi^2 - \chi^2_{\text{min}} = 13.28$ for 4 degrees of freedom). For this and the next part, let $c_i \to c_i/\Lambda^2$ and take $\Lambda \sim 1000$ GeV $=$ 1 TeV. *BE CAREFUL! If you use GeV somewhere and then interpret the results in TeV you will make a mistake unless you convert units.*

(c) Recently Tevatron data was reanalyzed, the authors found $m_W^{\text{exp}} = 80.4335 \pm 0.0094$ GeV. Redo 3b) for this new W-mass measurement.

7. Comparison of UV models mapped to the SMEFT and the fit from part (c) above.
We have the following mappings between UV models and the SMEFT:

Wilson Coefficient	$\mathbb{R}$ Scalar Singlet	$\mathbb{R}$ Scalar Triplet	XB mixing model
$c_H^{(4)}$	$\frac{g_{SH}^2}{2M^2}$	$\frac{g^2}{2M^2}\left(1-2\mu^2\right)$	0
$c_H^{(6)}$	$\frac{g_{SH}^2}{M^4}\left(\frac{g g_{SH}}{3!M^2}-\frac{\lambda_{SH}}{2}\right)$	$-\frac{g^2}{2M^4}\left(\lambda_{H\phi}-8\lambda\right)$	0
$c_{H\square}$	$-\frac{g_{SH}^2}{2M^4}$	$\frac{g^2}{2M^4}$	$-\frac{\kappa^2 g_1^2}{8M^2}$
c_{HD}	0	$-\frac{2g^2}{M^4}$	$-\frac{\kappa^2 g_1^2}{2M^2}$
c_{LL}	0	0	$-\frac{\kappa^2 g_1^2}{8M^2}$
$c_{Hl}^{(1)}$	0	0	$\frac{\kappa^2 g_1^2}{4M^2}$
$c_{He}^{(1)}$	0	0	$\frac{\kappa^2 g_1^2}{2M^2}$

(a) Comment on our fit's ability to constrain the $\mathbb{R}$ scalar singlet model.

(b) Comment on our fit's ability to constrain the $\mathbb{R}$ scalar triplet model. Taking your lowest 99% CL bound on c_{HD} and taking $g \sim 500$ GeV $= 0.5$ TeV derive an approximate value of the mass of the triplet.

(c) Comment on our fit's ability to constrain the XB mixing model. Taking your lowest 99% CL bound on c_{HD} and taking $\kappa \sim 1$ derive an approximate value of the mass of the X. Comment on whether your bounds on $c_{Hl}^{(1)}$, c_{He}, and c_{LL} are consistent with taking c_{HD} to be its lowest 99% CL bound.

$1 : X^3$

Q_G	$f^{ABC} G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$
$Q_{\widetilde G}$	$f^{ABC} \widetilde G_\mu^{A\nu} G_\nu^{B\rho} G_\rho^{C\mu}$
Q_W	$\epsilon^{IJK} W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$
$Q_{\widetilde W}$	$\epsilon^{IJK} \widetilde W_\mu^{I\nu} W_\nu^{J\rho} W_\rho^{K\mu}$

$2 : H^6$

Q_H	$(H^\dagger H)^3$

$3 : H^4 D^2$

$Q_{H\Box}$	$(H^\dagger H)\Box(H^\dagger H)$
Q_{HD}	$\left(H^\dagger D^\mu H\right)^* \left(H^\dagger D_\mu H\right)$

$5 : \psi^2 H^3 + \text{h.c.}$

Q_{eH}	$(H^\dagger H)(\bar l_p e_r H)$
Q_{uH}	$(H^\dagger H)(\bar q_p u_r \widetilde H)$
Q_{dH}	$(H^\dagger H)(\bar q_p d_r H)$

$4 : X^2 H^2$

Q_{HG}	$H^\dagger H\, G^A_{\mu\nu} G^{A\mu\nu}$
$Q_{H\widetilde G}$	$H^\dagger H\, \widetilde G^A_{\mu\nu} G^{A\mu\nu}$
Q_{HW}	$H^\dagger H\, W^I_{\mu\nu} W^{I\mu\nu}$
$Q_{H\widetilde W}$	$H^\dagger H\, \widetilde W^I_{\mu\nu} W^{I\mu\nu}$
Q_{HB}	$H^\dagger H\, B_{\mu\nu} B^{\mu\nu}$
$Q_{H\widetilde B}$	$H^\dagger H\, \widetilde B_{\mu\nu} B^{\mu\nu}$
Q_{HWB}	$H^\dagger \tau^I H\, W^I_{\mu\nu} B^{\mu\nu}$
$Q_{H\widetilde W B}$	$H^\dagger \tau^I H\, \widetilde W^I_{\mu\nu} B^{\mu\nu}$

$6 : \psi^2 X H + \text{h.c.}$

Q_{eW}	$(\bar l_p \sigma^{\mu\nu} e_r) \tau^I H W^I_{\mu\nu}$
Q_{eB}	$(\bar l_p \sigma^{\mu\nu} e_r) H B_{\mu\nu}$
Q_{uG}	$(\bar q_p \sigma^{\mu\nu} T^A u_r) \widetilde H\, G^A_{\mu\nu}$
Q_{uW}	$(\bar q_p \sigma^{\mu\nu} u_r) \tau^I \widetilde H\, W^I_{\mu\nu}$
Q_{uB}	$(\bar q_p \sigma^{\mu\nu} u_r) \widetilde H\, B_{\mu\nu}$
Q_{dG}	$(\bar q_p \sigma^{\mu\nu} T^A d_r) H\, G^A_{\mu\nu}$
Q_{dW}	$(\bar q_p \sigma^{\mu\nu} d_r) \tau^I H\, W^I_{\mu\nu}$
Q_{dB}	$(\bar q_p \sigma^{\mu\nu} d_r) H\, B_{\mu\nu}$

$7 : \psi^2 H^2 D$

$Q_{Hl}^{(1)}$	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar l_p \gamma^\mu l_r)$
$Q_{Hl}^{(3)}$	$(H^\dagger i \overleftrightarrow{D}^I_\mu H)(\bar l_p \tau^I \gamma^\mu l_r)$
Q_{He}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar e_p \gamma^\mu e_r)$
$Q_{Hq}^{(1)}$	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar q_p \gamma^\mu q_r)$
$Q_{Hq}^{(3)}$	$(H^\dagger i \overleftrightarrow{D}^I_\mu H)(\bar q_p \tau^I \gamma^\mu q_r)$
Q_{Hu}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar u_p \gamma^\mu u_r)$
Q_{Hd}	$(H^\dagger i \overleftrightarrow{D}_\mu H)(\bar d_p \gamma^\mu d_r)$
Q_{Hud} + h.c.	$i(\widetilde H^\dagger D_\mu H)(\bar u_p \gamma^\mu d_r)$

$8 : (\bar L L)(\bar L L)$

Q_{ll}	$(\bar l_p \gamma_\mu l_r)(\bar l_s \gamma^\mu l_t)$
$Q_{qq}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar q_s \gamma^\mu q_t)$
$Q_{qq}^{(3)}$	$(\bar q_p \gamma_\mu \tau^I q_r)(\bar q_s \gamma^\mu \tau^I q_t)$
$Q_{lq}^{(1)}$	$(\bar l_p \gamma_\mu l_r)(\bar q_s \gamma^\mu q_t)$
$Q_{lq}^{(3)}$	$(\bar l_p \gamma_\mu \tau^I l_r)(\bar q_s \gamma^\mu \tau^I q_t)$

$8 : (\bar R R)(\bar R R)$

Q_{ee}	$(\bar e_p \gamma_\mu e_r)(\bar e_s \gamma^\mu e_t)$
Q_{uu}	$(\bar u_p \gamma_\mu u_r)(\bar u_s \gamma^\mu u_t)$
Q_{dd}	$(\bar d_p \gamma_\mu d_r)(\bar d_s \gamma^\mu d_t)$
Q_{eu}	$(\bar e_p \gamma_\mu e_r)(\bar u_s \gamma^\mu u_t)$
Q_{ed}	$(\bar e_p \gamma_\mu e_r)(\bar d_s \gamma^\mu d_t)$
$Q_{ud}^{(1)}$	$(\bar u_p \gamma_\mu u_r)(\bar d_s \gamma^\mu d_t)$
$Q_{ud}^{(8)}$	$(\bar u_p \gamma_\mu T^A u_r)(\bar d_s \gamma^\mu T^A d_t)$

$8 : (\bar L L)(\bar R R)$

Q_{le}	$(\bar l_p \gamma_\mu l_r)(\bar e_s \gamma^\mu e_t)$
Q_{lu}	$(\bar l_p \gamma_\mu l_r)(\bar u_s \gamma^\mu u_t)$
Q_{ld}	$(\bar l_p \gamma_\mu l_r)(\bar d_s \gamma^\mu d_t)$
Q_{qe}	$(\bar q_p \gamma_\mu q_r)(\bar e_s \gamma^\mu e_t)$
$Q_{qu}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar u_s \gamma^\mu u_t)$
$Q_{qu}^{(8)}$	$(\bar q_p \gamma_\mu T^A q_r)(\bar u_s \gamma^\mu T^A u_t)$
$Q_{qd}^{(1)}$	$(\bar q_p \gamma_\mu q_r)(\bar d_s \gamma^\mu d_t)$
$Q_{qd}^{(8)}$	$(\bar q_p \gamma_\mu T^A q_r)(\bar d_s \gamma^\mu T^A d_t)$

$8 : (\bar L R)(\bar R L) + \text{h.c.}$

Q_{ledq}	$(\bar l_p^j e_r)(\bar d_s q_{tj})$

$8 : (\bar L R)(\bar L R) + \text{h.c.}$

$Q_{quqd}^{(1)}$	$(\bar q_p^j u_r) \epsilon_{jk} (\bar q_s^k d_t)$
$Q_{quqd}^{(8)}$	$(\bar q_p^j T^A u_r) \epsilon_{jk} (\bar q_s^k T^A d_t)$
$Q_{lequ}^{(1)}$	$(\bar l_p^j e_r) \epsilon_{jk} (\bar q_s^k u_t)$
$Q_{lequ}^{(3)}$	$(\bar l_p^j \sigma_{\mu\nu} e_r) \epsilon_{jk} (\bar q_s^k \sigma^{\mu\nu} u_t)$

$$H^\dagger i \overleftrightarrow{D}_\mu H \equiv H^\dagger i D_\mu H - (i D_\mu H^\dagger) H$$
$$H^\dagger i \overleftrightarrow{D}^I_\mu H \equiv H^\dagger i \tau^I D_\mu H - (i D_\mu \tau^I H^\dagger) H$$

Table 10. The $\mathcal{L}^{(6)}$ operators built from Standard Model fields which conserve baryon number, as given in Ref. [10, 19, 34, 35]. The operators are divided into eight Classes: X^3, H^6, etc. Operators with +h.c. in the table heading also have Hermitian conjugates, as does the $\psi^2 H^2 D$ operator Q_{Hud}. The subscripts p, r, s, t are flavour indices which are suppressed on the left hand sides of the sub-tables.