QED at 1 loop:

1) QED in d-dimensions, $\mathcal{L}_{Renorm}$

Our $\mathcal{L}$ is given by

$$\mathcal{L}_{QED} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \bar{\psi}(i\partial\!\!\!/ - e A\!\!\!/ - m)\psi$$

We want to know the dimensions of the fields:

$$[(\partial^\mu A^\nu)(\partial_\mu A_\nu)] = d = 4-2\epsilon$$

$$[A^2] = d-2 = 2(1-\epsilon)$$

$$[A^\mu] = 1-\epsilon$$

$$[\bar{\psi}\partial\!\!\!/\psi] = 4-2\epsilon$$

$$[\bar{\psi}\psi] = d-1 = 3-2\epsilon$$

$$[\bar{\psi}] = [\psi] = \frac{3}{2}-\epsilon$$

$$[e\bar{\psi}A\!\!\!/\psi] = d$$

$$[e] = 4-2\epsilon - 2(\tfrac{3}{2}-\epsilon) - (1-\epsilon)$$

$$= \epsilon$$

To maintain $[e]=0$, we will take

$$e \rightarrow e\mu^\epsilon$$

To renormalize the $\mathcal{L}$ we take:

$$\left.\begin{aligned} A_0^\mu &= Z_A^{1/2} A^\mu \\ \psi_0 &= Z_\psi^{1/2} \psi \\ \bar{\psi}_0 &= Z_\psi^{1/2} \bar{\psi} \end{aligned}\right\} \text{drop the "R" subscript on fields for simplicity}$$

Then:

$$\mathcal{L}_0 = -\frac{1}{4}\left(\partial_\mu A_{0\nu} - \partial_\nu A_{0\mu}\right)^2 + \bar{\psi}_0 \left(i\partial\!\!\!/ - e_0 A\!\!\!/_0 - m_0\right)\psi_0$$

$$= -\frac{1}{4} Z_A \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + Z_\psi \bar{\psi}\left(i\partial\!\!\!/ - m_0\right)\psi - \underbrace{e_0 Z_\psi Z_A^{1/2}}_{\equiv e Z_e = e(1+\delta_e)} \bar{\psi} A\!\!\!/ \psi$$

$$Z_\psi m_0 \equiv m Z_m = m(1+\delta_m)$$

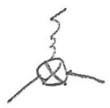
$$= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + \bar{\psi}\left(i\partial\!\!\!/ - m - eA\!\!\!/\right)\psi \qquad \leftarrow \mathcal{L}_R$$

$$-\frac{1}{4}\delta_A \left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + \bar{\psi}\left(i\partial\!\!\!/ \delta_\psi - m\delta_m - e\delta_e A\!\!\!/\right)\psi \qquad \leftarrow \mathcal{L}_{ct}$$

$\mathcal{L}_{ct}$ gives the ct Feynman Rules:

 $= -i\delta_A\left(p^2 \eta^{\mu\nu} - p^\mu p^\nu\right)$

 $= i\left(p\!\!\!/ \delta_\psi - m\delta_m\right)$

 $= ie\gamma^\mu \delta_e$

2) Ward Identities in QED (This discussion doesn't assume a pert. expansion)

recall our (local) gauge xform:

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

$$\psi \to e^{-ie\alpha(x)}\psi \sim \psi - ie\alpha(x)\psi \quad \text{(infinitesimally)}$$

From this we have:

$$\delta A_\mu = \partial_\mu \alpha$$

$$\delta\psi = -ie\alpha\psi$$

Without gauge fixing our effective action, Γ, is gauge invariant:

$$\delta\Gamma = 0$$

Applying the chain rule:

$$\delta\Gamma = 0 = \frac{\delta\Gamma}{\delta A_\mu}\delta A^\mu + \frac{\delta\Gamma}{\delta\psi_c}\delta\psi_c + \frac{\delta\Gamma}{\delta\bar{\psi}_c}\delta\bar{\psi}_c$$

← all fields have same spacetime argument, x

$$= \frac{\delta\Gamma}{\delta A_\mu}(\partial_\mu\alpha) - ie\frac{\delta\Gamma}{\delta\psi_c}\psi_c + ie\frac{\delta\Gamma}{\delta\bar{\psi}_c}\bar{\psi}_c$$

$$= \left[-\partial_\mu\frac{\delta\Gamma}{\delta A_\mu} - ie\left(\frac{\delta\Gamma}{\delta\psi_c}\psi_c - \bar{\psi}_c\frac{\delta\Gamma}{\delta\bar{\psi}_c}\right)\right]\alpha$$

anticommuting

Since α is arbitrary we conclude:

$$-\partial_\mu\frac{\delta\Gamma}{\delta A_\mu} - ie\left(\frac{\delta\Gamma}{\delta\psi_c}\psi_c - \bar{\psi}_c\frac{\delta\Gamma}{\delta\bar{\psi}_c}\right) = 0 \qquad *$$

This is our "master" Ward Identity, further Ward Identities are achieved by taking variations w/r to fields

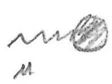
Setting the fields to their "vacuum expectation values", requires

$$A_\mu \to 0, \quad \psi_c \to 0$$

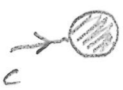
Otherwise the vaccum wouldn't be invariant under Lorentz transformations,

Similarly, $\frac{\delta\Gamma}{\delta A} = \frac{\delta\Gamma}{\delta\psi} = 0$, or something w/ a Lorentz index

could be generated from the vacuum:



or



(tadpoles)

Also violating Lorentz invariance.

Perturbatively, we cant source A or ψ from the vacuum:

$$\text{eg} \quad \mathcal{L}_{tadpole} = \underset{\substack{\uparrow \\ [\mu^3]=3}}{\mu^3} A_\mu \quad \text{or} \quad \mathcal{L}_{tadpole} = \mu^{5/2}\psi$$

So our master Ward ID is trivially satisfied:

$$\partial_\mu \frac{\delta\Gamma}{\delta A_\mu} = 0$$

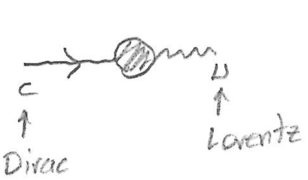
Taking a variation w/r to $A_\nu(y)$ and restoring the arguments:

$$-\partial_{x\mu}\frac{\delta^2\Gamma}{\delta A_\nu(y)\,\delta A_\mu(x)} - ie\left(\frac{\delta^2\Gamma}{\delta A_\nu(y)\,\delta\psi_c(x)}\psi_c(x) - \bar{\psi}_c\frac{\delta^2\Gamma}{\delta A_\nu(y)\,\delta\bar{\psi}_c(x)}\right)$$

$$= -\partial_{x\mu}\frac{\delta^2\Gamma}{\delta A_\nu(y)\,\delta A_\mu(x)} = 0 \qquad **$$

The fermionic terms vanish for $\overset{(-)}{\psi}_c \to 0$, but also

$\frac{\delta^2\Gamma}{\delta\psi\,\delta A} = 0$ otherwise we again violate Lorentz inv.



Transforming ** to mtm space, $\partial_\mu \to -ip_\mu$

$$p_\mu \frac{\delta^2 \Gamma}{\delta A_\nu(-p)\,\delta A_\mu(p)} = 0 = p_\mu$$ 

This means the photon is transvers.

The only Lorentz index carrying quantities we can write are p^μ & $\eta^{\mu\nu}$

So requiring the above be true means the 2pt function is prop to:

$$\Pi_T^{\mu\nu} = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{p^2}$$

This is the transverse projection operator.

But $\mathcal{L}_{GF}$ explicitly breaks the gauge symmetry:

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A_\mu)(\partial_\nu A_\nu)$$

$$\delta \mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial_\mu A_\mu)(\partial_\nu \partial_\nu \alpha) - \frac{1}{2\xi}(\partial_\mu \partial_\mu \alpha)(\partial_\nu A_\nu)$$

$$= -\frac{1}{\xi}(\Box \partial_\mu A_\mu)\alpha$$

$$\delta \Gamma_{GF} = i\int \mathcal{L}_{GF}\, d^4x = -\frac{i}{\xi}\int d^4x\, (\Box \partial_\mu A_\mu)\alpha$$

Recall the relation between our generating functional for connected diagrams & the 1PI diagrams

$$\underset{W}{\uparrow} \qquad\qquad \underset{\Gamma}{\uparrow}$$

$$\Gamma(\bar\varphi) = W[J] - \int d^4x\, J\bar\varphi \qquad w/\ \Phi \equiv \frac{\partial W}{\partial J}$$

$$\Gamma(A_\mu) = W[J_\mu] - \int d^4x\, J^\mu A_\mu \qquad w/\ A_\mu = \frac{\partial W}{\partial J^\mu}$$ ← A should be $\bar A$, but we were calling it A before when discussing Γ

$$\Rightarrow \frac{\partial \Gamma}{\partial A_\mu} = -J^\mu$$

So $$\delta\Gamma^{full} = \underbrace{\delta\Gamma}_{0} + \delta\Gamma_{GF}$$

$$= \delta\Gamma_{GF}$$

$$\frac{\delta\Gamma^{full}}{\delta\alpha} = \frac{\delta\Gamma^{full}}{\delta A_\mu}\delta A^\mu = \delta\Gamma_{GF}$$

$$\frac{\delta\Gamma^{full}}{\delta A_\mu}\delta A^\mu = -\int d^4x\, J^\mu \partial_\mu\alpha = -\frac{i}{\xi}\int d^4x\,(\Box\partial_\mu A^\mu)\alpha$$

$$\Rightarrow \partial_\mu J^\mu = -\frac{i}{\xi}\Box\,\partial_\mu \underset{\frac{\delta W}{\delta J_\mu}}{\underset{\uparrow}{A^\mu}}$$

$$\partial_\mu J^\mu = -\frac{i}{\xi}\Box\,\partial_\mu\left(\frac{\delta W}{\delta J_\mu}\right)$$

Taking a variation w/r to J_ν gives the connected 2pt function:

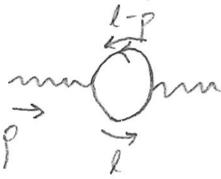
$$-\frac{i}{\xi}\Box\,\partial_\mu\left(\frac{\delta^2 W}{\delta J_\mu \delta J_\nu}\right) = \partial_\mu\left(\frac{\delta J^\mu}{\delta J_\nu}\right) = \partial_\mu(\eta^{\mu\nu}) = 0$$

So the photon connected 2pt function is transverse to all orders in perturbation thry, even though $\mathcal{L}_{GF}$ explicitly breaks gauge symmetry

3) Self Energies

corrections to the propagator are frequently referred to as "Self Energies"

The photon self energy is also referred to as the "vacuum polarization"



$$= (-1)\int \frac{d^d l}{(2\pi)^d} \frac{(ie\mu^\epsilon \gamma^\mu)_{ab}\, i(\not{l}+m)_{bc}\, (ie\mu^\epsilon \gamma^\nu)_{cd}\, i(\not{l}-\not{p}+m)_{da}}{[l^2-m^2+i\epsilon][(p-l)^2-m^2+i\epsilon]}$$

closed loop

$$= -e^2\mu^{2\epsilon}\int \frac{d^d l}{(2\pi)^d} \frac{Tr[\gamma^\mu(\not{l}+m)\gamma^\nu(\not{l}-\not{p}+m)]}{[l^2-m^2+i\epsilon][(p-l)^2-m^2+i\epsilon]}$$

Feynman parameterizing the denominator gives:

$$\frac{1}{[l^2-m^2+i\epsilon][(p-l)^2-m^2+i\epsilon]} = \int_0^1 \frac{dx}{[q^2-\Delta+i\epsilon]^2}$$

w/ $q \equiv l - xp$

$\Delta \equiv x(x-1)p^2 + m^2$

Evaluating the trace:

$$Tr[\gamma^\mu(\not{l}+m)\gamma^\nu(\not{l}-\not{p}+m)] = 8l^\mu l^\nu - 4l^\nu p^\mu - 4l^\mu p^\nu + 4(m^2 - l^2 + l\cdot p)g^{\mu\nu}$$

$$= 8x(x-1)p^\mu p^\nu + 8q^\mu q^\nu + 4(m^2 - x(x-1)p^2 - q^2)g^{\mu\nu} + (\text{odd in } q)$$

We make the replacement $q^\mu q^\nu \to \frac{1}{d} g^{\mu\nu} q^2$ (See Schwartz B.3.4)

This gives:

$$= -e^2\mu^{2\epsilon}\int_0^1 dx \int \frac{d^d q}{(2\pi)^d} \frac{[8x(x-1)p^\mu p^\nu + 4(m^2 - x(x-1)p^2)g^{\mu\nu}] - 4(1-\frac{2}{d})q^2 g^{\mu\nu}}{[q^2-\Delta+i\epsilon]^2}$$

The two resulting integrals are:

$$I_1 = \int \frac{d^dq}{(2\pi)^d}\frac{\mu^{2\epsilon}}{[q^2-\Delta+i\epsilon]^2} = \frac{i(-1)^{-2}}{(4\pi)^{2-\epsilon}}\frac{\mu^{2\epsilon}}{\Delta^{\epsilon}}\frac{\cancel{\Gamma(d/2)}\,\Gamma\left(\frac{4-d}{2}\right)}{\underbrace{\Gamma(2)}_{1}\,\cancel{\Gamma(d/2)}}$$

$$I_2 = \int \frac{d^dq}{(2\pi)^d}\frac{\mu^{2\epsilon}q^2}{[q^2-\Delta+i\epsilon]^2} = \frac{i(-1)^{1-2}}{(4\pi)^{2-\epsilon}}\frac{\mu^{2\epsilon}}{\Delta^{-1+\epsilon}}\frac{\Gamma\left(\frac{2+d}{2}\right)\Gamma\left(\frac{2-d}{2}\right)}{\Gamma(2)\,\Gamma(d/2)}$$

$$\Gamma(n+1) = n\Gamma(n) \rightarrow \Gamma\left(\frac{2+d}{2}\right) = \frac{d}{2}\Gamma(d/2)$$

$$\Gamma\left(\frac{2-d}{2}\right) = \left(\frac{2}{2-d}\right)\Gamma\left(\frac{4-d}{2}\right)$$

$$= \frac{-i}{(4\pi)^{2-\epsilon}}\frac{\Delta\mu^{2\epsilon}}{\Delta^{\epsilon}}\frac{d}{2}\left(\frac{2}{2-d}\right)\Gamma\left(\frac{4-d}{2}\right)$$

$$-4\left(\frac{2}{d}-1\right)I_2 = \frac{4i}{(4\pi)^{2-\epsilon}}\frac{\Delta\mu^{2\epsilon}}{\Delta^{\epsilon}}\underbrace{\left(1-\frac{2}{d}\right)\frac{d}{2}\left(\frac{2}{2-d}\right)}_{=-1}\Gamma\left(\frac{4-d}{2}\right)$$

$$[8x(x-1)p^\mu p^\nu + 4(m^2 - x(x-1)p^2)g^{\mu\nu}]I_1 - 4\left(1-\frac{2}{d}\right)g^{\mu\nu}I_2$$

$$= \frac{i}{(4\pi)^{2-\epsilon}}\frac{\mu^{2\epsilon}}{\Delta^{\epsilon}}\Gamma\left(\frac{4-d}{2}\right)\left[8x(x-1)p^\mu p^\nu + 4(m^2 - x(x-1)p^2)g^{\mu\nu} - 4(x(x-1)p^2+m^2)g^{\mu\nu}\right]$$

m^2 dep. vanishes p^2 terms add

 $= -\frac{8ie^2}{(4\pi)^2}\int_0^1 dx\; x(x-1)[p^\mu p^\nu - p^2 g^{\mu\nu}]\left(\frac{4\pi\mu^2}{\Delta}\right)^{\epsilon}\Gamma(\epsilon)$

$$= \frac{-8ie^2}{(4\pi)^2}[p^\mu p^\nu - p^2 g^{\mu\nu}]\int_0^1 dx\; x(x-1)\left(\frac{1}{\epsilon} - \gamma_E + \dots\right)\left(1 + \epsilon\ln\frac{4\pi\mu^2}{\Delta} + \dots\right)$$

$$= -\frac{8ie^2}{(4\pi)^2}\underbrace{[p^\mu p^\nu - p^2 g^{\mu\nu}]}_{\propto \Pi_T^{\mu\nu}}\left(-\frac{1}{6}\frac{1}{\epsilon} + \frac{\gamma_E}{6} + \int dx(x^2-x)\ln\frac{4\pi\mu^2}{(x^2-x)p^2+m^2} + O(\epsilon)\right)$$

$\propto \Pi_T^{\mu\nu}$, we confirm our Ward Identity

If we could perform the integral over the Feynman parameter we would find:

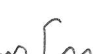
$$= \frac{4ie^2}{9(4\pi)^2}\Pi_T^{\mu\nu}p^2\left[5+\frac{3}{\epsilon}+3\ln\frac{\mu^2}{m^2}+12\frac{m^2}{p^2}+3\frac{2m^2+p^2}{p^2}\,\text{DiscB}[p^2,m,m]\right]$$

w/ $\text{DiscB}[p^2,m,m] \equiv \frac{\sqrt{p^2(p^2-4m^2)}}{p^2}\ln\underbrace{\frac{\sqrt{p^2(p^2-4m^2)}-p^2+2m^2}{2m^2}}_{\equiv z}$

The optical theorem in QFT (Schwartz 24.1) states:

$$\underbrace{\text{Im}\,\mathcal{M}(A\to A)}_{\text{Imaginary part of scattering } A\to A} = m_A \underbrace{\sum_X \Gamma(A\to X)}_{\text{is the some of decay products of } A\to X,\ X \text{ is therefore onshell}}$$

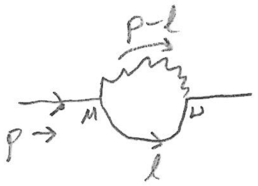
So an imaginary part corresponds to the particles in the loop going on-shell $\to$ the branch cut of the log

eg $\text{Im}\Big[$  $\Big] = \underbrace{\int d\Pi_{LIPS}}_{\text{Lorentz invariant Phase Space}}\Big|$  $\Big|^2$

In this case, since the photon is massless, we would have to think of this as a subprocess w/ an off-shell photon

w/ $p^2 = \mu^2$ ↑ an effective mass parameterizing how off-shell we are

For the electron self energy we have



$\equiv i\Sigma$

The photon propagator complicates this:

$$iG^{\mu\nu} = \frac{-i}{p^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

The extra term going as $\frac{1}{p^2}$ makes the algebra more difficult, so for brevity we take $\xi = 1$ to cancel this term (Feynman Gauge)

$$i\Sigma = \int \frac{d^d l}{(2\pi)^d} \frac{(ie\mu^\epsilon \gamma^\nu)\, i(\not{l}+m)(ie\mu^\epsilon\gamma^\mu)}{[l^2-m^2+i\epsilon][(p-l)^2+i\epsilon]}(-ig_{\mu\nu})$$

$$\gamma^\mu(\not{l}+m)\gamma_\mu = -(d-2)\not{l} + dm$$

$$= -e^2\mu^{2\epsilon}\int_0^1 dx \int\frac{d^dq}{(2\pi)^d}\frac{-(d-2)(\not{q}+x\not{p})+dm}{[q^2-\Delta]^2}$$

w/ $q \equiv l - xp$

$\Delta \equiv (x^2-x)p^2 + (1-x)m^2$

$$= -e^2\mu^{2\epsilon}\int dx\,[x(2-d)\not{p} + dm]\int\frac{d^dq}{(2\pi)^d}\frac{1}{[q^2-\Delta]^2} + (\text{odd in } q \to 0)$$

$$= -e^2\mu^{2\epsilon}\int dx\,[x(2-d)\not{p}+dm]\,\underbrace{\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Delta^{2-d/2}}}_{\frac{i}{(4\pi)^2}\left[\frac{1}{\epsilon}-\gamma_E+\ln\frac{4\pi}{\Delta}+O(\epsilon)\right]}$$

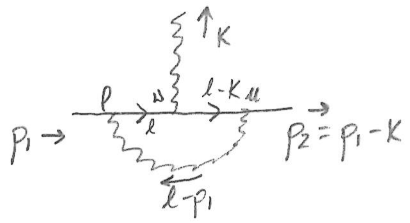
$$= \frac{-ie^2}{(4\pi)^2}\int dx\left[x(2\epsilon-2)\not{p} + (4-2\epsilon)m\right]\left[\frac{1}{\epsilon} - \gamma_E + \ln\frac{4\pi\mu^2}{\Delta} + O(\epsilon)\right]$$

$$= \frac{ie^2}{(4\pi)^2}\left[(\not{p}-4m)\frac{1}{\epsilon} - (\not{p}-2m) - (\not{p}-4m)\gamma_E + \int dx\,(2x\not{p}-4m)\ln\left(\frac{4\pi\mu^2}{\Delta}\right)\right]$$

unlike in ϕ^4 we have a mtm dep divergence!

4) Divergence in the interaction & $\overline{MS}$ renormalization of QED

The one-loop correction to the $\gamma\bar{e}e$ interaction is:



$$i\mathcal{M}^\nu = \int \frac{d^d l}{(2\pi)^d} \frac{(+ie\mu^\epsilon\gamma^\mu)\, i(\not l - \not k + m)(+ie\mu^\epsilon\gamma^\nu)\, i(\not l + m)(ie\mu^\epsilon\gamma^\rho)}{[l^2-m^2+i\epsilon][(l-k)^2-m^2+i\epsilon][(l-p_1)^2+i\epsilon]} \underbrace{(-ig_{\mu\rho})}_{\text{Feynman gauge}}$$

$$= \int_0^1 dx \int_0^{1-x} dy \int \frac{d^dq}{(2\pi)^d} \frac{\overset{\text{from Feyn param}}{2}\, e^3 \mu^{3\epsilon} \gamma^\mu(\not l - \not k + m)\gamma^\nu(\not l+m)\gamma_\mu}{[q^2-\Delta+i\epsilon]^3}$$

$$q \equiv l - yk - zp_1 \qquad\qquad z \equiv 1-x-y$$

$$\Delta = y(y-1)k^2 + z(1-z)p_1^2 + 2yz\, p_1\cdot k + m^2$$

For now we'll keep only the UV divergent part

Notice $\int \frac{d^4q}{(2\pi)^d} \frac{1}{[q^2]^3} \sim \frac{1}{\Lambda}$

So the divergence requires q^2 in the numerator:

$$\gamma^\mu(\not l - \not k + m)\gamma^\nu(\not l + m)\gamma_\mu = +\gamma^\mu(\not q + y\not k + z\not p_1 - \not k + m)\gamma^\nu(\not q + y\not k + z\not p_1 + m)\gamma_\mu$$

$$\rightarrow +\gamma^\mu \not q \gamma^\nu \not q \gamma_\mu$$

$$= +[\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma\gamma_\mu]\, q_\rho q_\sigma$$

$$= +[-2\gamma^\sigma\gamma^\nu\gamma^\rho + (4-d)\gamma^\rho\gamma^\nu\gamma^\sigma]\frac{1}{d} q^2 \eta_{\rho\sigma}$$

$$= -(d-2)[-2+(4-d)]\frac{q^2}{d}\gamma^\nu$$

$$= +\frac{(d-2)^2}{d} q^2 \gamma^\nu$$

Where we have freely used the following d-dimensional γ matrix IDs:
(Peskin A.4)

$$\gamma^\mu \gamma_\mu = d$$
$$\gamma^\mu \gamma^\nu \gamma_\mu = -(d-2)\gamma^\nu$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4\eta^{\nu\rho} - (4-d)\gamma^\nu \gamma^\rho$$
$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu + (4-d)\gamma^\nu \gamma^\rho \gamma^\sigma$$

Then we have:

$$i m^\nu_{UV} = \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d l}{(2\pi)^d} 2e^3 \mu^{3\epsilon}\left[+\frac{(d-2)^2}{d}\right]\gamma^\nu \frac{q^2}{[q^2-\Delta]^3}$$

$$= +\frac{(d-2)^2}{d} 2e^3 \mu^\epsilon \gamma^\nu \int dx dy \int \frac{d^d q}{(2\pi)^d} \mu^{2\epsilon} \frac{q^2}{[q^2-\Delta]^3}$$

$$= +\frac{(d-2)^2}{d} 2e^3 \mu^\epsilon \gamma^\nu \int dx dy \frac{i(-1)^{1-3}}{(4\pi)^{d/2}} \frac{\mu^{2\epsilon}}{\Delta^{3-1-d/2}} \underbrace{\frac{\Gamma(1+d/2)\Gamma(3-1-d/2)}{\Gamma(3)\Gamma(d/2)}}_{\frac{\Gamma(3+\epsilon)\Gamma(\epsilon)}{\Gamma(3)\Gamma(2-\epsilon)} \quad \Gamma(2-\epsilon)\to 1!}$$

$$= +\frac{(d-2)^2}{d} 2e^3 \mu^\epsilon \gamma^\nu \int dx dy \frac{i}{(4\pi)^2}\left[1+\epsilon \ln\frac{4\pi\mu^2}{\Delta}\right]\left[\frac{1}{\epsilon}+\gamma_E+\cdots\right]$$

$$= +\frac{4}{4} 2e^3 \mu^\epsilon \gamma^\nu \frac{i}{(4\pi)^2} \underbrace{\int_0^1 dx \int_0^{1-x} dy \frac{1}{\epsilon}}_{\frac{1}{2\epsilon}} \quad + \text{ finite}$$

$$= +\frac{i e^3 \mu^\epsilon \gamma^\nu}{(4\pi)^2} \frac{1}{\epsilon}$$

← take $\mu^\epsilon \to 1$ for convenience as the tree level diagram goes as $e\mu^\epsilon$; this is a convenient trick, not a proof. Keeping μ^ϵ will greatly complicate our calculations & give the same result
(or we include μ^ϵ in counter term)

In the MS renorm scheme we only remove the UV divergences.

Neglecting finite parts and including the counter terms:

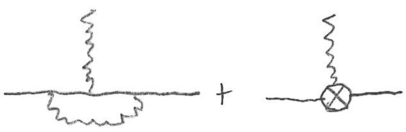
$$= \frac{+8ie^2}{(4\pi)^2}\left[p^\mu p^\nu - p^2 g^{\mu\nu}\right]\frac{1}{6\epsilon} - i\delta_A\left(p^2 g^{\mu\nu} - p^\mu p^\nu\right)$$

$$\Rightarrow \delta_A = -\frac{4e^2}{3(4\pi)^2}$$

$$= \frac{ie^2}{(4\pi)^2}(\not{p} - 4m)\frac{1}{\epsilon} + i(\not{p}\,\delta_\psi - m\,\delta_m)$$

$$\Rightarrow \delta_\psi = \frac{-e^2}{(4\pi)^2}$$

$$\delta_m = \frac{-4e^2}{(4\pi)^2}$$



$$= \frac{+ie^3\gamma^\mu}{(4\pi)^2}\frac{1}{\epsilon} + ie\gamma^\mu\delta_e$$

$$\Rightarrow \delta_e = \frac{-e^2}{(4\pi)^2}$$

5) RGEs in MS renorm scheme

recall we had

$$e_0 Z_\phi Z_A^{1/2} = e_R \mu^\epsilon Z_e$$

$$m_0 Z_\phi = m Z_m$$

Since the bare quantities can't depend on μ:

$$\mu \frac{d}{d\mu} e_0 = 0 = \mu \frac{d}{d\mu}\left(e_R Z_e Z_\phi^{-1} Z_A^{-1/2} \mu^\epsilon \right)$$

$$= \left(\mu \frac{d}{d\mu} e_R\right) Z_e Z_\phi^{-1} Z_A^{-1/2} \mu^\epsilon + e_R \left(\mu \frac{d}{d\mu} Z_e Z_\phi^{-1} Z_A^{-1/2}\right)\mu^\epsilon$$

$$+ e Z_e Z_\phi^{-1} Z_A^{-1/2} \epsilon \mu^\epsilon$$

dividing through by $e_R Z_e Z_\phi^{-1} Z_A^{-1/2} \mu^\epsilon$

$$0 = \mu \frac{d}{d\mu} \ln e_R + \mu \frac{d}{d\mu} \ln Z_e Z_\phi^{-1} Z_A^{-1/2} + \epsilon$$

$$= \frac{\mu}{e_R} \frac{d}{d\mu} e_R \left(1 + e_R \frac{\partial}{\partial e_R} \ln Z_e Z_\phi^{-1} Z_A^{-1/2} \right) + \epsilon$$

$$\uparrow$$

$$\frac{d}{d\mu} = \frac{de_R}{d\mu} \frac{\partial}{\partial e_R}$$

Giving

$$\mu \frac{d}{d\mu} e_R = -\epsilon e_R \left(1 + e_R \frac{\partial}{\partial e_R} \ln Z_e Z_\phi^{-1} Z_A^{-1/2} \right)^{-1}$$

$$\sim -\epsilon e_R + \epsilon e_R^2 \frac{\partial}{\partial e_R} \underbrace{\ln Z_e Z_\phi^{-1} Z_A^{-1/2}}_{= \ln Z_e - \ln Z_\phi - \frac{1}{2} \ln Z_A = -\frac{1}{2} \ln Z_A}$$

$$Z_e = Z_\phi = 1 - \frac{e^2}{(4\pi)^2}$$

$$= -\epsilon e_R - \frac{1}{2} e^2 \frac{\partial}{\partial e} \delta_A \qquad (\ln x \sim x + \cdots)$$

So we find

$$\mu \frac{d}{d\mu} e_R = -\epsilon e_R - \frac{1}{2}\epsilon e^2\left(-\frac{2e_R}{12\pi^2}\frac{1}{\epsilon}\right)$$

$$\rightarrow \frac{e_R^3}{12\pi^2}$$

So as μ is increased, e_R grows

$\alpha = \frac{e^2}{4\pi}$ at $\sqrt{s} \to 0$: $\alpha(0) = \frac{1}{137.035999084(21)}$

at $\sqrt{s} = m_Z \sim 91$ GeV: $\alpha(m_Z) \sim \frac{1}{145}$ ← note here we have to include all charged particles in the loop

Performing the same analysis for m_R we obtain:

$$\mu \frac{d}{d\mu} m_R = -m_R \epsilon e_R \frac{\partial}{\partial e_R} \ln Z_m Z_\phi^{-1}$$

$$= -m_R \epsilon e_R \frac{\partial}{\partial e_R}(\delta_m - \delta_\phi)$$

$$= m_R \frac{3e_R^3}{16\pi^2}$$

6) 3pt function, g-2, onshell renorm

Our fermions, ψ, classically obey the Dirac eqn

$(i\not\partial - m)\psi = 0$

Gauging the field gives:

$(i\not D - m)\psi = 0$

Multiplying by $(i\not D + m)$ gives:

$$(\not D^2 + m^2)\psi = 0$$

Recall $\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$

$$= \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu - \gamma_\mu\gamma_\nu + \gamma_\mu\gamma_\nu$$

$$= 2\gamma_\mu\gamma_\nu + [\gamma_\nu, \gamma_\mu] = 2\eta_{\mu\nu}$$

$$\gamma_\mu\gamma_\nu = \frac{1}{2}[\gamma_\mu, \gamma_\nu] + \eta_{\mu\nu}$$

$$\not D\not D = \frac{1}{2} D_\mu D_\nu \underbrace{[\gamma_\mu, \gamma_\nu]}_{\text{antisymmetric} \rightarrow \text{projects out antisymm part}} + D^2$$

So $D_\mu D_\nu [\gamma_\mu, \gamma_\nu] = \frac{1}{2}[D_\mu, D_\nu][\gamma_\mu, \gamma_\nu]$

$$= [\partial_\mu + ieA_\mu, \partial_\nu + ieA_\nu]$$

$$= \cancelto{0}{[\partial_\mu, \partial_\nu]} + ie[A_\mu, \partial_\nu] + ie[\partial_\mu, A_\nu] - \cancelto{0}{(ie)^2[A_\mu, A_\nu]}$$

$$= ie\left(A_\mu\partial_\nu - \partial_\nu A_\mu + \partial_\mu A_\nu - A_\nu\partial_\mu\right)$$

$$= ie F_{\mu\nu}$$

Note these terms are canceled by
$-\partial_\nu A_\mu + \partial_\mu A_\nu$
as the derivative acts on the field to the right as well:

$$(A_\mu\partial_\nu - \partial_\nu A_\mu)\psi = A_\mu\partial_\nu\psi - (\partial_\nu A_\mu)\psi - A_\mu\partial_\nu\psi$$
$$= (\partial_\nu A_\mu)\psi$$

Combining these gives:

$$\left(D^2 + \frac{e}{2}F_{\mu\nu}\underbrace{\frac{i}{2}[\gamma_\mu,\gamma_\nu]}_{\equiv\, i\sigma_{\mu\nu}} + m^2\right)\psi = 0$$

Using $F_{0i} = E_i$

$F_{ij} = -\epsilon_{ijk}B_k$

This gives:

$$\left[(\partial_\mu + ieA_\mu)^2 + m^2 + e\begin{pmatrix} (\vec{B}+i\vec{E})\cdot\vec{\sigma} & \\ & (\vec{B}-i\vec{E})\cdot\vec{\sigma}\end{pmatrix}\right]\psi = 0$$

Predicting $\mu_B = \frac{e}{2m}$

ie taking the nonrelativistic limit this gives:

$$i\partial_t|\psi\rangle = \left[\frac{1}{2m_e}(i\nabla - e\vec{A})^2 - eA_0 + \mu_B\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z\end{pmatrix}\right]|\psi\rangle$$

Taking the spin operator $\vec{S} = \frac{\vec{\sigma}}{2}$ (for $s = 1/2$) the $F_{\mu\nu}$ term looks like:

$$\frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} = -e\begin{pmatrix} (\vec{B}+i\vec{E})\vec{\sigma} & \\ & (\vec{B}-i\vec{E})\vec{\sigma}\end{pmatrix}$$

This gives the interaction of the e^- spin w/ the $\vec{B}$ field:

$$-2\frac{e}{2m}\vec{B}\cdot\vec{S} \equiv -g\mu_B\vec{B}\cdot\vec{S}$$

So the Dirac Eqn predicts $g = 2$ for the Magnetic Dipole moment

To obtain this from the Feynman Rule $ie\gamma^\mu$ consider:

we saw $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \frac{1}{2}[\gamma^\mu,\gamma^\nu] = \eta^{\mu\nu} - i\sigma^{\mu\nu}$

From the Dirac eqn we have

$$i\not\partial\psi = m\psi \qquad \Rightarrow \quad -i\not\partial\bar\psi = m\bar\psi$$

So $$m\bar\psi\gamma^\mu\psi = \frac{i}{2}\bar\psi\overleftarrow{\not\partial}\gamma^\mu\psi - \frac{i}{2}\bar\psi\gamma^\mu\not\partial\psi$$

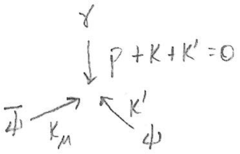
$$= \frac{i}{2}(\partial_\nu\bar\psi)\gamma^\nu\gamma^\mu\psi - \frac{i}{2}\bar\psi\gamma^\mu\gamma^\nu\partial_\nu\psi$$

$$= \frac{i}{2}\left[(\partial_\nu\bar\psi)(\eta^{\mu\nu}+i\sigma^{\mu\nu})\psi - \bar\psi(\eta^{\mu\nu}-i\sigma^{\mu\nu})\partial_\nu\psi\right]$$

$$\bar\psi\gamma^\mu\psi = \frac{i}{2m}\left[(\partial_\mu\bar\psi)\psi - \bar\psi(\partial_\mu\psi) + i(\partial_\nu\bar\psi)\sigma^{\mu\nu}\psi + i\bar\psi\sigma^{\mu\nu}(\partial_\nu\psi)\right]$$

In mtm space $(\partial_\mu\bar\psi)\to -ik_\mu\bar\psi$, $(\partial_\mu\psi)\to -ik'_\mu\psi$ (all incoming convention)

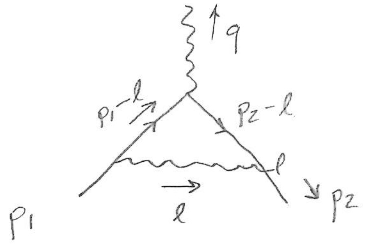
$$e\bar\psi\gamma^\mu\psi = \frac{e}{2m}(k_\mu-k'_\mu)\bar\psi\psi + \underbrace{\frac{e}{2m}\underbrace{(k_\mu+k'_\mu)}_{-p^\gamma_\mu}\bar\psi i\sigma^{\mu\nu}\psi}_{\text{this is our magnetic dipole moment}}$$



HW → Feynman Rule for $\bar\psi\sigma^{\mu\nu}\psi F_{\mu\nu}$ ⇒ this result

We expect at one-loop there will be corrections to this

These corrections come from (we'll take the electrons on-shell for simplicity)



Note: $p_1^2 = p_2^2 = m^2$

$p_1 \cdot p_2 = (-q^2 + 2m^2)/2$

Feynman Gauge

$$= e^3 \mu^{2\epsilon} \int \frac{d^d l}{(2\pi)^d} \frac{\bar{u}_2 \gamma_\rho (\not{p}_2 - \not{l} + m) \gamma_\mu (\not{p}_1 - \not{l} + m) \gamma_\nu u_1 \eta^{\nu\rho}}{[(l-p_2)^2 - m^2 + i\epsilon][(l-p_1)^2 - m^2 + i\epsilon][l^2 + i\epsilon]}$$

$$= \frac{i e^3}{(4\pi)^2} \Bigg[\left(1 - \frac{2\Lambda(q^2, m, m)(q^2 - 2m^2)}{q^2 - 4m^2}\right)\left(\frac{1}{\epsilon} - \gamma_E + \ln 4\pi + \ln \frac{\mu^2}{m^2}\right) \bar{u}_2 \gamma^\mu u_1$$

$$+ \frac{\Lambda(q^2, m, m)(3q^2 - 8m^2)}{q^2 - 4m^2} \bar{u}_2 \gamma^\mu u_1$$

$$+ 2(2m^2 - q^2) I_{C6}(q^2, m, m) \bar{u}_2 \gamma^\mu u_1$$

$$+ 2im \frac{\Lambda(q^2, m, m)}{q^2 - 4m^2} \bar{u}_2 \sigma^{\mu\nu} u_1 q_\nu \Bigg]$$

$$\Lambda(q^2, m, m) = \frac{\sqrt{q^2(q^2 - 4m^2)}}{q^2} \ln\left(\frac{2m^2 - q^2 + \sqrt{q^2(-4m^2 + q^2)}}{2m^2}\right)$$

$$I_{C6}(q^2, m, m) = -\frac{\Lambda(q^2, m, m)}{8m^2 - q^2} \ln\left(-\frac{m^2(q^2 - m^2 + \sqrt{q^2(q^2 - 4m^2)})}{2(q^2 - 4m^2)^2}\right)$$

$$- \frac{\pi^2 + 12\, \mathrm{Li}_2\left(\frac{q^2 - 2m^2 - \sqrt{q^2(q^2 - 4m^2)}}{2m^2}\right)}{6\sqrt{q^2(q^2 - 4m^2)}}$$

The correction to $g=2$ comes from the $\bar{u}_2 \sigma^{\mu\nu} u_1 q_\nu$ term as $q^2 \to 0$

$$\underset{\substack{\uparrow \\ \text{normalization}}}{\frac{2m}{e}} \left(\frac{ie^3}{(4\pi)^2}\right) 2im \underbrace{\frac{\Lambda(0,m,m)}{(-4m^2)}}_{\frac{1}{2m^2}} \underset{\substack{\uparrow \\ \text{our } q \text{ is outgoing}}}{(-1)} = \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}$$

The correction to g is twice this; $\left(ge\vec{B}\cdot\vec{S} = 2\frac{e}{2m}\vec{B}\cdot\vec{S}\right)$

$$g = 2 + \frac{\alpha}{\pi}$$

$$\sim 2.00232$$

The experimental value is: $2.0023193043617 \pm (3\times10^{-13})$

Until recently this was the best determination of α so couldn't be compared to experiment

For onshell renormalization we want:

(1) $\Sigma_R(m_p) =$  $= 0$ ← $m_p = m_R$

(2) $\frac{d}{d\not p}\Sigma_R(\not p)\Big|_{m_p} = 0$ ← residue $\Sigma = i$, remember $p^2 = \not p \not p$

(3) $e_R\Gamma^\mu(0) =$  $= e_R\gamma^\mu$ ← e_R is what is measured by coulombs law

(4) $\Pi(0) = 0$ ← residue

Looking at (1) & (2)

$$\Sigma_R(\not p) = \Sigma(\not p) + i(\not p\,\delta_\psi - m\,\delta m)$$

$$\Sigma_R(m) = 0 \rightarrow \Sigma(m) = im(\delta m - \delta_\psi)$$

$$\frac{d}{d\not p}\Sigma_R\Big|_{\not p \to m} = 0 \rightarrow \frac{d}{d\not p}\Sigma\Big|_{\not p \to m} = -i\,\delta_\psi$$

$$\frac{d}{d\not p}\Sigma\Big|_{\not p \to m} : \frac{ie^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + 4 + \ln\frac{\tilde{\mu}^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]$$

↑ introduce m_γ to regulate divergence as $p_\gamma^2 \to 0$

$$\Rightarrow \delta_\psi = \frac{-e^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + 4 + \ln\frac{\tilde{\mu}^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]$$

$$\Sigma(m) = im\,\delta m + im\frac{e^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + 4 + \ln\frac{\tilde{\mu}^2}{m^2} - \ln\frac{m^2}{m_\gamma^2}\right]$$

$$\Rightarrow \delta m = -\frac{4e^2}{(4\pi)^2}\left[\frac{1}{\epsilon} + 2 - \frac{1}{2}\ln\frac{m^2}{m_\gamma^2} + \ln\frac{\tilde{\mu}^2}{m^2}\right]$$

For (3) notice that the counter term

 $= ie\gamma^\mu\delta_e$

only has the Dirac form to cancel divergences going as γ^μ

Fortunately (but also as required by renormalizability) there is no divergence in the Magnetic dipole moment

Taking the limit $\Gamma(0)$:

$$e_R\Gamma^\mu(0) = ie_R\gamma^\mu + \frac{ie_R^3}{(4\pi)^2}\left[4+\frac{1}{\epsilon}+\ln\frac{\tilde{\mu}^2}{m^2} - 2\ln\frac{m^2}{m_\gamma^2}\right]\gamma^\mu + ie_R\gamma^\mu\delta_e = ie_R\gamma^\mu$$

$$\Rightarrow \delta_e = \frac{-e_R^2}{(4\pi)^2}\left[4+\frac{1}{\epsilon}+\ln\frac{\tilde{\mu}^2}{m^2} - 2\ln\frac{m^2}{m_\gamma^2}\right]\gamma^\mu$$

For the vacuum polarization we have:

$$\Pi^{\mu\nu}(0) = \frac{e_R^2}{(4\pi)^2}\left[\frac{4}{3}\frac{1}{\epsilon}+\frac{4}{3}\ln\frac{\tilde{\mu}^2}{m^2}\right]\Pi_T^{\mu\nu} + i\delta_A\Pi_T^{\mu\nu}$$

$$\Rightarrow \delta_A = \frac{-e_R^2}{(4\pi)^2}\frac{4}{3}\left[\frac{1}{\epsilon}+\ln\frac{\tilde{\mu}^2}{m^2}\right]$$

In the Homework you will derive the β function for QED and the anomalous dimension of m_R