Yang Mills Theory

1) Group thry & Lie Groups (Schwartz 25.1)

Recall the definition of a group G:

1) Closure: if $A, B \in G$, $(A * B) \in G$

$\uparrow$ group operation

2) Associativity: $(A*B)*C = A*(B*C)$

3) $\exists$ an identity st $A\mathbb{1} = \mathbb{1}A = A \quad \forall A \in G$

4) Inverse: $\exists$ an inverse $\forall A \in G$ st $AA^{-1} = A^{-1}A = \mathbb{1}$

Lie groups have an ∞ number of elements

Any element continuously connected to the identity can be written:

$$U = e^{i\theta^a T^a} \cdot \mathbb{1}$$

The T^a are called "generators"

The generators form a Lie Algebra:

$$[T^a, T^b] = i f^{abc} T^c$$

$\uparrow$ structure constants

For an Abelian group (ie all elements commute) it follows

$$f^{abc} = 0$$

Probably the most familiar Lie groups are rotations,

eg $O(2)$ — orthogonal group; in 2 dimensions

a general element can be written:

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

this is a single parameter group, w/ parameter θ

Taking a vector in 2D we have:

$$v' \equiv Rv = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix}$$

We also have

$$(v')^T v = v^T \underbrace{R^T R}_{=\mathbb{1}} v = v^T v$$

$R^T \cdot R = \mathbb{1}$ is why we call this an orthogonal group

If we consider a thry w/ rotational invar in x-y, but not in z, we can also think of

$$R = \begin{pmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}$$

$$v = (v_x, v_y, v_z)^T$$

In this case we still have $(v')^T v' = v v$ & $R^T R = \mathbb{1}$

We can then think of v as being in a higher dim representation of $O(2)$

Also note in both examples $\det R = 1$, this is because we haven't included improper rotations, eg $\bar{R} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} R$

O(3) is also familiar

Taking

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We find $[L_i, L_j] = \epsilon_{ijk} L_k$ ← Lie Algebra

Introducing 3 parameters $\theta_i = \{\theta_x, \theta_y, \theta_z\}$

We can find the elements of O(3) by exponentiating the generators

$$R = e^{i L_i \cdot \theta_i}$$

$$R^T R = \mathbb{1}$$

In this case we find, in contrast to O(2),

$$(\det R)^2 = 1 \rightarrow \det R = \pm 1$$

Rs w/ $\det R = 1$ → proper rotations

$= -1$ → improper rotations (proper x mirror reflection)

(Proper)x(Proper) = Proper

(Proper)x(Improper) = Improper

(Improper)x(Improper) = Proper

So improper rotations aren't closed ⇒ no subgroup

proper rotations are closed ⇒ subgroup

The proper subgroup is called SO(2)

↑ special, means $\det R = +1$

Orthogonal groups were characterized by $R^T R = \mathbb{1}$

We can generalize this to unitary groups w/ $U^\dagger U = \mathbb{1}$

The simplest is $U(1)$

unitary degree

These are phases: $U = e^{i\theta}$

This is a single parameter group, and is isomorphic to $SO(2)$

We already know $SU(2)$, the spin group, it is the <u>special</u> unitary group of degree 2, The unitary group of degree 2, $U(2)$ has $\det U = e^{i\theta'}$

Notice $\det U \sim U(1)$

So $SU(2) \sim U(1)\, SU(2)$ (semi-direct product, formally)

We are familiar w/ representations of $SU(2)$ from spin

the 2D irrep $\rightarrow$ fundamental irrep $\rightarrow \sigma$ Pauli matrices are generators

3D irrep $\rightarrow$ Adjoint irrep $\rightarrow \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}, \begin{pmatrix} 0 & & i \\ & 0 & \\ -i & & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix}$

For $SU(N)$ we have:

N dim irrep $\rightarrow$ fundamental irrep

$\underbrace{N^2-1}_{\text{this is the number of generators}}$ dim irrep $\rightarrow$ Adjoint irrep

We'll be particularly interested in SU(3):

generators: $\lambda^1 = \begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} \sigma_x & \\ & 0 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0 & -i & \\ i & 0 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} \sigma_y & \\ & 0 \end{pmatrix}$ $\lambda^3 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix} = \begin{pmatrix} \sigma_3 & \\ & 0 \end{pmatrix}$

$\lambda^4 = \begin{pmatrix} & & 1 \\ & 0 & \\ 1 & & \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} & & -i \\ & 0 & \\ i & & \end{pmatrix}$

$\lambda^6 = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}$

$\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$

The norm of generators is arbitrary,

$e^{iT^a\theta^a} \rightarrow e^{i\frac{1}{\alpha}T^a(\alpha\theta^a)}$ gives same group & θ's are arbitrary

In physics we often use

$$\sum_{c,d} f^{acd} f^{bcd} = N\delta^{ab}$$

$$\Rightarrow \quad \mathrm{Tr}[T_F^a T_F^b] = \tfrac{1}{2}\delta^{ab}$$

$$\mathrm{Tr}[T_A^a T_A^b] = N\delta^{ab}$$

In practice these identities occur in calculations, so we rarely need the explicit form of the generators

2) QED as a $U(1)$ gauge theory

We had the xformation of ψ in QED:

$$\psi \to e^{-ie\alpha(x)}\psi \equiv U\psi$$

This xform depends on a single parameter (local) $\alpha(x) \to U(1)$

How do we see the xform of A_μ in this way?

$$A_\mu \to A_\mu + \partial_\mu \alpha(x)$$

$$A_\mu \to U A_\mu U^{-1} + \frac{i}{e}(\partial_\mu U)U^{-1} = A_\mu + \frac{i}{e}\partial_\mu(-ie\alpha(x))$$

$$= A_\mu + \partial_\mu \alpha(x)$$

w/ this definition we have:

$$D_\mu\psi = (\partial_\mu + ieA_\mu)\psi \to U(\partial_\mu\psi) + (\partial_\mu U)\psi + ieUA_\mu U^{-1}U\psi + ie\frac{i}{e}(\partial_\mu U)U^{-1}U\psi$$

$$= U[\partial_\mu + ieA_\mu]\psi$$

$$= U D_\mu\psi$$

So $\bar\psi \not{D}\psi \to \bar\psi U^{-1}U\not{D}\psi = \bar\psi\not{D}\psi$

We also have $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu) \to \partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu\partial_\nu\alpha - \partial_\nu\partial_\mu\alpha$

$$= F_{\mu\nu}$$

The <u>field strength is gauge invariant</u>

3) Yang Mills theory

We want "to gauge" $SU(N)$

So we'll start w/ the Fermion field's xform:

$$\psi_i \to \underbrace{e^{-igT^a \cdot \alpha^a}}_{\text{matrix}} \underset{\substack{\uparrow \\ \text{vector}}}{\psi_i}$$

For QCD & the SM we'll take T^a in the fundamental irrep

$$\Rightarrow i \in (1, N) \text{ for } SU(N)$$
$$a \in (1, N^2-1)$$

Notice: $\bar\psi\psi \to \bar\psi\psi$ so our mass terms are invariant under $SU(N)$

Taking the kinetic term:

$$\bar\psi i \partial_\mu \gamma^\mu \psi$$

We have:

$$\partial_\mu \psi \to e^{-igT^a \cdot \alpha^a}\left[\partial_\mu - igT^a \partial_\mu \alpha^a\right]\psi \quad \leftarrow \text{indices supressed } \psi \to \psi_i,\ T^a \to T^a_{ij}$$

$$\bar\psi \partial_\mu \psi \to \bar\psi \partial_\mu \psi \underbrace{- ig(\partial_\mu \alpha^a)\bar\psi T^a \psi}$$

In QED we had $\alpha^a \to \alpha$ and needed only 1 vector

For $SU(N)$ we have $a \in (1, N)$ so need N vectors

So we take:

$$D_\mu \equiv \partial_\mu + igT^a \cdot A^a_\mu$$

$$D_\mu \psi \to \partial_\mu\left[e^{-igT^a\alpha^a}\psi\right] + igT^a \underset{\substack{\uparrow \\ \text{transformed } A^a_\mu}}{V^a_\mu} e^{-igT^a\alpha^a}\psi$$

$$= U\partial_\mu \psi + (\partial_\mu U)\psi + igT^a V^a_\mu U\psi$$

$$\text{w/ } U \equiv e^{-igT^a\alpha^a}$$

Taking: $T^a A'^a_\mu = U T^a A^a_\mu U^{-1} + \frac{i}{g}(\partial_\mu U)U^{-1}$

$$D_\mu \psi \to U(\partial_\mu \psi) + (\partial_\mu U)\psi + igUT^a A^a_\mu U^{-1}U\psi + ig\frac{i}{g}(\partial_\mu U)U^{-1}U\psi$$

$$= U D_\mu \psi$$

So we have:

$$i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi \to i\bar{\psi}\not{D}\psi - m\bar{\psi}\psi$$

If we take $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ we find: $\quad (A_\mu \equiv A^a_\mu T^a)$

$$\partial_\mu A_\nu \to \partial_\mu (U A_\nu U^{-1}) + \frac{i}{g} \partial_\mu [(\partial_\nu U) U^{-1}]$$

$$= (\partial_\mu U) A_\nu U^{-1} + U(\partial_\mu A_\nu) U^{-1} + U A_\nu \partial_\mu U^{-1} + \frac{i}{g} (\partial_\mu \partial_\nu U) U^{-1} + \frac{i}{g} (\partial_\nu U)(\partial_\mu U^{-1})$$

$$= U(\partial_\mu A_\nu) U^{-1} + \partial_\mu (U T^a U^{-1}) A^a_\nu + \frac{i}{g} (\partial_\mu \partial_\nu U) U^{-1} + \frac{i}{g} (\partial_\nu U)(\partial_\mu U^{-1})$$

$$\partial_\mu A_\nu - \partial_\nu A_\mu = U(\partial_\mu A_\nu - \partial_\nu A_\mu) U^{-1} + A^a_\nu \partial_\mu (U T^a U^{-1}) - A^a_\mu \partial_\nu (U T^a U^{-1}) + \frac{i}{g} (\partial_\nu U)(\partial_\mu U^{-1}) - \frac{i}{g} (\partial_\mu U)(\partial_\nu$$

So $F_{\mu\nu}$ isn't invariant under the gauge xform

What about $F_{\mu\nu} F^{\mu\nu}$?

$$F_{\mu\nu} F^{\mu\nu} \to U(\partial_\mu A_\nu - \partial_\nu A_\mu) U^{-1} U (\partial^\mu A^\nu - \partial^\nu A^\mu) U^{-1}$$

$$+ 4\left[A^a_\nu \partial_\mu (U T^a U^{-1}) + \frac{i}{g} (\partial_\nu U)(\partial_\mu U^{-1})\right] U (\partial^\mu A^\nu - \partial^\nu A^\mu) U^{-1} \quad \leftarrow \text{used antisymm}$$

$$+ 2\left[A^a_\nu \partial_\mu (U T^a U^{-1}) + \frac{i}{g} (\partial_\nu U)(\partial_\mu U^{-1})\right]\left[A^a_\nu \partial_\mu (U T^a U^{-1}) - \frac{i}{g} (\partial_\nu U)(\partial_\mu U^{-1}) - (\mu \leftrightarrow \nu)\right]$$

This also is not invariant!

If we took $Tr[F_{\mu\nu} F^{\mu\nu}]$ the first line would be

Our solution will be to amend $F_{\mu\nu}$

Taking $F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$

$$\left.\begin{aligned} &F^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu \\ &F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \end{aligned}\right\} \quad A_\mu \equiv A^a_\mu T^a$$

We can use a trick to simplify our life

Notice: $[D_\mu, D_\nu]\psi = [\partial_\mu + igA_\mu, \partial_\nu + igA_\nu]\psi$

$$= \Big(\underbrace{[\partial_\mu, \partial_\nu]}_{0} + ig\underbrace{[A_\mu, \partial_\nu]}_{A_\mu\partial_\nu\psi - \partial_\nu(A_\mu\psi) = -(\partial_\nu A_\mu)\psi} + ig[\partial_\mu, A_\nu] - g^2[A_\mu, A_\nu]\Big)\psi$$

$$= \left(-ig\partial_\nu A_\mu + ig\partial_\mu A_\nu - g^2[A_\mu, A_\nu]\right)\psi$$

$$= ig\underbrace{\left(\partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]\right)}_{= F_{\mu\nu}}\psi$$

We've seen $D_\mu\psi \to UD_\mu\psi$

So it follows $[D_\mu, D_\nu]\psi \to U[D_\mu, D_\nu]\psi$

$$= igUF_{\mu\nu}\psi$$

$$= igUF_{\mu\nu}U^{-1}U\psi$$

We conclude this new $F_{\mu\nu} \to UF_{\mu\nu}U^{-1}$

So $\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] \to \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}]$

We conclude:

$$\mathcal{L}_{YM} = -\frac{1}{2}\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$$

Or for YM coupled to fermions:

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$ ← For SU(3) this is Quantum Chromodynamics

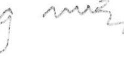
Notice in addition to the $\bar{\psi}\not{A}\psi$ interaction

 (most people use  for gluons)

We have:

$$F^a_{\mu\nu}F^{a\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu)^2$$

$= (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$ ← free part, propagator

$+2g(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) f^{abc} A^b_\mu A^c_\nu$ ← triple gauge coupling  $\propto g p^\mu$

$+g^2 f^{abc} f^{def} A^b_\mu A^c_\nu A^e_\mu A^f_\nu$ ← quartic gauge coupling  $\propto g^2$

4) Gauge fixing YM

"Faddeev - Popov" procedure

Recall when we gauge fixed the photon we took an overall function

$$f(\xi) = \int D\pi \, e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi)^2} = \int D\pi \, e^{-i\int d^4x \frac{1}{2\xi}(\Box\pi - \partial_\mu A_\mu)^2}$$

↑ shift symmetric integral

To rewrite the QED path integral, this resulted in an overall ∞ constant in the PI that drops out of calculations

$$\int DA_\mu \, D\bar{\psi} \, D\psi \, e^{i\int d^4x \, \mathcal{L}[A,\bar{\psi},\psi]} = \frac{1}{f(\xi)} \int D\pi \, DA_\mu \, D\bar{\psi} \, D\psi \, e^{i\int d^4x \left[\mathcal{L}[A,\bar{\psi},\psi] - \frac{1}{2\xi}(\Box\pi - \partial_\mu A_\mu)^2\right]}$$

$$= \left[\frac{1}{f(\xi)} \int D\pi\right] \int DA_\mu \, D\bar{\psi} \, D\psi \, e^{i\int d^4x \left[\mathcal{L}[A,\psi_i] - \frac{1}{2\xi}(\partial_\mu A_\mu)^2\right]}$$

↑ gauge xform

Now we do the same for YM, but this will be complicated by the fact YM fields are self interacting

We had: $T^a A_\mu^a = U T^a A_\mu^a U^{-1} + \frac{i}{g}(\partial_\mu U) U^{-1}$ w/ $U = e^{-igT^a\alpha^a}$

Infinitesimally: $T^a A_\mu^a \to (1 - igT^b\alpha^b) T^a A_\mu^a (1 + igT^c\alpha^c) + \frac{i}{g}\left[\partial_\mu(1 + igT^a\alpha^a)\right](1 - igT^b\alpha^b) + \ldots$

$$\sim T^a A_\mu^a \underbrace{- igT^bT^a A_\mu^a \alpha^b + igT^aT^c A_\mu^a \alpha^c}_{-ig[T^b,T^a]A_\mu^a\alpha^b = -ig(if^{bac}A_\mu^a\alpha^b)T^c} + \frac{i}{g}(ig)T^a \partial_\mu \alpha^a + O(\alpha^2)$$

$$= T^a A_\mu^a - T^a \partial_\mu \alpha^a + g f^{abc} \alpha^b A_\mu^c T^a$$

$$\Rightarrow \quad A_\mu^a \to A_\mu^a - \partial_\mu \alpha^a + g f^{abc} \alpha^b A_\mu^c$$

$$\equiv A_\mu^a - (D_\mu \alpha)^a$$

$(D_\mu \alpha)^a \equiv (\partial_\mu \delta^{ab} + g f^{abc} A_\mu^c)\alpha^b$ ← how a field in the Adjoint xforms

$$= D_\mu^{ab} \alpha^b$$

So now we'll define an analogue of $\xi[\xi]$:

$$\xi[A] = \int D\pi \, e^{-i\int d^4x \frac{1}{2\xi}(\partial_\mu D^\mu \pi^a)^2}$$

↑
Now a functional of A_μ

Shifting π^a by $\alpha^a[A]$:

$$\partial_\mu D_\mu \pi^a \to \partial_\mu D_\mu \pi^a + \partial_\mu D_\mu \alpha^a$$

Since we have

$$\partial_\mu A_\mu^a \to \partial_\mu [A_\mu^a + D_\mu \alpha^a]$$

$$= \partial_\mu A_\mu^a + \partial_\mu D_\mu \alpha^a$$

Again we want to impose $\partial_\mu A_\mu^a = 0$, so we have:

$$\partial_\mu A_\mu^a + \partial_\mu D_\mu \alpha^a = 0 \to \alpha = \frac{-1}{\partial_\mu D_\mu} \partial_\nu A_\nu$$

Then the shift (which doesn't change the integral) gives:

$$\xi[A] = \int D\pi \, e^{-i\int d^4x \frac{1}{2\xi}(\partial_\mu D^\mu \pi^a - \partial_\mu A_\mu)^2}$$

This let's us rewrite the YM path integral:

$$\int DA_\mu e^{i\int d^4x \mathcal{L}_{YM}}$$

$$= \int D\pi \, DA_\mu \frac{1}{\xi[A]} \exp\left[i\int d^4x \left(\mathcal{L}_{YM} - \frac{1}{2\xi}\underbrace{(\partial_\mu A_\mu^a - \partial_\mu D^\mu \pi^a)^2}_{\text{use gauge xform to write } A_\mu^a \to A_\mu^a + D_\mu \pi^a}\right)\right]$$

$$= \int D\pi \, DA_\mu \frac{1}{\xi[A]} \exp\left[i\int d^4x \left(\mathcal{L}_{YM} - \frac{1}{2\xi}(\partial_\mu A_\mu)^2\right)\right]$$

We've successfully gauge fixed, but since $\xi[A]$ is a functional of A_μ we can't pull it out of the DA_μ integral

Rewinding, we can perform the $f[A]$ integral, which is a gaussian:

$$f[A] = \int D\pi \, e^{-i\int d^4x \frac{1}{2\xi}(\partial_\mu D_\mu \pi^a)^2}$$

$$= \sqrt{\frac{1}{\det(\partial_\mu D_\mu)^2}} \times \text{constant}$$

So our path integral is:

$$\int DA_\mu \, e^{i\int d^4x \, \mathcal{L}_{YM}}$$

$$= \text{const} \int DA_\mu \det(\partial_\mu D_\mu) \, e^{i\int d^4x [\mathcal{L}_{YM} - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2]}$$

but recall for grassmann numbers

$$\det \hat{O} = \int D\bar{\psi} D\psi \, e^{-i\int d^4x \bar{\psi}\hat{O}\psi}$$

for our Dirac fermions $\hat{O}$ was $\not{\partial}$

So we rewrite:

$$\det(\partial^\mu D_\mu) = \int D\bar{c} Dc \, e^{i\int d^4x \, \bar{c}(-\partial^\mu D_\mu)c}$$

for some new grassmann fields $\bar{c}, c$

Putting this into our PI and itegrating by parts:

$$\int DA_\mu \, e^{i\int d^4x \, \mathcal{L}_{YM}} = (\text{const}) \int DA_\mu D\bar{c} Dc \exp\left[i\int d^4x \left(\mathcal{L}_{YM} - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2 + \underbrace{(\partial^\mu \bar{c}^a) D_\mu c^a}\right)\right]$$

quadratic in $\partial \to$ scalar
index $a \to$ charged under $SO(N)$
also # of particles $\to$ scalar

5) Feynman Rules in YM

Our $\mathcal{L}$ is:

$$\mathcal{L}_{YM} + \mathcal{L}_{GF} + \mathcal{L}_{gh} + \mathcal{L}_{\psi}$$

$$\mathcal{L}_{YM} + \mathcal{L}_{GF} = -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu\right)^2 - \frac{1}{2\xi}(\partial_\mu A^a_\mu)^2$$

$$= -\frac{1}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)(\partial_\nu A_\nu)$$

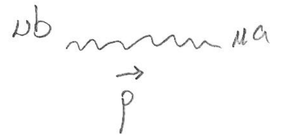
$$+ g f^{abc} A^b_\mu A^c_\nu (\partial_\mu A^a_\nu)$$ ← antisymm in b↔c, can sum

$$- \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu$$

The first term gives the same propagator as QED, except theres now a color adjoint index, notice that:

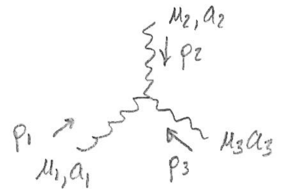
$$\frac{1}{2}\frac{\delta^2}{\delta A^b_\mu \, \delta A^c_\nu} A^a_\alpha A^a_\beta = \frac{1}{2}\left(\delta^b_a \delta^a_c \delta^\mu_\alpha \delta^\nu_\beta + \delta^a_b \delta^c_a \delta^\mu_\beta \delta^\nu_\alpha\right) = \frac{1}{2}\delta^b_c\left(\delta^\mu_\alpha \delta^\nu_\beta + \delta^\mu_\beta \delta^\nu_\alpha\right)$$

so we have a δ enforcing color is the same



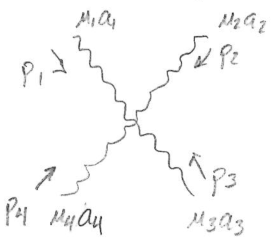
$$= -i\,\frac{\eta^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2}}{p^2 + i\epsilon}\,\delta^{ab}$$

The term involving 3 gluons gives:



$$-g f^{a_1a_2a_3}\left[(p_1-p_2)^{\mu_3}\eta_{\mu_1\mu_2} + (p_3-p_1)^{\mu_2}\eta_{\mu_1\mu_3} + (p_2-p_3)^{\mu_1}\eta_{\mu_2\mu_3}\right]$$

For 4 gluons we have:



$$ig^2\Big[f^{ca_1a_2}f^{ca_3a_4}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}-\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}) + f^{ca_1a_3}f^{ca_2a_4}(\eta_{\mu_1\mu_4}\eta_{\mu_2\mu_3}-\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4})$$
$$+ f^{ca_1a_4}f^{ca_2a_3}(\eta_{\mu_1\mu_3}\eta_{\mu_2\mu_4}-\eta_{\mu_1\mu_2}\eta_{\mu_3\mu_4})\Big]$$

The interactions w/ fermions are:

$$\mathcal{L}_\psi = \bar\psi(i\not\partial - g\not A^a T^a - m)\psi$$

Giving the same propagator, but the fermions xform in the Fundamental rep:

$$= \frac{i(\not p + m)}{p^2-m^2+i\epsilon}\,\delta_{ij}$$

The interaction is the same as in QED, but w/ the generator

$$= -ig\gamma^\mu T^a_{ji}$$

notice order, $j \to i$, follow fermion arrow backward

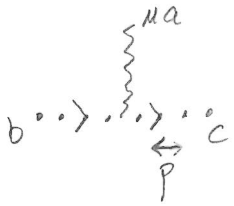
For the ghost $\mathcal{L}$:

$$\mathcal{L}_{Gh} = (\partial_\mu \bar{c}^a)(\delta^{ac}\partial_\mu - g f^{abc} A^b_\mu) c^c$$

The propagator is that of a massless scalar + adjoint representation

$$= \frac{i}{p^2 + i\epsilon}\delta^{ab}$$

The interaction contains $(\partial\bar{c})$ so is mtm dep:



$$= -g f^{cab} p^\mu$$

We also need to remember ghosts are grassmann variables

$\Rightarrow$ (-1) for closed loop

6) QCD at one-loop

we take all fields & parameters, rewrite them as bare "0", then renormalize as:

$$A^a_{\mu 0} \to Z_A^{1/2} A^a_\mu \quad \leftarrow \text{color adjoint}$$

$$\psi_{i0} \to Z_\psi \psi_i \quad \leftarrow \text{color fundamental}$$

$$c^a \to Z_c c^a \quad \leftarrow \text{adjoint}$$

This gives:

$$\mathcal{L} = -\frac{1}{4} Z_A (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 + Z_\psi \bar{\psi} (i\not{\partial} - m_0)\psi - Z_c \bar{c}^a \Box c^a$$
$$+ g_0 Z_A^{3/2} f^{abc} (\partial_\mu A^a_\nu) A^b_\mu A^c_\nu - \frac{1}{4} g_0^2 Z_A^2 f^{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu$$
$$- g_0 Z_A^{1/2} Z_\psi \bar{\psi} \gamma^\mu T^a \psi A^a_\mu - g_0 Z_A^{1/2} Z_c f^{abc} (\partial_\mu \bar{c}^a) A^b_\mu c^c$$

From this we define:

$$g_0 Z_A^{3/2} \equiv g Z_3 \qquad\qquad Z_\psi m_0 = Z_m m$$

$$g_0^2 Z_A^2 \equiv g^2 Z_4$$

$$g_0 Z_A^{1/2} Z_\psi \equiv g Z_1$$

$$g_0 Z_A^{1/2} Z_c = g Z_{1'}$$

From gauge symmetry we can infer (see Slavnov Taylor IDs, Srednicki 74)

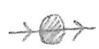
$$g_0^2 = \frac{g^2 Z_3^2}{Z_A^3} = g^2 \frac{Z_4}{Z_A^2} = \frac{g^2 Z_1^2}{Z_A Z_\psi^2} = \frac{g^2 Z_{1'}^2}{Z_A Z_c^2}$$

Any of these combinations of Z's is sufficient to determine

$$\mu \frac{d}{d\mu} g$$

We want the simplest loops to calculate

$\Rightarrow Z_1$  $\qquad Z_A$ 

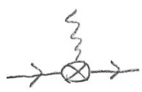
Z_ψ 

In addition to our usual FRs we need these counter terms:

 $= -i\delta_A(p^2\eta^{\mu\nu} - p^\mu p^\nu)\delta_{ab} = i\delta_A \Pi_T^{\mu\nu}\delta_{ab}$

color adjoint

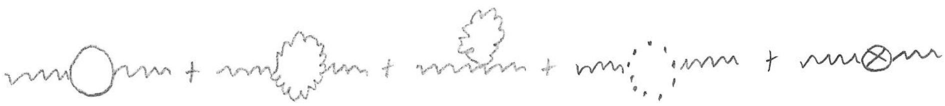
 $= i(\not{p}\,\delta_\psi - m\delta_m)\delta_{ij}$ ← color fundamental

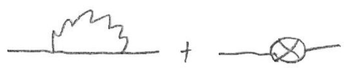
 $= -ig\gamma^\mu T^a_{ij}\delta_1$

adjoint

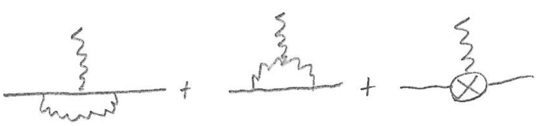
fundamental

For the 2pt functions we have:



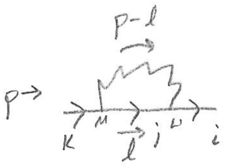


For the 3pt we have:



We will put the FRs together, the HW will be to identify the UV div parts

We will remove the divergences using MS scheme looking at the fermion 2pt function: (Feynman gauge)



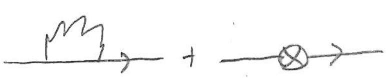
$$= (-ig\mu^{\epsilon})^2\int\frac{d^d\ell}{(2\pi)^d}\,\frac{\gamma^\nu T^a_{ij}\, i(\not{\ell}+m)\gamma^\mu T^b_{jk}}{[\ell^2-m^2+i\epsilon][(p-\ell)^2+i\epsilon]}\,(-ig_{\mu\nu})\,\delta^{ab}$$

$$T^a_{ij}T^a_{jk} = C_F\,\delta_{ik} \quad \text{w/} \quad C_F = \frac{N^2-1}{2N} \qquad \left(=\tfrac{4}{3} \text{ in QCD}\right)$$

This is the same as for QED w/ $e\to g$

plus a factor $C_F\,\delta_{ik}$ ← color fundamental

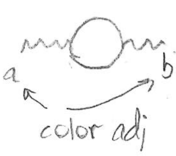
So the divergent part is:



$$= \frac{ig^2}{(4\pi)^2}C_F\,\delta_{ij}(\not{p}-4m)\frac{1}{\epsilon} + i(\not{p}\,\delta_4 - m\,\delta_m)\delta_{ij}$$

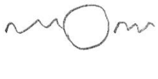
$$\to \delta_4 = -\frac{g^2}{(4\pi)^2}C_F \qquad \delta_m = -\frac{4g^2}{(4\pi)^2}C_F$$

For the gluon we have:



$$= (-ig\mu^{\epsilon})^2\int\frac{d^d\ell}{(2\pi)^d}\,\frac{\mathrm{Tr}[\gamma^\mu(\not{\ell}+m)\gamma^\nu(\not{\ell}-\not{p}+m)]}{[\ell^2-m^2+i\epsilon][(p-\ell)^2-m^2+i\epsilon]}\,\underbrace{T^a_{ij}T^b_{jk}}_{\mathrm{Tr}[T^aT^b]=\frac{1}{2}\delta^{ab}}$$

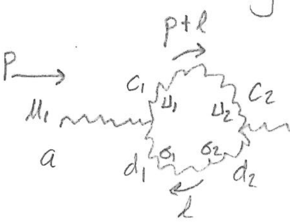
This is as in QED w/ $e\to g$ and a factor $\frac{1}{2}\delta^{ab}$



$$= +\frac{4ig^2}{3(4\pi)^2}\Pi_T^{\mu\nu}\frac{1}{2}\delta^{ab}\frac{1}{\epsilon}$$

Notice this is transverse!

The next gluon diagram is (Feynman gauge)



$$= \underset{\text{symm factor}}{\frac{(-g)^2}{2}} \int \frac{d^d l}{(2\pi)^d} \frac{(-i)^2}{[l^2+i\epsilon][(p+l)^2+i\epsilon]} N^{\mu_1\mu_2 ab}$$

The numerator requires care, we write out all the details:

prop num: $\eta^{\mu\nu}\delta^{ab}$

vertex: $-g f^{a_1a_2a_3}\left[(p_1-p_2)^{\mu_3}\eta_{\mu_1\mu_2} + (p_3-p_1)^{\mu_2}\eta_{\mu_1\mu_3} + (p_2-p_3)^{\mu_1}\eta_{\mu_2\mu_3}\right]$ ← all mta incoming, so needs care

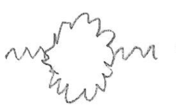
incoming! incoming!

$$f^{ac_1d_1}\left[(p+(p+l))^{\sigma_1}\eta_{\mu_1\nu_1} + (l-p)^{\nu_1}\eta_{\mu_1\sigma_1} + (-p-l-l)^{\mu_1}\eta_{\nu_1\sigma_1}\right]\eta_{\nu_1\nu_2}\delta^{c_1c_2}\eta_{\sigma_1\sigma_2}\delta^{d_1d_2}$$

$$\times f^{bc_2d_2}\left[(-p-(p+l))^{\sigma_2}\eta_{\mu_2\nu_2} + (-l+p)^{\nu_2}\eta_{\mu_2\sigma_2} + (p+l+l)^{\mu_2}\eta_{\nu_2\sigma_2}\right]$$

incoming! incoming! incoming!

$$= \underbrace{f^{ac_1d_1}f^{bc_1d_1}}_{N\delta^{ab}}\left[(3-2d)l_{\mu_2}p_{\mu_1} + 6p_{\mu_1}p_{\mu_2} - d\,p_{\mu_1}p_{\mu_2} - (-3+2d)l_{\mu_1}(2l_{\mu_2}+p_{\mu_2}) - 2l^2 g_{\mu_1\mu_2} - 2l\cdot p\, g_{\mu_1\mu_2} - 5p^2 g_{\mu_1\mu_2}\right]$$

 $= +\dfrac{ig^2 N\delta^{ab}}{(4\pi)^2}\left(\dfrac{19p^2g_{\mu_1\mu_2} - 22p_{\mu_1}p_{\mu_2}}{6}\right)\dfrac{1}{\epsilon}$ + finite

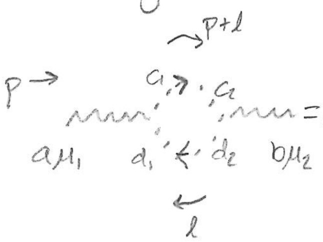
That this <u>is not</u> transverse is significant. there are nonphysical modes in the loops.

The 4pt gluon diagram vanishes as it is scaleless:

$$\int \frac{d^d l}{(2\pi)^d}\frac{1}{l^2-m^2+i\epsilon} = \frac{i(-1)^{-1}}{(4\pi)^{d/2}}\frac{1}{m^{1-d/2}}\frac{\Gamma(d/2)\Gamma(1-d/2)}{\Gamma(1)\Gamma(d/2)}$$

$$= \frac{-i}{(4\pi)^{d/2}}\, m^{1-\epsilon}\,\Gamma(1-d/2) \to 0 \quad \text{for } m\to 0$$

The ghost diagram:



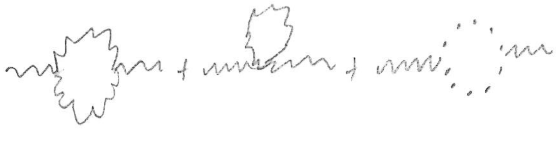
closed loop

$$= (-1)\, g^2 f^{c_1 a d_1} f^{d_2 b c_2}\int \frac{d^d l}{(2\pi)^d}\frac{(i)^2(-p-l)^{\mu_1}(-l)^{\mu_2}}{[l^2+i\epsilon][(p+l)^2+i\epsilon]}\,\delta^{c_1c_2}\delta^{d_1d_2}$$

$$= -g^2\underbrace{f^{a c_1 d_1} f^{b c_1 d_1}}_{N\delta^{ab}}\int \frac{d^d l}{(2\pi)^d}\frac{(p+l)^{\mu_1}\, l^{\mu_2}}{[l^2+i\epsilon][(p+l)^2+i\epsilon]}$$

$$= +\frac{ig^2N}{(4\pi)^2}\,\delta^{ab}\,\frac{p^2 g_{\mu_1\mu_2}+2p_{\mu_1}p_{\mu_2}}{12}\,\frac{1}{\epsilon}$$

Summing:



$$= -\frac{ig^2N}{(4\pi)^2}\,\Pi_T^{\mu\nu}\,\frac{5}{3}\,\frac{1}{\epsilon}$$

The ghosts cancel the nonphysical gluon modes, and our gluon remains transverse

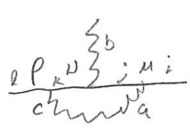
So the full correction to the gluon 4-pt function is (div parts only):
We'll allow for n_f fermions in the fundamental of $SU(N)$

$$= \frac{ig^2}{(4\pi)^2} \Pi_T^{\mu\nu} \delta^{ab} \left[\frac{2}{3} n_f - \frac{5}{3} N\right] \frac{1}{\epsilon} + i\delta_A \Pi_T \delta^{ab} \quad (+\text{ finite})$$

Which is rendered finite by choosing:

$$\delta_A = \frac{g^2}{(4\pi)^2} \left[\frac{5}{3} N - \frac{2}{3} n_f\right]$$

For the vertex correction we have the QED-like diagram:



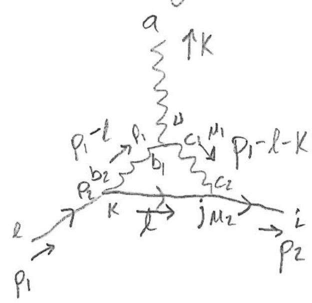
$$= (-ig)^3 \int \frac{d^d l}{(2\pi)^d} \frac{(\gamma^\mu T^a_{ij})\, i(\not{l}-\not{K}+m)\,\gamma^\nu T^b_{jk}\; i(\not{l}+m)\,\gamma^\rho T^c_{kl}}{[l^2-m^2+i\epsilon][(l-K)^2-m^2+i\epsilon][(l-p_1)^2+i\epsilon]} \;(-i g_{\mu\rho})\,\delta^{ac}$$

this differs from QED by $e \to g$

and $\quad T^a_{ij} T^b_{jk} T^a_{kl} = \left(C_F - \frac{N}{2}\right) T^b$

$$= -\frac{ig^3}{(4\pi)^2}\left(C_F - \frac{N}{2}\right)\gamma^\nu T^b \frac{1}{\epsilon}$$

The other diagram again involves the triple gauge coupling:

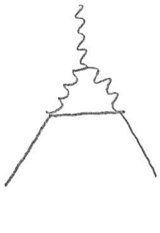


incoming! incoming incoming

$$= (-g)(-ig)^2 \int \frac{d^d l}{(2\pi)^d} \frac{\gamma^{\mu_2} T^{c_2}_{ij}\, i(\not{l}+m)\,\delta_{jk}\, \gamma^{\rho_2} T^{b_2}_{kl}\, f^{a c_1 b_1}\left[(-K+(p_2-l))^{\rho_1}\eta_{\nu\mu_1} + (p_1-l+K)^{\mu_1}\eta_{\rho_1\nu} + (-p_2+l-p_1+l)^{\nu}\eta_{\mu_1\rho_1}\right]}{[l^2-m^2+i\epsilon][(p_1-l)^2+i\epsilon][(p_2-l)+i\epsilon]}$$

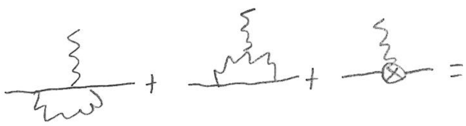
$$\times(-i\eta_{\rho_1\rho_2})\,\delta^{b_1 b_2}(-i\eta_{\mu_1\mu_2})\,\delta^{c_1 c_2}$$

The color factor is: $\quad T^c_{ij} T^b_{jl} f^{acb} = +\frac{i}{2} N T^a$



$$= \frac{-3g^3 i}{(4\pi)^2}\,\frac{N}{2}\,\frac{1}{\epsilon}\, T^a \gamma^\nu$$

Combining these we have:



$$= \frac{ig^3}{(4\pi)^2}\left[-C_F+\frac{N}{2}-\frac{3N}{2}\right]\frac{1}{\epsilon}T^a\gamma^\mu - ig\gamma^\mu T^a \delta_1$$

$$\rightarrow \delta_1 = \frac{+g^2}{(4\pi)^2}\left[-C_F - N\right]\frac{1}{\epsilon} = \frac{-g^2}{(4\pi)^2}\left[C_F + N\right]\frac{1}{\epsilon} \qquad \text{w/ } C_F = \frac{N^2-1}{2N}$$

We also had
$$\delta_\psi = \frac{-g^2}{(4\pi)^2} C_F$$

$$\delta_A = \frac{g^2}{(4\pi)^2}\left[\frac{5}{3}N - \frac{2}{3}n_f\right]$$

From our relation,

$$g_0 = \frac{g Z_1 \mu^\epsilon}{Z_A^{1/2} Z_\psi}$$ ← wasn't in our eqn, but required by dim reg

$$\mu\frac{d}{d\mu} g_0 = 0 = \left(\mu\frac{d}{d\mu}g\right) Z_1 Z_A^{-1/2} Z_\psi^{-1} + g\mu\frac{d}{d\mu} Z_1 Z_A^{-1/2} Z_\psi^{-1} + g Z_1 Z_A^{-1/2} Z_\psi^{-1}\epsilon$$

$$\rightarrow \mu\frac{d}{d\mu}\ln g = -\mu\frac{d}{d\mu}\ln Z_1 Z_A^{-1/2} Z_\psi^{-1} - \epsilon$$

$$= \left(-g\mu\frac{d}{d\mu}\ln g\right)\frac{\partial}{\partial g}\underbrace{\ln Z_1 Z_A^{-1/2} Z_\psi^{-1}}_{\delta_1 - \frac{1}{2}\delta_A - \delta_\psi + 2\text{loop}} - \epsilon$$

$$\left(\mu\frac{d}{d\mu}\ln g\right)\left[1 + \frac{g^2}{(4\pi)^2}\left(-2[\cancel{C}_F + N] - \frac{1}{2}2\left[\frac{5}{3}N - \frac{2}{3}n_f\right] + 2\cancel{C}_F\right)\frac{1}{\epsilon}\right]^{-1} = -\epsilon$$

$$\mu\frac{d}{d\mu}\ln g = -\epsilon\left[1 - \frac{g^2}{(4\pi)^2}\left(\frac{2}{3}n_f - \frac{11}{3}N\right)\frac{1}{\epsilon} + \mathcal{O}(g^4)\right]$$

$$\rightarrow -\frac{g^2}{(4\pi)^2}\left[\frac{11}{3}N - \frac{2}{3}n_f\right]$$

Notice $g(\mu)$ gets smaller w/ increasing μ for $\frac{11}{3}N > \frac{2}{3}n_f$, $\frac{11}{2}N > n_f$

In this case we say is is asymptotically free

QCD has $n_f = 6$ flavors and $N=3$, so $\frac{11}{3}N - \frac{2}{3}n_f = 7$, QCD is asymptotically free

But for small μ $g > 4\pi$ and perturbation theory breaks down, around $\Lambda_{QCD} \sim 200$ M

↳ occurs above Λ_{QCD}

7) QCD phenomenology

Fermions charged under $SU(3)_{color}$ → Quarks

There are 2 types of Quarks: up-like → $Q_u = +2/3$

down-like → $Q_d = -1/3$

Since quarks are charged under QED, they xform as:

$$u \to e^{-iQ_u e\alpha} e^{-igT^a \cdot \alpha^a} u$$

$$d \to e^{-iQ_d e\alpha} e^{-igT^a \cdot \alpha^a} d$$

Notice QCD doesn't "know" the difference between u & d but QED does

The relevant gauge group is $U(1)_{QED} \times SU(3)_c$

Since QCD is strongly (ie $g > 4\pi$) at low E quark masses are hard to define

The Particle Data Group gives:

$m_u = 2.16$ MeV

$m_d = 4.67$ MeV

Hadrons like the proton, neutron, and πs are composite & defined in terms of their "valence" quark structure:

$p \sim uud \qquad Q_p = 1 = Q_u + Q_u + Q_d$

$n \sim udd \qquad Q_n = 0 = Q_u + Q_d + Q_d$

That the neutron mass > proton is attributed to $m_d > m_u$

If $m_u = m_d \rightarrow m_p = m_n$ & neutrons would be stable

$m_p \sim m_n \sim 1$ GeV $\gg 3m_d$ → most of the proton/neutron mass comes from binding the quarks together *not* quark masses!

u & d make a generation, nature gave us 3 generations of quarks

u & d

s (strange) & c (charm)

b (bottom) & t (top)

$u, c, t \rightarrow$ up-like

$d, s, b \rightarrow$ down-like

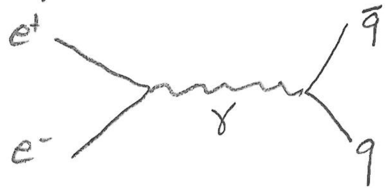
$m_s \sim 93.4$ MeV

$m_c \sim 1.27$ GeV

$m_b \sim 4.18$ GeV

$m_t \sim 172.69$ GeV $\leftarrow$ so heavy it behaves like a free quark (kinda)

Since quarks are electrically charged they can be produced in e^+e^-:



By varying $\sqrt{s}$ we can see as new quarks become kinematically accessible

$$R \equiv \frac{\sigma(e^+e^- \rightarrow \bar{q}q)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}$$

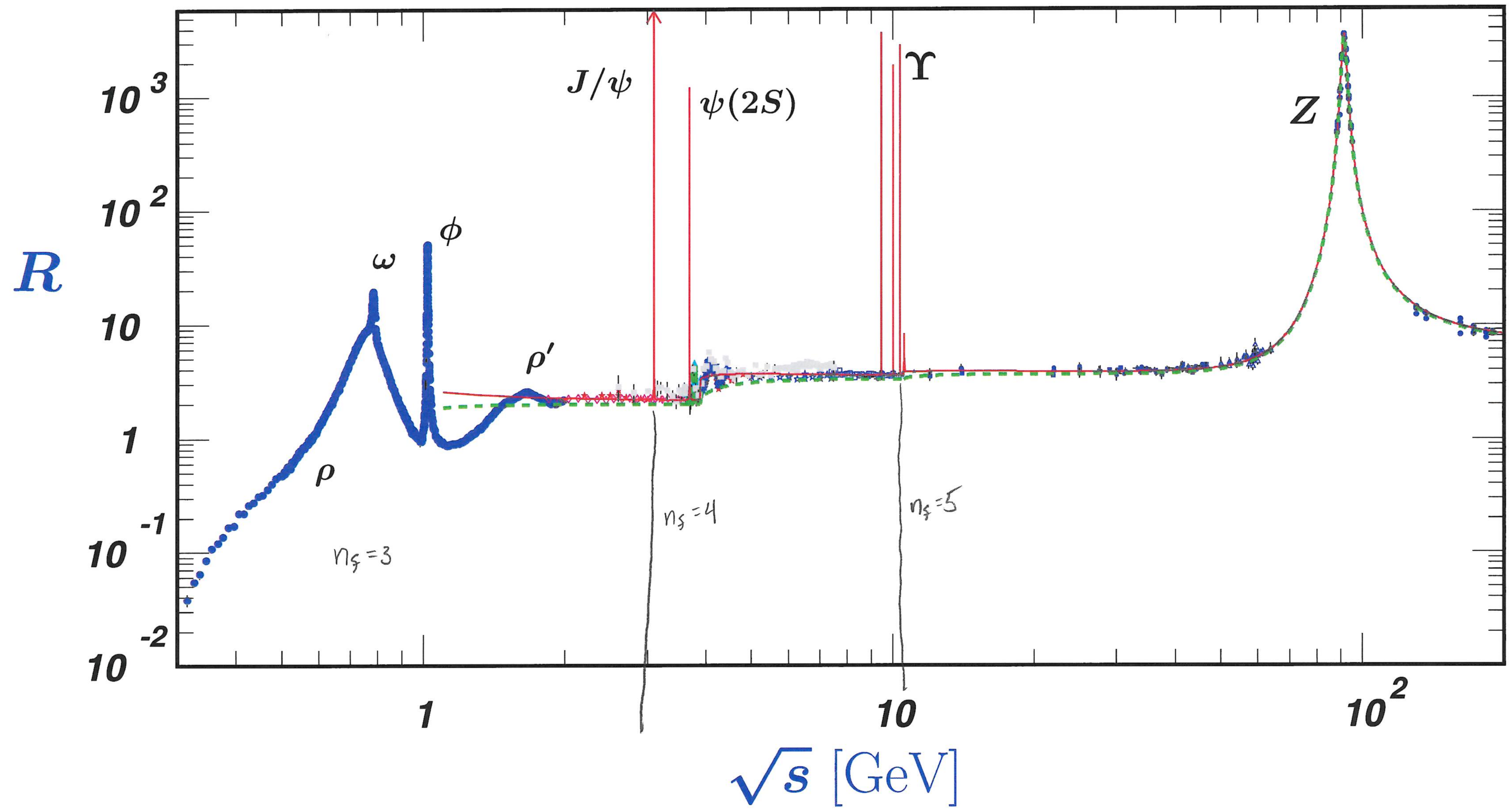
There are also many hadronic resonances along the way:

$\rho \sim \frac{u\bar{u} - d\bar{d}}{\sqrt{2}}$ $m_\rho \sim 775$ MeV $J/\psi \sim c\bar{c}$ $m_{J/\psi} \sim 3$ GeV

$\omega \sim \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}$ $m_\omega \sim 782$ MeV $\Upsilon \sim b\bar{b}$ $m_\Upsilon \sim 9.5$ GeV

upsilon

$\phi \sim s\bar{s}$ $m_\phi \sim 1$ GeV



R
10^3
10^2
10
1
10^{-1}
10^{-2}
ρ
ω
φ
ρ′
J/ψ
ψ(2S)
ϒ
Z
$n_f = 3$
$n_f = 4$
$n_f = 5$
1
10
10^2
√s [GeV]