1)

$$\mathcal{L} \propto F_{\mu\nu} F^{\mu\nu}$$

$$= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= (\partial_\mu A_\nu - \partial_\nu A_\mu + \partial_\mu \cancel{\partial_\nu} \alpha - \cancel{\partial_\nu \partial_\mu} \alpha)(\partial^\mu A^\nu - \partial^\nu A^\mu + \cancel{\partial^\mu \partial^\nu} \alpha - \cancel{\partial^\nu \partial^\mu} \alpha)$$

$$= F_{\mu\nu} F^{\mu\nu}$$

2) $\mathcal{L}_I = -e\bar{\psi}_a(x)\,\gamma^\mu_{ab}\,\psi_b(x)\,A_\mu(x)$

We derived

$$\frac{\partial^3\Gamma}{\partial\phi^3} = \frac{1}{(2\pi)^{2\cdot 4}}\,\delta^4(k_1+k_2+k_3)\; i\Gamma^{(3)}(k_1,k_2,k_3)\,\frac{N}{3!}$$

For

$$\frac{\partial^3\Gamma}{\partial\psi\,\partial\bar{\psi}\,\partial A_\mu} \qquad \begin{array}{l} N \to 1 \\ 3! \to 1 \quad \text{(all fields distinguishable)} \end{array}$$

So

$$\frac{\partial^3\Gamma}{\partial\psi_c\,\partial\bar{\psi}_d\,\partial A_\alpha} = \frac{1}{(2\pi)^{2\cdot 4}}\,\delta^4(k_1+k_2+k_3)\;\; i\Gamma^{(3)\alpha}_{cd}(k_1,k_2,k_3)$$

Interpreting $\Gamma = iS = i\int d^4x\,\mathcal{L}$:

$$\frac{\partial^3\Gamma}{\partial\psi_c\,\partial\bar{\psi}_d\,\partial A_\alpha} = i(-e)\,\frac{\partial^3}{\partial\psi_c\,\partial\bar{\psi}_d\,\partial A_\alpha}\int d^4x\,d^4y\,d^4z\;\delta(y-x)\,\delta(z-x)\,\bar{\psi}_a(x)\,\gamma^\mu_{ab}\,\psi_b(y)\,A_\mu(x)$$

$$= -ie\,\frac{\partial^3}{\partial\psi_c(p_1)\,\partial\bar{\psi}_d(p_2)\,\partial A_\alpha(p_3)}\int d^4x\,d^4y\,d^4z\,\delta(y-x)\,\delta(z-x)\int\frac{d^4k_1\,d^4k_2\,d^4k_3}{(2\pi)^{4\cdot 3}}\,e^{-ik_1\cdot x}\,e^{-ik_2\cdot y}\,e^{-ik_3\cdot z}$$

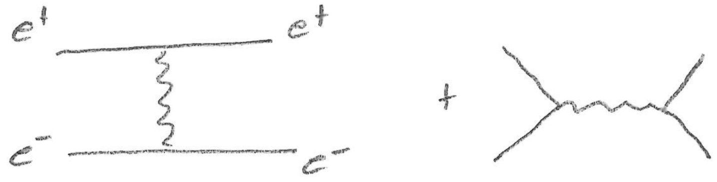
$$\times\,\bar{\psi}_a(k_1)\,\gamma^\mu_{ab}\,\psi_b(k_2)\,A_\mu(k_3)$$

$$= -ie\int\frac{d^4k_1\,d^4k_2\,d^4k_3}{(2\pi)^{4\cdot 2}}\,\delta^4(k_1+k_2+k_3)\,\delta(k_1-p_2)\,\delta(k_2-p_1)\,\delta(k_3-p_3)\,\delta_{bd}\,\delta_{ac}\,\delta^\alpha_\mu\,\gamma^\mu_{ab}$$

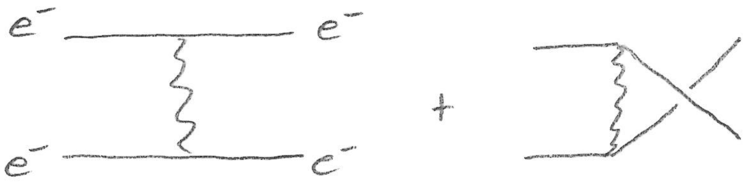
$$= \frac{1}{(2\pi)^{2\cdot 4}}\,\delta^4(k_1+k_2+k_3)\left[-ie\,\gamma^\alpha_{cd}\right]$$

$$i\Gamma^{(3)\alpha}_{cd} = -ie\,\gamma^\alpha_{cd}$$

4) a) $e^+e^- \rightarrow e^+e^-$

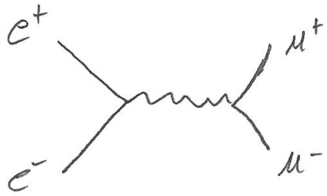


b) $e^-e^- \rightarrow e^-e^-$

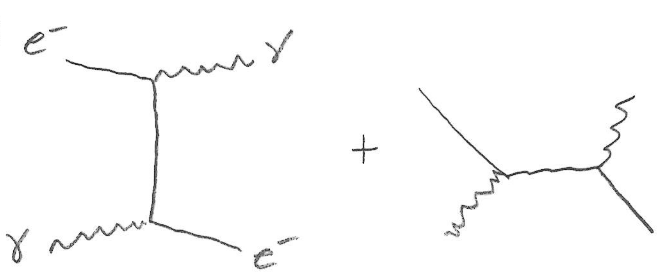


relative (-) Feynman Rule!

c) $e^+e^- \rightarrow \mu^+\mu^-$



only 1 diagram! first diagram from (a) doesn't contribute

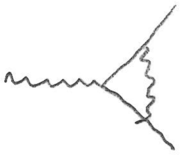
d) 

5)

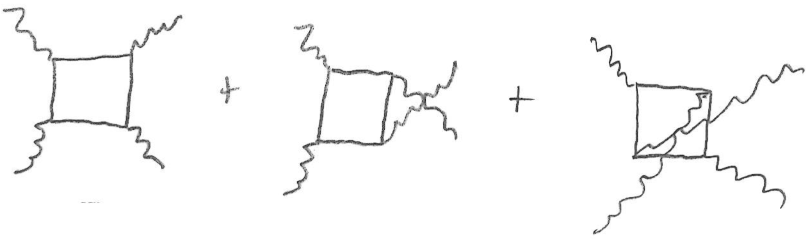
a)

b)

c)

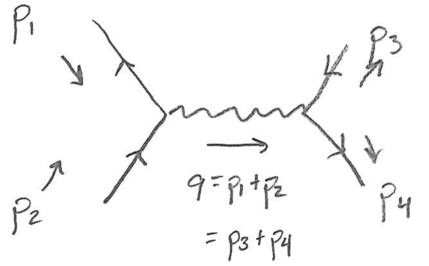


d)



S + T + U

6) $e^+e^- \to \mu^+\mu^-$



$$i\mathcal{M} = (ie)^2 \bar{v}_1 \gamma^\mu u_2 \left(\frac{-i\left(g_{\mu\nu} - (1-\xi)\frac{q_\mu q_\nu}{q^2}\right)}{q^2} \right) \bar{u}_4 \gamma^\nu v_3$$

$$= ie^2 \left[(\bar{v}_1 \gamma^\mu u_2)(\bar{u}_4 \gamma_\mu v_3) - (1-\xi)(\bar{v}_1 \not{q} u_2)(\bar{u}_4 \not{q} v_3)\frac{1}{q^2} \right] \frac{1}{q^2}$$

notice $\bar{v}_1 \not{q} u_2 = \bar{v}_1 \not{p}_1 u_2 + \bar{v}_1 \not{p}_2 u_2$

$= (-m_e + m_e)\bar{v}_1 u_2 = 0$ ← by Dirac Eqn

$\bar{u}_4 \not{q} v_3 = \bar{u}_4 \not{p}_3 v_3 + \bar{u}_4 \not{p}_4 v_3$

$= (-m_\mu + m_\mu)\bar{u}_4 v_3 = 0$

$$= ie^2 (\bar{v}_1 \gamma^\mu u_2)(\bar{u}_4 \gamma_\mu v_3)\frac{1}{q^2}$$ ← we're already gauge (parameter) indep

$$|i\mathcal{M}|^2 = e^4 (\bar{v}_1 \gamma^\mu u_2)(\bar{u}_4 \gamma_\mu v_3)(\bar{u}_2 \gamma^\nu v_1)(\bar{v}_3 \gamma_\nu u_4)\frac{1}{q^4}$$

$$\frac{1}{2^2}\sum_{s_1,s_2} |i\mathcal{M}|^2 = \frac{e^4}{4} \mathrm{Tr}\left[(\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu\right] \mathrm{Tr}\left[(\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\right]$$

$$\mathrm{Tr}\left[(\not{p}_1 - m)\gamma^\mu(\not{p}_2 + m)\gamma^\nu\right] = \mathrm{Tr}\left[\not{p}_1 \gamma^\mu \not{p}_2 \gamma^\nu\right] - m_e^2 \mathrm{Tr}\left[\gamma^\mu \gamma^\nu\right] + 0$$

$$= 4(p_1^\mu p_2^\nu - p_1 \cdot p_2 g^{\mu\nu} + p_1^\nu p_2^\mu) - 4m_e^2 g^{\mu\nu}$$

$$\mathrm{Tr}\left[(\not{p}_4 + m)\gamma_\mu(\not{p}_3 - m)\gamma_\nu\right] = \mathrm{Tr}\left[\not{p}_4 \gamma_\mu \not{p}_3 \gamma_\nu\right] - m_\mu^2 \mathrm{Tr}\left[\gamma_\mu \gamma_\nu\right] + 0$$

$$= 4(p_{4\mu} p_{3\nu} - p_3 \cdot p_4 g_{\mu\nu} + p_{4\nu} p_{3\mu}) - 4m_\mu^2 g_{\mu\nu}$$

$$\frac{1}{4}\sum_s |i\mathcal{M}|^2 = 4e^4\left[p_{4\mu}p_{3\nu} - p_3\cdot p_4 g_{\mu\nu} + p_{4\nu}p_{3\mu} - m_\mu^2 g_{\mu\nu}\right]\left[p_1^\mu p_2^\nu - p_1\cdot p_2 g^{\mu\nu} + p_1^\nu p_2^\mu - m_e^2 g^{\mu\nu}\right]\frac{1}{q^4}$$

$$= 4e^4\Big[\underline{p_1\cdot p_4\, p_3\cdot p_2} - \underline{p_1\cdot p_2\, p_3\cdot p_4} + p_1\cdot p_3\, p_2\cdot p_4 - m_e^2\, p_3\cdot p_4$$

$$- \underline{p_1\cdot p_2\, p_3\cdot p_4} + 4\,\underline{p_3\cdot p_4\, p_1\cdot p_2} - \underline{p_3\cdot p_4\, p_1\cdot p_2} + 4m_e^2\, p_3\cdot p_4$$

$$+ p_1\cdot p_3\, p_2\cdot p_4 - \underline{p_1\cdot p_2\, p_3\cdot p_4} + \underline{p_1\cdot p_4\, p_3\cdot p_2} - m_e^2\, p_3\cdot p_4$$

$$- m_\mu^2\, p_1\cdot p_2 + 4m_\mu^2\, p_1\cdot p_2 - m_\mu^2\, p_1\cdot p_2 + m_e^2 m_\mu^2\, 4\Big]\frac{1}{q^4}$$

$$= 4e^4\left[2p_1\cdot p_4\, p_2\cdot p_3 + 2p_1\cdot p_3\, p_2\cdot p_4 + 2m_e^2\, p_3\cdot p_4 + 2m_\mu^2\, p_1\cdot p_2 + 4m_e^2 m_\mu^2\right]\frac{1}{q^4}$$

$$S = (p_1+p_2)^2 = (p_3+p_4)^2$$

$$\rightarrow p_1\cdot p_2 = \frac{1}{2}(S - 2m_e^2)$$

$$p_3\cdot p_4 = \frac{1}{2}(S - 2m_\mu^2)$$

$$T = (p_1-p_3)^2 = (p_2-p_4)^2$$

$$\rightarrow p_1\cdot p_3 = \frac{-1}{2}(T - m_e^2 - m_\mu^2)$$

$$p_2\cdot p_4 = -\frac{1}{2}(T - m_e^2 - m_\mu^2)$$

$$U = (p_1-p_4)^2 = (p_2-p_3)^2$$

$$\rightarrow p_1\cdot p_4 = -\frac{1}{2}(U - m_e^2 - m_\mu^2)$$

$$p_2\cdot p_3 = -\frac{1}{2}(U - m_e^2 - m_\mu^2)$$

$$= \frac{2e^4}{S^2}\left[(T^2+U^2) + 2(S-T-U)(m_e^2+m_\mu^2) + 2(m_e^2+m_\mu^2)^2\right]$$

In the COM frame:

$$s = E_{COM}^2$$

$$T = (p_1 - p_3)^2 = m_e^2 + m_\mu^2 - 2p_1 \cdot p_3$$

$$= m_e^2 + m_\mu^2 - 2E^2 + 2\vec{p}_{in} \cdot \vec{p}_{out}$$

$$= m_e^2 + m_\mu^2 - 2E^2 + 2\sqrt{E^2 - m_e^2}\sqrt{E^2 - m_\mu^2}\cos\theta$$

$$U = (p_1 - p_4)^2 = m_e^2 + m_\mu^2 - 2E^2 - 2\vec{p}_{in} \cdot \vec{p}_{out}$$

$$= m_e^2 + m_\mu^2 - 2E^2 - 2\sqrt{E^2 - m_e^2}\sqrt{E^2 - m_\mu^2}\cos\theta$$

$$E^2 = s/4$$

$$\frac{1}{4}\sum_s |i\mathcal{M}|^2 = \frac{e^4}{s^2}\left[s^2(1+\cos^2\theta) + 4s(m_e^2 + m_\mu^2)(1 - \cos^2\theta) + 16m_e^2 m_\mu^2 \cos^2\theta\right]$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\bar{\beta}\left(\frac{1}{4}\sum_s |\mathcal{M}|^2\right)$$

$$\bar{\beta} = \sqrt{1 - \frac{4m_\mu^2}{s}}$$

$$= \frac{e^4}{64\pi^2 s}\sqrt{1 - \frac{4m_\mu^2}{s}}\left[(1+\cos^2\theta) + \frac{4(m_e^2 + m_\mu^2)}{s}(1 - \cos^2\theta) + \frac{16m_e^2 m_\mu^2}{s^2}\cos^2\theta\right]$$

$$\sigma(s) = \frac{e^4}{8\pi^2 s}\sqrt{1 - \frac{4m_\mu^2}{s}}\left[\frac{1}{3} + \frac{(m_e^2 + m_\mu^2)}{s} + \frac{4m_e^2 m_\mu^2}{3s^2}\right]$$