Path Integrals for Fermionic Fields:

1) Fermionic Lorentz invariants (Srednicki 34)

recall we wrote the irreps of the L group as: (LG = Lorentz Group)

$(2n+1, 2n'+1)$ w/ $n, n' \in \frac{1}{2}\mathbb{Z}$ or $\mathbb{Z}$

We had:

$(2,1)$ = Left handed spinor w/ 2 components
$(1,2)$ = Right handed spinor w/ 2 components

So we invent a LH & RH spinor as:

$\psi_a(x)$ ← LH index

$\psi^\dagger_{\dot{a}}(x)$ ← RH index has a dot

In terms of our generators of the L group:

$$[N_i, N_j] = i\epsilon_{ijk} N_k$$

$$[N_i^\dagger, N_j^\dagger] = i\epsilon_{ijk} N_k$$

$$[N_i, N_j^\dagger] = 0$$

a xforms according to N_i
$\dot{a}$ xforms according to $N_i^\dagger$

LH spinors then transform according to

$$\underbrace{U(\Lambda)^{-1}\psi_a(x)U(\Lambda)}_{\text{this is how ops xform in QM}} = L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x) \quad *$$

this is the Matrix that executes this xformation for the irrep ψ belongs to

For infinitesimal LT, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \delta\omega^\mu{}_\nu$, so we can write:

$$L_a{}^b(1+\delta\omega) = \delta_a{}^b + \frac{i}{2}\underbrace{\delta\omega_{\mu\nu}}_{\text{was antisymm}}\underbrace{(S_L^{\mu\nu})_a{}^b}_{\text{so this is as well}}$$

The S_L are the generators of the LG, so they have the same Lie Algebra as the $M^{\mu\nu}$ we had discussing LG:

$$[S_L^{\mu\nu}, S_L^{\rho\sigma}] = i\left(g^{\mu\rho}S_L^{\nu\sigma} - (\mu\leftrightarrow\nu)\right) - (\rho\leftrightarrow\sigma)$$

Taking $U(1+\delta\omega) = \mathbb{1} + \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}$

$$U(\Lambda)^{-1}\psi_a U(\Lambda) = \psi_a - \frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}\psi_a + \psi_a\frac{i}{2}\delta\omega_{\mu\nu}M^{\mu\nu}$$

$$= \psi_a + \frac{i}{2}\delta\omega_{\mu\nu}[\psi_a, M^{\mu\nu}]$$

$$L_a{}^b(\Lambda)\psi_b(\Lambda^{-1}x) = \psi_a(\Lambda^{-1}x) + \frac{i}{2}\delta\omega_{\mu\nu}(S_L^{\mu\nu})_a{}^b\psi_b(x)$$

$$\psi_a(\Lambda^{-1}x) = \psi_a(x_\mu + \delta\omega_{\mu\nu}x^\nu)$$

$$= \psi_a(x) + \underbrace{\frac{\partial\psi_a(x_\mu+\delta\omega_{\mu\nu}x^\nu)}{\partial(x_\mu+\delta\omega_{\mu\nu}x^\nu)}\left(\frac{\partial x_\mu+\delta\omega_{\mu\nu}x^\nu}{\partial x^\nu}\right)}_{\text{chain rule}}\Bigg|_{\omega\to0}\delta\omega_{\mu\nu}$$

$$= \psi_a(x) + \underbrace{\frac{\partial}{\partial x_\mu}}_{=\partial^\mu}\psi_a(x_\mu)\,x^\nu$$

$=\partial^\mu$ (review co- and contra-variant & derivatives for upper vs lower indices)

$$= \psi_a(x) + \delta\omega_{\mu\nu}x^\nu\partial^\mu\psi_a$$

$$= \psi_a + \frac{1}{2}\delta\omega_{\mu\nu}(x^\nu\partial^\mu - x^\mu\partial^\nu)\psi_a$$

Then equating the $\delta\omega$ terms in $*$ gives:

$$\frac{i}{2}[\psi_a, M^{\mu\nu}] = \frac{1}{2}(x^{\nu}\partial^{\mu} - x^{\mu}\partial^{\nu})\psi_a + \frac{i}{2}(S_L^{\mu\nu})_a{}^b\psi_b$$

For simplicity we choose a frame $x^{\mu}\to 0$

$$[\psi_a, M^{\mu\nu}] = (S_L)_a{}^b\psi_b$$

Recalling $M^{ij} = \epsilon^{ijk}J_K$:

$$\epsilon^{ijk}[\psi_a(0), J_K] = (S_L^{ij})_a{}^b\psi_b(0)$$

Also recall we decomposed the LG into what appeared to be 2 spin groups & our (2,1) irrep had spin ½, so we'll choose the Pauli matrices as generators. So for angular momentum J_K we can choose

$$(S_L^{ij})_a{}^b = \frac{1}{2}\epsilon^{ijk}\sigma_K$$

We also had: $J_K = N_K + N_K^{\dagger}$

$$K_K = i(N_K - N_K^{\dagger})$$

But $N_K^{\dagger}$ acting on a (2,1) rep vanishes: 1 indicates a singlet of $N^{\dagger}$

So the matrices for $K_K = M^{K0}$ are $i\times$ those of J_K:

$$(S_L^{K0})_a{}^b = \frac{1}{2}i\sigma_K$$

For the RH fields, the generators are $N_i^{\dagger}$ so hermitian conjugation swaps between the two Lie Algebras:

$$[\psi_a(x)]^{\dagger} = \psi_{\dot a}^{\dagger}(x)$$

Performing a similar calculation we could find:

$$[\psi^\dagger_{\dot{a}}, M^{\mu\nu}] = (S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}}\psi^\dagger_{\dot{b}}(0)$$

Taking the h.c.:

$$[M^{\mu\nu}, \psi_a(0)] = [(S_R^{\mu\nu})_a{}^b]^* \psi_b(0)$$

So we conclude:

$$(S_R^{\mu\nu})_{\dot{a}}{}^{\dot{b}} = -[(S_L^{\mu\nu})_a{}^b]^*$$

$\uparrow$ recall S_L was antisymm

Consider: $\Lambda_\mu{}^\rho \Lambda_\nu{}^\sigma g_{\rho\sigma} = g_{\mu\nu}$

$\uparrow$ $\nearrow$ g is a constant matrix that doesn't change under LTs: "invariant symbol"

We want invar. symbols for our spinors:

Recalling adding 2 spin ½ particles: $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0_A$

(or $2 \otimes 2 = 3 \oplus 1_A$)

So a 2 index object C_{ab} transforming as:

$$U(\Lambda)^{-1} C_{ab}\, U(\Lambda) = L_a{}^c(\Lambda) L_b{}^d(\Lambda) C_{cd}(\Lambda^{-1}x)$$

Should decompose as:

$$C_{ab}(x) = \epsilon_{ab} D(x) + G_{ab}(x)$$

$\uparrow$ totally antisymm, $\epsilon_{21} = -\epsilon_{12} = +1$

Since $D(x)$ is our Lorentz scalar we conclude:

$$L_a{}^c(\Lambda) L_b{}^d(\Lambda) \epsilon_{cd} = \epsilon_{ab}$$

ϵ is an invar symbol

This allows us to form LI quantities w/ our spinors:

$$\psi^a = \epsilon^{ab}\psi_b$$

$$\psi^a \chi_a = \epsilon^{ab}\psi_b\chi_a = -\epsilon^{ba}\psi_b\chi_a = -\psi_b\chi^b$$

Analogously for RH $\psi^{\dot{a}}\chi_{\dot{a}}$

We also concluded the $(2,2)$ rep was a vector, so we can write A^μ as:

$$A_{a\dot{a}} = \sigma^\mu_{a\dot{a}} A_\mu$$

we conclude this is another invar symbol

We choose:

$$\sigma^\mu_{a\dot{a}} = (\mathbb{1}, \vec{\sigma})$$

The validity of this choice can be seen by checking the invariances:

$$\sigma^\mu_{a\dot{a}}\sigma_{\mu b\dot{b}} = -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}$$

$$\epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^\mu_{a\dot{a}}\sigma^\nu_{b\dot{b}} = -2g^{\mu\nu}$$

In this way we demonstrate the invariance of σ in spinor and vector indices

We've defined vectors in the (2,1) and (1,2) spaces as:

$$\underbrace{\psi_a}_{\text{LH spinor}} \quad \dot{\xi} \quad \underbrace{\psi^{\dagger \dot{a}}}_{\text{RH spinor}}$$

So given our xformations

$$L_a{}^b \qquad R_{\dot{a}}{}^{\dot{b}}$$

And the invariant symbols

$$\epsilon_{ab} \qquad \epsilon_{\dot{a}\dot{b}}$$

$$\sigma^\mu_{a\dot{a}} \qquad \bar{\sigma}^{\mu \dot{a} a} = \epsilon^{ab} \epsilon^{\dot{a}\dot{b}} \sigma^\mu_{b\dot{b}}$$

We can form terms quadratic in $\psi, \psi^\dagger$ that are LI and will be our free $\mathcal{L}$ off which we base our pert. thry

$$\psi^a \psi_a = \psi_a \epsilon^{ab} \psi_b \rightarrow \underbrace{L_a{}^c L_b{}^d \epsilon^{ab}}\psi_c \psi_d$$

$$= \epsilon^{cd} \psi_c \psi_d$$

$$= \psi^c \psi_c$$

$$\psi^{\dagger \dot{a}} \sigma^\mu_{a\dot{a}} \partial_\mu \psi^a \rightarrow (\epsilon^{\dot{a}\dot{c}} R_{\dot{c}}{}^{\dot{d}} \psi^\dagger_{\dot{d}}) \underbrace{(\Lambda^\mu{}_\nu L_a{}^b R_{\dot{a}}{}^{\dot{b}} \sigma^\nu_{b\dot{b}})}_{\substack{\text{here we rewrite the invariant symbol in terms of xform} \\ \text{before we used the invariance to absorb the xform}}} (\Lambda_\mu{}^\rho \partial_\rho)(\epsilon^{ab} L_b{}^d \psi_d)$$

$$= \underbrace{(L_a{}^b \epsilon^{ac} L_c{}^d)}_{\epsilon^{bd}} \underbrace{(R_{\dot{a}}{}^{\dot{b}} \epsilon^{\dot{a}\dot{c}} R_{\dot{c}}{}^{\dot{d}})}_{\epsilon^{\dot{b}\dot{d}}} \psi^\dagger_{\dot{d}} \underbrace{\Lambda^\mu{}_\nu \Lambda_\mu{}^\rho}_{\delta_\nu{}^\rho} \sigma^\nu_{b\dot{b}} \partial_\rho \psi_d$$

$$= \psi^{\dagger \dot{b}} \sigma^\nu{}_{b\dot{b}} \partial_\nu \psi^b$$

With this we could define a QFT w/ a LH spinor & its h.c., but it is more common to use "Dirac" fields consisting of a LH & RH spinor:

$$\Psi \equiv \begin{pmatrix} \chi_a \\ \xi^{\dagger\dot{a}} \end{pmatrix} \qquad \Psi^\dagger = (\chi^\dagger_{\dot{a}}, \xi^a)$$

Introducing the matrix

$$\beta \equiv \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}$$

Allows us to write the Lorentz invariant:

$$\bar{\Psi}\Psi = \Psi^\dagger \beta \Psi = \xi^a \chi_a + \chi^\dagger_{\dot{a}} \xi^{\dot{a}}$$

Noting that $\bar{\sigma}^{\mu\dot{a}a} \equiv \epsilon^{ab}\epsilon^{\dot{a}\dot{b}}\sigma^\mu_{b\dot{b}} = (\mathbb{1}, -\vec{\sigma})$ is also an invariant symbol, we can also define:

$$\gamma^\mu \equiv \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix}$$

allowing us to write:

$$\bar{\Psi}\gamma^\mu \partial_\mu \Psi = \xi^a \sigma^\mu_{a\dot{c}} \partial_\mu \xi^{\dagger\dot{c}} + \chi^\dagger_{\dot{a}} \bar{\sigma}^{\mu\dot{a}c} \partial_\mu \chi_c$$

We could have considered

$$(\partial^\mu \psi^a)(\partial_\mu \psi_a)$$

but this leads to a Hamiltonian unbounded from below. See Advanced Particles

We need both ψ & $\psi^\dagger$ for a bounded Hamiltonian

This can be thought of as $\psi^a \psi_a = \epsilon^{ab}\psi_a\psi_b = \psi_1\psi_2 - \psi_2\psi_1 = 2\psi_1\psi_2$ which isn't positive definite like φ^2 because ψ has two components

2) Dirac Algebra

We want to use Dirac Fermions so we need to understand the γ^μ matrices better.

We will try to move away from a & $\dot{a}$ indices, so first notice:

$$\beta = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} = \begin{pmatrix} 0 & \delta^{\dot{a}}{}_{\dot{c}} \\ \delta_a{}^c & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0_{a\dot{c}} \\ \bar{\sigma}^{0\dot{a}c} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

And that $\beta^\dagger = \beta$, $\gamma^{0\dagger} = \gamma^0$ are similar, but their index structure differs. As long as β & γ^0 we understand $\bar{\Psi} = \psi^\dagger \beta \sim \psi^\dagger \gamma_0$ we can forget this

Next notice:

$$\gamma^\mu \gamma^\nu = \begin{pmatrix} 0 & \sigma^\mu_{a\dot{c}} \\ \bar{\sigma}^{\mu\dot{a}c} & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^\nu_{c\dot{d}} \\ \bar{\sigma}^{\nu\dot{c}d} & 0 \end{pmatrix} = \begin{pmatrix} \sigma^\mu_{a\dot{c}} \bar{\sigma}^{\nu\dot{c}d} & 0 \\ & \bar{\sigma}^{\mu\dot{a}c} \sigma^\nu_{c\dot{d}} \end{pmatrix}$$

$$\sigma^\mu \bar{\sigma}^\nu = \begin{cases} \mu\nu = 00 & \delta_a{}^d \\ \mu\nu = 0i & -\sigma^i{}_a{}^d \\ \mu\nu = i0 & +\sigma^i{}_a{}^d \\ \mu\nu = ij & -(\sigma^i \sigma^j)_a{}^d = -(\delta^{ij}\delta_a{}^d + i\epsilon^{ij}{}_k \sigma^k{}_a{}^d) \end{cases}$$

$$\bar{\sigma}^\mu \sigma^\mu = \begin{cases} \mu\nu = 00 & \delta^{\dot{a}}{}_{\dot{d}} \\ \mu\nu = 0i & \sigma^{i\dot{a}}{}_{\dot{d}} \\ \mu\nu = i0 & -\sigma^{i\dot{a}}{}_{\dot{d}} \\ \mu\nu = ij & -(\delta^{ij}\delta^{\dot{a}}{}_{\dot{d}} + i\epsilon^{ij}{}_k \sigma^{k\dot{a}}{}_{\dot{d}}) \end{cases}$$

$$\{\gamma^\mu, \gamma^\nu\} = \begin{cases} \mu\nu = 00 \to 2\begin{pmatrix} \delta_a{}^d & \\ & \delta^{\dot a}{}_{\dot d} \end{pmatrix} \\ \\ \mu\nu = 0i \to \begin{pmatrix} -\sigma^i{}_a{}^d + \sigma^i{}_a{}^d & 0 \\ 0 & \sigma^{i\dot a}{}_{\dot d} - \sigma^{i\dot a}{}_{\dot d} \end{pmatrix} = 0 \\ \\ \mu\nu = i0 \to 0 \\ \mu\nu = ij \to \begin{pmatrix} -2\delta^{ij}\delta_a{}^d & 0 \\ 0 & -2\delta^{ij}\delta^{\dot a}{}_{\dot d} \end{pmatrix} \end{cases}$$

$$= 2\eta^{\mu\nu}$$

This is the "Clifford Algebra"

We can also define:

$$\gamma_5 = \begin{pmatrix} -\delta_a{}^c & \\ & \delta^{\dot a}{}_{\dot c} \end{pmatrix}$$

$$P_L = P_- = \frac{1}{2}(\mathbb{1} - \gamma_5) = \begin{pmatrix} \delta_a{}^c & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_R = P_+ = \frac{1}{2}(\mathbb{1} + \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \delta^{\dot a}{}_{\dot c} \end{pmatrix}$$

Notice: $\left.\begin{aligned} & P_L + P_R = \mathbb{1} \\ & P_L^2 = P_L \\ & P_R^2 = P_R \\ & P_L P_R = 0 \end{aligned}\right\}$ the $P_\pm$ are projection operators

From this we see:

$$P_L \psi = \begin{pmatrix} \chi_c \\ 0 \end{pmatrix}$$

$$P_R \psi = \begin{pmatrix} 0 \\ \xi^{\dagger \dot{c}} \end{pmatrix}$$

The $P_\pm$ project out the Left & Right-handed spinors

Together w/ the identity and γ_5, the γ matrices form a basis:

$$\Gamma_{VA} = \{ \mathbb{1}, \gamma^\mu, \sigma^{\mu\nu} = \tfrac{i}{2}[\gamma^\mu, \gamma^\nu], \gamma_5, \gamma_5 \gamma^\mu \}$$

This is the V-A basis (vector - axial vector) as,

$\bar{\psi} \gamma^\mu \chi \rightarrow$ transforms as a vector, w/ χ another Dirac field

$\bar{\psi} \gamma_5 \gamma^\mu \chi \rightarrow$ transforms as an axial vector

Another basis is the Chiral basis:

$$\Gamma_{chiral} = \{ P_L, P_R, P_L \gamma^\mu, P_R \gamma^\mu, \sigma^{\mu\nu} \}$$

The V-A basis is useful for QCD and hadronic phenomenology

The Chiral is best for SM pheno

To study the Chiral basis we need some more identities:

consider $\gamma_5 = \begin{pmatrix} -\delta_a{}^c & \\ & +\delta^{\dot a}{}_{\dot c} \end{pmatrix}$

$$\gamma_5 \gamma^\mu = \begin{pmatrix} -\delta_a{}^c & \\ & +\delta^{\dot a}{}_{\dot c} \end{pmatrix} \begin{pmatrix} 0 & \sigma^\mu_{c\dot d} \\ \bar\sigma^{\mu \dot c d} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma^\mu_{a\dot d} \\ \bar\sigma^{\mu \dot a d} & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 0 & \sigma^\mu_{a\dot c} \\ \bar\sigma^{\mu \dot a c} & 0 \end{pmatrix} \begin{pmatrix} -\delta_c{}^d & 0 \\ 0 & \delta^{\dot c}{}_{\dot d} \end{pmatrix} = \gamma^\mu \gamma_5$$

From this we see:

$$\gamma^\mu P_\pm = \gamma^\mu \tfrac{1}{2}(\mathbb{1} \pm \gamma_5) = \tfrac{1}{2}(\mathbb{1} \mp \gamma_5)\gamma^\mu = P_\mp \gamma^\mu$$

In the exercises you will work out the Chiral structure of Bilinears of Dirac fields for all bilinears in the chiral basis.

As an example take ψ & χ to be Dirac fields, we saw

$$P_L \psi = \psi_L \quad \text{(the LH part of } \psi)$$

$$P_R \psi = \psi_R$$

and similarly for χ.

Using $P_L + P_R = \mathbb{1}$ and $P_L^2 = P_L$, $P_R^2 = P_R$ we see:

$$\psi = (P_L^2 + P_R^2)\psi = P_L \psi_L + P_R \psi_R$$

$$\chi = P_L \chi_L + P_R \chi_R$$

Since the $P_\pm$ are real & diagonal we also have

$$\psi^\dagger = \psi_L^\dagger P_L + \psi_R^\dagger P_R$$

Then the "scalar" bilinear $\bar{\Psi}\chi$ can be expanded in terms of Chiral components as:

$$\bar{\Psi}\chi = (\Psi_L^\dagger P_L + \Psi_R^\dagger P_R)\gamma_0 (P_L \chi_L + P_R \chi_R)$$

$$= (\bar{\Psi}_L P_R + \bar{\Psi}_R P_L)(P_L \chi_L + P_R \chi_R) \qquad (\{\gamma_0, P_\pm\} = 0)$$

$$= \bar{\Psi}_L \chi_R + \bar{\Psi}_R \chi_L \qquad (P_\pm^2 = P_\pm,\ P_+ P_- = 0)$$

So the "scalar" bilinear, which is also our mass term, mixes L & R chiral Dirac fields → this will be a problem for the SM which will be resolved by the Higgs mechanism

3) Path Integrals for fermions (Schwartz 14.6)

Since Fermions must anticommute, we will introduce Grassmann numbers $\{\theta_i\}$

w/ the rule $\theta_i\theta_j = -\theta_j\theta_i$

which commute additively $\theta_i + \theta_j = \theta_j + \theta_i$

and have a 0 element $\theta_i + 0 = \theta_i$

and can be multiplied by $\mathbb{C}$ numbers: $a\theta$ for $a \in \mathbb{C}$

For one θ the most general element is:

$$g = a + b\theta \qquad \text{because } \theta^2 = -\theta^2 = 0 \text{ since } \theta\text{'s anticommute}$$

For two:

$$g = A + B\theta_1 + C\theta_2 + F\theta_1\theta_2$$

and so on.

The Grassman numbers form an Algebra

The product of 2 θ's is bosonic, e.g. $(\theta_1\theta_2)(\theta_3\theta_4) = +\theta_3\theta_1\theta_2\theta_4$
$= +(\theta_3\theta_4)(\theta_1\theta_2)$

$\Rightarrow$ bosonic subalgebra

An odd # of θ's is fermionic

$\Rightarrow$ fermionic subalgebra, not closed as product of 2 elements is bosonic

Our fermions will be Grassmann variables: $\theta_1 = \psi(x_1)$

So we need to understand integration: $\int D\psi$

For a single θ the most general integral is:

$$\int d\theta (a + b\theta) = a\int d\theta + b\int d\theta\theta$$

Since we want our integral to map the Grassman numbers to $\mathbb{C}$ the first term must vanish, we will use the convention $\int d\theta\,\theta = 1$

So: $$\int d\theta (a + b\theta) = b$$

For derivatives we have the usual definition

$$\frac{d}{d\theta}(a + b\theta) = b$$

So Integration $\Leftrightarrow$ differentiation for Grassmann numbers

For many θ_i we define:

$$\int d\theta_1 \cdots d\theta_n X = \frac{\partial}{\partial\theta_1} \cdots \frac{\partial}{\partial\theta_n} X$$

$$\int d\theta_1 \cdots d\theta_n \theta_n \cdots \theta_1 = 1$$

Since order matters our convention is to integrate from inside out:

$$\int d\theta_1 d\theta_2 \theta_2 \theta_1 = -\int d\theta_1 d\theta_2 \theta_1 \theta_2 = 1$$

We have a shift symmety as with scalars:

$$\int_{-\infty}^{\infty} dx\, \xi(x) = \int_{-\infty}^{\infty} dx\, \xi(x+a) \quad \Leftrightarrow \quad \int d\theta (A + B\theta) = \int d\theta (A + B(\theta + X))$$

w/ X constant w/r to θ

For Path Integrals we will also need Gaussians:

$$\int d\theta_1 d\theta_2\, e^{-\theta_1 A_{12}\theta_2} = \int d\theta_1 d\theta_2 \underbrace{(1 - A_{12}\theta_1\theta_2)}_{\text{this is exact for anticommuting objects!}} = A_{12}$$

If we consider $n\ \theta_i$ & $n\ \bar{\theta}_i$:

$$\int d\bar{\theta}_1 \cdots d\bar{\theta}_n d\theta_n \cdots d\theta_1\, e^{-\bar{\theta}_i A_{ij}\theta_j} = \int d\bar{\theta}_1 \cdots d\theta_n \cdots \left[1 - \bar{\theta}_i A_{ij}\theta_j + \tfrac{1}{2}(\bar{\theta}_i A_{ij}\theta_j)(\bar{\theta}_k A_{k\ell}\theta_\ell)\cdots\right]$$

Notice the only term that survives involves <u>all</u> θ_i & $\bar{\theta}_i$

This will yield a sum over all $\{i,j\}$ where each $\bar{\theta}_i$ (column of A_{ij}) is chosen once as is each θ_i (row of A_{ij}) w/ a sign dictated by ordering,

$$123\cdots \to 1$$
$$2134\cdots \to -1$$

There are $n!$ such permutations, and this comes from the $\frac{1}{n!}$ term in the Taylor expansi

$$\int d\bar{\theta}_1 \cdots d\theta_n \cdots e^{-\bar{\theta}_i A_{ij}\theta_j} = \frac{1}{n!} \sum_{\substack{\text{perm.}\\ \{i_n\}}} (\pm)\, A_{i_1 i_2} \cdots A_{i_{n-1}, i_n}$$

The $\frac{1}{n!} \times n! = 1$ and we obtain the $\det A$:

$$\int d\bar{\theta}_1 \cdots d\theta_n \cdots e^{-\bar{\theta}_i A_{ij}\theta_j} = \det A$$

This differs from our bosonic case!

$$\int dx_1 \cdots dx_n\, e^{-\frac{1}{2}x_i A_{ij} x_j} = \sqrt{\frac{(2\pi)^n}{\det A}}$$

This difference <u>is important</u>, but we wont have time to discuss it:

See: Anomalies - Schwartz 30

- Srednicki 75, 77

w/ external currents we have:

$$\int d\bar{\theta}_1 \cdots d\bar{\theta}_n d\theta_n \cdots d\theta_1 e^{-\bar{\theta}_i A_{ij}\theta_j + \bar{\eta}_i\theta_i + \bar{\theta}_i\eta_i}$$

$$= e^{\bar{\eta}_i A^{-1}_{ij}\eta_j} \int d\bar{\theta}_1 \cdots d\theta_n \cdots e^{-(\bar{\theta}_i - \bar{\eta}_j A_{ji})A_{ik}(\theta_k - A_{kl}\eta_l)}$$

$$= \det A \, e^{\bar{\eta}_i A^{-1}_{ij}\eta_j}$$

In the continuum limit we conclude:

$$Z[\bar{\eta},\eta] = \int D\bar{\psi}(x) D\psi(x) e^{i\int d^4x [\bar{\psi}\underbrace{(i\not{\partial}-m)}_{A=-i(i\not{\partial}-m)}\psi + \bar{\eta}\psi + \bar{\psi}\eta]}$$

$$= N e^{-i\int d^4x d^4y \, \bar{\eta}(y)\underbrace{(i\not{\partial}-m)^{-1}}_{\frac{1}{i}A^{-1}}\eta(x)}$$

A^{-1} is the Green's function solution:

$$\underbrace{-i(i\not{\partial}-m)}_{A} i G_F(x-y) = +\delta(x-y)$$

From Advance particles (also inspection) the solution is:

$$iG_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{\not{p}-m} e^{-ip(x-y)}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{\not{p}+m}{\not{p}+m}\left(\frac{i}{\not{p}-m}\right) e^{-ip(x-y)} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2} e^{-ip(x-y)}$$

The two point function is:

$$\langle 0| T\{\psi(y'), \bar{\psi}(y)\}|0\rangle = \frac{1}{Z[0]} i\frac{\partial}{\partial\bar{\eta}} \frac{1}{i}\frac{\partial}{\partial\eta} \int D\bar{\psi}(x) D\psi(x) e^{i\int d^4x [\bar{\psi}(i\not{\partial}-m)\psi + \bar{\eta}\psi + \bar{\psi}\eta]}$$

$$= \frac{1}{Z[0]} i\frac{\partial}{\partial\bar{\eta}} \int D\bar{\psi}(x) D\psi(x) e^{i\int d^4x [\;]} \frac{1}{i}(-i\bar{\psi}(y))$$

anticommute ∂_η & $\bar{\psi}$

$$= \frac{1}{Z[0]} \int D\bar{\psi}(x) D\psi(x) e^{i\int d^4x [\;]} i(i\psi(y'))\frac{1}{i}(-i\bar{\psi}(y))$$

$$= \frac{1}{Z[0]} \int D\bar{\psi}(x) D\psi(x) e^{i\int d^4x [\;]} \psi(y')\bar{\psi}(y)$$

Notice we have $i\frac{\partial}{\partial\bar{\eta}}$ & $\frac{1}{i}\frac{\partial}{\partial\eta}$ to compensate the anticommutivity allowing us to obtain the correct sign on the RHS

We need to take care as we now have Dirac indices to track:

$$\langle 0|T\{\psi_a(x),\bar{\psi}_b(y)\}|0\rangle = \frac{1}{Z[0]}\, i\frac{\partial}{\partial\bar{\eta}_a(x)}\,\frac{1}{i}\frac{\partial}{\partial\eta_b(y)}\, Z[\bar{\eta},\eta]\Big|_{\eta,\bar{\eta}\to 0}$$

Dirac!

$$= \frac{1}{Z[0]}\, i\frac{\partial}{\partial\bar{\eta}_a(x)}\,\frac{1}{i}\frac{\partial}{\partial\eta_b(y)}\, \exp\left[-i\int d^4z\, d^4w\, \bar{\eta}_c(z)(+G_F(z-w))_{cd}\,\eta_d(w)\right]\Big|_{\eta\bar{\eta}\to 0}$$

have to anticommute through $\bar{\eta}$

$$= \frac{1}{Z[0]}\frac{\partial}{\partial\bar{\eta}_a(x)}\, Z[\bar{\eta},\eta](-1)(-i)\int d^4z'\, d^4w'\, \bar{\eta}_c(z')(+1)G_F(z'-w')_{cd}\,\delta(w'-y)\delta_{bd}\Big)\Big|_{\eta's\to 0}$$

$$= \frac{1}{Z[0]}\, Z[\bar{\eta},\eta]\Big[(-1)(-i)\int d^4z'\, d^4w'(+i)G_F(z'-w')_{cd}\,\delta(w'-y)\delta(z'-x)\delta_{bd}\delta_{ca} + \text{too many sources}\Big]\Big|_{\eta's\to 0}$$

$$= +i\,G_F(x-y)_{ab}$$

In mtm space the propagator is:

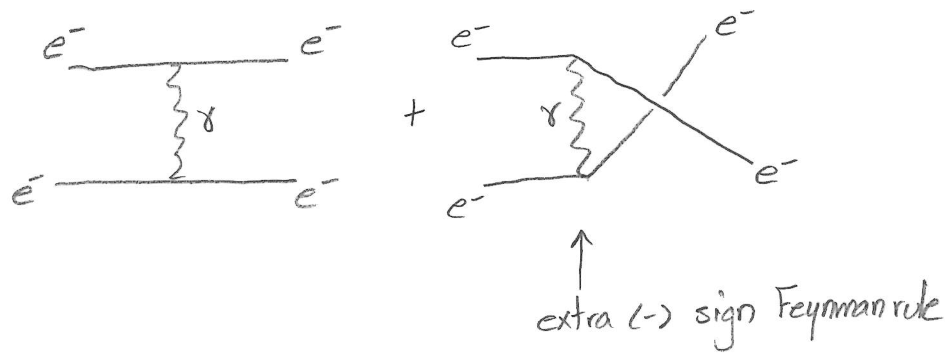
$$a \longrightarrow b \quad = \quad \frac{i(\not{p}+m)_{ab}}{p^2-m^2}$$

Extra Feynman Rules:

→ closed Fermion loops give a factor (−1) → Homework

→ crossed identical fermion lines give a factor (−1)

e.g.



For many permutations of identical fermions

→ close Fermion loops give a Tr[ ] over Dirac indices → Homework

→ when writing down an amplitude follow the propagators backward according to the arrow on the Ferm. line $\rightarrow$

This arrow points $\rightarrow$ for particles, $\leftarrow$ for antiparticles } external lines

$\underset{p \rightarrow}{\rightarrow}$ along momentum flow for internal lines