Problem Set 3

1. Fun with Gamma matrices:

   (a) Prove: $\gamma^\mu \gamma_\mu = 4\mathbb{1}$

   (b) Prove: $\gamma^\mu \gamma^\nu \gamma_\mu = -2\gamma^\nu$

   (c) Defining $\not{p} = p^\mu \gamma_\mu$, prove: $\not{p}\not{p} = p^2$

   hints: Use your knowledge of (anti-)symmetric matrices. Avoid using the dotted spinor index notation and just work from $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

2. Fun with chiral projections:

   (a) Recalling the $V - A$ basis, $\Gamma_{VA} = \{\mathbb{1}, \gamma^\mu, \sigma^{\mu\nu}, \gamma_5, \gamma_5\gamma^\mu\}$, write out all fermion bilinears $\bar{\psi}\Gamma\chi$ in terms of left- and right-handed projections of the ψ and χ.

   (b) Repeat for the chiral basis, $\Gamma_{\text{chiral}} = \{P_\pm, \gamma^\mu P_\pm, \sigma^{\mu\nu}\}$.

3. Hermiticity of the free Dirac Lagrangian:
   We found the free Dirac Lagrangian,

$$\mathcal{L}_0 = i\bar{\psi}\partial_\mu \gamma^\mu \psi \,. \tag{1}$$

   This Lagrangian is not Hermitian which is required for Lorentz Invariance (This is related to the CPT theorem). By taking the Hermitian conjugate of this equation, show that it is in fact hermitian up to a total divergence term $\partial_\mu(\text{stuff})$.

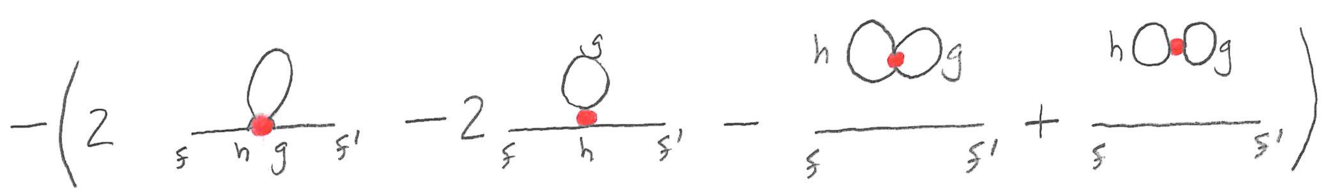
4. Feynman Rules:
   Closed fermion loops and four-fermion operators When we have a closed fermion loop we need an additional minus sign. To see this derive the one-loop correction to the free-theory two-point function in the presence of the interaction Lagrangian:

$$\mathcal{L}_I = c\left[\bar{\psi}_a(x)\psi_a(x)\right]\left[\bar{\psi}_b(x)\psi_b(x)\right]\,, \tag{2}$$

   where a, b are Dirac indices. Adapting the shorthand notation developed for path integrals for scalars you should evaluate:

$$\langle 0|T\{\psi_f(x_1), \bar{\psi}_f(x_2), \bar{\psi}_g(z)\psi_g(z)\bar{\psi}_h(z)\psi_h(z)\}|0\rangle \sim \partial_{\bar{\eta}_{f'x_2}}\partial_{\eta_{fx_1}}\partial_{\bar{\eta}_{gz}}\partial_{\eta_{gz}}\partial_{\bar{\eta}_{hz}}\partial_{\eta_{hz}} e^{\bar{\eta}_{cy}G_{cd,yx}\eta_{dx}} \tag{3}$$

   Here, $\{f, f', g, h, c, d\}$ are Dirac indices and x_1, x_2, z, y, x are spacetime coordinates.
   You should find:



   In the diagrams, the red point indicates that the interaction takes place at the spacetime point z, but that the Dirac indices are not contracted across this point.

   From this convince yourself that a *closed fermion loop*, i.e. a fermion loop where the Dirac indices are contracted within the loop, requires an additional Feynman rule giving a factor of (-1). Also that a closed Fermion loop results in a trace over Dirac Indices in the loop.

   hint/reminder: fermions are Grassmann variables and so anticommute!