The Z-pole in the SM:

1) Lagrangian parameters & input parameters, Fermi constant

Lagrangian parameters <u>are not</u> observable

→ this should be clear from our discussion of renormalization schemes

Different Renorm schemes ⇔ different determination of $\mathcal{L}$ parameters

In order to make predictions we need to make a determination of the $\mathcal{L}$ params

A common set are the observables:

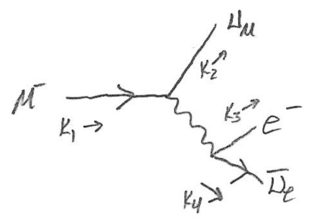
$$\alpha(p^2 \to 0) = \frac{e^2}{4\pi} \quad \text{"fine structure constant"} \quad \Leftrightarrow g_1, g_2$$

$$m_Z^2 = \frac{g_Z^2 v^2}{4 c_W^2} \quad \Leftrightarrow g_1, g_2, v$$

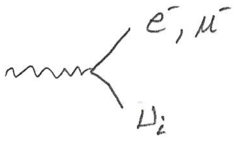
$$G_F = \frac{1}{\sqrt{2}\, v} \quad \text{"Fermi constant"} \quad \Leftrightarrow v$$

The first two are familiar from expanding $\mathcal{L}_{SM}$ (& QED)

The Fermi constant is determined from μ decay:



The relevent FRs are:



$$= \frac{i g_2}{\sqrt{2}} \gamma^\mu P_L$$

$$= \frac{-i}{k^2 - m_W^2}\left(\eta^{\mu\nu} - \frac{k^\mu k^\nu}{m_W^2}\right) \quad \leftarrow \text{unitary gauge } \xi \to \infty$$

The amplitude is then:

$$i\mathcal{M} = \left(\frac{ig_2}{\sqrt{2}}\bar{u}_2\gamma^\mu P_L u_1\right)\frac{-i}{(k_3+k_4)^2 - m_W^2}\left(\eta^{\mu\nu} - \frac{(k_3+k_4)^\mu (k_3+k_4)^\nu}{m_W^2}\right)\left(\frac{ig_2}{\sqrt{2}}\bar{u}_3\gamma^\nu P_L v_4\right)$$

consider: $(k_3+k_4)^\nu \bar{u}_3\gamma^\nu v_4 = \bar{u}_3 \not{k}_3 P_L v_4 + \bar{u}_3 \not{k}_4 P_L v_4$

$$= \underbrace{m_e}_{\ll m_\mu,\ \sim 0}\bar{u}_3 P_L v_4 + \underbrace{m_{\nu}}_{0}\bar{u}_3 P_L v_4$$

$$\sim \frac{ig_2^2}{2}\frac{[\bar{u}_2\gamma^\mu P_L u_1][\bar{u}_3\gamma^\mu P_L v_4]}{\underbrace{(k_3+k_4)^2} - m_W^2}$$

$(k_3+k_4)^2 \leq m_\mu^2 \ll m_W^2$, the muon decay sets m_μ^2 as the scale of the problem

$k_1^2 = m_\mu^2 = (k_2+k_3+k_4)^2$

$$\sim \frac{-ig_2^2}{2}\frac{1}{m_W^2}[\bar{u}_2\gamma^\mu P_L u_1][\bar{u}_3\gamma^\mu P_L v_4]$$

Notice: $m_W^2 = \frac{g_2^2 v^2}{4} \rightarrow \frac{g_2^2}{2m_W^2} = \frac{2}{v^2} \equiv \frac{4G_F}{\sqrt{2}}$ ← For Historical reasons, we will revisit this when we look at effective field thngs

Also: $i\mathcal{M} \sim \frac{1}{m_W^2}$, for $m_W^2 \to \infty$ $i\mathcal{M} \to 0$, we call this "decoupling"

this allowed us to remove the goldstones & ghosts from the spectrum by taking $\xi \to \infty$

Experimentally the input parameters are determined to be:

$$\hat{\alpha}^{-1} = 137.035999084(21) \qquad \sim 10^{-8}\% \text{ error}$$

$$\hat{m}_Z = 91.1876(21) \text{ GeV} \qquad \sim 10^{-3}\% \text{ error}$$

$$\hat{G}_F = 1.166364(5)\times 10^{-5}/\text{GeV}^2 \qquad \sim 10^{-4}\% \text{ error}$$

(the $\wedge$ notation is to indicate these are used as inputs)

We can solve for the $\mathcal{L}$ parameters:

$$\alpha = \frac{e^2}{4\pi} = \frac{(g_2 s_W)^2}{4\pi} \qquad\qquad \hat{g}_1 = \frac{\hat{e}}{\hat{c}_W} = \frac{\sqrt{4\pi\hat{\alpha}}}{\hat{c}_W}$$

$$m_Z = \frac{g_2^2 v^2}{4(1-s_W^2)} \quad \Longrightarrow \quad \hat{g}_2 = \frac{\hat{e}}{\hat{s}_W}$$

$$G_F = \frac{1}{\sqrt{2}\, v^2} \qquad\qquad \hat{v}^2 = \frac{\hat{G}_F^{-1}}{\sqrt{2}}$$

$$s_W^2 = \frac{g_1^2}{g_1^2+g_2^2} \qquad\qquad \hat{s}_W^2 = \frac{1}{2}\left[1-\sqrt{1-\frac{4\pi\hat{\alpha}}{\sqrt{2}\hat{G}_F \hat{M}_Z}}\,\right]$$

$$\hat{c}_W^2 = 1-\hat{s}_W^2$$

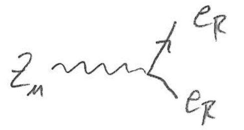
Notice this is really just 3 equations and 3 unknowns, $\hat{s}_W$ & $\hat{c}_W$ just make our expressions prettier

2) The Z-width into leptons

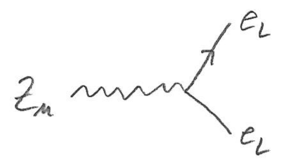
We had:

$$\mathcal{L}_{Z\bar\psi\psi} = -g_Z \bar\psi\left(\frac{\sigma_3}{2} - Q s_W^2\right) Z_\mu \gamma^\mu \psi \qquad \text{for } \psi \in \{L, Q, e, u, d\}$$

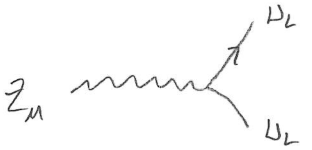
Considering only leptons for simplicity this gives:



$$= i \underbrace{Q_e g_Z s_W^2}_{\equiv g_R^e} \gamma^\mu P_R$$



$$= \underbrace{\frac{i g_Z}{2}(1 + 2Q_e s_W^2)}_{\equiv i g_L^e} \gamma^\mu P_L$$



$$= -\frac{i g_Z}{2} \gamma^\mu P_L$$

The Z-width is related to its lifetime

$$\Gamma = \frac{1}{\tau_{\text{lifetime}}}$$

The width Γ includes $Z \to$ everything, but is to a good degree is $\Gamma \to \sum_\psi \Gamma(Z \to \bar\psi\psi)$

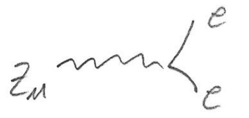
$\Gamma(Z \to \bar\psi\psi)$ is called the "partial width"

$\frac{\Gamma(Z \to \bar\psi\psi)}{\Gamma} \equiv BR(Z \to \bar\psi\psi)$ is the Branching ratio or Branching fraction
it is the % of Z decays into $\bar\psi\psi$

the partial width can be calculated as:

$$\Gamma_{Z\to\bar{f}f} = \int d\Pi_{LIPS}\, \frac{1}{2m_Z} |\mathcal{M}_{Z\to\bar{f}f}|^2$$

summing the FRs for e_L & e_R:



$$= i\bar{u}_1\gamma^\mu(g_R^e P_R + g_L^e P_L)v_2\epsilon_\mu \equiv i\mathcal{M}$$

$$i\mathcal{M}^\dagger = -i\bar{v}_2\gamma^\mu(g_R^e P_R + g_L^e P_L)u_1\epsilon_\mu^*$$

averaging over initial polarizations & summing over final spins:

$$\frac{1}{3}\sum_{p,s}|\mathcal{M}|^2 = \frac{1}{3}\sum_s\left(\bar{u}_1\gamma^\mu(g_R^e P_R + g_L^e P_L)v_2\right)\left(\bar{v}_2\gamma^\nu(g_R^e P_R + g_L^e P_L)u_1\right)\underbrace{\sum_p \epsilon_\mu\epsilon_\nu^*}_{=-\eta_{\mu\nu}+\frac{p_\mu^Z p_\nu^Z}{m_Z^2}}$$

← massive pol sum

these terms vanish for $m_e \ll m_Z$ just like in muon decay

recall: $\sum_s u_1\bar{u}_1 = \not{p}_1 + m$

$\sum_s v_2\bar{v}_2 = \not{p}_2 - m$

$\frac{m_e}{m_Z} \sim \frac{m_\mu}{m_Z} \sim 0$ so we neglect m

$\frac{m_\tau}{m_Z} \sim 2\%$

$$= \frac{1}{3}\mathrm{Tr}\left[\not{p}_1\gamma^\mu(c_R P_R + c_L P_L)\not{p}_2\gamma^\nu(c_R P_R + c_L P_L)\right](-\eta_{\mu\nu})$$

$\not{p}_2\gamma^\nu P_\pm = P_\pm \not{p}_2\gamma^\nu$

$$= \frac{-1}{3}\mathrm{Tr}\left[\not{p}_1\gamma^\mu(c_R^2 P_R + c_L^2 P_L)\not{p}_2\gamma^\mu\right]$$

$\gamma^\mu\not{p}_1\gamma^\mu = -2\not{p}_1$, use cyclicity of Tr[ ]

$$= \frac{2}{3}\mathrm{Tr}\left[(c_R^2 P_R + c_L^2 P_L)\not{p}_2\not{p}_1\right]$$

$P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$, $\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu] = 0$

$$= \frac{2}{3}\cdot\frac{1}{2}(c_R^2 + c_L^2)\underbrace{4p_1\cdot p_2}_{\mathrm{Tr}[\not{p}_1\not{p}_2]}$$

Notice that $p_Z = p_1 + p_2 \rightarrow p_Z^2 = m_Z^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2$

$$= m_e^2 + m_e^2 + 2p_1 \cdot p_2$$

$$\sim 2p_1 \cdot p_2$$

$$\frac{1}{3} \sum_{p,s} |m|^2 = \frac{1}{3}(c_R^2 + c_L^2) 2m_Z^2$$

$$\Gamma_{Z \to ee} = \int d\Pi_{LIPS} \frac{1}{\cancel{2}m_Z} \frac{\cancel{2}}{3} m_Z^2 (c_R^2 + c_L^2)$$

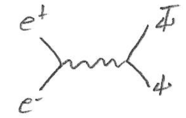
$$= \frac{1}{8\pi} \frac{m_Z}{3} (c_R^2 + c_L^2)$$

$$= \frac{g_z^2 m_Z}{96\pi} \left[4s_W^4 + (1 - 2s_W^2)^2\right] \qquad \text{note } g_Z = \frac{g_Z}{c_W} = \frac{2}{v}\frac{g_Z v}{2c_W} = \frac{2}{v} m_Z$$

$$= \frac{m_Z^3}{24v^2\pi} \left[4s_W^4 + (1 - 2s_W^2)^2\right]$$

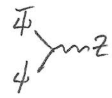
To extract s_W we can look at Forward-Backward Asymmetries:

$$A_{FB}^{\psi} = \frac{\sigma_{\cos\theta > 0} - \sigma_{\cos\theta < 0}}{\sigma_{\cos\theta > 0} + \sigma_{\cos\theta < 0}}$$

for σ: 

w/ θ angle between final state ψ's

$A_{FB}^{\psi} \sim \frac{3}{4} A_e A_\psi$ (+ nonresonant, eg )

$A_\psi \sim \frac{\sigma_L - \sigma_R}{\sigma_L + \sigma_R}$ w/ $\sigma_\pm$ the production xs: 

Since the coupling dep of $\Gamma_{Z \to \bar{\psi}\psi}$ is the same as $\sigma_{\bar{\psi}\psi \to Z}$ we have:

$$\sigma \propto c_L^2 + c_R^2 \rightarrow \sigma_L \propto c_L^2$$

$$\sigma_R \propto c_R^2$$

So $A_e \sim \frac{c_L^2 - c_R^2}{c_L^2 + c_R^2} = \frac{(1 + 2Q_e s_W^2)^2 - (2Q_e s_W^2)^2}{(1 + 2Q_e s_W^2)^2 + (2Q_e s_W^2)^2} = \frac{1 + 4Q_e s_W^2}{1 + 4Q_e s_W^2 + 8Q_e^2 s_W^4} = \frac{1 - 4s_W^2}{1 - 4s_W^2 + 8s_W^4}$

$A_{FB}^e = \frac{3}{4} A_e^2$ only depends on s_W, so indep measurement from $\Gamma_Z = \Gamma_Z(m_Z, v, s_W)$

3) Testing the tree level SM on the Z-pole

we have calculated:

$\Gamma_{Z\to e^+e^-}$

A_{FB}^{e}

we also know from the SM:

$m_W = m_Z c_W$

Using our input parameters $\hat{\alpha}, \hat{m}_Z, \hat{G}_F$, we predict:

$m_W = \hat{m}_Z \hat{c}_W = 80.94$ GeV

$$\Gamma_{Z\to e^+e^-} = \frac{\hat{m}_Z^3}{24\pi}\sqrt{2}\hat{G}_F\left[4\hat{s}_W^4 + (1-2\hat{s}_W^2)^2\right] = 84.84\times10^{-3}\,\text{GeV} = 84.84\ \text{MeV}$$

$$A_{FB}^{e} = \frac{3}{4}\frac{(1-4\hat{s}_W^2)^2}{(1-4\hat{s}_W^2+8\hat{s}_W^4)^2} = 0.0336$$

The values of these quantities (pre 2022):

$m_W = 80.379(12)$ GeV .01% error

$\Gamma_e = 83.92(12)$ MeV .14% error

$A_{FB}^{e} = 0.0171(10)$ 5.8% error

We can make a comparison w/ thry by making something similar to a χ^2:

$$X \equiv \left|\frac{\text{pred}-\text{exp}}{\sqrt{\delta\text{pred}^2+\delta\text{exp}^2}}\right| \sim \left|\frac{\text{pred}-\text{exp}}{\delta\text{exp}}\right|$$ bc $\hat{\alpha}, \hat{m}_Z, \hat{G}_F$ have very small errors

$X_{m_W} \sim 47$ $\rightarrow$ predicted value is about 47σ off from exp

$X_{\Gamma\to ee} \sim 6.6\sigma$

$X_{FBe} \sim 17\sigma$

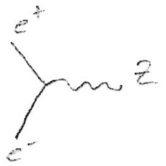
2 possible conclusions: 1) The SM is wrong/insufficient to explain the observations
2) Radiative corrections are necessary to correctly predict these observables w/ the precision of the experiment

4) Radiative corrections to Z-pole physics at LEP

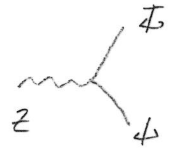
we wont calculate these as we need more QFT to do so:

Recall at tree level:

$O(g_Z)$:



&



$\psi \in \{e, \mu, \tau, u, d, s, c, b, \nu\}$

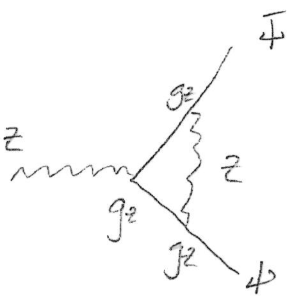
but not t as the COM Energy $= m_Z \ll 2m_t$

Vertex corrections: we can guess these using interactions like $\bar{\psi}\psi F$

w/ $F \in \{A, Z, G, W, h\}$

ψ is not in F as we have no $(\bar{\psi}\psi)^2$ interactions

$O(g_Z^3)$:



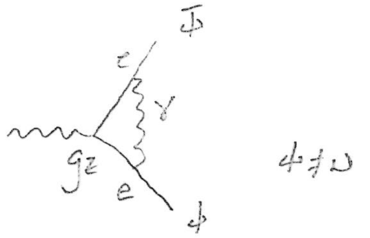
$O(g_Z g_Z^2)$

w/ $\left.\begin{matrix}\psi' \\ \psi\end{matrix}\right\} = \left\{\begin{matrix}\nu, e, u, d \\ e, \nu, d, u\end{matrix}\right.$

for all 3 generations

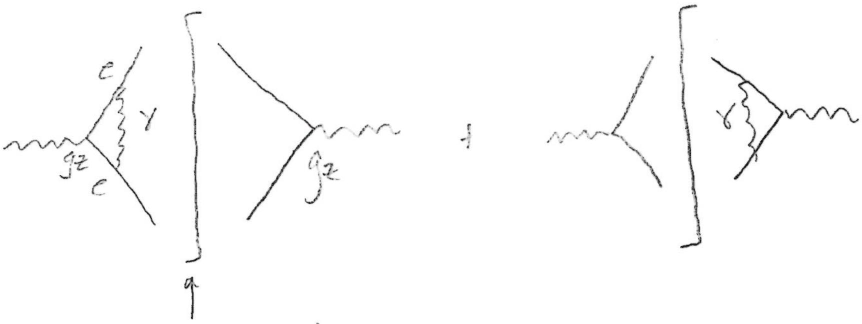
When we include massless gauge bosons we need to include real radiation too:

$g_Z e^2$:



when this loop is interfered w/ the tree level result:

$g_Z^2 e^2$

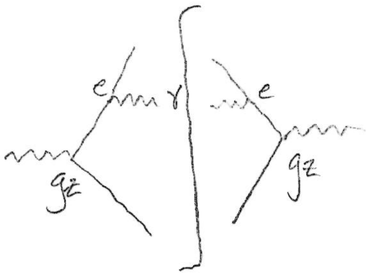


indicates interference

$\mathcal{M}_{loop} \times \mathcal{M}^*_{tree}$

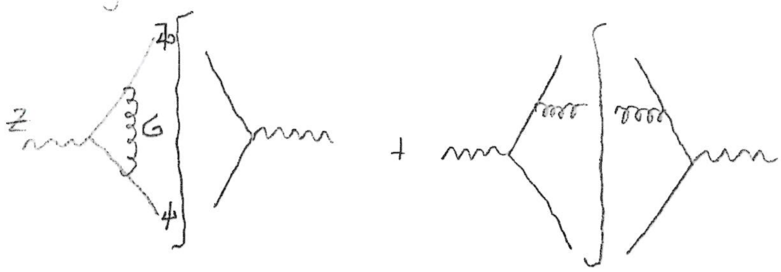
but at the same order we have:

$g_Z^2 e^2$



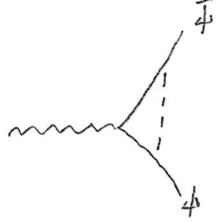
If the γ is too soft (ie $E \sim 0$) or too close to the lepton ($\theta_{e\gamma} \sim 0$) we can't tell the difference between $Z \to \bar{\psi}\psi$ & $Z \to \bar{\psi}\psi\gamma$.
This has consequences both theoretically & experimentally that we wont get in to.

Similarly for gluons we have

$g_Z^2 g_s^2$: 

$\psi \in$ Quarks

We also can have a higgs boson exchanged:

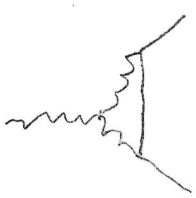
$g_Z Y_\psi^2$: 

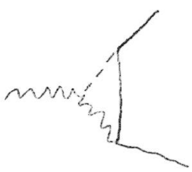
note $\quad Y_\psi = \dfrac{\sqrt{2}\, m_\psi}{v}$

$$Y_b = \frac{\sqrt{2}\, m_b}{v} = \frac{\sqrt{2} \times 4\,\text{GeV}}{246\,\text{GeV}} \sim 0.02$$

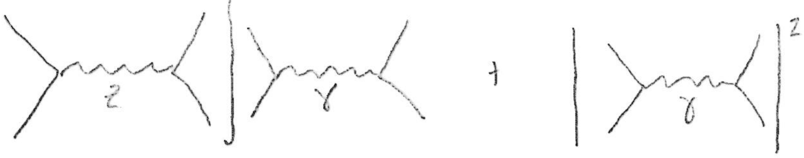
so these are very small corrections.

We can also have corrections like:

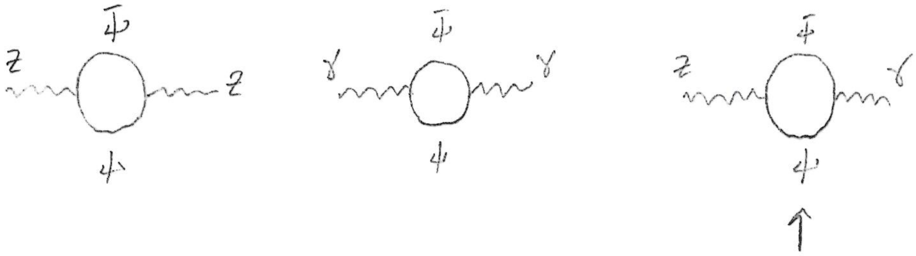
g^3: 

$g^3 Y$: 

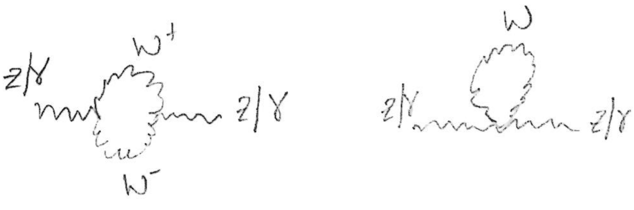
At tree level we can also have contributions from:



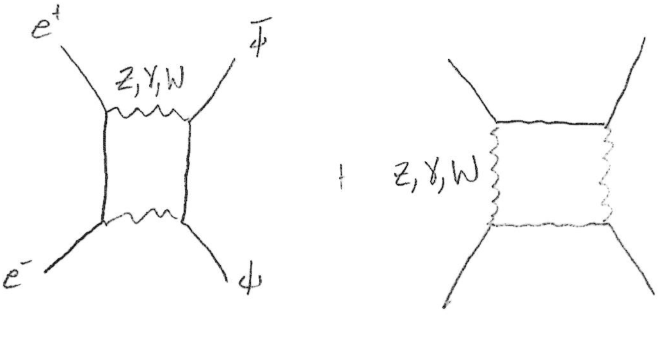
We therefore also need to consider propagator corrections:



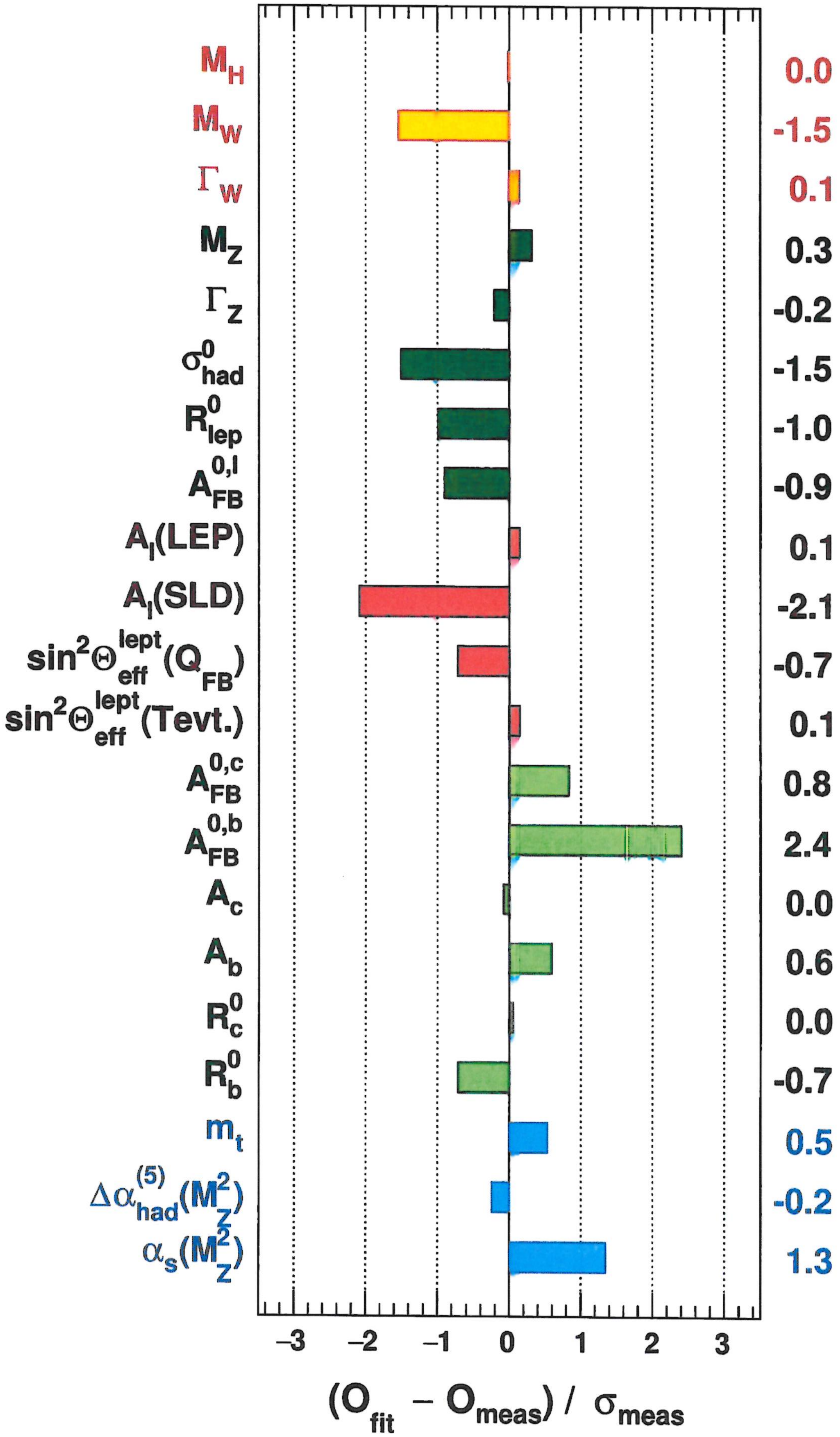
Z - γ mixing is reintroduced at one loop



We also have nonresonant box digrams:



The Z-pole observables have been calculated up to 2 loops and partially at 3 loops.



M_H 0.0
M_W -1.5
Γ_W 0.1
M_Z 0.3
Γ_Z -0.2
σ^0_{had} -1.5
R^0_{lep} -1.0
$A^{0,l}_{FB}$ -0.9
A_l(LEP) 0.1
A_l(SLD) -2.1
$\sin^2\Theta^{lept}_{eff}(Q_{FB})$ -0.7
$\sin^2\Theta^{lept}_{eff}$(Tevt.) 0.1
$A^{0,c}_{FB}$ 0.8
$A^{0,b}_{FB}$ 2.4
A_c 0.0
A_b 0.6
R^0_c 0.0
R^0_b -0.7
m_t 0.5
$\Delta\alpha^{(5)}_{had}(M^2_Z)$ -0.2
$\alpha_s(M^2_Z)$ 1.3
−3 −2 −1 0 1 2 3
$(O_{fit} - O_{meas}) / \sigma_{meas}$