1)

Let's check:

$$S^2 |\pm\pm\rangle = 2|\pm\pm\rangle \qquad S_z|\pm\pm\rangle = \pm|\pm\pm\rangle$$

$$S^2 \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) = \frac{1}{\sqrt{2}}\left[(|+-\rangle + |-+\rangle) + (|-+\rangle + |+-\rangle)\right]$$

$$= 2 \times \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)$$

$$S_z \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle) = \frac{1}{\sqrt{2}}(0 + 0) = 0$$

We can label these three states by

$$S^2: \quad 2 = s(s+1) \rightarrow s = 1$$

$$S_z: \quad m = 1, 0, -1$$

$$|s, m\rangle \rightarrow |1,1\rangle, \ |1,0\rangle, \ |1,-1\rangle$$

We're missing one state, which must be orthogonal to the other 3:

$$S^2 \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) = \frac{1}{\sqrt{2}}\left[(|+-\rangle + |-+\rangle) - (|-+\rangle + |+-\rangle)\right]$$

$$= 0$$

$$S_z \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) = \frac{1}{\sqrt{2}}(0 - 0) = 0$$

$$|s, m\rangle = |0, 0\rangle$$

So adding 2 spin ½ particles yields a spin one eigenket w/ s_z projections $\pm 1, 0$ and a spin zero eigenket.

How can we form eigenkets w/ simultaneous eigenvalues?

recall ladder operators for $\mathcal{H}_{1/2}$:

$$S_\pm = S_x \pm i S_y$$

$$S_- |+\rangle = |-\rangle$$

$$S_+ |-\rangle = |+\rangle$$

$$S_- |-\rangle = S_+ |+\rangle = 0$$

So for the addition of two spin ½ particles we have:

$$\mathcal{S}_\pm = S_\pm \otimes \mathbb{1} + \mathbb{1} \otimes S_\pm$$

Since $|\pm\pm\rangle$ has simultaneous eigenvalues of S^2 & S_z we can start "at the top (bottom) of the ladder and lower (raise) to the bottom"

$$\mathcal{S}_- |++\rangle = S_- \otimes \mathbb{1} |++\rangle + \mathbb{1} \otimes S_- |++\rangle$$

$$= |-+\rangle + |+-\rangle$$

$$\mathcal{S}_- (|-+\rangle + |+-\rangle) = S_- \otimes \mathbb{1} (|-+\rangle + |+-\rangle) + \mathbb{1} \otimes S_- (|-+\rangle + |+-\rangle)$$

$$= \phi + |--\rangle + |--\rangle + \phi$$

So we have 3 potential states, but we want them nicely normalized:

$$|++\rangle \rightarrow |++\rangle$$

$$|-+\rangle + |+-\rangle \rightarrow \frac{1}{\sqrt{2}}(|-+\rangle + |+-\rangle)$$

$$2|--\rangle \Rightarrow |--\rangle$$

Gaussian Integrals

$$I = \int_{-\infty}^{\infty} dp\, e^{-\frac{1}{2}ap^2 + Jp}$$

complete the square: $-\frac{1}{2}ap^2 + Jp = -\frac{1}{2}a(p-\frac{J}{a})^2 + X$

$$= -\frac{1}{2}ap^2 + Jp - \frac{1}{2}\frac{J^2}{a} + X \rightarrow X = \frac{1}{2}\frac{J^2}{a}$$

$$I = \int_{-\infty}^{\infty} dp\, e^{-\frac{1}{2}a(p-\frac{J}{a})^2 + \frac{J^2}{2a}}$$

let $p \rightarrow p + \frac{J}{a} \Rightarrow dp \rightarrow dp$

$$I = \int_{-\infty}^{\infty} dp\; e^{-\frac{1}{2}ap^2 + \frac{J^2}{2a}} = e^{J^2/2a} \int_{-\infty}^{\infty} dp\, e^{-\frac{1}{2}ap^2}$$

let $p \rightarrow p/\sqrt{a} \Rightarrow dp \rightarrow \frac{dp}{\sqrt{a}}$

$$I = \frac{1}{\sqrt{a}} e^{J^2/2a} \int dp\; e^{-\frac{1}{2}p^2}$$

Consider:

$$I^2 \propto \left[\int dp\, e^{-\frac{1}{2}p^2}\right]^2 = \int dx \int dy\; e^{-\frac{1}{2}(x^2+y^2)}$$

$$= \int r\,dr\,d\theta\; e^{-\frac{1}{2}r^2}$$

$$= 2\pi \int_0^{\infty} \left(\tfrac{1}{2}dr^2\right) e^{-\frac{1}{2}r^2}$$

$$= \pi \left(-2e^{-\frac{1}{2}r^2}\right\rvert_{r^2=0}^{r^2=\infty}$$

$$= 2\pi$$

$$I = \sqrt{\frac{2\pi}{a}}\, e^{J^2/2a}$$

For a multidimensional integral:

$$ap^2 \to \vec{p}^\dagger A \vec{p} = p_i^* A_{ij} p_j$$

$$I = \int_{-\infty}^{\infty} d\vec{p}\, e^{-\frac{1}{2} p_i^* A_{ij} p_j + J_i^* p_i}$$

Diagonalize A_{ij} w/ M_{ij} $\to$ $A = M_{ij} D_{jk} M_{kl}^{-1}$ ← diagonal

$$\Rightarrow I = \int_{-\infty}^{\infty} d\vec{p}\, e^{-\frac{1}{2} p_i^* M_{ij} D_{jk} M_{kl}^{-1} p_l + J_i^* p_i}$$

$$= \int_{-\infty}^{\infty} d\vec{p}'\, e^{\underbrace{-\frac{1}{2} p_i'^* D_{ii} p_i' + \underbrace{J_i^* M_{ij}}_{\equiv J_j'^*} p_j'}_{}} \qquad p_k' = M_{kl}^{-1} p_l$$

this is just a bunch of integrals over p_j' now w/ $a \to D_{jj}$, $J \to J_j'$

$$= \prod_j \sqrt{\frac{2\pi}{D_{jj}}}\, e^{\frac{1}{2} J_j'^* \frac{1}{D_{jj}} J_j'}$$

since D_{jj} is the diagonalized A_{ij}

$$\det A_{ij} = \prod_i \lambda_i = \prod_i D_{ii}$$

↑ eigenvalues

$$\frac{1}{D_{jj}} = D_{jj}^{-1}, \qquad A_{il}^{-1} = (M_{ij} D_{jk} M_{kl}^{-1})^{-1}$$

$$= M_{ij} D_{jk}^{-1} M_{kl}^{-1}$$

$$J_i'^* D_{ij}^{-1} J_j' = J_i^* M_{ij} D_{jk}^{-1} M_{kl}^{-1} J_l = J_i^* A_{ij}^{-1} J_j$$

$$I = \sqrt{\frac{(2\pi)^n}{\det A}}\, e^{\frac{1}{2} J^\dagger A^{-1} J} \qquad \text{for } A \text{ } n\times n \text{ matrix}$$