Homework # 6

1. Create for yourself a dataset containing $x_1, x_2, \ldots, x_n \in \mathbb{R}$ and $y_1, y_2, \ldots, y_n \in \mathbb{R}$ with $n = 1000$. Choose the x_i randomly with uniform distribution on the interval $[0, 10]$ and set $y_i = 1 + 2x_i + \epsilon_i$ where ϵ_i is chosen from a standard normal. Now given the data, consider the least squares problem,

$$\min_{\alpha \in \mathbb{R}^2} \sum_{i=1}^{n} |y_i - \alpha_1 - \alpha_2 x_i|^2 \tag{1}$$

   (a) Let X be an $n \times 2$ matrix with $X_{i1} = 1$ and $X_{i2} = x_i$ for $i = 1, 2, \ldots, n$. Show that the solution to the minimization is given by $\alpha = (X^T X)^{-1} X^T y$.

   (b) Compute the optimal α and plot the line $y = \alpha_1 + \alpha_2 x$ and the data points (x_i, y_i) on one graph to show the fit.

2. This is essentially problem $11.2a$ in Trefethon and Bau. Suppose we would like to approximate the function $f(x) = 1/x$ on the interval $[1, 2]$ using a linear combination of the functions $\sin(x)$, e^x and the gamma function $\Gamma(x)$ by finding $\alpha \in \mathbb{R}^3$ that minimizes,

$$\int_1^2 (f(x) - (\alpha_1 \sin(x) - \alpha_2 \mathrm{e}^x - \alpha_3 \Gamma(x))^2 \mathrm{d}x \tag{2}$$

   Write code that estimates the answer using a discretization of $[1, 2]$ and a least squares problem. Given your estimate of the optimal α, create a plot comparing $f(x)$ to the approximating linear combination. See

   `https://en.wikipedia.org/wiki/Gamma_function`

   for a definition of $\Gamma(x)$. The library scipy.special has a gamma function.

3. From homework 5, consider again the sparse matrix X of problem 4. Use a random matrix approach to compute the first 2 singular values of X. Experiment with different values for the q. You may also find that by trying to compute more than 2 singular values, you get a more accurate estimate of the first 2. You can check your answer using eigh or your orthogonalized power iteration from homework 5.