Homework #2

1. Let V be an $n \times k$ orthonormal matrix. (Recall, this means the columns of V are orthonormal vectors.)
   (a) Show that $V^T V = I$ where I is the $k \times k$ identity matrix.
   (b) Show that VV^T is a projection matrix.
   (c) Show that if $n > k$ then VV^T is not the identity matrix and if $n = k$ then $VV^T = I$ where I is the $n \times n$ identity matrix.
   (d) Let $x \in \mathbb{R}^k$. Show $\|Vx\| = \|x\|$.

2. Let P be a symmetric, projection matrix in $\mathbb{R}^n$. Show that $P = VV^T$ for some orthonormal matrix V. This result is a converse to 1b. (Hint: Consider the linear space $\mathcal{S} = \{x \mid Px = x\}$ and its orthogonal complement $\mathcal{S}^\perp$. If you construct V as a basis for $\mathcal{S}$ and use the symmetry of P to show that $Py = 0$ for $y \in \mathcal{S}^\perp$ then you'll be able to verify $P = VV^T$.)

3. Let M be an $p \times p$ matrix. M is positive definite if $x^T Mx > 0$ for all $x \in \mathbb{R}^p$ and $x \neq 0$.
   (a) Let M be a symmetric matrix. Use the spectral theorem to show that M is positive definite if and only if all its eigenvalues are positive.
   (b) Suppose $M = X^T X$ for some matrix X. Show that M is positive definite as long as the columns of X are linearly independent.

4. Attached you will find the senators dataset. There are 100 rows corresponding to 100 senators. The first column contains senator names. The rest of the columns contain how the senator voted with $1, -1, 0$ meaning for, against, and abstain.
   (a) In class we derived the formula for computing a projection onto a linaer space as well as showing that the principle components should be the eigenvectors of the covariance matrix. Write out a derivation of these results.
   (b) Compute a 2-dimensional PCA of the data restricted to the vote columns. Before applying the PCA, center each column by subtracting off its mean. Form a scatter plot the scores, which will be two dimensional, with the color of the points given according to the senator's party.
   (c) PCA is an unsupervised dimension reduction, meaning that we don't have any labeling associated with the samples. In this case, we do know the senator's party affiliations. Let v be the vector given by the difference between the mean Republican vote vector and the mean Democrat vote vector. Perform a 1-d dimension reduction by projecting onto v and compare to a 1-d PCA dimension reduction.