Homework #3

1. Let X be an $n \times p$ matrix. Below you'll prove the svd $X = USV^T$ with the number of columns in U and V will be restricted to $\min(n, p)$.

   (a) Assume $n \leq p$. Prove the svd. (To do this construct the v_i from $X^T X$, but keep just n of them. Then you'll be able to define $s_i u_i = X v_i$ for $i = 1, 2, \ldots, n$ and the rest of the proof will follow.)

   (b) Assume $n > p$. Prove the svd. (This time use all the v_i from $X^T X$. Construct $s_i u_i = X v_i$ for $i = 1, 2, \ldots, p$ and that's all you'll need for the rest of the proof.)

2. Here are some odds and ends to prove.

   (a) Let A be $n_1 \times p$ and B be $n_2 \times p$. Show,

$$AB^T = \sum_{i=1}^{p} a_i b_i^T \tag{1}$$

   where a_i and b_i are the ith columns of A, B respectively.

   (b) Let X be $n \times n$ and symmetric. Prove the spectral decomposition $X = QDQ^T$ and $X = \sum_{i=1} \lambda_i q_i q_i^T$. (Here the q_i are the eigenvectors of X and λ_i are the eigenvalues. Q is the matrix with the q_i as columns and D is diagonal with $D_{ii} = \lambda_i$. Hint: show $X = QDQ^T$ on the basis of the q_i.)

   (c) (Not sure if you've seen this before.) Assume X is symmetric and positive semidefinite (i.e. $v^T X v \geq 0$ for all $v \neq 0$.). Let $X = QDQ^T$ be the spectral decomposition of X. Show that $M = \sqrt{D} Q^T$ satisfies $X = M^T M$ where $\sqrt{D}$ is D with its diagonal elements replaced by their square root. Loosely, we can view M as the square root of X. Is M unique? Show that there is no square root if X is not positive semidefinite.

3. Let X be $n \times p$. Show that $\|X\|_F^2 = \sum_{i=1}^{\min(n,p)} s_i^2$ where s_i is the ith singular value of X. (Hint: Use the svd $X = \sum s_i u_i v_i^T$ and $\|X\|_F^2 = \mathrm{trace}(X^T X)$.)

4. Let A be an $n \times p$ matrix. All columns of A have one entry equal to 1 and all other entries equal to 0. Let $\sum_{j=1}^{p} A_{ij} = \tau_i$ for $i = 1, 2, \ldots, n$. What are the singular values of A? What are the left singular vectors? What can you say about the right singular vectors? (Hint: AA^T has a simple form. That'll give you information about the svd of A^T.)

5. Go back to your code for the senators data from last weeks homework. Rewrite your computations by computing the svd of the data matrix X rather than using the covariance matrix $X^T X$.