Problem Set #3

PSE Masters in Economics
Quantitative Macro, FALL 2023

Due Date: 23 Oct Sep 12noon

Please hand in your answers by email using file names that contain all group members' last names (e.g. BROER_ELINA_PS_1.pdf). Hand-written answers are also fine.

Consider the following income fluctuation problem for $t = 0, 1, ..., T$.

$$\max_{\{c_t, k_{t+1}, i_t\}_{t=1}^T} E_0 \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

$$\text{s.t.}$$

$$c_t + i_t \leq z_t k_t^\theta$$

$$k_{t+1} = (1-\delta)k_t + i_t$$

$$\log(z_{t+1}) = \rho \log(z_t) + \varepsilon_{t+1} \text{ where } \varepsilon_{t+1} \sim i.i.d.N(0, \sigma^2)$$

$$k_0 \text{ and } z_0 \text{ given}$$

where c_t is consumption, k_t capital, z_t productivity, i_t investment and ε_t an exogenous shock.

In this problem set we consider the case without shocks, so assume $z_0 = 1$ and $\epsilon_t = 0$, $t = 0, 1, 2, ...$

Problem 1 The Bellman equation and its properties.

1. Simplify the problem as a choice of $\{k_{t+1}\}$. Define the "givens" (X,Γ, F, β) seen in class for this problem.

2. Write down the functional equation (FE) for this problem that defines the value function v.

3. The return function F is not bounded, but (FE) nevertheless maps $C(X)$, the space of continuous bounded functions on $\mathbb{R}$, into itself. (Make sure you understand why). (FE) defines a contraction mapping using Blackwell's sufficient conditions. (Again, make sure you understand why). Can you further characterise v given the functional forms of F and Γ?

4. Assume that $\delta = 1$ and $\sigma = 1$ (s.t. the utility function is $log(c_t)$).

   (a) Consider the finite-horizon problem, where T denotes the final period. Solve for the optimal path of the endogenous variables for $T = 2$ by backward induction.

(b) Infer the optimal policy rule for the finite-horizon problem from this (or do one more iteration to $T = 3$).

(c) Infer the value function for the infinite horizon problem from $V_2(k_0)$. (*Hint*: From the value functions V_0, V_1 and V_2, you can guess the pattern of the finite horizon value functions. Then take the limit as $T \to \infty$ to obtain the infinite horizon value function.) Solve for the optimal stationary policy function that attains the maximum in (FE).

(d) Alternatively, you can find the value function with the method of "guess and verify" with $V(k) = \alpha_0 + \alpha_k \log(k))$ as an initial guess and determining the unknown coefficients θ_0, θ_k in terms of the coefficients of the model.

Problem 2 Numerical solution I: Discrete-grid value function iteration

Consider $T = \infty$ from now on and the following parameter values:

β	θ	σ	δ
0.99	0.4	2	0.1

Calculate analytically the steady state level of capital k^* of this economy and calculate its value given parameters.

(a) Rewrite the recursive continuous dynamic programming from Problem 1.1. as a discrete one (by constraining the state space to equal a discrete grid.)

(b) Now solve the above problem by "discrete" value function iterations using matlab or some other program. In particular,

   i. Choose a criterion ε for convergence of the value function (choose e.g. 10^{-6}).

   ii. Choose an equally-spaced grid $\mathbb{K} = \{k_1 < k_2 < ...k_N\}$ with $k_1 = 3/4k^*$ and $k_N = 5/4k^*$, where k^* denotes the steady-state value of the capital stock.

   iii. For $i, l = 1, ..., N$, calculate the one-period return function $F(k_i, k'_l)$.

   iv. Choose an initial value function $v_0(k_i)$, an $N \times 1$ vector.

   v. Calculate for all $i = 1, .., N$

$$v_{s+1}[i] = \max_l \{F(k_i, k_l) + \beta v_s[l]\} \tag{1}$$

   for $s = 0, 1, 2, ...$, where $X[j]$ denotes the j'th element of X.

   vi. If $\max_{i,j} |v_{s,i,j} - v_{s-1,i,j}| \geq \varepsilon$ go back to b). Otherwise stop.

(c) Plot the policy function $k'(k)$.

(d) Evaluate the Euler equation errors. In particular, for every value $k_i, i = 1, .., N$, calculate the maximum percentage difference between the consumption according to your policy function $c(k_i)$, and that which makes the Euler equation hold with equality given tomorrows consumption and marginal productivity

$$c(k_i)^{imp} = [\beta(\theta * k'(k_i)^{\theta-1} + 1 - \delta)c(k'(k_i))^{-\sigma}]^{-1/\sigma} \tag{2}$$

(e) Bonus: Add a policy function iteration step, i.e. after k initial iterations on the Bellman equation, alternate between i) one iteration step on the Bellman equation ii) iteration using the resulting policy (without changing it) until the value function converges. Go back to i) until the value function approximately solves the Bellman Equation. Does this speed up the solution? (You can experiment with k.)

Problem 3 Numerical solution II: Value function iteration with interpolation

(a) Now reformulate the maximisation step by allowing choices of k' that are not on the grid. Use linear interpolation (matlab function interp1.m) to evaluate the Value function between grid points. Use the golden-search algorithm to find the optimal choice k' for each level of k. To find the bracket more efficiently, you can exploit that $k'(k)$ is an increasing function, such that $k^{i\star}(k_{i+j}) > k^{i\star}(k_i)$.

(b) Again plot the policy function $k'(k)$.

(c) Again evaluate the Euler equation errors. Comment on how they relate to those found in Problem 1.