Quantitative Macro Problem Set 1

Bingxue Li, Zhihang Liang, Tuba Seker

PSE-APE, Fall 2023

Consider the following problem solved by a social planner for $t = 0, 1, \ldots, T$.

$$\max_{\{c_t, k_{t+1}, i_t\}_{t=1}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$
$$\text{s.t.}$$
$$c_t + i_t \leq k_t^\alpha$$
$$k_{t+1} = (1-\delta)k_t + i_t$$
$$k_0 \text{ given}$$
$$\text{for } u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

1. Simplify the problem as a choice of $\{k_{t+1}\}$, and write down the first-order condition, the feasibility constraint, and the transversality conditions for both finite and infinite T.

Optimality requires resource constraints to be binding, ie. using the all production to finance consumption and investment. Thus, the resource constraint becomes $c_t + i_t = k_t^a$. Then, we substitute i_t into the law of motion and get:

$$k_{t+1} = (1-\delta)k_t + k_t^a - c_t$$
$$c_t = (1-\delta)k_t + k_t^a - k_{t+1} \tag{1}$$

where equation 1 defines feasibility constraint.
Then, we replace 1 into the utility function to simplify the problem as a choice of k_{t+1}. Note that since there is no uncertainty in our problem, we do not write expectation operator.

$$\max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t \frac{[(1-\delta)k_t + k_t^a - k_{t+1}]^{1-\sigma} - 1}{1-\sigma} \tag{2}$$
$$\text{s.t. } k_0 \text{ is given.} \tag{3}$$

Then, the first-order condition with respect to k_{t+1} gives us the intertemporal optimality condition:

$$[(1-\delta)k_t + k_t^a - k_{t+1}]^{-\sigma} = \beta[(1-\delta)k_{t+1} + k_{t+1}^a - k_{t+2}]^{-\sigma}(1 - \delta + \alpha k_{t+1}^{\alpha-1}) \tag{4}$$

Regarding the transversality condition in equation 1,

- If T is finite, the social planner does not want to leave some capital stock at time k_{T+1}, ie. $k_{T+1} = 0$.
- If T is infinite, the optimal decision will require that the present discounted value of capital to converge zero in limit, ie. $lim_{n\to\infty} \lambda_T k_{T+1} = 0$ where $\lambda_T = \beta^T [k_T^\alpha + (1-\delta)k_T - k_{T+1}]^{-\sigma}$.

2. Calculate analytically the constant "steady-state" levels of capital $\bar{k}$ and consumption $\bar{c}$ such that $x_s = \bar{x}, = t \ldots \infty, x = c, k$ satisfy feasibility and optimality.

To find the steady-state level of consumption, rewrite the feasibility constraint as

$$c = (1-\delta)\bar{k} + \bar{k}^a - \bar{k} \tag{5}$$

Then, by plugging $\bar{k}$ into the intertemporal optimality condition (4), we get steady-state level of capital, $\bar{k}$:

$$\begin{aligned} [(1-\delta)\bar{k} + \bar{k}^a - \bar{k}]^{-\sigma} &= \beta[(1-\delta)\bar{k} + \bar{k}^a - \bar{k}]^{-\sigma}(1-\delta+\alpha\bar{k}^{\alpha-1}) \\ \bar{k} &= \Big[\frac{\alpha\beta}{1-\beta(1-\delta)}\Big]^{\frac{1}{1-\alpha}} \end{aligned} \tag{6}$$

Then, we plug equation 6 into 5 and get the steady-state level of consumption, $\bar{c}$:

$$\bar{c} = \Big[\frac{\alpha\beta}{1-\beta(1-\delta)}\Big]^{\frac{\alpha}{1-\alpha}} - \delta\Big[\frac{\alpha\beta}{1-\beta(1-\delta)}\Big]^{\frac{1}{1-\alpha}} \tag{7}$$

Consider the following parameter values:

β	α	σ	δ
0.99	0.4	1.0001	0.025

3. Consider a deterministic path of the infinite-horizon problem starting at $k_0 = 0.9\bar{k}$. Consider T large enough, and assume the economy converges to a steady state in at most T periods.

(a) Write a function that takes paths of capital $\{k_{t+1}\}_{t=0}^{T}$ and consumption $\{c_{t+1}\}_{t=0}^{T}$ as inputs and outputs a $2T \times 1$ vector with errors of the Euler equation for capital investment and the feasibility constraint. Write a program that uses Broyden's method to solve for the optimal path starting at k_0. Use the formula for the recursive updating of the Jacobian from the lecture slides.

For Broyden's method to work, we need to be aware of how to inverse the Jacobian at each step of the iteration. The formulas for iteration are:

$$x^{n+1} = x^n - Z_n^{-1}F(x^n) \tag{8}$$

, where Z_n is the estimated Jacobian from iteration:

$$\mathbb{Z}^{n+1} = \mathbb{Z}^n + \frac{F\left(x^{n+1}\right)s^{n\prime}}{s^{n\prime}s^n} \tag{9}$$

The most computation-demanding part is to inverse the Jacobian. The computed Z_n seems to be very much ill-conditioned. To deal with this problem, we use the `pinv` function in Matlab to compute the inverse, which uses the Moore-Penrose pseudo-inverse. We don't want to dig deep into the details of the comparison of different numerical methods dealing with ill-conditioned matrices. For robustness, we will compare the result calculated from Broyden's method and the result calculated from the built-in root-finding function `fsolve` in Matlab.
The results are shown below in figure 1. It takes for more than 100 periods for consumption and capital to converge.

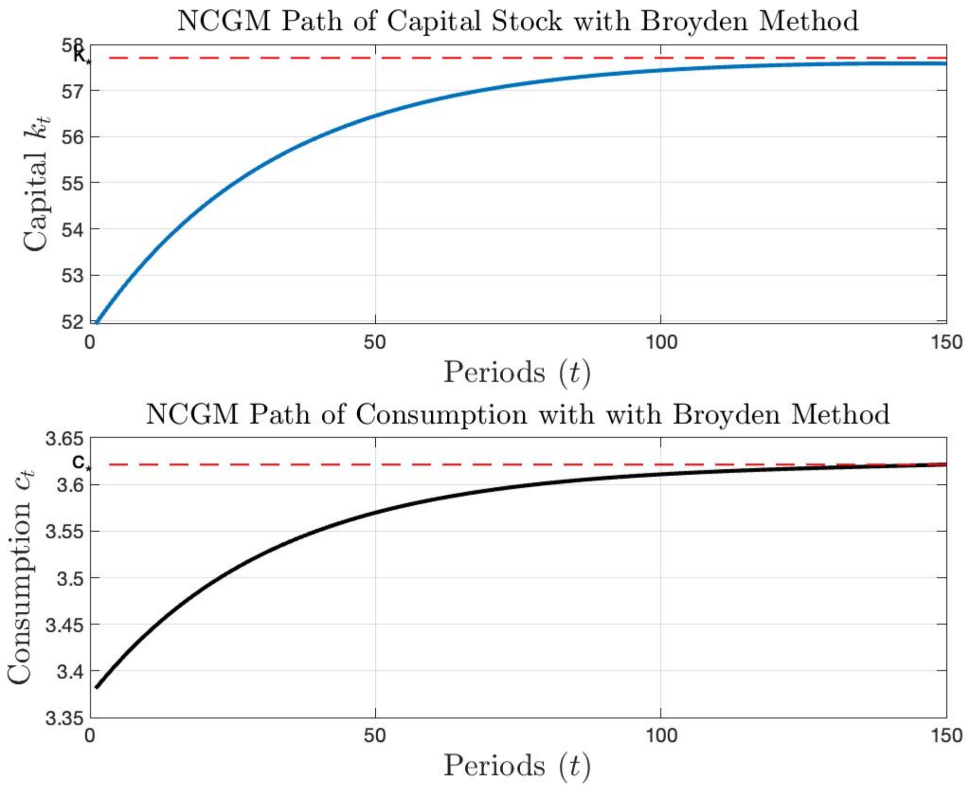

Figure 1: Infinitely-lived Agent, Broyden

(b) Write a program that solves for k_T as a function of c_0 using the Euler equation for capital investment and the feasibility constraint. Use a multiple shooting algorithm (you can use for example the bisection method) to solve for c_0 by imposing convergence to $c_T = \bar{c}, k_{T+1} = \bar{k}$. Plot the sequences $\{c_t, k_t\}$.

The key issue of the multiple shooting algorithm is that it is not stable in the sense that the consumption and the capital accumulation path might diverge with different initial guesses.For instance, if we guess initial consumption a bit higher than the solution, we might have problems such as complex numbers for consumption values or negative capital stock in the last period due to very high initial consumption. To deal with this problem, we simply restrict the capital and the consumption to be real and non-negative values, as shown below:

```
function [terminal_val_c, terminal_val_k] = cT(c0, k0,T,alpha,delta,sigma,beta)
    c = c0;
    k = k0;
    for i = 1:T
       k_p =k;
       c_p = c;
       k = k_p^alpha+(1-delta) * k_p - c_p;
       if k <=0
           k = k_p;
           break
       end
       c = ( 1/beta* c_p^(-sigma)/ (alpha * k^(alpha-1) +(1-delta)))^(-1/sigma);
       if ~isreal(c)
           c = c_p;
          break
       end
```

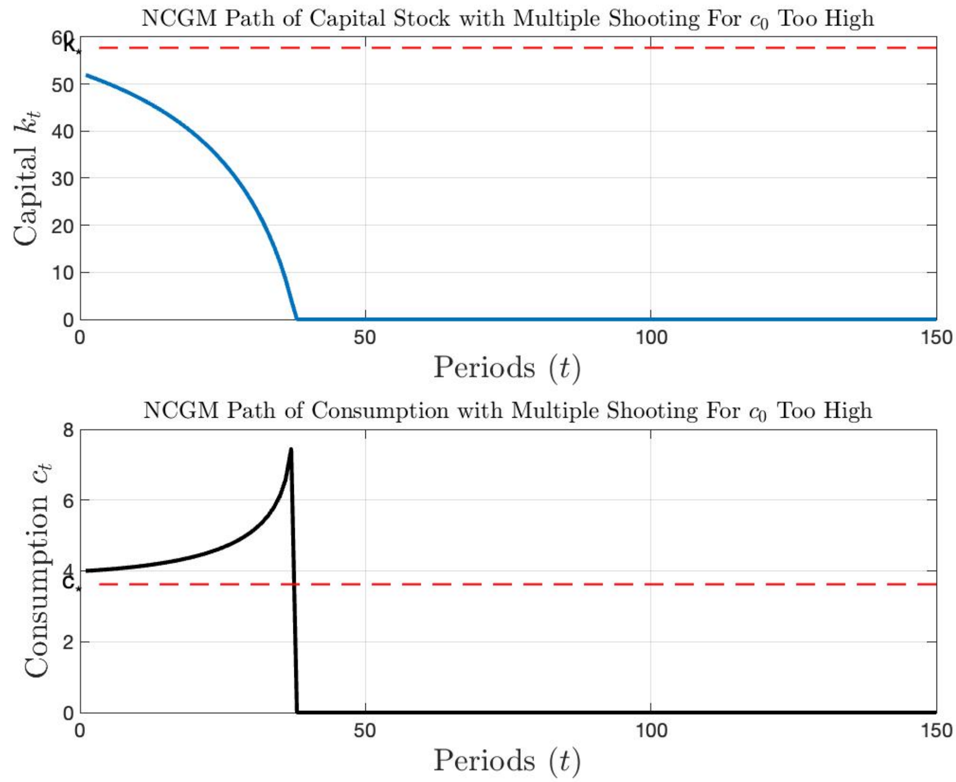

Figure 2: Multiple Shooting, c_0 High

```
    end
    terminal_val_c = c;
    terminal_val_k = k;
end
```

With different guesses, the consumption path and the capital path can diverge. We demonstrate this be showing the paths for guesses of a high c_0 and a low c_0. See figure 2 and figure 3
As instructed, we combine the bisection method and the multiple shooting method. Here, we need to notice that for a high c_0, the computed terminal value for consumption can be pretty high, as shown in figure 2. For low c_0, the terminal value c_T can be low, displayed in figure 3. This is intuitive in the sense that if one consumes too slowly, the capital can accumulate too fast, and the low return for capital will force the consumption to goes down. If one consumes too quickly, she can enjoy a huge flow of consumption at first, but soon the capital will deplete, leaving nothing to consume.
The result for the steady-state path is shown in figure 4, which is of the same shape as figure 1.

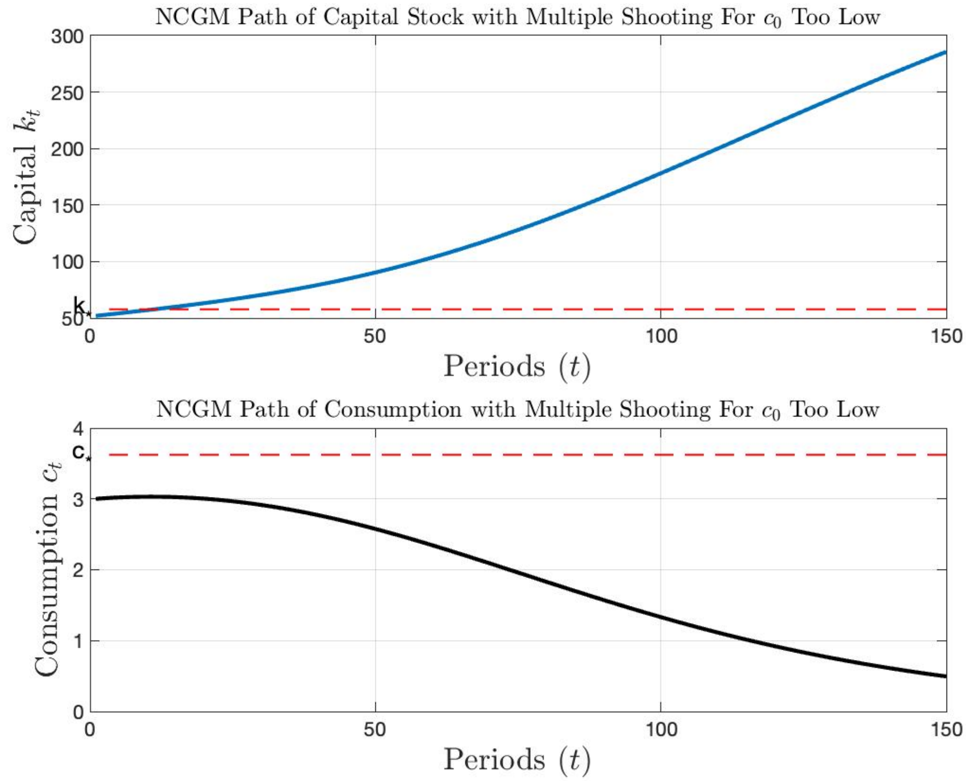

Figure 3: Multiple Shooting, c_0 Low

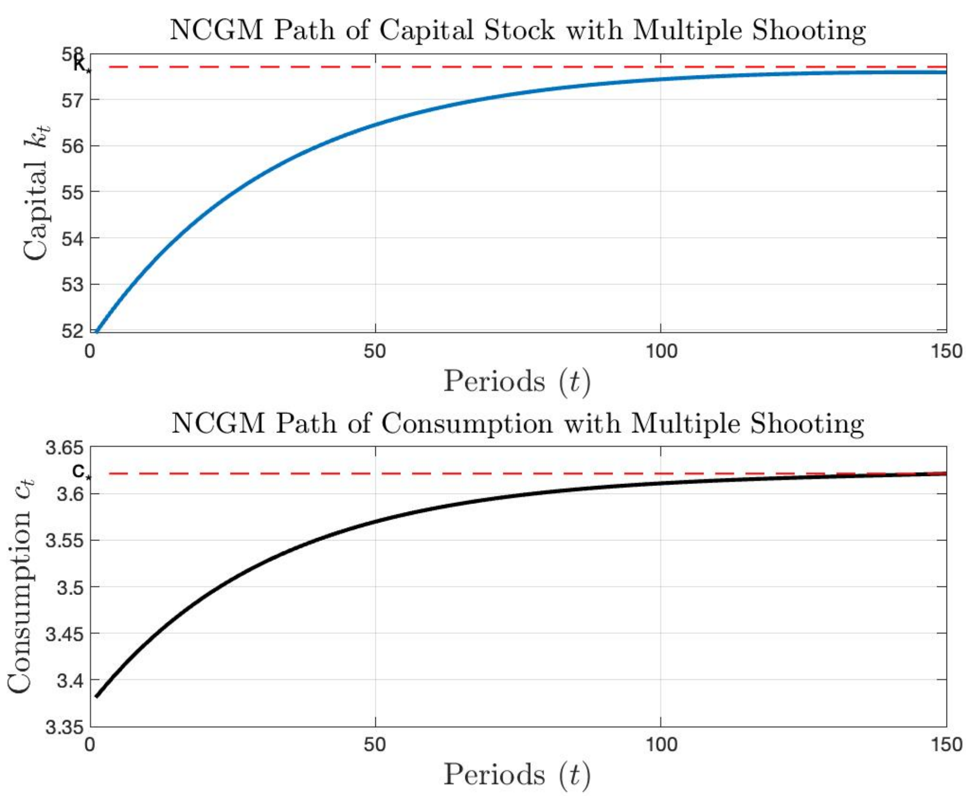

Figure 4: Infinite-lived Agent, Multiple Shooting

4.Use the programs considered above, but with finitely-lived agents, alive for T periods, and compute a deterministic path of the finite-horizon problem starting at $k_0 = 0.9\bar{k}$ with $T = 10, 100, 200$. Plot the sequences $\{c_t, k_t\}$.

We show the result using the `fsolve` function in matlab. In finite periods, c_0 should be such that we consume all our capital stock and production in the last period. In other words, the difference between an infinitely living agent and a finitely living agent is that a finitely living agent can consume all the capital on the last day of her life. This generates a different pattern of consumption, which combines the pattern of over-consumption and steady-state consumption.

Depletion of K And A Surge in C in Late Stages: As seen in figures 5, 6, and 7, the capital accumulates towards steady state level in the initial periods, then declines, starting from $t = 30$ in the 100-period case and $t = 87$ in the 200-period case. In very short horizon $T = 10$, k_t begin to deplete from the very beginning. In all finite horizon cases, k_t never reaches or exceeds the steady-state level k_* computed in section 1. As capital stock declines, consumption shoots up, which shows a similar pattern as figure 2, where the consumer eats up the all the resources without considering any far future consequences. In the case of $T = 10$ figure 5, consumption peaks in period $t = 10$. The surge in consumption is by the fact that the consumer expects to die in the late periods, thus consuming as fast as possible.

Finite Horizon with $T \to \infty$: Comparing figure 5, 6, and 7 with growing length of periods, for large T and small t, longer time horizon and early stages, the optimal rule for consumption and capital accumulation are close to infinite horizon ones, as in 1 and 4. For $\{k_{t+1}\}_{t=0}^{m}$ and $\{c_t\}_{t=0}^{m}$, with m small compared to T, these truncated paths resembles convergence paths of infinitely living agents. To conclude, although taking limit of our finite horizon solutions does not necessarily coincide with solutions of an infinite horizon one, the early stage solutions for T large enough should be an informative approximation for the infinite case.

In the appendix, we include results generated by Broyden's method and the multiple shooting method. All methods exhibit same patterns over time for capital stock and consumption paths. The capital first rises to the steady state level, then declines.

Note that for the multiple shooting algorithm, we decide to do more exercise: we shoot from the last period to try to hit the level of the capital and consumption in the first period, which are given. To do this, we need to be aware that the higher the guess for k_T, the higher the computed $\hat{k}_0$ is. This kind of understanding helps us to better apply the bisection method. Moreover, to avoid the computed $\hat{k}_0$ to be negative, we impose capital lower than $1e-3$ to be zero. This can seen in our function in the `reverse_ms.m`, as shown below.

```
function [k_path, c_path] = reverse_ms(kT, T,alpha,delta,sigma,beta)
    % kT+1 = 0
    k_path = zeros(T,1);
    c_path = zeros(T,1);
    c_path(T) = kT^alpha+(1-delta)*kT;
    k_path(T) = kT;

    for t = flip(1:T-1)
        c_path(t) = (beta* c_path(t+1)^(-sigma)...
            * (1-delta + alpha * k_path(t+1)^(alpha-1)))^(-1/sigma);
        k_path(t) = fzero(@(x)( x^alpha + (1-delta)*x - k_path(t+1)- c_path(t)),...
                    [0,2*k_path(t+1)]  );
        if k_path(t) <1e-3
            break
        end
    end
end
```

A small remark for this function: iterating over Euler equation and budget constraint instead of iterating just one Euler equation with consumption substituted by capital can be beneficial in

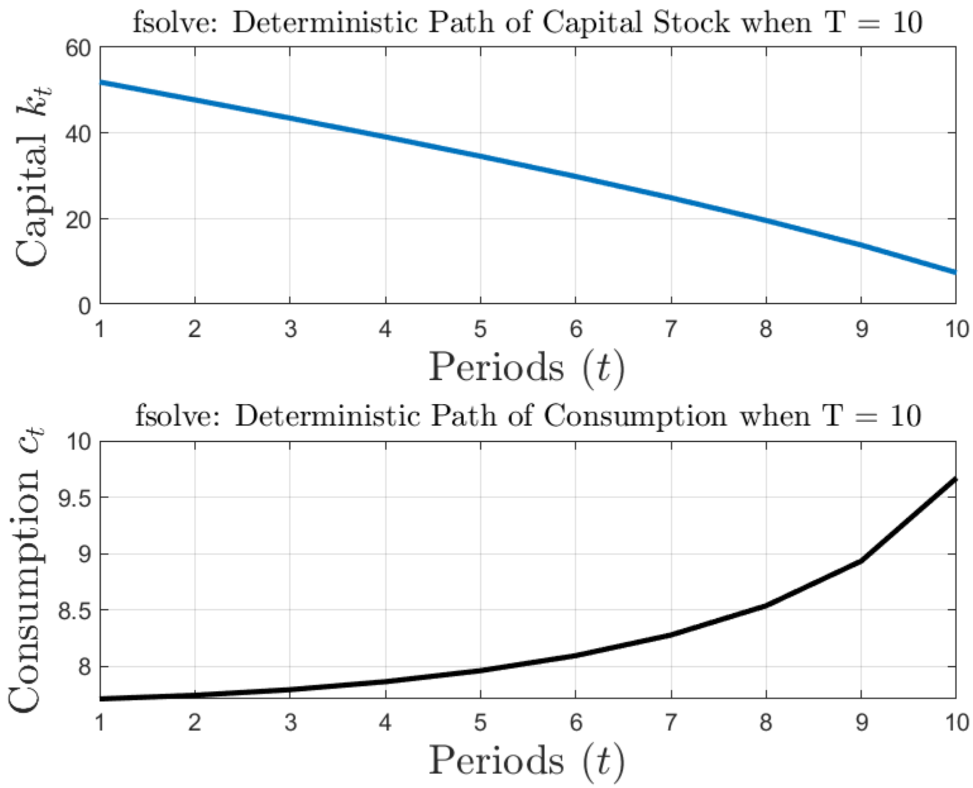

Figure 5: Deterministic Path of Consumption when T = 10

reverse multiple shooting. As shown above, solving k_t from k_{t+1} and c_t involves using a nonlinear solver. Keeping the budget constraint makes the function monotonic, ensuring stability of `fzero` over iterations.

1 Appendix

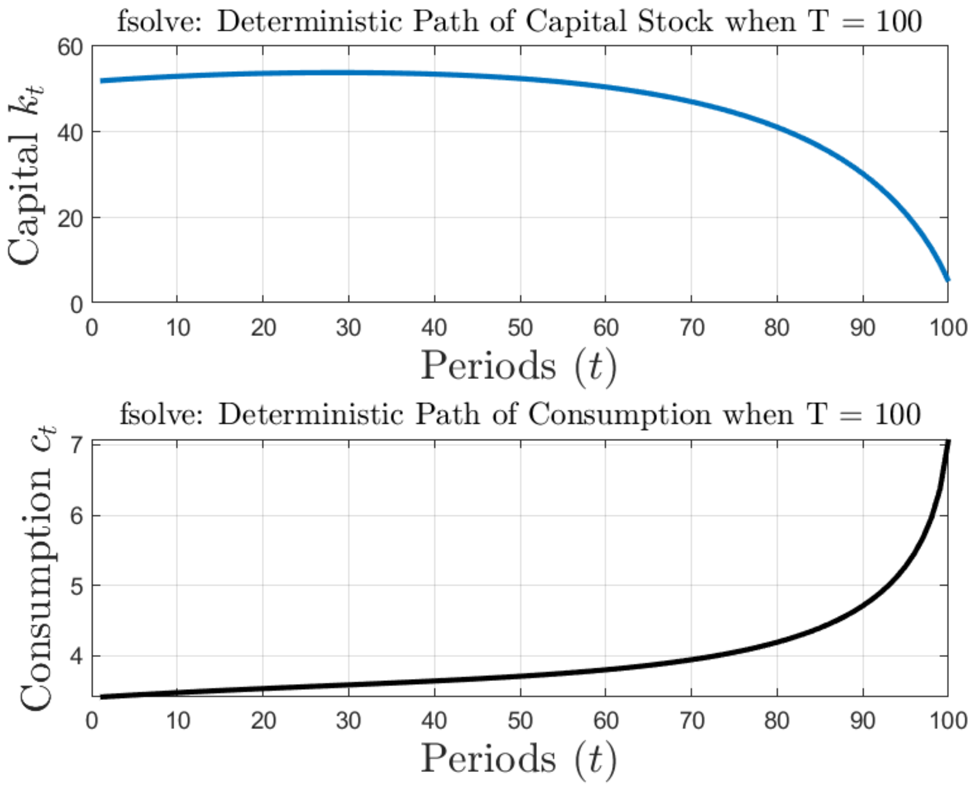

Figure 6: Deterministic Path of Consumption when T = 100

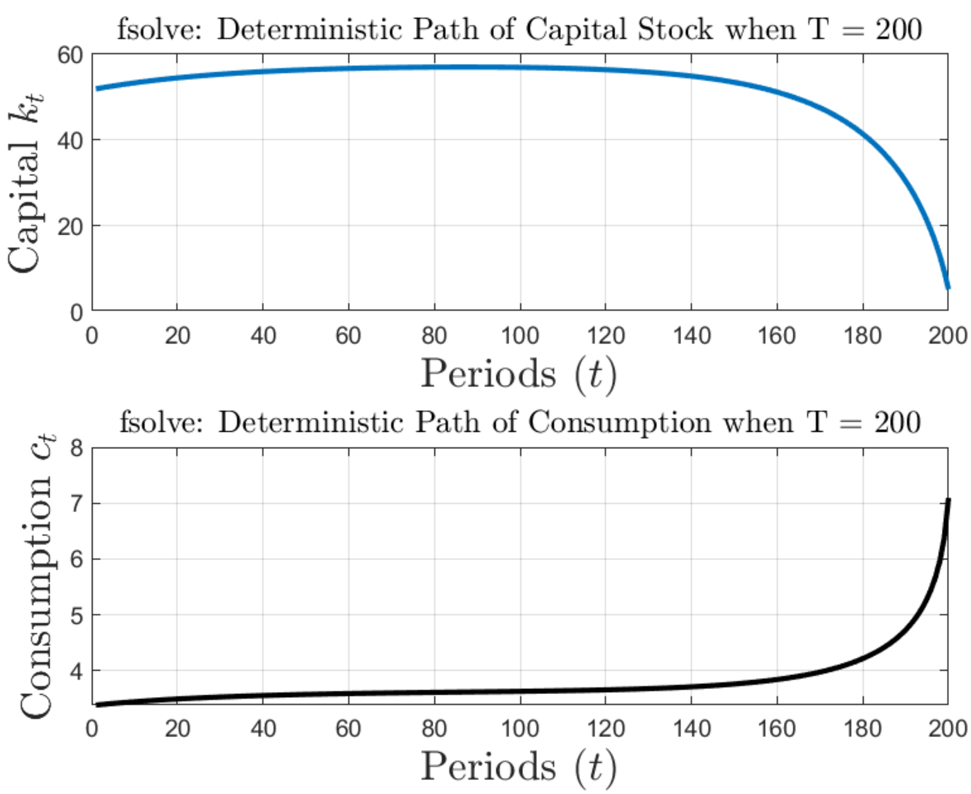

Figure 7: Deterministic Path of Consumption when T = 200

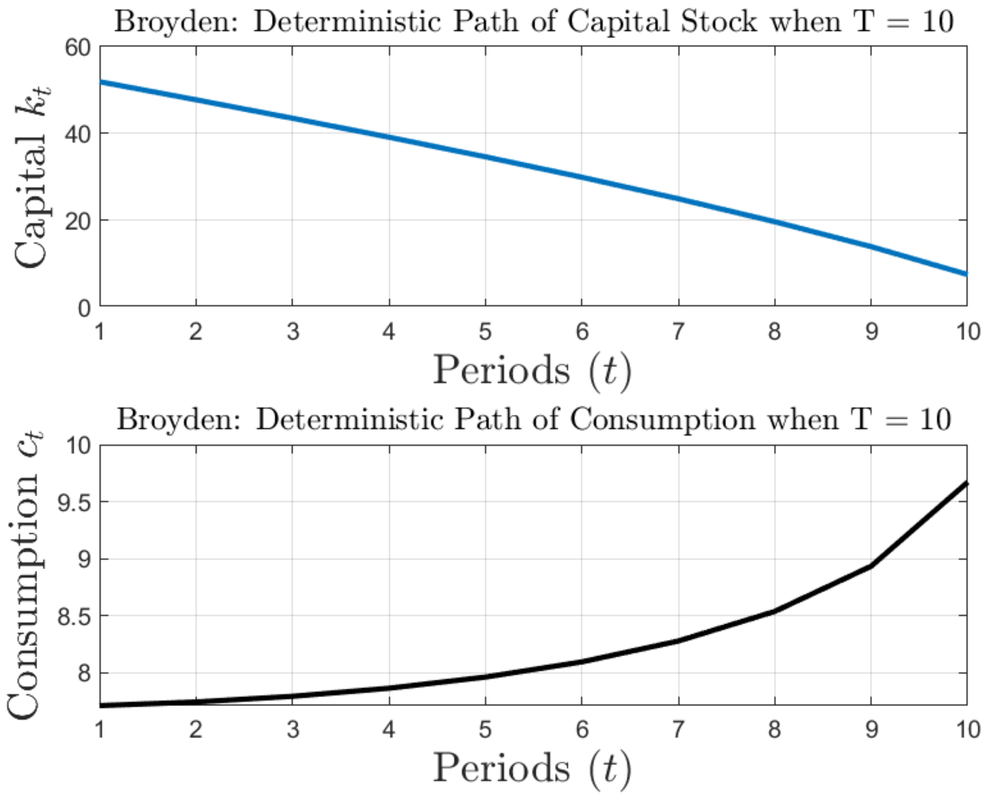

Figure 8: Finitely-Lived Agent, Broyden's Method, T =10

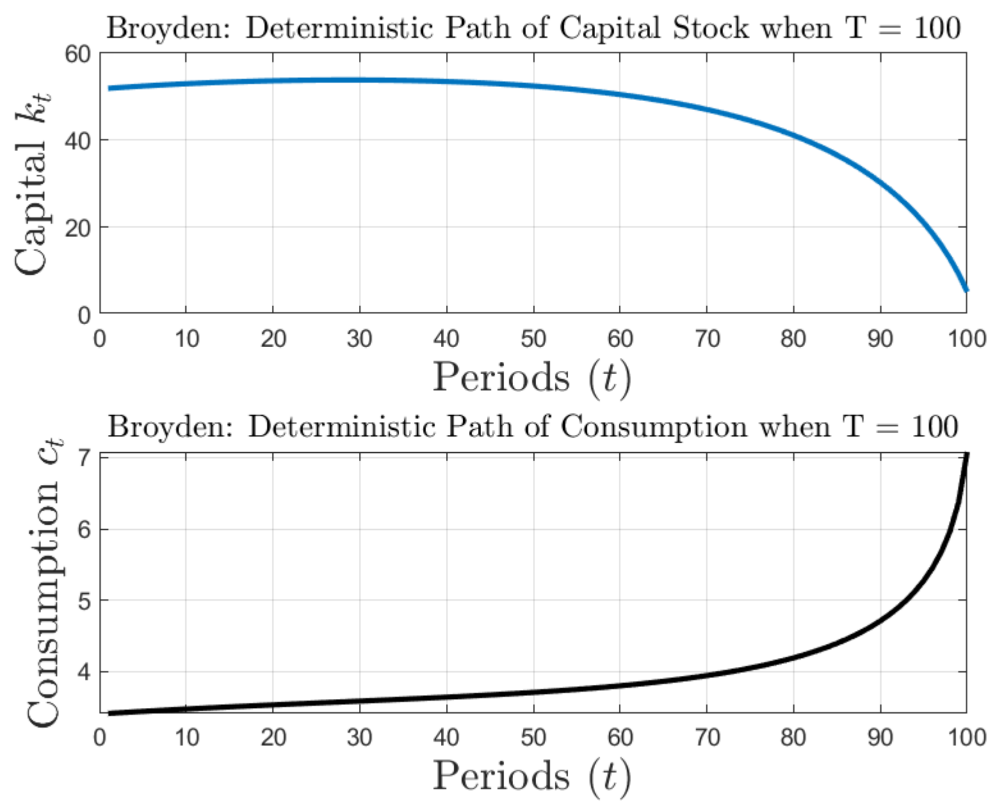

Figure 9: Finitely-Lived Agent, Broyden's Method, T =100

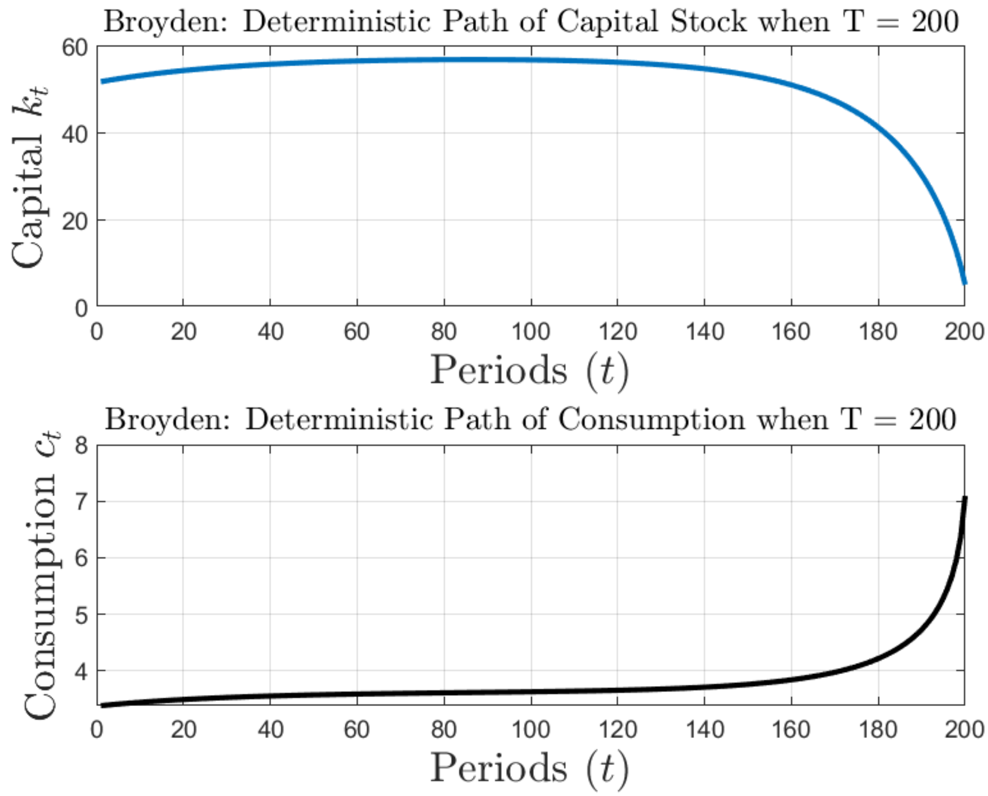

Figure 10: Finitely-Lived Agent, Broyden's Method, T =200

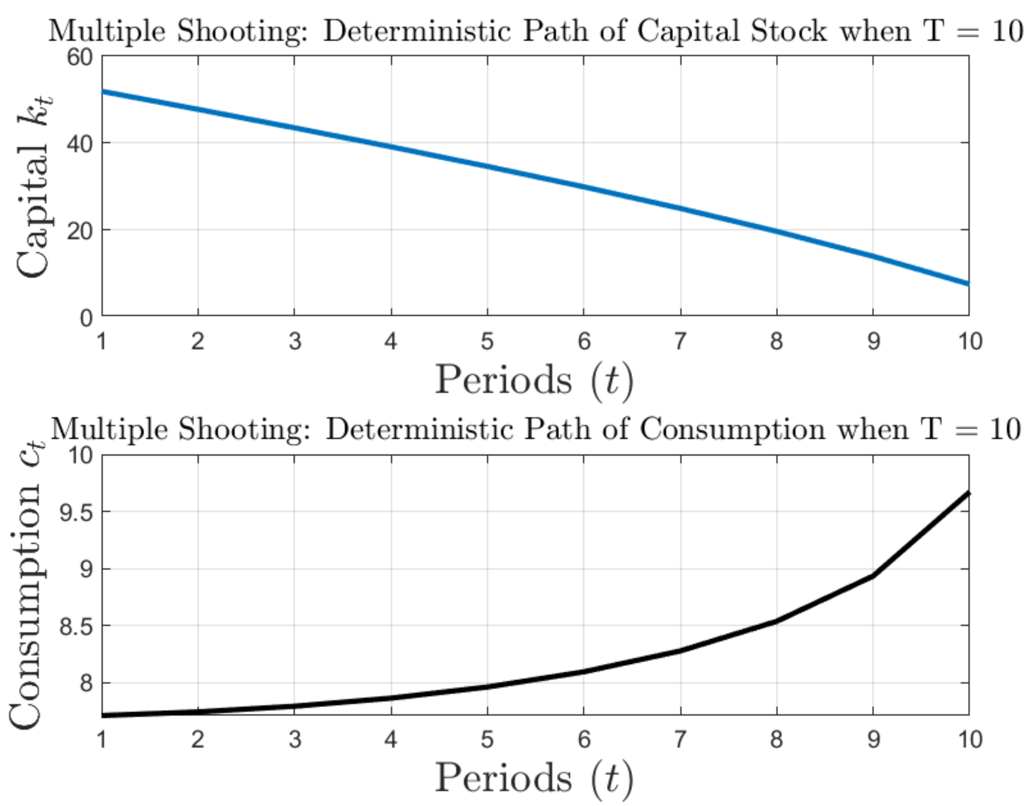

Figure 11: Finitely-Lived Agent, Reverse Multiple Shooting, T =10

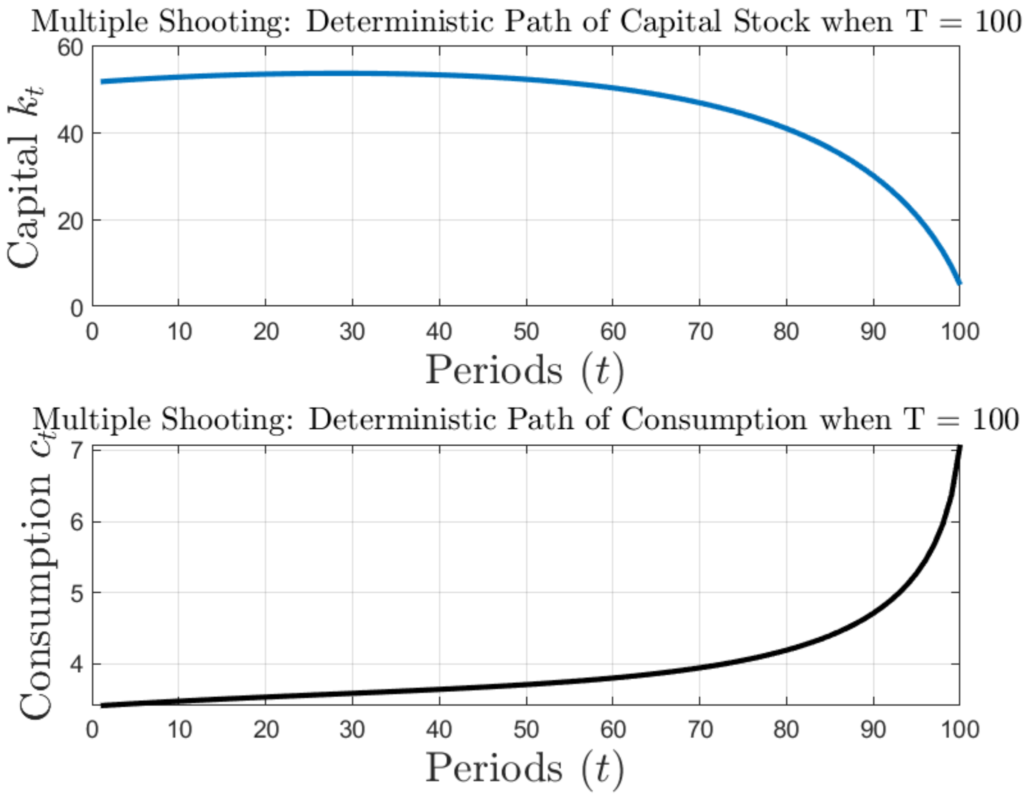

Figure 12: Finitely-Lived Agent, Reverse Multiple Shooting, T =100

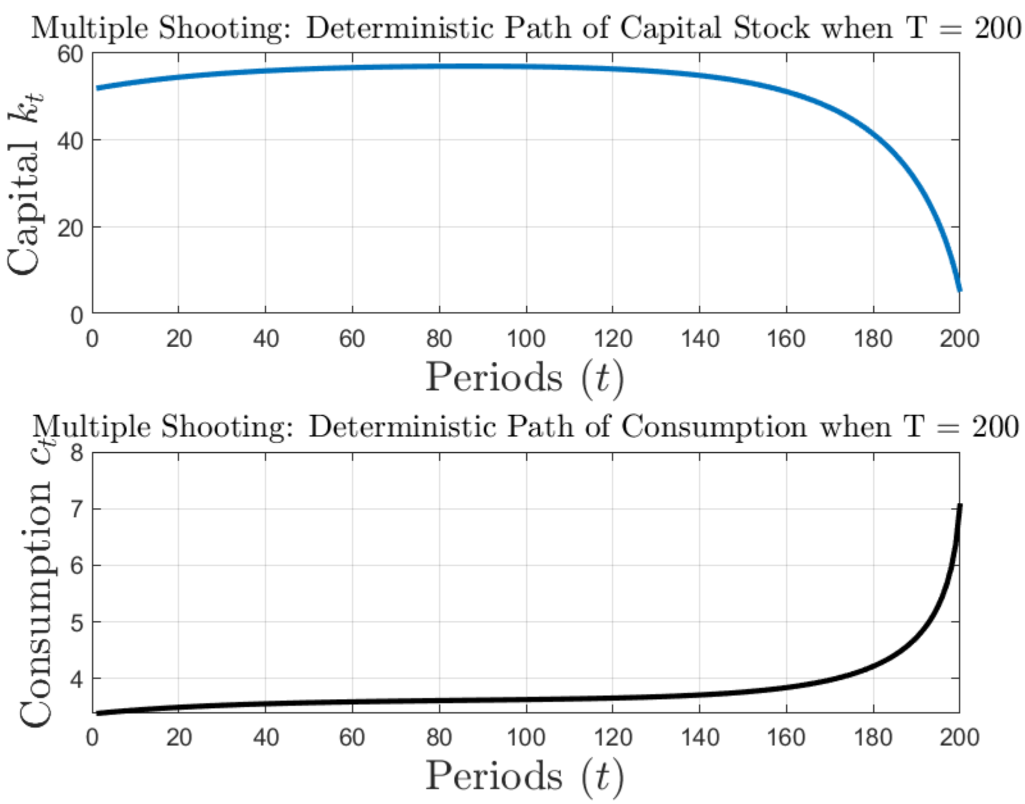

Figure 13: Finitely-Lived Agent, Reverse Multiple Shooting, T =200