Quantitative Macro Problem Set 2

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Consider the following problem solved by a social planner for $t = 0, 1, \ldots, T$.

$$\max_{\{c_t, k_{t+1}, i_t\}_{t=1}^{T}} E_0 \sum_{t=0}^{T} \beta^t u(c_t)$$
$$\text{s.t.}$$
$$c_t + i_t \leq k_t^\alpha$$
$$k_{t+1} = (1-\delta)k_t + i_t$$
$$k_0 \text{ given}$$
$$\text{for } u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

1. Calculate analytically the constant "steady-state" levels of capital $\bar{k}$, consumption $\bar{c}$, the net interest rate defined as the marginal product of capital minus depreciation, and the capital-output ratio. "Calibrate" the parameters β, α, and δ at a quarterly frequency, such that the steady state of the model replicates long-run features of developed economies, in particular an annualized net interest rate equal to 4 percent, an income share of capital (defined as the marginal product of capital times the stock of capital, divided by total output) equal to 1/3, and a quarterly capital-output ratio of 10. Set γ to 1.0001.

As we solved in the 1st problem set, the problem can be reduced to choice of $\{c_t, k_{t+1}\}_{t=1}^{T}$ with the solution characterized by the following equations system consist of Euler Equation and Budget Constraint:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}(1 - \delta + \alpha k_{t+1}^{\alpha-1}) \tag{1}$$
$$c_t + k_{t+1} = k_t^\alpha + (1-\delta)k_t \tag{2}$$

where t=1,2,...,T. Then, we can derive the closed form solutions that characterize steady state level of consumption ($\bar{c}$) and capital stock ($\bar{k}$) from:

$$1 = \beta(1 - \delta + \alpha \bar{k}^{\alpha-1}) \tag{3}$$
$$\bar{c} = \bar{k}^\alpha - \delta \bar{k} \tag{4}$$

$$\bar{k} = \left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}} \tag{5}$$
$$\bar{c} = \left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} - \delta\left[\frac{\alpha\beta}{1-\beta(1-\delta)}\right]^{\frac{1}{1-\alpha}} \tag{6}$$
$$\bar{r} = \alpha \bar{k}^{\alpha-1} - \delta \tag{7}$$
$$\frac{\bar{k}}{\bar{y}} = \bar{k}^{1-\alpha} \tag{8}$$

where net interest rate is defined as marginal product of capital minus depreciation rate. To calibrate the model at quarterly frequency, we have the following statistics that describe the real-world features of macroeconomics: The annualized interest rate is 4%, i.e. $(1+\bar{r})^4 - 1 = 0.04$. The income share of capital is $\frac{1}{3}$, i.e. $\frac{\alpha\bar{k}^{\alpha-1}\bar{k}}{\bar{y}} = \frac{1}{3}$, and capital output ratio is 10, i.e. $\frac{\bar{k}}{\bar{y}} = 10$.

$$\hat{\alpha} = \frac{\alpha\bar{k}^{\alpha}}{\bar{k}^{\alpha}} = \frac{\alpha\bar{k}^{\alpha}}{\bar{y}} = \frac{1}{3} \tag{9}$$

$$\hat{\delta} = \alpha\bar{k}^{\alpha-1} - \bar{r} = \frac{\frac{\alpha\bar{k}^{\alpha-1}\bar{k}}{\bar{y}}}{\frac{\bar{k}}{\bar{y}}} - \bar{r} = \frac{1}{30} - ((1+0.04)^{\frac{1}{4}} - 1) = 0.02347... \tag{10}$$

$$\hat{\beta} = (\alpha\bar{k}^{\alpha-1} + 1 - \hat{\delta})^{-1} = (1/30 + 1 - 0.02347)^{-1} = 0.99023... \tag{11}$$

$$\hat{\sigma} = 1.0001 \tag{12}$$

2. Now solve the model using log-linear techniques seen in class:

(a) Log-linearise the Euler equation for capital investment and the feasibility constraint around the deterministic steady state, and write the resulting system as

$$Ax_{t+1} = Bx_t$$

where $x_t = [c_t \ k_t]'$ is a column vector.

Recall that we calculated the Euler equation in problem set 1 as follows:

$$\begin{aligned} [(1-\delta)k_t + k_t^a - k_{t+1}]^{-\sigma} &= \beta[(1-\delta)k_{t+1} + k_{t+1}^a - k_{t+2}]^{-\sigma}(1 - \delta + \alpha k_{t+1}^{\alpha-1}) \\ c_t^{-\sigma} &= \beta c_{t+1}^{-\sigma}(1 - \delta + \alpha k_{t+1}^{\alpha-1}) \end{aligned} \tag{13}$$

By taking the log, we get

$$-\sigma Inc_t = In\beta - \sigma Inc_{t+1} + In(1 - \delta + \alpha k_{t+1}^{\alpha-1}) \tag{14}$$

Then, we apply first-order Taylor series expansion around the steady-state,

$$-\sigma In(\bar{c}) - \sigma\frac{c_t - \bar{c}}{\bar{c}} = In\beta - \sigma In(\bar{c}) - \sigma\frac{c_{t+1} - \bar{c}}{\bar{c}} + In(1 - \delta + \alpha\bar{k}^{\alpha-1}) + \frac{\alpha(\alpha-1)\bar{k}^{\alpha-2}}{1 - \delta + \alpha\bar{k}^{\alpha-1}}(k_{t+1} - \bar{k}) \tag{15}$$

and we know that 14 holds in every period. Thus, $-\sigma In(\bar{c}) = In\beta - \sigma In\bar{c} + In(1 - \delta + \alpha\bar{k}^{\alpha-1})$. Then, we can simplify 15 as

$$-\sigma\frac{c_t - \bar{c}}{\bar{c}} = -\sigma\frac{c_{t+1} - \bar{c}}{\bar{c}} + \alpha(\alpha-1)\bar{k}^{\alpha-2}\frac{k_{t+1} - \bar{k}}{1 - \delta + \alpha\bar{k}^{\alpha-1}} \tag{16}$$

Also, we can simplify the equation one more time by using the Euler equation at the steady state and get,

$$\beta = \frac{1}{1 - \delta + \alpha\bar{k}^{\alpha-1}}$$

Then, by using β and with some algebraic manipulation of 16, we obtain

$$-\sigma\frac{c_t - \bar{c}}{\bar{c}} = -\sigma\frac{c_{t+1} - \bar{c}}{\bar{c}} + \beta\alpha(\alpha-1)\bar{k}^{\alpha-1}\frac{k_{t+1} - \bar{k}}{\bar{k}} \tag{17}$$

Let's define $\hat{x}$ as a deviation from the steady state. Then, we can rewrite 17 as,

$$-\sigma\hat{c}_{t+1} + \beta\alpha(\alpha-1)\bar{k}^{\alpha-1}\hat{k}_{t+1} = -\sigma\hat{c}_t \tag{18}$$

Now, we will look at the feasibility constraint, $c_t = k_t^{\alpha} + (1-\delta)k_t - k_{t+1}$. Let's rewrite the constraint by taking the log,

$$In(c_t + k_{t+1}) = In(k_t^{\alpha} + (1-\delta)k_t) \tag{19}$$

Then, apply the first-order Taylor expansion,

$$In(\bar{c}+\bar{k}) + \frac{c_t - \bar{c}}{\bar{c}+\bar{k}} + \frac{k_{t+1}-\bar{k}}{\bar{c}+\bar{k}} = In(\bar{k}^{\alpha} + (1-\delta)\bar{k}) + \frac{\alpha\bar{k}^{\alpha-1}+1-\delta}{\bar{k}^{\alpha}+(1-\delta)\bar{k}}(k_t - \bar{k}) \tag{20}$$

Simplify 20 as we did before for steady state, and multiply by $\frac{\bar{c}+\bar{k}}{\bar{k}}$ and get

$$\frac{\bar{c}}{\bar{k}}\frac{c_t-\bar{c}}{\bar{c}} + \frac{k_{t+1}-\bar{k}}{\bar{k}} = (\alpha\bar{k}^{\alpha-1}+1-\delta)\frac{k_t-\bar{k}}{\bar{k}}$$

and by using the notation for steady-state deviations, we obtain

$$\hat{k}_{t+1} = -\frac{\bar{c}}{\bar{k}}\hat{c}_t + (\alpha\bar{k}^{\alpha-1}+1-\delta)\hat{k}_t \tag{21}$$

Now, let's return to our problem of representing the system as $Ax_{t+1} = Bx_t$. By using equations 18 and 21, our system becomes,

$$\begin{bmatrix} -\sigma & \beta\alpha(\alpha-1)\bar{k}^{\alpha-1} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} \hat{c}_{t+1} \\ \hat{k}_{t+1} \end{bmatrix} = \begin{bmatrix} -\sigma & 0 \\ -\frac{\bar{c}}{\bar{k}} & \alpha\bar{k}^{\alpha-1}+1-\delta \end{bmatrix} \times \begin{bmatrix} \hat{c}_t \\ \hat{k}_t \end{bmatrix} \tag{22}$$

where

$$A = \begin{bmatrix} -\sigma & \beta\alpha(\alpha-1)\bar{k}^{\alpha-1} \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -\sigma & 0 \\ -\frac{\bar{c}}{\bar{k}} & \alpha\bar{k}^{\alpha-1}+1-\delta \end{bmatrix}$$

(b) Transform this equation to

$$x_{t+1} = Dx_t$$

We can see that $x_{t+1} = A^{-1}Bx_t$ where $D = A^{-1}B$. We find the inverse of A as

$$A^{-1} = \begin{bmatrix} -\frac{1}{\sigma} & \frac{\beta\alpha(\alpha-1)\bar{k}^{\alpha-1}}{\sigma} \\ 0 & 1 \end{bmatrix}$$

Then, applying $A^{-1}B$, we can get the D. On the other hand, we can apply another method to obtain D, which is Schur decomposition such that $D = QUQ^{-1}$ where U is the diagonalized and upper triangular matrix whose diagonal entries are eigenvalues, and Q is an orthogonal matrix.

(c) Diagonalise the system and solve for the policy rule for consumption $\hat{c}_t = \alpha\hat{k}_t$.

As mentioned above, to diagonalize the system, we need to find the eigenvalues of D and its eigenvectors such that $D = QUQ^{-1}$ where $U = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

Let's rewrite,

$$\begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} = QU^tQ^{-1}\begin{pmatrix} \hat{c}_0 \\ \hat{k}_0 \end{pmatrix} \tag{23}$$

Then, we show,

$$\begin{aligned} Q^{-1}\begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} &= U^tQ^{-1}\begin{pmatrix} \hat{c}_0 \\ \hat{k}_0 \end{pmatrix} \\ Q^{-1}\begin{pmatrix} \hat{c}_t \\ \hat{k}_t \end{pmatrix} &= \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix} Q^{-1}\begin{pmatrix} \hat{c}_0 \\ \hat{k}_0 \end{pmatrix} \end{aligned} \tag{24}$$

Assume $Q^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$. Then,

$$\begin{pmatrix} q_{11}\hat{c}_t + q_{12}\hat{k}_t \\ q_{21}\hat{c}_t + q_{22}\hat{k}_t \end{pmatrix} = \begin{pmatrix} \lambda_1^t & 0 \\ 0 & \lambda_2^t \end{pmatrix}\begin{pmatrix} q_{11}\hat{c}_0 + q_{12}\hat{k}_0 \\ q_{21}\hat{c}_0 + q_{22}\hat{k}_0 \end{pmatrix} \tag{25}$$

When time goes to infinity, $q_{11}\hat{c}_t + q_{12}\hat{k}_t$ will go to infinity if $q_{11}\hat{c}_0 + q_{12}\hat{k}_0 > 0$. Note that according to Blanchard-Kahn condition, for the system to have a unique, stable solution, number of eigenvalues for matrix D with an absolute value above 1 should equal to number of non-predetermined control variables, which is 1 in the above system. Therefore, while one of the eigenvalues is greater than one, the other one should be less than one. Intuitively, there exists only one c_0 such that we end up in the saddle path. Then, see that

$$q_{11}\hat{c}_t + q_{12}\hat{k}_t = \lambda_1^t(q_{11}\hat{c}_0 + q_{12}\hat{k}_0) \tag{26}$$

Whenever time goes to infinity, this linear combination goes to infinity when $q_{11}\hat{c}_0 + q_{12}\hat{k}_0 > 0$ assuming $\lambda_1^t > 1$ and Blanchard-Kahn condition is satisfied. On the other hand, if we choose c_0 optimally, this system goes to a steady state. In other words, if we want our system to converge to a steady state, the following equation should be satisfied $q_{11}\hat{c}_0 + q_{12}\hat{k}_0 = 0$, ie. there exists a unique c_0 such that we end up at saddle path. Then, we can get our policy function for time t as

$$\hat{c}_t = -\frac{q_{12}}{q_{11}}\hat{k}_t \tag{27}$$

The results of calculation in matlab are as follow:

$$Q = \begin{pmatrix} -0.4894 & 0.4296 \\ -0.8720 & -0.9030 \end{pmatrix}, Q^{-1} = \begin{pmatrix} -1.1058 & -0.5261 \\ 1.0679 & -0.5993 \end{pmatrix}, \tag{28}$$

And thus we have $\hat{c}_t = 0.5613\hat{k}_t$

(d) Draw the time series of c_t and k_t when the initial level of capital equals k_0 $= 0.9\bar{k}$. Comment. (Feel free to compare to that you solved for in the previous problem set).

The transition paths generated by the log-linearization method is shown in figure 1. We also include figure 2 generated by the multiple shooting algorithm. The two figures are very similar in terms of general transition path and convergence speed. In fact, the results generated by these two methods do not differ much from each other in this deterministic model with a nicely chosen initial condition, yet log-linearization could be computationally easier and less demanding for trials of different c_0s. Overall, the error between these two methods is at the level of $1e-3$.
Why does log-linearization generate such a small error? Log-linearization performs well with small deviations from steady state by the nature of taylor approximation. In our setting, k_0 is only slightly deviated from $\bar{k}$ with $k_0 = 0.9\bar{k}$, thus $k_t = \bar{k}e^{\hat{k_t}}$ doesn't deviate much from $k_t = \bar{k}(\hat{k_t}+1)$, as $\hat{k_t} = log(\frac{k_t-\bar{k}}{\bar{k}}) \approx -0.1$ is close to zero.
In addition, the results show the typical pattern of consumption smoothing under risk-averse utility structure: the consumer saves by accumulating capital, which maximizes the discounted utility of all future consumption and enables an increasing consumption path for the consumer in return. Both capital stock and consumption levels converge to steady state levels around the 110th period. With the log-linearization method that we can derive by hand, λs provide informative insights on the convergence speed of the model without even solving the policy function through matlab.

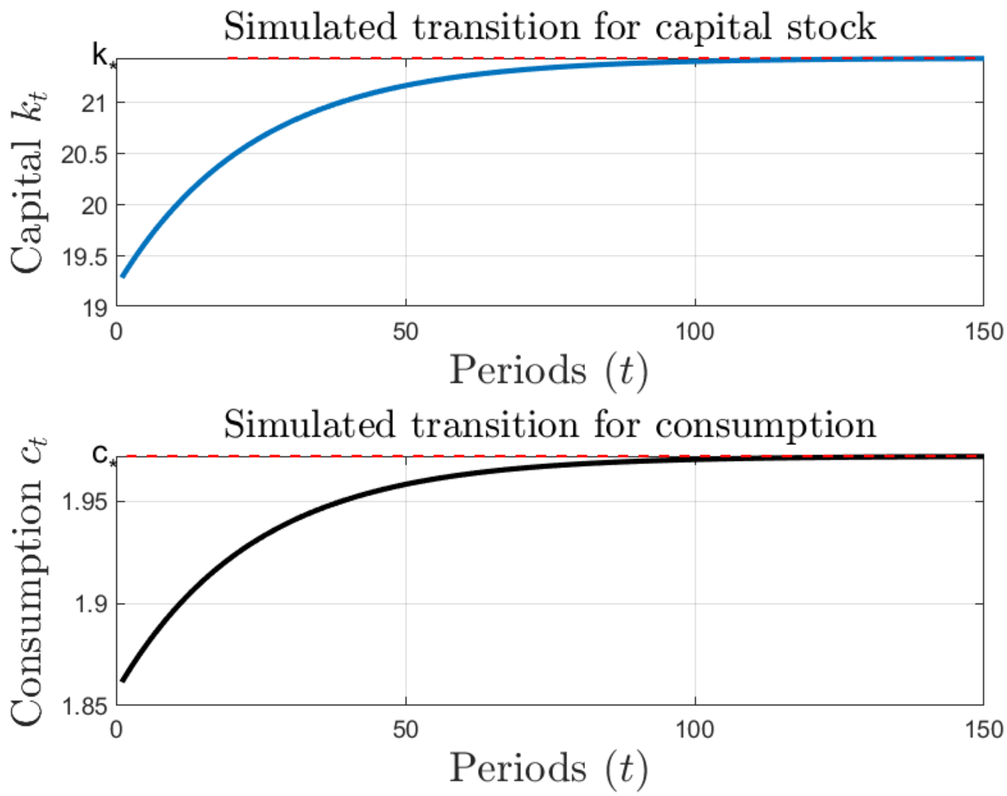

Figure 1: Time Series, Inelastic Labor Supply, Log-Linearization

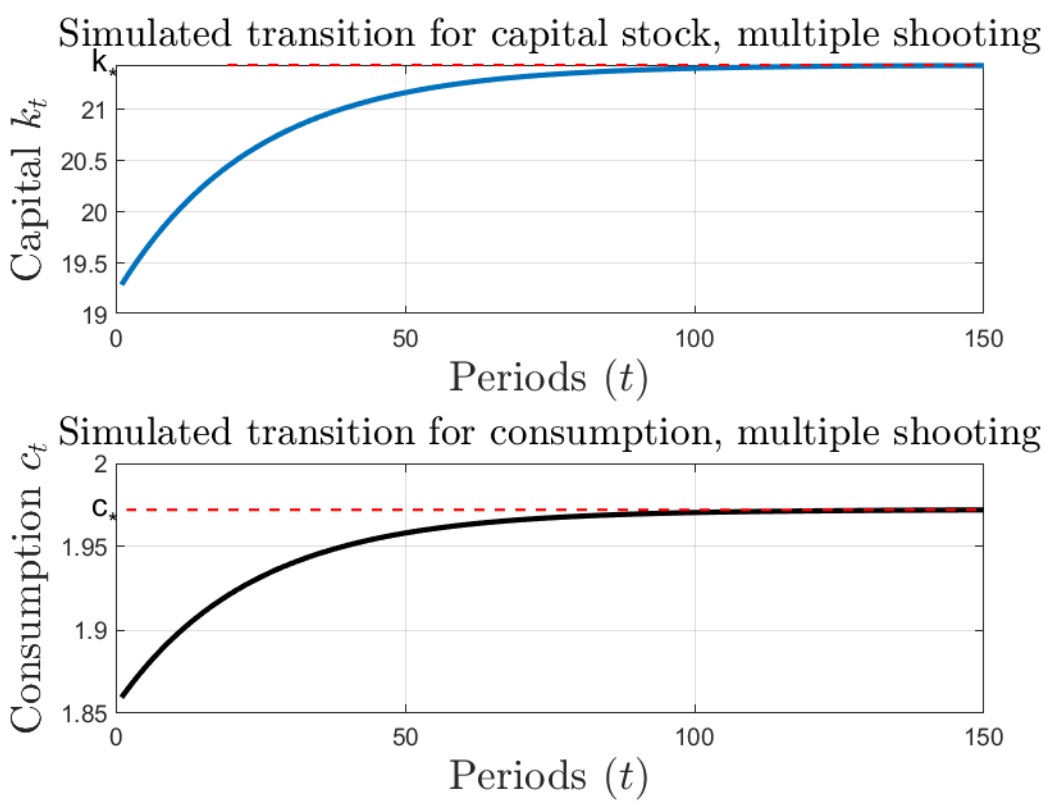

Figure 2: Time Series, Inelastic Labor Supply, Multiple Shooting

3. Now consider an alternative version of the model, where labor supply l_t is time-varying. In particular, consider a period utility function that also takes labor as an argument

$$u(c_t, l_t) = u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \theta \frac{l_t^{1+\mu}}{1+\mu}$$

(a) Derive the first-order condition for labor supply. Calculate the steady- state labor supply $\bar{l}$.

We use the optimal control method:

$$\begin{aligned} V(k_t) &= \max_{c_t, l_t} [u(c_t, l_t) + \beta V(k_{t+1})] \\ &= \max_{c_t, l_t} [u(c_t, l_t) + \beta V((1-\delta)k_t + k_t^{\alpha} l_t^{1-\alpha} - c_t)] \end{aligned} \tag{29}$$

FOCs with respect to c_t and l_t give:

$$\begin{aligned} c_t^{-\sigma} &= \beta V'(k_{t+1}) \\ -\theta l_t^{\mu} + \beta V'(k_{t+1}) \cdot k_t^{\alpha} l_t^{\alpha} (1-\alpha) &= 0 \end{aligned}$$

Envelope theorem:

$$V'(k_t) = \beta V'(k_{t+1})[(1-\delta) + \alpha k_t^{\alpha-1} l_t^{1-\alpha}]$$

Thus, with the budget constraint, we have the three equations for the dynamic system:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}[(1-\delta) + \alpha k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha}] \tag{30}$$

$$\theta l_t^{\mu+\alpha} = (1-\alpha) c_t^{-\sigma} k_t^{\alpha} \tag{31}$$

$$k_{t+1} = (1-\delta)k_t + k_t^{\alpha} l_t^{1-\alpha} - c_t \tag{32}$$

In the steady-state, we have

$$1 = \beta[(1-\delta) + \alpha \bar{k}^{\alpha-1} \bar{l}^{1-\alpha}] \tag{33}$$

$$\theta \bar{l}^{\mu+\alpha} = (1-\alpha) \bar{c}^{-\sigma} \bar{k}^{\alpha} \tag{34}$$

$$\bar{k} = (1-\delta)\bar{k} + \bar{k}^{\alpha} \bar{l}^{1-\alpha} - \bar{c} \tag{35}$$

We denote $m \equiv \frac{\bar{k}}{\bar{l}}$. From equation 33, we know that $m = \left(\frac{\alpha}{1/\beta - (1-\delta)}\right)^{1/(1-\alpha)}$.
Substitute $\bar{c}$ in equation 34 using equation 35. And we get:

$$\theta \bar{l}^{\mu+\sigma} = (1-\alpha)(m^{\alpha} - \delta m)^{-\sigma} m^{\alpha} \tag{36}$$

$\bar{l}$ is the solution to the above equation.

(b) Calibrate θ to have $\bar{l} = 1/3$. Use $\mu = 1$.

We set $\bar{l}$ to 1/3 and let $\mu = 1$, and then we can solve or θ in equation 36. In matlab, we get $\hat{\theta} = 8.0122$. A positive value of θ characterizes disutility with labor supply as expected.

(c) Log-linearise the first-order condition for labor supply in period t. Write the resulting system as $Ax_{t+1} = Bx_t$.

Log-linearize equation 30 to equation 32, we get:

$$\hat{k}_{t+1} = [(1-\delta) + \alpha \bar{k}^{\alpha-1} \bar{l}^{1-\alpha}]\hat{k}_t + (1-\alpha)\bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \hat{l}_t - \frac{\bar{c}}{\bar{k}} \hat{c}_t \tag{37}$$

$$-\sigma \hat{c}_t = -\sigma \hat{c}_{t+1} + \beta\alpha(\alpha-1)\bar{l}^{1-\alpha} \bar{k}^{\alpha-1} \hat{k}_{t+1} + \beta\alpha(1-\alpha)\bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \hat{l}_{t+1} \tag{38}$$

$$(\mu+\alpha)\hat{l}_t = -\sigma \hat{c}_t + \alpha \hat{k}_t \tag{39}$$

Rewrite in the form of $Ax_{t+1} = Bx_t$, where $x_t = \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \\ \hat{l}_t \end{pmatrix}$. We get:

$$\begin{pmatrix} 1 & 0 & 0 \\ \beta\alpha(\alpha-1)\bar{l}^{1-\alpha}\bar{k}^{\alpha-1} & -\sigma & \beta\alpha(1-\alpha)\bar{k}^{\alpha-1}\bar{l}^{1-\alpha} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \\ \hat{l}_{t+1} \end{pmatrix}$$
$$= \begin{pmatrix} (1-\delta) + \alpha\bar{k}^{\alpha-1}\bar{l}^{1-\alpha} & -\frac{\bar{c}}{\bar{k}} & (1-\alpha)\bar{k}^{\alpha-1}\bar{l}^{1-\alpha} \\ 0 & -\sigma & 0 \\ -\alpha & \sigma & \mu+\alpha \end{pmatrix} \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \\ \hat{l}_t \end{pmatrix} \tag{40}$$

(d) Use the Schur decomposition of A and B to solve the system. For this, you can use pre-programmed routines, like Paul Klein's solab.m (which you can download from his website).

For this part, our main reference is *Economic Dynamics in Discrete Time* by Miao Jianjun.
We use the complex generalized Shur form, or the "QZ decomposition", to deal with the singularity problem. When A and B are regular, we can use the QZ decomposition, meaning that there is α such that the determinant:

$$det(A\alpha - B) = 0 \tag{41}$$

Theorem 1 *(**complex generalized Schur form**) Let the $n \times n$ matrices A and B be regular. Then there exist $n \times n$ unitary matrices of complex numbers Q and Z such that:*

1. *$QAZ = S$ is upper triangular;*
2. *$QBZ = T$ is upper triangular;*
3. *For each i, s_{ii} and t_{ii} are not both zero;*
4. *$\lambda(A,B) = \{t_{ii}/s_{ii} : s_{ii} \neq 0\}$;*
5. *The pairs (s_{ii}, t_{ii}), $i = 1, ..., n$ can be arranged in any order.*

Thus, we have $A = Q^H S Z^H$ and $B = Q^H T Z^H$, where $Q^H = Q^{-1}$ and $Z^H = Z^{-1}$ are the conjugate transpose of Q and Z respectively. Denote $x_t^* = Z^H x_t$.
We use the following partition:

$$x_t = \begin{bmatrix} k_t \\ y_t \end{bmatrix} \begin{matrix} n_k \times 1 \\ n_y \times 1 \end{matrix}, \quad x_t^* = \begin{bmatrix} s_t \\ u_t \end{bmatrix} \begin{matrix} n_s \times 1 \\ n_u \times 1 \end{matrix},$$

where k_t is the vector of the state variables, y_t is the vector of the control variables.
Also, we partition Z accordingly:

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix},$$

We have:

$$Sx_{t+1}^* = Tx_t^*$$

that is,

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix}, \tag{42}$$

where s_t corresponds to the rows of state variables in x_t, and u_t corresponds to the rows of control variables in x_t.

By the ordering of stable generalized eigenvalues, S_{11} is invertible because all diagonal elements s_{ii} of S_{11} cannot be zero. In addition, T_{22} is also invertible because all diagonal elements t_{ii} of T_{22} cannot be zero. Thus, we can derive the formula:

$$
\begin{aligned}
u_t &= T_{22}^{-1} S_{22} u_{t+1} \\
S_{11} s_{t+1} + S_{12} u_{t+1} &= T_{11} s_t + T_{12} u_t.
\end{aligned}
$$

Since $\lim_t u_t = 0$ (convergence back to steady state), we know that $u_t = 0, \forall t$. This echoes with the case of non-singular A, where we set the variables corresponding to non-stable eigenvalues to zero.
Thus, we have

$$
u_t = 0, \forall t \tag{43}
$$

$$
s_{t+1} = S_{11}^{-1} T_{11} s_t \tag{44}
$$

Then we can solve the whole system using $x_t = Z x_t^*$:

$$
\begin{aligned}
k_{t+1} &= Z_{11} s_{t+1} = Z_{11} S_{11}^{-1} T_{11} s_t = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} k_t, \\
y_t &= Z_{21} s_t = Z_{21} Z_{11}^{-1} k_t.
\end{aligned}
$$

Note that we need to check whether Z_{11} is invertible or not first.
We use the `solab.m` as recommended. The result is shown in figure 3

(f) Draw the time series of c_t, k_t and l_t when the initial level of capital equals k0 $=$ 0.9 $\bar{k}$. Compare to that in 2.

We use the `solab.m` as recommended. One could plug the three equation linearized difference system into Dynare as well to check the answer. The result is shown in figure 3.
Consumption c_t and capital stock k_t are rising as time goes by, which is consistent with the case without endogenous labor supply. The consumption smoothing motive exists as before since we preserve risk-aversion in the utility specification. The agent accumulates capital and smooths his consumption over time to benefit from a stable return from capital stock. At the same time, the working hour is declining. The agent does no need to work as much as before in later periods, as he has already accumulated enough capital for a smoothed consumption path and stable level of c_t.
Note that the steady-state level of capital now is 7.1454, substantially lower than the steady-steady capital in question 2. Consumption level $\bar{c}$ is also lower under elastic labour supply. This is because the consumer dislikes working, and labour enters into the budget constraints simultaneously. Thus, he would consume less and accumulate less capital to accommodate his distaste for working. One can modify variable choice of Frisch elasticity μ, relative disutility control θ, and labour return rate $1 - \alpha$ to see how elastic labour choice decreases steady state level $\bar{c}$. This is also a result of our calibration: $\bar{l}$ is set to be $1/3$. In addition, by merit of labour entering both utility and production function in nonlinear manners, while preserving parametrization of patience, risk aversion, and diminishing return level, the paths converge to steady state around the 88th period, which is faster than inelastic-labour model.

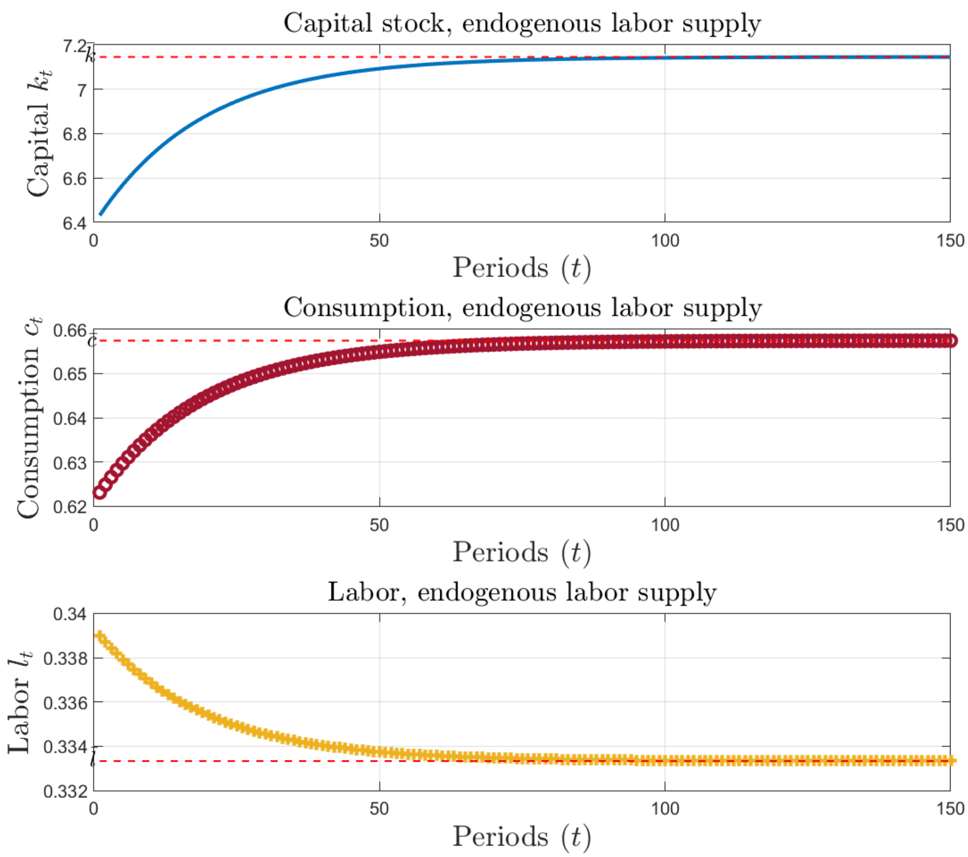

Figure 3: Time Series, Endogenous Labor Supply