Quantitative Macroeconomics II

Solving HANK models

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HANK: Main ingredients

- Optimising household and firms
- Frictions: incomplete markets and nominal rigidity
- Idiosyncratic and aggregate shocks
- Equilibrium: Labor, goods and capital markets clear

HANK: Solution methods

- Infinite-dimensional state variable (cross-sectional distribution) in decision problem
- Requires approximation:
  1. Reduced-information methods (following Krusell-Smith 1998, Algan et al 2014)
  2. State-space approach with linearization (following Reiter 2009)
  3. Sequence-based method (following Boppart et al 2018)

Solution methods 1: Reduced-information methods (KS 98)

- Summarise cross-sectional distribution by a small number of moments M_t
- Solve individual problem given LOM $M_{t+1} = \Gamma(M_t)$, update Γ
- Update can be through simulation, explicit aggregation of pol function, etc.
- $M_t = \{\bar{k}_t\}$ suffices in original KS 98 model ("near aggregation")
- Other models require more moments (e.g. bond economy with constant B)
- See Algan et al 2014 for a comparision of different versions.
- Recently: Machine-learning techniques can endogenously condense information in distribution (Fernandez-Villaverde et al. 2021; Maliar et al. 2021)

Solution method II: State-space approach with linearization

- Preston & Roca (06): Perturb around deterministic SS (no agg, no idios risk - so difficult with e.g. unemployment)
- Reiter (09): ... around stat. distribution, aggregate shock ϵ_t
  1. Discretize pol fct C (spline on n_p pts) and distribitution Ψ (histogram with $n_d = n'_d \times n_z$ bins)
  2. Use param.s $\Phi_{c,t}$ of $C/\Phi_{\Psi,t}$ of Ψ as jump/state variables in
  $$H(\Phi_{c,t}, \Phi_{c,t+1}, \Phi_{d,t-1}, \Phi_{d,t}, \epsilon_t) = 0 \tag{1}$$
  consisting of optimality cond.s and LOM for distribution
  3. Solve for SS.
  4. For H_i^{SS} the partial derivate w.r.t. to ith argument at SS, solve
  $$\begin{aligned} & H_1^{SS}(\Phi_{c,t} - \Phi_{c,SS}) + H_2^{SS}(\Phi_{c,t+1} - \Phi_{c,SS}) \\ & + H_3^{SS}(\Phi_{d,t-1} - \Phi_{d,SS}) + H_4^{SS}(\Phi_{d,t} - \Phi_{d,SS}) + H_5^{SS}(\epsilon_t - \bar{\epsilon}) = 0 \end{aligned}$$
- More recently: Ahn et al. (2018); Bayer and Luetticke (2020); Bhandari et al. (2023); Bilal (2023)

Solution Methods III: Sequence-based

- Deterministic, perfect-foresight path of end. variables given agg. shock $\{\epsilon\}_{t=0}^{T}$
- Non-linear equation system in N variables $Z_t, t = 1, ..., T$
- Boppart et al (18): IRF as numerical derivative
  1. Solve for $\mathrm{Z} = [Z_1', Z_2', ..., Z_T']'$ using relaxation algorithm
  2. Check linearity: $\mathrm{Z}(\epsilon + b\epsilon') \stackrel{?}{=} \mathrm{Z}(\epsilon) + b\mathrm{Z}(\epsilon')$
  3. If passed, can simulate model
- Auclert et al (21): Solve efficiently using sequence-space Jacobian

Solution Methods compared

1. Sequence-based:
    - Abstracts from aggregate uncertainty
    - But allows simulation of (approx) linear models
2. Perturbation:
    - Equivalent to 1. if first-order
    - But 2nd order possible
3. Information-reduction (KS 98)
    - Impossible for many / continuous exogenous shocks
    - Active research area using machine-learning
4. Key challenge: choose method adequate for purpose

Sequence-based solution to dynamic models: Details

- Model maps n_x $1 \times T$ paths $\boldsymbol{X}$ of unknowns to $n_x \times T$ equation errors / targets given n_z $1 \times T$ paths of shocks $\boldsymbol{Z}$
- Solution given by $n_x \times T$ vector of equations

$$F(\boldsymbol{X}, \boldsymbol{Z}) = 0 \tag{2}$$

- In principle: many algorithms for solving for unknowns $\boldsymbol{X}$
- First step: reduce number of unknowns:
    1. Arrange model "blocks" as "Directed acyclical graph" (DAG)
    2. Block i maps inputs (shocks, unknowns, outputs X^j from blocks $j < i$) to outputs
    3. Partition $F = [F^1, F^2, F^3, ...F^M]'$, where F^j directly maps $[X^1, ..., X^{j-1}]'$ into block j'th outputs X^j
    4. Solve $F^M(X^1, \boldsymbol{Z}) = F^M(X^1, X^2(X^1), ..., \boldsymbol{Z}) = 0$ instead
    5. (Similar to direct substitution of variables)

Sequence-based solution to dynamic models: NC growth model

1. Unknowns $\{\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{r}, \boldsymbol{w}, \boldsymbol{Y}\}$
2. NC firm block $\{\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{Z}\} \longrightarrow \{\boldsymbol{Y}, \boldsymbol{r}, \boldsymbol{w}\}$
3. HH block $\{\boldsymbol{r}, \boldsymbol{w}\} \longrightarrow \{\mathbb{L}, \mathbb{K}\}$
4. Simple Market-clearing block ($\epsilon^K = \boldsymbol{K} - \mathbb{K}$, $\epsilon^L = \boldsymbol{L} - \mathbb{L}$)
5. Produces a $2xT$ system

$$H(\boldsymbol{K}, \boldsymbol{Z}) = \boldsymbol{K} - \mathbb{K} = 0 \quad (3)$$

$$\boldsymbol{L} - \mathbb{L} = 0$$

$$[\boldsymbol{r}, \boldsymbol{w}, \boldsymbol{Y}]' = M(\boldsymbol{K}, \boldsymbol{L}, \boldsymbol{Z}) \quad (4)$$

Sequence-based solution to HA models

- Typically can use aggregate inputs $\boldsymbol{X}$ and outputs $\boldsymbol{Y}$ only
- HA-block often maps prices & exogenous shocks (& non-labor income components) into goods & asset demand
- E.g. KS98 model combines
  1. Simple NC firm block $(\{\boldsymbol{K}, \boldsymbol{Z}\} \longrightarrow \{\boldsymbol{Y}, \boldsymbol{r}, \boldsymbol{w}\})$,
  2. HA block $(\{\boldsymbol{r}, \boldsymbol{w}\} \longrightarrow \{\mathbb{K}, \boldsymbol{r}, \boldsymbol{w}\})$
  3. Simple Market-clearing block $(\epsilon = \boldsymbol{K} - \mathbb{K})$
- Direct substitution of $\boldsymbol{r}, \boldsymbol{w}$ produces a system

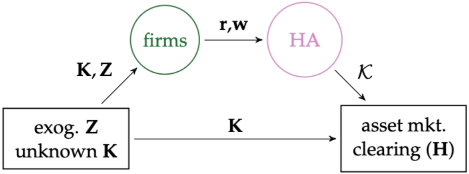
$$H(\boldsymbol{K}, \boldsymbol{Z}) = \boldsymbol{K} - \mathbb{K} = 0 \tag{5}$$

$$[\boldsymbol{r}, \boldsymbol{w}, \boldsymbol{Y}]' = M(\boldsymbol{K}, \boldsymbol{Z}) \tag{6}$$

- Solution only requires solving (5).

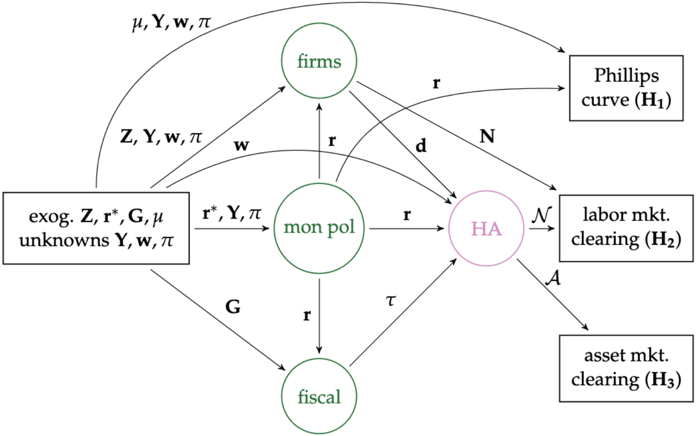
Auclert et al (21): KS 98 as DAG

Figure 3: DAG representation of Krusell-Smith economy



Auclert et al (21): HANK as DAG

Figure C.1: DAG representation of one-asset HANK economy



Auclert et al (21): Efficient 1st order solution

- Want: 1st order GE impulse response in SS: $d\boldsymbol{X} = F_1^{-1} F_2 d\boldsymbol{Z}$
- Requires: HA-block Jacobian to solve $d\boldsymbol{U} = H_1^{-1} H_2 d\boldsymbol{Z}$ w. H_1 $nTxnT$
- E.g. KS98: $d\boldsymbol{K} = H_1^{-1} H_2 d\boldsymbol{Z}$
- How to find Jacobian H_1? (Also for nonlinear, e.g. Broyden).
- Direct computation
  - Compute responses to $n_u \times T$ paths with single, scalar shock in s $e^{j,s}$, $j = 1, .., n_u$, $s = 1, ..., T$ of individual pol. fcts $\boldsymbol{y}_t^{j,s}$ and distribution $\boldsymbol{D}_t^{j,s}$.
  - For each path:
    1. Solve HH policy backwards from T to $t = 1$
    2. Simulate HH distribution forward from $t = 1$ to T.
    3. Aggregation gives element $H_1^{1:nT,(j-1)*T+s}$
  - Requires T^2 solutions and simulations.

Auclert et al (21): Fake-news algorithm to compute H_1

Auclert et al (21): Fake-news algorithm - Two insights

Response $d\boldsymbol{X}_t^{j,s}$ to paths with single shock in s $e^{j,s}$, j=1, .., J

1. **Forward-looking policies**: $d\boldsymbol{y}_t^{j,s}$ only depends on $s-t \geq 0$

$\Rightarrow$ Compute one sequence of policy changes $d\boldsymbol{y}_t^{j,T}$ t=0, ..., T

2. **Distrib. & policy changes don't interact for small $e^{j,s}$**
   $d\mathrm{D}_t^{j,s}$ sums t paths following one-time policy innovations $d\boldsymbol{y}_k^{j,s}$, k=0, 1, ..., s to D_{ss}
   - $\Rightarrow$ Compute T paths $d\hat{\boldsymbol{D}}_t^{j,s}=(\Lambda_{ss}')^{t-1}d\boldsymbol{y}_0^{j,s}\boldsymbol{D}_{ss}$,$s$, t=0, ..., T
   - $\Rightarrow$ $d\boldsymbol{Y}_t^{j,s}$ just the sum of $s-t+1$ paths

Auclert et al (21): Fake-news algorithm - detail

Write HH block as mapping $h()$ from agg inputs X to agg outputs Y, for distribution D_t, Value Fct v_t, and indiv. outputs y

$$\boldsymbol{v}_t = v\left(\boldsymbol{v}_{t+1}, \boldsymbol{X}_t\right) \tag{7}$$

$$\boldsymbol{D}_{t+1} = \Lambda\left(\boldsymbol{v}_{t+1}, \boldsymbol{X}_t\right)' \boldsymbol{D}_t \tag{8}$$

$$\boldsymbol{Y}_t = y\left(\boldsymbol{v}_{t+1}, \boldsymbol{X}_t\right)' \boldsymbol{D}_t \tag{9}$$

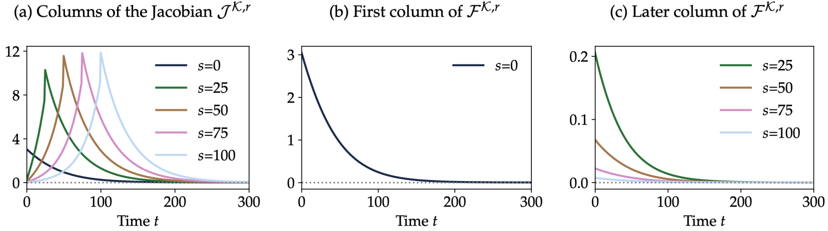
- Assume v, Y, Γ differentiable at Y^{SS}, v^{SS}, D^{SS}
- Defines $\boldsymbol{Y} = h(\boldsymbol{X})$ for stacked inputs / outputs

Auclert et al (21): "Fake-news shock"

- Shock $\hat{e}_{j,t+s}$, retracted in $t+1$, implies reaction $d\hat{\boldsymbol{X}}_t^{j,s}$
- For $t=0$, only behavioral change from SS is $d\boldsymbol{y}_0^{j,s}$
- Implies $d\hat{\boldsymbol{D}}_1^{j,s} = \boldsymbol{\Gamma}_0^s \boldsymbol{D}_0$
- Propagates as $d\hat{\boldsymbol{D}}_{t+1}^{j,s} = \Lambda'_{ss} d\hat{\boldsymbol{D}}_t = (\Lambda'_{ss})^{t-1} d\hat{\boldsymbol{D}}_1$
- Aggregate output change to fake-news shock is
  1. $t=0$: $d\hat{\boldsymbol{Y}}_t^{j,s} = d\boldsymbol{y}_0^{j,s}\boldsymbol{D}_0$
  2. $1 \leq t \leq s$: $d\hat{\boldsymbol{Y}}_t^{j,s} = \boldsymbol{y}'_{ss} d\hat{\boldsymbol{D}}_t^{j,s} = (\Lambda'_{ss})^{t-1} d\boldsymbol{y}_0^{j,s}\boldsymbol{D}_{ss}$
- Aggregate reaction in t to $e^{j,s}$ (not retracted) is sum of paths $d\boldsymbol{Y}_t^{j,s} = \sum_{k=1}^{min\{s,t\}} d\hat{\boldsymbol{Y}}_t^{j,s}$
- Yields $(i-1)*T+t, (j-1)*T+s$ entry of HH jacobian $\mathbb{I}$, for i=1, ... outputs i

Auclert et al (21): Jacobian H_1 and its building blocks

Figure 2: Jacobian $\mathcal{J}^{\mathcal{K},r}$ and fake news matrix $\mathcal{F}^{\mathcal{K},r}$ in the Krusell-Smith model.



HANK: a simple rigid-wage example

- Standard: sticky-price, flex-wage $\Rightarrow$ markups and profits fluctuate
- Here: sticky-wage, flex-price $\Rightarrow$ no profits, zero markup
- "Union" acts as representative wage setter
- Households: supply identical labor l_t, taken as given
- Firms: maximise profits from production of final goods using labor

Household problem

$$
\begin{aligned}
v_t(z_t, a_{t-1}) &= \max_{c_t} \frac{c_t^{1-\sigma}}{1-\sigma} - \varphi \frac{\ell_t^{1+\nu}}{1+\nu} + \beta \mathbb{E}_t \left[v_{t+1}(z_{t+1}, a_t) \right] \\
\text{s.t. } a_t + c_t &= (1 + r_t^a) a_{t-1} + (1 - \tau_t) w_t \ell_t z_t + \chi_t \\
\log z_{t+1} &= \rho_z \log z_t + \psi_{t+1} \quad , \psi_t \sim \mathcal{N}(\mu_\psi, \sigma_\psi), \; \mathbb{E}[z_t] = 1 \\
a_t &\geq 0
\end{aligned}
$$

- Household pol fct $C_t^{hh} = C^{hh}\left(\{r_s^a, \tau_s, w_s, \ell_s, \chi_s\}_{s \geq 0}\right)$

Firms

- Production and profits

$$Y_t = \Gamma_t L_t$$

$$\Pi_t = P_t Y_t - W_t L_t$$

- FOC

$$\frac{\partial \Pi_t}{\partial L_t} = 0 \Leftrightarrow P_t \Gamma_t - W_t = 0 \Leftrightarrow w_t \equiv W_t / P_t = \Gamma_t$$

- Zero profits $\Pi_t = 0$
- Wage and price inflation

$$\pi_t^w \equiv W_t / W_{t-1} - 1$$

$$\pi_t \equiv \frac{P_t}{P_{t-1}} - 1 = \frac{W_t / \Gamma_t}{W_{t-1} / \Gamma_{t-1}} - 1 = \frac{1 + \pi_t^w}{\Gamma_t / \Gamma_{t-1}} - 1$$

Unions

- Everybody works the same:

$$\ell_t = L_t^{hh}$$

- Unspecified wage adjustment costs imply NKWPC

$$\pi_t^w = \kappa \left(\varphi \left(L_t^{hh} \right)^{\nu} - \frac{1}{\mu} \left(1 - \tau_t \right) w_t \left(C_t^{hh} \right)^{-\sigma} \right) + \beta \pi_{t+1}^w$$

Government

- Expenditure G_t (exogenous)
- Raises taxes on labor

$$T_t = \int \tau_t w_t \ell_t z_t d\boldsymbol{D}_t = \tau_t \Gamma_t L_t = \tau_t Y_t$$

- Government budget constraint

$$B_t = (1 + r_t^b) B_{t-1} + G_t + \chi_t - T_t$$

- Tax rule

$$\tau_t = \tau_{ss} + \omega q_{ss} \frac{B_{t-1} - B_{ss}}{Y_{ss}}$$

Central Bank

- Taylor rule

$$1 + i_t = (1 + i_{t-1})^{\rho_i} \left((1 + r_{ss}) (1 + \pi_t)^{\phi_\pi} \right)^{1-\rho_i}$$

- Fisher-equation:

$$1 + r_t = \frac{1 + i_t}{1 + \pi_{t+1}}$$

Market clearing

- Asset market $B_t = A_t^{hh}$
- Labor market $L_t = L_t^{hh}$
- Goods market: $Y_t = C_t^{hh} + G_t$

Equation system

$$
\begin{bmatrix}
w_t - \Gamma_t \\
Y_t - \Gamma_t L_t \\
1 + \pi_t - \frac{1+\pi_t^w}{\Gamma_t / \Gamma_{t-1}} \\
1 + i_t - (1 + i_{t-1})^{\rho_i} \left((1 + r_{ss}) (1 + \pi_t)^{\phi_\pi} \right)^{1-\rho_i} \\
1 + r_t - \frac{1+i_t}{1+\pi_{t+1}} \\
\tau_t - \left[\tau_{ss} + \omega q_{ss} \frac{B_{t-1} - B_{ss}}{Y_{ss}} \right] \\
B_t - A_t^{hh} \\
\pi_t^w - \left[\kappa \left(\varphi \left(L_t^{hh} \right)^\nu - \frac{1}{\mu} (1 - \tau_t) w_t \left(C_t^{hh} \right)^{-\sigma} \right) + \beta \pi_{t+1}^w \right]
\end{bmatrix} = 0
$$

Equation system in blocks

$$\boldsymbol{H}(\boldsymbol{\pi^w},\boldsymbol{L},\boldsymbol{G},\chi,\boldsymbol{\Gamma}) = \left[\begin{array}{c} B_t - A_t^{hh} \\ \pi_t^w - \left[\kappa\left(\varphi\left(L_t^{hh}\right)^{\nu} - \frac{1}{\mu}(1-\tau_t)w_t\left(C_t^{hh}\right)^{-\sigma}\right) + \beta\pi_{t+1}^W\right] \end{array}\right] = 0$$

$$Production : w_t = \Gamma_t$$

$$Y_t = \Gamma_t L_t$$

$$\pi_t = \frac{1+\pi_t^w}{\Gamma_t/\Gamma_{t-1}} - 1$$

$$Central\ \ bank : i_t = (1+i_{t-1})^{\rho_i}\left((1+r_{ss})(1+\pi_t)^{\phi_\pi}\right)^{1-\rho_i} - 1(forwa$$

$$r_t = \frac{1+i_t}{1+\pi_{t+1}} - 1$$

$$Government : \left[\begin{array}{c} \tau_t \\ B_t \end{array}\right] = \left[\begin{array}{c} \tau_{ss} + \omega q_{ss}\frac{B_{t-1}-B_{ss}}{Y_{ss}} \\ \frac{(1+\delta q_t)B_{t-1}+G_t+\chi_t-\tau_t Y_t}{q_t} \end{array}\right] (forwards)$$

Simplified: $\xi_t = G = 0$, $B_t = B$, $\rho_i = 0$

$$\boldsymbol{H}(\boldsymbol{\pi}^{\boldsymbol{w}}, \boldsymbol{L}, \boldsymbol{\Gamma}) = \left[\begin{array}{c} B - A_t^{hh} \\ \pi_t^w - \left[\kappa \left(\varphi \left(L_t^{hh} \right)^{\nu} - \frac{1}{\mu} (1 - \tau_t) w_t \left(C_t^{hh} \right)^{-\sigma} \right) + \beta \pi_{t+1}^W \right] \end{array} \right] = 0$$

$$\begin{aligned} Production: w_t &= \Gamma_t \\ Y_t &= \Gamma_t L_t \\ \pi_t &= \frac{1 + \pi_t^w}{\Gamma_t / \Gamma_{t-1}} - 1 \\ Central\ \ bank: i_t &= (1 + r_{ss}) (1 + \pi_t)^{\phi_\pi} - 1 \\ r_t &= \frac{1 + i_t}{1 + \pi_{t+1}} - 1 \end{aligned}$$

Algorithm

1. Solve for steady state (by bisection on r^{ss}).
2. Choose T and *crit*.
3. Solve for transition with exogenous $\{\Gamma_t\}_0^T$
   3.1 Guess for path of $\{\pi_t\}^i$.
   3.2 Taylor rule implies $\{i_t\}^i$.
   3.3 Fisher equation implies $\{r_t\}^i$.
   3.4 Firm optimality implies $\{\pi_t^W\}^i$.
   3.5 NKPC implies $\{L_t\}^i$.

$$\pi_t^w = \left[\kappa\left(\varphi\left(L_t^{hh}\right)^\nu - \frac{1}{\mu}\left(1-\tau_t\right)w_t\left(w_t L_t^{hh}\right)^{-\sigma}\right) + \beta\pi_{t+1}^W\right]$$

   3.6 Given $\{r_t\}^i, \{L_t\}^i$ HH block implies $\{A_t^{hh,i}\}$
   3.7 If $norm(\{A_t^{hh,i}\} - B) > crit$, choose new $\{\pi_t\}^{i+1}$ and go back to 1.

How to update $\{\pi_t\}^i$

1. Multiples shooting not applicable (guess on whole path).
2. Relaxation algorithm
3. Newton / Broyden
   - Calculate steady state Jacobian.
   - Use Broyden or no jupdate at all.

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Problem of intermediate goods firms

- Dynamic problem

$$J_t(p_{jt-1}) = \max_{y_{jt}, p_{jt}, n_{jt}} \left\{ \frac{p_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Omega(p_{jt}, p_{jt-1}) Y_t + \frac{J_{t+1}(p_{jt})}{1 + r_{t+1}} \right\}$$

$$\text{s.t. } y_{jt} = \Gamma_t n_{jt}, \quad y_{jt} = \left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} Y_t$$

$$\Omega(p_{jt}, p_{jt-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} \left[\log\left(\frac{p_{jt}}{p_{jt-1}}\right)\right]^2$$

- Symmetry: In equilibrium all firms set the same price, $p_{jt} = P_t, \; Y_t = y_{jt}$
- NKPC derived from FOC wrt. pjt and envelope condition

$$\log(1+\pi_t) = \kappa \left(\frac{w_t}{\Gamma_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t} \frac{\log(1 + \pi_{t+1})}{1 + r_{t+1}}, \quad \pi_t \equiv P_t / P_{t-1} - 1$$

Derivation of the NKPC

FOC

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{\Gamma_t}\left(\frac{p_{jt}}{P_t}\right)^{-\frac{\mu}{\mu-1}} \frac{Y_t}{p_{jt}}$$

$$- \frac{\mu}{\mu - 1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt}}{p_{jt-1}}\right)}{p_{jt}} Y_t + \frac{J'_{t+1}(p_{jt})}{1 + r_{t+1}}$$

Env Cond $J'_{t+1}(p_{jt}) = \frac{\mu}{\mu-1}\frac{1}{\kappa}\frac{\log\left(\frac{p_{jt+1}}{p_{jt}}\right)}{p_{jt}} Y_{t+1}$

$$0 = \left(1 - \frac{\mu}{\mu - 1}\right)\frac{Y_t}{P_t} + \frac{\mu}{\mu - 1}\frac{w_t}{\Gamma_t}\frac{Y_t}{P_t}$$

$$+ \frac{\mu}{\mu - 1}\frac{1}{\kappa}\log\left(1 + \pi_t\right)\frac{Y_t}{P_t} + \frac{\frac{\mu}{\mu-1}\frac{1}{\kappa}\log\left(1 + \pi_{t+1}\right)\frac{Y_{t+1}}{P_t}}{1 + r_{t+1}}$$

$$\log(1 + \pi_t) = \kappa\left(\frac{w_t}{Z_t} - \frac{1}{\mu}\right) + \frac{Y_{t+1}}{Y_t}\frac{\log\left(1 + \pi_{t+1}\right)}{1 + r_{t+1}}$$