Quant Mass QM2

January 24

Jan 23 - Class I

Feb: 6, 7, 13

ADMIN — Jan: 23, 26, 30,

3 PS

1/29

2/5

2/16

12 pm (mid-day) Eastern time

Teams of 2

50% grade + 50% exam

SEND ME AN EMAIL

TOMORROW MAX

FRIDAY 4 pm

~~THU 6.15 pm~~

Ch 1 _ Introduction Heterogeneity in macroeconomics

Ch 2 _ $\boxed{\text{Steady-states}}$ in Aiyagari models

A] Eq B] Algorithm C] Application

Ch 3 _ Deterministic TRANSITION

Ch 4 _ Aggregate SHOCKS

B] Algorithm

⋮

HANK session

Ch1 _ Heterogeneity

A] History of thoughts (Ben Moll)

Modern Macro _ 70s / 80s

"First generation" Models ... Sargent Lucas

Rational Expectations

└ eq / solve / calibrate

RA {HA + complete markets}

cycle, shocks, ...

$\boxed{\text{. no data,}}$ $\boxed{\text{no computers}}$

"Second generation" ... 2000s ~ / 90s

▷ widening of inequality

▷ computational power

HA + IM

+ welfare .. MACRO ⇒ welfare poor → vs rich

⇒ inequality ↑ ?

Bewley

Aiyagari, Huggett, Imrohoroglu

Violante, Storesletten, Heathcote, Rios-Rull, ...

MACRO → HA

rich ~ RA

(linear) ..

Ineq has no effect on macro dynamics

(3) "Third generation" of HA macro models

Moll, Violante, Auclert, ...

Two-asset models

Second + curvature → HA ⇒ Macro Dynamics

$E[c^i]$

(B) Topics

(C) Facts (US)

Ch 2 _ Bewley - Huggett - Aiyagari models in steady state

Incompleteness

STATIONARY RECURSIVE COMP EQUILIBRIUM

$K_{t+1} = K_t = K$

$N, r, \ldots$

value function

$\{r, w\}$

GE

A] Equilibrium

A1] Income Fluctuation Problem

HH problem with IM

A2] RCE exchange economy

A3] RCE production (RBC + IM)

A1) income fluctuation problem

IM ... a' is not state-contingent

- PIH ∴ IM + $\boxed{\text{quadratic}}$

↳ consume all of permanent shock

r of temporary shock

<u>utility</u> $u' > 0$

$u'' < 0$

Inada condition

PE <u>HH</u>

$$\max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t. $A_0 = 0$ $\quad c_t + \dfrac{A_{t+1}}{1+r} = A_t + y_t$

$$A_{t+1} \geq \phi$$

$\{y_t\}$: $y_t \in \{\bar{y}_1, \bar{y}_2, \dots, \bar{y}_N\}$

$\bar{y}_N < \infty$

deterministic , stochastic {iid y}

$-y$ iid

$$V(A_t + y_t) = \max_{c_t,\ A_{t+1}} u(c_t) + \beta \sum_{y_{t+1}|y_t} \pi(y_{t+1}|\cancel{y_t})\, V(A_{t+1} + y_{t+1})$$

$$A_{t+1} \geq \phi \ , \quad c_t + \frac{A_{t+1}}{1+r} = A_t + y_t$$

$$a \equiv A + y$$

$$V(a) = \max u(c) + \beta \sum_{s'} \pi(y_{s'})\, v(a'_{s'})$$

$$\frac{A'}{1+r} = \underbrace{(A+y)}_{a} - c$$

$$A' = (1+r)(a-c)$$

$$\boxed{a'_{s'} - y_{s'} = (1+r)(a-c)} \longleftarrow \{ a'_{s'}$$

$$V(a) = \max u(c) + \beta \sum_{s'} \pi(y_{s'})\, V\big(\overbrace{(1+r)(a-c) + y'_{s'}}^{a'_{s'}}\big)$$

BC

$$A' \geq \phi$$

$$(1+r)(a-c) \geq \phi$$

$$c \leq -\frac{\phi}{1+r} + a$$

$$V(a) = \max_{0 \leq c \leq a - \frac{\phi}{1+r}} u(c) + \beta \sum_{s'} n(y_{s'}) V\left[(1+r)(a-c) + y_{s'}\right]$$

λ

$$\boxed{a} = A + \boxed{y}$$

FOC

$$u_c + \beta \sum_{s} n(y_{s'}) V'\left((1+r)(a-c) + y_{s'}\right) \cdot \left(-(1+r)\right) - \lambda = 0$$

Env $\quad V'(a) = \beta^{(1+r)} \sum_{s'} n(y_{s'}) V'(\ldots) + \lambda$

FOC: $\quad u_c = \beta \sum_{s'} n(y_{s'}) V'(a_{s'}) (1+r) + \lambda$

$$= V'(a)$$

$$u_c = \cancel{\beta(1+r)} \sum_{s'} \pi(y_{s'}) \, u_{c_{s'}} + \lambda$$

$$\boxed{\beta(1+r) = 1} \qquad u_{c_t} = E_t \, u_{c,t+1} + \lambda$$

$$\lambda \geq 0$$

$$\boxed{U_{c,t} \geq E_t U_{c_{t+1}} \geq 0}$$

Path for $\{c_t\}$ in LR ?

3 cases

1. deterministic path for $\{y_t\}$
   $+ \ \phi \ldots$ natural borrowing limit

$$\lambda = 0 \qquad U_{c_t} = U_{c_{t+1}}$$

$$c_t = c_{t+1} = \bar{c}$$

BC $\boxed{\sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j c_j = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_j}$

$$\bar{c} = \frac{r}{1+r} \underbrace{\left\{ \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_j \right\}}_{\text{TOTAL "PERMANENT" wealth}}$$

"interest rate"

② determine but $\phi = 0$

$$U_{c_t} = U_{c_{t+1}} + \lambda$$

$$\lambda = 0 \Rightarrow \quad c < \frac{\phi}{1+r} \quad \dots .$$

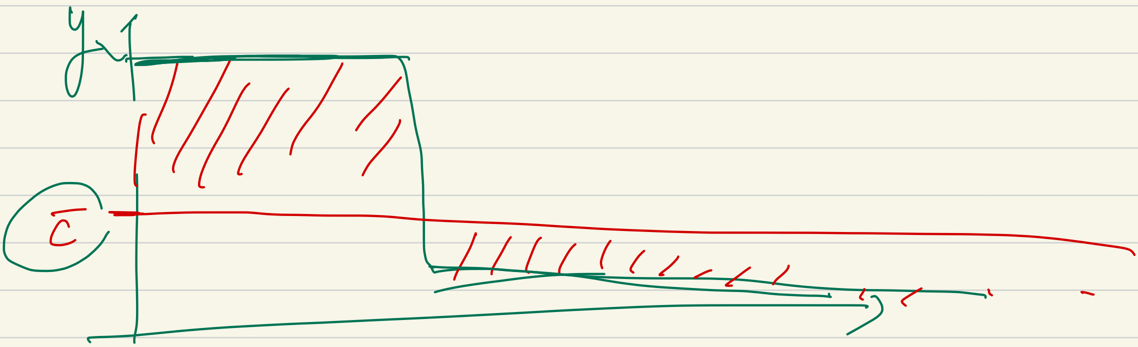
$$A' > 0 \longrightarrow \boxed{c_t = c_{t+1}}$$

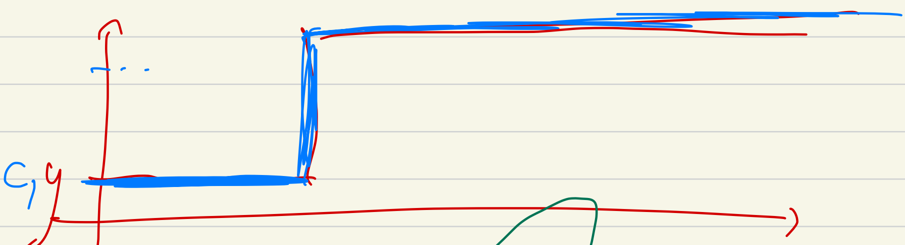
$$\lambda > 0 \rightarrow A' = 0 \rightarrow U_{c_t} > U_{c_{t+1}}$$

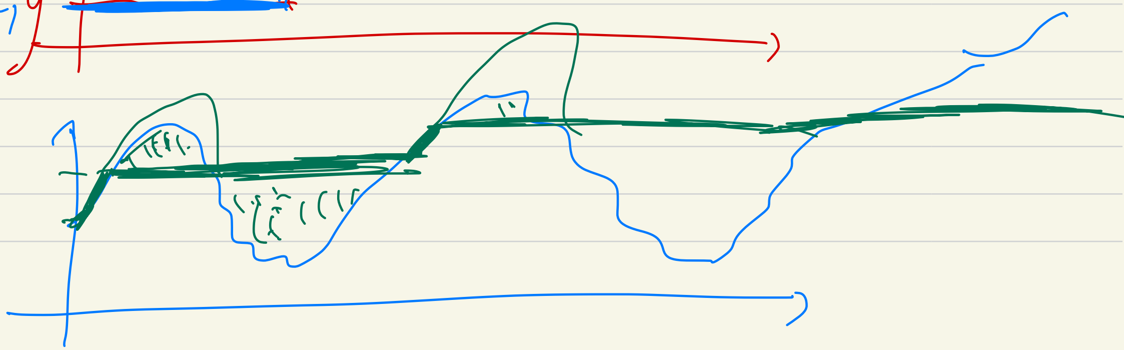
$$\boxed{c_t < c_{t+1}}$$

$$\lim_{t\to\infty} c_t = \sup_t \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^j y_{t+j}$$

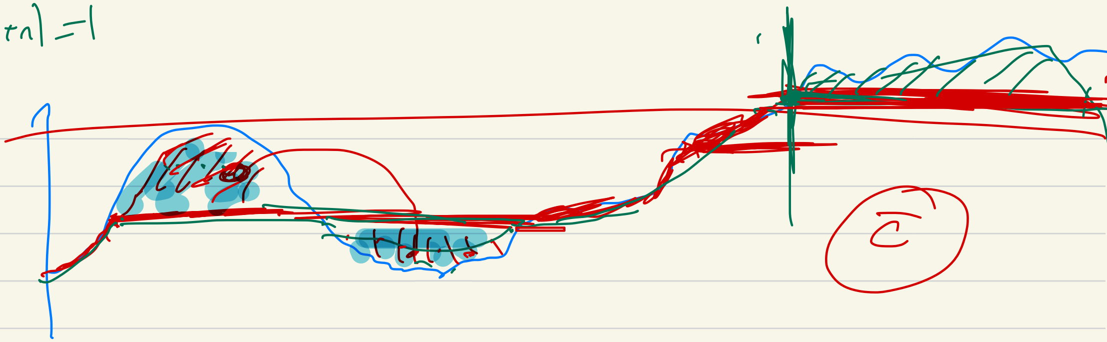
$$< \infty \qquad \{y \text{ has finite support...}\}$$







$\beta(1+r) = 1$



③ stochastic case

$$u'(c_t) \geq E_t u'(c_{t+1}) \geq 0$$

$\boxed{\text{DOOB (53)}}$ — Let M_t supermartingale bounded below

$M_t = E_t M_{t+1}$

by 0

$$\boxed{M_t \geq E_t M_{t+1}} \geq 0$$

$$M_t \to \bar{M} < \infty$$

$u'(c_t)$ RV $\quad u'(c_t) \geq E_t u'(c_{t+1}) \geq 0$

$$u'(c_t) = u'(a_t)$$

Doob $\to$ $u'(c_t) \to \alpha \geq 0$

$\alpha < \infty$

$\alpha \{0, \alpha^+\}$

$V'(a_t) \rightarrow V'(a) = \alpha \geq 0$

$< \infty$

if $\alpha > 0 \rightarrow$ no

$V'(a) \rightarrow \alpha > 0$

$a_t \rightarrow V'^{-1}(\alpha)$

$\hookrightarrow \alpha$

$\boxed{a' = (1+r)(a - c) + y_s'}$

$\alpha \qquad \alpha$

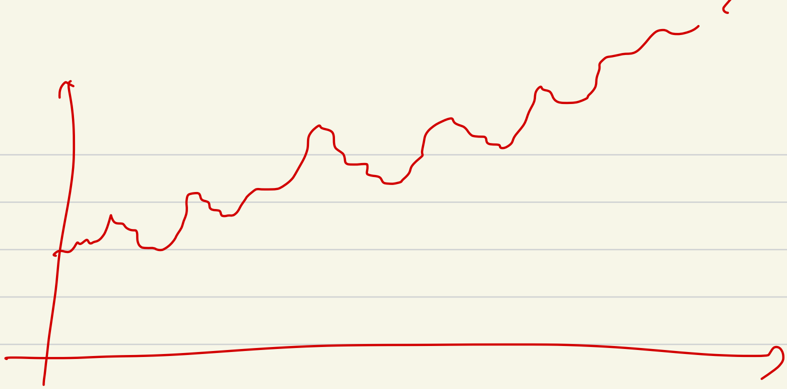
$\rightarrow a' \rightarrow +\infty$

c

$+\infty \qquad +\infty$

$V'(a) \rightarrow 0 \Rightarrow u'(c) \rightarrow 0$

$c \rightarrow +\infty$



Chamberlain Wilson (2000)

IM + $\beta(1+r) = 1$

stochastic $\{y_t\}$ $\longrightarrow$ precautionary

$c_t \to +\infty$

$a_t \to +\infty$

Asset M $\Rightarrow$ $\boxed{1+r < \frac{1}{\beta}}$

$$\boxed{r^{IM} < r^{CM}}$$

Ch 11.A.2 RCE in an Exchange Economy

$$\beta(1+r) < 1$$

$$r < \frac{1}{\beta} - 1$$

HH $\quad \varepsilon_t \in E = \{\varepsilon^1, \varepsilon^2, \dots, \varepsilon^N\}$ endowment

Markov $\quad \pi_{\varepsilon\varepsilon'} = P(\varepsilon_{t+1} = \varepsilon' \mid \varepsilon_t = \varepsilon)$

Unique stationary distribution: $\Pi^* = [\pi_1^* \; \pi_2^* \dots \pi_n^*]$

Mass 1 of hh

Total output / endowment $\quad \boxed{Y = \sum_{k=1}^{N} \pi_k^* \varepsilon^k}$

HH's problem with idiosyncratic shocks

$$v(a, \varepsilon) = \max_{c, a'} \; u(c) + \beta \sum_{\varepsilon'} \pi_{\varepsilon\varepsilon'} v(a', \varepsilon')$$

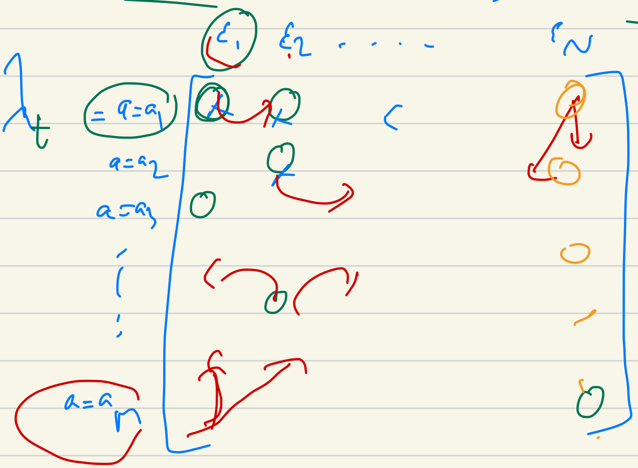
$$\text{s.t.} \quad c + a' = \varepsilon + (1+r)a, \quad a' \geq b.$$

policy function: $c(a,\varepsilon)$, $a'(a,\varepsilon)$

distribution $\lambda(a,\varepsilon)$ $\leftarrow \{\varepsilon^1, \varepsilon^2, \dots, \varepsilon^N\}$

continuum $[b, \bar{a}]$

$\{a_1, a_2, \dots, a_M\}$



$$\sum_a \sum_\varepsilon \lambda(a,\varepsilon) = 1$$

$$\int_a \sum_\varepsilon \lambda(a,\varepsilon) = 1$$

Λ_t move ?

Transition function

$S = A \times E$ $\quad \boxed{A = [\underline{b}, \bar{a}]}$

$(\mathcal{A} \times \mathcal{E})$ subset of σ-algebra defined by $\mathcal{B}_A \times P(E)$

$\{\phi, \{\varepsilon^1\}, \{\varepsilon^2\}, \ldots$
$\{\varepsilon^1, \varepsilon^2\}, \ldots \}$

$Q((a, \varepsilon), \mathcal{A} \times \mathcal{E})$: prob that someone with state (a, ε) ends up next period with $a' \in \mathcal{A}$ and $\varepsilon' \in \mathcal{E}$

$$\lambda(\mathcal{A} \times \mathcal{E}) = \int_{A \times E} Q((a, \varepsilon), \mathcal{A} \times \mathcal{E}) \, d\lambda(a, \varepsilon)$$

$$Q((a, \varepsilon), \mathcal{A} \times \mathcal{E}) = \mathbb{1}_{\{a'(a,\varepsilon) \in \mathcal{A}\}} \sum_{\varepsilon' \in \mathcal{E}} \pi_{\varepsilon\varepsilon'}$$

$$Q((a, \varepsilon), A \times E) = 1$$

stationarity $\boxed{\lambda' = \lambda}$ (*)

Definition RCE: a VF $v: S \to \mathbb{R}$

PF $c: S \to \mathbb{R}^+$

$a': S \to A$

price r, and a distribution λ^* st:

(1) given r, VF + PF solve HH's problem

(2) Good market must clear

$$\int c(a,\varepsilon)\, d\lambda(a,\varepsilon) = C = Y = \sum_{k=1}^{N} n_k^* \varepsilon^k$$

(3) Assets market have to clear

$$\int a'(a,\varepsilon)\, d\lambda(a,\varepsilon) = 0$$

(4) λ^* stationary (*)

$$\int a\, d\lambda(a,\varepsilon) = 0$$

class #2

Assets are in zero-net supply

$b = 0 \Rightarrow$ "$a'(a, \varepsilon)$" $= 0$

Ch 2 A3] RBC model

Jan 24

Friday, Jan 26: 16:15 – 17:15

Feb 2, 16:45 – 17:45

Feb 16 – 16:45 – 18:45

ch 2 A3] RBC model

labor, capital – HH, firm

$\boxed{\text{hh}}$ continuum of mass 1

consumption, leisure β

ε labor productivity $\pi_{\varepsilon\varepsilon'}$

$$E = \{ \varepsilon^1, \varepsilon^2, \dots, \varepsilon^N \}$$

$$\sum_{k=1}^{N} \Pi^*_k = 1 , \quad \Pi^* \text{ stationary distribution.}$$

$$c, n, a' \qquad (a, \varepsilon)$$

labor income : $\boxed{w \, \varepsilon \, n(a, \varepsilon)}$

Firms : $Y = F(K, N)$ CRS

CE $\to$ no profits

HH: $$v(a, \varepsilon) = \max_{c, n, a'} u(c, n) + \beta \sum_{\varepsilon'} \pi_{\varepsilon\varepsilon'} v(a', \varepsilon')$$

$$c + a' = (1+r)a + w\varepsilon n , \quad a' \geq b$$

foc

$$u_{c_t} = \beta(1+r) E_t u_{c_{t+1}} + \lambda_t \qquad \text{(Euler)}$$

$$u_{c_t} w_t \varepsilon_t = - u_{n_t} \qquad \text{(intratemp FOC)}$$

Firm: $\max_{K_t, N_t} K_t^{\alpha} N_t^{1-\alpha} - w_t N_t - (r_t + \delta) K_t$

$$\begin{cases} w_t = (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha} \\ r_t + \delta = \alpha \left(\frac{K_t}{N_t}\right)^{\alpha-1} \end{cases}$$

$$w_t = (1-\alpha)\left[\frac{r_t+\delta}{\alpha}\right]^{\frac{\alpha}{\alpha-1}} \quad (*)$$

STATIONARY RCE:

- VF $v : A \times E \to \mathbb{R}$
- PF $a' : A \times E \to A$
  - $n : A \times E \to \mathbb{R}^+$
  - $c : A \times E \to \mathbb{R}^+$

PF for the firm $\{N, K\}$,

prices $\{r, w\}$, and a distribution λ^* s.t.:

$A \times E$

- Given (r, w), hhs behave optimally
- " " " , firm behave optimally

- Asset markets clear:

$$K' = \int a'(a,\varepsilon)\, d\lambda^*(a,\varepsilon)$$
$$= K = \int a\, d\lambda^*(a,\varepsilon)$$

- Labor market clears:

$$N = \int \varepsilon n(a,\varepsilon)\, d\lambda^*(a,\varepsilon)$$

- λ^* stationary ... (transition function Q)

Ch II A4] Comments on the RBC model

- bounds on r, existence, uniqueness
- key properties of this model

Ch II B] ALGORITHM

- inelastic labor supply

Structure

$\boxed{\text{Step 0.}}$ ε ... AR(1) in logs... $\rho = 0.9$ ... variance

discretize ... $\{\varepsilon^1, \varepsilon^2, \dots, \varepsilon^N\}$

TAUCHEN ... ROUWENHORST –

Discretize assets ... $[b, \bar{a}]$

discretize $\{a^1, a^2, \dots, a^M\}$

$a^1 = b, \dots\dots, a^M = \bar{a}$

Step 1: Given r_0 [bounds ... $r_0 \geq -\delta$, $r_0 \leq \frac{1}{\beta} - 1$]

Restart a new r_1

Compute w_0 $\left[(1-\alpha) \dots \left(\frac{r_0 + \delta}{+ \dots}\right) \right]$ (* Firm)

Step 2: $\{r_0, w_0\}$ VFI ... v^0, c^0, n^0, a^0
(a, ε)

$\boxed{n^0 = 1}$

$V(a, \varepsilon)$ ... VF, policy function

Step 3 : distribution

$a'^{0} \rightarrow \varphi^{0}$

$\lambda^{0} = \int \varphi^{0} \, d\lambda^{0}$

Step 4 : Check whether markets clear.

$$\boxed{\bar{N} = \sum_{n=1}^{N} \varepsilon^{n} \, \pi_n^{*}}$$ (inelastic labor supply)

Capital demand firm K^{0} ... given $r_0, w_0, \bar{N}$

$$r_0 + \delta = \alpha \left(\frac{K}{N_0} \right)^{\alpha - 1}$$

$$\left(\frac{r_0 + \delta}{\alpha} \right)^{\frac{1}{\alpha - 1}} \bar{N} = K_0$$

Capital supplied by hh

$K_1 = \int a^0(a,\varepsilon)\, d\lambda^0(a,\varepsilon)$

... I_1 $K_0 = K_1$? No...

$K_0 > K_1$ ... r_0 was too low

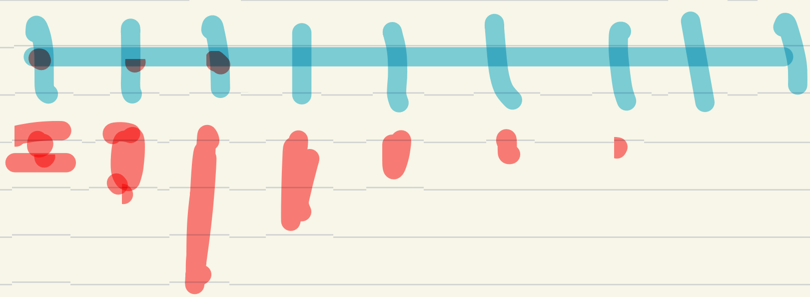
→ increase r

↓

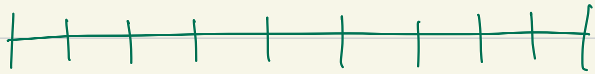
$r_1 > r_0$

$K_0 < K_1$ ... r_0 was too high ...

decrease r, and call that r_1

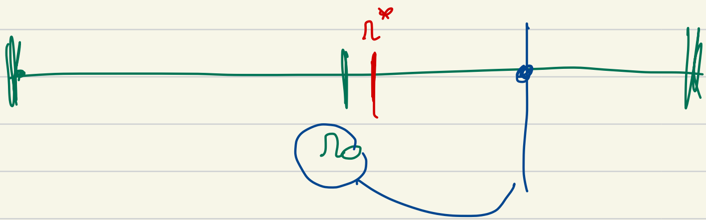


- TAUCHEN ... (Rouwenhorst)
- $N=5$, grid on a



equally spaced

- r_0 ... $\longrightarrow$ r_1 bisection



- $\boxed{\text{VFI}}$ $\quad v_0 = 0 \quad \forall (a, \varepsilon)$

grid search

$\forall (a, \varepsilon)$, ... compute all a'

$$\hat{V}(a, \varepsilon; a') = u(c, \bar{n}) + \beta \cancel{\sum} \Pi_{\varepsilon\varepsilon'} v_0(a, \varepsilon, a')$$

$$c = w \varepsilon \bar{n} + (1+r) a - a'$$

$\forall a' \rightarrow$ Pick the best a' on the grid

update V ... until converges.

- DISTRIBUTION ... simulate a'?

$$\begin{pmatrix} a, \varepsilon \\ \\ \\ \\ \end{pmatrix}$$

7500

$\varepsilon = \varepsilon^3$

$a = 2$

$$\frac{\sum a}{75000} \ \ldots \quad K_1 \text{ stable}$$

↳ $K_1 \ldots K_0 \ldots \longrightarrow \Omega$? (bisection)

ϕ One guy j

a_1^j, ε_1^j

$a_1^j = 1$ $\quad\quad$ $\varepsilon_1^j = 1$

$\forall j$ ... Random draw to compt ε'

Random draw $\pi_{\varepsilon\varepsilon'} \rightarrow \varepsilon_2^j$

$a_2^j = a_0'(a_1^j, \varepsilon_1^j)$

a_2^j, ε_2^j

Rand draw to compt $\varepsilon' \rightarrow \varepsilon_3^j$

$a_3^j = a_0'(a_2^j, \varepsilon_j^2)$

$\mathcal{E} = \mathcal{E}^1$

$$\begin{bmatrix} 0.85 & 0.10 & 0.05 & 0 & 0 \\ & & & & \\ & & & & \end{bmatrix}$$

RD..

$0.4 \longrightarrow \mathcal{E}^1$

$0.8 \longrightarrow \mathcal{E}^2$

$0.99 \longrightarrow \mathcal{E}^3$ ____ - - -

↳ $K^1 = K^0$? ... n —

Initial guess

V_0 increasing in a and in ε

Assume $\boxed{a' = a}$

$$V_0(a, \varepsilon) = u(c) \ldots$$

$$c = (\cancel{1}+r)a + w\varepsilon n - \cancel{a}$$

$$= ra + w\varepsilon n$$

$$v_0(a, \varepsilon) = \frac{[ra + w\varepsilon n]^{1-\mu}}{1-\mu} + \beta \sum \pi_{\varepsilon\varepsilon'} \cancel{v_0(a, \varepsilon')}$$

$$v_0(a, \varepsilon)$$

$$v_0(a, \varepsilon) = \frac{1}{1-\beta} \frac{[ra + w\varepsilon \bar{n}]^{1-\mu}}{1-\mu}$$

~~$v = 0$~~

Howard Step -

VFI ① $v^k(a,\varepsilon) = \max_{c,a'} u(c) + \beta \sum_{\varepsilon'} \pi_{\varepsilon\varepsilon'} v^{k-1}(a',\varepsilon')$

k is for given

$c^k(a,\varepsilon), \quad a'^k(a,\varepsilon)$

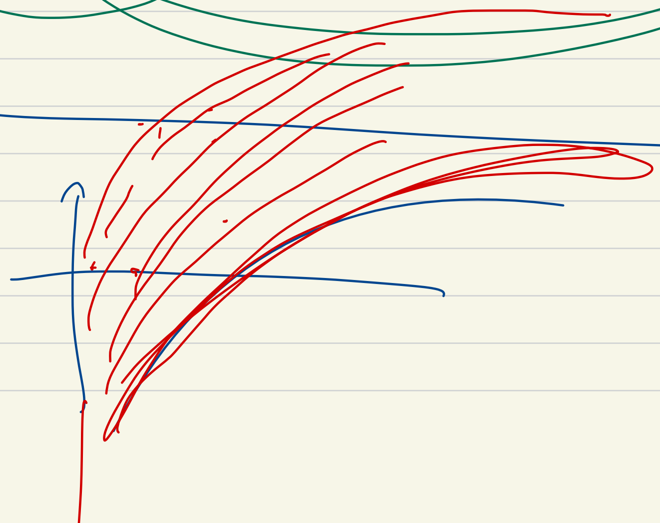
② Howard step

h

$\hat{v}^{h+1}(a,\varepsilon) = u(c^k(a,\varepsilon)) + \beta \sum_{\varepsilon'} \pi_{\varepsilon\varepsilon'} \hat{v}^{h}(a'^k(a,\varepsilon),\varepsilon')$

iterate until

new $v^k(a,\varepsilon)$



simulation

1) matrix $\cancel{(\varepsilon \times a)}$ $[N_m \times N_n]$ $(a \times \varepsilon)$

$$\begin{bmatrix} (a=1, \varepsilon=1) & \cdots & (a=1, \varepsilon=n) \\ (a=2, \varepsilon=1) & & \vdots \\ \vdots & & \vdots \\ (a=m, \varepsilon=1) & & (a=m, \varepsilon=n) \end{bmatrix}$$

vector $[N_m \times N_n, 1]$ = Γ

$$\begin{bmatrix} \lambda(a=1, \varepsilon=1) \\ \lambda(a=2, \varepsilon=1) \\ \lambda(a=3, \varepsilon=1) \\ \vdots \\ \lambda(a=m, \varepsilon=1) \\ \lambda(a=1, \varepsilon=2) \\ \vdots \\ \lambda(a=m, \varepsilon=n) \end{bmatrix}$$

$$\Gamma = Q \cdot \Gamma$$

$$[N_m \times N_n, 1] \quad [N_m \times N_n, \quad N_m \times N_n]$$

$$\begin{bmatrix} \lambda^1(a=1, \varepsilon=1) \\ \lambda^2(a=2, \varepsilon=1) \\ \vdots \\ \lambda^1(a=m, \varepsilon=n) \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} & q_{13} & \cdots & q_{1\,m\times n} \\ q_{21} & q_{22} & & & \\ & & Q & & \end{bmatrix} \begin{bmatrix} \lambda(a=1, \varepsilon=1) \\ \lambda(a=2, \varepsilon=1) \\ \vdots \\ \lambda(a=m, \varepsilon=n) \end{bmatrix}$$

$q_{11} = P($hh in state $(a=1, \varepsilon=1)$ goes to $(a'=1, \varepsilon'=1)$ next period$)$

Q

$$q_{11} = \mathbb{1}\left[a'(a, \varepsilon) = a_1\right] \pi(\varepsilon' = \varepsilon_1 \mid \varepsilon = \varepsilon_1)$$

$$q_{12} = \mathbb{1}\{a'(a=2, \varepsilon=1) = a_1\} \, \pi(\varepsilon'=\varepsilon_1 \mid \varepsilon=\varepsilon_1)$$

given Γ ...

$Q\Gamma$ ...

$\boxed{\text{until converges}}$

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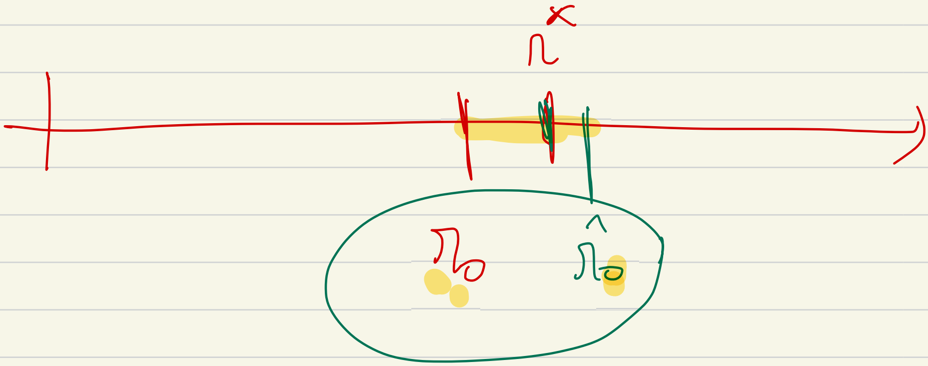
4) $\pi_0 \rightarrow K_1$

$\bar{N}$

$$\hat{\pi}_0 = \alpha \left(\frac{K_1}{\bar{N}}\right)^{\alpha-1} - \delta$$

$$r_1 = 0.9 \times r_0 + 0.1 \, \hat{r}_0$$

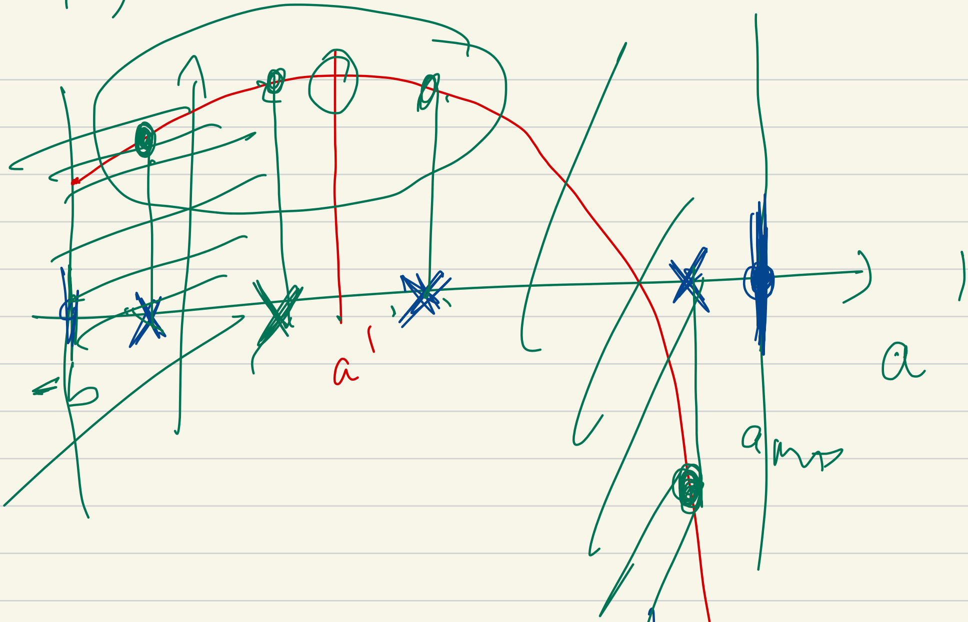


Slow weights

Golden search optimal a'

VF concave

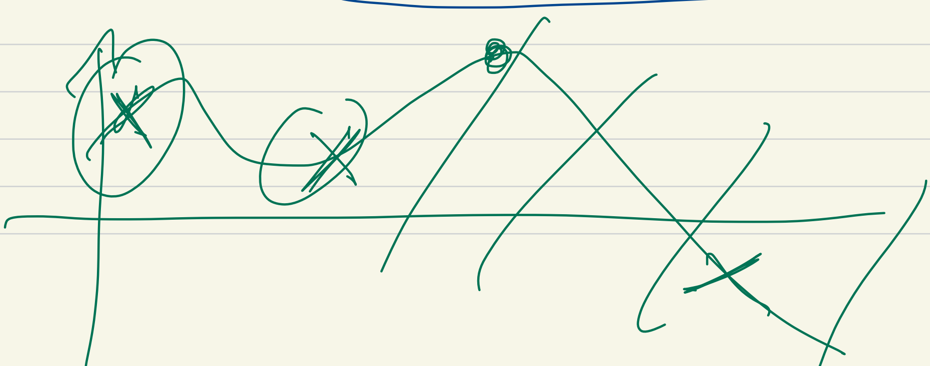
(a, ε)

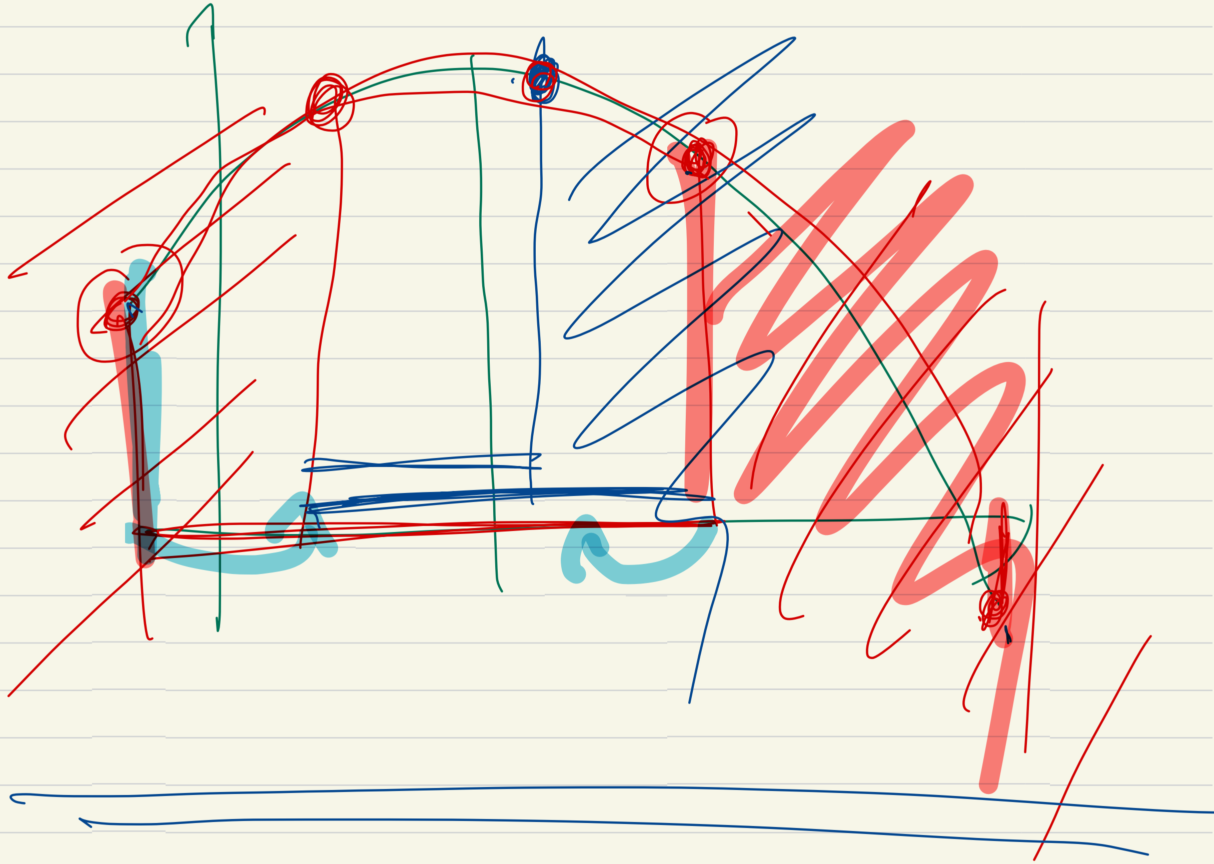


bounds: $[-b, a_{max}$ s.t. $c > 0$

$$c \leq (1+n)a + w\varepsilon n - \hat{a} > 0$$

$$\boxed{\hat{a} < (1+n)a + w\varepsilon n}$$





VF_5

v_0 → a' golden

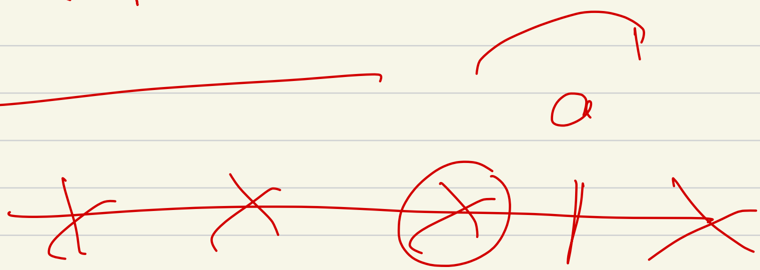
(bounds ... golden)

$a'(a, \varepsilon)$

$$u(c(a, \varepsilon, a')) + \beta \sum_{\varepsilon'} \pi_{\varepsilon\varepsilon'} v_0(a', \varepsilon')$$

INTERPOLATE

linear interpolation



FIND a_i st $a_i < a' \le a_j$

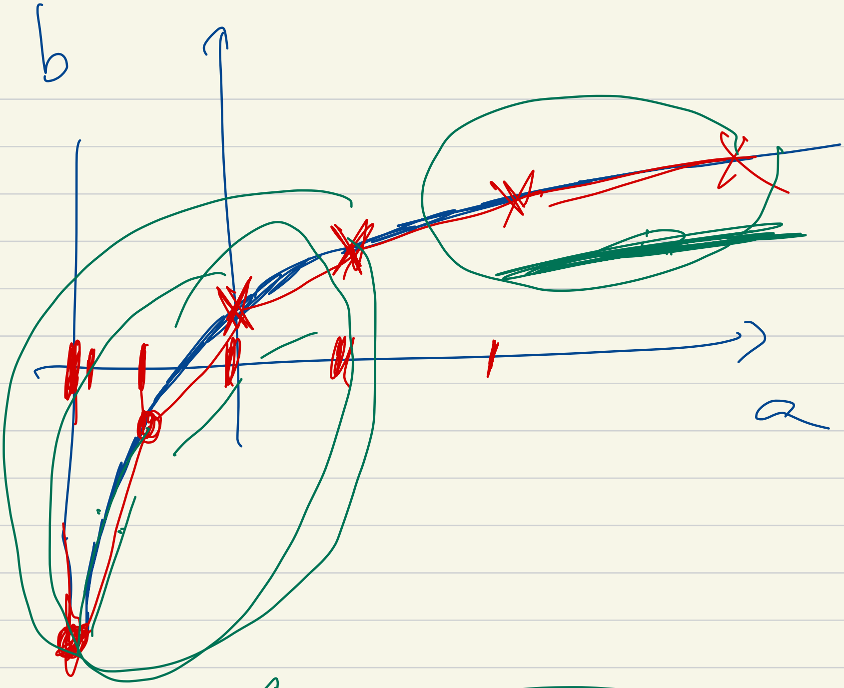
$$v_0(a', \varepsilon') = v_0(a_i, \varepsilon') + (a' - a_i) \ldots$$

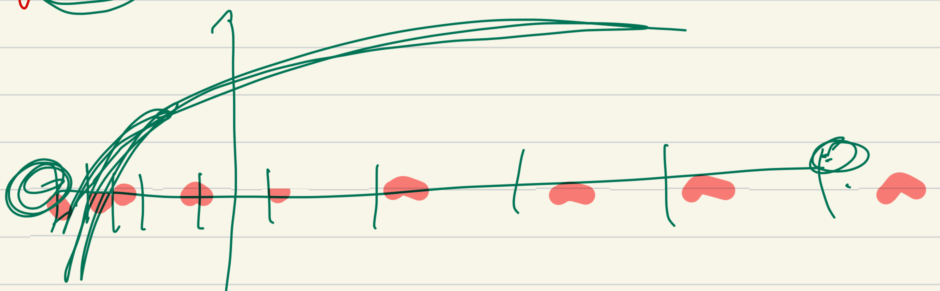
TRANSITION MATRIX

$$\mathbb{1}\{a' \mid a = a_m, \varepsilon = \varepsilon_n) = a_k\}$$

$$\pi(\varepsilon' = \varepsilon_s \mid \varepsilon = \varepsilon_n)$$

"stupid"





equally spaced log grid → exp