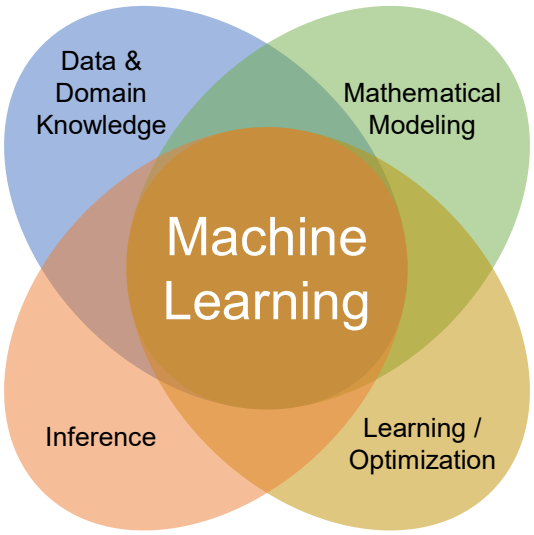
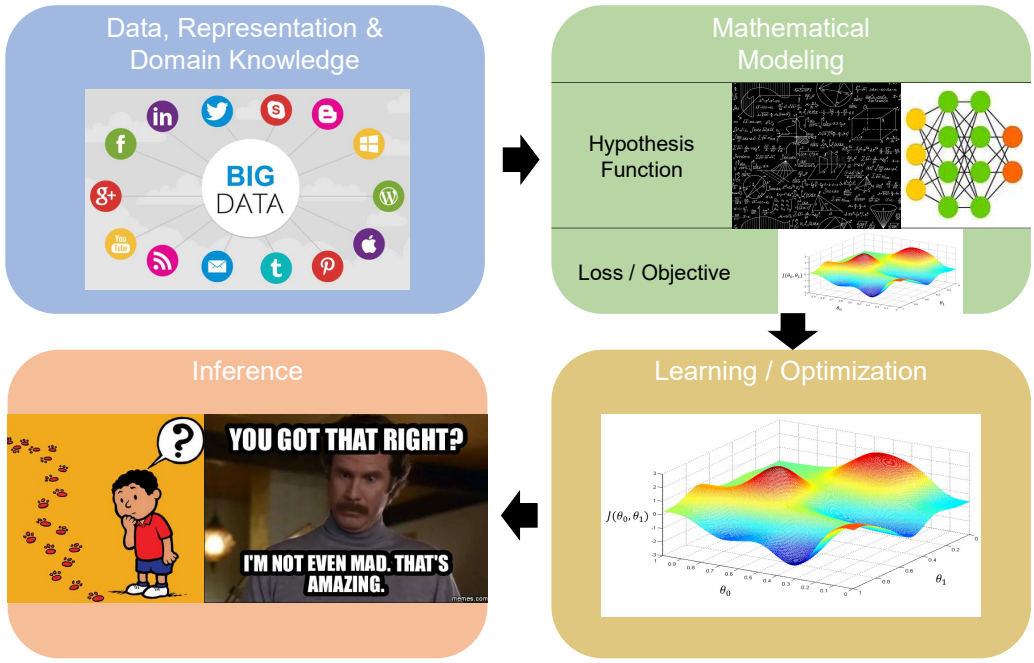
Linear Models for Regression

Kai-Lung Hua (花凱龍)

Basic Machine Learning Recipe / Pipeline





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Data, Representation & Domain Knowledge

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Linear Regression

Similar to k-NN, we assume that our input data is a vector $x \in \mathbb{R}^D$.

We can express the whole dataset as a matrix $X \in \mathbb{R}^{N \times D}$

Since this is a regression task, our target variable is a continuous scalar value $y \in \mathbb{R}$.

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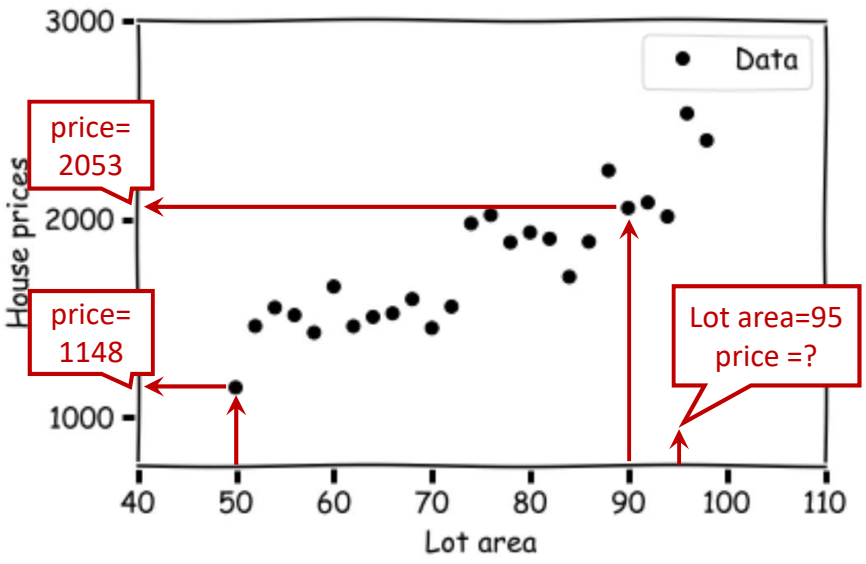
Lot area	House prices
50	1148
52	1458
54	1551
56	1513
58	1425
60	1657
62	1457
64	1504
66	1522
68	1594
70	1448
72	1556
74	1975



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Mathematical Modeling

Our Hypothesis

Lot area	House prices
50	1148
52	1458
54	1551
56	1513
58	1425
60	1657
62	1457
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66	1522
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72	1556
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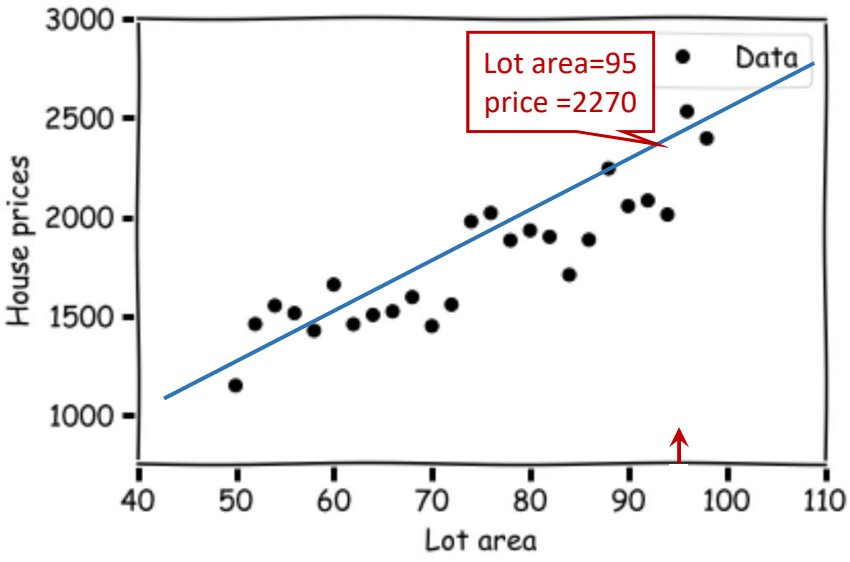
Hypothesis function



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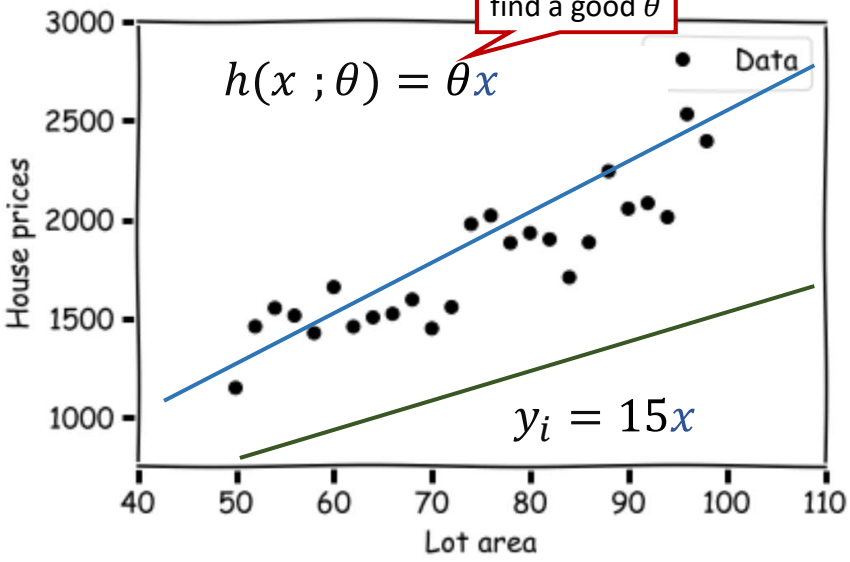
Hypothesis function

We need to find a good θ



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Linear Regression

For linear regression, we assume that the input x and the output y are related with a linear function with some noise as shown in the equation below

$$y = h(x;\theta) + \epsilon$$

$$h(x;\theta) = \sum_{i=1}^{D} \theta^j x^j = \theta^T x$$

x_i^j **Superscript** – denotes feature dimension j
Subscript – denotes sample i

where $h(x;\theta)$ is our hypothesis function, $\theta \in \mathbb{R}^D$ are the parameters (weights) and $\epsilon \sim N(0,\sigma^2) \in \mathbb{R}^D$ represents measurement error or some other random noise.

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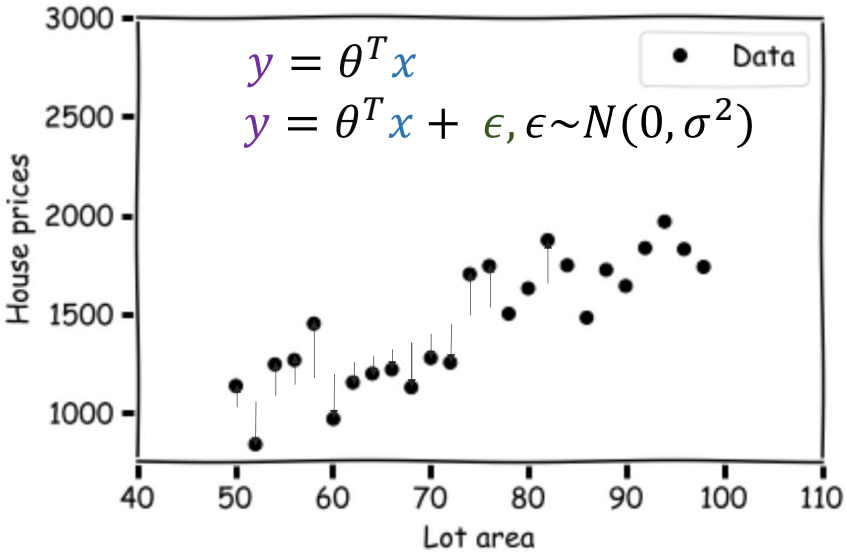
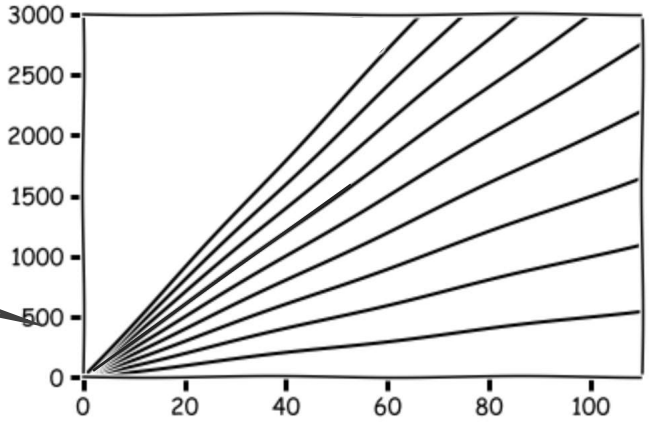
Lot area	House prices	ϵ
50	1000	-47
52	1040	-376
54	1080	193
56	1120	-293
58	1160	383
60	1200	-73
62	1240	363
64	1280	-502
66	1320	312
68	1360	473
70	1400	-3
72	1440	-433
74	1480	-43
76	1520	-609
78	1560	-419
80	1600	-423
82	1640	-325
84	1680	143
86	1720	-483
88	1760	701
90	1800	-497



Isn't a line (1D) / hyperplane $y = \theta^T x + b$?

Where did the b go?

$y = \theta^T x$



Isn't a line (1D) / hyperplane $y = \theta^T x + b$?

Where did the b go?

$$\theta = [\theta_1, \theta_2, \dots \theta_D]$$
$$x = [x_1, x_2, \dots, x_D]$$
$$b \in \mathbb{R}$$
$$\theta^T x = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_D x_D$$
$$\theta^T x + b = b + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_D x_D$$

If we add another element $\theta_0 = b$ and $x_0 = 1$, we can express this as

$$\theta^T x = \theta_0 x_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_D x_D$$
$$= \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_D x_D$$
$$= b + \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_D x_D$$

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Mathematical Modeling

The Objective

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Lot area	House prices
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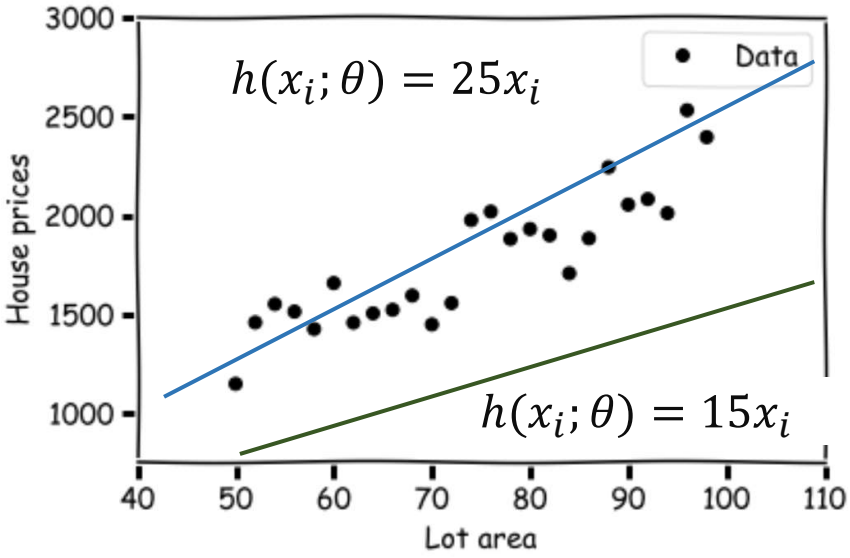
Remember this?



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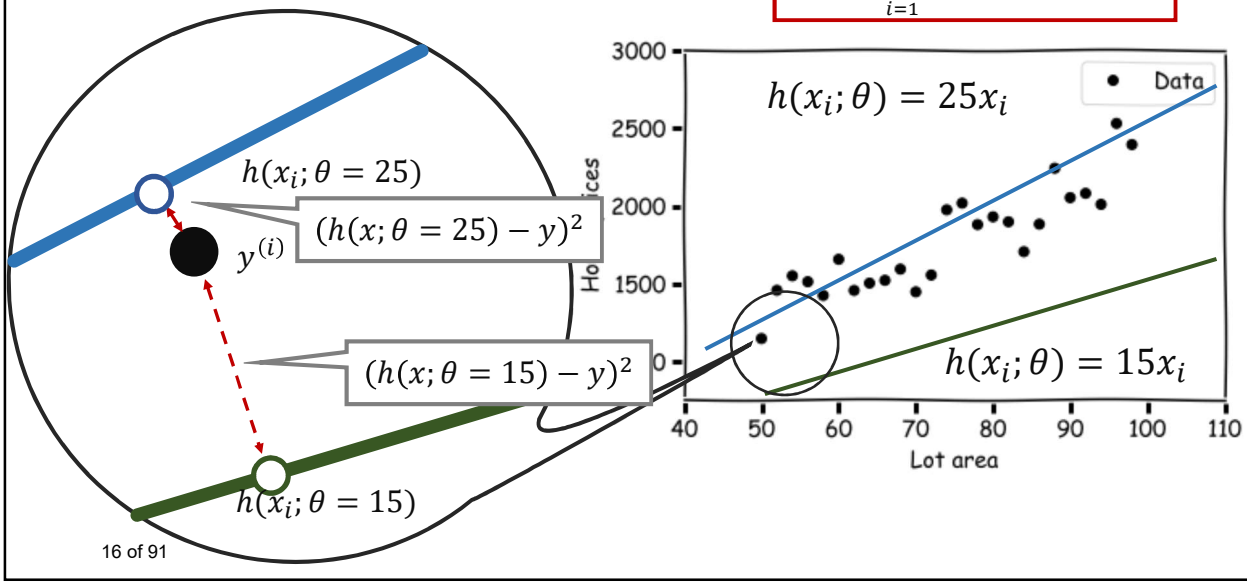
Objective/Loss function

How is $\theta = 25$ better than $\theta = 15$?

$$\frac{1}{2N}\sum_{i=1}^{N}(h(x_i;\theta) - y)^2$$



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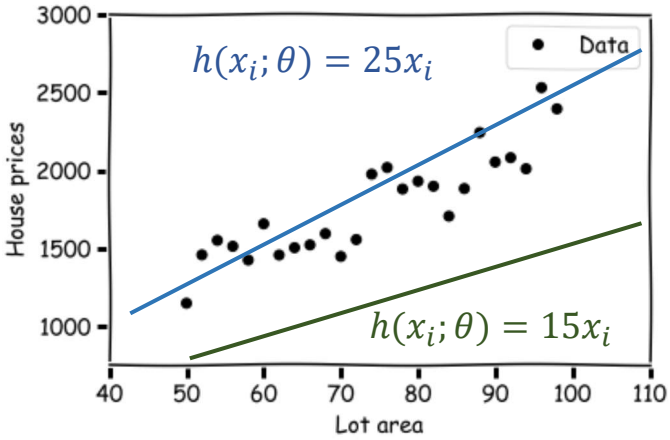
Objective/Loss function

Will explain this later

$$L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(h(x_i;\theta) - y)^2$$

$$L(\theta = 25) = 16463$$
$$L(\theta = 15) = 237255$$

$$\underset{\theta}{\operatorname{argmin}}\, L(\theta)$$



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Linear Regression

- Why Euclidean Loss?
  - It has nice probabilistic interpretations (Maximum Likelihood Estimation)

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Derivations

Recall:

(random) (random / stochastic)

$$y = \theta^T x + \epsilon, \epsilon \sim N(0, \sigma^2)$$

(deterministic / "constant")

So what will be the distribution of y?

$$y \sim N(\theta^T x, \sigma^2)$$

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Derivations MLE to Euclidean Loss

$$P(y_i|x_i;\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{y_i-\theta^T x_i}{\sigma^2}\right)^2}$$

$$\text{likelihood } \mathcal{L}(\theta) = P(y_1, y_2, \dots y_N|x_1, x_2, \dots, x_N;\theta) = \prod_{i=1}^{N} P(y_i|x_i;\theta)$$

$$\log \mathcal{L}(\theta) = \sum_{i=1}^{N} \log P(y_i|x_i;\theta) = \sum_{i=1}^{N} \log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}(y_i - \theta^T x_i)^2$$

$$\approx -\sum_{i=1}^{N} (y_i - \theta^T x_i)^2$$

$$\underset{\theta}{\text{argmax}} \log \mathcal{L}(\theta) = \underset{\theta}{\text{argmin}} - \log \mathcal{L}(\theta)$$

$$\underset{\theta}{\text{argmin}} - \log \mathcal{L}(\theta) = \underset{\theta}{\text{argmin}} \sum_{i=1}^{N} (y_i - \theta^T x_i)^2$$ Least Squares Objective

Additive constants and positive multiplicative constants do not affect the optimization problem

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Optimization

Learning the optimal parameters for our model

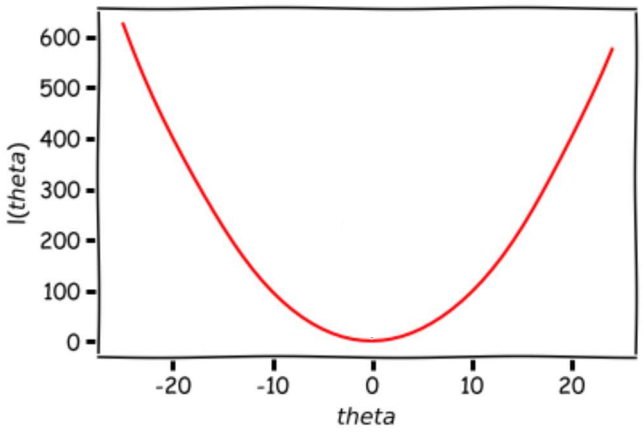
Linear Regression

Hypothesis function

$h(x_i; \theta) = \theta^T x_i$

Objective function

$$L(\theta) == \frac{1}{2N} \sum_{i=1}^{N} (h(x_i; \theta) - y)^2$$

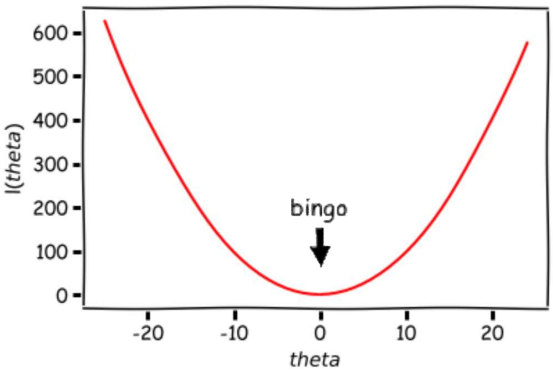


How can we get a θ that will yield a low $L(\theta)$?

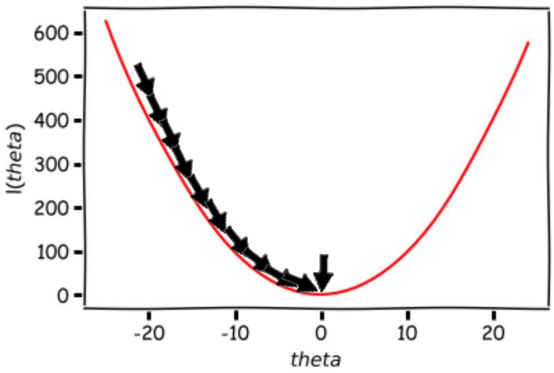
Which part of the graph yields the lowest $l(\theta)$/loss?

Two approaches to get to minima

Analytical Solution



Gradient Descent



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Analytical Solution

Normal equation

$$\theta = (X^T X)^{-1} X^T y$$

Using this equation we can solve for the optimal θ directly

Problem! This is an incredibly expensive computation!

Inverse operation is approximately $O(N^3)$

$X^T X$ uses all of your data which may not fit on your memory if data is large

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Derivation of the normal equation

From single instance to batch

$\theta^T x_i$ → $X\theta$

$(1 \times D)$ $(D \times 1)$ $(N \times D)$ $(D \times 1)$

Scalar output

Vector output
$(N \times 1)$

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Derivation of the normal equation

Least Squares Objective

$$\sum_{i=1}^{N} (y_i - \theta^T x_i)^2 = (X\theta - y)^T (X\theta - y)$$

To optimize we get the derivative with respect to θ and set to 0.

$$\frac{\partial}{\partial \theta} (X\theta - y)^T (X\theta - y) = 0$$
$$2X^T(X\theta - y) = 0$$
$$X^T(X\theta - y) = 0$$
$$X^T X\theta - X^T y = 0$$
$$X^T X\theta = X^T y$$
$$\theta = (X^T X)^{-1} X^T y$$

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Gradient Descent Solution

Idea: Start with a random guess then iteratively improve your solution by stepping towards the opposite direction of the gradient

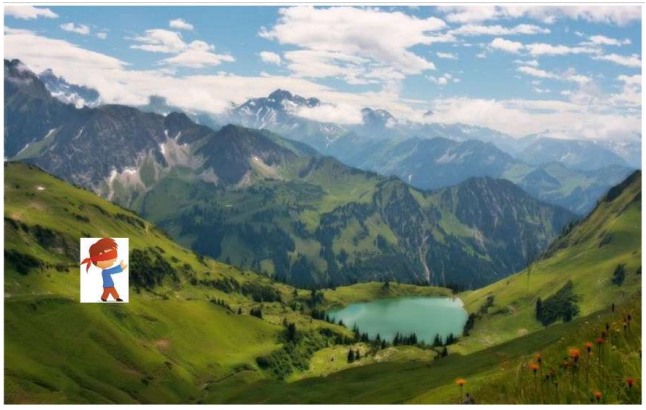

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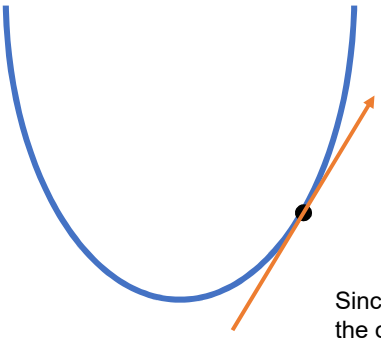
Image Credit: Fei-Fei Li, Andrej Karpathy, Justin Johnson - CS231N

Gradients

Why do we want to go to the opposite direction of the gradient?



Gradient - “slope of the tangent line”
- direction of steepest ascent

Since we are minimizing, we want to go to the opposite (negative) direction

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Gradient Descent

procedure gradient_descent(X, θ_i):
initialize θ randomly
while not converged do:
$\theta := \theta - \alpha \frac{\partial}{\partial \theta} L(\theta)$
return θ

α is the learning rate, determines how large the update will be

$\frac{\partial}{\partial \theta_i} L(\theta)$ is the gradient of the loss with respect to θ

$$L(\theta) = \frac{1}{2}\sum_{i=1}^{n}(h(x_i;\theta) - y_i)^2$$

$$= \frac{1}{2}\sum_{i=1}^{n}(\theta^T x_i - y_i)^2$$

$$\frac{\partial}{\partial \theta} L(\theta) = \frac{\partial}{\partial \theta}\frac{1}{2}\sum_{i=1}^{n}(\theta^T x_i - y_i)^2$$

$$= 2 * \frac{1}{2}\sum_{i=1}^{n}(\theta^T x_i - y_i)\frac{\partial}{\partial \theta_j}(\theta^T x_i - y_i)$$

$$= 1\sum_{i=1}^{n}(\theta^T x_i - y_i)\, x_i$$

$$= \sum_{i=1}^{n}(\theta^T x_i - y_i)\, x_i$$

Gradient Descent

$$\underset{\theta}{\operatorname{argmin}}\, L(\theta) = \frac{1}{2n}\sum_{i=1}^{n}(h(x_i;\theta) - y_i)^2$$

So the loss doesn't scale with the size of the batch

Gradient Descent	Stochastic GD	Mini-Batch GD
Updates θ based on gradient of the **whole dataset**	Updates θ based on gradient of **one data instance**	Updates θ based on gradient of a **subset of the whole dataset**
Runs slow if dataset is very large	Runs fast, but jittery	Runs faster than GD, and path is not as jittery as stochastic GD
1 iter = N instances N = size of whole dataset	1 iter = 1 instance	1 iter = n instances, n = size of batch

$$\theta := \theta - \alpha\underbrace{(\theta^T x_1 - y_1)\, x_1}$$

Assuming $x \geq 0$

1 If $(\theta^T x_1 - y_1)x_1 > 0,\ \theta$ will go down

2 If $(\theta^T x_1 - y_i)x_1 = 0,\ \theta$ will remain the same

3 If $(\theta^T x_1 - y_1)x_1 < 0,\ \theta$ will go up

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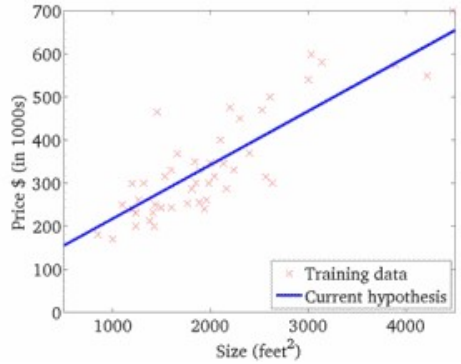
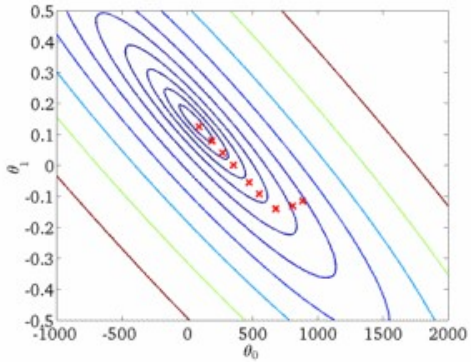
$h_\theta(x)$

(for fixed θ_0, θ_1, this is a function of x)



$L(\theta_0, \theta_1)$

(function of the parameters θ_0, θ_1)



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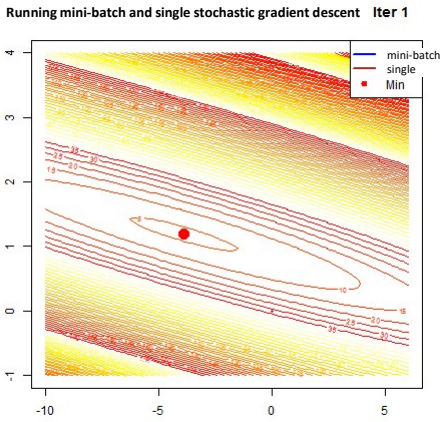

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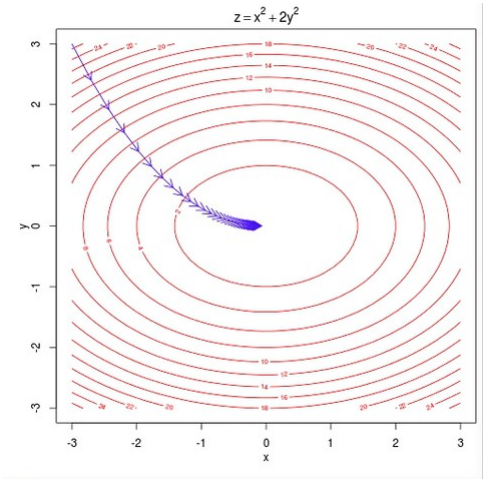
Gradient Descent

- Pros
  - It is very simple and quite effective in many ML tasks
  - Very scalable
- Cons
  - Only applies to smooth/differentiable functions
  - Can get stuck in local minimum



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Inference

Predict the most likely values for new data

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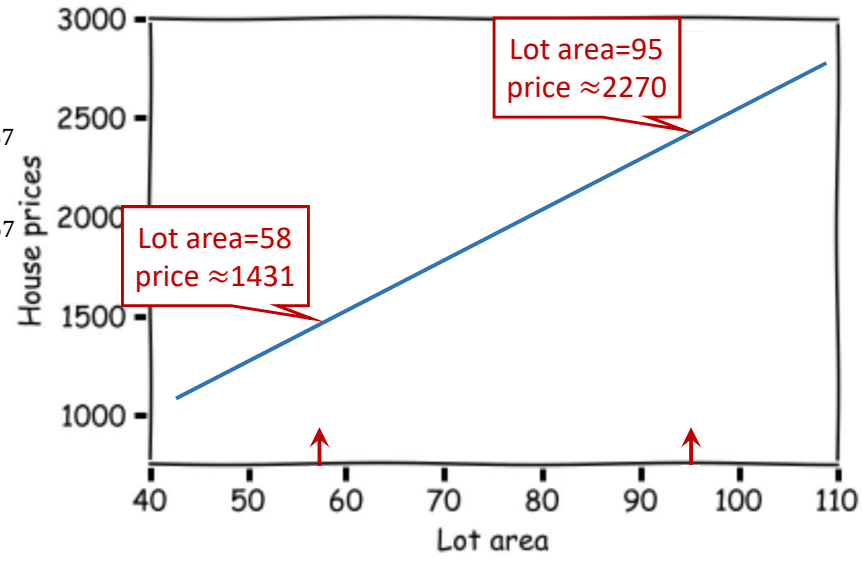
Suppose these are the learned parameters

$$\theta = \begin{bmatrix} 115.81081 \\ 22.67567 \end{bmatrix}$$

$\theta^T x_1 = 115.81081 + 95 * 22.67567$
$= 2269.99$

$\theta^T x_2 = 115.81081 + 58 * 22.67567$
$= 1430.99$

Lot area	House prices
95	?
58	?



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Disadvantages of LR

- Limited to Linear function / Linear Relationships
- Assumes data is independent and identically distributed (i.i.d.)

Advantages of LR

- Relatively fast to train
- Fast test time (just plug in to the equation)
- Do not need to store all the data just the weights (in contrast to kNN)

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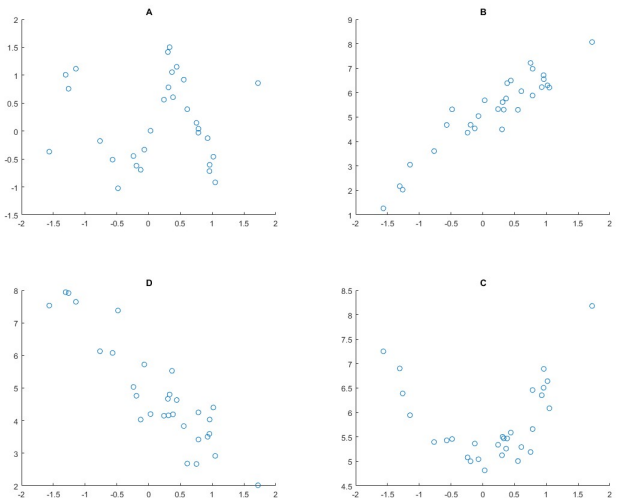
Which of these data will Linear Regression be suitable with?



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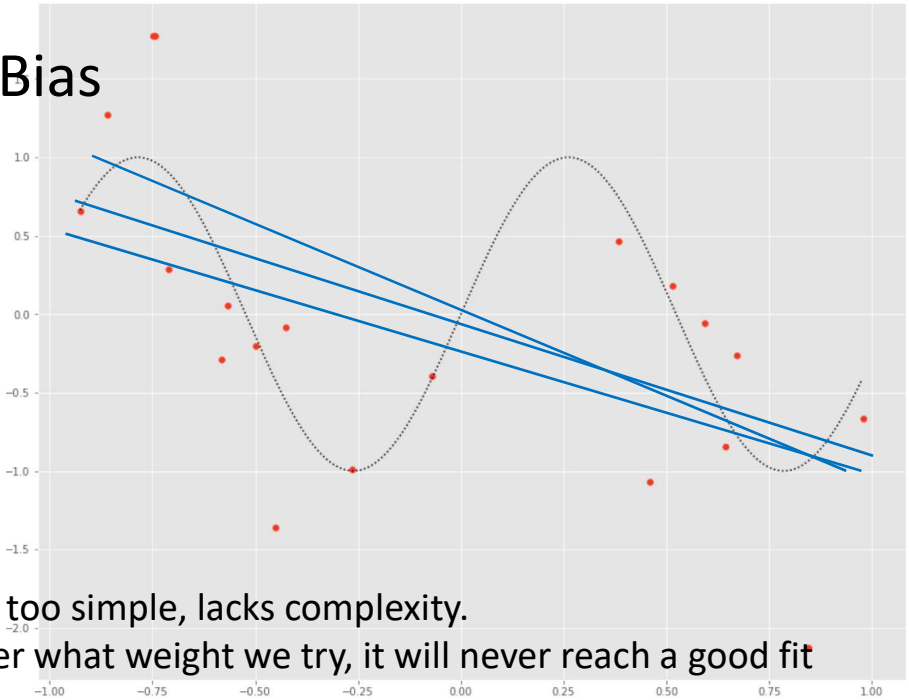
How do we generate non-linear regression curves / decision boundaries?

And, why do we want to do that anyway?

High Bias



Model is too simple, lacks complexity.

No matter what weight we try, it will never reach a good fit

How do we generate non-linear regression curves / decision boundaries?

Solution: Transform features

By Linear Models, we mean that the hypothesis function $h(x;\theta)$ is a linear function of the parameters θ, not the input vector x.

$$y = \theta_0 + \theta_1 x + \theta_2 x^2$$

We don't define linear by the requirement that the graph is a line.

As long as these functions can be directly computed from the observed values and the **parameters are still linear in the data**, then the problem is still a Linear Regression problem.

This is related to the kernel trick that is usually used for SVM.

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How do we generate non-linear regression curves / decision boundaries?

So far we only used the original observed values / features $[x_1, x_2, \ldots x_D]$. However, Linear Regression can be applied in the same way to **functions of these values.**

$$\phi(x) = [x_1, x_2, \ldots x_D, \quad x_1x_2, \quad x_1^2]$$

Common feature transforms $\phi(x)$:

- Polynomial
- Gaussian
- Exponential

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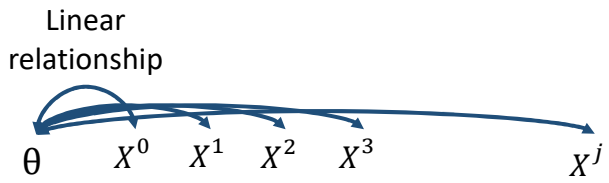
Let's do Polynomial Feature Transform with order j.



$\theta \in \mathbb{R}^{D\times 1}$

Append θ to accommodate more features

θ_0	1	1	1	1	...	x_0^j
θ_1	1	3	9	27		x_1^j
	1	4	16	64		x_2^j
	1	6	36	216		x_3^j
	1	7	49	343		x_4^j
	1	8	64	512		x_5^j

$X \in \mathbb{R}^{N\times D}$

Append *X* in the *D dimension*
With feature transformed X

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How do we generate non-linear regression curves / decision boundaries?

Before:

After:

Everything else in the procedure is the same except we replace the original input x with its transformed version $\phi(x)$.

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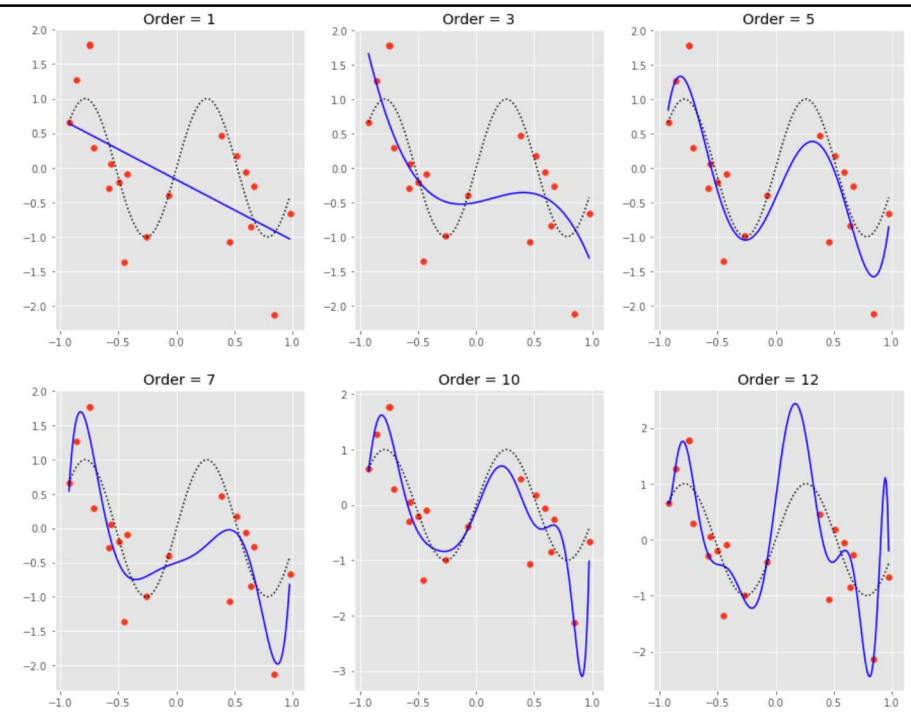

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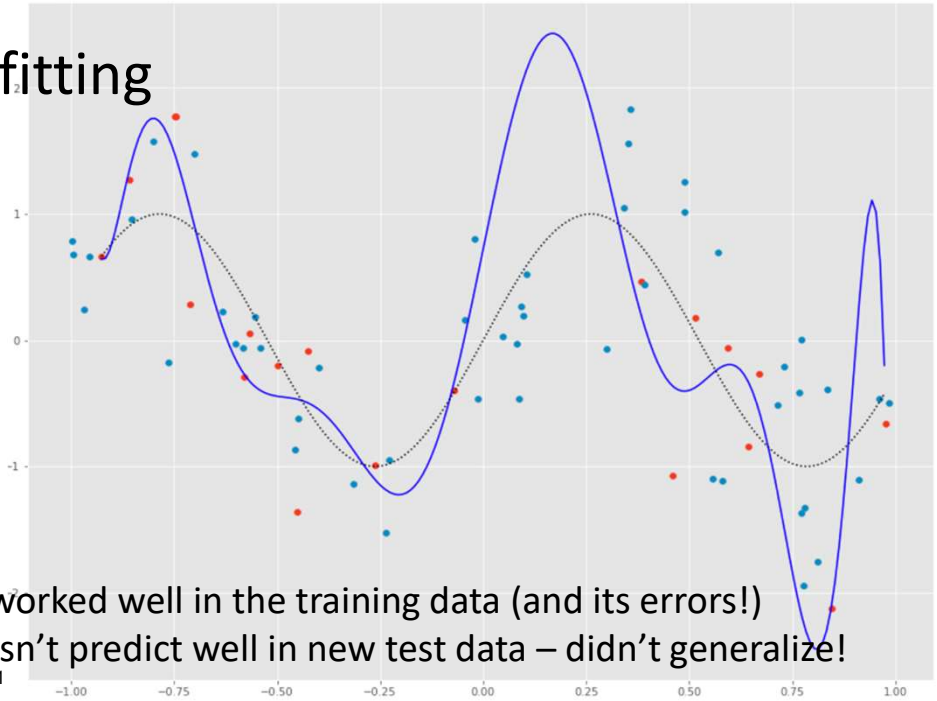
Overfitting



Model worked well in the training data (and its errors!)
but doesn't predict well in new test data – didn't generalize!

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Anatomy of the error of an estimator

Anatomy of the error of an estimator

- $y = f(x) + \epsilon$ and $\epsilon \sim N(0, \sigma^2)$

For linear regression, we assume $h(x) = \theta^T x$

- $\underset{\theta}{\operatorname{argmin}}\ L(\theta) = \sum_{i=1}^{N} (h(x_i; \theta) - y)^2$

Possible Sources Errors:

- (bias) Systematic prediction error
- Variance
- Irreducible error (noise)

Anatomy of the error of an estimator

- Assuming that the data points are drawn independently and identically distributed (i.i.d.) from a unique underlying probability distribution.
- Suppose we have a model trained on training data X. Given a new test data point x, what is the expected prediction error?

$$E\left[\left(y - h(x)\right)^2 \mid X\right]$$

- We will decompose this expectation into three components that characterizes our errors.

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Recall: Statistics

- Let X be a random variable with possible values $x_i, i = 1 \ldots N$ and with probability distribution $P(X)$.
- The **expected value** or **mean** of X is

$$E[X] = \sum_{i=1}^{N} x_i P(x_i)$$

The **variance** of X is:

$$\begin{aligned} Var[X] &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + E[X]^2] \\ &= E[X^2] - 2E[X]E[X] + E[X]^2 \\ &= E[X^2] - 2E[X]^2 + E[X]^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

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Bias-Variance Decomposition

$$E\left[\left(y-h(x)\right)^2\middle|X\right] = E\left[\left(h(x)\right)^2 - 2yh(x) + y^2\middle|X\right]$$
$$= E[(h(x)^2|X] + E[y^2|X] - 2E[y|X]E[h(x)|X]$$

$E[h(x)\|X] = \overline{h(x)}$	denotes the mean prediction of the hypothesis at x.
$E[y\|X] = E[f(x)+\epsilon\|X]$ $= E[f(x)] + E[\epsilon\|X]$ $= f(x)$	Linearity of expectation $\epsilon \sim N(0,\sigma^2)$
$E[h(x)^2\|X] = E\left[\left(h(x)-\overline{h(x)}\right)^2\|X\right] + \overline{h(x)}^2$	Based on $E[X^2] = Var[X] + (E[X])^2$
$E[y^2\|X] = E\left[\left(y-f(x)\right)^2\middle\|X\right] + \left(f(x)\right)^2$	Based on $E[X^2] = Var[X] + (E[X])^2$

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Substituting back everything

$$E\left[\left(y-h(x)\right)^2\middle|X\right] = E\left[\left(h(x)-\overline{h(x)}\right)^2|X\right] + \overline{h(x)}^2 - 2f(x)\overline{h(x)} + \left(f(x)\right)^2 + E\left[\left(y-f(x)\right)^2\middle|X\right]$$
$$= E\left[\left(h(x)-\overline{h(x)}\right)^2|X\right] + (\overline{h(x)} - f(x))^2 + E\left[\left(y-f(x)\right)^2\middle|X\right]$$

$E\left[\left(h(x)-\overline{h(x)}\right)^2\|X\right]$	Variance of the hypothesis
$(\overline{h(x)} - f(x))^2$	Squared bias (or systematic error) associated with the class of hypothesis we are considering
$E\left[\left(y-f(x)\right)^2\middle\|X\right]$ $y = f(x)+\epsilon$	Irreducible noise / error (cannot be avoided)

These are the three components that characterizes our errors.

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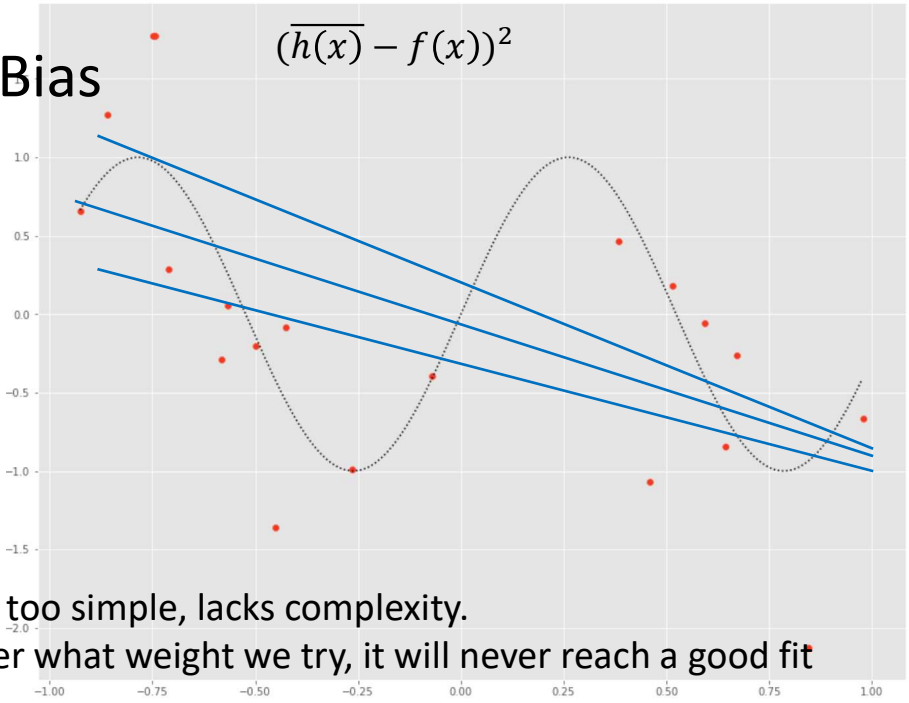
High Bias

$$(\overline{h(x)} - f(x))^2$$



Model is too simple, lacks complexity.
No matter what weight we try, it will never reach a good fit

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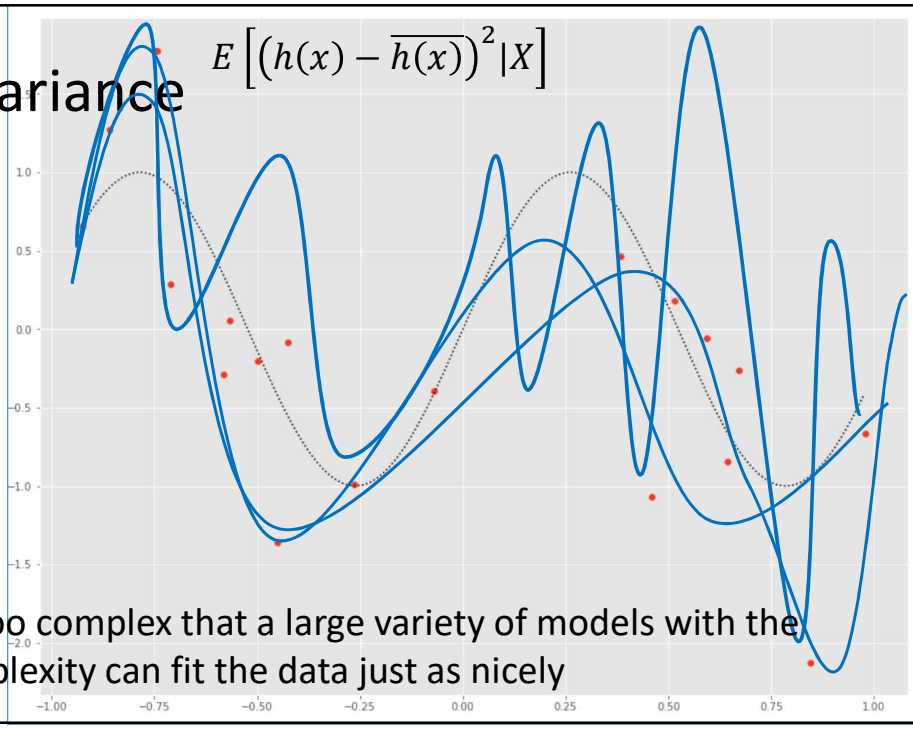
High Variance

$$E\left[\left(h(x) - \overline{h(x)}\right)^2 | X\right]$$



Model is too complex that a large variety of models with the same complexity can fit the data just as nicely

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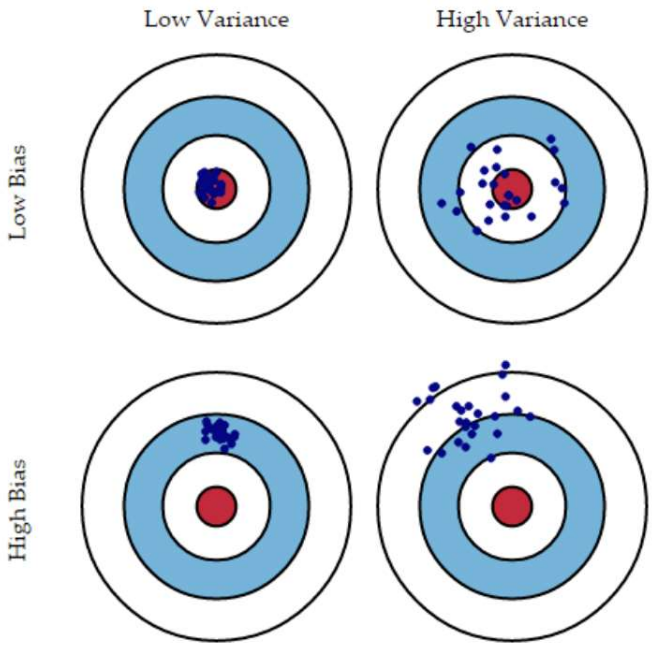

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	Training error	Test error
Underfit model	HIGH	HIGH
Overfit model	LOW	HIGH

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Problem with complex functions.. Overfitting!

- More complex functions have more expressive power and can fit your data better. However, they fit the noise better as well. **Overfitting** is when we are fitting to the noise rather than the general trends of the data.
- Typical **overfitting** means that error on the training data is very low, but error on new instance is high.
- Typical **underfitting** means that error on the training data is very high.

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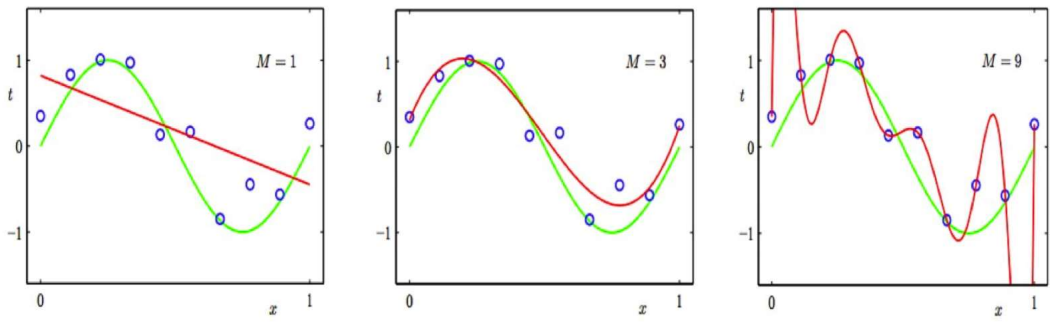

predictor too inflexible: cannot capture pattern

predictor too flexible: fits noise in the data

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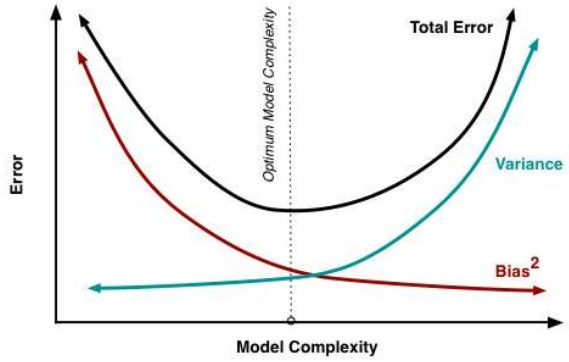
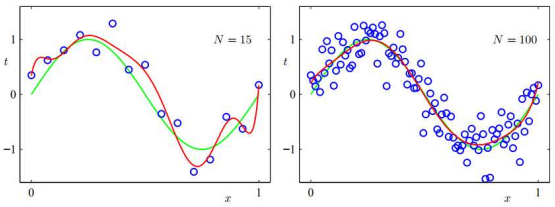
Bias-Variance Trade-off



Hypothesis complexity (decreasing left-to-right)



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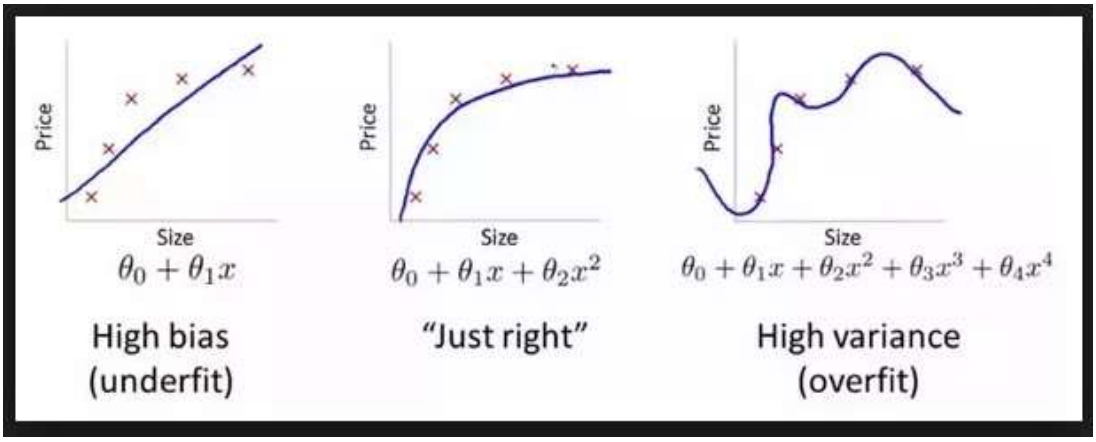
Bias-Variance Trade-off



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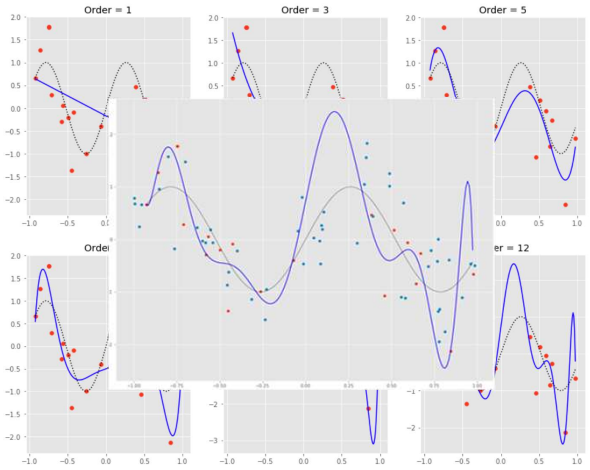
What can we do?

Definitely not underfit!

Stop before it gets too weird!
(Stop at order = 5)

Get more data!
(This way we just overfit on all of them)

Make the algo more robust to overfitting.
(But … how?)



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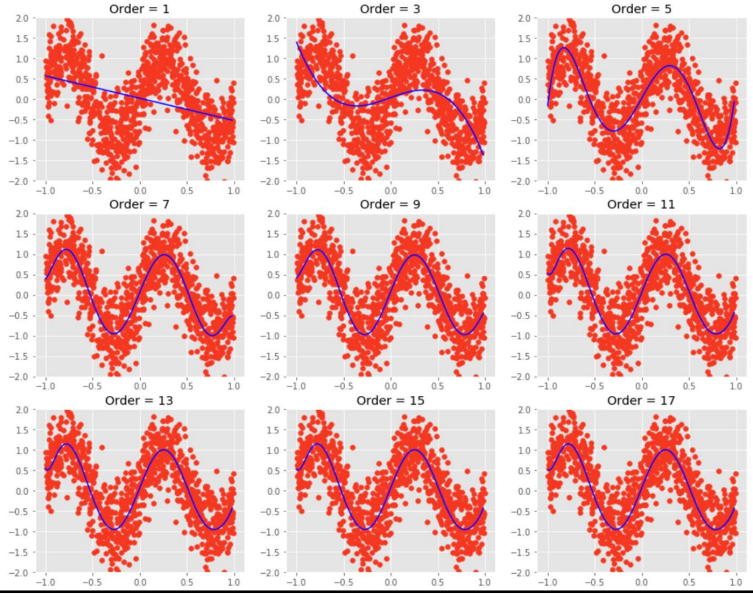
If we had more data..



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Can we reduce variance (i.e. overfitting) without collecting more data or stopping the optimization early?

Yes! Add Regularization!

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What happens when we increase the degree

poly degree	weight_1	weight_2	weight_3	weight_4	weight_5
1	-0.87	-0.17			
3	-2.15	0.89	0.34	-0.50	
5	13.40	-1.31	-17.28	1.36	3.79
7	31.30	-5.53	-37.29	4.48	6.87
10	179.74	94.89	-378.87	-185.43	259.32
12	-540.45	-1015.67	1487.13	2623.60	-1490.51
30	-14156072.98	16329720.76	21300299.24	-14967132.99	13066492.56

Big weights mean feature is important, but these are too large!

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$$\underset{\theta}{\operatorname{argmin}} L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(\theta^T x_i - y_i)^2 + R(\theta)$$

+ a high number if weights are far from 0

$$\underset{\theta}{\operatorname{argmin}} L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(\theta^T x_i - y_i)^2 + \lambda\sum_{j=1}^{D}\theta^2$$

or

$$\underset{\theta}{\operatorname{argmin}} L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(\theta^T x_i - y_i)^2 + \lambda\sum_{j=1}^{D}|\theta|$$

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$$\underset{\theta}{\operatorname{argmin}} L(\theta) = \underbrace{\frac{1}{2N}\sum_{i=1}^{N}(\theta^T x_i - y_i)^2}_{\text{Least Squares error}} + \underbrace{\lambda R(\theta)}_{\text{Regularization error}}$$

Regularization error
θ near 0 is better

Contradicting each other. They must find a balance!

Can you think of how λ will affect this?

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What does lambda have to do with it?

$$L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(\theta^T x_i - y)^2 + \frac{1}{2}\lambda||\theta||_2^2$$

If λ is large (e.g. 1000)	Least Squares error has little impact on loss
If λ is 0	No effect, it's like removing regularization
If λ is negative	The higher the weights, the lower the loss!

Need to find λ that is not too restrictive (large) and not too flexible (very small)

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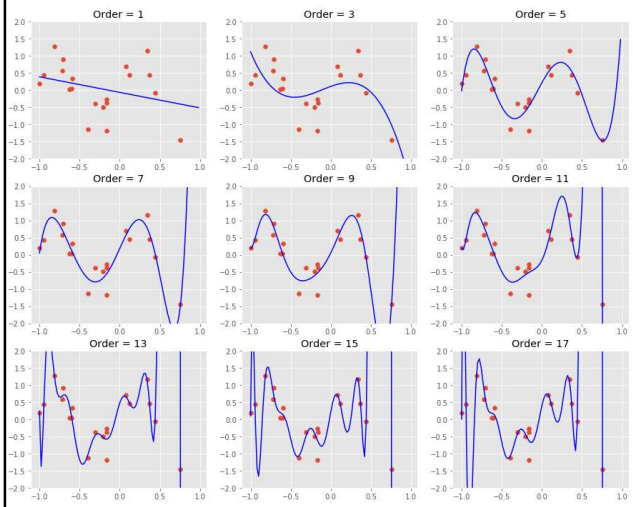
Without Regularization



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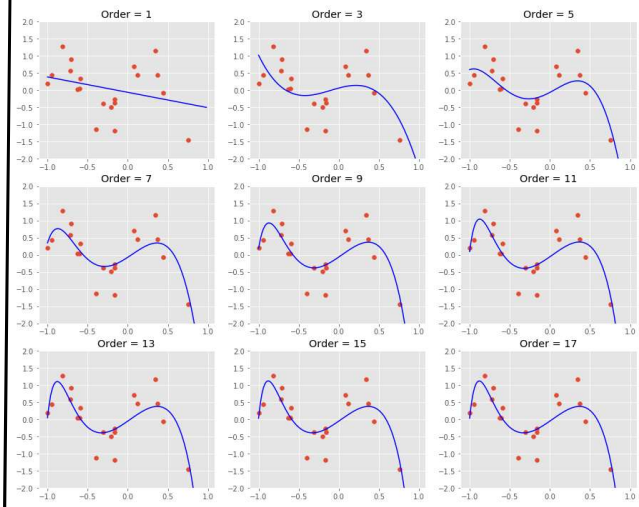
With Regularization



Two common regularization methods

Ridge Regression

L2 Regularization

$$L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(\theta^T x_i - y)^2 + \frac{1}{2}\lambda||\theta||_2^2$$

Lasso Regression

L1 Regularization

$$L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(\theta^T x_i - y)^2 + \lambda||\theta||_1$$

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Remember this?

poly degree	weight_1	weight_2	weight_3	weight_4	weight_5
1	-0.87	-0.17			
3	-2.15	0.89	0.34	-0.50	
5	13.40	-1.31	-17.28	1.36	3.79
7	31.30	-5.53	-37.29	4.48	6.87
10	179.74	94.89	-378.87	-185.43	259.32
12	-540.45	-1015.67	1487.13	2623.60	-1490.51
30	-14156072.98	16329720.76	21300299.24	-14967132.99	13066492.56

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This is what happens with Ridge (L_2)

poly degree	weight_1	weight_2	weight_3	weight_4	weight_5	weight_6	weight_7	weight_8	weight_9	weight_10
1	-0.44									
3	0.17	0.22	-0.27							
5	1.90	0.34	-2.16	-0.08	0.42					
7	0.30	0.03	0.84	0.28	-1.16	-0.09	0.24			
10	1.78	-0.46	-3.06	-0.59	2.11	1.35	-0.82	-0.61	0.12	0.08
12	1.95	0.06	-3.43	-1.35	2.22	1.31	-0.70	-0.15	0.05	-0.12
20	2.78	-1.04	-6.02	1.21	4.23	-0.73	-0.68	0.15	-0.27	0.06

Small weights!

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This is what happens with Lasso (L_1)

poly degree	weight_1	weight_2	weight_3	weight_4	weight_5	weight_6	weight_7	weight_8	weight_9	weight_10
1	-0.44									
3	0.17	0.22	-0.27							
5	1.90	0.34	-2.16	-0.08	0.42	0.00				
7	0.29	0.03	0.85	0.28	-1.16	-0.09	0.24			
10	2.44	0.57	-4.84	-3.22	3.63	3.49	-1.32	-1.30	0.17	0.16
12	4.99	-1.21	-15.19	0.00	18.06	0.60	-10.16	0.00	2.61	-0.13
20	3.97	-1.58	-9.05	-0.03	5.82	2.95	0.00	-2.38	-0.77	0.26

Also small weights! And some of them are zero! What do zero weights imply?

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What's with this behavior?

Why does Lasso zero out weights, while Ridge does not?

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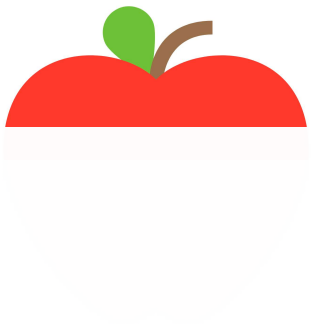
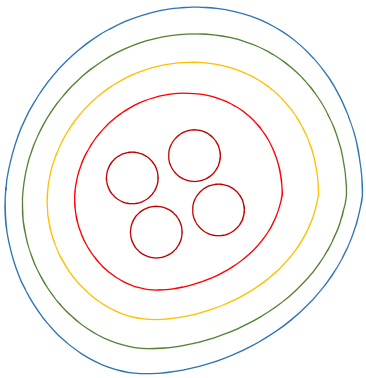
Watch this video for contour plots

https://www.khanacademy.org/math/multivariable-calculus/thinking-about-multivariable-function/visualizing-scalar-valued-functions/v/contour-plots





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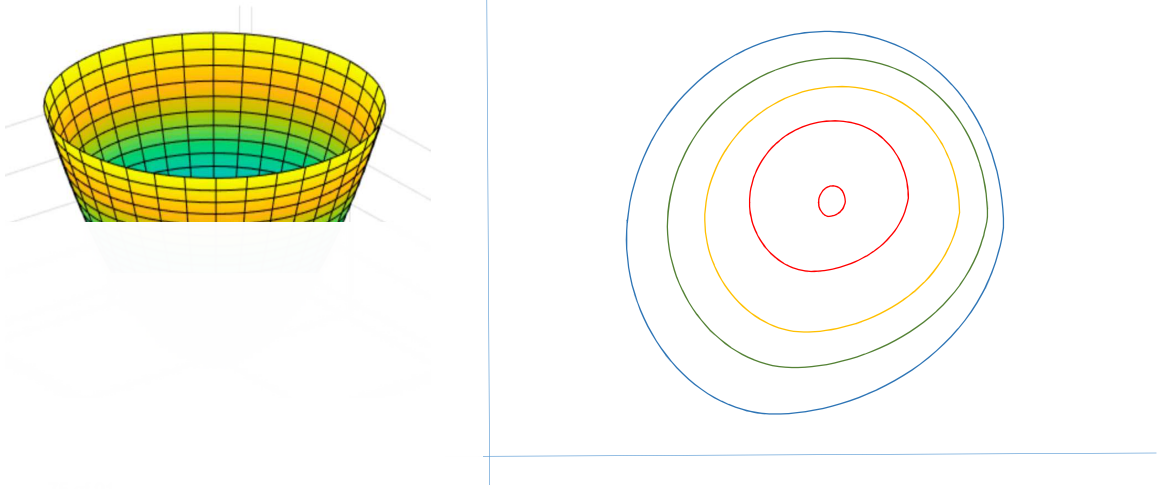
Watch this video for contour plots

https://www.khanacademy.org/math/multivariable-calculus/thinking-about-multivariable-function/visualizing-scalar-valued-functions/v/contour-plots



$$\underset{\theta}{\operatorname{argmin}} L(\theta) = \frac{1}{2N}\sum_{i=1}^{N}(\theta^T x_i - y_i)^2 + \lambda R(\theta)$$

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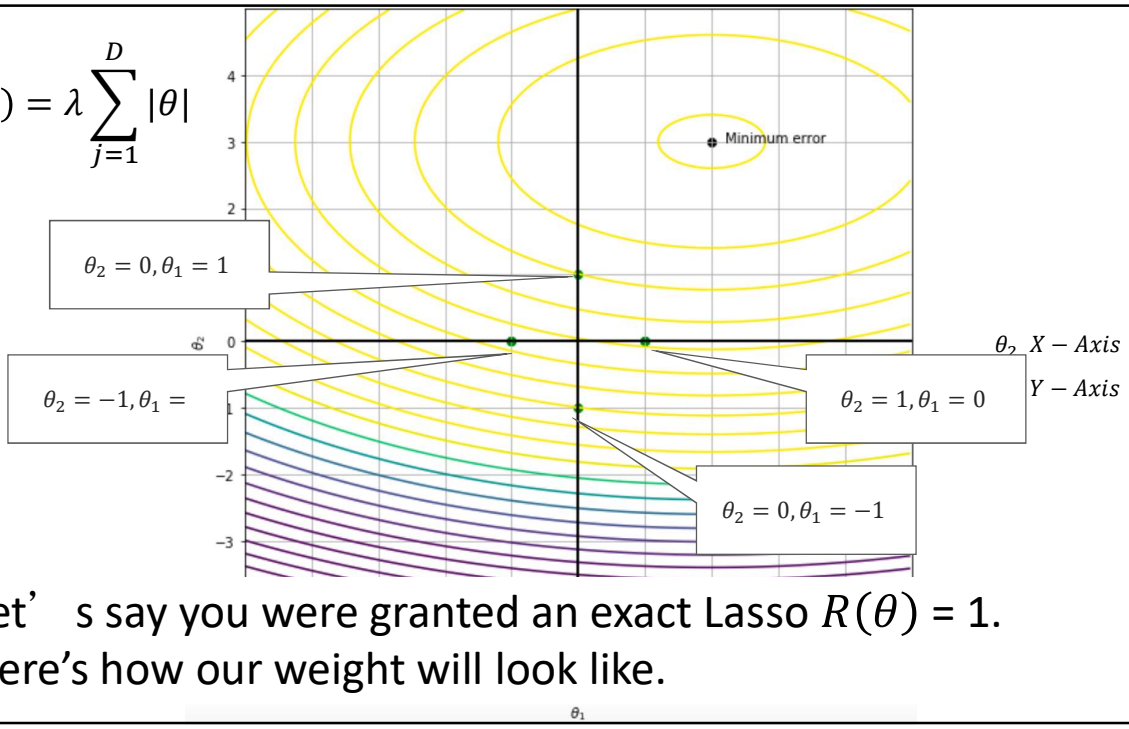
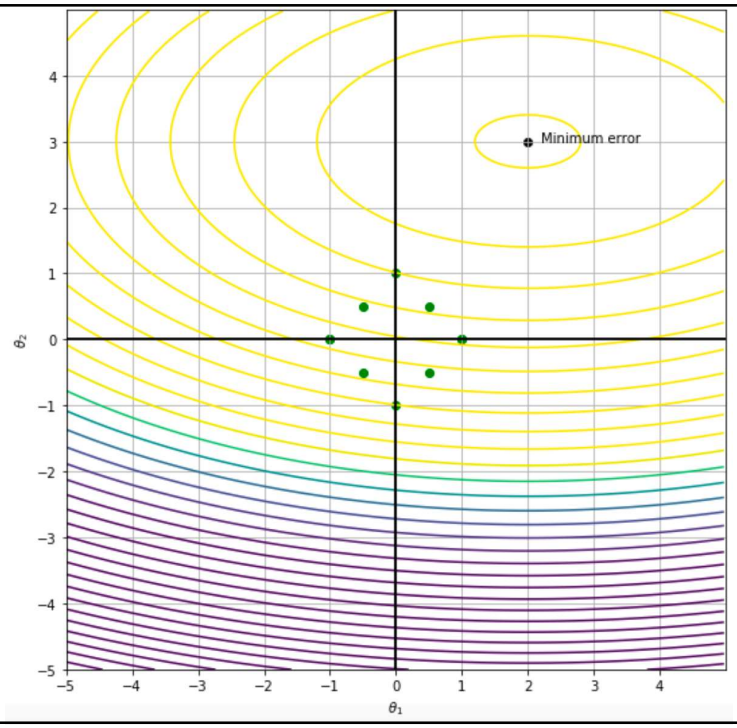
$$R(\theta) = \lambda \sum_{j=1}^{D} |\theta|$$



Let's say you were granted an exact Lasso $R(\theta)$ = 1. Here's how our weight will look like.



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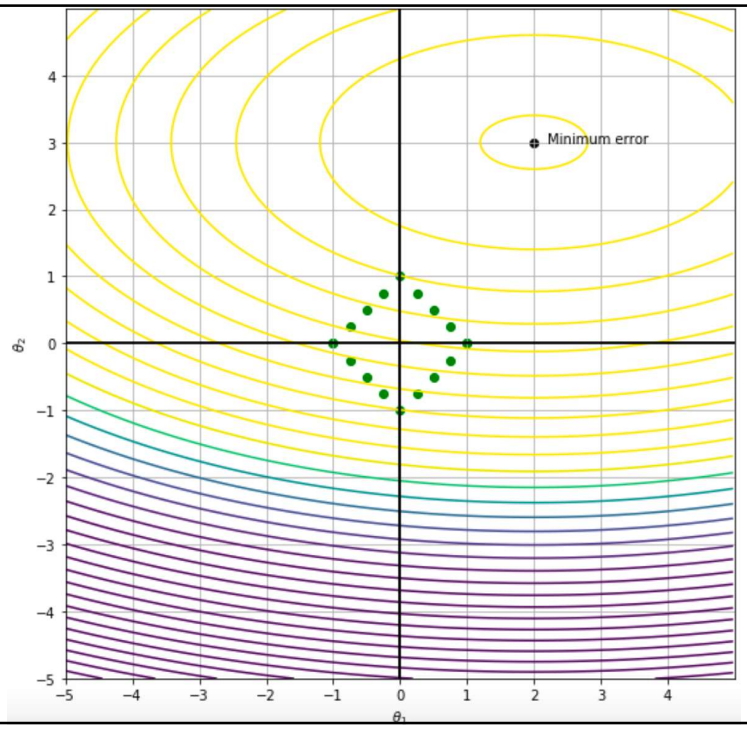

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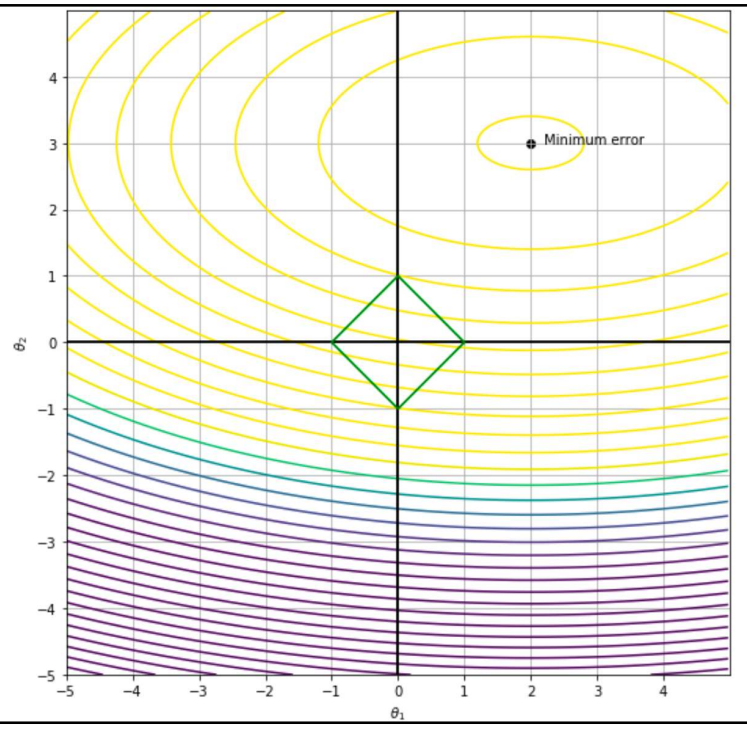

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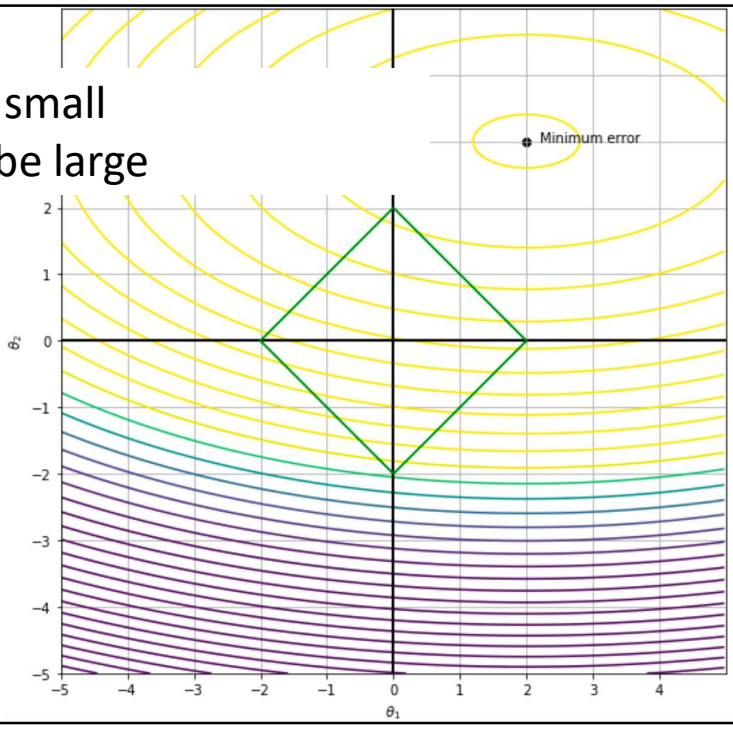
Here is if λ is small
Weights can be large



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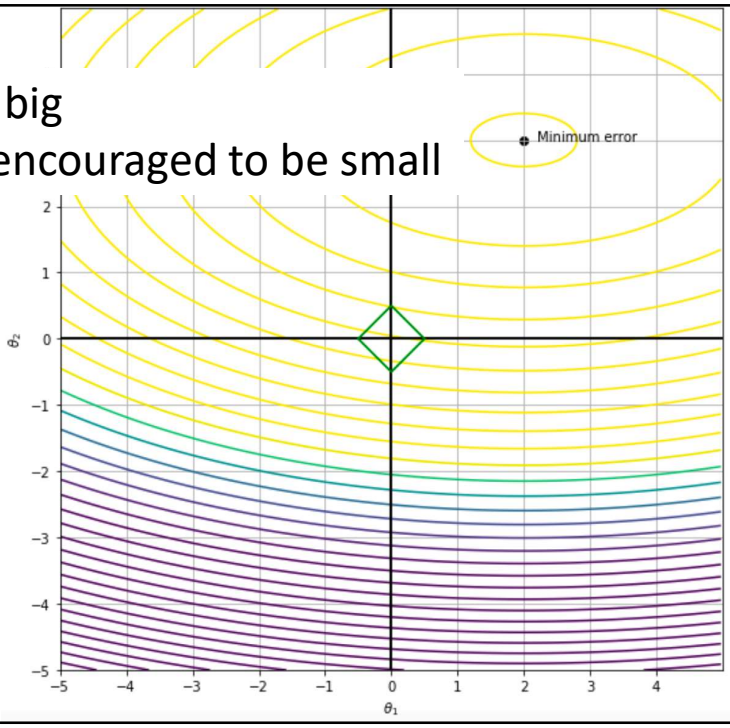
Here is if λ is big
Weights are encouraged to be small



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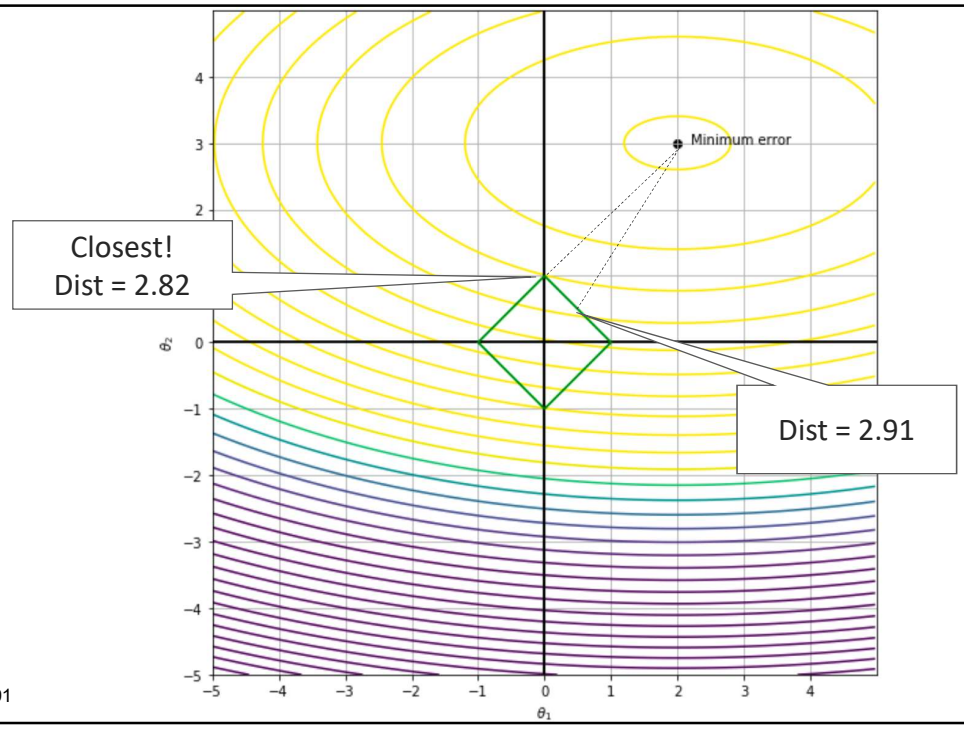

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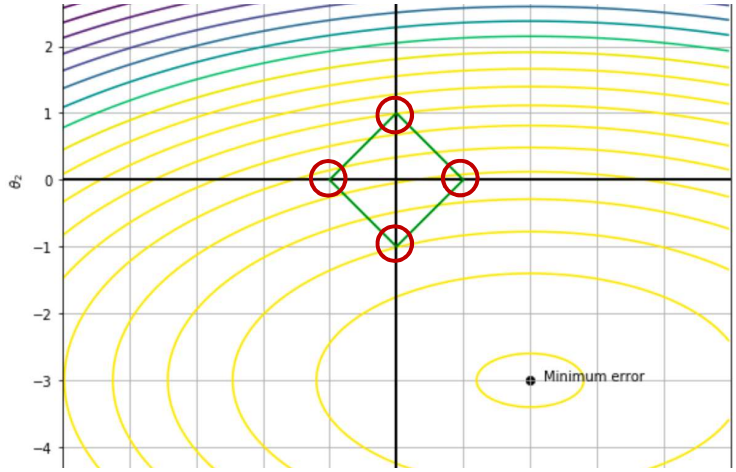
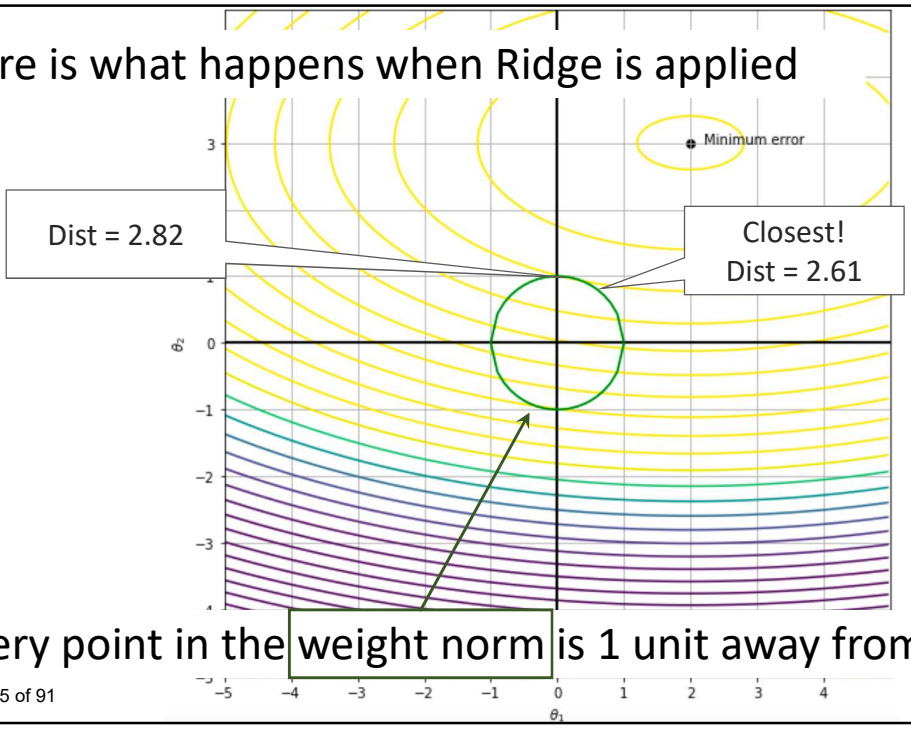
Weights most likely reach the min error when θ is along the axis
These are where its reach is farthest when using Lasso



These are also where some weights are 0

Here is what happens when Ridge is applied



Every point in the weight norm is 1 unit away from 0

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Two common regularization methods

Ridge Regression

L2 Regularization

Weights approach 0, but almost always never reaches 0

Lasso Regression

L1 Regularization
Least Absolute Shrinkage and Selection Operator (LASSO)

Weights approach 0
This makes the feature useless

Good for removing features from dataset with a large number of features

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Summary

- Transform features to generate more complex functions
- Simple functions are prone to under-fitting (high bias, low variance)
- Complex functions are prone to over-fitting (low bias, high variance)
- Overfitting depends on the amount of data relative to the complexity of the hypothesis.
- A good way to prevent overfitting is to use regularization
  - L_2 regularization (Ridge regression) – likely to make weights of irrelevant features small
  - L_1 regularization (LASSO) – likely to make weights of irrelevant features 0

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Next meeting

Moving from regression to classification

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fin

References / Slide Credits

- Andrew Ng – CS229
- Eric Xing – CMU 10-701

Homework -2

- For the next homework, we will implement our own linear regression
- We will be working on a house price prediction case study
- Will be posted tonight
- Deadline – Next Week Friday (March 20)