Complex Analysis

A Concise Review of Complex Analysis

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Chapter 1

Introduction

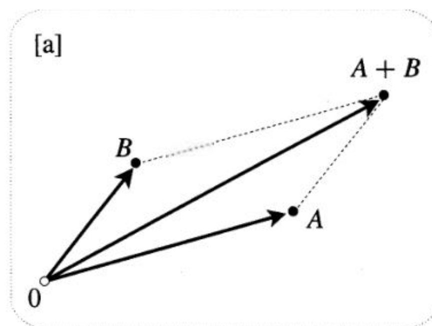
1.1 Introduction

The Historical context for complex numbers is actually quite interesting, considering that complex arise with trivial calculations. Complex numbers were first discovered in around the 1500s, but for the first two and a half centuries much was not accomplished. To make one feel better in 1770 even Euler argued that $\sqrt{-2}\sqrt{-3} = \sqrt{6}$. Consider with your current knowledge why this is not the case.

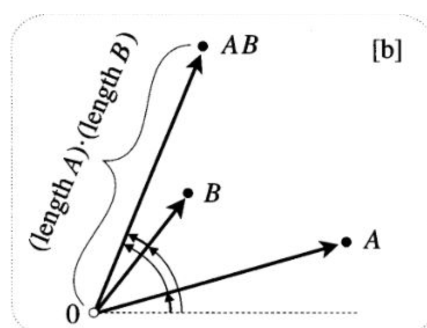
Later people recognized it was convenient to represent complex numbers as vectors in the complex plane. The plane is denoted $\mathbb{C}$. The form for complex plane followed that of the $x - y$ axis where the form of a complex number is $a + bi$

The operations of adding and multiplying (by extension subtraction and division) can be now given geometric meaning.

The sum of two complex numbers $A + B$ is given by the parallelogram rule of ordinary vector addition.



The length of AB is the product of the lengths of A and B, and the angle of AB is the sum of the angles of A and B.



It would take until people figured out how to do calculus with complex numbers would the field of complex analysis emerge. In summary people were not that concerned about complex numbers initially because when solving there quadratic functions

$$x^2 = mx + c$$

and the quadratic formula

$$x = \frac{1}{2}[m \pm \sqrt{m^2 + 4c}]$$

the simply discounted when $m^2 + 4c$ was negative. And using there geometric intuitions of the intersection of a parabola and a line, it also showed that there was no solution.

Now we enter Bombelli who considers the the cubic function and the solution to the cubic functions. In the case of a cubic function. We know that a cubic curve and a line must intersect, but when their formula resulted in complex numbers Bombelli figured the solution must exist. Hence, arrived the theory for adding complex numbers.

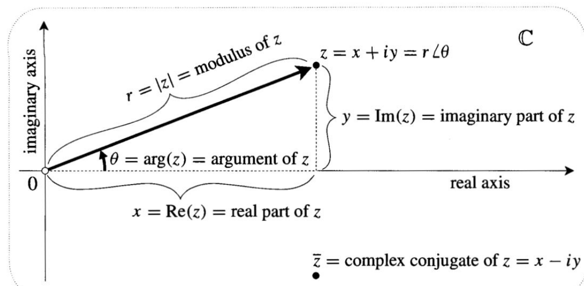
It is important to understand that a complex number is a single, indivisible entity - a point in the plane.

The rule for addition of complex numbers is simple to understand. The multiplication rule is just as simple when considered in polar coordinates. In place of $z = x + iy$ we may write $z = r\angle\theta$. The geometric multiplication now shows

$$(R\angle\phi)(r\angle\theta) = (Rr)\angle(\phi + \theta)$$

Though the polar and Cartesian labels may seem similar consider that θ is periodical thus the representation of z is not unique in the polar form. Consider the following table and diagram to further understand the notation and terms.

Name	Meaning	Notation
modulus of z	length r of z	$\lvert z \rvert$
argument of z	angle θ of z	$\arg(z)$
real part of z	x coordinate of z	$\mathrm{Re}(z)$
imaginary part of z	y coordinate of z	$\mathrm{Im}(z)$
imaginary number	real multiple of i	
real axis	set of real numbers	
imaginary axis	set of imaginary numbers	
complex conjugate of z	reflection of z in the real axis	$\overline{z}$



1.2 Euler's Formula

We replace the $r\angle\theta$ notation with a really dope one

$$e^{i\theta} = \cos\theta + i\sin\theta$$

The derivation for this formula can easily be found, and one should already be familiar with. The Taylor Series expansion is a useful insight for this.
Now we can represent $z = re^{i\theta}$. This leads to the geometric rule for multiplying complex numbers:

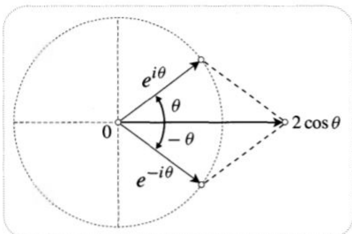
$$(Re^{i\phi})(re^{i\theta}) = Rre^{i(\phi+\theta)}$$

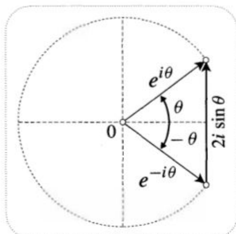
Which makes the geometric property so obvious.
Another useful property from Euler's formula is derivation of sine and cosine function with respect to the exponential.

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \quad and \quad e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$

which gives us

$$\frac{e^{i\theta}+e^{-i\theta}}{2}=\cos\theta \quad \textit{and} \quad \frac{e^{i\theta}-e^{-i\theta}}{2i}=\sin\theta$$





1.3 Exercises

1. Express sine and cosine as an expression with the exponential

2. Use the above expression to derive angle addition properties

3. What is $\sqrt{-2}\sqrt{-3}$

4. Express the angle addition property in polar form

5. Express the multiplicative properties in polar form

6. What is $z+w$ where

$$z = 5+4i$$

and

$$w = -4+7i$$

Chapter 2

Complex Numbers

2.1 Sums and Products

Complex numbers can be represented as ordered pairs (x, y) where the complex number is given by $z = x + iy$. Two complex numbers are said to be equal iff their respective real and imaginary parts are equal

Example 2.1.1.

$$z = 4 + 3i \tag{2.1}$$
$$w = 5 + 3i \tag{2.2}$$
$$z \neq w \tag{2.3}$$

The sum of two complex numbers is defined by the sum of the respective real and imaginary parts. That is

Example 2.1.2.

$$z = (x, y) \quad w = (u, v) \tag{2.4}$$
$$z + w = (x + u, y + v) \tag{2.5}$$

2.2 Basic Algebraic Properties

Most of the properties for addition and multiplication are same for complex numbers

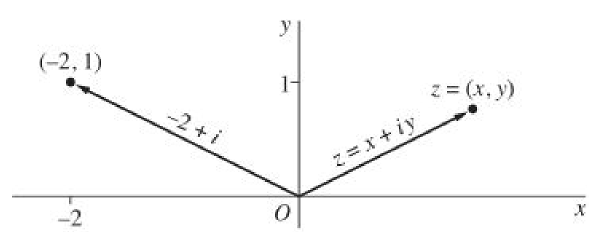
The communitive properties still hold

$$z + w = w + z \tag{2.6}$$
$$zw = wz \tag{2.7}$$

Similarly, the associative and distributive properties hold as well. This will be verified in the exercises section.

2.3 Vectors and Moduli

Complex numbers can be represented as a vector from the origin to the point (x, y). Consider the figure below showing the point $-2 + i$



Treating complex numbers as vectors allows us to take their distance from the origin, or the modulus. Denoted by $|z|$. We can now

compare two complex numbers by their modulus.
The modulus can be computed using Pythagorean theorem.

Example 2.3.1. Describe the figure represented by the equation $|z - 1 + 3i| = 2$
This represents a circle whose center $z_0 = (1, -3)$ and whose radius is $R = 2$

2.4 Complex Conjugates

The complex conjugate or simply the conjugate of a complex number

$$z = x + iy$$

is define by

$$\bar{z} = x - iy$$

and denoted by $\bar{z}$. Considering the ordered pairs it is apparent that the conjugate is a reflection of the complex number over the real axis.

2.5 Exercises

1. Show that complex numbers are distributive and associative

2. Show the following

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
2. derive the modulus of a complex number
3. $\Re z = \dfrac{z + \bar{z}}{2}$
4. $\Im z = \dfrac{z - \bar{z}}{2i}$
5. z^{-1}

Chapter 3

Analytic Functions

This section will introduce analytic functions and the differentiation for complex valued functions.

3.1 Functions of a Complex Variable

The convention is that a complex function, $f(z)$ may have values of $w = u + vi$ this form follows the input giving us

$$f(x + iy) = w = u + iv$$

where the two parts of the output are functions with respect to x and y such as

$$u(x, y) \text{ and } v(x, y)$$

This gives us the notation

$$f(z) = u(x, y) + iv(x, y)$$

This shows that a complex function can be considered a two-dimensional mapping, which can be difficult to visualize.
It may also be useful to consider the function in polar form. For this we consider Euler's formula and find

$$f(re^{i\theta}) = u + iv$$

in the case that $z = re^{i\theta}$ we can express the function as

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Example 3.1.1. if $f(z) = z^2$, then

$$f(x + iy) = u + iv \tag{3.1}$$
$$z = x + iy \tag{3.2}$$

applying the function we find

$$f(z) = (x + iy)^2 \tag{3.3}$$
$$f(z) = x^2 - y^2 + 2ixy \tag{3.4}$$

This gives us the values for u and v

$$u(x, y) = x^2 - y^2 \tag{3.5}$$
$$v(x, y) = 2xy \tag{3.6}$$

In polar coordinates this is given by

$$z = re^{i\theta} \tag{3.7}$$
$$f(z) = z^2 = r^2e^{i2\theta} \tag{3.8}$$

if we express this in terms of sine and cosine

$$r^2e^{i2\theta} = r^2\cos 2\theta + r^2 i\sin 2\theta \tag{3.9}$$
$$u(r,\theta) = r^2\cos 2\theta \tag{3.10}$$
$$v(r,\theta) = r^2\sin 2\theta \tag{3.11}$$

3.2 Limits

The definition for limits of complex function is similar to that of a multivariable function. The limit must exist from all directions for the limit to exist.

Example 3.2.1.

$$f(z) = \frac{z}{\bar{z}} \tag{3.12}$$

the limit

$$\lim_{z\to 0} f(z) \tag{3.13}$$

does not exist because the limit does not agree from all directions. We can consider two directions from the real axis and from imaginary axis. From the real axis the values for $z = x + i0$ so the limit becomes

$$\lim_{x\to 0}\frac{x}{x} = 1 \tag{3.14}$$

From the imaginary axis it is

$$\lim_{y\to 0}\frac{y}{-y} = -1 \tag{3.15}$$

The limit for a complex number w of the form

$$w = u + iv$$

exists if and only if the limits for u and v exist.
If a function is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that

$$|f(z)| \leq M$$

for all points z in R. We prove this by finding the maximum distance a point is from a point.

$$\sqrt{[u(x,y)]^2 + [v(x,y)]^2}$$

Since f is continuous, the function above is continuous as well. Showing that there must be a maximum value M

3.3 Continuity

A function f is continuous at a point z_0 if the following three statements are satisfied

1. $\lim\limits_{z\to z_0} f(z)$ exists,
2. $f(z_0)$ exists,
3. $\lim\limits_{z\to z_0} f(z) = f(z_0)$

Note that the last condition also says, that for each positive number ϵ, there is a positive number δ such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta$$

Theorem 3.3.1. A composition of continuous functions is itself continuous

Theorem 3.3.2. if a function $f(z)$ is continuous and nonzero at a point z_0, then $f(z) \neq 0$ throughout some neighborhood of that point.

Assuming that $f(z)$ is, in fact, continuous and nonzero at z_0, we can prove Theorem 2 by assigning the positive value $|f(z_0)/2$ to the number ϵ in statement considering the limit. This tells us that there is a positive number δ such that

$$|f(z) - f(z_0) < \frac{|f(z_0)|}{2} \text{ whenever } |z - z_0| < \delta$$

So if the point z is in the neighborhood $|z - z_0| < \delta$ at which $f(z) = 0$, we have the contradiction

$$|f(z_0)| < \frac{|f(z_0)|}{2}$$

Theorem 3.3.3. if a function f is continuous throughout a region R that is both closed and bounded, there exists a nonnegative real number M such that

$$|f(z)| \leq M \text{ for all points } z \text{ in } R$$

where equality holds for at least on such z

3.4 Derivatives

The derivative for complex functions can be similarly evaluated.
The derivative of f at z_0 is the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

and the function f is differentiable at z_0 when $f'(0)$ exists.
By expressing $\Delta z = z - z_0$ where $(z \neq z_0)$ we can write the derivative as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

because f is defined throughout a neighborhood of z_0.
We often drop the subscript on z_0 and introduce

$$\Delta w = f(z + \Delta z) - f(z)$$

which denotes a change in the value of $w = f(z)$ of f corresponding to a change Δz in the point at which f is evaluated. Then, if we write dw/dz for $f'(z)$, equation becomes

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

Example 3.4.1. Suppose that $f(z) = z^2$. At any point z,

$$\lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} (2z + \Delta z) = 2z$$

Example 3.4.2. Consider the real-valued function $f(z) = |z|^2$. Here

$$\frac{\Delta w}{\Delta z} = \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} \tag{3.16}$$

using the fact $|z|^2 = z\overline{z}$ and that the conjugate is distribuitve over addition.

$$\frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z} \tag{3.17}$$

$$\frac{\Delta w}{\Delta z} = \overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z} \tag{3.18}$$

We can now consider the limit from two directions: the real axis and the imaginary axis. Considering each of these axis, would mean setting Δx and Δy to zero respectively. From the real axis $\Delta y = 0$, so the limit of the last term

$$\frac{\overline{\Delta z}}{\Delta z} = 1 \tag{3.19}$$

With respect to the imaginary axis, $\Delta x = 0$, so the limit is

$$\frac{\overline{\Delta z}}{\Delta z} = -1 \tag{3.20}$$

Now finding the limit gives two expressions

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \overline{z} + z \tag{3.21}$$

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \overline{z} - z \tag{3.22}$$

The only point when both limits are true is for the values $z = 0$, thuse the derivative does exist there and the derivative is 0

$$\tag{3.23}$$

Example 3 shows that a function $f(z) = u + iv$ can be differentiable at a point z but nowhere else in any neighborhood of that point.
Since

$$u(x, y) = x^2 + y^2 \text{ and } v(x, y) = 0$$

when $f(z) = |z|^2$, it also shows that the real imaginary components of a functions of a complex variable can have continuous partial derivatives of all order at a point $z = (x, y)$ and yet the function may not be differentiable there.
The function $f(z) = |z|^2$ is continuous at each point in the plane since its components are continuous at each point. So the continuity of a function at a point does not imply the existence of a derivative there. It is, however, true that *the existence of the derivative of a function at a point implies the continuity of the function at that point.*
Most formulas for differentiating extend to complex functions as well, such as: power rule, rules for constants, addition, multiplication, division and chain rule.

Example 3.4.3. The derivative of $(2z^2+i)^5$, write $w = 2z^2 + i$ and $W = w^5$. Then

$$\frac{d}{dz}(2z^2 + i)^5 = 20z(2z^2 + i)^4$$

3.5 Cauchy-Riemann Equations

In this section, we will discuss the conditions required to show that a complex valued function is differentiable.
These conditions are given as follows

Theorem 3.5.1. Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

there. Also, $f'(z_0)$ can be written

$$f'(z_0) = u_x + iv_x$$

where these partial derivatives can be evaluated at (x_0, y_0).

We will now arrive at the Cauchy-Riemann conditions, consider

$$f(z) = u(x, y) + iv(x, y)$$

which must satisfy at a point $z_0 = (x_0, y_0)$ when the derivative of f exists here. Assuming the derivative does in fact exist

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \quad (3.24)$$

we know the limit can be expressed as

$$f'(z_0) = \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\mathrm{Re} \frac{\Delta w}{\Delta z} \right) + i \lim_{(\Delta x, \Delta y) \to (0,0)} \left(\mathrm{Im} \frac{\Delta w}{\Delta z} \right) \quad (3.25)$$

It is important to note that the expression can approach $(0, 0)$ in any manner. In particular, it is useful to consider the limit from the vertical $(0, \Delta y)$ and the horizontal $(\Delta x, 0)$

Specifically from horizontal, when $\Delta y = 0$ gives

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \quad (3.26)$$

Thus

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\mathrm{Re} \frac{\Delta w}{\Delta z} \right) = u_x(x_0, y_0) \quad (3.27)$$

and

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \left(\mathrm{Im} \frac{\Delta w}{\Delta z} \right) = v_x(x_0, y_0) \quad (3.28)$$

where $u(x_0, y_0)$ and $v(x_0, y_0)$ are the first order partial derivatives with respect to x of the functions u and v. substitution of these limits give us

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0) \quad (3.29)$$

We might have let Δz tend to zero vertically in which case $\Delta x = 0$

$$\frac{\Delta w}{\Delta z} = \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i \Delta y} + \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i \Delta y}$$

$$\frac{\Delta w}{\Delta z} = \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} - \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} \quad (3.30)$$

which gives

$$f'(z_0) = v_y(x_0, y_0) - i u_x(x_0, y_0) \quad (3.31)$$

Comparing the two different expressions for the derivative of the same function, we can find the conditions. Remember that for two complex numbers to equal the real and imaginary parts must be equal respectively. Thus, we can set the two respective parts equal to each other and find the conditions.

Example 3.5.1. In Example 3.4.1 we showed that the derivative of the function $f(z) = z^2$ is equal to $2z$. Here we show that the derivative exists everywhere using the Cauchy-Riemann equations

$$u(x, y) = x^2 - y^2 \quad (3.32)$$
$$v(x, y) = 2xy \quad (3.33)$$

Thus

$$u_x = 2x = v_y, \quad (3.34)$$
$$u_y = -2y = -v_x \quad (3.35)$$

Moreover, according to the Theorem 3.5.1

$$f'(z) = u_x + iu_y$$
$$f'(z) = 2x + i2y = 2(x + iy) = 2z \quad (3.36)$$

Cauchy-Riemann equations can also be used to determine where the derivate of a function does not exist.

Example 3.5.2. If we consider the equation from Example 3.4.2, $f(z) = |z|^2$ we can arrive at our previous result, that f is only differentiable when $z = 0$

$$|z|^2 = z\overline{z} = (x + iy)(x - iy) = x^2 + y^2 \quad (3.37)$$
$$u_x = 2x \quad v_y = 0 \quad (3.38)$$
$$u_y = 2y \quad -v_x = 0 \quad (3.39)$$

The following conditions are only true if $z = 0$ which follows the solution we found earlier

$$f'(z) = 0 \quad (3.40)$$

3.6 Polar Coordinates

Assuming that $z_0 \neq 0$, we shall show the Cauchy-Riemann conditions in polar coordinates, where

$$x = r\cos\theta, \quad y = r\sin\theta$$

Depending on whether we write

$$z = x + iy \quad \text{or} \quad z = re^{i\theta}$$

The Cauchy-Riemann condition for polar coordinates are given such:

Theorem 3.6.1. Let the function

$$f(z) = u(r, \theta) + iv(r, \theta)$$

be defined through some ϵ neighborhood of a nonzero point $z_0 = r_0e^{i\theta_0}$ and suppose that

1. the first-order partial derivatives of the functions u and v with respect to r and θ exist everywhere in the neighborhood;
2. those partial derivatives are continuous at (r_0, θ_0) and satisfy the polar form

   $$ru_r = v_\theta,$$

   $$u_\theta = -rv_r$$

   of the Cauchy-Riemann equations at (r_0, θ_0)

Then $f'(z_0)$ exists, its value being

$$f'(z_0) = e^{-i\theta}(u_r + iv_r),$$

where the right hand side is the be evaluated at (r_0, θ_0)

We can prove the theorem, by using the chain rule and definition of x and y in polar coordinates. Rememver that r is a function of x and y, and so is θ.

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad (3.41)$$
$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} \quad (3.42)$$

Which gives

$$u_r = u_x \cos\theta + u_y \sin\theta, \tag{3.43}$$
$$u_\theta = -u_x r \sin\theta + u_y r \cos\theta \tag{3.44}$$

likewise

$$v_r = v_x \cos\theta + v_y \sin\theta, \tag{3.45}$$
$$v_\theta = -v_x r \sin\theta + v_y r \cos\theta \tag{3.46}$$

Since, $u_x = v_y$ and $u_y = -v_x$ we can make this substitution

$$v_\theta = u_y r \sin\theta + u_x r \cos\theta \tag{3.47}$$

Which is equal to

$$v_\theta = r u_r \tag{3.48}$$

likewise

$$v_r = -u_y \cos\theta + u_x \sin\theta \tag{3.49}$$
$$-r v_r = u_\theta \tag{3.50}$$

Example 3.6.1. Consider the function

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$$

For when $z \neq 0$. The conditions for the Cauchy-Riemann equations are satisfied everywhere in the domain of the function.

$$r u_r = -r\frac{\cos\theta}{r^2} = v_\theta = -\frac{\cos\theta}{r} \tag{3.51}$$
$$-r v_r = -r\frac{\sin\theta}{r^2} = u_\theta = -\frac{\sin\theta}{r} \tag{3.52}$$

The conditions hold, and the function is differentiable everywhere in its domain.

3.7 Analytic Functions

A function f is said to be analytic if the complex variable z is analytic at a point z_0 if it has derivative at each point in the neighborhood of z_0

Theorem 3.7.1. If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant throughout D.

Example 3.7.1. if

$$f(z) = \cosh x \cos y + i \sinh x \sin y$$

The component functions u and v follow the Cauchy-Riemann conditions. Therefor the function is analytic.

3.8 Harmonic Functions

A function is said to be harmonic if it has partial derivatives of the first and second order that satisfy the PDE

$$H_{xx}(x, y) + H_{yy}(x, y) = 0$$

known as Laplace's Equation. Harmonic functions play an important role in applied mathematics.

3.9 Reflection Principle

Some analytic functions have the property that

$$\overline{f(z)} = f(\overline{z})$$

for all points z in certain domains.

Theorem 3.9.1. Suppose that a function f is analytic in some domain D which contains a segment of the x axis and whose lower half is the reflection of the upper half with respect to that axis. Then

$$\overline{f(z)} = f(\overline{z})$$

for each point z in the domain if and only if $f(x)$ is real for each point x on the segment. 2

Prove the theorem above as an exercise.

Chapter 4

Elementary Functions

We define analytic functions of a complex variable z that reduce to the elementary functions in calculus when $z = x + i0$.

4.1 The Exponential Function

We define the exponential function e^z

$$e^z = e^x + e^{iy}$$

since $z = x = iy$.
The addition and multiplication formulas for real-valued exponential functions also hold for complex-valued. In fact

$$\frac{d}{dz}e^z = e^z$$

everywhere in the z plane. This also shows that

$$e^z \neq 0 \quad \text{for any complex number } z$$

This can be shown, writing

$$e^z = \rho e^{i\phi}$$

where $p = e^x$ and $\phi = y$ which tells us that

$$|e^z| = e^x$$

and

$$arg(e^z) = y + 2n\pi$$

Some properties of e^z are interesting. For example, since

$$e^{z+2\pi i} = e^z e^{2\pi i} \quad \text{and} \quad e^{2\pi i} = 1,$$

we find that e^z is periodic, with a pure imaginary period $2\pi i$:

$$e^{z+2\pi i} = e^z.$$

e^z can also be negative, recall Euler's identity.

Example 4.1.1. In order to find the numbers $z = x + iy$ such that

$$e^z = 1 + i$$

we write the equation as

$$e^x e^{iy} = \sqrt{2}e^{i\pi/4}$$

Then, we can express it in the form

$$e^x = \sqrt{2} \quad \text{and} \quad y = \frac{\pi}{4} + 2n\pi$$

thus

$$x = \ln\sqrt{2} = \frac{1}{2}\ln 2$$

and so

$$z = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i$$

4.2 The Logarithmic Function

The definition of the logarithmic function is based on solving the equation

$$e^w = z$$

for when w is a nonzero complex number and $w = u + iv$ and $z = re^{i\theta}$.

$$e^u e^{iv} = re^{i\theta}$$

based on the equality for two complex numbers

$$e^u = r \text{ and } v = \theta + 2n\pi$$

. The equation is satisfied iff

$$w = \ln r + i(\theta + 2n\pi)$$

Thus, if we write

$$\log z = \ln r + i(\theta + 2n\pi)$$

equation tells us that

$$e^{\log z} = z$$

which serves as the definition for the logarithmic function.

The expression van be written as

$$\log z = \ln|z| + i\arg z$$

The principal value of $\log z$ is when $n = 0$.

4.3 Branches and Derivatives of Logarithms

The logarithm function can be written as

$$\log z = \ln r + i\theta$$

the complex function has components

$$u(r,\theta) = \ln r \quad \text{and} \quad v(r,\theta) = \theta$$

We can find that these components satisfy the Cauchy-Riemann conditions for polar functions. Furthermore we can find that

$$\frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Chapter 5

Integrals

The theorems are generally concise and powerful, and many of the proofs are short.

5.1 Derivatives of Functions $w(t)$

First consider derivatives of complex-valued functions w of a real variable t.

$$w(t) = u(t) + iv(t)$$

, where the functions u and v are real valued functions of t. The derivative $w'(t)$, of the function at a point t is defined as

$$w'(t) = u'(t) = iv'(t)$$

provided the respective derivatives exist.
Also

$$\frac{d}{dt}[z_0 w(t)] = z_0 w'(t).$$

Another expected rule is

$$\frac{d}{dt}e^{z_0 t} = z_0 e^{z_0 t}$$

One can prove this as a simple exercise.

5.2 Definite Integrals of Functions $w(t)$

For a complex valued functions of a real variable t

$$w(t) = u(t) + iv(t)$$

the definite integral of $w(t)$ over an interval

$$a \leq t \leq b$$

is defined as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the integrals exist, the real and imaginary parts of the integral are given by just taking the integral of the real and imaginary components of the function.

Example 5.2.1. For and illustration

$$\int_0^1 (1+it)^2 dt = \int_0^1 (1-t^2)dt + i\int_1^0 2tdt \tag{5.1}$$

$$\int_0^1 (1+it)^2 dt = \frac{2}{3} + i \tag{5.2}$$

5.3 Contours

Integrals of a complex functions of a complex variable are defined on curves in the complex plane, rather than on just intervals of the real line. Classes of curves that are adequate for the study of such integrals are introduced in this sections.

A set of points $z = (x, y)$ in the complex plane is said to be an arc if

$$x = x(t), \quad y = y(t) \quad (a \le t \le b),$$

where $x(t)$ and $y(t)$ are continuous functions of the parameter t. The definition establishes a continuous mapping in the interval. It is convenient to describe the points of C by means of the equation $z = z(t)$ where

$$z(t) = x(t) + iy(t).$$

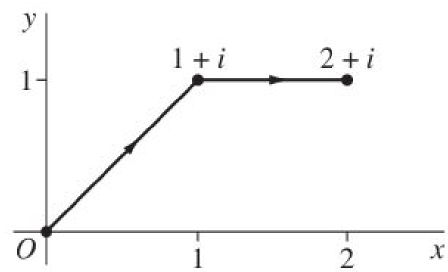
The arc C is a simple arc, or a Jordan arc, if it does not cross itself; that is, C is simple if $z(t_1) \neq z(t_2)$ when $t_1 \neq t_2$. When the arc C is simple except for the fact that $z(b) = z(a)$, we say that C is a simple closed curve, or Jordan curve. Such a curve is positively oriented when it is in the counterclockwise direction.

The geometric nature of a particular arc often suggests different notation for the parameter t. This is, in fact, the case in the following examples.

Example 5.3.1. The polygonal line defined by means of the equations

$$z = \begin{cases} x + ix & when \quad 0 \le x \le 1, \\ x + i & when \quad 1 \le x \le 2 \end{cases} \tag{5.3}$$

and consisting of a line segment from 0 to $1+i$ followed by one from $1+i$ to $2+i$ is a simple arc



Example 5.3.2. The unit circle

$$z = e^{i\theta} \quad (0 \le \theta \le 2\pi)$$

about the origin is a simple closed curve, oriented in the counterclockwise direction centered at the point z_0 and with radius R.

5.4 Contour Integrals

We turn now to integrals of complex valued functions f of the complex variable z. This integral is defined along a contour C, extending from z_1 to z_2, in the complex plane. It is therefore a line integral, written

$$\int_C f(z)dz \quad or \quad \int_{z_1}^{z_2} f(z)dz,$$

the latter notation is used when the value of the integral is independent of the choice of contour taken between two fixed end points.

For the equation

$$z = z(t) \quad (a \le t \le b)$$

representing a contour C, extending from points z_1 to z_2 the integral

$$\int_C f(z)dz \quad = \int_a^b f[z(t)]z'(t)dt$$

we also find the property that

$$\int_{-C} f(z)dz \quad = -\int_C f(z)dz$$

5.5 Some Examples

Example 5.5.1. Let us find the value of the integral

$$I = \int_C \overline{z}dz \tag{5.4}$$

when C is the right-hand half

$$z = 2e^{i\theta} \quad \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right) \tag{5.5}$$

of the circle with radius 2

$$I = \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}}(2e^{i\theta})'d\theta = 4\int_{-\pi/2}^{\pi/2} \overline{e^{i\theta}}(e^{i\theta})'d\theta \tag{5.6}$$

since

$$\overline{e^{i\theta}} = e^{-i\theta} \tag{5.7}$$

we find that the value of the integral is $4\pi i$

$$\tag{5.8}$$

5.6 Upper Bounds for Moduli of Contour Integrals

Theorem 5.6.1. Let C denote a contour of length L, and suppose that a funciton $f(z)$ is piecewise continuous on C. if M is a nonnegative constant such that

$$|f(z)| \le M$$

for all points z on C at which $f(z)$ is defined, then

$$\left|\int_C f(z)dz\right| \le ML$$

Example 5.6.1. Let C be the arc of the circle $|z| = 2$ from 2 to 2i, in the first quadrant. We can show that

$$\left|\int_C \frac{z+4}{z^3-1}dz\right| \le \frac{6\pi}{7}$$

5.7 Antiderivatives

Although the value of a contour integral of a function $f(z)$ from a fixed point z_1 to a fixed point z_2 depends, in general, on the path taken, there are certain functions whose integrals from these points have values independent of path. This also leads to the fact that such integrals exist of closed paths of value zero. This leads to the following theorem

Theorem 5.7.1. Suppose that a function $f(z)$ is continuous on a domain D. If any one of the following statements is true, then so are the others:

1. $f(z)$ has an antiderivative $F(z)$ throughout D;

2. the integrals of $f(z)$ along contours lying entirely in D and extending from any fixed point z_1 to any fixed point z_2 all have the same value, namely

$$\int_{z_1}^{z_2} f(z)dz = F(z)]_{z_1}^{z_2} = F(z_2) - F(z_1)$$

where $F(z)$ is the antiderivative in statement (a);

3. the integrals of $f(z)$ around closed contours lying entirely in D all have value zero.

It should be emphasized that the theorem does not claim that any of these statements is true for a given function $f(z)$. It says only that all of them are true or that none of them is true.

Example 5.7.1. The continuous function $f(z) = z^2$ has an antiderivative $F(z) = z^3/3$ throughout the plane. Hence

$$\int_0^{1+i} z^2 dz = \left[\frac{z^3}{3}\right]_0^{1+i}$$

$$= \frac{1}{3}(1+i)^3 = \frac{2}{3}(-1+i)$$

for every contour from $z = 0$ to $z = 1 + i$

5.8 Cauchy-Goursat Theorem

Suppose that simple function f is analytic in a simply connected domain D. Then for every simple closed contour C in D,

$$\oint f(z)dz = 0$$

The domain is just an open set where two points can be joined by a polygonal path inside. A simply connected domain exists when all the points in the contour are in the domain and when the domain has no holes.
Another way to state the theorem is: if f is analytic everywhere within and on C, which is simple and closed, then the same theorem holds.
consider the following example

Example 5.8.1. C is a circle $|z| = 1$

$$\oint_C \frac{e^z}{3z+4} dz =$$

Since the point when this is not defined, $z = -4/3$ but that point is outside the contour we know it will be 0 within the contour.

Now lets consider an example where the theorem may seem to fail (spoiler: it does not).

Example 5.8.2. Compute the exact value of $\oint_C \frac{1}{z} dz$, where C is the circle with radius 1 centered at 0. We can do this by first parameterizing the equation

$$\oint_C \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta = 2\pi i$$

We can see that $2\pi i \neq 0$. This happens because, the condition that the function is analytic on and within the contour C does not hold. The function is not analytic on the origin, which is within the circle of radius 1.

5.9 Simply Connected Domains

A simply connected domain D is a domain such that every simple closed contour within it encloses only points of D. The set of points interior to a simple closed contour is an example. The annular domain between two concentric circles is, however, not simply connected.

Theorem 5.9.1. If a function is analytic throughout a simply connected domain D, then

$$\oint_C f(z)dz = 0$$

for every closed contour C lying in D.

Example 5.9.1. If C denotes any closed contour lying in the open disk $|z| < 2$, then

$$\oint_C \frac{ze^z}{(z^2+9)^5}dz = 0.$$

Because of the theorem above. The theorem holds because the function is not analytic for the values $z = \pm 3i$ which lies outside the open disk.

Example 5.9.2. Compute

$$\oint_C ze^z dz$$

where C is the square with vertices $z = 0, 1, 1+i, i$.

Consider calculating this without the Cauchy-Goursat Theorem to confirm the results from the theorem.

5.10 Multiply Connected Domains

A domain that is not simply connected is said to be multiply connected. The following theorem is an adaptation of the Cauchy-Goursat theorem to multiply connected domains.

Theorem 5.10.1. Suppose that

1. C is a simple closed contour, described in the counterclockwise direction;

2. $C_k (k = 1, 2, ..., n)$ are simple closed contours interior to C, all described in the clockwise direction, that are disjoint and whose interiors have no points in common

If a function f is analytic on all of these contours and throughout the multiply connected domain consisting of the points inside C and exterior to each C_k , then

$$\oint_C f(z)dz + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0.$$

Note that in equation, the direction of each path of integration is such that the multiply connected domain lies to the left of that path.

An obvious cororllary from this is that if a contour C_1 lies entirely within another C_2. Then,

$$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz.$$

5.11 Cauchy Integral Formula

Another fundamental boiii shall be established in this review.

Theorem 5.11.1. Let f be analytic everywhere inside and on a simple closed contour C, taken in the positive sense. If z_0 is any point interior to C, then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

The following equation above is called the *Cauchy integral formula.* It tells us that if a function f is to be analytic within and on a simple closed contour C, then the values of f interior to C are completely determined by the values of f on C.

The formula is often written as

$$\int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0),$$

it can be used to evaluate certain integrals along simple closed contours.

Example 5.11.1. Let C be the positively oriented circle $|z| = 2$. Since the function

$$f(z) = \frac{z}{9 - z^2}$$

is analytic within and on C and since the point $z_0 = -i$ is interior to C, we can find that

$$\int_C \frac{z dz}{(9 - z^2)(z + i)} = \int_C \frac{z/(9 - z^2)}{z - (-i)} dz =$$

$$= 2\pi i \left(\frac{-i}{10} \right) = \frac{\pi}{5}$$

5.12 Liouville's Theorem and The Fundamental Theorem of Algebra

Theorem 5.12.1. If a function f is entire and bounded in the complex plane, then $f(z)$ is constant throughout the plane.

Theorem 5.12.2. Any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + ... + a_n z^n$$

where $a_n \neq 0$ of degree $n(n \geq 1)$ has at least one zero.

Chapter 6

Residues and Poles

The Cauchy-Goursat theorem states that if a function is analytic at all points interior to and on a simple closed contour C, then the value of the integral of the function around that contour is zero. If, however, the function fails to be analytic at a finite number of points interior to C, there is, as we shall see in this chapter, a specific number, called a residue, which each of those points contributes to the value of the integral. We develop here the theory of residues

6.1 Isolated Singular Points

A point z_0 is called a singular point of a function f if f fails to be analytic at z_0 byt it analytica t some point in every neiborhood of z_0. A singular point z_0 is said to be isolated.

Example 6.1.1. the function

$$\frac{z+1}{z^3(z^2+1)}$$

has the thee isolated singular points $z = 0$ and $z = \pm i$.

Example 6.1.2. The origin is a singular point of the principal branch

$$Logz = \ln r + i\theta \quad (r > 0, -\pi < \theta < \pi)$$

of the logarithmic function. It is not, however, an isolated singular point since every deleted ϵ neighborhood of it contains points on the negative real axis and the branch is not even defined there. Similar remarks can be made regarding any branch

6.2 Residues

When z_0 is an isolated singular point of the function, there is a positive number R_2 such that f is analytic at each point z for which $0 < |z - z_0| < R_2$. Consequently, $f(z)$ has a Laurent series representation

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + ... + \frac{b_n}{(z-z_0)^n}$$

where the coefficients have integral representations. In particular

$$b_n = \frac{1}{2\pi i}\int_C \frac{f(zdz)}{(z-z_0)^{-n+1}}$$

where C is any positively oriented simple closed contour around z_0 that lies in the punctured disk.
We find the equation

$$\int_C f(z)dz = 2\pi i Res_{z=z_0} f(z).$$

Sometimes we simply use B to denote the residue when the function f and the point z_0 are clearly indicated.

Example 6.2.1. Consider the integral

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz$$

where C is the positively oriented unit circle. Since the integrand is analytic everywhere in the plane except at the origin, it has a laruent series representation $0 < |z| < \infty$. Thus, according to equation, the value of integral is $2\pi i$ times the residue of its integrand at $z = 0$.
To determine that residue, we recall the Maclaurn series representation

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...$$

and use it to write for $\frac{1}{z}$ and find the desired residue

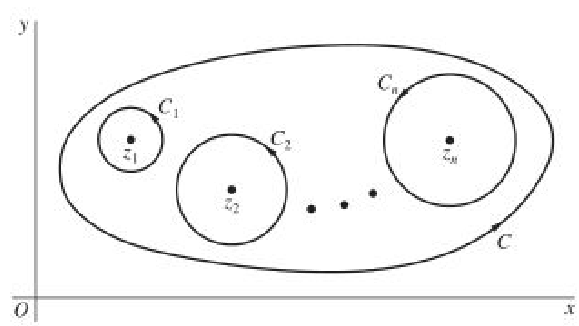
$$= 2\pi i \left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$$

6.3 Cauchy's Residue Theorem

If, except for a finite number of singular points, a function f is analytic inside a simple closed contour C, those singular points must be isolated. The following theorem, which is known as Cauchy's residue theorem, is a precise statement of the fact that if f is also analytic on C and if C is positively oriented, then the value of the integral of f around C is $2\pi i$ times the sum of the residues of f at the singular points inside C.

Theorem 6.3.1. Let C be a simple closed contour, described in the positive sense. If a function f is analytic inside and on C except for a finite number of singular points $z_k(k = 1, 2, ..., n)$ inside C, then

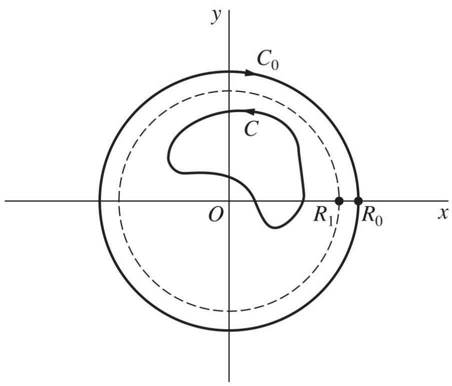
$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z).$$



6.4 Residue at Infinity

Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C. Next, let R_1 denote a positive number which is large enough that C lies inside that circle

$|z| = R_1$. The function f is evidently analytic throughout the domain $R_1 < |z| < \infty$ and, the point at infinity is then said to be an isolated singular point of f.



Now let C_0 denote a circle $|z| = R_0$, oriented clockwise. The residue at infinity is defined by means of the equation

$$\int_{C_0} f(z)dz = 2\pi i Res_{z=\infty} f(z)$$

Theorem 6.4.1. If a function f is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour C, then

$$\int_C f(z)dz = 2\pi \underset{z=0}{Res}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

Example 6.4.1. The integral

$$f(z) = \frac{5z-2}{z(z-1)}$$

around the circle $|z| = 2$, described counterclockwise, by finding the residues of $f(z)$ at $z = 0$ and $z = 1$. Since

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{5-2z}{z(1-z)} = \frac{5-2z}{z} \cdot \frac{1}{1-z}$$

$$= \left(\frac{5}{z} - 2\right)(1 + z + z^2 + ...)$$

$$= \frac{5}{2} + 3 + 3z + ... \quad (0 < |z| < 1)$$

we see that the theorem here can be used where the desired residue is 5.

$$= 2\pi(5) = 10\pi i$$

where C is the circle in question.

6.5 The Three Types of Isolated Singular Points

We saw that the theory is based on the fact that if f has an isolated singular point at z_0, then $f(z)$ has a Laurent series representation involving negative powers of $z - z_0$, is called the principal part of f at z_0. We now use the principal part to identify the isolated singular point z_0 as one of three special types. This classification will aid us in the development of residue theory.

If the principal part of f at z_0 contains at least one nonzero term but the number of such terms is only finite, then there exists a positive integer $m (m \geq 1)$ such that

$$b_m \neq 0$$

and

$$b_{m+1} = b_{m+2} = ... = 0$$

That is, expansion takes the form

$$f(z = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + ... + \frac{b_m}{z - z_0}^m$$

where $b_m \neq 0$. In this case, the isolated singular point z_0 is called a pole of order m. A pole of order $m = 1$ is usually reffered to as a simple pole.
There remain two extremes, the case in which every coefficient in the principal part is zero and the one in which an infinie number of them are nonzero. z_0 is known as a removable singular point. Note that the residue at a removable singular point is always zero. If we define, or possibly redefine, f at z_0 so that $f(z_0) = a_0$, expansion becomes valid throughout the entire disk $|z - z_0| < R_2$.
If an infinite number of the coefficients b_n in the principal part are nonzero, z_0 is said to be an essential singular point of f.

6.6 Residue at Poles

When a function f has an isolated singularity at a point z_0 , the basic method for identifying z_0 as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of $\dfrac{1}{z-z_0}$. The following theorem provides an alternative characterization of poles and a way of finding residues at poles that is often more convenient.

Theorem 6.6.1. An isolated singular point z_0 of a function f is a pole of order m if and only if $f(z)$ can be written in the form

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $\phi(z)$ is analytic and nonzero at z_0. Moreover,

$$\underset{z=z_0}{Res} f(z) = \phi(z_0) \quad \text{if } m = 1$$

and

$$\underset{z=z_0}{Res} f(z) = \frac{\phi^{m-1}(z_0)}{(m-1)!} \text{ if } m \geq 2.$$

6.7 Zeros of Analytic Functions

Zeros and poles are closely related. Zeros can be a source of poles.

Theorem 6.7.1. Let a function f be analytic at a point z_0. It has a zero of order m at z_0 if and only if there is a function g, which is analytic and nonzero at z_0, such that

$$f(z) = (z - z_0)^m g(z)$$

Both parts of the proof follows the fact the the function is analytic at z_0

Example 6.7.1. The polynomial $f(z) = z^3 - 8$ has a zero of order $m = 1$ at $z_0 = 2$ since

$$f(z) = (z - 2)g(z),$$

where $g(z) = z^2 + 2z + 4$, and because f and g are entire and $g(2) = 12 \neq 0$. Note how the fact that $z_0 = 2$ is a zero fo order $m = 1$ of f also follows from the observations that f is entire and that

$$f(2) = 0$$

and

$$f'(2) = 12 \neq 0$$

Our next theorem tells us that the zeros of an analytic function are isolated when the function is not identically equal to zero.

Theorem 6.7.2. Given a function f and a point z_0, suppose that

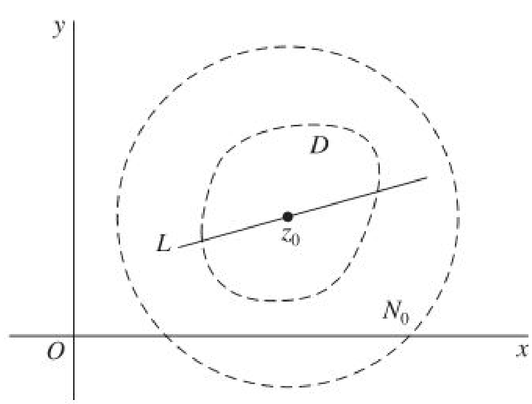
1. f is analytic at z_0;
2. $f(z_0) = 0$ but $f(z)$ is not identically equal to zero in any neighborhood of z_0. Then $f(z) \neq 0$ throughout some deleted neighborhood $0 < |z - z_0| < \epsilon$ of z_0.

Our final theorem here concerns functions with zeros that are not all isolated. It was referred to earlier and makes an interesting contrast to Theorem 2 just above.

Theorem 6.7.3. Given a function f and a point z_0, suppose that

1. f is analytic throughout a neighborhood N_0 of z_0;
2. $f(z) = 0$ at each point z of a domain D or line segment L containing z_0

Then $f(z) = 0$ in N_0; that is, $f(z)$ is identically equal to zero throughout N_0.



6.8 Zeros and Poles

The following theorem shows how zeros of order m can create poles of order m.

Theorem 6.8.1. Suppose that

1. two functons p and q are analytic at a point z_0;
2. $p(z_0) \neq 0$ and q has a zero of order m at z_0.

Then the quotient $p(z)/q(z)$ has a pole of order m at z_0

Theorem 1 leads us to another method for identifying simple poles and finding the corresponding residues. This method, stated just below as Theorem 2, is sometimes easier to use

Theorem 6.8.2. Let two functions p and q be analytic at a point z_0. If $p(z_0) \neq 0$, $q(z_0) = 0$ and $q'(z_0) \neq 0$ then z_0 is a simple pole of the quotient $p(z)/q(z)$ and

$$\underset{z=z_0}{Res}\frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

Example 6.8.1. Consider the function

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

which is a quotient of the entire functions. Its singularities occur at the zeros of $q(z) = \sin z$, or at the points

$$z = n\pi$$

Since

$$p(n\pi) = (-1)^n \neq 0$$

$$q(n\pi) = 0$$
$$q'(n\pi) = -1^n \neq 0$$

each singular point is a simple pole, with residue

$$B_n = 1.$$

Chapter 7

Application of Residues

We turn now to some important applications of the theory of residues, which was developed in the previous chapter.

7.1 Evaluation of Improper Integrals

In calculus, the improper integral of a continuous function $f(x)$ over the semiinfinite interval $0 \leq x < \infty$ is defined by means of the equation

$$\int_0^\infty f(x)dx = \lim \lim_{R\to\infty} \int_0^R f(x)dx$$

7.2 Improper Integrals From Fourier Analysis

Residue theory can be useful in evaluating convergent improper integrals of the form of trig identities

$$\int_{-\infty}^\infty f(x)\sin ax dx$$

$$\int_{-\infty}^\infty f(x)\cos ax dx$$