1 Three Tails

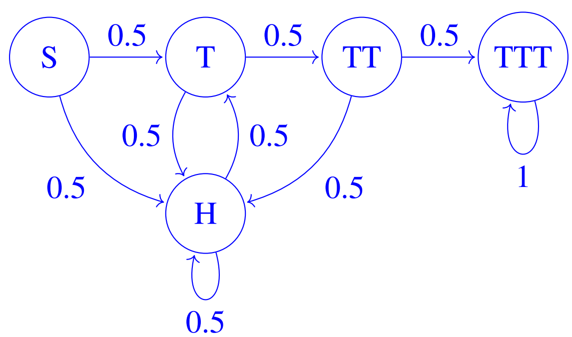
You flip a fair coin until you see three tails in a row. What is the average number of heads that you'll see until getting TTT?

Hint: How is this different than the number of *coins* flipped until getting TTT?

Solution:

We can model this problem as a Markov chain with the following states:

- S: Start state, which we are only in before flipping any coins.
- H: We see a head, which means no streak of tails currently exists.
- T: We've seen exactly one tail in a row so far.
- TT: We've seen exactly two tails in a row so far.
- TTT: We've accomplished our goal of seeing three tails in a row and stop flipping.



We can write the first step equations and solve for $\beta(S)$, only counting heads that we see since we are not looking for the total number of flips. The equations are as follows:

$$\beta(S) = 0.5\beta(T) + 0.5\beta(H) \quad (1)$$
$$\beta(H) = 1 + 0.5\beta(H) + 0.5\beta(T) \quad (2)$$
$$\beta(T) = 0.5\beta(TT) + 0.5\beta(H) \quad (3)$$
$$\beta(TT) = 0.5\beta(H) + 0.5\beta(TTT) \quad (4)$$
$$\beta(TTT) = 0 \quad (5)$$

From equation (2), we see that

$$0.5\beta(H) = 1 + 0.5\beta(T)$$

and can substitute that into equation (3) to get

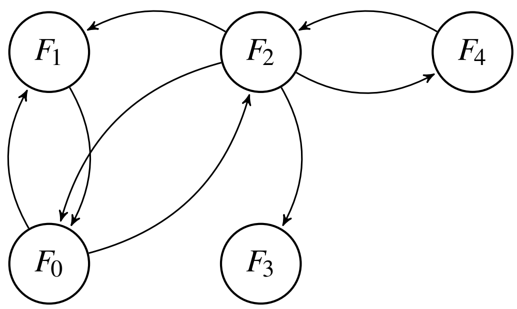
$$0.5\beta(T) = 0.5\beta(TT) + 1.$$

Substituting this into equation (4), we can deduce that $\beta(TT) = 4$. This allows us to conclude that $\beta(T) = 6$, $\beta(H) = 8$, and $\beta(S) = 7$. On average, we expect to see 7 heads before flipping three tails in a row.

2 The Dwinelle Labyrinth

You have decided to take a humanities class this semester, a French class to be specific. Instead of a final exam, your professor has issued a final paper. You must turn in this paper *before* noon to the professor's office on floor 3 in Dwinelle, and it's currently 11:48 a.m.

Let Dwinelle be modeled by the following Markov chain. Instead of rushing to turn it in, we will spend valuable time computing whether or not we *could have* made it. Suppose walking between floors takes 1 minute.



(a) Will you make it in time if you choose a floor to transition to uniformly at random? (If T_i is the number of steps needed to get to F_3 starting from F_i, where $i \in \{0,1,2,3,4\}$, is $\mathbb{E}[T_0] < 12$?)

(b) Would you expect to make it in time, if for every floor, you order all accessible floors and are twice as likely to take higher floors? (If you are considering 1, 2, or 3, you will take each with probabilities $1/7$, $2/7$, $4/7$, respectively.)

Solution:

(a) Write out all of the first-step equations.

$$\begin{aligned}
\mathbb{E}[T_0] &= 1 + \frac{1}{2}\mathbb{E}[T_1] + \frac{1}{2}\mathbb{E}[T_2] \\
\mathbb{E}[T_1] &= 1 + \mathbb{E}[T_0] \\
\mathbb{E}[T_2] &= 1 + \frac{1}{4}\mathbb{E}[T_0] + \frac{1}{4}\mathbb{E}[T_1] + \frac{1}{4}\mathbb{E}[T_3] + \frac{1}{4}\mathbb{E}[T_4] \\
\mathbb{E}[T_3] &= 0 \\
\mathbb{E}[T_4] &= 1 + \mathbb{E}[T_2]
\end{aligned}$$

Let us rewrite these equations, before placing it in matrix form.

$$\begin{aligned}
-1 &= -\mathbb{E}[T_0] + \frac{1}{2}\mathbb{E}[T_1] + \frac{1}{2}\mathbb{E}[T_2] \\
-1 &= -\mathbb{E}[T_1] + \mathbb{E}[T_0] \\
-1 &= -\mathbb{E}[T_2] + \frac{1}{4}\mathbb{E}[T_0] + \frac{1}{4}\mathbb{E}[T_1] + \frac{1}{4}\mathbb{E}[T_3] + \frac{1}{4}\mathbb{E}[T_4] \\
0 &= \mathbb{E}[T_3] \\
-1 &= -\mathbb{E}[T_4] + \mathbb{E}[T_2]
\end{aligned}$$

We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/2 & 1/2 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/4 & 1/4 & -1 & 1/4 & 1/4 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We can now reduce the matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 15 \\ 0 & 1 & 0 & 0 & 0 & 16 \\ 0 & 0 & 1 & 0 & 0 & 12 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 13 \end{bmatrix}$$

We see that $\mathbb{E}[T_0] = 15$, meaning it will take 15 minutes for us to get to floor 3. Unfortunately, we only have 12 minutes.

(b) Write out all of the first-step equations.

$$\begin{aligned} \mathbb{E}[T_0] &= 1 + \frac{1}{3}\mathbb{E}[T_1] + \frac{2}{3}\mathbb{E}[T_2] \\ \mathbb{E}[T_1] &= 1 + \mathbb{E}[T_0] \\ \mathbb{E}[T_2] &= 1 + \frac{1}{15}\mathbb{E}[T_0] + \frac{2}{15}\mathbb{E}[T_1] + \frac{4}{15}\mathbb{E}[T_3] + \frac{8}{15}\mathbb{E}[T_4] \\ \mathbb{E}[T_3] &= 0 \\ \mathbb{E}[T_4] &= 1 + \mathbb{E}[T_2] \end{aligned}$$

Let us rewrite these equations, before placing it in matrix form.

$$\begin{aligned} -1 &= -\mathbb{E}[T_0] + \frac{1}{3}\mathbb{E}[T_1] + \frac{2}{3}\mathbb{E}[T_2] \\ -1 &= -\mathbb{E}[T_1] + \mathbb{E}[T_0] \\ -1 &= -\mathbb{E}[T_2] + \frac{1}{15}\mathbb{E}[T_0] + \frac{2}{15}\mathbb{E}[T_1] + \frac{4}{15}\mathbb{E}[T_3] + \frac{8}{15}\mathbb{E}[T_4] \\ 0 &= \mathbb{E}[T_3] \\ -1 &= -\mathbb{E}[T_4] + \mathbb{E}[T_2] \end{aligned}$$

We can rewrite this in matrix form.

$$P = \begin{bmatrix} -1 & 1/3 & 2/3 & 0 & 0 & -1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 1/15 & 2/15 & -1 & 4/15 & 8/15 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

We row reduce to get the following.

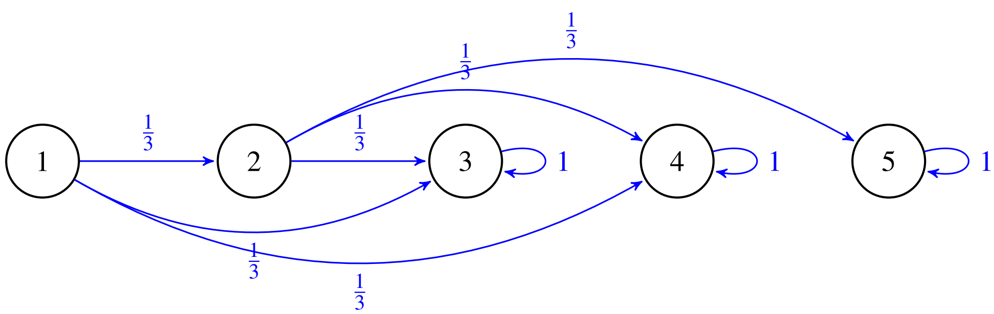
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 9.75 \\ 0 & 1 & 0 & 0 & 0 & 10.75 \\ 0 & 0 & 1 & 0 & 0 & 7.75 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 8.75 \end{bmatrix}$$

We see that $\mathbb{E}[T_0] = 9.75$, meaning it will take 9.75 minutes for us to get to floor 3. That's fewer than 12 minutes, so if you finished this computation in less than 2 minutes and 15 seconds, you could make it!

3 Skipping Stones

We consider a simple Markov chain model for skipping stones on a river, but with a twist: instead of trying to make the stone travel as far as possible, you want the stone to hit a target. Let the set of states be $\mathscr{X} = \{1, 2, 3, 4, 5\}$. State 3 represents the target, while states 4 and 5 indicate that you have overshot your target. Assume that from states 1 and 2, the stone is equally likely to skip forward one, two, or three steps forward. If the stone starts from state 1, compute the probability of reaching our target before overshooting, i.e. the probability of $\{3\}$ before $\{4, 5\}$.

Solution: Here is the Markov Chain we are working with:



Let $\alpha(i)$ denote the probability of reaching the target before overshooting, starting at state i. Then:

$$\begin{aligned} \alpha(5) &= 0 \\ \alpha(4) &= 0 \\ \alpha(3) &= 1 \\ \alpha(2) &= \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) + \frac{1}{3}\alpha(5) = \frac{1}{3} \\ \alpha(1) &= \frac{1}{3}\alpha(2) + \frac{1}{3}\alpha(3) + \frac{1}{3}\alpha(4) = \frac{1}{9} + \frac{1}{3} \end{aligned}$$

Therefore, $\alpha(1) = 1/9 + 1/3 = 4/9$.