Due: Saturday 11/13, 4:00 PM
Grace period until Saturday 11/13, 5:59 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Exploring Concepts

The following snippet illustrates a concept about polynomials.

Polynomial?
Interpolation gives one!
Few roots says; No more!

Yet another concept.

Lo! Delta sub i.
Behold! One at x sub i.
Sub j? Such empty.

And finally.

Polynomial in x.
A zero at r.
Cleft by x minus r.
And nothing remains.

1. You should write a poem or snippet to illustrate a concept from this weeks content. We are certain you can do better, but no pressure. It's all good.
2. In terms of the staff using your content for fun and "profit". Do you wish to (1) allow us to share to the class without atttribution (2) allow us to share to the class with attribution or (3) please, please do not share!

This problem will be sampled but only adds to the numerator not the “out of” (or denominator). Also, it will be generously graded and worth 10 points. If we feel this is useful to students (based on the connection to concepts in the course), we may continue it in future problem sets.

2 Will I Get My Package?

A delivery guy in some company is out delivering n packages to n customers, where $n \in \mathbb{N}$, $n > 1$. Not only does he hand each customer a package uniformly at random from the remaining packages, he opens the package before delivering it with probability $1/2$. Let X be the number of customers who receive their own packages unopened.

(a) Compute the expectation $\mathbb{E}(X)$.

(b) Compute the variance $\mathrm{Var}(X)$.

3 Diversify Your Hand

You are dealt 5 cards from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

(a) Calculate $E[X]$.

(b) Calculate $Var[X]$.

4 Fishy Computations

Assume for each part that the random variable can be modelled by a Poisson distribution.

(a) Suppose that on average, a fisherman catches 20 salmon per week. What is the probability that he will catch exactly 7 salmon this week?

(b) Suppose that on average, you go to Fisherman’s Wharf twice a year. What is the probability that you will go at most once in 2018?

(c) Suppose that in March, on average, there are 5.7 boats that sail in Laguna Beach per day. What is the probability there will be *at least* 3 boats sailing throughout the *next two days* in Laguna?

5 Unreliable Servers

A Google competitor owns a warehouse consisting of a large number of servers (a server farm). On any given day, each server in the farm is equally likely to go down or to stay online, independently of all other servers, and independently of what happens on any number of other days. On average, 4 servers go down in the cluster per day.

(a) What is an appropriate distribution to model the number of servers that crash on any given day for a certain cluster? What is its parameter?

(b) Compute the expected value and variance of the number of crashed servers on a given day for a certain cluster.

(c) Compute the probability that fewer than 3 servers crashed on a given day for a certain cluster.

(d) Compute the probability at least 3 servers crashed on a given day for a certain cluster.

6 Subset Card Game

Jonathan and Yiming are playing a card game. Jonathan has $k \geq 2$ cards, and each card has a real number written on it. Jonathan tells Yiming (truthfully), that the sum of the card values is 0, and that the sum of squares of the values on the cards is 1. Specifically, if the card values are $c_1, c_2, \ldots, c_k$, then we have $\sum_{i=1}^{k} c_i = 0$ and $\sum_{i=1}^{k} c_i^2 = 1$. Jonathan and Yiming also agree on a positive target value of α.

The cards are then going to be dealt randomly in the following fashion: for each card in the deck, a fair coin is flipped. If the coin lands heads, then the card goes to Yiming, and if the coin lands tails, the card goes to Jonathan. Note that it is possible for either player to end up with no cards/all the cards.

A player wins the game if the sum of the card values in their hand is at least α, otherwise it is a tie. Prove that the probability that Yiming wins is at most $\frac{1}{8\alpha^2}$.

7 Optimal Gambling

Jonathan has a coin that may be biased, but he doesn't think so. You disagree with him though, and he challenges you to a bet. You start off with X_0 dollars. You and Jonathan then play multiple rounds, and each round, you bet an amount of money of your choosing, and then coin is tossed. Jonathan will match your bet, no matter what, and if the coin comes up heads, you win and you take both yours and Jonathan's bet, and if it comes up tails, then you lose your bet.

(a) Now suppose you actually secretly know that the bias of the coin is $\frac{1}{2} < p < 1$! You use the following strategy: on each round, you will bet a fraction q of the money you have at the start of the round. Let's say you play n rounds. What is the probability that you win exactly k of the rounds? What is the amount of money you would have if you win exactly k rounds? [*Hint*: Does the order in which you win the games affect your profit?]

(b) Let X_n denote the amount of money you have on round n. X_0 represents your initial assets and is a constant value. Show that $\mathbb{E}[X_n] = ((1-p)(1-q) + p(1+q))^n X_0$.

You may use the binomial theorem in your answer:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{(n-k)}$$

[*Hint*: Try computing a sum over the number of rounds you win out of the n rounds you play - use your answers from the previous part.]

(c) What value of q will maximize $\mathbb{E}[X_n]$? For this value of q, what is the distribution of X_n? Can you predict what will happen as $n \to \infty$? [*Hint*: Under this betting strategy, what happens if you ever lose a round?]

(d) The problem with the previous approach is that we were too concerned about expected value, so our gambling strategy was too extreme. Let's start over: again we will use a gambling strategy in which we bet a fraction q of our money at each round. Express X_n in terms of n, q, X_0, and W_n, where W_n is the number of rounds you have won up until round n. [*Hint*: Does the order in which you win the games affect your profit?]

(e) By the law of large numbers, what does W_n/n converge to as $n \to \infty$? Using this fact, what does $(\ln X_n)/n$ converge to as $n \to \infty$?

(f) The rationale behind $(\ln X_n)/n$ is that if $(\ln X_n)/n \to c$, where c is a constant, then that means for large n, X_n is roughly e^{cn}. Therefore, c is the asymptotic growth rate of your fortune! Find the value of q that maximizes your asymptotic growth rate. (*Hint*: Use calculus!)

(g) Using the value of q you found in the previous part, compute $\mathbb{E}[X_n]$.

(h) Say Jonathan wishes to estimate the bias of the coin - he may want to use the value W_n/n as his estimate. What is the expectation of this value? What is a bound on the variance of this value? (Your bound should not include p.)

(i) Let's say after playing 100 rounds, Jonathan observes that 74 heads have appeared. Help Jonathan construct a 75% confidence interval using Chebyshev's inequality on the estimator from the previous part.