Today

Probability:
  Keep building it formally..
  And our intuition.

Poll: blows my mind.

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(C) Well. Quantum. IDK- TBH.

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Also, “cuz” $==$ “because”

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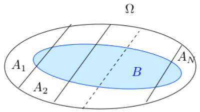

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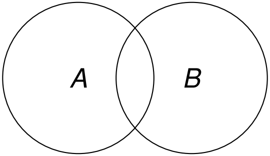

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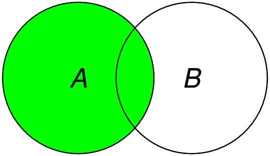

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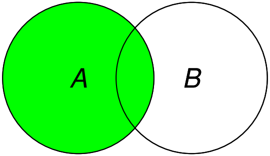

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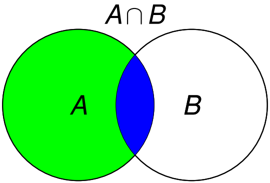

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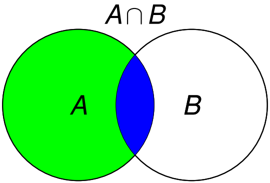

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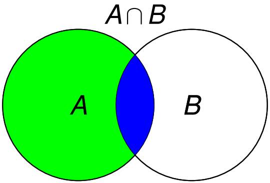

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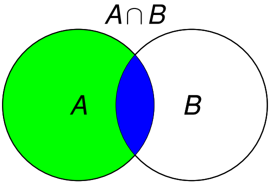

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Thus,

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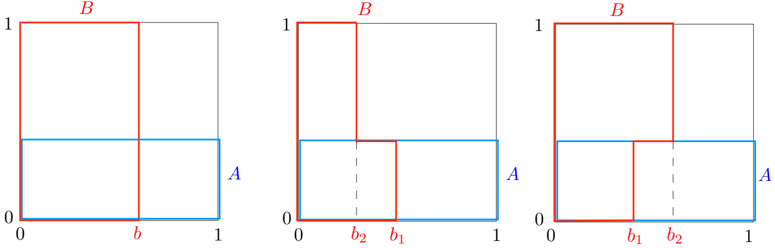

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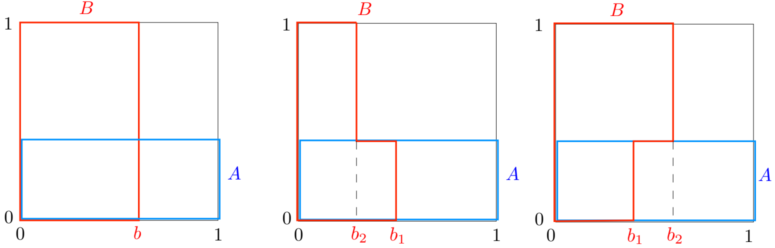

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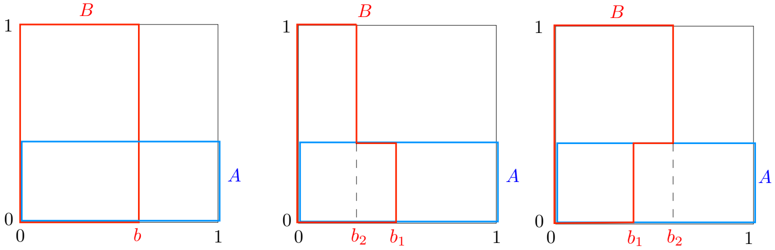

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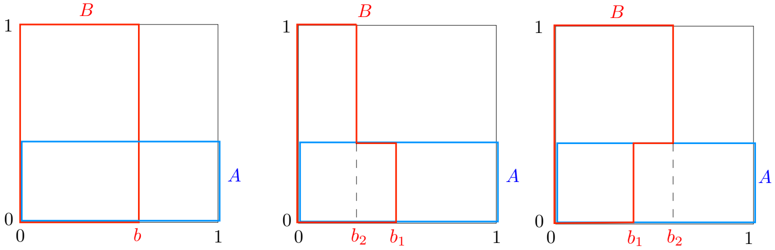

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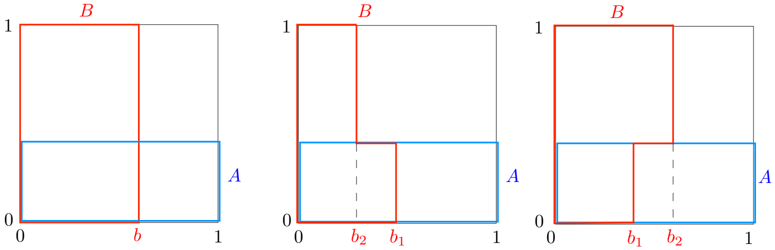

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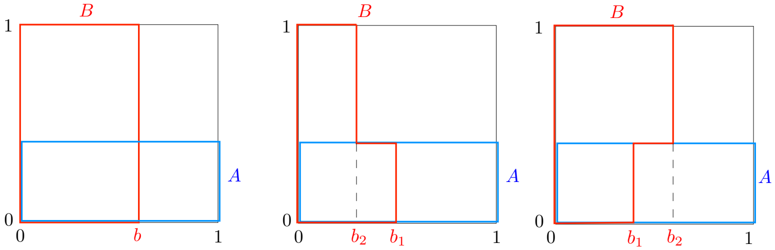

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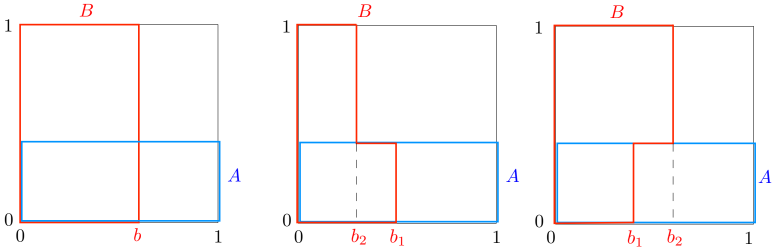

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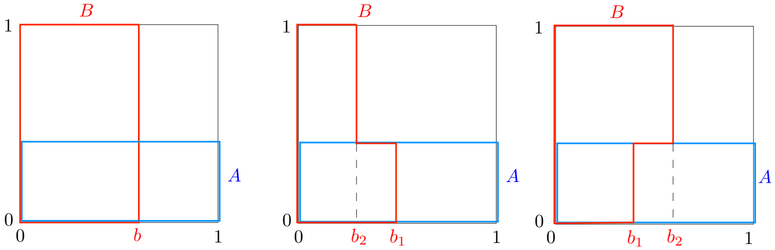

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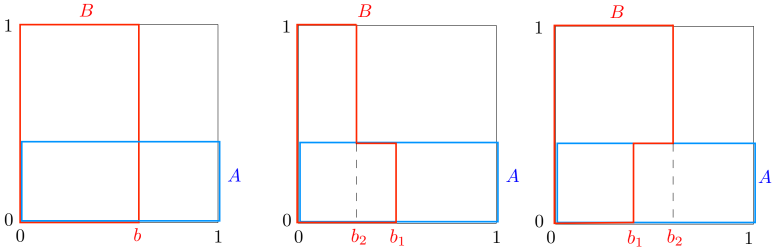

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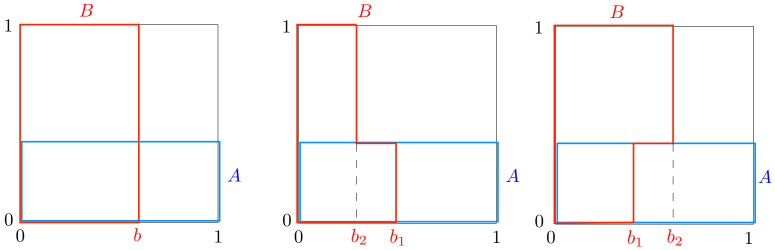

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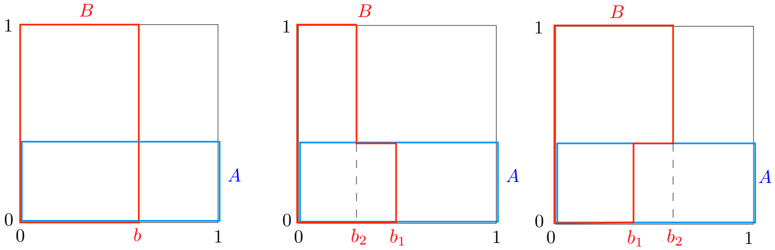

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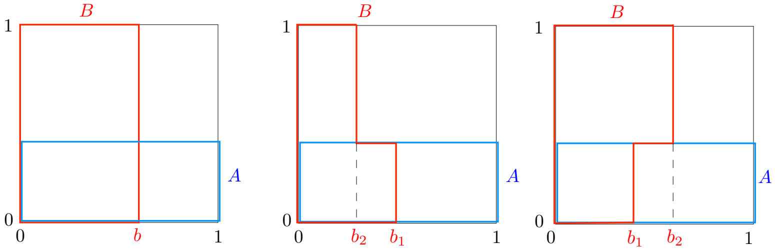

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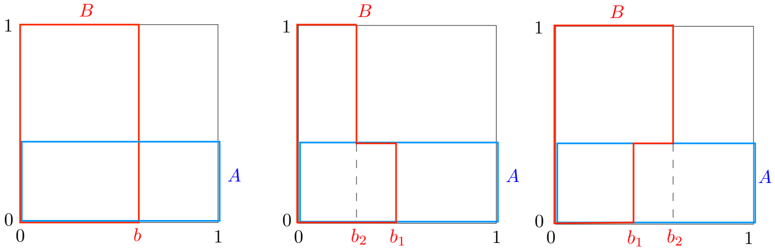

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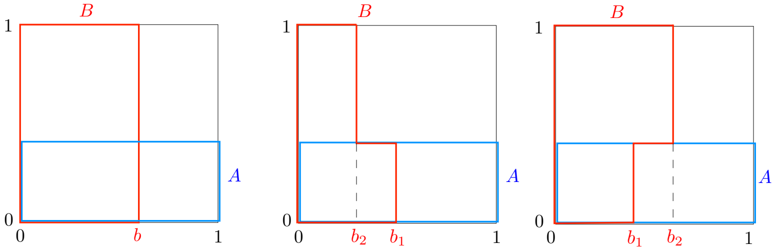

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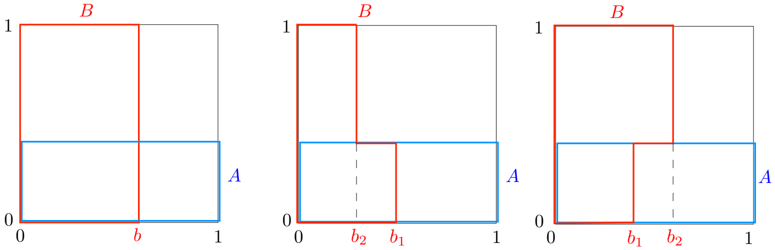

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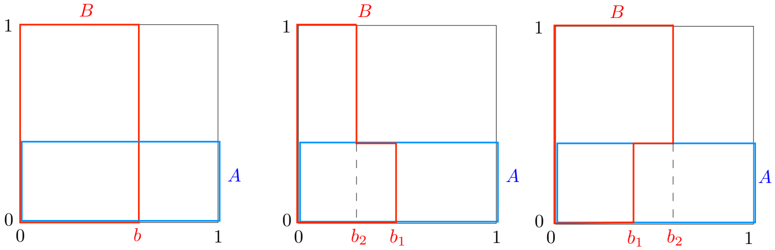

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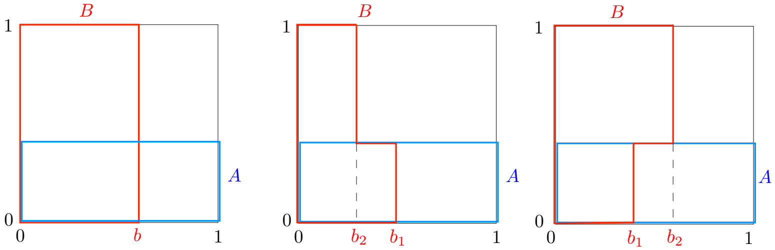

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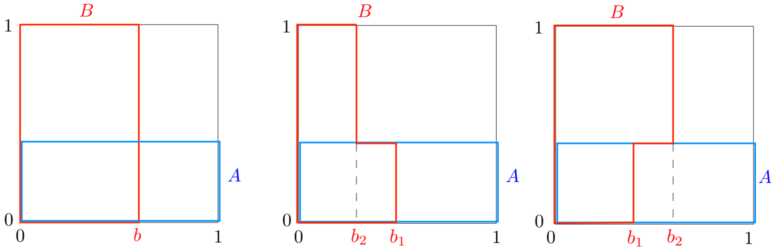

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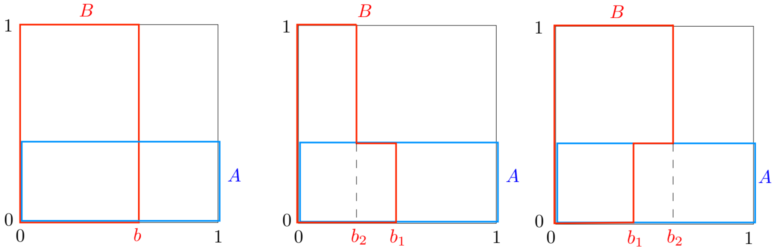

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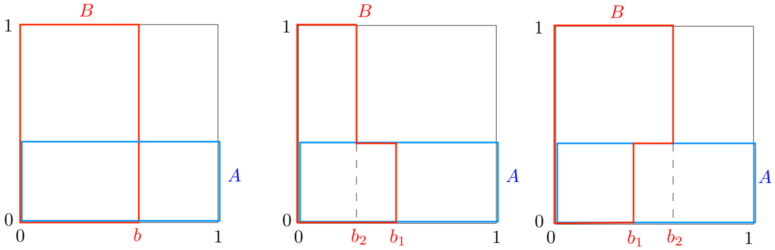

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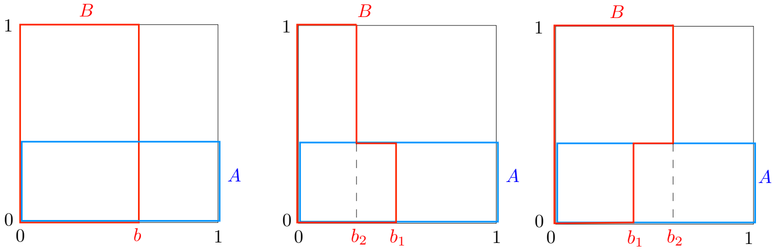

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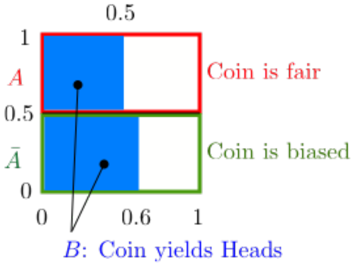
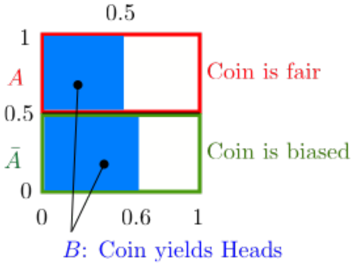


Bayes and Biased Coin



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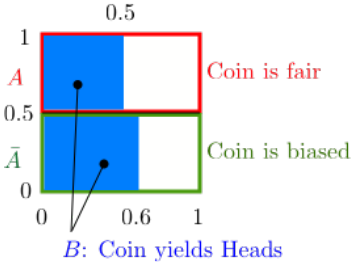
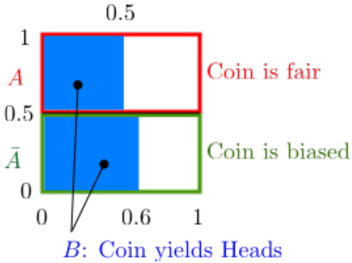
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] =$$

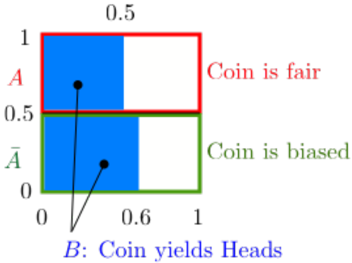
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5;$

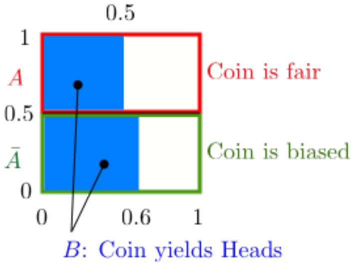
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] =$$

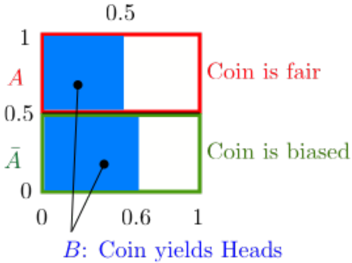
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$

Bayes and Biased Coin

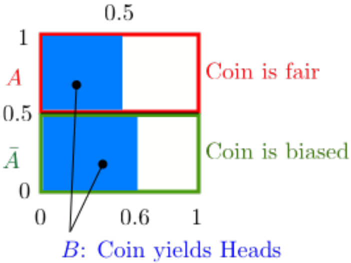


Pick a point uniformly at random in the unit square. Then

$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$

$Pr[B|A] =$

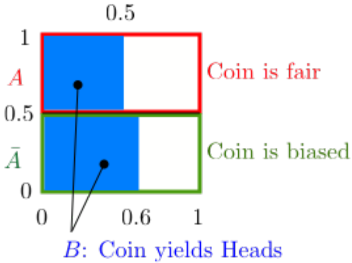
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5;$$

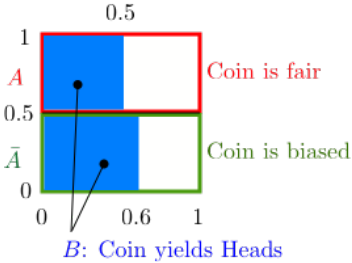
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] =$$

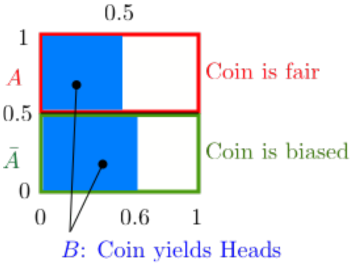
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6;$$

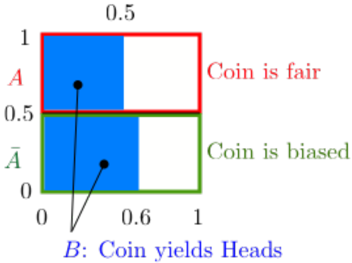
Bayes and Biased Coin



Pick a point uniformly at random in the unit square. Then

$$Pr[A] = 0.5; Pr[\bar{A}] = 0.5$$
$$Pr[B|A] = 0.5; Pr[B|\bar{A}] = 0.6; Pr[A \cap B] =$$

Bayes and Biased Coin



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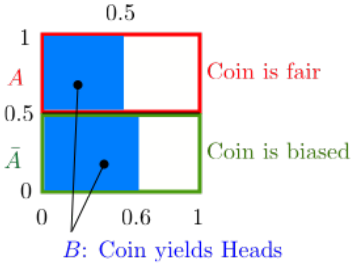
Bayes and Biased Coin



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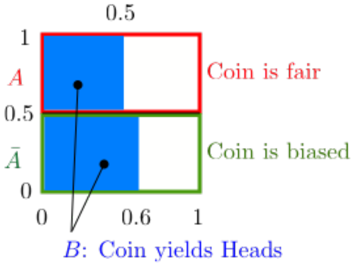
$$Pr[B] =$$

Bayes and Biased Coin



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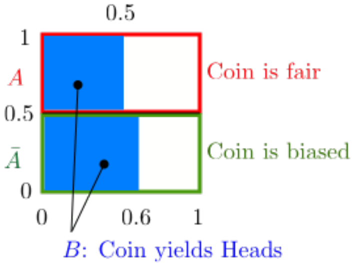
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6$$

Bayes and Biased Coin



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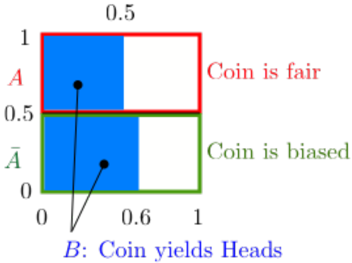
$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$

Bayes and Biased Coin



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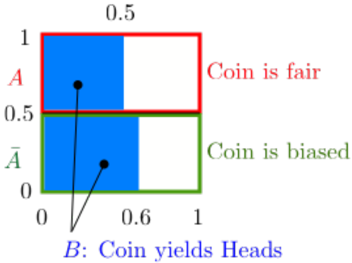
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6}$$

Bayes and Biased Coin



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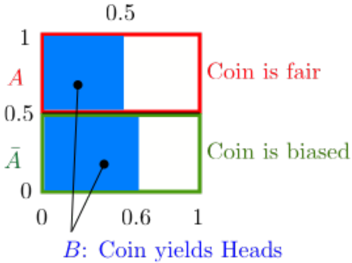
Bayes and Biased Coin



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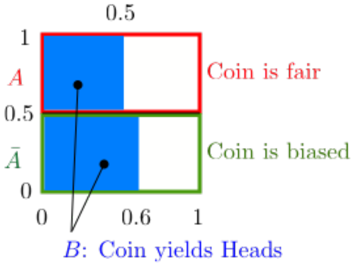
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$

$$\approx 0.46$$

Bayes and Biased Coin



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$$Pr[B] = 0.5 \times 0.5 + 0.5 \times 0.6 = Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]$$
$$Pr[A|B] = \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.5 \times 0.6} = \frac{Pr[A]Pr[B|A]}{Pr[A]Pr[B|A] + Pr[\bar{A}]Pr[B|\bar{A}]}$$
$$\approx 0.46 = \text{fraction of } B \text{ that is inside } A$$

Bayes: General Case

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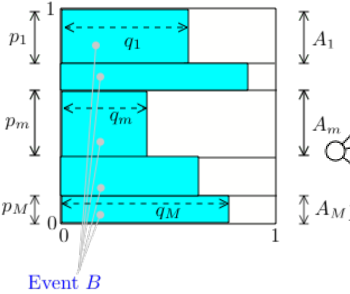
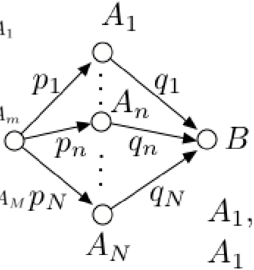




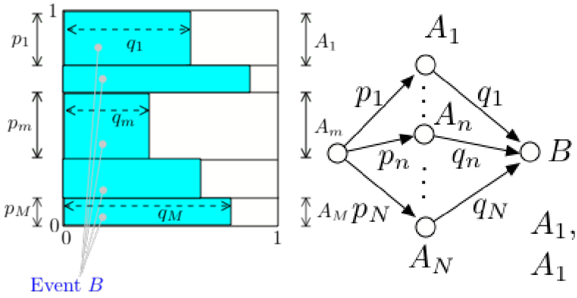
$$p_n = Pr[A_n]$$

$$q_n = Pr[B|A_n]$$

$A_1, \ldots, A_N$ disjoint

$A_1 \cup \cdots \cup A_N = \Omega$

Bayes: General Case



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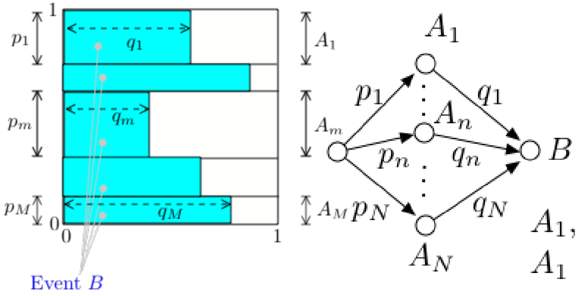

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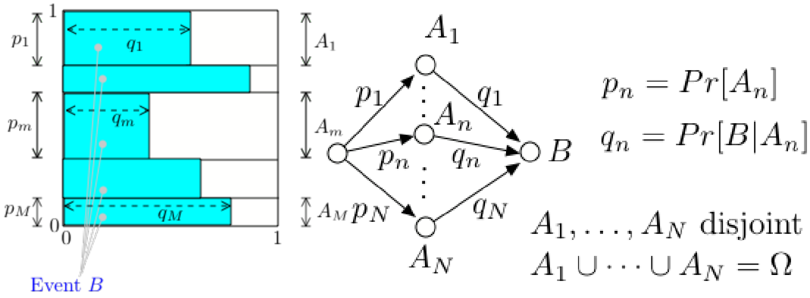
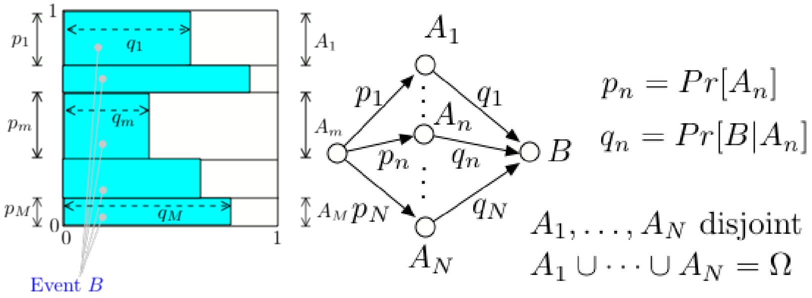


Pick a point uniformly at random in the unit square. Then

$$Pr[A_n] = p_n, n = 1, \ldots, N$$

Bayes: General Case



Pick a point uniformly at random in the unit square. Then

$$Pr[A_n] = p_n, n = 1, \ldots, N$$
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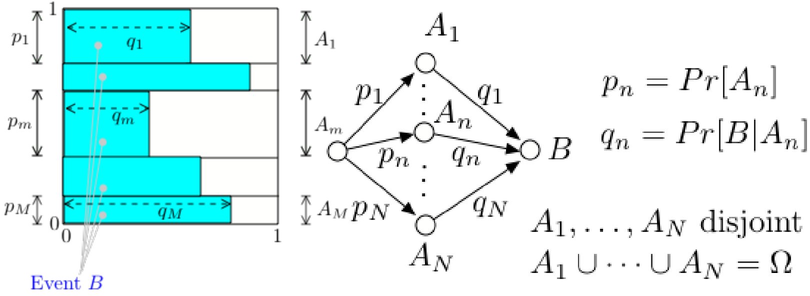
Bayes: General Case



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$$Pr[A_n] = p_n, n = 1, \dots, N$$
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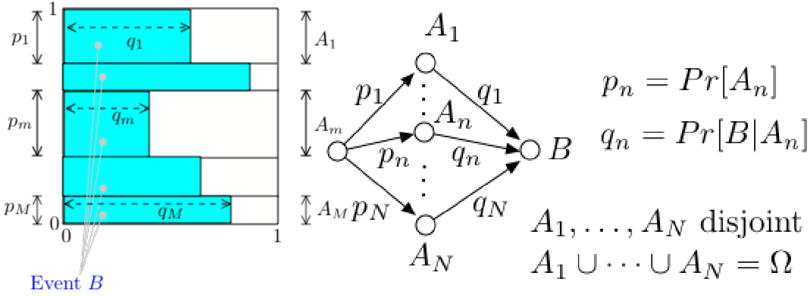
Bayes: General Case



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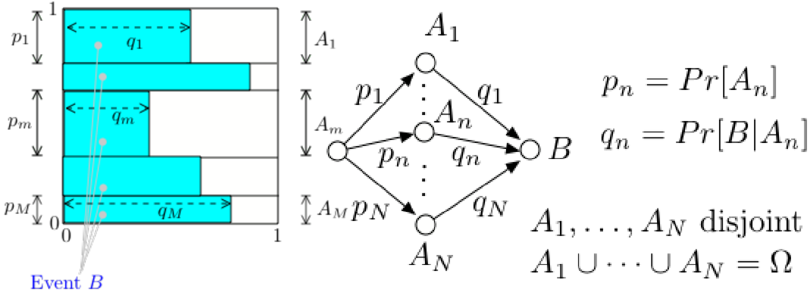

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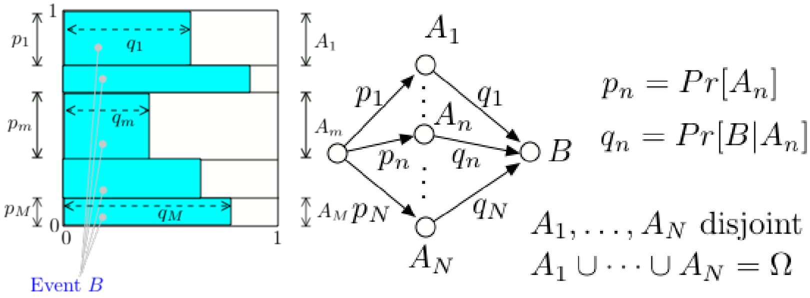

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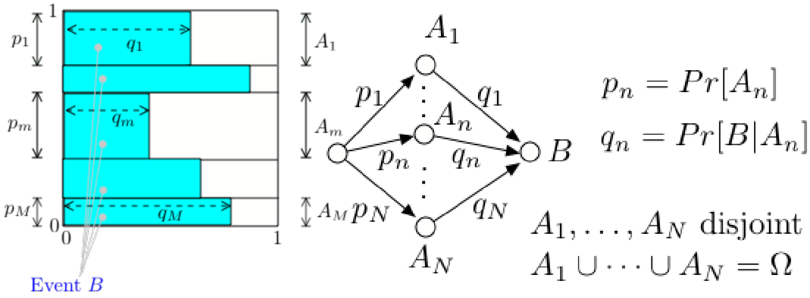
Bayes: General Case



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Bayes Rule

A general picture: We imagine that there are N possible causes $A_1, \ldots, A_N$.

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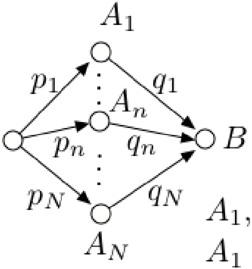

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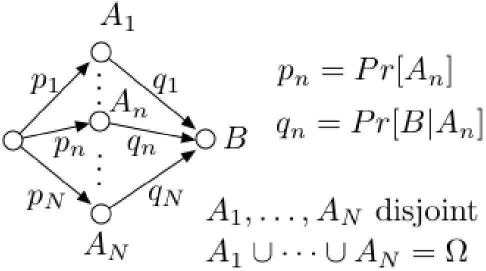

100 situations: $100 p_n q_n$ where A_n and B occur, for $n = 1, \ldots, N$.

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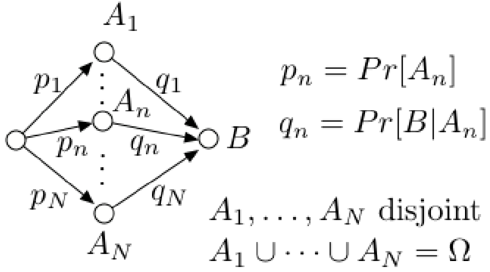

100 situations: $100 p_n q_n$ where A_n and B occur, for $n = 1, \ldots, N$.
In $100 \sum_m p_m q_m$ occurences of B, $100 p_n q_n$ occurrences of A_n.

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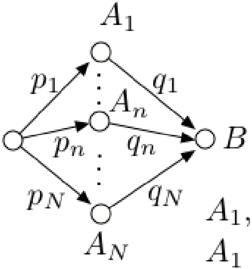

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Hence,

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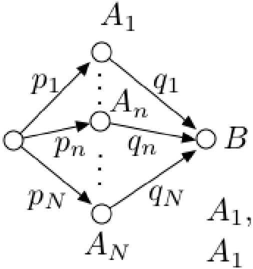

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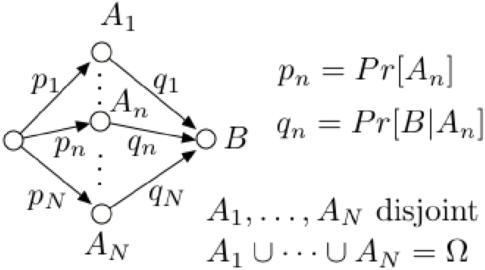

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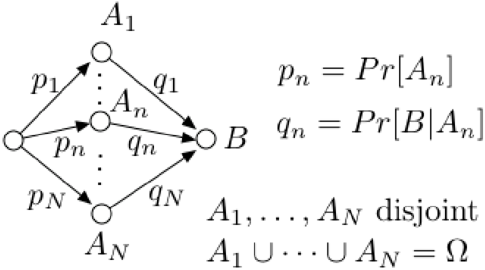

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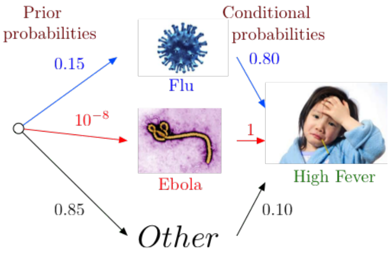
$$Pr[A_n|B] = \frac{Pr[B|A_n] Pr[A_n]}{Pr[B]}.$$

Why do you have a fever?



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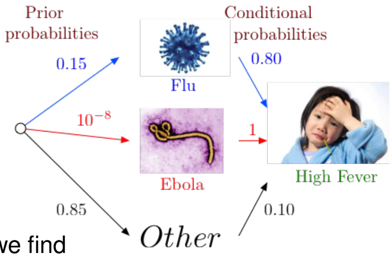

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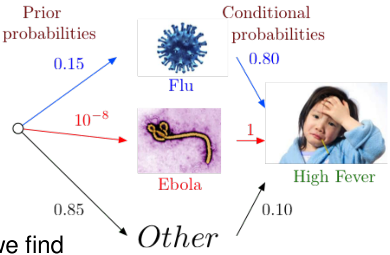
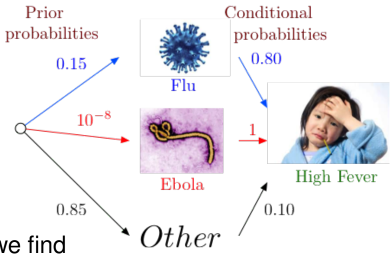


Using Bayes' rule, we find

$$Pr[\text{Flu}|\text{High Fever}] = \frac{0.15 \times 0.80}{0.15 \times 0.80 + 10^{-8} \times 1 + 0.85 \times 0.1} \approx 0.58$$

Why do you have a fever?



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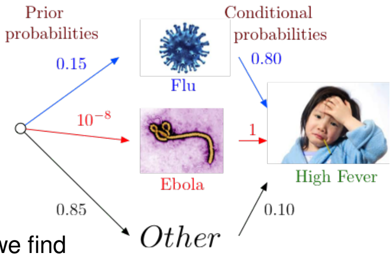

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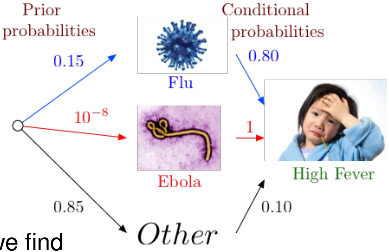

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The values $0.58, 5 \times 10^{-8}, 0.42$ are the posterior probabilities.

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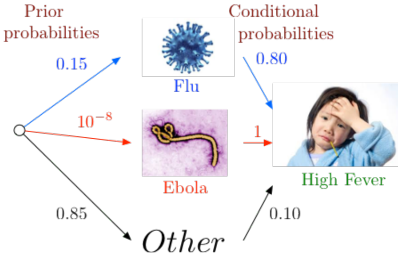

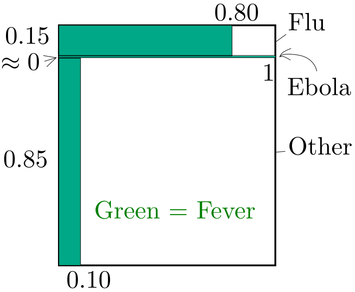

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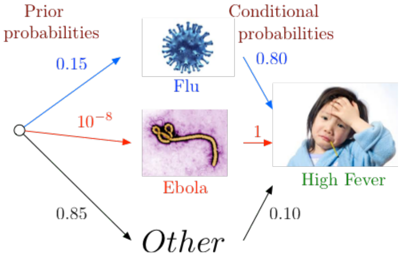

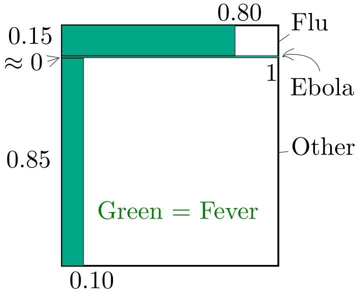

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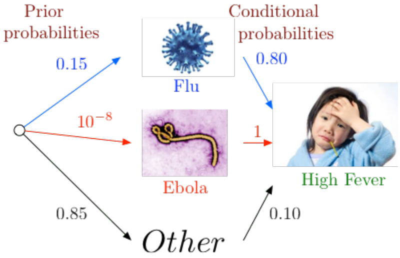

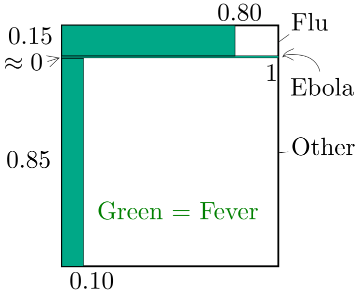

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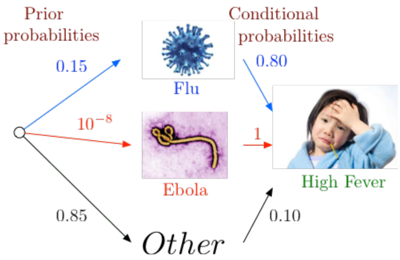

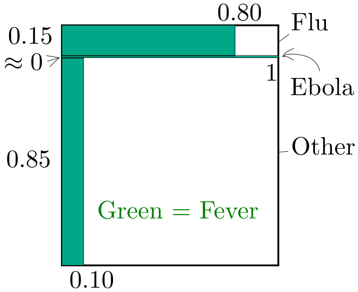

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This example shows the importance of the prior probabilities.

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Bayes' Rule Operations

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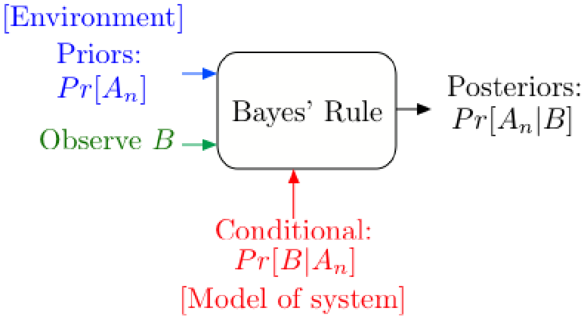

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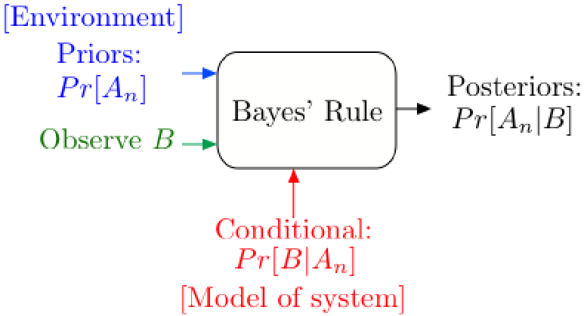

Bayes' Rule: canonical example of how information changes our opinions.

Thomas Bayes

Thomas Bayes

Portrait used of Bayes in a 1936 book,[1] but it is doubtful whether the portrait is actually of him.[2] No earlier portrait or claimed portrait survives.

Born	c. 1701 London, England
Died	7 April 1761 (aged 59) Tunbridge Wells, Kent, England
Residence	Tunbridge Wells, Kent, England
Nationality	English
Known for	Bayes' theorem

Source: Wikipedia.

Thomas Bayes

A

FIG. 3. *Joshua Bayes (1671–1746).*

$$\xrightarrow{Pr[B|A]}$$

Max
over
B

B

Thomas Bayes?

A Bayesian picture of Thomas Bayes.

Testing for disease.

Random Experiment: Pick a random male.

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$$Pr[A|B]???$$

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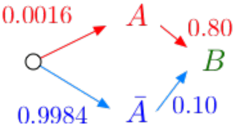

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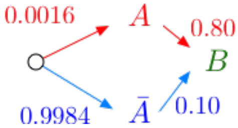

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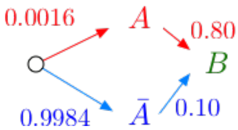

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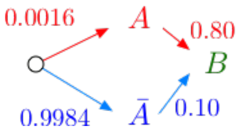

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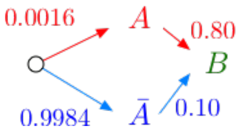

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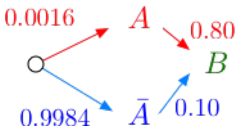

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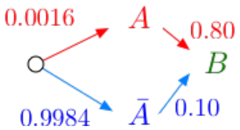

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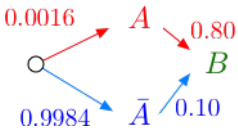

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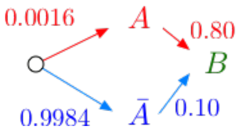

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$\boxed{\text{Events, Conditional Probability, Independence, Bayes' Rule}}$

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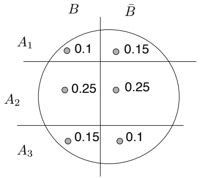

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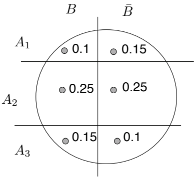

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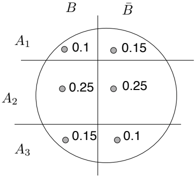

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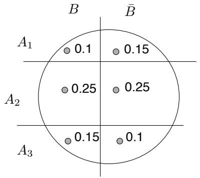

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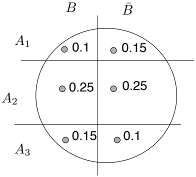

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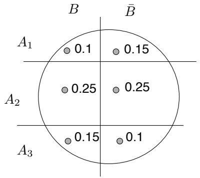

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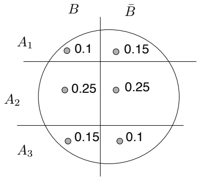

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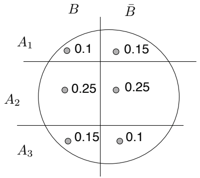

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Pairwise Independence

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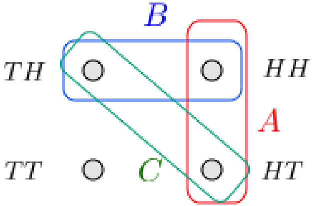

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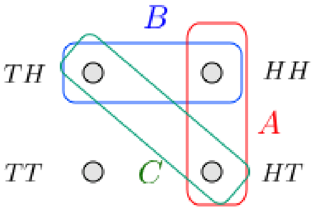

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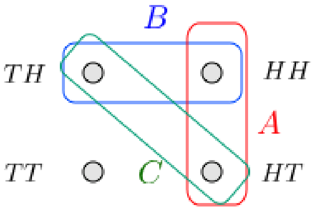

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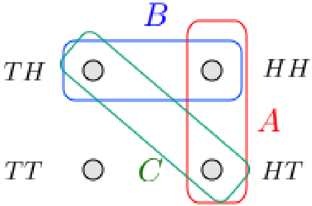

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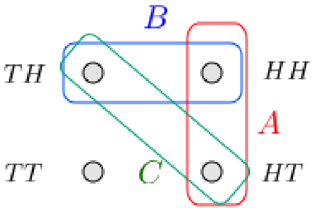

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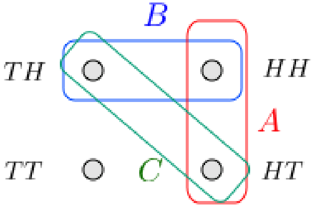

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False: If A did not say anything about C and B did not say anything about C, then $A \cap B$ would not say anything about C.

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Example: Flip a fair coin forever. Let A_n = 'coin n is H.' Then the events A_n are mutually independent.

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