CS 70 Summer 2022 — Discrete Mathematics and Probability Theory — HW 2

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1 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

Solution: We give a proof by contraposition. Suppose there does not exist an all-red column. This means that, in each column, we can find a blue pebble. Therefore, if we take one blue pebble from each column, we have a way of choosing one pebble from each column without any red pebbles. This is the negation of the original hypothesis, so we are done.

2 Contraposition

Prove the statement "if $x+y<z+w$, then $x<z$ or $y<w$".

(a) Prove this statement with a direct proof.

(b) Prove this statement via contraposition.

Solution:

(a) We can prove this with a bit of casework. First, in the case where $x<z$, we're immediately done, since we just need one of $x<z$ or $y<w$ to be true. The only other case is if $x \geq z$. In this case, we can say that $y=(x+y)-x<(z+w)-x \leq (z+w)-z=w$, and so we have that $y<w$. Thus, no matter which case we're in, we have that either $x<z$ or $y<w$.

(b) The implication we're trying to prove is $(x+y<z+w) \implies ((x<z) \vee (y<w))$, so the contrapositive is $((x \geq z) \wedge (y \geq w)) \implies (x+y \geq z+w)$. The proof of this is quite straightforward: since we have both that $x \geq z$ and that $y \geq w$, we can just add these two inequalities together, giving us $x+y \geq z+w$, which is exactly what we wanted.

3 Fibonacci Proof

Let F_i be the i^{th} Fibonacci number, defined by $F_{i+2}=F_{i+1}+F_i$ and $F_0=0$, $F_1=1$. Prove that

$$\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}.$$

Solution:

We proceed by induction on n.

Base case: $\sum_{i=0}^{0} F_i^2 = F_0^2 = 0 = F_0 F_1$.

Inductive hypothesis: Assume $\sum_{i=0}^{n} F_i^2 = F_n F_{n+1}$.

Inductive step: We have

$$\begin{aligned}\sum_{i=0}^{n+1} F_i^2 &= F_{n+1}^2 + \sum_{i=0}^{n} F_i^2 \\ &= F_{n+1}^2 + F_n F_{n+1} \\ &= F_{n+1}(F_n + F_{n+1}) \\ &= F_{n+1} F_{n+2}\end{aligned}$$

where the second equality is the inductive hypothesis and the last equality is the definition of the Fibonacci numbers.

4 Airport

Suppose that there are $2n+1$ airports where n is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

Solution:

For $n = 1$, let the 3 airports be A, B, C and let their distance be $|AB|, |AC|, |BC|$. Without loss of generality suppose B, C is the closest pair of airports (which is well defined since all distances are different). Then the airplanes departing from B and C are flying towards each other. Since the airplane from A must fly to somewhere else, no airplanes are destined for airport A.

Now suppose the statement is proven for $n = k$, i.e. when there are $2k+1$ airports. For $n = k+1$, i.e. when there are $2k+3$ airports, the airplanes departing from the closest two airports must be destined for each other's starting airports. Removing these two airports reduce the problem to $2k+1$ airports. From the inductive hypothesis, we know that among the $2k+1$ airports remaining, there is an airport with no incoming flights which we call airport Z. When we add back the two airports that we removed, the airplane flights may change; in particular, it is possible that an airplane will now choose to fly to one of these two airports (because the airports that were added may be closer than the airport to which the airplane was previously flying), but observe that none of the airplanes will is destined for the airport Z. Also, the two airports that were added back will have airplanes destined for each other, so they too will not be destined for airport Z. We conclude that the airport Z will continue to have no incoming flights when we add back the two airports, and so the statement holds for $n = k+1$. By induction, the claim holds for all $n \geq 1$.

5 Coloring Trees

Prove that all trees with at least 2 vertices are *bipartite*: the vertices can be partitioned into two groups so that every edge goes between the two groups.

[*Hint:* Use induction on the number of vertices.]

Solution:

Proof using induction on the number of vertices n.

Base case $n = 2$. A tree with two vertices has only one edge and is a bipartite graph by partitioning the two vertices into two separate parts.

Inductive hypothesis. Assume that all trees with k vertices for an arbitrary $k \geq 2$ is bipartite.

Inductive step. Consider a tree $T = (V, E)$ with $k+1$ vertices. We know that every tree must have at least two leaves, so remove one leaf u and the edge connected to u, say edge e. The resulting graph $T - u$ is a tree with k vertices and is bipartite by the inductive hypothesis. Thus there exists a partitioning of the vertices $V = R \cup L$ such that there does not exist an edge that connects two vertices in L or two vertices in R. Now when we add u back to the graph. If edge e connects u with a vertex in L then let $L' = L$ and $R' = R \cup \{u\}$. On the other hand if edge e connects u with a vertex in R then let $L' = L \cup \{u\}$ and $R' = R$. L' and R' gives us the required partition to show that T is bipartite. This completes the inductive step and hence by induction we get that all trees with at least 2 vertices are bipartite.

6 Planarity

(a) Prove that $K_{3,3}$ is nonplanar.

(b) Consider graphs with the property T: For every three distinct vertices v_1, v_2, v_3 of graph G, there are at least two edges among them. Use a proof by contradiction to show that if G is a graph on ≥ 7 vertices, and G has property T, then G is nonplanar.

Solution:

(a) Assume toward contradiction that $K_{3,3}$ were planar. In $K_{3,3}$, there are $v = 6$ vertices and $e = 9$ edges. If $K_{3,3}$ were planar, from Euler's formula we would have $v - e + f = 2 \Rightarrow f = 5$. On the other hand, each region is bounded by at least four edges, so $4f \leq 2e$, i.e., $20 \leq 18$, which is a contradiction. Thus, $K_{3,3}$ is not planar.

(b) In this problem, we use proof by contradiction. Assume G is planar. Select any five vertices out of the seven. Consider the subgraph formed by these five vertices. They cannot form K_5, since G is planar. So some pair of vertices amongst these five has no edge between them. Label these vertices v_1 and v_2. The remaining five vertices of G besides v_1 and v_2 cannot form K_5 either, so there is a second pair of vertices amongst these new five that has no edge between them. Label these v_3 and v_4. Label the remaining three vertices v_5, v_6 nd v_7. Since $v_1 v_2$ is not an edge, by property T (which states any three vertices must have at least two edges between them)it must be that $\{v_1, v\}$ and $\{v_2, v\}$ are edges, where $v \in \{v_3, v_4, v_5, v_6, v_7\}$. Similarly for v_3, v_4 we have that $\{v_3, v\}$ and $\{v_4, v\}$ are edges, where $v \in \{v_1, v_2, v_5, v_6, v_7\}$. Now consider the subgraph induced by $\{v_1, v_2, v_3, v_5, v_6, v_7\}$. With the three vertices $\{v_1, v_2, v_3\}$ on one side and $\{v_5, v_6, v_7\}$ on the other, we observe that $K_{3,3}$ is a subgraph of this induced graph. This contradicts the fact that G is planar.

The above shows that any graph with 7 vertices and property T is non-planar. Any graph with greater than 7 vertices and property T will also be non-planar because it will contain a subgraph with 7 vertices and property T.

7 Bipartite Graphs

An undirected graph is bipartite if its vertices can be partitioned into two disjoint sets L, R such that each edge connects a vertex in L to a vertex in R (so there does not exist an edge that connects two vertices in L or two vertices in R).

(a) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Prove that $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$.

(b) Suppose that a graph G is bipartite, with L and R being a bipartite partition of the vertices. Let s and t denote the average degree of vertices in L and R respectively. Prove that $s/t = |R|/|L|$.

(c) Prove that a graph is bipartite if and only if it can be 2-colored. (A graph can be 2-colored if every vertex can be assigned one of two colors such that no two adjacent vertices have the same color).

Solution:

(a) Since G is bipartite, each edge connects one vertex in L with a vertex in R. Since each edge contributes equally to $\sum_{v \in L} \deg(v)$ and $\sum_{v \in R} \deg(v)$, we see that these two values must be equal.

(b) By part (a), we know that $\sum_{v \in L} \deg(v) = \sum_{v \in R} \deg(v)$. Thus $|L| \cdot s = |R| \cdot t$. A little algebra gives us the desired result.

(c) Given a bipartite graph, color all of the vertices in L one color, and all of the vertices in R the other color. Conversely, given a 2-colored graph (call the colors red and blue), there are no edges between red vertices and red vertices, and there are no edges between blue vertices and blue vertices. Hence, take L to be the set of red vertices and R to be the set of blue vertices. We see that the graph is bipartite.

8 Build-Up Error?

What is wrong with the following "proof"? In addition to finding a counterexample, you should explain what is fundamentally wrong with this approach, and why it demonstrates the danger of build-up error.

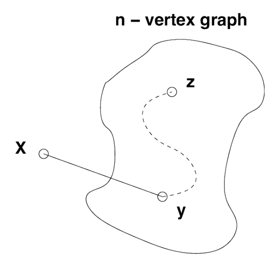
False Claim: If every vertex in an undirected graph has degree at least 1, then the graph is connected.

Proof: We use induction on the number of vertices $n \geq 1$.

Base case: There is only one graph with a single vertex and it has degree 0. Therefore, the base case is vacuously true, since the if-part is false.

Inductive hypothesis: Assume the claim is true for some $n \geq 1$.

Inductive step: We prove the claim is also true for $n+1$. Consider an undirected graph on n vertices in which every vertex has degree at least 1. By the inductive hypothesis, this graph is connected. Now add one more vertex x to obtain a graph on $(n+1)$ vertices, as shown below.



All that remains is to check that there is a path from x to every other vertex z. Since x has degree at least 1, there is an edge from x to some other vertex; call it y. Thus, we can obtain a path from x to z by adjoining the edge $\{x,y\}$ to the path from y to z. This proves the claim for $n+1$.

Solution:

The mistake is in the argument that *"every $(n+1)$-vertex graph with minimum degree 1 can be obtained from an n-vertex graph with minimum degree 1 by adding 1 more vertex"*. Instead of starting by considering an arbitrary $(n+1)$-vertex graph, this proof only considers an $(n+1)$-vertex graph that you can make by starting with an n-vertex graph with minimum degree 1. As a counterexample, consider a graph on four vertices $V = \{1,2,3,4\}$ with two edges $E = \{\{1,2\},\{3,4\}\}$. Every vertex in this graph has degree 1, but there is no way to build this 4-vertex graph from a 3-vertex graph with minimum degree 1.

More generally, this is an example of *build-up error* in proof by induction. Usually this arises from a faulty assumption that every size $n+1$ graph with some property can be "built up" from a size n graph with the same property. (This assumption is correct for some properties, but incorrect for others, such as the one in the argument above.)

One way to avoid an accidental build-up error is to use a *"shrink down, grow back"* process in the inductive step: start with a size $n+1$ graph, remove a vertex (or edge), apply the inductive hypothesis $P(n)$ to the smaller graph, and then add back the vertex (or edge) and argue that $P(n+1)$ holds.

Let's see what would have happened if we'd tried to prove the claim above by this method. In the inductive step, we must show that $P(n)$ implies $P(n+1)$ for all $n \geq 1$. Consider an $(n+1)$- vertex graph G in which every vertex has degree at least 1. Remove an arbitrary vertex v, leaving an n-vertex graph G' in which every vertex has degree... uh-oh! The reduced graph G' might contain a vertex of degree 0, making the inductive hypothesis $P(n)$ inapplicable! We are stuck —– and properly so, since the claim is false!