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Integration by Parts: $\int udv = uv - \int vdu$.

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6. Joint pdf: $Pr[X \in (x, x+\delta), Y = (y, y+\delta)) = f_{X,Y}(x,y)\delta^2$.
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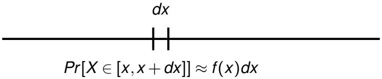

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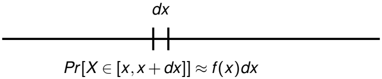

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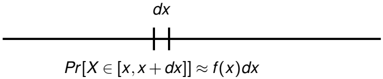

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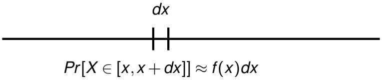

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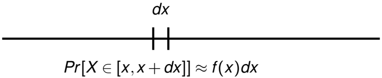

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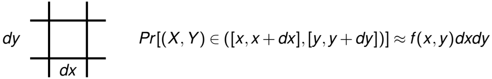

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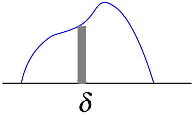

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Continuous: X in $[.2, .3]$. $X \in [.2, .3]$ or $X \in [.4, .6]$.

Conditional Probability: $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$

$Pr[\text{"Second Heads"}|\text{"First Heads"}]$,
$Pr[X \in [.2, .3] | X \in [.2, .3] \text{ or } X \in [.5, .6]]$.

Total Probability Rule: $Pr[A] = Pr[A \cap B] + Pr[A \cap \overline{B}]$
$Pr[\text{"Second Heads"}] = Pr[HH] + Pr[HT]$
B is First coin heads.
$Pr[X \in [.45, .55]] = Pr[X \in [.45, .50]] + Pr[X \in (.50, .55]]$
B is $X \in [0, .5]$

Product Rule: $Pr[A \cap B] = Pr[A|B]Pr[B]$.
Bayes Rule: $Pr[A|B] = Pr[B|A]Pr[A]/Pr[B]$.

All work for continuous with intervals as events.

Conditional density.

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Corollary: For independent random variables, $f_{X|Y}(x,y) = f_X(x)$.

Expo(λ)

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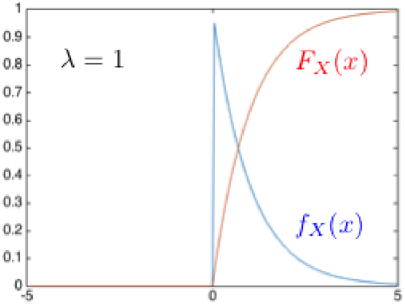
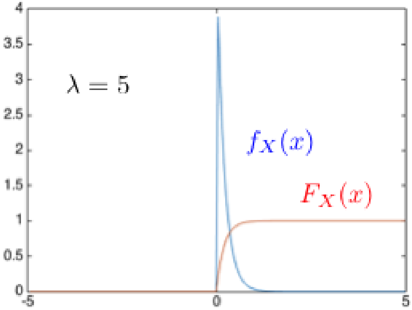
$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-\lambda x}, & \text{if } x \geq 0. \end{cases}$$





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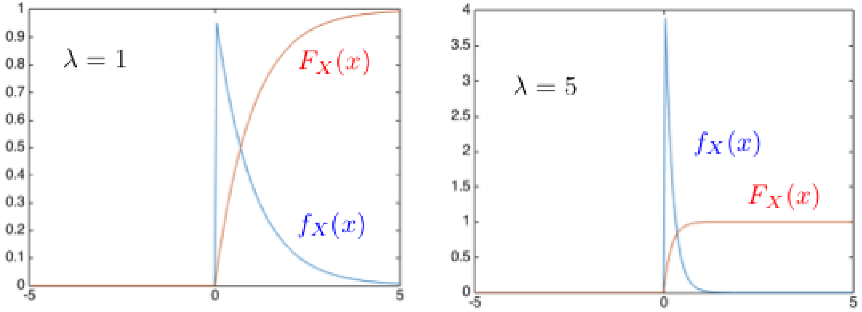

Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

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Replace b by $b-a$, use $X = U[0,1]$, then $Y = a + (b-a)X$ is $U[a,b]$.

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Definition: The **expectation** of a random variable X with pdf $f(x)$ is *defined* as

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx.$$

Justification: Say $X = n\delta$ w.p. $f_X(n\delta)\delta$ for $n \in \mathbb{Z}$. Then,

$$E[X] = \sum_n (n\delta) Pr[X = n\delta] = \sum_n (n\delta) f_X(n\delta)\delta = \int_{-\infty}^{\infty} x f_X(x) dx.$$

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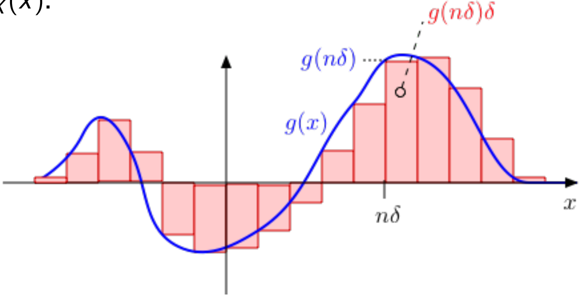

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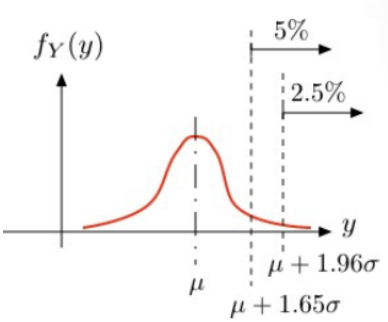

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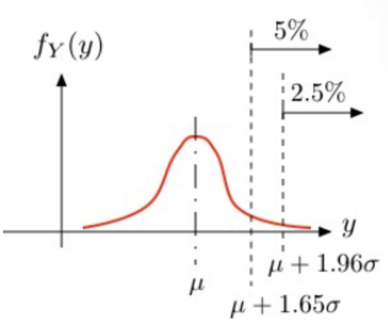

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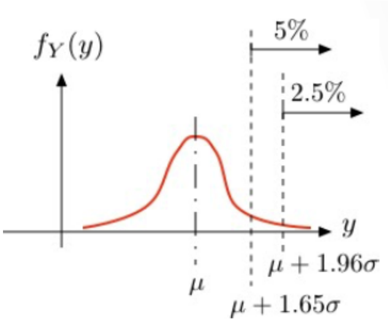

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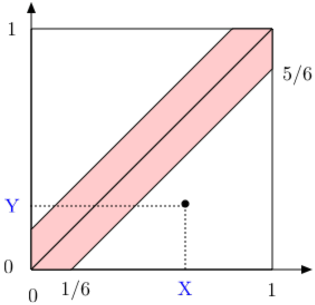

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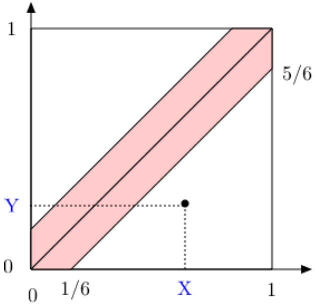

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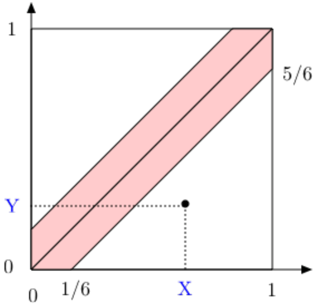

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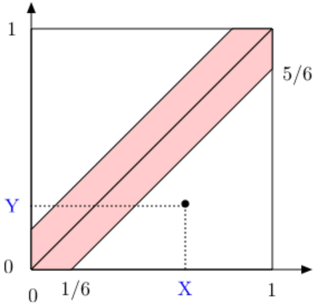

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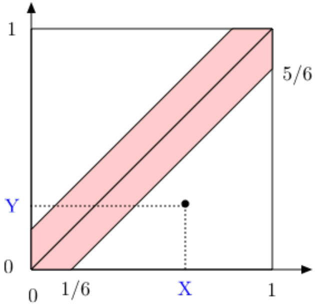

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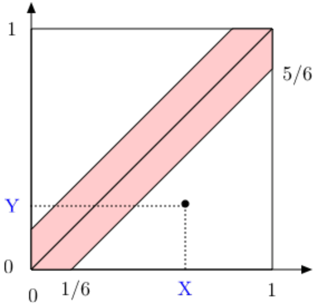

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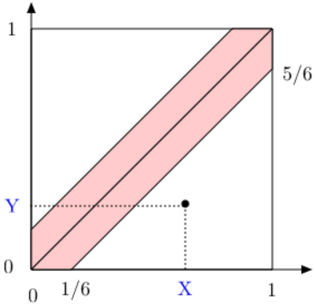

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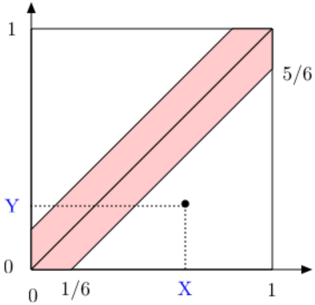

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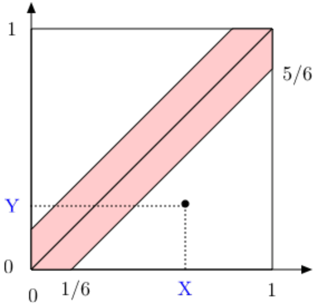

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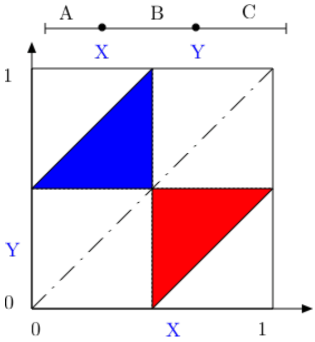

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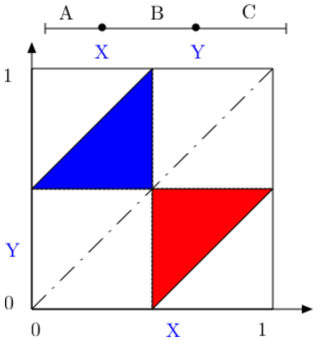

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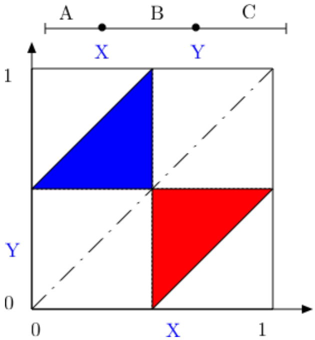

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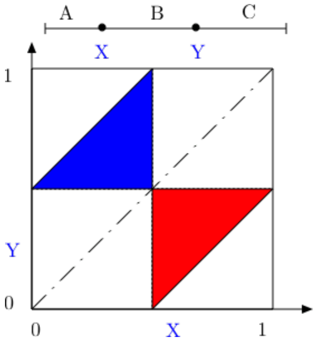

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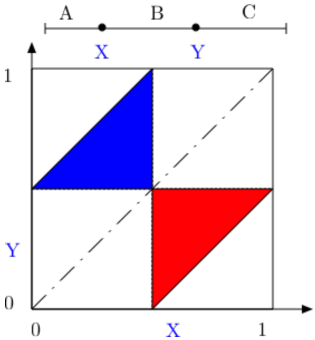

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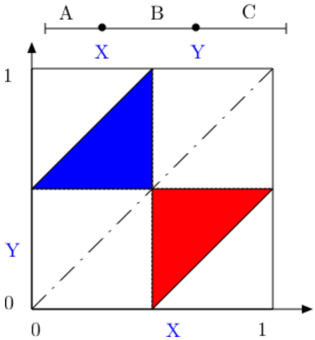

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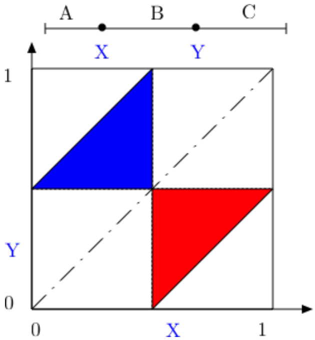

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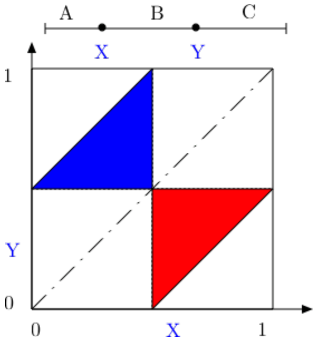

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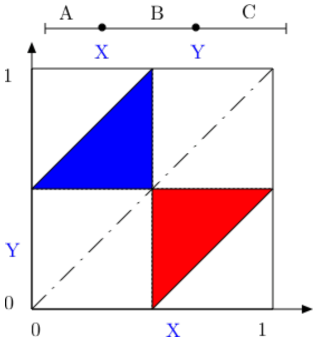

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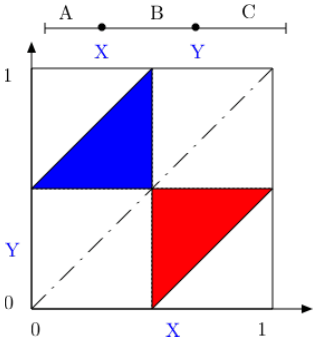

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Thus, $Pr[\text{make triangle}] = 1/4$.

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Hence,

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For instance, if $n = 16$, then $SNR(dB) \approx 112dB$.

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