

1 Counting, Counting, and More Counting

The only way to learn counting is to practice, practice, practice, so here is your chance to do so. Although there are many subparts, each subpart is fairly short, so this problem should not take any longer than a normal CS70 homework problem. You do not need to show work, and **Leave your answers as an expression** (rather than trying to evaluate it to get a specific number).

- (a) How many ways are there to arrange n 1s and k 0s into a sequence?
- (b) How many 7-digit ternary (0,1,2) bitstrings are there such that no two adjacent digits are equal?
- (c) A bridge hand is obtained by selecting 13 cards from a standard 52-card deck. The order of the cards in a bridge hand is irrelevant.
 - i. How many different 13-card bridge hands are there?
 - ii. How many different 13-card bridge hands are there that contain no aces?
 - iii. How many different 13-card bridge hands are there that contain all four aces?
 - iv. How many different 13-card bridge hands are there that contain exactly 6 spades?
- (d) Two identical decks of 52 cards are mixed together, yielding a stack of 104 cards. How many different ways are there to order this stack of 104 cards?
- (e) How many 99-bit strings are there that contain more ones than zeros?
- (f) An anagram of ALABAMA is any re-ordering of the letters of ALABAMA, i.e., any string made up of the letters A, L, A, B, A, M, and A, in any order. The anagram does not have to be an English word.
 - i. How many different anagrams of ALABAMA are there?
 - ii. How many different anagrams of MONTANA are there?
- (g) How many different anagrams of ABCDEF are there if:
 - i. C is the left neighbor of E
 - ii. C is on the left of E (and not necessarily E's neighbor)
- (h) We have 9 balls, numbered 1 through 9, and 27 bins. How many different ways are there to distribute these 9 balls among the 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).

- (i) How many different ways are there to throw 9 identical balls into 27 bins? Assume the bins are distinguishable (e.g., numbered 1 through 27).
- (j) We throw 9 identical balls into 7 bins. How many different ways are there to distribute these 9 balls among the 7 bins such that no bin is empty? Assume the bins are distinguishable (e.g., numbered 1 through 7).
- (k) There are exactly 20 students currently enrolled in a class. How many different ways are there to pair up the 20 students, so that each student is paired with one other student? Solve this in at least 2 different ways. **Your final answer must consist of two different expressions.**
- (l) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a non-negative integer?
- (m) How many solutions does $x_0 + x_1 = n$ have, if each x must be a *strictly positive* integer?
- (n) How many solutions does $x_0 + x_1 + \cdots + x_k = n$ have, if each x must be a *strictly positive* integer?

Solution:

- (a) $\binom{n+k}{k}$
- (b) There are 3 options for the first digit. For each of the next digits, they only have 2 options because they cannot be equal to the previous digit. Thus, $3 \cdot 2^6$
- (c)
 - i. We have to choose 13 cards out of 52 cards, so this is just $\binom{52}{13}$.
 - ii. We now have to choose 13 cards out of 48 non-ace cards. So this is $\binom{48}{13}$.
 - iii. We now require the four aces to be present. So we have to choose the remaining 9 cards in our hand from the 48 non-ace cards, and this is $\binom{48}{9}$.
 - iv. We need our hand to contain 6 out of the 13 spade cards, and 7 out of the 39 non-spade cards, and these choices can be made separately. Hence, there are $\binom{13}{6} \binom{39}{7}$ ways to make up the hand.
- (d) If we consider the $104!$ rearrangements of 2 identical decks, since each card appears twice, we would have overcounted each distinct rearrangement. Consider any distinct rearrangement of the 2 identical decks of 52 cards and see how many times this appears among the rearrangement of 104 cards where each card is treated as different. For each identical pair (such as the two Ace of spades), there are two ways they could be permuted among each other (since $2! = 2$). This holds for each of the 52 pairs of identical cards. So the number $104!$ overcounts the actual number of rearrangements of 2 identical decks by a factor of 2^{52} . Hence, the actual number of rearrangements of 2 identical decks is $\frac{104!}{2^{52}}$.
- (e) **Answer 1:** There are $\binom{99}{k}$ 99-bit strings with k ones and $99 - k$ zeros. We need $k > 99 - k$, i.e. $k \geq 50$. So the total number of such strings is $\sum_{k=50}^{99} \binom{99}{k}$.

This expression can however be simplified. Since $\binom{99}{k} = \binom{99}{99-k}$, we have

$$\sum_{k=50}^{99} \binom{99}{k} = \sum_{k=50}^{99} \binom{99}{99-k} = \sum_{l=0}^{49} \binom{99}{l}$$

by substituting $l = 99 - k$.

Now $\sum_{k=50}^{99} \binom{99}{k} + \sum_{l=0}^{49} \binom{99}{l} = \sum_{m=0}^{99} \binom{99}{m} = 2^{99}$. Hence, $\sum_{k=50}^{99} \binom{99}{k} = \frac{1}{2} \cdot 2^{99} = 2^{98}$.

Answer 2: Symmetry Since the answer from above looked so simple, there must have been a more elegant way to arrive at it. Since 99 is odd, no 99-bit string can have the same number of zeros and ones. Let A be the set of 99-bit strings with more ones than zeros, and B be the set of 99-bit strings with more zeros than ones. Now take any 99-bit string x with more ones than zeros i.e. $x \in A$. If all the bits of x are flipped, then you get a string y with more zeros than ones, and so $y \in B$. This operation of bit flips creates a one-to-one and onto function (called a bijection) between A and B . Hence, it must be that $|A| = |B|$. Every 99-bit string is either in A or in B , and since there are 2^{99} 99-bit strings, we get $|A| = |B| = \frac{1}{2} \cdot 2^{99}$. The answer we sought was $|A| = 2^{98}$.

- (f) **ALABAMA:** The number of ways of rearranging 7 distinct letters and is $7!$. In this 7 letter word, the letter A is repeated 4 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $4!$ (which is the number of ways of permuting the 4 A's among themselves). Aka, we only want $1/4!$ out of the total rearrangements. Hence, there are $\frac{7!}{4!}$ anagrams.

MONTANA: In this 7 letter word, the letter A and N are each repeated 2 times while the other letters appear once. Hence, the number $7!$ overcounts the number of different anagrams by a factor of $2! \times 2!$ (one factor of $2!$ for the number of ways of permuting the 2 A's among themselves and another factor of $2!$ for the number of ways of permuting the 2 N's among themselves). Hence, there are $\frac{7!}{(2!)^2}$ different anagrams.

- (g) i. We consider CE is a new letter X, then the question becomes counting the rearranging of 5 distinct letters, and is $5!$.
- ii. Symmetry: Let A be the set of all the rearranging of ABCDEF with C on the left side of E, and B be the set of all the rearranging of ABCDEF with C on the right side of E. $|A \cup B| = 6!$, $|A \cap B| = 0$. There is a bijection between A and B by construct a operation of exchange the position of C and E. Thus $|A| = |B| = \frac{6!}{2}$.
- (h) Each ball has a choice of which bin it should go to. So each ball has 27 choices and the 9 balls can make their choices separately. Hence, there are 27^9 ways.
- (i) Since there is no restriction on how many balls a bin needs to have, this is just the problem of throwing k identical balls into n distinguishable bins, which can be done in $\binom{n+k-1}{k}$ ways. Here $k = 9$ and $n = 27$, so there are $\binom{35}{9}$ ways.
- (j) **Answer 1:** Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. There are 2 cases to consider:

Case 1: The 2 balls land in the same bin. This gives 7 ways.

Case 2: The 2 balls land in different bins. This gives $\binom{7}{2}$ ways of choosing 2 out of the 7 bins for the balls to land in. Note that it is *not* 7×6 since the balls are identical and so there is no order on them.

Summing up the number of ways from both cases, we get $7 + \binom{7}{2}$ ways.

Answer 2: Since each bin is required to be non-empty, let's throw one ball into each bin at the outset. Now we have 2 identical balls left which we want to throw into 7 distinguishable bins. From class (see note 11), we already saw that the number of ways to put k identical balls into n distinguishable bins is $\binom{n+k-1}{k}$. Taking $k = 2$ and $n = 7$, we get $\binom{8}{2}$ ways to do this.

EXERCISE: Can you give an expression for the number of ways to put k identical balls into n distinguishable bins such that no bin is empty?

- (k) **Answer 1:** Let's number the students from 1 to 20. Student 1 has 19 choices for her partner. Let i be the smallest index among students who have not yet been assigned partners. Then no matter what the value of i is (in particular, i could be 2 or 3), student i has 17 choices for her partner. The next smallest indexed student who doesn't have a partner now has 15 choices for her partner. Continuing in this way, the number of pairings is $19 \times 17 \times 15 \times \cdots \times 1 = \prod_{i=1}^{10} (2i - 1)$.

Answer 2: Arrange the students numbered 1 to 20 in a line. There are $20!$ such arrangements. We pair up the students at positions $2i - 1$ and $2i$ for i ranging from 1 to 10. You should be able to see that the $20!$ permutations of the students doesn't miss any possible pairing. However, it counts every different pairing multiple times. Fix any particular pairing of students. In this pairing, the first pair had freedom of 10 positions in any permutation that generated it, the second pair had a freedom of 9 positions in any permutation that generated it, and so on. There is also the freedom for the elements within each pair i.e. in any student pair (x, y) , student x could have appeared in position $2i - 1$ and student y could have appeared in position $2i$ and also vice versa. This gives 2 ways for each of the 10 pairs. Thus, in total, these freedoms cause $10! \times 2^{10}$ of the $20!$ permutations to give rise to this particular pairing. This holds for each of the different pairings. Hence, $20!$ overcounts the number of different pairings by a factor of $10! \times 2^{10}$. Hence, there are $\frac{20!}{10! \cdot 2^{10}}$ pairings.

Answer 3: In the first step, pick a pair of students from the 20 students. There are $\binom{20}{2}$ ways to do this. In the second step, pick a pair of students from the remaining 18 students. There are $\binom{18}{2}$ ways to do this. Keep picking pairs like this, until in the tenth step, you pick a pair of students from the remaining 2 students. There are $\binom{2}{2}$ ways to do this. Multiplying all these, we get $\binom{20}{2} \binom{18}{2} \cdots \binom{2}{2}$. However, in any particular pairing of 20 students, this pairing could have been generated in $10!$ ways using the above procedure depending on which pairs in the pairing got picked in the first step, second step, ..., tenth step. Hence, we have to divide the above number by $10!$ to get the number of different pairings. Thus there are $\binom{20}{2} \binom{18}{2} \cdots \binom{2}{2} / 10!$ different pairings of 20 students.

You may want to check for yourself that all three methods are producing the same integer, even though they are expressed very differently.

- (l) $\binom{n+k}{k}$. This is just n indistinguishable balls into $k+1$ distinguishable bins (stars and bars). There is a bijection between a sequence of n ones and k pluses and a solution to the equation: x_0 is the number of ones before the first plus, x_1 is the number of ones between the first and second plus, etc. A key idea is that if a bijection exists between two sets they must be the same size, so counting the elements of one tells us how many the other has. Note that this is the exact same answer as part (a) — make sure you understand why!
- (m) $n-1$. It's easiest just to enumerate the solutions here. x_0 can take values $1, 2, \dots, n-1$ and this uniquely fixes the value of x_1 . So, we have $n-1$ ways to do this. But, this is just an example of the more general question below.
- (n) $\binom{(n-(k+1))+k}{k} = \binom{n-1}{k}$. This is just $n-(k+1)$ indistinguishable balls into distinguishable $k+1$ bins. By subtracting 1 from all $k+1$ variables, and $k+1$ from the total required, we reduce it to problem with the same form as the previous problem. Once we have a solution to that we reverse the process, and adding 1 to all the non-negative variables gives us positive variables.

2 Grids and Trees!

Suppose we are given an $n \times n$ grid, for $n \geq 1$, where one starts at $(0,0)$ and goes to (n,n) . On this grid, we are only allowed to move left, right, up, or down by increments of 1.

- (a) How many shortest paths are there that go from $(0,0)$ to (n,n) ?
- (b) How many shortest paths are there that go from $(0,0)$ to $(n-1, n+1)$?

Now, consider shortest paths that meet the conditions where we can only visit points (x,y) where $y \leq x$. That is, the path cannot cross line $y = x$. We call these paths n -legal paths for a maze of side length n . Let F_n be the number of n -legal paths.

- (c) Compute the number of shortest paths from $(0,0)$ to (n,n) that cross $y = x$. (Hint: Let (i,i) be the first time the shortest path crosses the line $y = x$. Then the remaining path starts from $(i, i+1)$ and continues to (n,n) . If in the remainder of the path one exchanges y -direction moves with x -direction moves and vice versa, where does one end up?)
- (d) Compute the number of shortest paths from $(0,0)$ to (n,n) that do not cross $y = x$. (You may find your answers from parts (a) and (c) useful.)
- (e) A different idea is to derive a recursive formula for the number of paths. Fix some i with $0 \leq i \leq n-1$. We wish to count the number of n -legal paths where the last time the path touches the line $y = x$ is the point (i,i) . Show that the number of such paths is $F_i \cdot F_{n-i-1}$. (Hint: If $i = 0$, what are your first and last moves, and where is the remainder of the path allowed to go?)
- (f) Explain why $F_n = \sum_{i=0}^{n-1} F_i \cdot F_{n-i-1}$.

- (g) Create and explain a recursive formula for the number of trees with n vertices ($n \geq 1$), where each non-root node has degree at most 3, and the root node has degree at most 2. Two trees are different if and only if either left-subtree is different or right-subtree is different.

(Notice something about your formula and the grid problem. Neat!)

Solution: Let $(x, y) \rightarrow (x+1, y)$ be a move 'right' command, and $(x, y) \rightarrow (x, y+1)$ be a move 'up' command.

- (a) There are $\binom{2n}{n}$ paths, as there are total number of $2n$ moves, and n of them must be move 'right' command, the rest of them must be the move 'up' command.
- (b) There are $\binom{2n}{n-1}$ paths as there are now $n-1$ move 'right' command.
- (c) We will argue that the set of all paths that cross $y = x$, form a bijection with the set of all paths from $(0, 0)$ to $(n-1, n+1)$. Suppose we have a path that crosses $y = x$. Once a path crosses $y = x$, we can flip the later portion of the path. Let the first time the invalid path crosses $y = x$ be at (i, i) and arrives at $(i, i+1)$. Then if we do not flip the path, it will arrive at (n, n) by taking $n-i$ "right" commands, and $n-1-i$ "up" commands. If we flip these commands, it will go to $(i + (n-1-i), (i+1) + (n-i)) = (n-1, n+1)$. So all invalid paths map to a path from $(0, 0)$ to $(n-1, n+1)$. Next we argue that all path from $(0, 0)$ to $(n-1, n+1)$ maps to a invalid path. Paths from $(0, 0)$ to $(n-1, n+1)$ must cross the line $y = x$, let it first cross the line at (i, i) and arrives $(i, i+1)$. Then it must take $(n-1) - i$ "right" commands, and $(n+1) - (i+1)$ "up" commands. We flip these commands and so we now have, $n-1-i$ "up" commands, $n-i$ "right" commands. Then the path will arrive $(i + (n-i), (i+1) + (n-1-i)) = (n, n)$ and this new path is considered as invalid path since it crosses $y = x$ at point (i, i) . So all paths from $(0, 0)$ to $(n-1, n+1)$ can be mapped to an invalid paths.

So there is a bijective mapping between invalid paths and paths from $(0, 0)$ to $(n-1, n+1)$. Hence, the number of invalid paths is $\binom{2n}{n-1}$.

- (d) The number of paths that don't cross $y = x$ is given by the number of paths minus the number of paths that do cross $y = x$, or $\binom{2n}{n} - \binom{2n}{n-1}$.
- (e) Let F_n be the total number of different ways from $(0, 0)$ to (n, n) satisfies the condition above. We know $F_1 = 1$. Let (i, i) be the last point on line $y = x$ that a path touches except for (n, n) . Then total number of such path is $F_i \cdot F_{n-1-i}$ where F_i is the total number of paths from $(0, 0)$ to (i, i) . Since (i, i) is the last boundry point it touches, so for all later steps, it must not cross the line $y = x - 1$, it's equivalent to say the total number of paths from $(i+1, i)$ to $(n, n-1)$, it's F_{n-1-i} .
- (f) From the previous part, the number of n -legal paths where the last time the path touches the line $y = x$ at the point (i, i) is $F_i \cdot F_{n-1-i}$. To get the total number of paths that cross the $y = x$, we simply need to sum across every possible point (i, i) that the path could have passed through. Hence, $F_n = \sum_{i=0}^{n-1} F_i \cdot F_{n-1-i}$.

- (g) Let T_n be the total number of different trees with n nodes. The number of different trees when the left subtree has size i and right subtree has size $n - i - 1$ is $T_i \cdot T_{n-i-1}$. If we sum over all possible sizes of left subtrees, we can get the total number of different trees: $T_n = \sum_{i=0}^{n-1} T_i T_{n-i-1}$, with the base cases $T_0 = 1, T_1 = 1$. Note that a similar counting argument captures totally different objects (mazes and trees)!

If you are interested in why these problems are related, check out the [Catalan numbers](#).

3 Fermat's Wristband

Let p be a prime number and let k be a positive integer. We have beads of k different colors, where any two beads of the same color are indistinguishable.

- We place p beads onto a string. How many different ways are there to construct such a sequence of p beads with up to k different colors?
- How many sequences of p beads on the string are there that use at least two colors?
- Now we tie the two ends of the string together, forming a wristband. Two wristbands are equivalent if the sequence of colors on one can be obtained by rotating the beads on the other. (For instance, if we have $k = 3$ colors, red (R), green (G), and blue (B), then the length $p = 5$ necklaces RGGBG, GGBGR, GBGRG, BGRGG, and GRGGB are all equivalent, because these are all rotated versions of each other.)

How many non-equivalent wristbands are there now? Again, the p beads must not all have the same color. (Your answer should be a simple function of k and p .)

[Hint: Think about the fact that rotating all the beads on the wristband to another position produces an identical wristband.]

- Use your answer to part (c) to prove Fermat's little theorem.

Solution:

- k^p . For each of the p beads, there are k possibilities for its colors. Therefore, by the first counting principle, there are k^p different sequences.
- $k^p - k$. You can have k sequences of a beads with only one color.
- Since p is prime, rotating any sequence by less than p spots will produce a new sequence. As in, there is no number x smaller than p such that rotating the beads by x would cause the pattern to look the same. So, every pattern which has more than one color of beads can be rotated to form $p - 1$ other patterns. So the total number of patterns equivalent with some bead sequence is p . Thus, the total number of non-equivalent patterns are $(k^p - k)/p$.
- $(k^p - k)/p$ must be an integer, because from the previous part, it is the number of ways to count something. Hence, $k^p - k$ has to be divisible by p , i.e., $k^p \equiv k \pmod{p}$, which is Fermat's Little Theorem.

4 Counting on Graphs + Symmetry

- (a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.
- (b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1) , (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).
- (c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.
- (d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

Solution:

- (a) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$. (Recall that $2^m = \sum_{k=0}^m \binom{m}{k}$, where $m = n(n-1)/2$ in this case.)
- (b) The number k of vertices in a cycle is at least 3 and at most n . Without accounting for duplicates, there are $n!/(n-k)!$ cycles. Due to inversions (read from front or end doesn't matter, $abc = cba$) we divide by 2, and rotations (where we start reading from doesn't matter $abc = bca = cab$) we divide by k . Hence the total number of distinct cycles is

$$\sum_{k=3}^n \frac{n!}{(n-k)! \cdot 2k}.$$

- (c) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.
- (d) Without considering symmetries there are $6!$ ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.

5 Proofs of the Combinatorial Variety

Prove each of the following identities using a combinatorial proof.

- (a) For every positive integer $n > 1$,

$$\sum_{k=0}^n k \cdot \binom{n}{k} = n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k}.$$

- (b) For each positive integer m and each positive integer $n > m$,

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c} = \binom{3n}{m}.$$

(Notation: the sum on the left is taken over all triples of nonnegative integers (a, b, c) such that $a + b + c = m$.)

Solution:

- (a) Suppose we have n people and want to pick some of them to form a special committee. Moreover, suppose we want to pick a leader from among the committee members - how many ways can we do this?

We can do so by first picking the committee members, and then choosing the leader from among the chosen members. We can pick a committee of size k in $\binom{n}{k}$ ways, and once we have picked the committee, we have k choices for which member becomes the leader. In order to account for all possible committee sizes, we need to sum over all valid values of k , hence we get the expression

$$\sum_{k=0}^n k \cdot \binom{n}{k},$$

which is exactly the left hand side of the identity we want to prove.

Now, we can also count this set by first picking the leader for the committee, then choosing the rest of committee. We have n choices for the leader, and then among the remaining $n - 1$ people, we can pick any subset to form the rest of the committee. Picking a subset of size k can be done in $\binom{n-1}{k}$ ways, hence summing over k , we get the expression

$$n \cdot \sum_{k=0}^{n-1} \binom{n-1}{k},$$

which is exactly the right hand side of the identity we want to prove.

- (b) Suppose we have n distinguishable red pencils, n distinguishable blue pencils, and n distinguishable green pencils ($3n$ pencils total), and want to choose m of these pencils to bring to class. How many ways can we do this?

We can do so by just picking the m pencils without considering color, as they are all distinguishable. There are $\binom{3n}{m}$ ways of doing this, which is exactly the right hand side of the identity we want to prove.

We can also count this set by picking some red pencils, then picking some blue pencils, and then finally picking some green pencils. We can pick a red pencils, b blue pencils, and c green pencils (with the tacit assumption that $a + b + c = m$) in $\binom{n}{a} \cdot \binom{n}{b} \cdot nc$ ways. Finally, in order to account for all possible distributions of pencils, we need to sum over all valid triples (a, b, c) , which gives us the expression

$$\sum_{a+b+c=m} \binom{n}{a} \cdot \binom{n}{b} \cdot \binom{n}{c},$$

which is exactly the left hand side of the identity we want to prove.

6 Fibonacci Fashion

You have n accessories in your wardrobe, and you'd like to plan which ones to wear each day for the next t days. As a student of the Elegant Etiquette Charm School, you know it isn't fashionable to wear the same accessories multiple days in a row. (Note that the same goes for clothing items in general). Therefore, you'd like to plan which accessories to wear each day represented by subsets S_1, S_2, \dots, S_t , where $S_1 \subseteq \{1, 2, \dots, n\}$ and for $2 \leq i \leq t$, $S_i \subseteq \{1, 2, \dots, n\}$ and S_i is disjoint from S_{i-1} .

- (a) For $t \geq 1$, prove that there are F_{t+2} binary strings of length t with no consecutive zeros (assume the Fibonacci sequence starts with $F_0 = 0$ and $F_1 = 1$).
- (b) Use a combinatorial proof to prove the following identity, which, for $t \geq 1$ and $n \geq 0$, gives the number of ways you can create subsets of your n accessories for the next t days such that no accessory is worn two days in a row:

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t} = (F_{t+2})^n.$$

(You may assume that $\binom{a}{b} = 0$ whenever $a < b$.)

Solution:

- (a) We will prove this by strong induction.

Base cases: For $k = 1$, the only binary strings possible are 0 and 1. Therefore, there are two possible binary strings, and $F_{k+2} = F_3 = 2$. For $k = 2$, the binary strings possible are 11, 01, and 10, and we have $F_{k+2} = F_4 = 3$, so the identity holds.

Inductive hypothesis: For $k \geq 2$, assume that for all $1 \leq x \leq k$, there are F_{x+2} binary strings of length x with no consecutive zeros.

Inductive step: Consider the set of binary strings of length $k+1$ with no consecutive zeros. We can group these into two sets: those which end with 0, and those which end with 1.

For those that end with a 0, these can be constructed by taking the set of binary strings of length $k-1$ with no consecutive zeros and appending 10 to the end of them. Then by the inductive hypothesis, this set is of size F_{k+1} . For those that end with a 1, these can be constructed by taking the set of binary strings of length k with no consecutive zeros and appending a 1 to the end of them. Then by the inductive hypothesis, this set is of size F_{k+2} .

Since the union of these two subsets (those which end with 0 and those which end with 1) cover all possible elements in the set of binary strings of length $k+1$ with no consecutive zeros, the size of this set will be $F_{k+1} + F_{k+2} = F_{k+3}$. This thus proves the inductive hypothesis.

- (b) We first consider the left-hand-side of the identity. To create subsets of accessories that are consecutively disjoint with sizes $x_i = |S_i|$, $1 \leq i \leq n$, there are $\binom{n}{x_1}$ ways to create S_1 , the subset of accessories you will wear on the first day. Then since S_2 must be disjoint from S_1 , there are $\binom{n-x_1}{x_2}$ ways choose accessories to create S_2 . Since S_3 must be disjoint from S_2 , there are $\binom{n-x_2}{x_3}$ ways choose accessories to create S_3 , and so on. Thus there are $\binom{n}{x_1} \binom{n-x_1}{x_2} \cdots \binom{n-x_{t-1}}{x_t}$ ways to create subsets of accessories S_1, \dots, S_t with respective sizes x_1, \dots, x_t . Then altogether, S_1, \dots, S_t can be created in

$$\sum_{x_1 \geq 0} \sum_{x_2 \geq 0} \cdots \sum_{x_t \geq 0} \binom{n}{x_1} \binom{n-x_1}{x_2} \binom{n-x_2}{x_3} \cdots \binom{n-x_{t-1}}{x_t}$$

ways.

Now, consider the right-hand-side of the identity. Now for each accessory $i \in \{1, \dots, n\}$, we will first decide which subsets S_1, \dots, S_t will contain accessory i , where we can't assign item i to consecutive subsets. For each accessory, we create a binary string of length t , where the leading digit represents S_1 , the next digit represents S_2 , and so on. We will say that a 0 in digit k means that we will wear the accessory on day k . Therefore, the number of ways we can assign accessory i to subsets S_1, \dots, S_t such that no two consecutive subsets both have accessory i is the same as the number of binary strings of length t with no consecutive zeros. Thus using the result in part (a), there are F_{t+2} ways to select the nonconsecutive subsets containing i among S_1, \dots, S_t . Since we have n accessories, accessories $1, \dots, n$ can be placed into subsets S_1, \dots, S_t in $(F_{t+2})^n$ ways.

This thus proves the identity.