

## 1 Solving a System of Equations

Alice wants to buy apples, beets, and carrots. An apple, a beet, and a carrot cost 16 dollars, two apples and three beets cost 23 dollars, and one apple, two beets, and three carrots cost 35 dollars. What are the prices for an apple, for a beet, and for a carrot, respectively? Set up a system of equations and show your work.

### Solution:

Letting  $a, b$ , and  $c$  be the dollar cost of an apple, beet, and carrot, respectively, we get the system of equations

$$\begin{aligned}a + b + c &= 16 \\2a + 3b &= 23 \\a + 2b + 3c &= 35.\end{aligned}$$

There are many approaches to solving this system (Gaussian Elimination, substitution, etc.). Here we show a solution via substitution.

Subtracting the third equation from three times the first equation gives

$$2a + b = 3(a + b + c) - (a + 2b + 3c) = 3 \cdot 16 - 35 = 13.$$

Subtracting this equation from the second equation gives

$$2b = (2a + 3b) - (2a + b) = 23 - 13 = 10,$$

so  $b = 5$ . Backsolving gives  $a = 4$  and  $c = 7$ .

## 2 Calculus Review

(a) Compute the following integral:

$$\int_0^\infty \sin(t)e^{-t} dt.$$

(b) Find the minimum value of the following function over the reals and determine where it occurs.

$$f(x) = \int_0^{x^2} e^{-t^2} dt.$$

Show your work.

(c) Compute the double integral

$$\iint_R 2x + y \, dA,$$

where  $R$  is the region bounded by the lines  $x = 1$ ,  $y = 0$ , and  $y = x$ .

**Solution:**

(a) Let  $I = \int \sin(t)e^{-t} \, dt$ .

Use integration by parts, with  $u = \sin(t)$  and  $dv = e^{-t}$ .

This means  $du = \cos(t)$  and  $v = -e^{-t}$ .

$$\begin{aligned} I &= \int \sin(t)e^{-t} \, dt = uv - \int v \cdot du \\ &= -\sin(t)e^{-t} + \int e^{-t} \cos(t) \, dt \end{aligned}$$

Use integration by parts again on  $\int e^{-t} \cos(t) \, dt$ , with  $u = \cos(t)$  and  $dv = e^{-t}$ . This means  $du = -\sin(t)$  and  $dv = -e^{-t}$ .

$$\begin{aligned} \int e^{-t} \cos(t) \, dt &= uv - \int v \cdot du \\ &= -\cos(t)e^{-t} - \int e^{-t} \cdot \sin(t) \, dt \\ &= -\cos(t)e^{-t} - I \end{aligned}$$

Combining these results:

$$\begin{aligned} I &= -\sin(t)e^{-t} - \cos(t)e^{-t} - I \\ \Rightarrow 2I &= -\sin(t)e^{-t} - \cos(t)e^{-t} \\ \Rightarrow I &= \frac{-\sin(t)e^{-t} - \cos(t)e^{-t}}{2} \end{aligned}$$

Finally, we have:

$$I \Big|_0^\infty = \frac{0-0}{2} - \frac{0-1}{2} = \frac{1}{2}.$$

(b) Compute the derivative of the function, and set it equal to 0. Let  $y = x^2$ . By the Chain Rule and the Fundamental Theorem of Calculus,

$$\begin{aligned} \frac{df}{dx} &= \frac{df}{dy} \cdot \frac{dy}{dx} \\ &= e^{-y^2} \cdot 2x \\ &= 2xe^{-x^4} = 0 \end{aligned}$$

Since  $e^{-x^4}$  is always positive, we get that the derivative is 0 only at  $x^* = 0$ , with corresponding value of the function  $f(x^*) = 0$ .

We double check that the found point is a minima by checking the second derivative:

$$\frac{d^2 f}{dx^2} = \frac{d}{dx} 2xe^{-x^4} = 2e^{-x^4} - 8x^4 e^{-x^4}.$$

The second derivative evaluated at  $x^* = 0$  is  $2e^{-0^4} - 8 \cdot 0^4 e^{-0^4} = 2$ , which is positive, so our found point is a minima, as desired.

Hence, the minimum value of  $f(x)$  will be

$$f(x^*) = f(0) = \int_0^0 e^{-t^2} dt = 0.$$

(c) We may set up the integral over the region  $R$  as follows:

$$\int_0^1 \int_0^x 2x + y \, dy \, dx.$$

Evaluating this integral gives

$$\begin{aligned} \int_0^1 \int_0^x 2x + y \, dy \, dx &= \int_0^1 2xy + \frac{y^2}{2} \Big|_0^x \, dx \\ &= \int_0^1 \frac{5x^2}{2} \, dx \\ &= \frac{5x^3}{6} \Big|_0^1 \\ &= \frac{5}{6}. \end{aligned}$$

### 3 Prove or Disprove

For each of the following, either prove the statement, or disprove by finding a counterexample.

- (a)  $(\forall n \in \mathbb{N})$  if  $n$  is odd then  $n^2 + 4n$  is odd.
- (b)  $(\forall a, b \in \mathbb{R})$  if  $a + b \leq 15$  then  $a \leq 11$  or  $b \leq 4$ .
- (c)  $(\forall r \in \mathbb{R})$  if  $r^2$  is irrational, then  $r$  is irrational.
- (d)  $(\forall n \in \mathbb{Z}^+) 5n^3 > n!$ . (Note:  $\mathbb{Z}^+$  is the set of positive integers)

**Solution:**

(a) **Answer:** True.

**Proof:** We will use a direct proof. Assume  $n$  is odd. By the definition of odd numbers,  $n = 2k + 1$  for some natural number  $k$ . Substituting into the expression  $n^2 + 4n$ , we get  $(2k + 1)^2 + 4 \cdot (2k + 1)$ . Simplifying the expression yields  $4k^2 + 12k + 5$ . This can be rewritten as

$2 \cdot (2k^2 + 6k + 2) + 1$ . Since  $2k^2 + 6k + 2$  is a natural number, by the definition of odd numbers,  $n^2 + 4n$  is odd.

Alternatively, we could also factor the expression to get  $n(n + 4)$ . Since  $n$  is odd,  $n + 4$  is also odd. The product of 2 odd numbers is also an odd number. Hence  $n^2 + 4n$  is odd.

(b) **Answer:** True.

**Proof:** We will use a proof by contraposition. Suppose that  $a > 11$  and  $b > 4$  (note that this is equivalent to  $\neg(a \leq 11 \vee b \leq 4)$ ). Since  $a > 11$  and  $b > 4$ ,  $a + b > 15$  (note that  $a + b > 15$  is equivalent to  $\neg(a + b \leq 15)$ ). Thus, if  $a + b \leq 15$ , then  $a \leq 11$  or  $b \leq 4$ .

(c) **Answer:** True.

**Proof:** We will use a proof by contraposition. Assume that  $r$  is rational. Since  $r$  is rational, it can be written in the form  $\frac{a}{b}$  where  $a$  and  $b$  are integers with  $b \neq 0$ . Then  $r^2$  can be written as  $\frac{a^2}{b^2}$ . By the definition of rational numbers,  $r^2$  is a rational number, since both  $a^2$  and  $b^2$  are integers, with  $b \neq 0$ . By contraposition, if  $r^2$  is irrational, then  $r$  is irrational.

(d) **Answer:** False.

**Proof:** We will show a counterexample. Let  $n = 7$ .  $5 \cdot 7^3 = 1715$ .  $7! = 5040$ . Since  $5n^3 < n!$ , the claim is false.

## 4 Rationals and Irrationals

Prove that the product of a non-zero rational number and an irrational number is irrational.

**Solution:** We prove the statement by contradiction. Suppose that  $ab = c$ , where  $a \neq 0$  is rational,  $b$  is irrational, and  $c$  is rational. Since  $a$  and  $b$  are not zero (because 0 is rational),  $c$  is also non-zero. Thus, we can express  $a = \frac{p}{q}$  and  $c = \frac{r}{s}$ , where  $p, q, r$ , and  $s$  are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

which is the ratio of two nonzero integers, giving that  $b$  is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational.

## 5 Twin Primes

(a) Let  $p > 3$  be a prime. Prove that  $p$  is of the form  $3k + 1$  or  $3k - 1$  for some integer  $k$ .

(b) *Twin primes* are pairs of prime numbers  $p$  and  $q$  that have a difference of 2. Use part (a) to prove that 5 is the only prime number that takes part in two different twin prime pairs.

**Solution:**

- (a) First we note that any integer can be written in one of the forms  $3k$ ,  $3k + 1$ , or  $3k + 2$ . (Note that  $3k + 2$  is equal to  $3(k + 1) - 1$ . Since  $k$  is arbitrary, we can treat these as equivalent forms).

We can now prove the contrapositive: that any integer  $m > 3$  of the form  $3k$  must be composite. Any such integer is divisible by 3, so this is true right away. Thus our original claim is true as well.

- (b) We can check all the primes up to 5 to see that of these, only 5 takes part in two twin prime pairs (3,5 and 5,7). What about primes  $> 5$ ?

For any prime  $m > 5$ , we can check if  $m + 2$  and  $m - 2$  are both prime. Note that if  $m > 5$ , then  $m + 2 > 3$  and  $m - 2 > 3$  so we can apply part (a) and we can do a proof by cases based on the two forms from part (a).

Case 1:  $m$  is of the form  $3k + 1$ . Then  $m + 2 = 3k + 3$ , which is divisible by 3. So  $m + 2$  is not prime.

Case 2:  $m$  is of the form  $3k - 1$ . Then  $m - 2 = 3k - 3$ , which is divisible by 3. So  $m - 2$  is not prime.

So in either case, at least one of  $m + 2$  and  $m - 2$  is not prime.

## 6 Airport

Suppose that there are  $2n + 1$  airports where  $n$  is a positive integer. The distances between any two airports are all different. For each airport, exactly one airplane departs from it and is destined for the closest airport. Prove by induction that there is an airport which has no airplanes destined for it.

### Solution:

For  $n = 1$ , let the 3 airports be  $A, B, C$  and let their distance be  $|AB|, |AC|, |BC|$ . Without loss of generality suppose  $B, C$  is the closest pair of airports (which is well defined since all distances are different). Then the airplanes departing from  $B$  and  $C$  are flying towards each other. Since the airplane from  $A$  must fly to somewhere else, no airplanes are destined for airport  $A$ .

Now suppose the statement is proven for  $n = k$ , i.e. when there are  $2k + 1$  airports. For  $n = k + 1$ , i.e. when there are  $2k + 3$  airports, the airplanes departing from the closest two airports must be destined for each other's starting airports. Removing these two airports reduce the problem to  $2k + 1$  airports. From the inductive hypothesis, we know that among the  $2k + 1$  airports remaining, there is an airport with no incoming flights which we call airport  $Z$ . When we add back the two airports that we removed, the airplane flights may change; in particular, it is possible that an airplane will now choose to fly to one of these two airports (because the airports that were added may be closer than the airport to which the airplane was previously flying), but observe that none of the airplanes will be destined for the airport  $Z$ . Also, the two airports that were added back will have airplanes destined for each other, so they too will not be destined for airport  $Z$ . We conclude that the airport  $Z$  will continue to have no incoming flights when we add back the two airports, and so the statement holds for  $n = k + 1$ . By induction, the claim holds for all  $n \geq 3$ .

## 7 A Coin Game

Your "friend" Stanley Ford suggests you play the following game with him. You each start with a single stack of  $n$  coins. On each of your turns, you select one of your stacks of coins (that has at least two coins) and split it into two stacks, each with at least one coin. Your score for that turn is the product of the sizes of the two resulting stacks (for example, if you split a stack of 5 coins into a stack of 3 coins and a stack of 2 coins, your score would be  $3 \cdot 2 = 6$ ). You continue taking turns until all your stacks have only one coin in them. Stan then plays the same game with his stack of  $n$  coins, and whoever ends up with the largest total score over all their turns wins.

Prove that no matter how you choose to split the stacks, your total score will always be  $\frac{n(n-1)}{2}$ . (This means that you and Stan will end up with the same score no matter what happens, so the game is rather pointless.)

### Solution:

We can prove this by strong induction on  $n$ .

**Base Case:** If  $n = 1$ , you start with a stack of one coin, so the game immediately terminates. Your total score is zero—and indeed,  $\frac{n(n-1)}{2} = \frac{1 \cdot 0}{2} = 0$ .

**Inductive Step:** Suppose that if you start with  $i$  coins (for  $i$  between 1 and  $n$  inclusive), your score will be  $\frac{i(i-1)}{2}$  no matter what strategy you employ. Now suppose you start with  $n + 1$  coins. In your first move, you must split your stack into two smaller stacks. Call the sizes of these stacks  $s_1$  and  $s_2$  (so  $s_1 + s_2 = n + 1$  and  $s_1, s_2 \geq 1$ ). Your end score comes from three sources: the points you get from making this first split, the points you get from future splits involving coins from stack 1, and the points you get from future splits involving coins from stack 2. From the rules of the game, we know you get  $s_1 s_2$  points from the first split. From the inductive hypothesis (which we can apply because  $s_1$  and  $s_2$  are between 1 and  $n$ ), we know that the total number of points you get from future splits of stack 1 is  $\frac{s_1(s_1-1)}{2}$  and similarly that the total number of points you get from future splits of stack 2 is  $\frac{s_2(s_2-1)}{2}$ , regardless of what strategy you employ in splitting them. Thus, the total number of points we score is

$$\begin{aligned} s_1 s_2 + \frac{s_1(s_1-1)}{2} + \frac{s_2(s_2-1)}{2} &= \frac{s_1(s_1-1) + 2s_1 s_2 + s_2(s_2-1)}{2} \\ &= \frac{(s_1(s_1-1) + s_1 s_2) + (s_2(s_2-1) + s_1 s_2)}{2} \\ &= \frac{s_1(s_1 + s_2 - 1) + s_2(s_1 + s_2 - 1)}{2} \\ &= \frac{(s_1 + s_2)(s_1 + s_2 - 1)}{2} \end{aligned}$$

Since  $s_1 + s_2 = n + 1$ , this works out to  $\frac{(n+1)(n+1-1)}{2}$ , which is what we wanted to show your total number of points came out to. This completes our proof by induction.

## 8 AM-GM

For nonnegative real numbers  $a_1, \dots, a_n$ , the arithmetic mean, or average, is defined by

$$\frac{a_1 + \dots + a_n}{n},$$

and the geometric mean is defined by

$$\sqrt[n]{a_1 \cdots a_n}.$$

In this problem, we will prove the “AM-GM” inequality. More precisely, for all positive integers  $n \geq 2$ , given any nonnegative real numbers  $a_1, \dots, a_n$ , we will show that

$$\frac{a_1 + \dots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

We will do so by induction on  $n$ , but in an unusual way.

- (a) Prove that the inequality holds for  $n = 2$ . In other words, for nonnegative real numbers  $a_1$  and  $a_2$ , show that

$$\frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}.$$

(This equation might be of use:  $(\sqrt{a} - \sqrt{b})^2 = a - 2\sqrt{ab} + b$ )

- (b) For some positive integer  $k$ , suppose that the AM-GM inequality holds for  $n = 2^k$ . Show that the AM-GM inequality holds for  $n = 2^{k+1}$ . (Hint: Think about how the AM-GM inequality for  $n = 2$  could be used here.)
- (c) For some positive integer  $k \geq 2$ , suppose that the AM-GM inequality holds for  $n = k$ . Show that the AM-GM inequality holds for  $n = k - 1$ . (Hint: In the AM-GM expression for  $n = k$ , consider substituting  $a_k = \frac{a_1 + \dots + a_{k-1}}{k-1}$ .)
- (d) Argue why parts a) - c) imply that the AM-GM inequality holds for all positive integers  $n \geq 2$ .

### Solution:

- (a) Since the  $a_1$  and  $a_2$  are nonnegative and real,  $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$  since the square of a real number is nonnegative. We can manipulate the equation:

$$\begin{aligned} & (\sqrt{a_1} - \sqrt{a_2})^2 \geq 0 \\ \iff & a_1 - 2\sqrt{a_1 a_2} + a_2 \geq 0 \\ \iff & a_1 + a_2 \geq 2\sqrt{a_1 a_2} \\ \iff & \frac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2} \end{aligned}$$

Thus, our initial inequality holds. Note that since every step was bidirectional, performing the proof in the reverse direction is valid but in general you should not assume what you are proving to be true.

- (b) Let  $a_1, \dots, a_{2^k}, a_{2^k+1}, \dots, a_{2^{k+1}}$  be any nonnegative real numbers. By our hypothesis, we know that

$$l_1 = \frac{a_1 + \dots + a_{2^k}}{2^k} \geq \sqrt[2^k]{a_1 \dots a_{2^k}} = r_1$$

and

$$l_2 = \frac{a_{2^k+1} + \dots + a_{2^{k+1}}}{2^k} \geq \sqrt[2^k]{a_{2^k+1} \dots a_{2^{k+1}}} = r_2.$$

For ease of notation, we define  $l_1, l_2, r_1$ , and  $r_2$  as above. Then applying part a), we know that

$$\frac{l_1 + l_2}{2} \geq \sqrt{l_1 l_2} \geq \sqrt{r_1 r_2}.$$

On the left hand side, we have that

$$\frac{l_1 + l_2}{2} = \frac{\frac{a_1 + \dots + a_{2^k}}{2^k} + \frac{a_{2^k+1} + \dots + a_{2^{k+1}}}{2^k}}{2} = \frac{a_1 + \dots + a_{2^{k+1}}}{2^{k+1}},$$

and on the right hand side, we have that

$$\sqrt{r_1 r_2} = \sqrt{\sqrt[2^k]{a_1 \dots a_{2^k}} \sqrt[2^k]{a_{2^k+1} \dots a_{2^{k+1}}}} = \sqrt[2^{k+1}]{a_1 \dots a_{2^{k+1}}},$$

so we conclude that

$$\frac{a_1 + \dots + a_{2^{k+1}}}{2^{k+1}} \geq \sqrt[2^{k+1}]{a_1 \dots a_{2^{k+1}}},$$

which is AM-GM for  $n = 2^{k+1}$ , as desired.

- (c) Let  $a_1, \dots, a_k$  be any nonnegative real numbers. By our hypothesis, we know that

$$\frac{a_1 + \dots + a_k}{k} \geq \sqrt[k]{a_1 \dots a_k}.$$

Let  $a_k = \frac{a_1 + \dots + a_{k-1}}{k-1}$ . Then we have that

$$\begin{aligned} \frac{a_1 + \dots + \frac{a_1 + \dots + a_{k-1}}{k-1}}{k} &= \frac{(a_1 + \dots + a_{k-1}) + \frac{a_1 + \dots + a_{k-1}}{k-1}}{k} \\ &= \frac{k \frac{a_1 + \dots + a_{k-1}}{k-1}}{k} \\ &= \frac{a_1 + \dots + a_{k-1}}{k-1}, \end{aligned}$$

and

$$\sqrt[k]{a_1 \dots a_k} = \sqrt[k]{a_1 \dots a_{k-1} \frac{a_1 + \dots + a_{k-1}}{k-1}},$$

so

$$\frac{a_1 + \dots + a_{k-1}}{k-1} \geq \sqrt[k]{a_1 \dots a_{k-1} \frac{a_1 + \dots + a_{k-1}}{k-1}}.$$

Raising both sides to the  $k$  power and dividing both sides by  $\frac{a_1 + \dots + a_{k-1}}{k-1}$  gives

$$\left( \frac{a_1 + \dots + a_{k-1}}{k-1} \right)^{k-1} \geq a_1 \dots a_{k-1},$$



so taking the  $k - 1$ th root of both sides gives

$$\frac{a_1 + \cdots + a_{k-1}}{k-1} \geq \sqrt[k-1]{a_1 \cdots a_{k-1}},$$

which is AM-GM for  $n = k - 1$ , as desired.

- (d) For any positive integer  $k$ , there exists a power of 2  $2^\ell$  greater or equal to  $k$ . Thus, starting at the base case in part a), we can use part b) to show that AM-GM applies for  $n = 2^\ell$ , then use part c) to show that AM-GM applies to  $n = k$ .