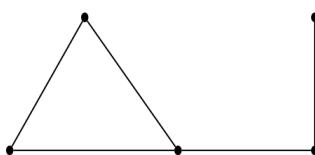


1 Degree Sequences

The *degree sequence* of a graph is the sequence of the degrees of the vertices, arranged in descending order, with repetitions as needed. For example, the degree sequence of the following graph is $(3, 2, 2, 2, 1)$.



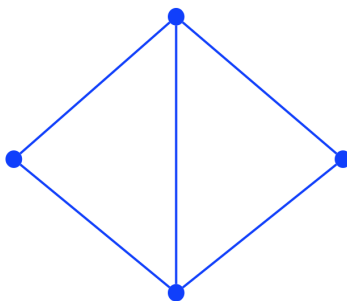
For each of the parts below, determine if there exists a simple undirected graph G (i.e. a graph without self-loops and multiple-edges) having the given degree sequence. Justify your claim.

- (a) $(3, 3, 2, 2)$
- (b) $(3, 2, 2, 2, 2, 1, 1)$
- (c) $(6, 2, 2, 2)$
- (d) $(4, 4, 3, 2, 1)$

Solution:

- (a) **Yes**

The following graph has degree sequence $(3, 3, 2, 2)$.



(b) **No**

For any graph G , the number of vertices that have odd degree is even (since the sum of degrees is twice the number of edges). The given degree sequence has 3 odd degree vertices.

(c) **No**

The total number of vertices is 4. Hence there cannot be a vertex with degree 6.

(d) **No**

The total number of vertices is 5. Hence, any degree 4 vertex must have an edge with every other vertex. Since there are two degree 4 vertices, there cannot be a vertex with degree 1.

2 Bipartite Graph

A bipartite graph consists of 2 disjoint sets of vertices (say L and R), such that no 2 vertices in the same set have an edge between them. For example, here is a bipartite graph (with $L = \{\text{green vertices}\}$ and $R = \{\text{red vertices}\}$), and a non-bipartite graph.



Figure 1: A bipartite graph (left) and a non-bipartite graph (right).

Prove that a graph has no tours of odd length if it is a bipartite (This is equivalent to proving that, a graph G being a bipartite implies that G has no tours of odd length).

Solution:

Suppose there is a tour in the bipartite graph. Let us start traveling the tour from a node n_0 in L . Since each edge in the graph connects a vertex in L to one in R , the 1st edge in the tour connects our start node n_0 to a node n_1 in R . The 2nd edge in the tour must connect n_1 to a node n_2 in L . Continuing on, the $(2k+1)$ -th edge connects node n_{2k} in L to node n_{2k+1} in R , and the $2k$ -th edge connects node n_{2k-1} in R to node n_{2k} in L . Since only even numbered edges connect to vertices in L , and we started our tour in L , the tour must end with an even number of edges.

3 Not everything is normal: Odd-Degree Vertices

Claim: Let $G = (V, E)$ be an undirected graph. The number of vertices of G that have odd degree is even.

Prove the claim above using:

- (i) Direct proof (e.g., counting the number of edges in G). *Hint: in lecture, we proved that $\sum_{v \in V} \deg v = 2|E|$.*
- (ii) Induction on $m = |E|$ (number of edges)
- (iii) Induction on $n = |V|$ (number of vertices)

Solution:

Let $V_{\text{odd}}(G)$ denote the set of vertices in G that have odd degree. We prove that $|V_{\text{odd}}(G)|$ is even.

- (i) Let d_v denote the degree of vertex v (so $d_v = |N_v|$, where N_v is the set of neighbors of v). Observe that

$$\sum_{v \in V} d_v = 2m$$

because every edge is counted exactly twice when we sum the degrees of all the vertices. Now partition V into the odd degree vertices $V_{\text{odd}}(G)$ and the even degree vertices $V_{\text{odd}}(G)^c$, so we can write

$$\sum_{v \in V_{\text{odd}}(G)} d_v = 2m - \sum_{v \notin V_{\text{odd}}(G)} d_v.$$

Both terms in the right-hand side above are even ($2m$ is even, and each term d_v is even because we are summing over even degree vertices $v \notin V_{\text{odd}}(G)$). So for the left-hand side $\sum_{v \in V_{\text{odd}}(G)} d_v$ to be even, we must have an even number of terms, since each term in the summation is odd. Therefore, there must be an even number of odd-degree vertices, namely, $|V_{\text{odd}}(G)|$ is even.

- (ii) We use induction on $m \geq 0$.

Base case $m = 0$: If there are no edges in G , then all vertices have degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with m edges.

Inductive step: Let G be a graph with $m + 1$ edges. Remove an arbitrary edge $\{u, v\}$ from G , so the resulting graph G' has m edges. By the inductive hypothesis, we know $|V_{\text{odd}}(G')|$ is even. Now add the edge $\{u, v\}$ to get back the original graph G . Note that u has one more edge in G than it does in G' , so $u \in V_{\text{odd}}(G)$ if and only if $u \notin V_{\text{odd}}(G')$. Similarly, $v \in V_{\text{odd}}(G)$ if and only if $v \notin V_{\text{odd}}(G')$. The degrees of all other vertices are unchanged in going from G' to G . Therefore,

$$V_{\text{odd}}(G) = \begin{cases} V_{\text{odd}}(G') \cup \{u, v\} & \text{if } u, v \notin V_{\text{odd}}(G') \\ V_{\text{odd}}(G') \setminus \{u, v\} & \text{if } u, v \in V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{u\}) \cup \{v\} & \text{if } u \in V_{\text{odd}}(G'), v \notin V_{\text{odd}}(G') \\ (V_{\text{odd}}(G') \setminus \{v\}) \cup \{u\} & \text{if } u \notin V_{\text{odd}}(G'), v \in V_{\text{odd}}(G') \end{cases}$$

so we see that $|V_{\text{odd}}(G)| - |V_{\text{odd}}(G')| \in \{-2, 0, 2\}$. Since $|V_{\text{odd}}(G')|$ is even, we conclude $|V_{\text{odd}}(G)|$ is also even.

(iii) We use induction on $n \geq 1$.

Base case $n = 1$: If G only has 1 vertex, then that vertex has degree 0, so $V_{\text{odd}}(G) = \emptyset$.

Inductive hypothesis: Assume $|V_{\text{odd}}(G)|$ is even for all graphs G with n vertices.

Inductive step: Let G be a graph with $n + 1$ vertices. Remove a vertex v and all edges adjacent to it from G . The resulting graph G' has n vertices, so by the inductive hypothesis, $|V_{\text{odd}}(G')|$ is even. Now add the vertex v and all edges adjacent to it to get back the original graph G . Let $N_v \subseteq V$ denote the neighbors of v (i.e., all vertices adjacent to v). Among the neighbors N_v , the vertices in the intersection $A = N_v \cap V_{\text{odd}}(G')$ had odd degree in G' , so they now have even degree in G . On the other hand, the vertices in $B = N_v \cap V_{\text{odd}}(G')^c$ had even degree in G' , and they now have odd degree in G . The vertex v itself has degree $|N_v|$, so $v \in V_{\text{odd}}(G)$ if and only if $|N_v|$ is odd. We now consider two cases:

(a) Suppose $|N_v|$ is even, so $v \notin V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B$$

so $|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B|$. Note that A and B are disjoint and their union equals N_v , so $|A| + |B| = |N_v|$. Therefore, we can write $|V_{\text{odd}}(G)|$ as

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| + |N_v| - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is even by assumption.

(b) Suppose $|N_v|$ is odd, so $v \in V_{\text{odd}}(G)$. Then

$$V_{\text{odd}}(G) = (V_{\text{odd}}(G') \setminus A) \cup B \cup \{v\}$$

so, again using the relation $|A| + |B| = |N_v|$, we can write

$$|V_{\text{odd}}(G)| = |V_{\text{odd}}(G')| - |A| + |B| + 1 = |V_{\text{odd}}(G')| + (|N_v| + 1) - 2|A|$$

which is even, since $|V_{\text{odd}}(G')|$ is even by the inductive hypothesis, and $|N_v|$ is odd by assumption.

This completes the inductive step and the proof.

Note how this proof is more complicated than the proof in part (ii), even though they are both using induction. This tells you that choosing the right variable to induct on can simplify the proof.