CS 70 Discrete Mathematics and Probability Theory
Fall 2021 HW 1

1 Administrivia

(a) Make sure you are on the course Piazza (for Q&A) and Gradescope (for submitting homeworks, including this one). Find and familiarize yourself with the course website. What is its homepage's URL?

(b) Read the policies page on the course website.

   (i) What is the breakdown of how your grade is calculated?
   (ii) What is the grade breakdown for the "no-homework" option?
   (iii) What is the attendance policy for discussions?
   (iv) When are vitamins released and when are they due?
   (v) How many "drops" do you get for vitamins? For homework? (Hint: it's the same number)
   (vi) What percentage score is needed to earn full credit on a homework or vitamin?

Solution:

(a) The course website is located at `http://www.eecs70.org/`.

(b) (i) Discussion Attendance: 5%, Vitamins: 5%, Homework: 20%, Midterm: 30%, Final: 40%.
   (ii) Discussion Attendance: 6.25%, Vitamins: 6.25%, Midterm: 37.5%, Final: 50%.
   (iii) You will receive 1 attendance point for every discussion, and will need at least 13 points in order to receive full credit for discussion attendance. You are welcome to attend other discussion sections, but your attendance will only be counted for the section you are actually assigned.
   (iv) The vitamin for the current week is released on Gradescope on Monday by 10:00 am. The vitamin is due on Gradescope the following Friday at 10:00 pm (grace period until Friday midnight); the solutions for that vitamin will be released on Saturday.
   (v) You can drop the lowest 3 homework scores and vitamin scores. However, please save these drops for emergencies. We do not have the bandwidth to make personalized exceptions to this rule.
   (vi) 73%

2 Course Policies

Go to the course website and read the course policies carefully. Leave a followup on Piazza if you have any questions. Are the following situations violations of course policy? Write "Yes" or "No", and a short explanation for each.

(a) Alice and Bob work on a problem in a study group. They write up a solution together and submit it, noting on their submissions that they wrote up their homework answers together.

(b) Carol goes to a homework party and listens to Dan describe his approach to a problem on the board, taking notes in the process. She writes up her homework submission from her notes, crediting Dan.

(c) Erin comes across a proof that is part of a homework problem while studying course material. She reads it and then, after she has understood it, writes her own solution using the same approach. She submits the homework with a citation to the website.

(d) Frank is having trouble with his homework and asks Grace for help. Grace lets Frank look at her written solution. Frank copies it onto his notebook and uses the copy to write and submit his homework, crediting Grace.

(e) Heidi has completed her homework using LaTeX. Her friend Irene has been working on a homework problem for hours, and asks Heidi for help. Heidi sends Irene her PDF solution, and Irene uses it to write her own solution with a citation to Heidi.

(f) Joe found homework solutions before they were officially released, and every time he got stuck, he looked at the solutions for a hint. He then cited the solutions as part of his submission.

Solution:

(a) Yes, this is a violation of course policy. All solutions must be written entirely by the student submitting the homework. Even if students collaborate, each student must write a unique, individual solution. In this case, both Alice and Bob would be culpable.

(b) No, this is not a violation of course policy. While sharing *written solutions* is not allowed, sharing *approaches* to problems is allowed and encouraged. Because Carol only copied down *notes*, not *Dan's solution*, and properly cited Dan's contribution, this is an actively encouraged form of collaboration.

(c) No, this is not a violation of course policy. Using external sources to help with homework problems, while less encouraged than peer collaboration, is fine as long as (i) the student makes sure to understand the solution; (ii) the student uses understanding to write a new solution, and does not copy from the external source; and (iii) the student credits the external source. However, looking up a homework problem online is a violation of course policies; the correct course of action upon finding homework solutions online is to close the tab.

(d) Yes, this is a violation of course policy, and both Frank and Grace would be culpable. Even though Frank credits Grace, written solutions should never be shared in the first place, and certainly not copied down. This is to ensure that each student learns how to write and present clear and convincing arguments. To be safe, try not to let anybody see your written solutions at any point in the course—restrict your collaboration to *approaches* and *verbal communication*.

(e) Yes, this is a violation of course policy. Once again, a citation does not make up for the fact that written solutions should never be shared, in written or typed form. In this case, both Heidi and Irene would be culpable.

(f) Yes, this is a violation of course policy. Joe should not be reading solutions before they are officially released. Instead, Joe should ask for help when he is stuck through Piazza or Office Hours.

3 Use of Piazza

Piazza is incredibly useful for Q&A in such a large-scale class. We will use Piazza for all important announcements. You should check it frequently. We also highly encourage you to use Piazza to ask questions and answer questions from your fellow students.

(a) Read the Piazza Etiquette section of the course policies and explain what is wrong with the following hypothetical student question: "Can someone explain the proof of Theorem XYZ to me?" (Assume Theorem XYZ is a complicated concept.)

(b) When are the weekly posts released? Are they required reading?

(c) If you have a question or concern not directly related to the course content, where should you direct it?

Solution:

(a) There are two things wrong with this question. First, this question does not pass the 5-minute test. This concept takes longer than 5 minutes to explain, and therefore is better suited to Office Hours. Second, this question does not hone in on a particular concept with which the student is struggling. Questions on Piazza should be narrow, and should include every step of reasoning that led up to the question. A better question in this case might be: "I understood every step of the proof of Theorem XYZ in Note 2, except for the very last step. I tried to reason it like this, but I didn't see how it yielded the result. Can someone explain where I went wrong?"

(b) The weekly posts are released every Sunday. They're required reading.

(c) Please send an email to fa21@eecs70.org.

4 Discussion Assignment

Please confirm that you have signed up for one of the discussion section at https://tinyurl.com/cs70fa21-signup. What is the name of your GSI and the time of your discussion section?

Solution: Ensure that they have signed up for a valid discussion section.

5 Remote Course Option

Although this class is designated to be in-person, we are facilitating remote exams for those who are not physically at Berkeley. We ask everyone to fill out this form.

(a) Do you acknowledge that the above form is binding, and you will not be able to change your decision except in rare extenuating circumstances (i.e., you're out sick with COVID the day of the exam)?

(b) What is the secret phrase?

Solution: The form is located here. The secret phrase is "I love CS70".

6 Academic Integrity

Please write or type out the following pledge in print, and sign it.

> I pledge to uphold the university's honor code: to act with honesty, integrity, and respect for others, including their work. By signing, I ensure that all written homework I submit will be in my own words, that I will acknowledge any collaboration or help received, and that I will neither give nor receive help on any examinations.

7 Propositional Practice

In parts (a)-(c), convert the English sentences into propositional logic. In parts (d)-(f), convert the propositions into English. In part (f), let $P(a)$ represent the proposition that a is prime.

(a) There is one and only one real solution to the equation $x^2 = 0$.

(b) Between any two distinct rational numbers, there is another rational number.

(c) If the square of an integer is greater than 4, that integer is greater than 2 or it is less than -2.

(d) $(\forall x \in \mathbb{R})\ (x \in \mathbb{C})$

(e) $(\forall x, y \in \mathbb{Z})(x^2 - y^2 \neq 10)$

(f) $(\forall x \in \mathbb{N})\ [\ (x > 1) \implies (\exists a, b \in \mathbb{N})\ ((a+b=2x) \wedge P(a) \wedge P(b))\ ]$

Solution:

(a) Let $p(x) = x^2$. The sentence can be read: "There is a solution x to the equation $p(x) = 0$, and any other solution y is equal to x". Or,

$$(\exists x \in \mathbb{R})\,((p(x)=0) \wedge ((\forall y \in \mathbb{R})(p(y)=0) \implies (x=y)))\,.$$

(b) The sentence can be read "If x and y are distinct rational numbers, then there is a rational number z between x and y." Or,

$$(\forall x, y \in \mathbb{Q})((x \neq y) \implies ((\exists z \in \mathbb{Q})(x < z < y \vee y < z < x))).$$

Equivalently,

$$(\forall x, y \in \mathbb{Q})((x = y) \vee (\exists z \in \mathbb{Q})(x < z < y \vee y < z < x)).$$

Note that $x < z < y$ is mathematical shorthand for $(x < z) \wedge (z < y)$, so the above statement is equivalent to

$$(\forall x, y \in \mathbb{Q})(x = y) \vee ((\exists z \in \mathbb{Q})\,((x < z) \wedge (z < y)) \vee ((y < z) \wedge (z < x))).$$

(c) $(\forall x \in \mathbb{Z})\ ((x^2 > 4) \implies ((x > 2) \vee (x < -2)))$

(d) All real numbers are complex numbers.

(e) There are no integer solutions to the equation $x^2 - y^2 = 10$.

(f) For any natural number greater than 1, there are some prime numbers a and b such that $2x = a + b$.

In other words: Any even integer larger than 2 can be written as the sum of two primes.

Aside: This statement is known as Goldbach's Conjecture, and it is a famous unsolved problem in number theory (https://xkcd.com/1310/).

8 Implication

Which of the following assertions are true no matter what proposition Q represents? For any false assertion, state a counterexample (i.e. come up with a statement $Q(x,y)$ that would make the implication false). For any true assertion, give a brief explanation for why it is true.

(a) $\exists x \exists y Q(x,y) \implies \exists y \exists x Q(x,y)$.

(b) $\forall x \exists y Q(x,y) \implies \exists y \forall x Q(x,y)$.

(c) $\exists x \forall y Q(x,y) \implies \forall y \exists x Q(x,y)$.

(d) $\exists x \exists y Q(x,y) \implies \forall y \exists x Q(x,y)$.

Solution:

(a) True. There exists can be switched if they are adjacent; $\exists x, \exists y$ and $\exists y, \exists x$ means there exists x and y in our universe.

(b) False. Let $Q(x,y)$ be $x < y$, and the universe for x and y be the integers. Or let $Q(x,y)$ be $x = y$ and the universe be any set with at least two elements. In both cases, the antecedent is true and the consequence is false, thus the entire implication statement is false.

(c) True. The first statement says that there is an x, say x' where for every y, $Q(x,y)$ is true. Thus, one can choose $x = x'$ for the second statement and that statement will be true again for every y. Note: 4c and 4d are not logically equivalent. In fact, the converse of 4d is 4c, which we saw is false.

(d) False. Suppose Q is the statement "y is 5, and x is any integer". The antecedent is true when $y = 5$, but for $y \neq 5$, there is no x that will make it true.

9 Logical Equivalence?

Decide whether each of the following logical equivalences is correct and justify your answer.

(a) $\forall x \left(P(x) \wedge Q(x)\right) \equiv \forall x\, P(x) \wedge \forall x\, Q(x)$

(b) $\forall x \left(P(x) \vee Q(x)\right) \equiv \forall x\, P(x) \vee \forall x\, Q(x)$

(c) $\exists x \left(P(x) \vee Q(x)\right) \equiv \exists x\, P(x) \vee \exists x\, Q(x)$

(d) $\exists x \left(P(x) \wedge Q(x)\right) \equiv \exists x\, P(x) \wedge \exists x\, Q(x)$

Solution:

(a) **Correct**.

Assume that the left hand side is true. Then we know for an arbitrary x $P(x) \wedge Q(x)$ is true. This means that both $\forall x P(x)$ and $\forall x Q(x)$. Therefore the right hand side is true. Now for the other direction assume that the right hand side is true. Since for any x $P(x)$ and for any y $Q(y)$ holds, then for an arbitrary x both $P(x)$ and $Q(x)$ must be true. Thus the left hand side is true.

(b) **Incorrect**.

Note, there are many possible counterexamples - here we present only one. Suppose that the universe (i.e. the values that x can take on) is $\{1,2\}$ and that P and Q are truth functions defined on this universe. If we set $P(1)$ to be true, $Q(1)$ to be false, $P(2)$ to be false and $Q(2)$ to be true, the left-hand side will be true, but the right-hand side will be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.

(c) **Correct**

Assuming that the left hand side is true, we know there exists some x such that one of $P(x)$ and $Q(x)$ is true. Thus $\exists x P(x)$ or $\exists x Q(x)$ and the right hand side is true. To prove the other direction, assume the left hand side is false. Then there does not exists an x for which $P(x) \vee Q(x)$ is true, which means there is no x for which $P(x)$ or $Q(x)$ is true. Therefore the right hand side is false.

(d) **Incorrect**.

Note, there are many possible counterexamples - here we present only one. Suppose that the universe (i.e. the values that x can take on) is the natural numbers $\mathbb{N}$, and that P and Q are truth functions defined on this universe. If we set $P(1)$ to be true and $P(x)$ to be false for all other x, and $Q(2)$ to be true and $Q(x)$ to be false for all other x, then the right hand side would be true. However, there would be no value of x at which both $P(x)$ and $Q(x)$ would be simultaneously true, so the left hand side would be false. Hence, we can find a universe and truth functions P and Q for which these two expressions have different values, so they must be different.

10 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets. By doing so, you will show that inverse images preserve set operations, but images typically do not.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(b) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$.

(c) $f(A \cap B) \subseteq f(A) \cap f(B)$, and give an example where equality does not hold.

(d) $f(A \setminus B) \supseteq f(A) \setminus f(B)$, and give an example where equality does not hold.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose x is such that $f(x) \in A \cap B$. Then $f(x)$ lies in both A and B, so x lies in both $f^{-1}(A)$ and $f^{-1}(B)$, so $x \in f^{-1}(A) \cap f^{-1}(B)$. So $f^{-1}(A \cap B) \subseteq f^{-1}(A) \cap f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cap f^{-1}(B)$. Then, x is in both $f^{-1}(A)$ and $f^{-1}(B)$, so $f(x) \in A$ and $f(x) \in B$, so $f(x) \in A \cap B$, so $x \in f^{-1}(A \cap B)$. So $f^{-1}(A) \cap f^{-1}(B) \subseteq f^{-1}(A \cap B)$.

(b) Suppose x is such that $f(x) \in A \setminus B$. Then, $f(x) \in A$ and $f(x) \notin B$, which means that $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, which means that $x \in f^{-1}(A) \setminus f^{-1}(B)$. So $f^{-1}(A \setminus B) \subseteq f^{-1}(A) \setminus f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \setminus f^{-1}(B)$. Then, $x \in f^{-1}(A)$ and $x \notin f^{-1}(B)$, so $f(x) \in A$ and $f(x) \notin B$, so $f(x) \in A \setminus B$, so $x \in f^{-1}(A \setminus B)$. So $f^{-1}(A) \setminus f^{-1}(B) \subseteq f^{-1}(A \setminus B)$.

(c) Suppose $x \in A \cap B$. Then, x lies in both A and B, so $f(x)$ lies in both $f(A)$ and $f(B)$, so $f(x) \in f(A) \cap f(B)$. Hence, $f(A \cap B) \subseteq f(A) \cap f(B)$.

Consider when there are elements $a \in A$ and $b \in B$ with $f(a) = f(b)$, but A and B are disjoint. Here, $f(a) = f(b) \in f(A) \cap f(B)$, but $f(A \cap B)$ is empty (since $A \cap B$ is empty).

(d) Suppose $y \in f(A) \setminus f(B)$. Since y is not in $f(B)$, there are no elements in B which map to y. Let x be any element of A that maps to y; by the previous sentence, x cannot lie in B. Hence, $x \in A \setminus B$, so $y \in f(A \setminus B)$. Hence, $f(A) \setminus f(B) \subseteq f(A \setminus B)$.

Consider when $B = \{0\}$ and $A = \{0,1\}$, with $f(0) = f(1) = 0$. One has $A \setminus B = \{1\}$, so $f(A \setminus B) = \{0\}$. However, $f(A) = f(B) = \{0\}$, so $f(A) \setminus f(B) = \varnothing$.

11 Proof by?

(a) Prove that if for any two integers x and y, if 10 does not divide xy, then 10 does not divide x and 10 does not divide y. In notation: $(\forall x, y \in \mathbb{Z})\ (10 \nmid xy) \implies ((10 \nmid x) \ \wedge \ (10 \nmid y))$. What proof technique did you use?

(b) Prove or disprove the contrapositive.

(c) Prove or disprove the converse.

Solution:

(a) We will use proof by contraposition. For any arbitrary given x and y, the statement $(10 \nmid xy) \implies ((10 \nmid x) \wedge (10 \nmid y))$ is equivalent using contraposition to $\neg(10 \nmid x \wedge 10 \nmid y) \implies \neg(10 \nmid xy)$. Moving the negations inside, this becomes equivalent to $(10 \mid x \vee 10 \mid y) \implies 10 \mid xy$.

Now for this part, we give a proof by cases. Assuming that $10 \mid x \vee 10 \mid y$, one of the two cases must be true.

  (a) $10 \mid x$: in this case $x = 10k$ for some $k \in \mathbb{Z}$. Therefore $xy = 10ky$ which is a multiple of 10. So $10 \mid xy$.

  (b) $10 \mid y$: in this case $y = 10k$ for some $k \in \mathbb{Z}$. Therefore $xy = 10kx$ which is a multiple of 10. So $10 \mid xy$.

Therefore assuming $10 \mid x \vee 10 \mid y$ we proved $10 \mid xy$.

We used proof by cases and proof by contraposition.

(b) We proved the statement. The contrapositive of a statement is logically equivalent to the statement. So we are done.

(c) It's not true! The converse is that if 10 does not divide x and does not divide y then 10 does not divide xy. We can choose $x = 2$ and $y = 5$ and see a counterexample to the statement.

12 Rationals and Irrationals

Prove that the product of a non-zero rational number and an irrational number is irrational.

Solution: We prove the statement by contradiction. Suppose that $ab = c$, where $a \neq 0$ is rational, b is irrational, and c is rational. Since a and b are not zero (because 0 is rational), c is also non-zero. Thus, we can express $a = \frac{p}{q}$ and $c = \frac{r}{s}$, where $p, q, r,$ and s are nonzero integers. Then

$$b = \frac{c}{a} = \frac{rq}{ps},$$

which is the ratio of two nonzero integers, giving that b is rational. This contradicts our initial assumption, so we conclude that the product of a nonzero rational number and an irrational number is irrational.