CS 70 Discrete Mathematics and Probability Theory
Fall 2021 DIS 13A

1 Short Answer

(a) Let X be uniform on the interval $[0,2]$, and define $Y = 2X+1$. Find the PDF, CDF, expectation, and variance of Y.

(b) Let X and Y have joint distribution

$$f(x,y) = \begin{cases} cxy + 1/4 & x \in [1,2] \text{ and } y \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c. Are X and Y independent?

(c) Let $X \sim \text{Exp}(3)$.

(i) Find probability that $X \in [0,\, 1]$.

(ii) Let $Y = \lfloor X \rfloor$. For each $k \in \mathbb{N}$, what is the probability that $Y = k$? Write the distribution of Y in terms of one of the famous distributions; provide that distribution's name and parameters.

(d) Let $X_i \sim \text{Exp}(\lambda_i)$ for $i = 1,\ldots,n$ be mutually independent. It is a (very nice) fact that $\min(X_1,\ldots,X_n) \sim \text{Exp}(\mu)$. Find μ.

Solution:

(a) Let's begin with the CDF. It will first be useful to recall that

$$F_X(t) = \mathbb{P}(X \le t) = \begin{cases} 0 & t \le 0 \\ \frac{t}{2} & t \in [0,2] \\ 1 & t \ge 2 \end{cases}.$$

Since Y is defined in terms of X, we can compute that

$$\begin{aligned} F_Y(t) = \mathbb{P}(Y \le t) &= \mathbb{P}[2X+1 \le t] \\ &= \mathbb{P}\left[X \le \frac{t-1}{2}\right] \\ &= F_X\left(\frac{t-1}{2}\right) \\ &= \begin{cases} 0 & t \le 1 \\ \frac{t-1}{4} & t \in [1,5] \\ 1 & t \ge 5 \end{cases} \end{aligned}$$

where in the third line we have used the PDF for X. We know that the PDF can be found by taking the derivative of the CDF, so

$$f_Y(t) = \frac{dF_Y(t)}{dt} = \begin{cases} \frac{1}{4} & t \in [1,5] \\ 0 & \text{else} \end{cases}.$$

By linearity of expectation $\mathbb{E}[Y] = \mathbb{E}[2X+1] = 2\,\mathbb{E}[X]+1 = 3$, and similarly

$$\mathrm{Var}(Y) = \mathrm{Var}(2X+1) = 4\,\mathrm{Var}(X) = 4 \cdot \frac{4}{12} = \frac{4}{3}.$$

(b) To find the correct constant, we use the fact that a PDF must integrate to one. In particular,

$$1 = \int_1^2 \int_0^2 (cxy + 1/4)\,dy\,dx = 3c + 1/2,$$

so $c = 1/6$. In order to check independence, we need to first find the marginal distributions of X and Y:

$$f_X(x) = \int_0^2 f(x,y)\,dy = 1/2 + x/3$$
$$f_Y(y) = \int_1^2 f(x,y)\,dx = 1/4 + y/4.$$

Since $f_X(x)f_Y(y) = 1/8 + y/8 + x/12 + xy/12 \neq 1/4 + xy/6 = f(x,y)$, the random variables are not independent.

(c) (i) Since $X \sim \mathrm{Exp}(3)$, the CDF of X is $F(x) = 1 - e^{-3x}$. Thus we have

$$\mathbb{P}[X \in [0,1]] = \int_0^1 f(x)\,dx = F(1) - F(0) = (1 - e^{-3}) - (1 - e^0) = 1 - e^{-3}.$$

(ii) Similarly, if $Y = \lfloor X \rfloor$, then $Y = k$ exactly when $X \in [k, k+1)$, so

$$\begin{aligned}
\mathbb{P}[Y = k] &= \mathbb{P}[X \in [k, k+1)] \\
&= \int_k^{k+1} f(x)\,dx \\
&= F(k+1) - F(k) \\
&= (1 - e^{-3(k+1)}) - (1 - e^{-3k}) \\
&= e^{-3k} - e^{-3(k+1)} \\
&= e^{-3k}\left(1 - e^{-3}\right) = (e^{-3})^k\left(1 - e^{-3}\right).
\end{aligned}$$

In other words, $Y = W - 1$ for $W \sim \mathrm{Geometric}(1 - e^{-3})$.

(d) Since the X_i are independent,

$$\begin{aligned}
\mathbb{P}[\min(X_1,\ldots,X_n) \le t] &= 1 - \mathbb{P}[X_1 > t, X_2 > t, \cdot \ldots \cdot X_n > t] \\
&= 1 - \mathbb{P}[X_1 > t] \cdot \mathbb{P}[X_2 > t] \cdot \ldots \cdot \mathbb{P}[X_n > t] \qquad \text{by independence} \\
&= 1 - e^{-\lambda_1 t} e^{-\lambda_2 t} \cdot \ldots \cdot e^{-\lambda_n t} \\
&= 1 - e^{-(\lambda_1 + \lambda_2 + \ldots + \lambda_n)t}.
\end{aligned}$$

This is exactly the CDF of an $\mathrm{Exp}(\lambda_1 + \lambda_2 + \ldots + \lambda_n)$ random variable, so $\mu = \lambda_1 + \ldots + \lambda_n$.

2 First Exponential to Die

Let X and Y be Exponential(λ_1) and Exponential(λ_2) respectively, independent. What is

$$\mathbb{P}\big(\min(X,Y) = X\big),$$

the probability that the first of the two to die is X?

Solution:

Recall that the CDF of an exponential is $\mathbb{P}[X \le x] = 1 - \exp(-\lambda x)$ for $x \ge 0$.

$$\begin{aligned}\mathbb{P}\big(\min(X,Y) = X\big) = \mathbb{P}(Y > X) = \int_0^\infty \mathbb{P}(Y > X \mid X = x) f_X(x)\,\mathrm{d}x = \int_0^\infty \mathrm{e}^{-\lambda_2 x} \cdot \lambda_1 \,\mathrm{e}^{-\lambda_1 x}\,\mathrm{d}x \\ = -\frac{\lambda_1}{\lambda_1 + \lambda_2}\,\mathrm{e}^{-(\lambda_1+\lambda_2)x}\Big|_{x=0}^{\infty} = \frac{\lambda_1}{\lambda_1 + \lambda_2}.\end{aligned}$$