CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Koushik Sen and Satish Rao — DIS 0B

1 Contraposition

Prove the statement "if $a+b<c+d$, then $a<c$ or $b<d$".

Solution:

The implication we're trying to prove is $(a+b<c+d) \implies ((a<c) \vee (b<d))$, so the contrapositive is $((a \geq c) \wedge (b \geq d)) \implies (a+b \geq c+d)$. The proof of this is quite straightforward: since we have both that $a \geq c$ and that $b \geq d$, we can just add these two inequalities together, giving us $a+b \geq c+d$, which is exactly what we wanted.

2 Numbers of Friends

Prove that if there are $n \geq 2$ people at a party, then at least 2 of them have the same number of friends at the party. Assume that friendships are always reciprocated: that is, if Alice is friends with Bob, then Bob is also friends with Alice.
(Hint: The Pigeonhole Principle states that if n items are placed in m containers, where $n > m$, at least one container must contain more than one item. You may use this without proof.)

Solution:

We will prove this by contradiction. Suppose the contrary that everyone has a different number of friends at the party. Since the number of friends that each person can have ranges from 0 to $n-1$, we conclude that for every $i \in \{0,1,\ldots,n-1\}$, there is exactly one person who has exactly i friends at the party. In particular, there is one person who has $n-1$ friends (i.e., friends with everyone), is friends with a person who has 0 friends (i.e., friends with no one). This is a contradiction since friendship is mutual.
Here, we used the pigeonhole principle because assuming for contradiction that everyone has a different number of friends gives rise to n possible containers. Each container denotes the number of friends that a person has, so the containers can be labelled $0,1,...,n-1$. The objects assigned to these containers are the people at the party. However, containers 0, $n-1$ or both must be empty since these two containers cannot be occupied at the same time. This means that we are assigning n people to at most $n-1$ containers, and by the pigeonhole principle, at least one of the $n-1$ containers has to have two or more objects i.e. at least two people have to have the same number of friends.

3 Pebbles

Suppose you have a rectangular array of pebbles, where each pebble is either red or blue. Suppose that for every way of choosing one pebble from each column, there exists a red pebble among the chosen ones. Prove that there must exist an all-red column.

Solution: We give a proof by contraposition. Suppose there does not exist an all-red column. This means that, in each column, we can find a blue pebble. Therefore, if we take one blue pebble from each column, we have a way of choosing one pebble from each column without any red pebbles. This is the negation of the original hypothesis, so we are done.

4 Preserving Set Operations

For a function f, define the image of a set X to be the set $f(X) = \{y \mid y = f(x) \text{ for some } x \in X\}$. Define the inverse image or preimage of a set Y to be the set $f^{-1}(Y) = \{x \mid f(x) \in Y\}$. Prove the following statements, in which A and B are sets.

Recall: For sets X and Y, $X = Y$ if and only if $X \subseteq Y$ and $Y \subseteq X$. To prove that $X \subseteq Y$, it is sufficient to show that $(\forall x)\ ((x \in X) \implies (x \in Y))$.

(a) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) $f(A \cup B) = f(A) \cup f(B)$.

Solution:

In order to prove equality $A = B$, we need to prove that A is a subset of B, $A \subseteq B$ and that B is a subset of A, $B \subseteq A$. To prove that LHS is a subset of RHS we need to prove that if an element is a member of LHS then it is also an element of the RHS.

(a) Suppose $x \in f^{-1}(A \cup B)$ which means that $f(x) \in A \cup B$. Then either $f(x) \in A$, in which case $x \in f^{-1}(A)$, or $f(x) \in B$, in which case $x \in f^{-1}(B)$, so in either case we have $x \in f^{-1}(A) \cup f^{-1}(B)$. This proves that $f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)$.

Now, suppose that $x \in f^{-1}(A) \cup f^{-1}(B)$. Suppose, without loss of generality, that $x \in f^{-1}(A)$. Then $f(x) \in A$, so $f(x) \in A \cup B$, so $x \in f^{-1}(A \cup B)$. The argument for $x \in f^{-1}(B)$ is the same. Hence, $f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)$.

(b) Suppose that $x \in A \cup B$. Then either $x \in A$, in which case $f(x) \in f(A)$, or $x \in B$, in which case $f(x) \in f(B)$. In either case, $f(x) \in f(A) \cup f(B)$, so $f(A \cup B) \subseteq f(A) \cup f(B)$.

Now, suppose that $y \in f(A) \cup f(B)$. Then either $y \in f(A)$ or $y \in f(B)$. In the first case, there is an element $x \in A$ with $f(x) = y$; in the second case, there is an element $x \in B$ with $f(x) = y$. In either case, there is an element $x \in A \cup B$ with $f(x) = y$, which means that $y \in f(A \cup B)$. So $f(A) \cup f(B) \subseteq f(A \cup B)$.

The purpose of this problem is to gain familiarity to naming things precisely. In particular, we named an element in the LHS (or the pre-image of the LHS) and then argued about whether that element or its image was in the right hand side. By explicitly naming an element generically where it could be *any element in the set, we could argue about its membership in a set and or its image or preimage. With these different concepts floating around it is helpful to be clear in the argument.*