1 Sum of Poisson Variables

Assume that you were given two independent Poisson random variables X_1, X_2. Assume that the first has mean λ_1 and the second has mean λ_2. Prove that $X_1 + X_2$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$.

Hint: Recall the binomial theorem.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Solution:

To show that $X_1 + X_2$ is a Poisson random variable with mean $\lambda_1 + \lambda_2$, we have show that

$$\mathbb{P}[(X_1 + X_2) = i] = \frac{(\lambda_1 + \lambda_2)^i}{i!} \mathrm{e}^{-(\lambda_1+\lambda_2)}.$$

We proceed as follows:

$$\begin{aligned}
\mathbb{P}[(X_1 + X_2) = i] &= \sum_{k=0}^{i} \mathbb{P}[X_1 = k, X_2 = (i-k)] = \sum_{k=0}^{i} \frac{\lambda_1^k}{k!} \mathrm{e}^{-\lambda_1} \cdot \frac{\lambda_2^{i-k}}{(i-k)!} \mathrm{e}^{-\lambda_2} \\
&= \mathrm{e}^{-\lambda_1} \mathrm{e}^{-\lambda_2} \sum_{k=0}^{i} \frac{1}{k!(i-k)!} \lambda_1^k \lambda_2^{i-k} = \frac{\mathrm{e}^{-\lambda_1} \mathrm{e}^{-\lambda_2}}{i!} \sum_{k=0}^{i} \frac{i!}{k!(i-k)!} \lambda_1^k \lambda_2^{i-k} \\
&= \frac{\mathrm{e}^{-(\lambda_1+\lambda_2)}}{i!} \sum_{k=0}^{i} \binom{i}{k} \lambda_1^k \lambda_2^{i-k} = \frac{\mathrm{e}^{-(\lambda_1+\lambda_2)}}{i!} (\lambda_1 + \lambda_2)^i
\end{aligned}$$

In the last line, we use the binomial expansion.

2 Variance

(a) Let X be a random variable representing the outcome of the roll of one fair 6-sided die. What is $\mathrm{Var}(X)$?

(b) Let Z be a random variable representing the average of n rolls of a fair 6-sided die. What is $\mathrm{Var}(Z)$?

Solution:

(a) Recall that $\text{Var}(()X) = \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2$. We can compute each of the individual terms using the definition of expectation:

$$\begin{aligned}\mathbb{E}[X] &= \frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}\\ \mathbb{E}\left[X^2\right] &= \frac{1}{6}\left(1^2+2^2+3^2+4^2+5^2+6^2\right)\\ &= \frac{1}{6}(1+4+9+16+25+36) = \frac{91}{6}\end{aligned}$$

Now, we plug back into the variance expression:

$$\begin{aligned}\text{Var}(X) &= \mathbb{E}\left[X^2\right] - \mathbb{E}[X]^2\\ &= \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}\end{aligned}$$

(b) Because each die roll is independent of the others, we can utilize the fact that for independent random variables X and Y, $\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$. Let X_i be a random variable representing the outcome of the ith dice roll. We now have:

$$\begin{aligned}\text{Var}(Z) &= \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\\ &= \left(\frac{1}{n}\right)^2 \text{Var}\left(\sum_{i=1}^{n} X_i\right)\\ &= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \text{Var}(X_i) && \text{All } X_i\text{'s are independent.}\\ &= \left(\frac{1}{n}\right)^2 \sum_{i=1}^{n} \frac{35}{12} && \text{We computed the variance of one roll in part (a).}\\ &= \left(\frac{1}{n}\right)^2 \cdot n \cdot \frac{35}{12} = \frac{35}{12n}\end{aligned}$$

3 Covariance

(a) We have a bag of 5 red and 5 blue balls. We take two balls uniformly at random from the bag without replacement. Let X_1 and X_2 be indicator random variables for the events of the first and second ball being red, respectively. What is $\text{cov}(X_1, X_2)$? Recall that $\text{cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$.

(b) Now, we have two bags A and B, with 5 red and 5 blue balls each. Draw a ball uniformly at random from A, record its color, and then place it in B. Then draw a ball uniformly at random from B and record its color. Let X_1 and X_2 be indicator random variables for the events of the first and second draws being red, respectively. What is $\text{cov}(X_1, X_2)$?

Solution:

(a) We can use the formula $\text{cov}(X_1,X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2]$.

$$\begin{aligned}
\mathbb{E}[X_1] &= \frac{5}{10}\times 1 + \frac{5}{10}\times 0 = \frac{1}{2},\\
\mathbb{E}[X_2] &= \frac{5}{10}\times 1 + \frac{5}{10}\times 0 = \frac{1}{2},\\
\mathbb{E}[X_1X_2] &= \frac{5}{10}\cdot\frac{4}{9}\times 1 + \left(1 - \frac{5}{10}\cdot\frac{4}{9}\right)\times 0 = \frac{2}{9}.
\end{aligned}$$

Therefore,

$$\text{cov}(X_1,X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2] = \frac{2}{9} - \frac{1}{2}\times\frac{1}{2} = -\frac{1}{36}.$$

(b) Again, we use the formula $\text{cov}(X_1,X_2) = \mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2]$.

$$\begin{aligned}
\mathbb{E}[X_1] &= \frac{5}{10}\times 1 + \frac{5}{10}\times 0 = \frac{1}{2}\\
\mathbb{E}[X_2] &= \left(\frac{5}{10}\times\frac{6}{11} + \frac{5}{10}\times\frac{5}{11}\right)\times 1 + \left(\frac{5}{10}\times\frac{5}{11} + \frac{5}{10}\times\frac{6}{11}\right)\times 0 = \frac{1}{2}\\
\mathbb{E}[X_1X_2] &= \frac{5}{10}\times\frac{6}{11}\times 1 = \frac{30}{110}.
\end{aligned}$$

Therefore,

$$\mathbb{E}[X_1X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = \frac{30}{110} - \frac{1}{4} = \frac{1}{44}.$$

Note that in part (a), if one event happened, the other would be less likely to happen, and thus the covariance was negative. Similarly, in part (b), if one event happened, the other would be more likely to happen, and thus the covariance was positive.