CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Satish Rao and Koushik Sen — DIS 13A

1 Continuous Intro

(a) Is

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

a valid density function? Why or why not? Is it a valid CDF? Why or why not?

(b) Calculate $\mathbb{E}[X]$ and $\mathrm{Var}(X)$ for X with the density function

$$f(x) = \begin{cases} \frac{1}{\ell}, & 0 \le x \le \ell, \\ 0, & \text{otherwise.} \end{cases}$$

(c) Suppose X and Y are independent and have densities

$$f_X(x) = \begin{cases} 2x, & 0 \le x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

$$f_Y(y) = \begin{cases} 1, & 0 \le y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

What is their joint distribution? (Hint: for this part and the next, we can use independence in much the same way that we did in discrete probability)

(d) Calculate $\mathbb{E}[XY]$ for the above X and Y.

Solution:

(a) Yes, it is a valid density function; it is non-negative and integrates to 1.

No, it is not a valid CDF; a CDF should go to 1 as x goes to infinity and be non-decreasing.

(b) We have

$$\mathbb{E}[X] = \int_{x=0}^{\ell} x \cdot \frac{1}{\ell}\,\mathrm{d}x = \frac{\ell}{2}$$
$$\mathbb{E}[X^2] = \int_{x=0}^{\ell} x^2 \cdot \frac{1}{\ell}\,\mathrm{d}x = \frac{\ell^2}{3}$$
$$\mathrm{Var}(X) = \frac{\ell^2}{3} - \frac{\ell^2}{4} = \frac{\ell^2}{12}$$

This is known as the continuous uniform distribution over the interval $[0,\ell]$, sometimes denoted $\mathrm{Uniform}[0,\ell]$.

(c) Note that due to independence,

$$\begin{aligned} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y &= \mathbb{P}[X \in [x, x+\mathrm{d}x], Y \in [y, y+\mathrm{d}y]] \\ &= \mathbb{P}[X \in [x, x+\mathrm{d}x]]\mathbb{P}[Y \in [y, y+\mathrm{d}y]] \\ &\approx f_X(x) f_Y(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

so their joint distribution is $f(x,y) = 2x$ on the unit square $0 \le x \le 1$, $0 \le y \le 1$.

(d) We have

$$\mathbb{E}[XY] = \int_{x=0}^{1} \int_{y=0}^{1} xy \cdot 2x\,\mathrm{d}y\,\mathrm{d}x = \int_{x=0}^{1} x^2\,\mathrm{d}x = \frac{1}{3}.$$

Alternatively, since X and Y are independent, we can compute $\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$. Note that

$$\mathbb{E}[X] = \int_0^1 x \cdot 2x\,\mathrm{d}x = \frac{2}{3}x^3\Big|_0^1 = \frac{2}{3},$$

and $\mathbb{E}[Y] = \frac{1}{2}$ since the density of Y is symmetric around $\frac{1}{2}$. Hence,

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] = \frac{1}{3}.$$

2 Uniform Distribution

You have two fidget spinners, each having a circumference of 10. You mark one point on each spinner as a needle and place each of them at the center of a circle with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinner's mark and Y be the position of the second spinner's mark, what is the probability that $X \ge 5$, given that $Y \ge X$?

Solution:

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \ge 5 \mid Y \ge X] = \frac{\mathbb{P}[Y \ge X \cap X \ge 5]}{\mathbb{P}[Y \ge X]}.$$

$\mathbb{P}[Y \ge X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \ge X \cap X \ge 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

$$\mathbb{P}[Y \ge X \cap X \ge 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\mathbb{P}[X \ge 5 \mid Y \ge X] = (1/8)/(1/2) = 1/4$.

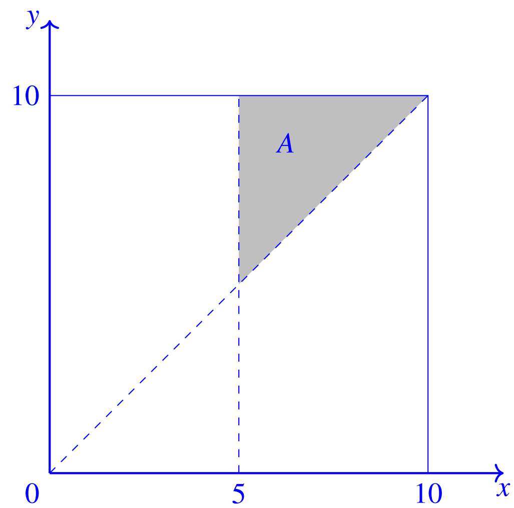

Figure 1: Joint probability density for the spinner.

3 Darts Again

Edward and Khalil are playing darts on a circular dartboard.

Edward's throws are uniformly distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter $\frac{1}{2}$.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}[X < Y]$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0,10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}[X < x]$.]

Solution: We are given that $Y \sim \text{Exponential}(1/2)$. We now find the distribution of X by solving for the CDF of X, $\mathbb{P}[X < x]$. To get this, we'll consider the ratio of the area where the distance to the center is less than x, compared to the entire available area. This gives us the following expression:

$$\mathbb{P}[X < x] = \frac{\pi x^2}{\pi 10^2} = \frac{x^2}{100}.$$

Differentiating gives us the PDF of X, which is given by $f_X(x) = \frac{x}{50}$. Now, we solve for $\mathbb{P}[X < Y]$ with total probability:

$$\begin{aligned}
\mathbb{P}[X < Y] &= \int_0^{10} \mathbb{P}[Y > X \mid X = x] f_X(x)\, \mathrm{d}x \\
&= \int_0^{10} \mathbb{P}[Y > x] f_X(x)\, \mathrm{d}x \\
&= \int_0^{10} \frac{x}{50} e^{-0.5x}\, \mathrm{d}x
\end{aligned}$$

Evaluating this integral gives us $\mathbb{P}[X < Y] \approx 0.0767$.