Outline

Linear Regression: wrapup.

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How do I love e?

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- Birthday.
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Poisson Distribution: Sum of two Poissons is Poisson.

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Dividing by $var(Y)$, one gets reduction: $\frac{(cov(X,Y))^2}{var(Y)var(Y)} = (corr(X,Y))^2 = r^2$.

LR: Another Figure

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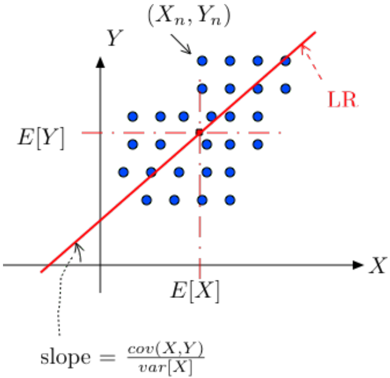

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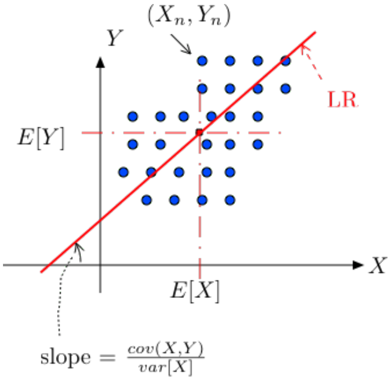

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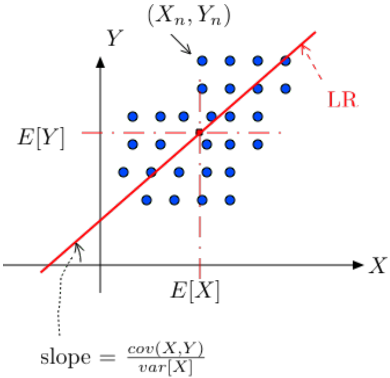

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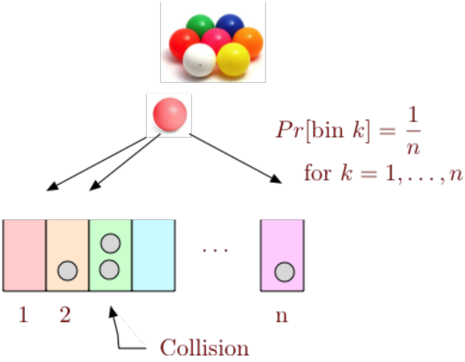

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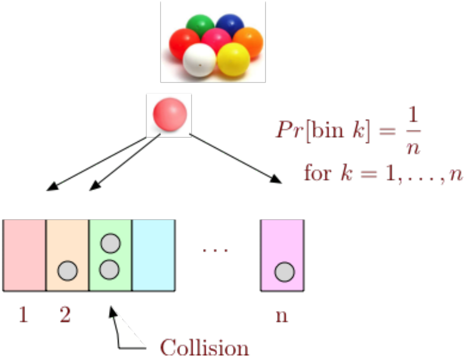

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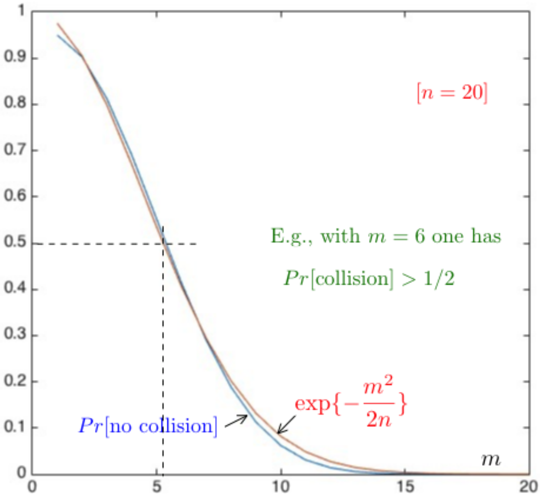

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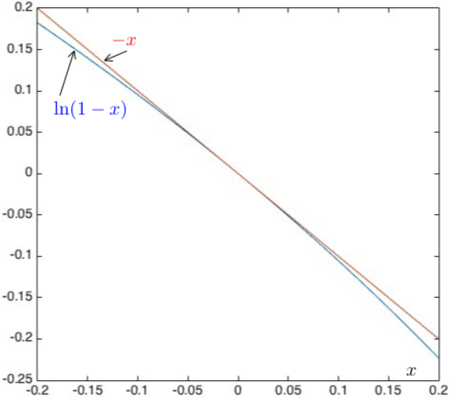
$^{(\dagger)}$ $1 + 2 + \cdots + m - 1 = (m-1)m/2$.

Approximation



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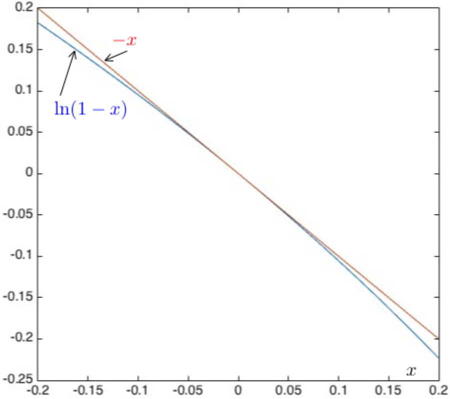

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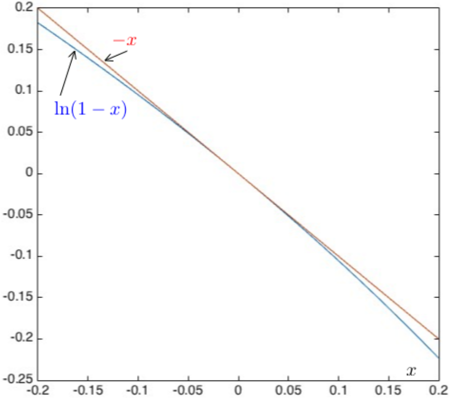

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Hence, $-x \approx \ln(1-x)$ for $|x| \ll 1$.

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Note: $\log_2(x) = \log_2(e)\ln(x) \approx 1.44\ln(x)$.

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Theorem: If you buy m boxes,

(a) Pr[miss one specific item] $\approx e^{-\frac{m}{n}}$

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For $p_m = \frac{1}{2}$, we need around $n\ln 2 \approx 0.69n$ boxes.

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Experiment: Choose m cards at random with replacement.

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$$p := Pr[E_1 \cup E_2 \cdots \cup E_n]$$

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Plug in and get

$$p \leq ne^{-\frac{m}{n}}.$$

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Time to collect coupons

X-time to get n coupons.

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Pr[“get second coupon”|“got milk ”]

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Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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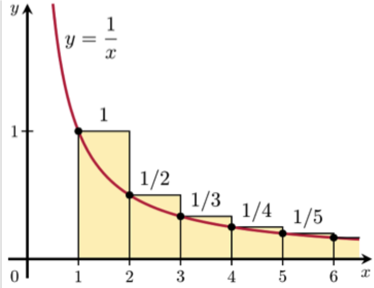

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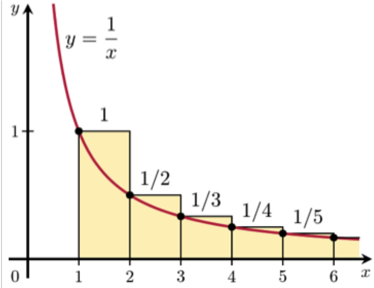

A good approximation is

$$H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant).}$$

Simplest..

Load balance: m balls in n bins.

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$Pr[X_i \geq k] \leq \sum_{S \subseteq [n], |S|=k} Pr[\text{balls in } S \text{ chooses bin } i]$

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Details: both could arrive with probability $\lambda\mu/n^2$.
But this goes to zero as $n \to \infty$.
(Like λ^2/n^2 in previous derivation)

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Inference:
Have one of two coins. Flip coin, which coin do you have?
Got positive test result. What is probability you have disease?

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$\varepsilon \approx \frac{\sigma}{\sqrt{n}} \log \frac{1}{\delta}$ using “Chernoff.”

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Chebyshev: For a random variable X: $Pr[|X - E(X)| > \varepsilon] \leq \frac{Var(X)}{epsilon^2}$

For $X = \frac{X_1 + \cdots + X_n}{n}$, where X_i are indentical and independent.
$Var(X) = \frac{var(X_i)}{n}$.

Law of Large Numbers: $A_n = \frac{X_1 + \cdots + X_n}{n}$.
$\lim_{n \to} A_n = E[X_1]$.
Cuz:
$Pr[|A_n - E[A_n]| \geq \varepsilon] \leq \frac{var A_n}{\varepsilon^2} = \frac{var(X_1)}{n\varepsilon^2}$.

For X_i with $Var(X_i) = \sigma^2$.
What is the confidence interval for A_n for confidence .95?
For what ε is $Pr[|A_n - E[A_n]| \geq \varepsilon] \leq .05 = \delta$?
$\varepsilon = \frac{\sigma}{\sqrt{n\delta}}$ using Chebyshev.
$\varepsilon \approx \frac{\sigma}{\sqrt{n}} \log \frac{1}{\delta}$ using "Chernoff."
"z-score" from AP statistics.

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