

You don't have any summer plans, so you decide to spend a few months working for a magical hotel with a countably infinite number of rooms. The rooms are numbered according to the natural numbers, and all the rooms are currently occupied. Assume that guests don't mind being moved from their current room to a new one, so long as they can get to the new room in a finite amount of time (i.e. guests can't be moved into a room infinitely far from their current one).

- (a) A new guest arrives at the hotel. All the current rooms are full, but your manager has told you never to turn away a guest. How could you accommodate the new guest by shuffling other guests around? What if you instead had k guests arrive, for some fixed, positive $k \in \mathbb{Z}$?

- (b) Unfortunately, just after you've figured out how to accommodate your first $k + 1$ guests, a countably infinite number of guests arrives in town on an infinitely long train. The guests on the train are sitting in seats numbered according to the natural numbers. How could you accommodate all the new guests?

- (c) Thanks to a (literally) endless stream of positive TripAdvisor reviews, word of the infinite hotel gets around quickly. Soon enough you find out that a countably infinite number of trains have arrived in town. Each is of infinite length, and carries a countably infinite number of passengers. How would you accommodate all the new passengers?

2 Counting Functions

Are the following sets countable or uncountable? Prove your claims.

- (a) The set of all functions f from \mathbb{N} to \mathbb{N} such that f is non-decreasing. That is, $f(x) \leq f(y)$ whenever $x \leq y$.

- (b) The set of all functions f from \mathbb{N} to \mathbb{N} such that f is non-increasing. That is, $f(x) \geq f(y)$ whenever $x \leq y$.

3 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

- (a) You want to determine whether a program P on input x prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.
- (b) You want to determine whether a program P prints "Hello World!" before running the k th line in the program. Is there a computer program that can perform this task? Justify your answer.
- (c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer.