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Integration by Parts: $\int udv = uv - \int vdu$.

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5. Target: $f_X(x) = 2x \cdot 1\{0 \leq x \leq 1\}; F_X(x) = x^2$ for $0 \leq x \leq 1$.
6. Joint pdf: $Pr[X \in (x, x+\delta), Y = (y, y+\delta)) = f_{X,Y}(x,y)\delta^2$.
   6.1 Conditional Distribution: $f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$.
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What is true?
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(C) $Y = 2X$, $f_Y(y) = 2f(y)$.
(D) $Y = 2X$, $F_Y(y) = F(y/2)$.
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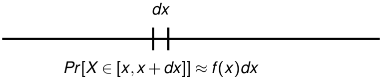

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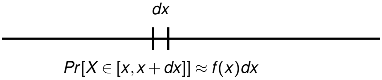


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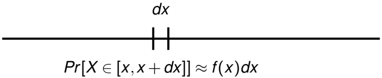


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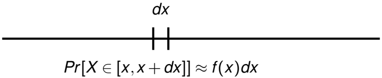


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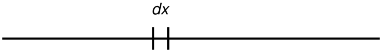


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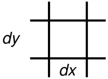


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Distribution: $Pr[X = x]$
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CDF: $F(x) = Pr[X \leq x]$.
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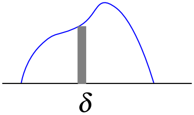

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# Probability Rules are all good.

Conditional Probability.

Events: $A, B$

Discrete: "Heads", "Tails", $X = 1$, $Y = 5$.

Continuous: $X$ in $[.2, .3]$. $X \in [.2, .3]$ or $X \in [.4, .6]$.

Conditional Probability: $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$

$Pr[\text{"Second Heads"}|\text{"First Heads"}]$,
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Corollary: For independent random variables, $f_{X|Y}(x,y) = f_X(x)$.

# *Expo*($\lambda$)

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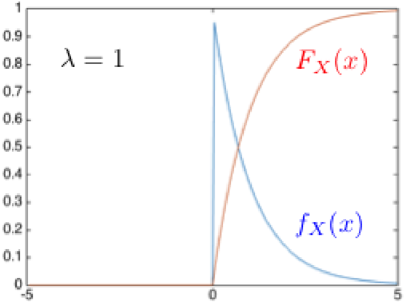


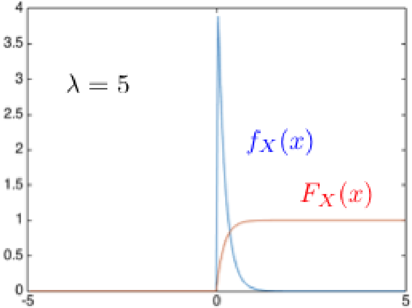

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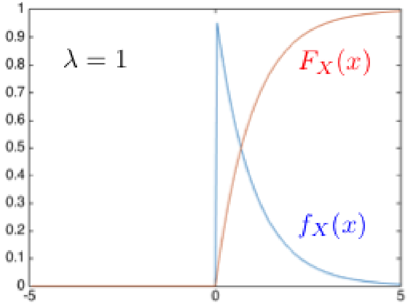


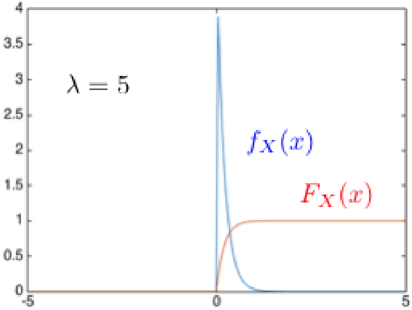


Note that $Pr[X > t] = e^{-\lambda t}$ for $t > 0$.

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Also, $Expo(\lambda) = \frac{1}{\lambda} Expo(1)$.

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Thus, $f_Y(y) = \frac{1}{b}$ for $a < y < a + b$. Hence, $Y = U[a, a+b]$.

Replace $b$ by $b - a$, use $X = U[0,1]$, then $Y = a + (b-a)X$ is $U[a,b]$.

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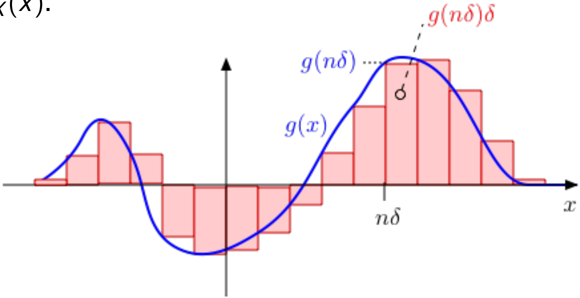

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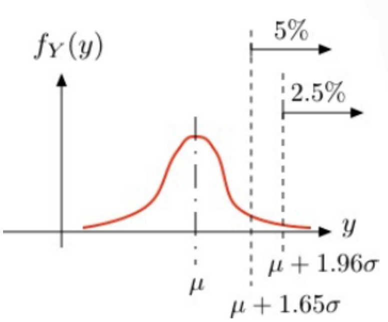

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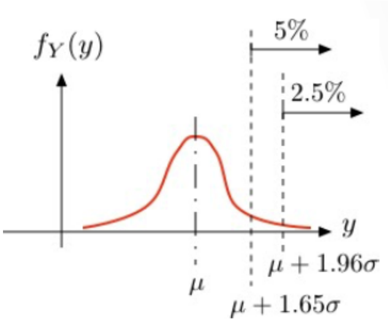


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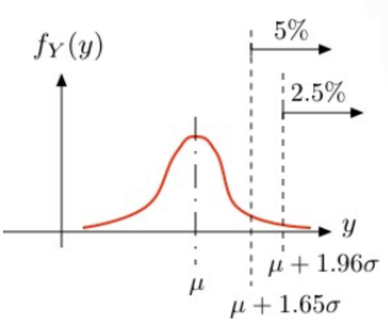


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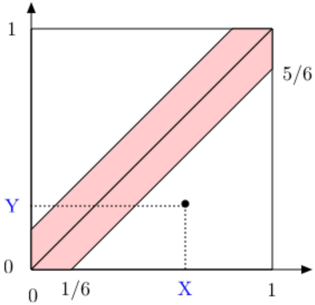

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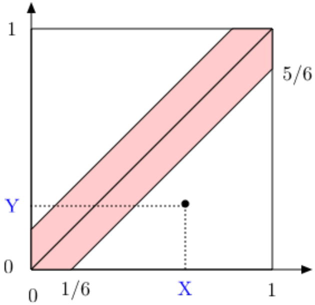


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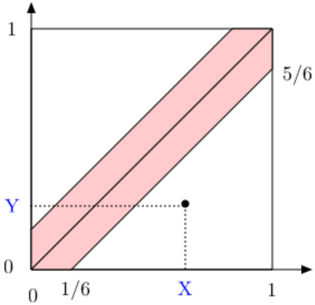


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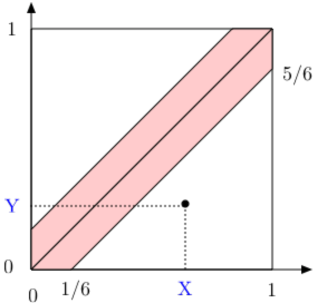


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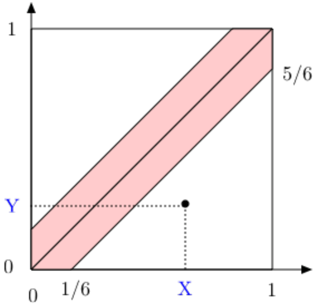


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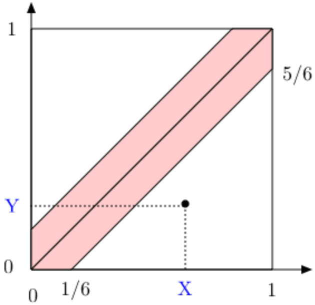


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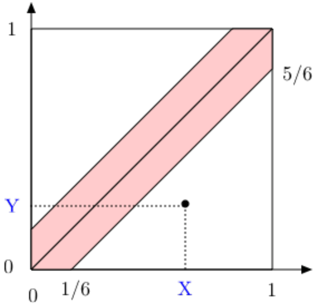


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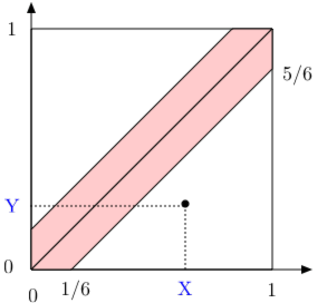


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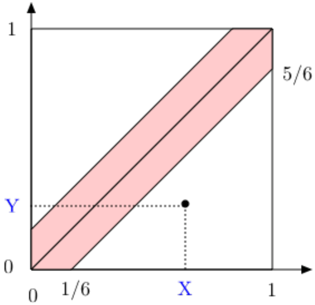


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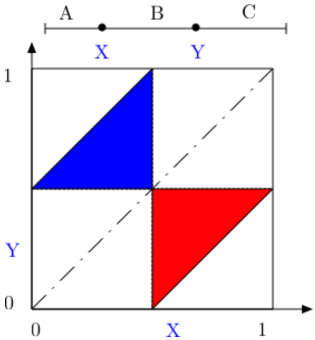

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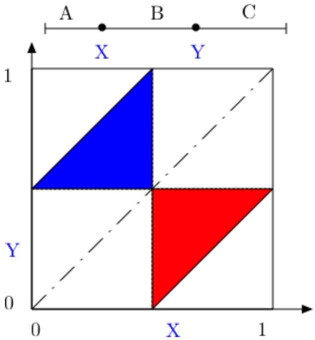


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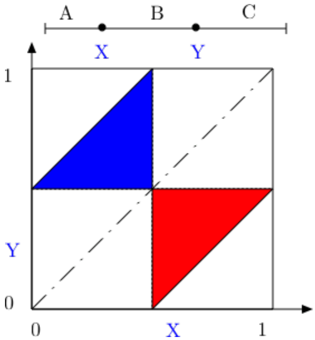


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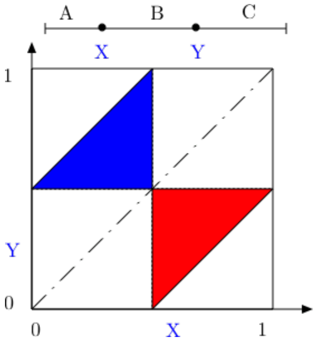


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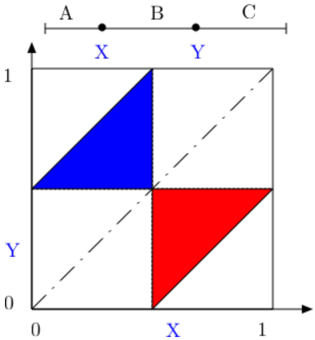


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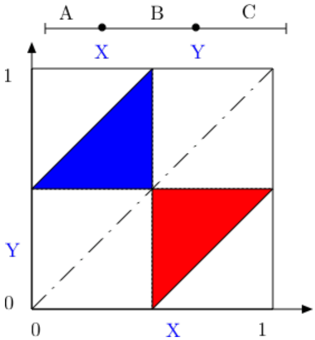


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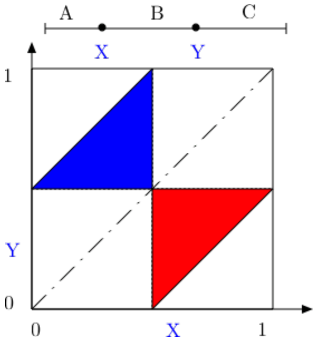


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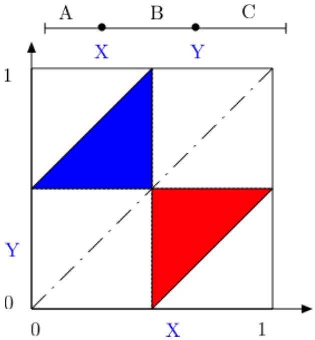


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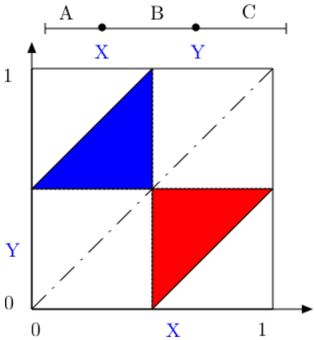


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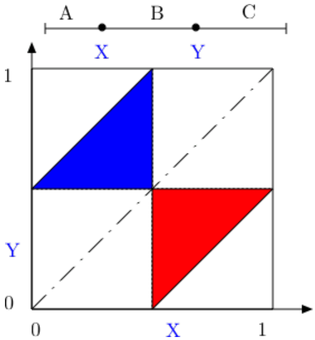


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(C) is an argument for (A), (B) and (D).

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Hence,

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For instance, if $n = 16$, then $SNR(dB) \approx 112dB$.

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**Problem 3:**

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**Problem 1:** Pick two points $X$ and $Y$ independently and uniformly at random in $[0,1]$.

What is $E[(X-Y)^2]$?

**Analysis:** One has

$$\begin{aligned}
E[(X-Y)^2] &= E[X^2+Y^2-2XY] \\
&= \frac{1}{3}+\frac{1}{3}-2\frac{1}{2}\frac{1}{2} \\
&= \frac{2}{3}-\frac{1}{2}=\frac{1}{6}.
\end{aligned}$$

**Problem 2:** What about in a unit square?

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$$\begin{aligned}
E[||\mathbf{X}-\mathbf{Y}||^2] &= E[(X_1-Y_1)^2]+E[(X_2-Y_2)^2] \\
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