

1 Independent Complements

Let Ω be a sample space, and let $A, B \subseteq \Omega$ be two independent events.

- (a) Prove or disprove: \bar{A} and \bar{B} must be independent.
- (b) Prove or disprove: A and \bar{B} must be independent.
- (c) Prove or disprove: A and \bar{A} must be independent.
- (d) Prove or disprove: It is possible that $A = B$.

Solution:

- (a) True. \bar{A} and \bar{B} must be independent:

$$\begin{aligned}\mathbb{P}[\bar{A} \cap \bar{B}] &= \mathbb{P}[\overline{A \cup B}] && \text{(by De Morgan's law)} \\ &= 1 - \mathbb{P}[A \cup B] && \text{(since } \mathbb{P}[\bar{E}] = 1 - \mathbb{P}[E] \text{ for all } E) \\ &= 1 - (\mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cap B]) && \text{(union of overlapping events)} \\ &= 1 - \mathbb{P}[A] - \mathbb{P}[B] + \mathbb{P}[A]\mathbb{P}[B] && \text{(using our assumption that } A \text{ and } B \text{ are independent)} \\ &= (1 - \mathbb{P}[A])(1 - \mathbb{P}[B]) \\ &= \mathbb{P}[\bar{A}]\mathbb{P}[\bar{B}] && \text{(since } \mathbb{P}[\bar{E}] = 1 - \mathbb{P}[E] \text{ for all } E)\end{aligned}$$

- (b) True. A and \bar{B} must be independent:

$$\begin{aligned}\mathbb{P}[A \cap \bar{B}] &= \mathbb{P}[A - (A \cap B)] \\ &= \mathbb{P}[A] - \mathbb{P}[A \cap B] \\ &= \mathbb{P}[A] - \mathbb{P}[A]\mathbb{P}[B] \\ &= \mathbb{P}[A](1 - \mathbb{P}[B]) \\ &= \mathbb{P}[A]\mathbb{P}[\bar{B}]\end{aligned}$$

- (c) False in general. If $0 < \mathbb{P}[A] < 1$, then $\mathbb{P}[A \cap \bar{A}] = \mathbb{P}[\emptyset] = 0$ but $\mathbb{P}[A]\mathbb{P}[\bar{A}] > 0$, so $\mathbb{P}[A \cap \bar{A}] \neq \mathbb{P}[A]\mathbb{P}[\bar{A}]$; therefore A and \bar{A} are not independent in this case.
- (d) True. To give one example, if $\mathbb{P}[A] = \mathbb{P}[B] = 0$, then $\mathbb{P}[A \cap B] = 0 = 0 \times 0 = \mathbb{P}[A]\mathbb{P}[B]$, so A and B are independent in this case. (Another example: If $A = B$ and $\mathbb{P}[A] = 1$, then A and B are independent.)

2 Symmetric Marbles

A bag contains 4 red marbles and 4 blue marbles. Leanne and Sylvia play a game where they draw four marbles in total, one by one, uniformly at random, without replacement. Leanne wins if there are more red than blue marbles, and Sylvia wins if there are more blue than red marbles. If there are an equal number of marbles, the game is tied.

- (a) Let A_1 be the event that the first marble is red and let A_2 be the event that the second marble is red. Are A_1 and A_2 independent?
- (b) What is the probability that Leanne wins the game?
- (c) Given that Leanne wins the game, what is the probability that all of the marbles were red?

Now, suppose the bag contains 8 red marbles and 4 blue marbles. Moreover, if there are an equal number of red and blue marbles among the four drawn, Leanne wins if the third marble is red, and Sylvia wins if the third marble is blue.

- (d) What is the probability that the third marble is red?
- (e) Given that there are k red marbles among the four drawn, where $0 \leq k \leq 4$, what is the probability that the third marble is red? Answer in terms of k .
- (f) Given that the third marble is red, what is the probability that Leanne wins the game?

Solution:

- (a) They are not independent; removing one red marble lowers the probability of the next marble being red.
- (b) Let p be the probability that Leanne wins. Since there are an equal number of red and blue marbles, by symmetry, the probability that Leanne wins and the probability that Sylvia wins is the same. Thus, the probability that there is a tie is $1 - p - p = 1 - 2p$.

We now compute the probability that there is a tie. For there to be a tie, two of the four marbles need to be red. There are $\binom{8}{4}$ ways to pick 4 marbles, and $\binom{4}{2}\binom{4}{2}$ to pick 2 red and blue marbles, respectively, giving a probability of $\frac{\binom{4}{2}\binom{4}{2}}{\binom{8}{4}} = \frac{36}{70} = \boxed{\frac{18}{35}}$.

We conclude that $1 - 2p = \frac{18}{35}$. Solving for p gives $p = \boxed{\frac{17}{70}}$.

- (c) Let A be the event that there are 3 red marbles drawn, and let B be the event that there are 4 red marbles drawn. We wish to compute $P[B|(A \cup B)] = \frac{P[B \cap (A \cup B)]}{P[A \cup B]} = \frac{P[B]}{P[A] + P[B]}$. Similar to the calculation in part (a), the probability that there are 3 red marbles drawn is $\frac{\binom{4}{3}\binom{4}{1}}{\binom{8}{4}} = \frac{16}{70}$,

and the probability that there are 4 red marbles drawn is $\frac{\binom{4}{4}\binom{0}{4}}{\binom{8}{4}} = \frac{1}{70}$, giving a final answer of

$$\frac{\frac{1}{70}}{\frac{16}{70} + \frac{1}{70}} = \boxed{\frac{1}{17}}.$$

- (d) By symmetry, the probability that the third marble is red is the same as the probability that the first marble is red, or the same as any marble being red. One way to see this is to imagine drawing the four marbles in order, then moving the first marble drawn to the third position. This is another way to draw four marbles that yields the same distribution.

There are 8 red marbles, and 12 marbles in total. Thus, the probability that the third marble is red is $\frac{8}{12} = \boxed{\frac{2}{3}}$.

- (e) We are given that there are k red marbles among the 4 drawn. By symmetry, each marble has the same probability of being red, so the probability that the third marble is red is $\boxed{\frac{k}{4}}$.

- (f) The only way for Leanne to lose the game given that the third marble is red is if all the other marbles are blue. The probability that the third marble is red and all the other marbles are blue is $\frac{4}{12} \cdot \frac{3}{11} \cdot \frac{8}{10} \cdot \frac{2}{9} = \frac{8}{495}$, and the probability that the third marble is red is $\frac{8}{12} = \frac{2}{3}$, so the probability that Leanne loses given that the third marble is red is $\frac{\frac{8}{495}}{\frac{2}{3}} = \frac{4}{165}$, and the probability

that Leanne wins given that the third marble is red is $\boxed{\frac{161}{165}}$.

3 Cliques in Random Graphs

Consider the graph $G = (V, E)$ on n vertices which is generated by the following random process: for each pair of vertices u and v , we flip a fair coin and place an (undirected) edge between u and v if and only if the coin comes up heads.

- What is the size of the sample space?
- A k -clique in graph is a set S of k vertices which are pairwise adjacent (every pair of vertices is connected by an edge). For example a 3-clique is a triangle. Let's call the event that S forms a clique E_S . What is the probability of E_S for a particular set S of k vertices?
- Suppose that $V_1 = \{v_1, \dots, v_\ell\}$ and $V_2 = \{w_1, \dots, w_k\}$ are two arbitrary sets of vertices. What conditions must V_1 and V_2 satisfy in order for E_{V_1} and E_{V_2} to be independent? Prove your answer.
- Prove that $\binom{n}{k} \leq n^k$. (You might find this useful in part (e))
- Prove that the probability that the graph contains a k -clique, for $k \geq 4\log_2 n + 1$, is at most $1/n$.

Solution:

- (a) Between every pair of vertices, there is either an edge or not. Since there are two choices for each of the $\binom{n}{2}$ pairs of vertices, the size of the sample space is $2^{\binom{n}{2}}$.
- (b) For a fixed set of k vertices to be a k -clique, all of the $\binom{k}{2}$ pairs of those vertices have to be connected by an edge. The probability of this event is $1/2^{\binom{k}{2}}$.
- (c) E_{V_1} and E_{V_2} are independent if and only if V_1 and V_2 share at most one vertex: If V_1 and V_2 share at most one vertex, then since edges are added independently of each other, we have

$$\begin{aligned}\mathbb{P}[E_{V_1} \cap E_{V_2}] &= \mathbb{P}[\text{all edges in } V_1 \text{ and all edges in } V_2 \text{ are present}] \\ &= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2}} \\ &= \mathbb{P}[E_{V_1}] \cdot \mathbb{P}[E_{V_2}].\end{aligned}$$

Conversely, if V_1 and V_2 share at least two vertices, then their intersection $V_3 = V_1 \cap V_2$ has at least 2 elements, and whence

$$\begin{aligned}\mathbb{P}[E_{V_1} \cap E_{V_2}] &= \left(\frac{1}{2}\right)^{\binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} - \binom{|V_3|}{2}} \cdot \left(\frac{1}{2}\right)^{\binom{|V_2|}{2} - \binom{|V_3|}{2}} \\ &= \left(\frac{1}{2}\right)^{\binom{|V_1|}{2} + \binom{|V_2|}{2} - \binom{|V_3|}{2}} \neq \mathbb{P}[E_{V_1}] \cdot \mathbb{P}[E_{V_2}].\end{aligned}$$

- (d) The algebraic solution is an application of the definition of $\binom{n}{k}$:

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{(n-k)!k!} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \\ &\leq n \cdot (n-1) \cdots (n-k+1) \\ &\leq n^k\end{aligned}$$

- (e) Let A_S denote the event that S is a k -clique, where $S \subseteq V$ is of size k . Then, the event that the graph contains a k -clique can be described as the union of A_S 's over all $S \subseteq V$ of size k . Using the union bound,

$$\mathbb{P}\left[\bigcup_{S \subseteq V, |S|=k} A_S\right] \leq \sum_{S \subseteq V, |S|=k} \mathbb{P}[A_S] = \sum_{S \subseteq V, |S|=k} \frac{1}{2^{\binom{k}{2}}}.$$

Now, since there are $\binom{n}{k}$ ways of choosing a subset $S \subseteq V$ of size k , the right-hand side of the above equality is

$$\frac{\binom{n}{k}}{2^{\binom{k}{2}}} = \frac{\binom{n}{k}}{2^{k(k-1)/2}} \leq \frac{n^k}{(2^{(k-1)/2})^k} \leq \frac{n^k}{(2^{(4 \log n + 1 - 1)/2})^k} = \frac{n^k}{(2^{2 \log n})^k} = \frac{n^k}{n^{2k}} = \frac{1}{n^k} \leq \frac{1}{n}.$$

4 (Un)conditional (In)equalities

Let us consider a sample space $\Omega = \{\omega_1, \dots, \omega_N\}$ of size $N > 2$ and two probability functions \mathbb{P}_1 and \mathbb{P}_2 on it. That is, we have two probability spaces: (Ω, \mathbb{P}_1) and (Ω, \mathbb{P}_2) .

- (a) Suppose that for every subset $A \subseteq \Omega$ of size $|A| = 2$ and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1[\omega | A] = \mathbb{P}_2[\omega | A]$.
- (i) Let $A = \{\omega_i, \omega_j\}$ for some $i, j \in \{1, \dots, N\}$. What can you say about $\frac{\mathbb{P}_1[\omega_i]}{\mathbb{P}_1[\omega_j]}$ and $\frac{\mathbb{P}_2[\omega_i]}{\mathbb{P}_2[\omega_j]}$?
 - (ii) Is it necessarily true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$? That is, if \mathbb{P}_1 and \mathbb{P}_2 are equal conditional on events of size 2, are they equal unconditionally? (*Hint*: Remember that probabilities must add up to 1.)
- (b) Suppose that for every subset $A \subseteq \Omega$ of size $|A| = k$, where k is some fixed element in $\{2, \dots, N\}$, and for every outcome $\omega \in \Omega$, it is true that $\mathbb{P}_1[\omega | A] = \mathbb{P}_2[\omega | A]$. Is it necessarily true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$? (*Hint*: Use part (a).)

For the following two parts, assume that $\Omega = \{(a_1, \dots, a_k) \mid \sum_{j=1}^k a_j = n\}$ is the set of configurations of n balls into k labeled bins, and let \mathbb{P}_1 be the probabilities assigned to these configurations by throwing the balls independently one after another and they will land into any of the k bins uniformly at random, and let \mathbb{P}_2 be the probabilities assigned to these configurations by uniformly sampling one of these configurations.

As an example, suppose $k = 6$ and $n = 2$. \mathbb{P}_1 is equivalent to rolling 2 six-sided dice, and letting a_i be the number of i s that appear. \mathbb{P}_2 is equivalent to sampling uniformly from all unordered pairs (i, j) with $1 \leq i, j \leq 6$.

- (c) Let A be the event that all n balls are in exactly one bin.
- (i) What are $\mathbb{P}_1[\omega | A]$ and $\mathbb{P}_2[\omega | A]$ for any $\omega \in A$?
 - (ii) Repeat part (i) for $\omega \in \Omega \setminus A$.
 - (iii) Is it true that $\mathbb{P}_1[\omega] = \mathbb{P}_2[\omega]$ for all $\omega \in \Omega$?
- (d) For the special case of $n = 9$ and $k = 3$, provide two outcomes B and C , so that $\mathbb{P}_1[B] < \mathbb{P}_2[B]$ and $\mathbb{P}_1[C] > \mathbb{P}_2[C]$. Provide justification.

Solution:

- (a) (i) For any $k \in \{1, 2\}$, we have $\mathbb{P}_k[\omega_i | A] = \frac{\mathbb{P}_k[\omega_i]}{\mathbb{P}_k[A]}$. Since this expression (by assumption) is the same for $k = 1$ and $k = 2$, we conclude that

$$\frac{\mathbb{P}_1[\omega_i]}{\mathbb{P}_2[\omega_i]} = \frac{\mathbb{P}_1[A]}{\mathbb{P}_2[A]}.$$

Repeating the reasoning for ω_j , we similarly find that

$$\frac{\mathbb{P}_1[\omega_j]}{\mathbb{P}_2[\omega_j]} = \frac{\mathbb{P}_1[A]}{\mathbb{P}_2[A]},$$

and whence

$$\frac{\mathbb{P}_1[\omega_i]}{\mathbb{P}_1[\omega_j]} = \frac{\mathbb{P}_2[\omega_i]}{\mathbb{P}_2[\omega_j]}.$$

- (ii) Yes, this is indeed true. Since the result from (a) holds for any $i, j \in \{1, \dots, N\}$, we can sum over i to get

$$\frac{1}{\mathbb{P}_1[\omega_j]} = \sum_{i=1}^N \frac{\mathbb{P}_1[\omega_i]}{\mathbb{P}_1[\omega_j]} = \sum_{i=1}^N \frac{\mathbb{P}_2[\omega_i]}{\mathbb{P}_2[\omega_j]} = \frac{1}{\mathbb{P}_2[\omega_j]},$$

which shows that $\mathbb{P}_1[\omega_j] = \mathbb{P}_2[\omega_j]$ for all $j \in \{1, \dots, N\}$.

- (b) Yes, it indeed would. There are two ways of verifying this. The first way to convince ourselves that part (b) is true, is to observe that none of the arguments used in part (a) really relied on A having size 2, and so the very same reasoning carries through for A of size k .

The second (more rigorous) one is to observe that if $A' \subset A$ and $\omega \in A'$, then

$$\mathbb{P}_1[\omega | A'] = \mathbb{P}_1[\omega | A' \cap A] = \frac{\mathbb{P}_1[\omega | A]}{\mathbb{P}_1[A' | A]} = \frac{\mathbb{P}_2[\omega | A]}{\mathbb{P}_2[A' | A]} = \mathbb{P}_2[\omega | A'],$$

where the second equality follows from the product rule:

$$\mathbb{P}_1[A] \cdot \mathbb{P}_1[A' | A] \cdot \mathbb{P}_1[\omega | A \cap A'] = \mathbb{P}_1[\{\omega\} \cap A \cap A'] = \mathbb{P}_1[\omega] = \mathbb{P}_1[A] \mathbb{P}_1[\omega | A].$$

That is, if \mathbb{P}_1 and \mathbb{P}_2 coincide conditional on some event A , they also coincide conditional on any smaller event A' . In particular, if they coincide on all events of size k , they also coincide on all events of size 2, which we have already dealt with in part (a).

- (c) There are exactly k outcomes in A (namely, $(n, 0, \dots, 0), (0, n, 0, \dots), \dots, (0, \dots, 0, n)$; i.e. each bin could be the full one), and all of them are equally likely under either \mathbb{P}_1 or \mathbb{P}_2 . That is, if $\omega \in A$, then $\mathbb{P}_1[\omega] = \left(\frac{1}{k}\right)^n$, and $\mathbb{P}_2[\omega] = \left(\frac{1}{k}\right)^{n+k-1}$. Consequently, for $\omega \in A$,

$$\begin{aligned} \mathbb{P}_1[\omega | A] &= \frac{k^{-n}}{k \cdot k^{-n}} = \frac{1}{k} \\ \mathbb{P}_2[\omega | A] &= \frac{\left(\frac{1}{k}\right)^{n+k-1}}{k \cdot \left(\frac{1}{k}\right)^{n+k-1}} = \frac{1}{k}. \end{aligned}$$

If $\omega \notin A$, then $\mathbb{P}_1[\omega | A] = \mathbb{P}_2[\omega | A] = 0$, and so $\mathbb{P}_1[\omega | A]$ and $\mathbb{P}_2[\omega | A]$ coincide for all $\omega \in \Omega$. This, however, does *not* imply that \mathbb{P}_1 and \mathbb{P}_2 are the same!

Indeed, when computing the probability of $\omega \in A$ above, we saw that $\mathbb{P}_1[\omega] \neq \mathbb{P}_2[\omega]$ (remember that the assumption of part (b) was that the conditional probabilities coincide for *all* events of size k , here we have only shown equality conditional on *one* such event).

- (d) Intuitively, throwing balls independently one after another makes it much less likely that all balls stack up in one bin as opposed to spreading out more evenly.

This suggests taking, e.g., $A = \{\text{all balls land in bin 1}\}$, whose probability we already computed in part (c). That is, to show that $\mathbb{P}_1[A] < \mathbb{P}_2[A]$, we need to show that $k^{-n} < \binom{n+k-1}{k-1}^{-1}$. Plugging in $k = 3$ and $n = 9$, we have

$$k^{-n} = 3^{-9} = 3^{-2} \cdot 3^{-3} \cdot 3^{-4} = \frac{1}{9 \cdot 27 \cdot 3^4} < \frac{1}{5 \cdot 11} = \frac{2}{11 \cdot 10} = \binom{11}{2}^{-1} = \binom{n+k-1}{k-1}^{-1},$$

as desired.

Conversely, the same reasoning suggests that evenly distributed balls are much more likely under \mathbb{P}_1 than under \mathbb{P}_2 . And indeed, letting $B = \{\text{each bin has exactly three balls}\}$, we have

$$\mathbb{P}_1[B] = \binom{9}{3} \binom{6}{3} \binom{3}{3} \left(\frac{1}{3}\right)^9 = \frac{9!}{(3!)^3 \cdot 3^9} = \frac{7!}{3^{10}} > \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{3^6} \cdot \frac{2}{11 \cdot 10} = \frac{7}{3} \cdot \frac{6 \cdot 5}{3^3} \cdot \frac{4}{3} \cdot \mathbb{P}_2[B],$$

and since the last factor is bigger than 1, we have $\mathbb{P}_1[B] > \mathbb{P}_2[B]$ as promised (of course, we could have also just plugged all these powers and factorials into a calculator to get the same result).

5 Cookie Jars

You have two jars of cookies, each of which starts with n cookies initially. Every day, when you come home, you pick one of the two jars randomly (each jar is chosen with probability $1/2$) and eat one cookie from that jar. One day, you come home and reach inside one of the jars of cookies, but you find that is empty! Let X be the random variable representing the number of remaining cookies in non-empty jar at that time. What is the distribution of X ?

Solution: Assume that you found jar 1 empty. The probability that $X = k$ and you found jar 1 empty is computed as follows. In order for there to be k cookies remaining, you must have eaten a cookie for $2n - k$ days, and then you must have chosen jar 1 (to discover that it is empty). Within those $2n - k$ days, exactly n of those days you chose jar 1. The probability of this is $\binom{2n-k}{n} 2^{-(2n-k)}$. Furthermore, the probability that you then discover jar 1 is empty the day after is $1/2$. So, the probability that $X = k$ and you discover jar 1 empty is $\binom{2n-k}{n} 2^{-(2n-k+1)}$. However, we assumed that we discovered jar 1 to be empty; the probability that $X = k$ and jar 2 is empty is the same by symmetry, so the overall probability that $X = k$ is:

$$\mathbb{P}[X = k] = \binom{2n-k}{n} \frac{1}{2^{2n-k}}, \quad k \in \{0, \dots, n\}.$$

6 Maybe Lossy Maybe Not

Let us say that Alice would like to send a message to Bob, over some channel. Alice has a message of length 4.

- (a) Packets are dropped with probability p . If Alice sends 5 packets, what is probability that Bob can successfully reconstruct Alice's message using polynomial interpolation?
- (b) Again, packets can be dropped with probability p . The channel may additionally corrupt 1 packet after deleting packets. Alice realizes this and sends 8 packets for a message of length 4. What is the probability that Bob receives enough packets to successfully reconstruct Alice's message using Berlekamp-Welch?
- (c) Again, packets can be dropped with probability p . This time, packets may be corrupted with probability q . A packet being dropped is independent of whether or not is corrupted (i.e. a packet may be both corrupted and dropped). Consider the original scenario where Alice sends 5 packets for a message of length 4. What is probability that Bob can correctly reconstruct Alice's message using polynomial interpolation on all of the points he receives?

Solution:

- (a) Alice's message requires a polynomial of degree 3, which can be uniquely identified by 4 points. Thus, at least 4 points need to make it across the channel. The probability that Bob can recover the message is thus the probability that at most one packet is lost. Since the packets are lost with probability with probability p , we have the probability of losing 1 packet is

$$\binom{5}{1}(1-p)^4p.$$

The probability of losing 0 packets is $(1-p)^5$. Thus, the probability of losing 0 or 1 packets is

$$\binom{5}{1}(1-p)^4p + (1-p)^5.$$

This is the probability that Bob receives 4 packets, meaning he can successfully reconstruct the 3-degree polynomial.

- (b) Bob needs $n + 2k = 6$ packets to guarantee successful reconstruction of Alice's message. There are a total of 8 packets sent, so this guarantee occurs only if 0 packets, 1 packet or 2 packets are lost. The probability of 0 packets lost is

$$(1-p)^8.$$

The probability of one packet lost is

$$\binom{8}{1}p(1-p)^7.$$

The probability of two packets lost is

$$\binom{8}{2}p^2(1-p)^6.$$

Thus, the probability of success is

$$(1-p)^8 + \binom{8}{1}p(1-p)^7 + \binom{8}{2}p^2(1-p)^6.$$

- (c) Again, Bob can reconstruct the message if none of the packets are corrupted. We use the same idea as in Part (a). The probability that none of the packets are corrupted is $(1 - q)^5$. We know that *on top of* being uncorrupted, we can only at lose at most 1 packet. Thus, we can either lose one packet, which has probability

$$\binom{5}{1} p(1 - p)^4.$$

Or, we can lose no packets, which has probability $(1 - p)^5$. Yet another possibility is if exactly one packet is corrupted, but that packet is also dropped; in this case, we can recover the message, so long as no other packets are corrupted or dropped. This occurs with probability

$$\binom{5}{1} pq(1 - p)^4(1 - q)^4.$$

As a result, we have the following.

$$(1 - q)^5(5p(1 - p)^4 + (1 - p)^5) + 5pq(1 - p)^4(1 - q)^4.$$