# CS 70      Discrete Mathematics and Probability Theory
Summer 2022    Jingjia Chen, Michael Psenka and Tarang Srivastava     HW 4

Due: Thursday 7/14, 11:59 PM
Grace period until Sunday 7/18, 8:00 PM

## Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

## 1 Lagrange? More like Lamegrange.

In this problem, we walk you through an alternative to Lagrange interpolation.

(a) Let's say we wanted to interpolate a polynomial through a single point, $(x_0, y_0)$. What would be the polynomial that we would get? (This is not a trick question.)

(b) Call the polynomial from the previous part $f_0(x)$. Now say we wanted to define the polynomial $f_1(x)$ that passes through the points $(x_0, y_0)$ and $(x_1, y_1)$. If we write $f_1(x) = f_0(x) + a_1(x - x_0)$, what value of $a_1$ causes $f_1(x)$ to pass through the desired points?

(c) Now say we want a polynomial $f_2(x)$ that passes through $(x_0, y_0)$, $(x_1, y_1)$, and $(x_2, y_2)$. If we write $f_2(x) = f_1(x) + a_2(x - x_0)(x - x_1)$, what value of $a_2$ gives us the desired polynomial?

(d) Suppose we have a polynomial $f_i(x)$ that passes through the points $(x_0, y_0)$, ..., $(x_i, y_i)$ and we want to find a polynomial $f_{i+1}(x)$ that passes through all those points and also $(x_{i+1}, y_{i+1})$. If we define $f_{i+1}(x) = f_i(x) + a_{i+1} \prod_{j=0}^{i}(x - x_j)$, what value must $a_{i+1}$ take on?

## 2 How Many Polynomials?

Let $P(x)$ be a polynomial of degree at most 2 over GF(5). As we saw in lecture, we need $d + 1$ distinct points to determine a unique $d$-degree polynomial, so knowing the values for say, $P(0)$, $P(1)$, and $P(2)$ would be enough to recover $P$. (For this problem, we consider two polynomials to be distinct if they return different values for any input.)

(a) Assume that we know $P(0) = 1$, and $P(1) = 2$. Now consider $P(2)$. How many values can $P(2)$ have? How many distinct possibilities for $P$ do we have?

(b) Now assume that we only know $P(0) = 1$. We consider $P(1)$ and $P(2)$. How many different $(P(1), P(2))$ pairs are there? How many distinct possibilities for $P$ do we have?

(c) Now, let $P$ be a polynomial of degree at most $d$ on $\mathrm{GF}(p)$ for some prime $p$ with $p > d$. Assume we only know $P$ evaluated at $k \leq d+1$ different values. How many different possibilities do we have for $P$?

(d) A polynomial with integer coefficients that cannot be factored into polynomials of lower degree on a finite field, is called an irreducible or prime polynomial.

Show that $P(x) = x^2 + x + 1$ is a prime polynomial on $\mathrm{GF}(5)$.

# 3 Monke

In the Monke Kingdom, there are seven bald uakaris, four black howlers, two mandrills, two gelada and one baboon. The Monke King has told the five species about an uncountably large supply of bananas; however, Monke doesn't want to reveal where the bananas are unless at least two species can cooexist in harmony.

(a) Help Monke create a scheme that allows the monkeys to find the key only if all members of one species in addition to at least one member from a different specie agree to work together.

For example, if all seven bald uakaris and one mandrill agree, they can find the key. However, if six bald uakaris and one mandrill agree, or if only seven bald uakaris agree, then the key cannot be found.

(b) If the monkeys of each species are distinct individuals, how many different groups of monkeys can be formed be such that that group can access the infinite bananas?

(c) Seeing that all the uakaris and all howlers have teamed up, Monke wants to encourage even more monkey friendship. He wants to set up an additional scheme so that, if at least one mandrill, one gelada and one baboon agree, the five parties can uncover exactly two more supplies of bananas, one countably infinite and one finite. How can Monke implement part 2 of their scheme?

# 4 That's Numberwang!

Congratulations! You've earned a spot on the game show "Numberwang".

(a) How many permutations of NUMBERWANG contain "GAMER" as a substring? How about as a subsequence (meaning the letters of "GAMER" have to appear in that order, but not necessarily next to each other)?

(b) In round 1 of Numberwang, each player chooses an ordered sequence of 5 digits. A valid sequence must have the property that it is non-increasing when read from left to right. For example, 99621 is a valid sequence, but 43212 is not. How many choices of valid sequences are there? (**Hint:** Relate the problem to balls and bins.)

(c) To win round 2 of Numberwang, a contestant must choose five nonnegative integers $x_0, x_1, x_2, x_3, x_4$ such that $x_0 + x_1 + x_2 + x_3 + x_4 = 100$, and $x_i \equiv i \pmod 5$. How many ways are there to pick a winning set of integers?

# 5 M, Mi, Mic, Mich, Micha, Michae, Michael!

In the vowel substring problem, we are tasked to find the number of vowels in every substring of an input string $S$. A substring is defined as any contingious set of characters in the original order of $S$. For example, $S =$ "aba", the substrings of $S$ would be "a", "ab", "aba", "b", "ba", "a", counting a total of 6 vowels. We will make a combinatorial argument to count the number of vowels in all substrings.

- How many substrings are there in a string of length $n$?

- Suppose there is a vowel at 1st position of $S$ (the beginning). Given that $S$ has length $n$, how many substrings contain that vowel at the beginning of $S$?

- Suppose there is a vowel at the $k^{th}$ position of $S$. How many substrings of $S$ contain that vowel at the $k^{th}$ position?

- Given that there are vowels in positions $i_1, i_2, ..., i_k$ what are the total number of vowels in all substrings of $S$?

# 6 Maze

Let's assume that Tom is located at the bottom left corner of the $9 \times 9$ maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.
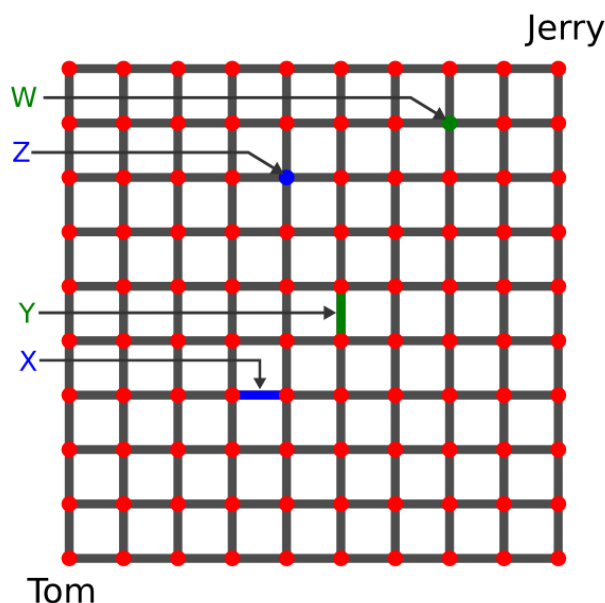
(a) What is the length of the shortest path from Tom to Jerry?

(b) How many such shortest paths exist?

(c) How many shortest paths pass through the edge labeled $X$? The edge labeled $Y$? Both the edges $X$ and $Y$? Neither edge $X$ nor edge $Y$?