Today.

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Polynomials.

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Secret Sharing.

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Correcting for loss or even corruption.

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Two points make a line.

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The idea of the day.

Two points make a line.
Lots of lines go through one point.

Polynomials

A **polynomial**

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0.$$

is specified by **coefficients** $a_d, \ldots a_0$.

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Polynomials $P(x)$ with arithmetic modulo p: [1] $a_i \in \{0, \ldots, p-1\}$ and

$$P(x) = a_d x^d + a_{d-1} x^{d-1} \cdots + a_0 \pmod p,$$

for $x \in \{0, \ldots, p-1\}$.

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Line:$P(x) = a_1 x + a_0$

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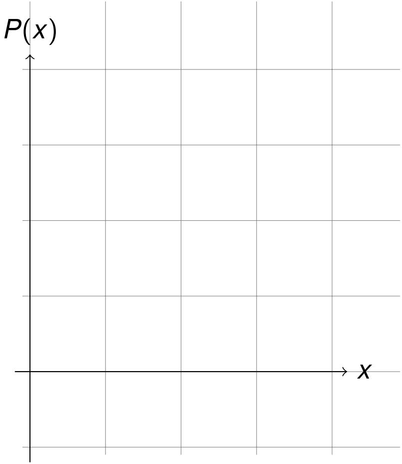

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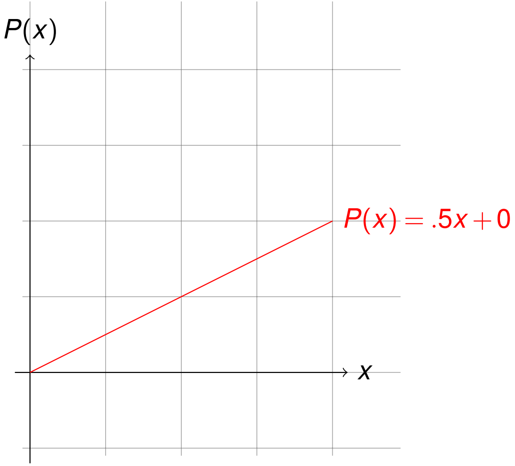

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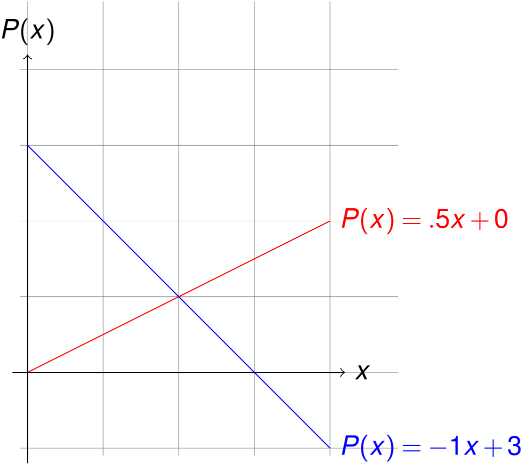

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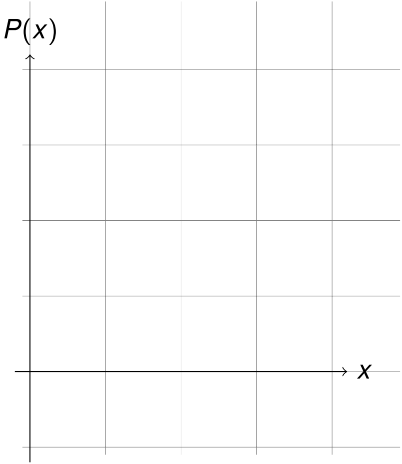

Parabola: $P(x) = a_2 x^2 + a_1 x + a_0$

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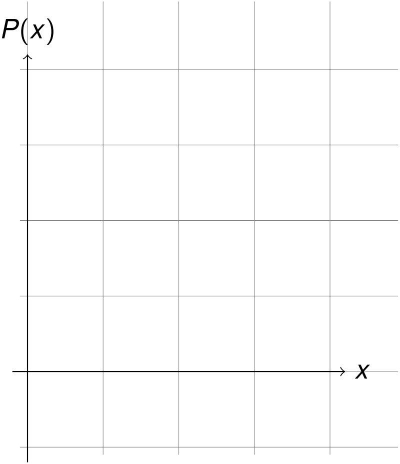

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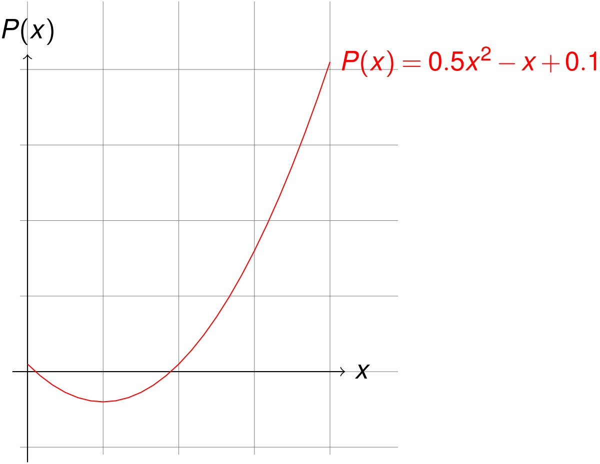

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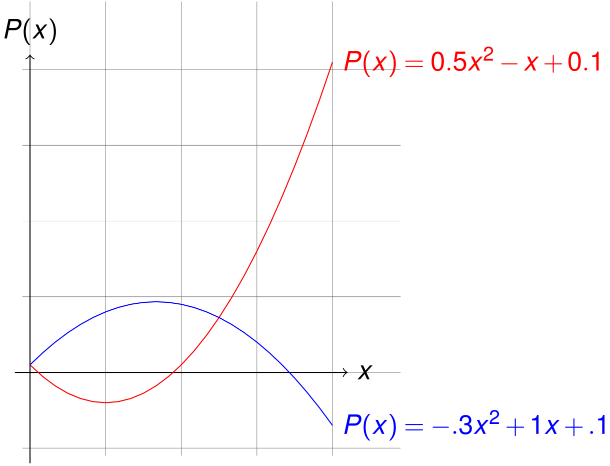

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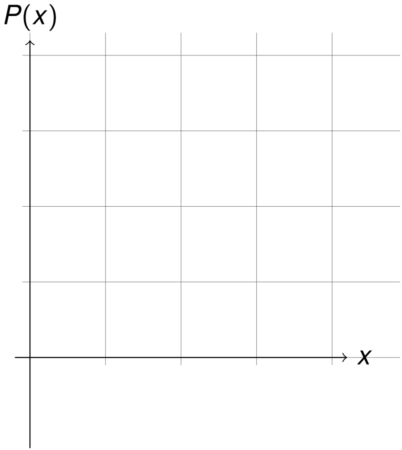

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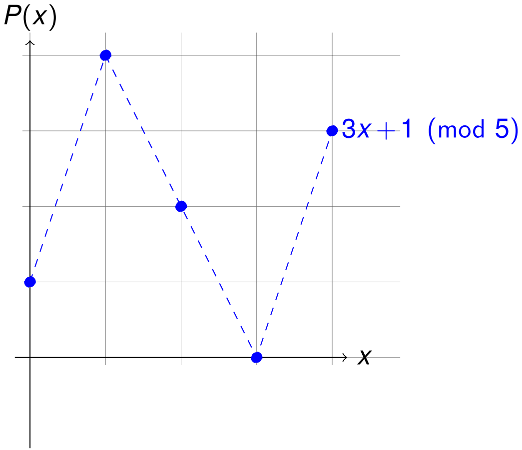

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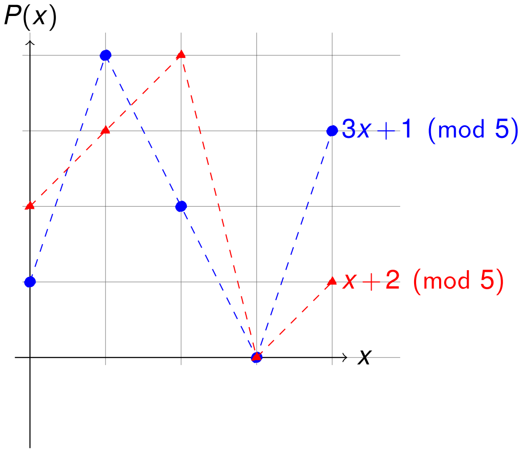

Finding an intersection.
$x+2 \equiv 3x+1 \pmod{5}$
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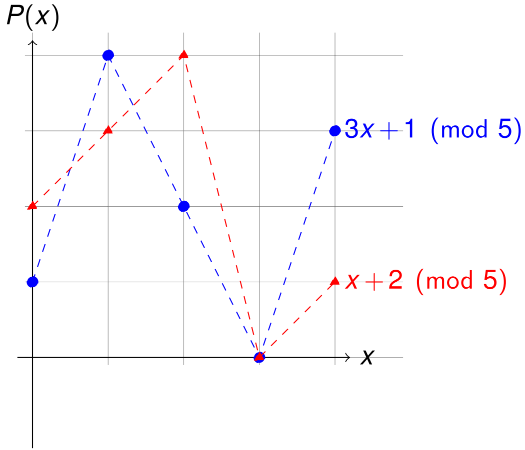

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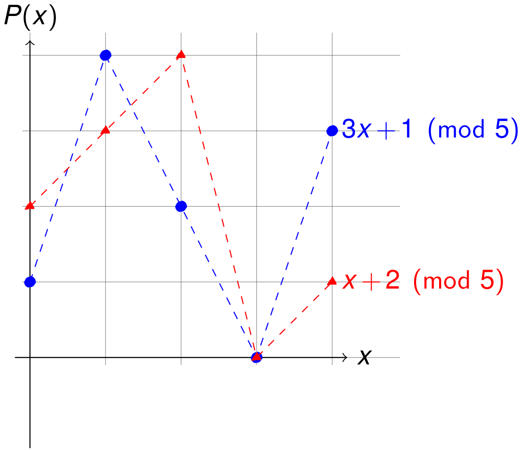

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Good when modulus is prime!!

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Fact: Exactly 1 degree $\leq d$ polynomial contains $d+1$ points. [2]

[2] Points with different x values.

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Modular Arithmetic Fact: Exactly 1 degree $\leq d$ polynomial with arithmetic modulo prime p contains $d+1$ pts.

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Poll.

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What facts below tell you this?

Say points are $(x_1, y_1), (x_2, y_2)$.

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(A) Line is $y = mx + b$.
(B) Plug in a point gives an equation: $y_1 = mx_1 + b$
(C) The unknowns are m and b.
(D) If equations have unique solution, done.

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All true.

Flow Poll.

Why solution? Why unique?

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(A) Solution cuz: $m = (y_2 - y_1)/(x_2 - x_1), b = y_1 - m(x_1)$
(B) Unique cuz, only one line goes through two points.
(C) Try: $(m'x + b') - (mx + b) = (m' - m)x + (b - b') = ax + c \neq 0$.
(D) Either $ax_1 + c \neq 0$ or $ax_2 + c \neq 0$.
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Flow poll. (All true. (B) is not a proof, it is restatement.)

Notation: two points on a line.

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(A) and (D)

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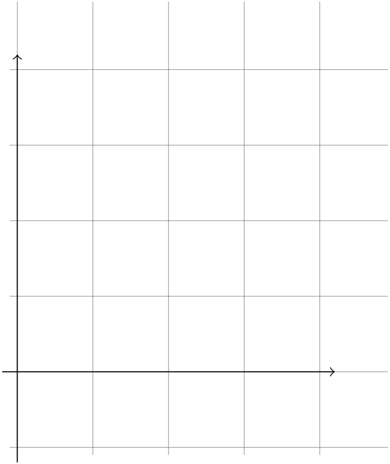

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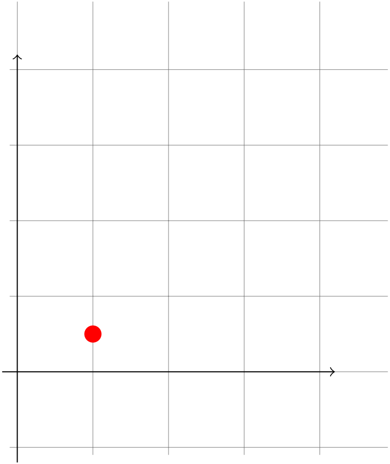

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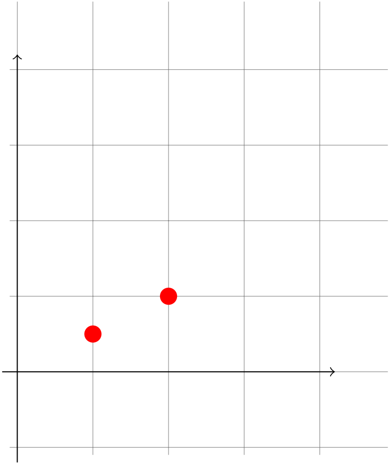

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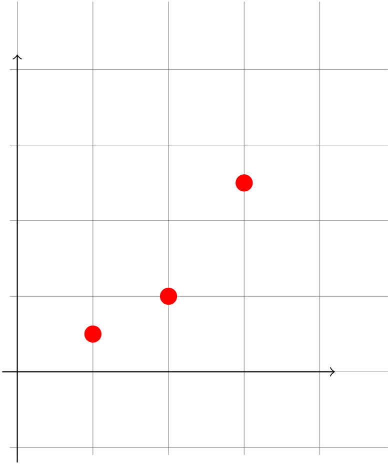

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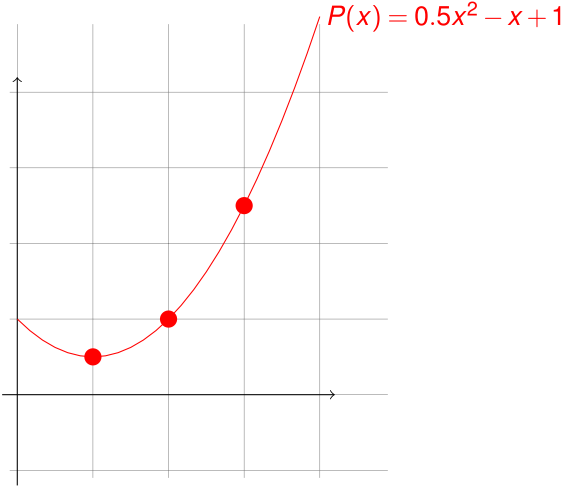

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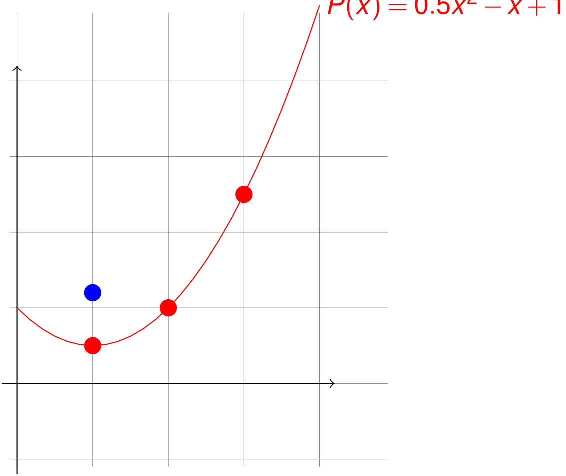

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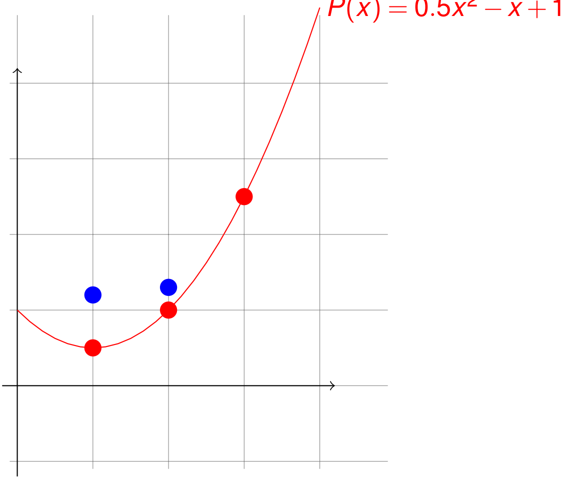

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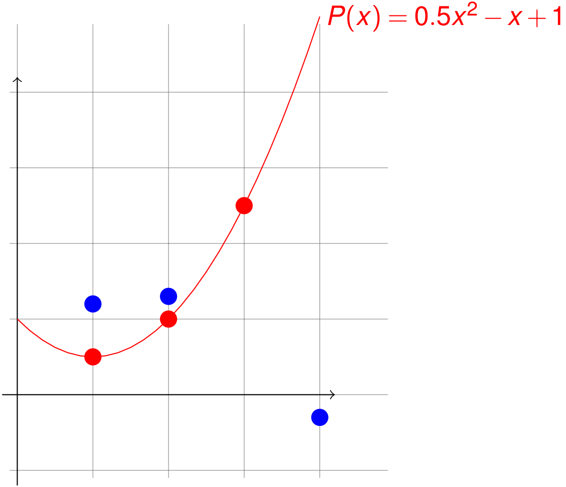

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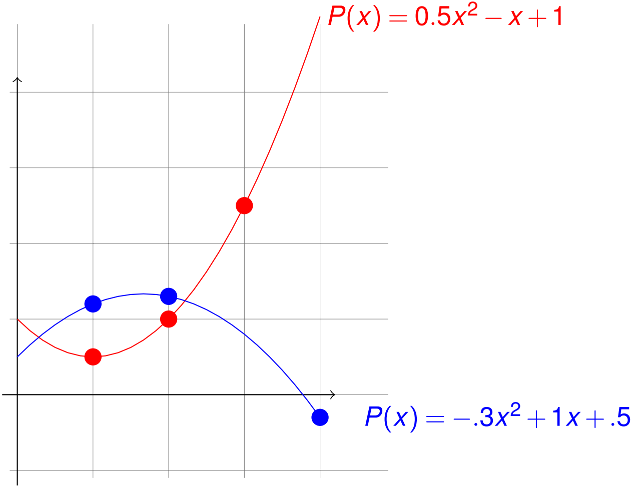

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[3] Points with different x values.

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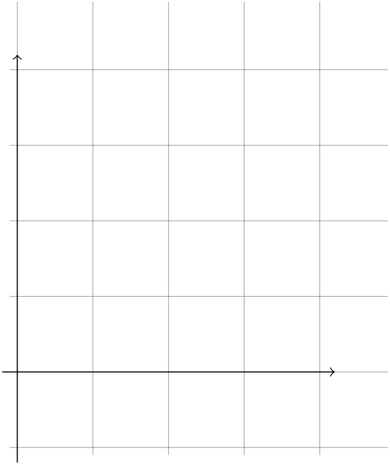

There is $P(x)$ contains blue points and *any* $(0, y)$!

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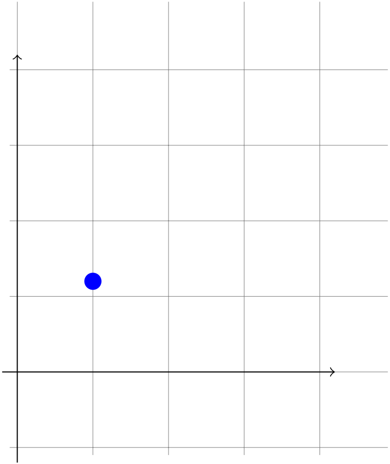

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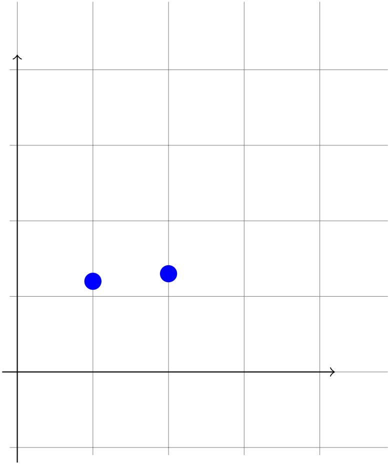

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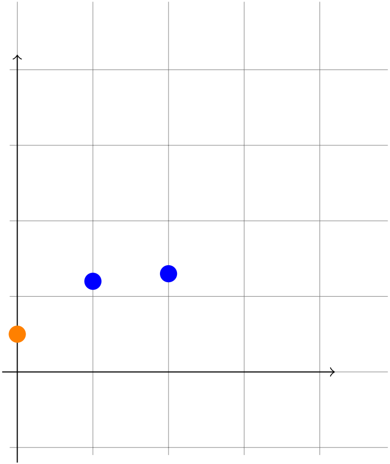

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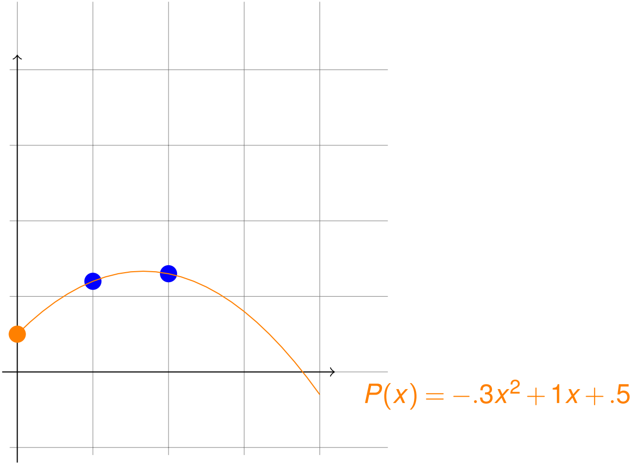

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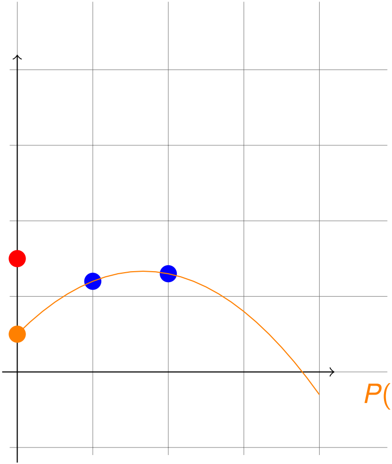

$P(x) = -.3x^2 + 1x + .5$

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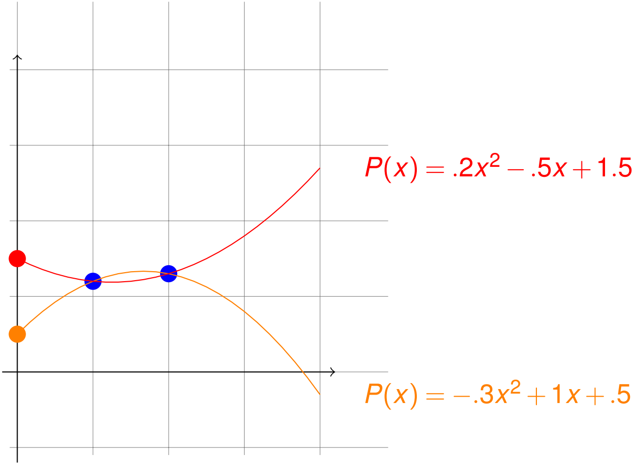

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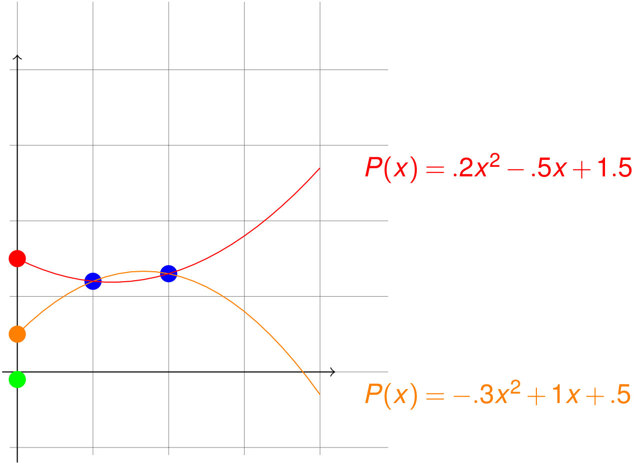

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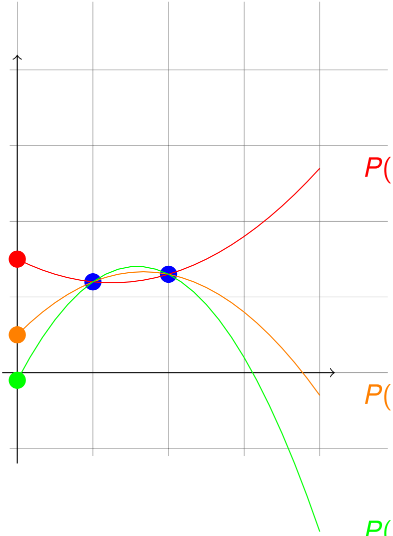

$P(x) = .2x^2 - .5x + 1.5$

$P(x) = -.3x^2 + 1x + .5$

$P(x) = -.6x^2 + 1.9x - .1$

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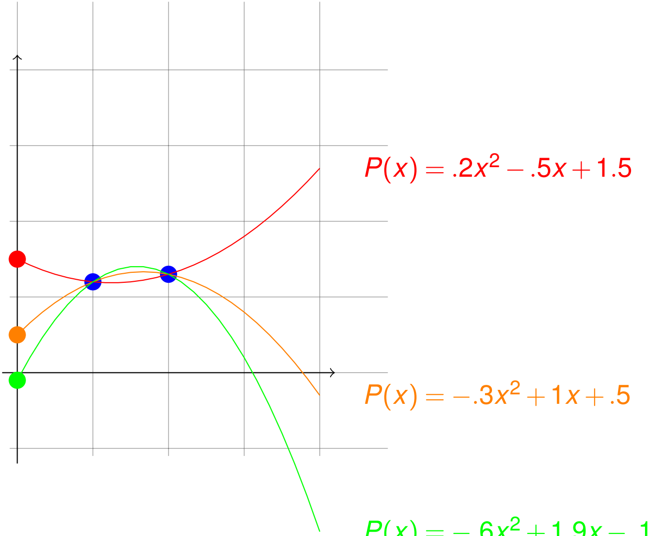

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Roubustness: Any k shares gives secret.

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(A) The secret is “2”.
(B) The secret is “3”.
(C) A share could be $(1,5)$ cuz $P(1) = 5$
(D) A share could be $(2,4)$
(E) A share could be $(0,3)$

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Multiplicative inverses due to $gcd(x,p) = 1$, forall $x \in \{1, \ldots, p-1\}$

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For set of x-values, $x_1, \ldots, x_{d+1}$.

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Construction proves the existence of a polynomial!

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Put the delta functions together.

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Construction proves the existence of the polynomial!

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Uniqueness Fact. At most one degree d polynomial hits $d+1$ points.

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Must prove **Roots fact.**

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That is, $P(x) = (x-a)Q(x) + r$

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$d+1$ roots implies degree is at least $d+1$.

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Lemma 1: $P(x)$ has root a iff $P(x)/(x-a)$ has remainder 0:
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Proof: $P(x) = (x-a)Q(x) + r$.
Plugin a: $P(a) = r$.
It is a root if and only if $r = 0$. $\square$

Lemma 2: $P(x)$ has d roots; $r_1, \ldots, r_d$ then
$P(x) = c(x-r_1)(x-r_2)\cdots(x-r_d)$.
Proof Sketch: By induction.

Induction Step: $P(x) = (x-r_1)Q(x)$ by Lemma 1. $Q(x)$ has smaller degree so use the induction hypothesis. $\square$

$d+1$ roots implies degree is at least $d+1$.

Roots fact: Any degree d polynomial has at most d roots.

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Intuitively, a field is a set with operations corresponding to addition, multiplication, and division.

Secret Sharing

Modular Arithmetic Fact: Exactly one polynomial degree $\leq d$ over $GF(p)$, $P(x)$, that hits $d+1$ points.

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(Almost) the same as what is missing: one $P(i)$.

Runtime.

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Runtime: polynomial in k, n, and $\log p$.

1. Evaluate degree $k-1$ polynomial n times using $\log p$-bit numbers.
2. Reconstruct secret by solving system of k equations using $\log p$-bit arithmetic.

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Infinite number for reals, rationals, complex numbers!

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k points on degree $k-1$ polynomial is great!
Can hand out n points on polynomial as shares.