

## 1 Short Answers

- (a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?
- (b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

### Solution:

- (a) **7.**  
Use Euler's formula  $v + f = e + 2$ .
- (b) **5.**  
The 3-dimensional hypercube has  $3(2^3)/2 = 12$  edges and  $2^3 = 8$  vertices. A tree on 8 vertices has 7 edges, so one needs to remove 5 edges.

## 2 Always, Sometimes, or Never

In each part below, you are given some information about the so-called original graph,  $OG$ . Using only the information in the current part, say whether  $OG$  will always be planar, always be non-planar, or could be either. If you think it is always planar or always non-planar, prove it. If you think it could be either, give a planar example and a non-planar example.

- (a)  $OG$  can be vertex-colored with 4 colors.
- (b)  $OG$  requires 7 colors to be vertex-colored.
- (c)  $e \leq 3v - 6$ , where  $e$  is the number of edges of  $OG$  and  $v$  is the number of vertices of  $OG$ .
- (d)  $OG$  is connected, and each vertex in  $OG$  has degree at most 2.
- (e) Each vertex in  $OG$  has degree at most 2.

### Solution:

- (a) Either planar or non-planar. By the 4-color theorem, any planar graph can provide the planar example. The easiest non-planar example is  $K_{3,3}$ , which can be 2-colored because it is bipartite. (Certainly, any graph which can be colored using only 2 colors can also be colored using 4 colors.)

- (b) Always non-planar. The 4-color theorem tells us that if a graph is planar, it can be colored using only 4 colors. The contrapositive of this is that if a graph requires more than 4 colors to vertex-color, it must be non-planar. (Using the 5- or 6-color theorem would also work.)
- (c) Either planar or non-planar. From the notes, we know that every planar graph follows this formula, so any planar graph is a valid planar example. The easiest non-planar example is again  $K_{3,3}$ , which has  $e = 9$  and  $v = 6$ , meaning our formula becomes  $9 \leq 3(6) - 6 = 12$ , which is certainly true.
- (d) Always planar. There are two cases to deal with here: either  $G$  is a tree, or  $G$  is not a tree and so contains at least one cycle. In the former case, we're immediately done, since all trees are planar. In the latter case, consider any cycle in  $G$ . We know that every vertex in that cycle is adjacent to the vertex to its left in the cycle and to the vertex to its right in the cycle. But we also know that no vertex can be connected to more than two other vertices, so the cycle isn't connected to anything else. But  $G$  is a connected graph, so we must have that  $G$  is just a single large cycle. And we can certainly draw a simple cycle on a plane without crossing any edges, so even in this case  $G$  is still planar.
- (e) Always planar. Each of  $G$ 's connected components is connected and has no vertex of degree more than 2, so by the previous part, each of them must be planar. Thus, each of  $G$ 's connected components must be planar, so  $G$  itself must be planar.

### 3 Trees and Components

- (a) Bob removed a degree 3 node from an  $n$ -vertex tree. How many connected components are there in the resulting graph? Please provide an explanation.
- (b) Given an  $n$ -vertex tree, Bob added 10 edges to it and then Alice removed 5 edges. If the resulting graph has 3 connected components, how many edges must be removed in order to remove all cycles from the resulting graph? Please provide an explanation.

#### Solution:

- (a) **3.**

Let the original graph be denoted by  $G = (V, E)$  and the resulting graph after Bob removes the node be denoted by  $G' = (V', E')$ . Let  $|V| = n$  and hence  $|E| = n - 1$  by definition of a tree. Also,  $|V'| = n - 1$  and  $|E'| = n - 4$ . Let  $k$  denote the number of connected components in  $G'$ . Since removing vertices and edges should not give rise to cycles, we know that the graph  $G'$  is acyclic. Hence each of the connected components is a tree. Let  $n_1, n_2, \dots, n_k$  denote the number of nodes in each of the  $k$  connected components respectively. Again, by definition of a tree, we have that each of the component consists of  $n_1 - 1, n_2 - 1, \dots, n_k - 1$  edges respectively. Thus

the total number of edges in  $G'$  is

$$n - 4 = |E'| = \sum_{i=1}^k (n_i - 1) = \left(\sum_{i=1}^k n_i\right) - k = (n - 1) - k.$$

Hence  $k = 3$ .

*Alternate Solution.* Here we use the fact that removing an edge from a forest (i.e an acyclic graph) increases the number of components by exactly 1. If we remove the three edges incident on the vertex removed by Bob, we get 4 components. However, Bob has also removed the degree 3 vertex which itself is one of the four connected components. Hence we are left with 3 connected components.

(b) 7.

We first note that in any connected graph if we remove an edge belonging to a cycle, then the resulting graph is still connected. Hence for any connected graph, we can repeatedly remove edges belonging to cycles, until no more cycles remain. This process will give rise to a connected acyclic graph, i.e., a tree.

Since the final graph we wish to obtain is acyclic, each of its connected component must be a tree. Thus the components should have  $n_1 - 1$ ,  $n_2 - 1$  and  $n_3 - 1$  edges each, where  $n_1, n_2, n_3$  are the number of vertices in each of these components. Let  $n$  denote the total number of vertices and hence  $n = n_1 + n_2 + n_3$ . As a result, the total number of edges in the final graph is  $n - 3$ . The total number of edges after Bob and Alice did their work was  $n - 1 + 10 - 5 = n + 4$ . Thus one needs to remove 7 edges.

## 4 Hypercubes

The vertex set of the  $n$ -dimensional hypercube  $G = (V, E)$  is given by  $V = \{0, 1\}^n$  (recall that  $\{0, 1\}^n$  denotes the set of all  $n$ -bit strings). There is an edge between two vertices  $x$  and  $y$  if and only if  $x$  and  $y$  differ in exactly one bit position. These problems will help you understand hypercubes.

- (a) Draw 1-, 2-, and 3-dimensional hypercubes and label the vertices using the corresponding bit strings.
- (b) Show that for any  $n \geq 1$ , the  $n$ -dimensional hypercube is bipartite.

### Solution:

- (a) The three hypercubes are a line, a square, and a cube, respectively. See also note 5 for pictures.
- (b) Consider the vertices with an even number of 0 bits and the vertices with an odd number of 0 bits. Each vertex with an even number of 0 bits is adjacent only to vertices with an odd number

of 0 bits, since each edge represents a single bit change (either a 0 bit is added by flipping a 1 bit, or a 0 bit is removed by flipping a 0 bit). Let  $L$  be the set of the vertices with an even number of 0 bits and let  $R$  be the vertices with an odd number of 0 bits, then no two adjacent vertices will belong to the same set.

Alternate solution (using induction and coloring):

It may be simpler to that a graph being 2-colorable is the same as being bipartite. Now, the argument is easier to state. First the base case is a hypercube with two vertices which is clearly two-colorable. Then notice, switching the colors in a two-coloring is still valid as if endpoints are differently colored, switching leaves them differently colored. Now, recursively one two colors the two subcubes the same, and then switches the colors in one subcube. The internal to subcube edges are fine by induction. The edges across are fine as the corresponding vertices are differently colored due to the switching.