CS 70 Spring 2022 — Discrete Mathematics and Probability Theory — Satish Rao and Koushik Sen — HW 11

1 Class Enrollment

Lydia has just started her CalCentral enrollment appointment. She needs to register for a marine science class and CS 70. There are no waitlists, and she can attempt to enroll once per day in either class or both. The CalCentral enrollment system is strange and picky, so the probability of enrolling successfully in the marine science class on each attempt is μ and the probability of enrolling successfully in CS 70 on each attempt is λ. Also, these events are independent.

(a) Suppose Lydia begins by attempting to enroll in the marine science class everyday and gets enrolled in it on day M. What is the distribution of M?

(b) Suppose she is not enrolled in the marine science class after attempting each day for the first 5 days. What is $\mathbb{P}[M = i | M > 5]$, the conditional distribution of M given $M > 5$?

(c) Once she is enrolled in the marine science class, she starts attempting to enroll in CS 70 from day $M + 1$ and gets enrolled in it on day C. Find the expected number of days it takes Lydia to enroll in both the classes, i.e. $\mathbb{E}[C]$.

Suppose instead of attempting one by one, Lydia decides to attempt enrolling in both the classes from day 1. Let M be the number of days it takes to enroll in the marine science class, and C be the number of days it takes to enroll in CS 70.

(d) What is the distribution of M and C now? Are they independent?

(e) Let X denote the day she gets enrolled in her first class and let Y denote the day she gets enrolled in both the classes. What is the distribution of X?

(f) What is the expected number of days it takes Lydia to enroll in both classes now, i.e. $\mathbb{E}[Y]$.

(g) What is the expected number of classes she will be enrolled in by the end of 14 days?

Solution:

(a) $M \sim \text{Geometric}(\mu)$.

(b) Given that $M > 5$, the random variable M takes values in $\{6, 7, \dots\}$. For $i = 6, 7, \dots,$

$$\mathbb{P}[M = i | M > 5] = \frac{\mathbb{P}[M = i \wedge M > 5]}{\mathbb{P}[M > 5]} = \frac{\mathbb{P}[M = i]}{\mathbb{P}[M > 5]} = \frac{\mu(1-\mu)^{i-1}}{(1-\mu)^5} = \mu(1-\mu)^{i-6}.$$

If K denotes the additional number of days it takes to get enrolled in the marine science class after day 5, i.e. $K = M - 5$, then conditioned on $M > 5$, the random variable K has the geometric distribution with parameter μ. Note that this is the same as the distribution of M. This is known as the memoryless property of geometric distribution.

(c) We have $C - M \sim \text{Geometric}(\lambda)$. Thus $\mathbb{E}[M] = 1/\mu$ and $\mathbb{E}[C - M] = 1/\lambda$. And hence $\mathbb{E}[C] = \mathbb{E}[M] + \mathbb{E}[C - M] = 1/\mu + 1/\lambda$.

(d) $M \sim \text{Geometric}(\mu)$, $C \sim \text{Geometric}(\lambda)$. Yes they are independent.

(e) We have $X = \min\{M, C\}$ and $Y = \max\{M, C\}$. We also use the following definition of the minimum:

$$\min(m, c) = \begin{cases} m & \text{if } m \leq c; \\ c & \text{if } m > c. \end{cases}$$

Now, for all $k \in \{1, 2, \ldots\}$, $\min(M, C) = k$ is equivalent to $(M = k) \cap (C \geq k)$ or $(C = k) \cap (M > k)$. Hence,

$$\begin{aligned} \mathbb{P}[X = k] &= \mathbb{P}[\min(M, C) = k] \\ &= \mathbb{P}[(M = k) \cap (C \geq k)] + \mathbb{P}[(C = k) \cap (M > k)] \\ &= \mathbb{P}[M = k] \cdot \mathbb{P}[C \geq k] + \mathbb{P}[C = k] \cdot \mathbb{P}[M > k] \end{aligned}$$

(since M and C are independent)

$$= [(1 - \mu)^{k-1} \mu](1 - \lambda)^{k-1} + [(1 - \lambda)^{k-1} \lambda](1 - \mu)^k$$

(since M and C are geometric)

$$\begin{aligned} &= ((1 - \mu)(1 - \lambda))^{k-1} (\mu + \lambda(1 - \mu)) \\ &= (1 - \mu - \lambda + \lambda\mu)^{k-1} (\mu + \lambda - \mu\lambda). \end{aligned}$$

But this final expression is precisely the probability that a geometric r.v. with parameter $\mu + \lambda - \mu\lambda$ takes the value k. Hence $X \sim \text{Geom}(\mu + \lambda - \mu\lambda)$.

An alternative, slightly cleaner approach is to work with the *tail probabilities* of the geometric distribution, rather than with the usual point probabilities as above. In other words, we can work with $\mathbb{P}[X \geq k]$ rather than with $\mathbb{P}[X = k]$; clearly the values $\mathbb{P}[X \geq k]$ specify the values $\mathbb{P}[X = k]$ since $\mathbb{P}[X = k] = \mathbb{P}[X \geq k] - \mathbb{P}[X \geq (k+1)]$, so it suffices to calculate them instead. We then get the following argument:

$$\begin{aligned} \mathbb{P}[X \geq k] &= \mathbb{P}[\min(M, C) \geq k] \\ &= \mathbb{P}[(M \geq k) \cap (C \geq k)] \\ &= \mathbb{P}[M \geq k] \cdot \mathbb{P}[C \geq k] && \text{since } M, C \text{ are independent} \\ &= (1 - \mu)^{k-1}(1 - \lambda)^{k-1} && \text{since } M, C \text{ are geometric} \\ &= ((1 - \mu)(1 - \lambda))^{k-1} \\ &= (1 - \mu - \lambda + \mu\lambda)^{k-1}. \end{aligned}$$

This is the tail probability of a geometric distribution with parameter $\mu + \lambda - \mu\lambda$, so we are done.

(f) From part (e) we get $\mathbb{E}[X] = 1/(\mu + \lambda - \mu\lambda)$. From part (d) we have $\mathbb{E}[M] = 1/\mu$ and $\mathbb{E}[C] = 1/\lambda$. We now observe that $\min\{m,c\} + \max\{m,c\} = m + c$. Using linearity of expectation we get $\mathbb{E}[X] + \mathbb{E}[Y] = \mathbb{E}[M] + \mathbb{E}[C]$. Thus $\mathbb{E}[Y] = 1/\mu + 1/\lambda - 1/(\mu + \lambda - \mu\lambda)$.

(g) Let I_M and I_C be the indicator random variables of the events "$M \leq 14$" and "$C \leq 14$" respectively. Then $I_M + I_C$ is the number of classes she will be enrolled in within 14 days. Hence the answer is $\mathbb{E}[I_M] + \mathbb{E}[I_C] = \mathbb{P}[M \leq 14] + \mathbb{P}[C \leq 14] = 1 - (1-\mu)^{14} + 1 - (1-\lambda)^{14}$

2 Geometric and Poisson

Let $X \sim \text{Geo}(p)$ and $Y \sim \text{Poisson}(\lambda)$ be independent. random variables. Compute $\mathbb{P}[X > Y]$. Your final answer should not have summations.

Hint: Use the total probability rule.

Solution: We condition on Y so we can use the nice property of geometric random variables that $\mathbb{P}[X > k] = (1-p)^k$. This gives

$$\begin{aligned}
\mathbb{P}[X > Y] &= \sum_{y=0}^{\infty} \mathbb{P}[X > Y | Y = y] \cdot \mathbb{P}[Y = y] \\
&= \sum_{y=0}^{\infty} (1-p)^y \cdot \frac{e^{-\lambda}\lambda^y}{y!} \\
&= e^{-\lambda p} e^{\lambda p} \sum_{y=0}^{\infty} \frac{e^{-\lambda}(\lambda(1-p))^y}{y!} \\
&= e^{-\lambda p} \sum_{y=0}^{\infty} \frac{e^{-\lambda(1-p)}(\lambda(1-p))^y}{y!} \\
&= e^{-\lambda p}
\end{aligned}$$

To simplify the last summation, we observe that the sum could be interpreted as the sum of the probabilities for a Poisson($\lambda(1-p)$) random variable, which is equal to 1. Alternatively, you can use the Taylor series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ to simplify the sum.

3 Two Sides of a Coin

(a) Alice has 1 fair coin. She tosses the coin until she sees both sides. In expectation, how many tosses does this take?

(b) Bob has 2 fair coins. He tosses the first coin until he sees both sides of it, then tosses the second coin until he sees both sides of it. In expectation, how many total tosses does this take?

(c) Charlie has 2 fair coins. He repeatedly tosses the pair of coins simultaneously (i.e., two tosses at a time), until he has seen both sides of both coins. In expectation, how many total tosses does this take?

Solution:

(a) 3. After the first toss, the number of remaining tosses needed (to see the side opposite to the first toss) is distributed as $X \sim \text{Geometric}\left(\frac{1}{2}\right)$. This means that the expected number of tosses is $1 + \mathbb{E}[X] = 1 + \frac{1}{\left(\frac{1}{2}\right)} = 3$.

(b) This is doing the experiment from the above part twice (one after another), so we have by linearity $3 + 3 = 6$ tosses in expectation.

(c) Define a "round" to be a simultaneous toss of the pair of coins, and let X represent the number of rounds needed.

There are two possible solutions here; one with the maximum of geometric random variables, and another with the tail sum formula.

Solution 1 (Tail sum): We have

$$\mathbb{P}[X \geq 1] = 1$$

since a coin has two sides. Then, observe that for all $k \geq 2$,

$$\begin{aligned}\mathbb{P}[X \geq k] &= \frac{2}{2^{k-1}} + \frac{2}{2^{k-1}} - \frac{4}{4^{k-1}} \\ &= \frac{1}{2^{k-2}} + \frac{1}{2^{k-2}} - \frac{1}{4^{k-2}}\end{aligned}$$

since for there to be k or more rounds, at least one of the coins must have yielded the same result in all of the first $k-1$ rounds (i.e., all heads or all tails).

Thus, by tail sum

$$\begin{aligned}\mathbb{E}[X] &= \sum_{k=1}^{\infty} \mathbb{P}[X \geq k] \\ &= 1 + \sum_{k=2}^{\infty} \left(\frac{1}{2^{k-2}} + \frac{1}{2^{k-2}} - \frac{1}{4^{k-2}} \right) \\ &= 1 + 2\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k - \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \\ &= 1 + 2\left(\frac{1}{1-\frac{1}{2}}\right) - \frac{1}{1-\frac{1}{4}} \\ &= \frac{11}{3}\end{aligned}$$

The total number of tosses $T = 2X$, so $\mathbb{E}[T] = 2\,\mathbb{E}[X] = \frac{22}{3}$.

Solution 2 (Maximum of Geometric RVs): We are guaranteed at least one round, so let us consider the rounds *after* the first.

Let X_1 be the number of tosses (after the first toss) before we see both sides of the first coin, regardless of the second coin, and let X_2 be the number of tosses (after the first toss) before we see both sides of the second coin, regardless of the first coin. Observe that $X = 1 + \max(X_1, X_2)$; we keep flipping both coins until we've seen both sides of *both* coins.

This means that we want to calculate $\mathbb{E}[X] = 1 + \mathbb{E}[\max(X_1, X_2)]$. To do this, we can express

$$\max(X_1, X_2) = X_1 + X_2 - \min(X_1, X_2).$$

This is because $\max(X_1, X_2) + \min(X_1, X_2) = X_1 + X_2$, as one RV must be the maximum and the other must be the minimum, and we can rearrange to get the above equation.

By linearity, we know $\mathbb{E}[\max(X_1, X_2)] = \mathbb{E}[X_1] + \mathbb{E}[X_2] - \mathbb{E}[\min(X_1, X_2)] = 2 + 2 - \mathbb{E}[\min(X_1, X_2)]$. Further, we can see that $\min(X_1, X_2) \sim \text{Geometric}(\frac{3}{4})$:

$$\begin{aligned}
\mathbb{P}[\min(X_1, X_2) \geq k] &= \mathbb{P}[X_1 \geq k, X_2 \geq k] \\
&= \mathbb{P}[X_1 \geq k]\mathbb{P}[X_2 \geq k] \\
&= \left(\frac{1}{2}\right)^{k-1} \cdot \left(\frac{1}{2}\right)^{k-1} \\
&= \left(\frac{1}{4}\right)^{k-1}
\end{aligned}$$

This is also equal to $\mathbb{P}[Y \geq k]$ for $Y \sim \text{Geometric}(\frac{3}{4})$.

This means that the expectation is $\mathbb{E}[\max(X_1, X_2)] = 4 - \frac{4}{3} = \frac{8}{3}$, and $\mathbb{E}[X] = 1 + \frac{8}{3} = \frac{11}{3}$. The total number of tosses is just twice this value, or $\frac{22}{3}$.

4 Swaps and Cycles

We'll say that a permutation $\pi = (\pi(1), \ldots, \pi(n))$ contains a *swap* if there exist $i, j \in \{1, \ldots, n\}$ so that $\pi(i) = j$ and $\pi(j) = i$, where $i \neq j$.

(a) What is the expected number of swaps in a random permutation?

(b) What about the variance?

(c) In the same spirit as above, we'll say that π contains a *s-cycle* if there exist $i_1, \ldots, i_s \in \{1, \ldots, n\}$ with $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_s) = i_1$. Compute the expectation of the number of s-cycles.

Solution:

(a) As a warm-up, let's compute the probability that 1 and 2 are swapped. There are $n!$ possible permutations, and $(n-2)!$ of them have $\pi(1) = 2$ and $\pi(2) = 1$. This means

$$\mathbb{P}[(1,2) \text{ are a swap}] = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

There was nothing special about 1 and 2 in this calculation, so for any $\{i,j\} \subset \{1,\ldots,n\}$, the probability that i and j are swapped is the same as above. Let's write $I_{i,j}$ for the indicator that i and j are swapped, and N for the total number of swaps, so that

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{\{i,j\}\subset\{1,\ldots,n\}} I_{i,j}\right] = \sum_{\{i,j\}\subset\{1,\ldots n\}} \mathbb{P}[(i,j) \text{ are swapped}] = \frac{1}{n(n-1)}\binom{n}{2} = \frac{1}{2}.$$

(b) For the variance, when we expand N^2 as a sum over pairs $\{i,j\},\{k,l\}$, we'll need to know $\mathbb{E}[I_{i,j}I_{k,l}]$, which is just the probability that i and j are swapped *and* k and l are swapped as well. There are three cases to consider. If $\{i,j\} = \{k,l\}$, then the probability is exactly what we computed above. If the two pairs share one element, then it is impossible that they are both swaps. If the pairs are disjoint, then of the $n!$ possible permutations, $(n-4)!$ include the two swaps we are concerned with. Thus

$$\begin{aligned}
\mathbb{E}[N^2] &= \sum_{\{i,j\}\subset\{1,\ldots,n\}} \mathbb{E}[I_{i,j}^2] + \sum_{\{i,j\}\cap\{k,l\}=\emptyset} \mathbb{E}[I_{i,j}I_{k,l}] \\
&= \binom{n}{2}\frac{1}{n(n-1)} + \binom{n}{2}\binom{n-2}{2}\frac{1}{n(n-1)(n-2)(n-3)} \\
&= \frac{1}{2} + \frac{n(n-1)}{2}\frac{(n-2)(n-3)}{2}\frac{1}{n(n-1)(n-2)(n-3)} = \frac{1}{2}+\frac{1}{4},
\end{aligned}$$

and $\mathrm{Var}(N^2) = \mathbb{E}[N^2] - \mathbb{E}[N]^2 = \frac{1}{2}+\frac{1}{4}-\frac{1}{4} = \frac{1}{2}$.

(c) The idea here is quite similar to the above, so we'll be a little less verbose in the exposition. However, as a first aside we need the notion of a *cyclic ordering* of s elements from a set $\{1,\ldots n\}$. We mean by this a labelling of the s beads of a necklace with elements of the set, where we say that labelings of the beads are the same if we can move them along the string to turn one into the other. For example, $(1,2,3,4)$ and $(1,2,4,3)$ are different cyclic orderings, but $(1,2,3,4)$ and $(2,3,4,1)$ are the same. There are

$$\binom{n}{s}\frac{s!}{s} = \frac{n!}{(n-s)!}\frac{1}{s}$$

possible cyclic orderings of length s from a set with n elements, since if we first count all subsets of size s, and then all permutations of each of those subsets, we have overcounted by a factor of s.

Now, let N be a random variable counting the number of s-cycles, and for each cyclic ordering $(i_1,\ldots,i_s)$ of s elements of $\{1,\ldots,n\}$, let $I_{(i_1,\ldots,i_s)}$ be the indicator that $\pi(i_1) = i_2, \pi(i_2) = i_3, \ldots, \pi(i_s) = i_1$. There are $(n-s)!$ permutations in which $(i_1,\ldots,i_s)$ form an s-cycle (since we are free to do whatever we want to the remaining $(n-s)$ elements of $\{1,\ldots,n\}$), so the probability that $(i_1,\ldots,i_s)$ are such a cycle is $\frac{(n-s)!}{n!}$, and

$$\mathbb{E}[N] = \mathbb{E}\left[\sum_{(i_1,\ldots,i_s)\text{ cyclic ordering}} I_{(i_1,\ldots,i_s)}\right] = \frac{n!}{(n-s)!}\frac{1}{s}\frac{(n-s)!}{n!} = \frac{1}{s}.$$

5 Will I Get My Package?

A delivery guy in some company is out delivering n packages to n customers, where n is a natural number greater than 1. Not only does he hand each customer a package uniformly at random from the remaining packages, he opens the package before delivering it with probability $1/2$. Let X be the number of customers who receive their own packages unopened.

(a) Compute the expectation $\mathbb{E}[X]$.

(b) Compute the variance $\text{Var}(X)$.

Solution:

(a) Define

$$X_i = \begin{cases} 1, & \text{if the } i\text{-th customer gets his/her package unopened,} \\ 0, & \text{otherwise.} \end{cases}$$

By linearity of expectation, $\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} \mathbb{E}[X_i]$. We have

$$\mathbb{E}[X_i] = \mathbb{P}[X_i = 1] = \frac{1}{2n},$$

since the ith customer will get his/her own package with probability $1/n$ and it will be unopened with probability $1/2$ and the delivery guy opens the packages independently.
Hence,

$$\mathbb{E}[X] = n \cdot \frac{1}{2n} = \boxed{\frac{1}{2}}.$$

(b) To calculate $\text{Var}(X)$, we need to know $\mathbb{E}[X^2]$.
By linearity of expectation,

$$\mathbb{E}[X^2] = \mathbb{E}\left[(X_1 + X_2 + \cdots + X_n)^2\right] = \mathbb{E}\left[\sum_{i,j} X_i X_j\right] = \sum_{i,j} \mathbb{E}[X_i X_j].$$

Then we consider two cases, either $i = j$ or $i \neq j$.
Hence $\sum_{i,j} \mathbb{E}[X_i X_j] = \sum_{i} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j]$.

$$\mathbb{E}[X_i^2] = \mathbb{E}[X_i] = \frac{1}{2n}$$

for all i. To find $\mathbb{E}[X_i X_j]$, we need to calculate $\mathbb{P}[X_i X_j = 1]$:

$$\mathbb{P}[X_i X_j = 1] = \mathbb{P}[X_i = 1]\mathbb{P}[X_j = 1 \mid X_i = 1] = \frac{1}{2n} \cdot \frac{1}{2(n-1)}$$

since if customer i has received his/her own package, customer j has $n-1$ choices left.

Hence,

$$\mathbb{E}[X^2] = n \cdot \frac{1}{2n} + n \cdot (n-1) \cdot \frac{1}{2n} \cdot \frac{1}{2(n-1)} = \frac{3}{4},$$
$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{3}{4} - \frac{1}{4} = \boxed{\frac{1}{2}}.$$

6 Diversify Your Hand

You are dealt 5 cards from a standard 52 card deck. Let X be the number of distinct values in your hand. For instance, the hand (A, A, A, 2, 3) has 3 distinct values.

(a) Calculate $\mathbb{E}[X]$.

(b) Calculate $\mathrm{Var}(X)$.

Solution:

(a) Let X_i be the indicator of the ith value appearing in your hand. Then, $X = X_1 + X_2 + \ldots + X_{13}$. (Here we let 13 correspond to K, 12 correspond to Q, and 11 correspond to J.) By linearity of expectation, $\mathbb{E}[X] = \sum_{i=1}^{13} \mathbb{E}[X_i]$.

We can calculate $\mathbb{P}[X_i = 1]$ by taking the complement, $1 - \mathbb{P}[X_i = 0]$, or 1 minus the probability that the card does not appear in your hand. This is $1 - \frac{\binom{48}{5}}{\binom{52}{5}}$.

Then, $\mathbb{E}[X] = 13\mathbb{P}[X_1 = 1] = 13\left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right)$.

(b) To calculate variance, since the indicators are not independent, we have to use the formula $\mathbb{E}[X^2] = \sum_{i=j} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i X_j]$.

First, we have

$$\sum_{i=j} \mathbb{E}[X_i^2] = \sum_{i=j} \mathbb{E}[X_i] = 13\left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right).$$

Next, we tackle $\sum_{i \neq j} \mathbb{E}[X_i X_j]$. Note that $\mathbb{E}[X_i X_j] = \mathbb{P}[X_i X_j = 1]$, as $X_i X_j$ is either 0 or 1.

To calculate $\mathbb{P}[X_i X_j = 1]$, we note that $\mathbb{P}[X_i X_j = 1] = 1 - \mathbb{P}[X_i = 0] - \mathbb{P}[X_j = 0] + \mathbb{P}[X_i = 0, X_j = 0]$. Then

$$\begin{aligned}
\sum_{i \neq j} \mathbb{E}[X_i X_j] &= 13 \cdot 12 \mathbb{P}[X_i X_j = 1] \\
&= 13 \cdot 12 (1 - \mathbb{P}[X_i = 0] - \mathbb{P}[X_j = 0] + \mathbb{P}[X_i = 0, X_j = 0]) \\
&= 156\left(1 - 2\frac{\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{44}{5}}{\binom{52}{5}}\right)
\end{aligned}$$

Putting it all together, we have

$$\begin{aligned}
\operatorname{Var}(X) &= \mathbb{E}[X^2] - \mathbb{E}[X]^2 \\
&= 13\left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right) + 156\left(1 - 2\frac{\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{44}{5}}{\binom{52}{5}}\right) - \left(13\left(1 - \frac{\binom{48}{5}}{\binom{52}{5}}\right)\right)^2.
\end{aligned}$$

7 Double-Check Your Intuition Again

(a) You roll a fair six-sided die and record the result X. You roll the die again and record the result Y.

  (i) What is $\operatorname{cov}(X+Y, X-Y)$?

  (ii) Prove that $X+Y$ and $X-Y$ are not independent.

For each of the problems below, if you think the answer is "yes" then provide a proof. If you think the answer is "no", then provide a counterexample.

(b) If X is a random variable and $\operatorname{Var}(X) = 0$, then must X be a constant?

(c) If X is a random variable and c is a constant, then is $\operatorname{Var}(cX) = c\operatorname{Var}(X)$?

(d) If A and B are random variables with nonzero standard deviations and $\operatorname{Corr}(A,B) = 0$, then are A and B independent?

(e) If X and Y are not necessarily independent random variables, but $\operatorname{Corr}(X,Y) = 0$, and X and Y have nonzero standard deviations, then is $\operatorname{Var}(X+Y) = \operatorname{Var}(X) + \operatorname{Var}(Y)$?

(f) If X and Y are random variables then is $\mathbb{E}[\max(X,Y)\min(X,Y)] = \mathbb{E}[XY]$?

(g) If X and Y are independent random variables with nonzero standard deviations, then is

$$\operatorname{Corr}(\max(X,Y), \min(X,Y)) = \operatorname{Corr}(X,Y)?$$

Solution:

(a) (i) $\operatorname{cov}(X+Y, X-Y) = \operatorname{cov}(X,X) + \operatorname{cov}(X,Y) - \operatorname{cov}(Y,X) - \operatorname{cov}(Y,Y) = \operatorname{cov}(X,X) - \operatorname{cov}(Y,Y) = 0$.

  (ii) Observe that $\mathbb{P}[X+Y=7, X-Y=0] = 0$ because if $X-Y=0$, then the sum of our two dice rolls must be even. However, both $\mathbb{P}[X+Y=7]$ and $\mathbb{P}[X-Y=0]$ are nonzero, so $\mathbb{P}[X+Y=7, X-Y=0] \neq \mathbb{P}[X+Y=7] \cdot \mathbb{P}[X-Y=0]$.

(b) Yes. If we write $\mu = \mathbb{E}[X]$, then $0 = \operatorname{Var}(X) = \mathbb{E}[(X-\mu)^2]$ so $(X-\mu)^2$ must be identically 0 since perfect squares are non-negative. Thus $X = \mu$.

(c) No. We have $\operatorname{Var}(cX) = \mathbb{E}[(cX - \mathbb{E}[cX])^2] = c^2\,\mathbb{E}[(X - \mathbb{E}[X])^2] = c^2\operatorname{Var}(X)$ so if $\operatorname{Var}(X) \neq 0$ and $c \neq 0$ or $c \neq 1$ then $\operatorname{Var}(cX) \neq c\operatorname{Var}(X)$. This does prove that $\sigma(cX) = c\sigma(X)$ though.

(d) No. Let $A = X+Y$ and $B = X-Y$ from part (a). Since A and B are not constants then part (b) says they must have nonzero variances which means they also have nonzero standard deviations. Part (a) says that their covariance is 0 which means they are uncorrelated, and that they are not independent.

Recall from lecture that the converse is true though.

(e) Yes. If $\text{Corr}(X,Y) = 0$, then $\text{cov}(X,Y) = 0$. We have $\text{Var}(X+Y) = \text{cov}(X+Y, X+Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{cov}(X,Y) = \text{Var}(X) + \text{Var}(Y)$.

(f) Yes. For any values x, y we have $\max(x,y)\min(x,y) = xy$. Thus, $\mathbb{E}[\max(X,Y)\min(X,Y)] = \mathbb{E}[XY]$.

(g) No. You may be tempted to think that because $(\max(x,y), \min(x,y))$ is either (x,y) or (y,x), then $\text{Corr}(\max(X,Y), \min(X,Y)) = \text{Corr}(X,Y)$ because $\text{Corr}(X,Y) = \text{Corr}(Y,X)$. That reasoning is flawed because $(\max(X,Y), \min(X,Y))$ is not always equal to (X,Y) or always equal to (Y,X) and the inconsistency affects the correlation. It is possible for X and Y to be independent while $\max(X,Y)$ and $\min(X,Y)$ are not.

For a concrete example, suppose X is either 0 or 1 with probability $1/2$ each and Y is independently drawn from the same distribution. Then $\text{Corr}(X,Y) = 0$ because X and Y are independent. Even though X never gives information about Y, if you know $\max(X,Y) = 0$ then you know for sure $\min(X,Y) = 0$.

More formally, $\max(X,Y) = 1$ with probability $3/4$ and 0 with probability $1/4$, and $\min(X,Y) = 1$ with probability $1/4$ and 0 with probability $3/4$. This means

$$\mathbb{E}[\max(X,Y)] = 1 * 3/4 + 0 * 1/4 = 3/4$$

and

$$\mathbb{E}[\min(X,Y)] = 1 * 1/4 + 0 * 3/4 = 1/4.$$

Thus,

$$\text{cov}(\max(X,Y), \min(X,Y)) = \mathbb{E}[\max(X,Y)\min(X,Y)] - 3/16 = 1/4 - 3/16 = 1/16 \neq 0.$$

We conclude that $\text{Corr}(\max(X,Y), \min(X,Y)) \neq 0 = \text{Corr}(X,Y)$.