$Pr[A \cap B] = Pr[A|B]Pr[B] = Pr[B|A]Pr[A]$.

Bayes Rule: $Pr[A|B] = \frac{Pr[B|A]Pr[A]}{Pr[B]}$.

Is you coin loaded?

Your coin is fair w.p. 1/2 or such that $Pr[H] = 0.6$, otherwise.

You flip your coin and it yields heads.

What is the probability that it is fair?

Analysis:

We want to calculate $P[A|B]$.

We know $P[B|A] = 1/2, P[B|\bar{A}] = 0.6, Pr[A] = 1/2 = Pr[\bar{A}]$

Now,

Thus,

Independence

Definition: Two events A and B are independent if

Examples:

When rolling two dice, A = sum is 7 and B = red die is 1 are independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{1}{6}\right)\left(\frac{1}{6}\right)$.

When rolling two dice, A = sum is 3 and B = red die is 1 are blue not independent; $Pr[A \cap B] = \frac{1}{36}$, $Pr[A]Pr[B] = \left(\frac{2}{36}\right) \big(\frac{1}{6}$

When flipping coins, A = coin 1 yields heads and B = coin 2 yields tails are independent; $Pr[A \cap B] = \frac{1}{4}$, $Pr[A]Pr[B] = \big(\frac{1}{2}$

When throwing 3 balls into 3 bins, A = bin 1 is empty and B = bin 2 is empty are blue not independent; $Pr[A \cap B] = \frac{1}{27}$, $Pr[A]Pr[B] = \left(\frac{8}{27}\right)\left(\frac{8}{27}\right)$.

Independence and conditional probability

Fact: Two events A and B are independent if and only if

Indeed: $Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$, so that

Conditional Probability: Review

Recall:

$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$.

Hence, $Pr[A \cap B] = Pr[B]Pr[A|B] = Pr[A]Pr[B|A]$.

A and B are *positively correlated* if $Pr[A|B] > Pr[A]$,

i.e., if $Pr[A \cap B] > Pr[A]Pr[B]$.

A and B are *negatively correlated* if $Pr[A|B] < Pr[A]$,

i.e., if $Pr[A \cap B] < Pr[A]Pr[B]$.

A and B are *independent* if $Pr[A|B] = Pr[A]$,

i.e., if $Pr[A \cap B] = Pr[A]Pr[B]$.

Note: $B \subset A \Rightarrow A$ and B are positively correlated. ($Pr[A|B] = 1 > Pr[A]$)

Note: $A \cap B = \emptyset \Rightarrow A$ and B are negatively correlated. ($Pr[A|B] = 0 < Pr[A]$)

Conditional Probability: Pictures/Poll.

Illustrations: Pick a point uniformly in the unit square

Which A and B are independent?

(A) Left.
(B) Middle.
(B) Right.

See next slide.

Conditional Probability: Pictures

Illustrations: Pick a point uniformly in the unit square

Left: A and B are independent. $Pr[B] = b; Pr[B|A] = b$.

Middle: A and B are positively correlated. $Pr[B|A] = b_1 > Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_2, b_1)$.

Right: A and B are negatively correlated. $Pr[B|A] = b_1 < Pr[B|\bar{A}] = b_2$. Note: $Pr[B] \in (b_1, b_2)$.

Bayes and Biased Coin