Today.

Quick review.

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Finish Graphs (maybe.)

Proof of “handshake” lemma.

Lemma: The sum of degrees is $2|E|$, for a graph $G = (V, E)$.
What’s true?

(A) The number of edge-vertex incidences for an edge e is 2.
(B) The total number of edge-vertex incidences is $|V|$.
(C) The total number of edge-vertex incidences is $2|E|$.
(D) The number of edge-vertex incidences for a vertex v is its degree.
(E) The sum of degrees is $2|E|$.
(F) Total number of edge-vertex incidences is sum of vertex degrees.

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(E) The sum of degrees is $2|E|$.
(F) Total number of edge-vertex incidences is sum of vertex degrees.

(B) is false. The others are statements in the proof.

Poll: Euler concepts.

A graph is Euleurian if it is connected and has even degree.

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(A) There is no Hotel California in this graph.
(B) Walking on unused edges, starting at v, eventually “stuck” at v.
(C) Removing a tour leaves a graph of even degree.
(D) Removing a tour leaves a connected graph.
(E) Remove set of edges E' in connected graph, connected component is incident to edge in E'
(F) A tour connecting a set of connected components, each with a Eulerian tour is really cool! This implies the graph is Eulerian.

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Only (C) is false. The rest are steps in the proof.

Lecture 6.

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Euler's Formula.

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Planar Six and then Five Color theorem.

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Types of graphs.

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- Complete Graphs.
- Trees (a little more.)
- Hypercubes.

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A graph that can be drawn in the plane without edge crossings.

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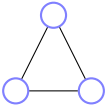

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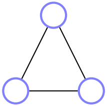

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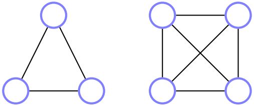

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Four node complete?

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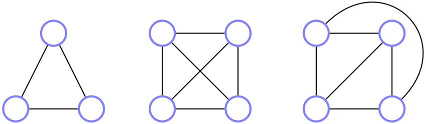

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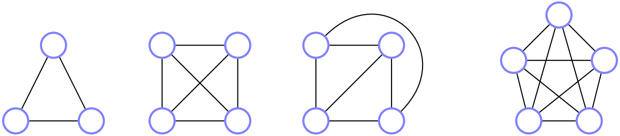

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Five node complete or K_5 ?

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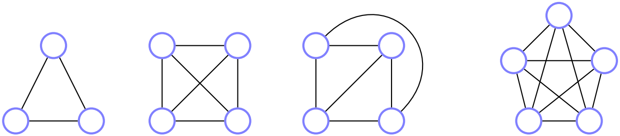

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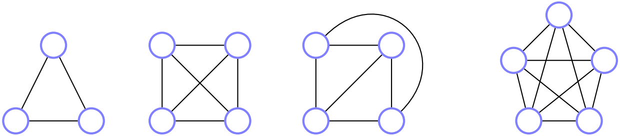

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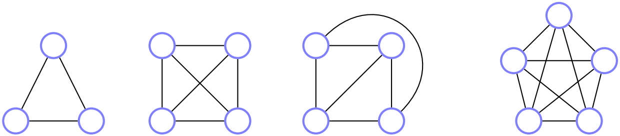

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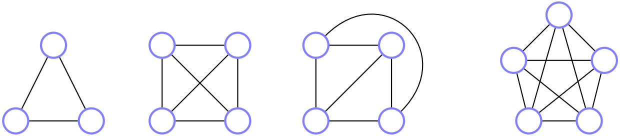

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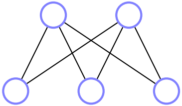

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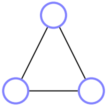 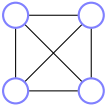 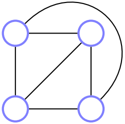 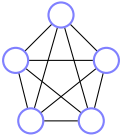

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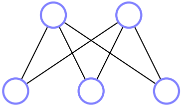

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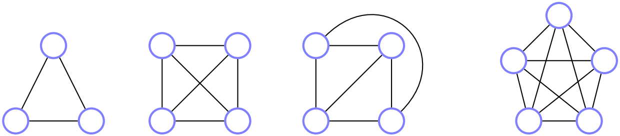

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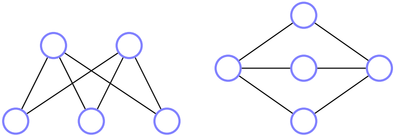

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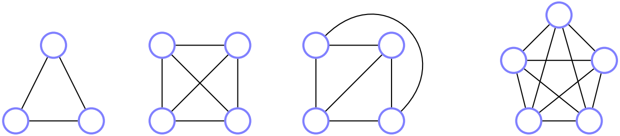

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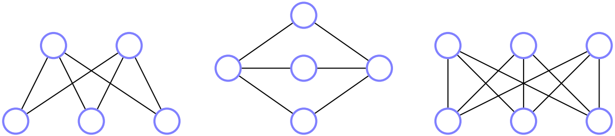

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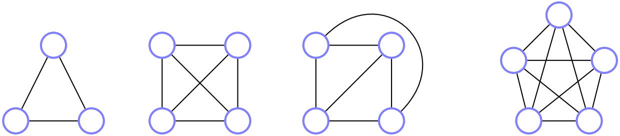

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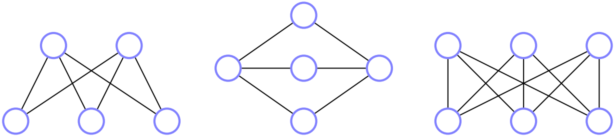

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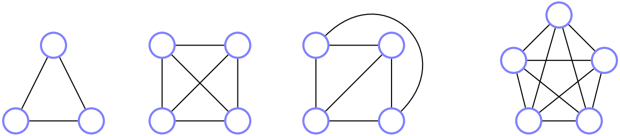

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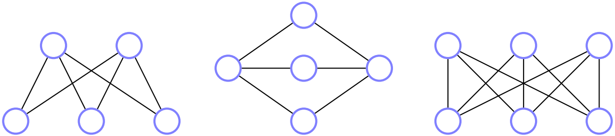

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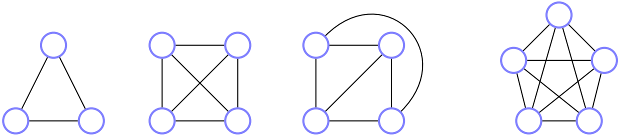

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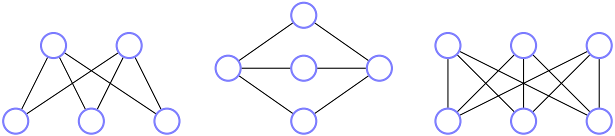

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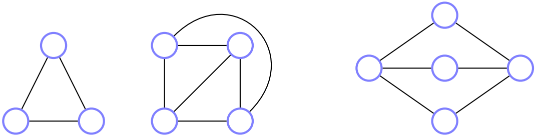

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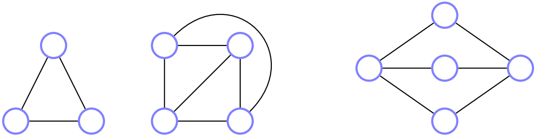

Faces: connected regions of the plane.

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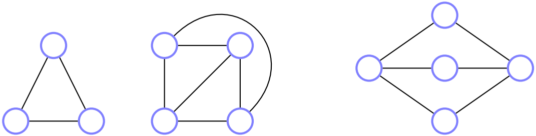

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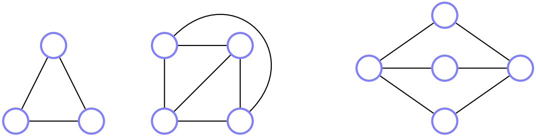

Faces: connected regions of the plane.

How many faces for
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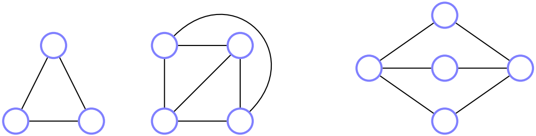

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How many faces for

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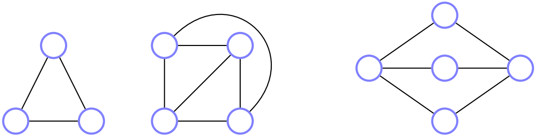

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How many faces for

triangle? 2

complete on four vertices or K_4?

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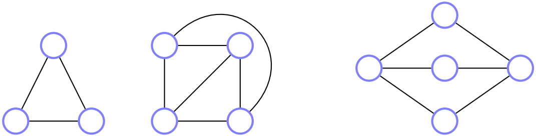

Faces: connected regions of the plane.

How many faces for

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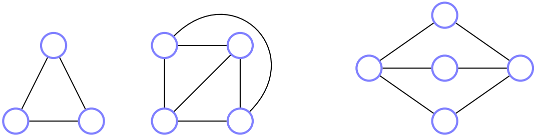

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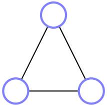

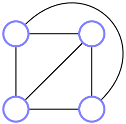

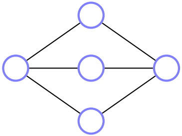

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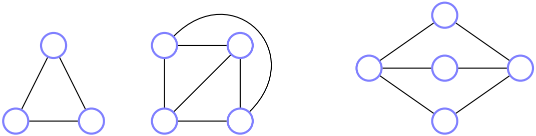

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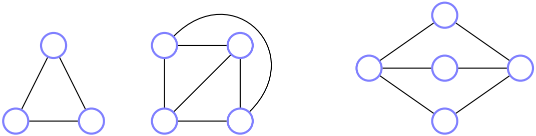

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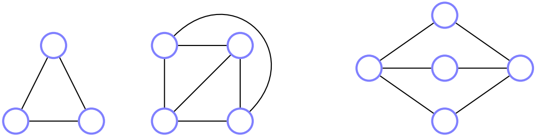

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Euler's Formula: Connected planar graph has $v + f = e + 2$.

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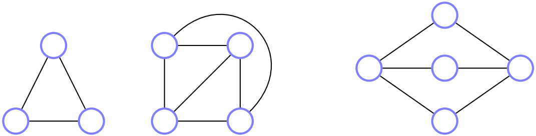

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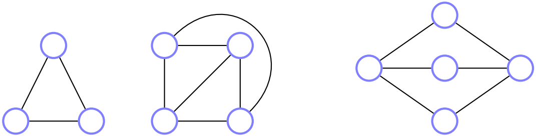

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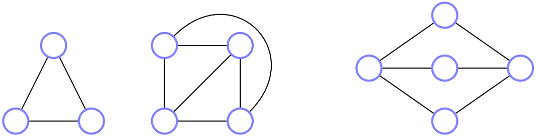

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Triangle: $3+2=3+2$!

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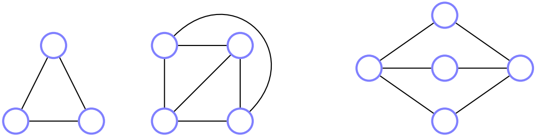

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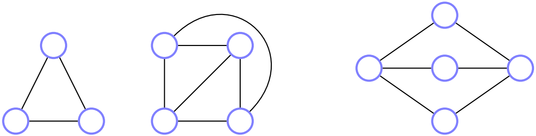

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K_4: $4+4=6+2$!

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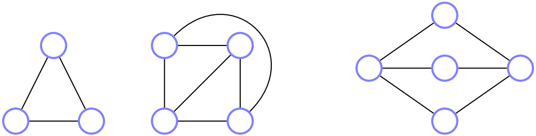

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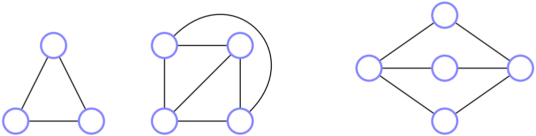

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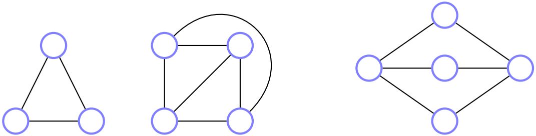

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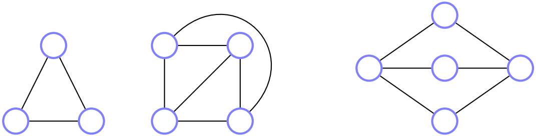

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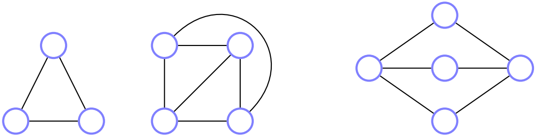

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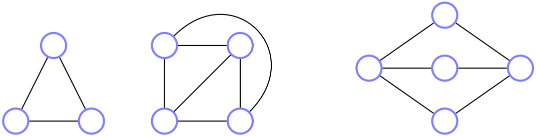

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Euler and Polyhedron.

Greeks knew formula for polyhedron.

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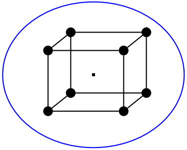

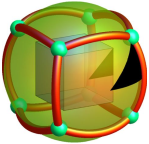

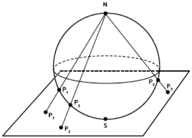

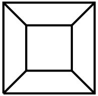

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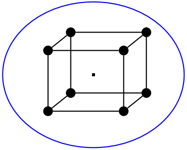

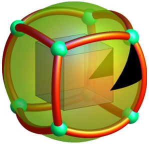

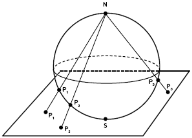

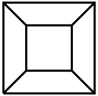

Faces?

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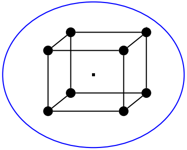

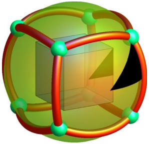

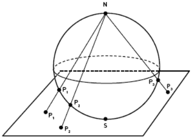

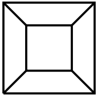

Faces? 6. Edges?

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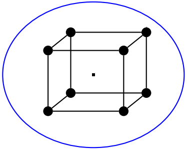

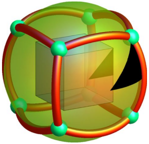

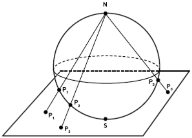

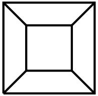

Faces? 6. Edges? 12.

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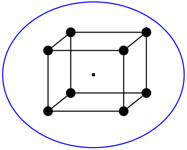

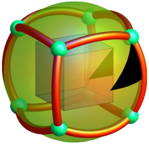

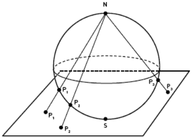

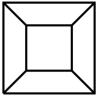

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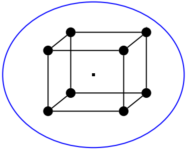

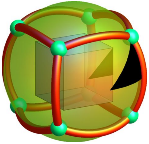

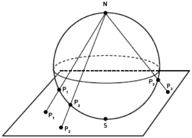

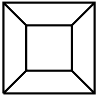

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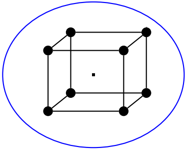

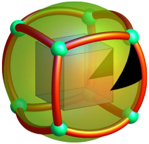

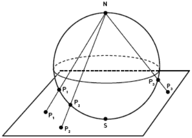

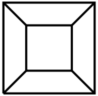

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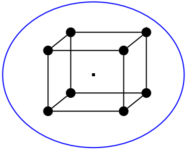
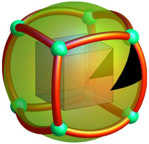
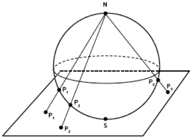

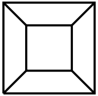

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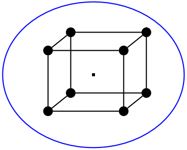
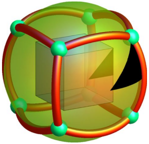
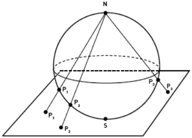

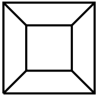

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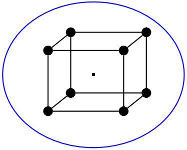 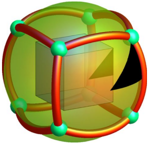 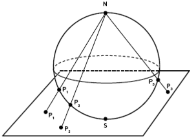

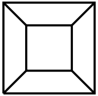

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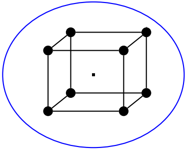
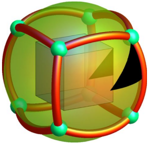
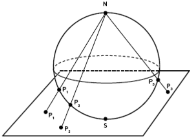

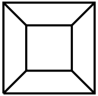

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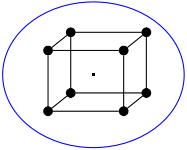
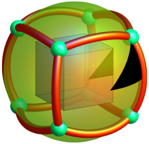
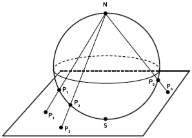

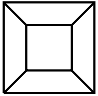

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Greeks couldn't prove it. Induction? Remove vertice for polyhedron?

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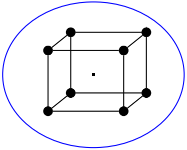
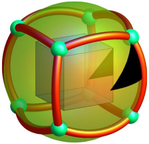
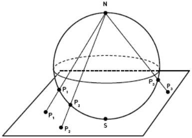

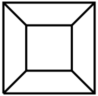

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Polyhedron without holes

Euler and Polyhedron.

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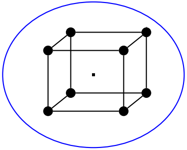
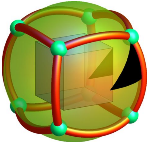
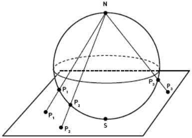

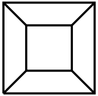

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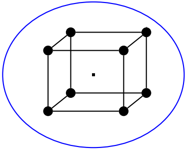
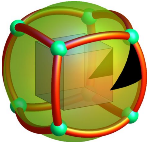
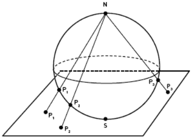

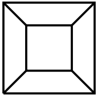

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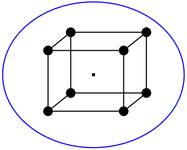 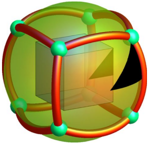 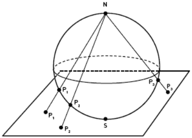

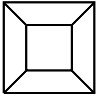

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For Convex Polyhedron:

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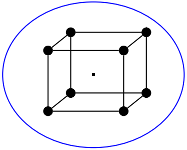
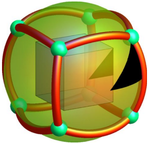
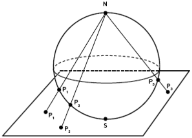

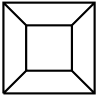

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For Convex Polyhedron:

Surround by sphere.

Euler and Polyhedron.

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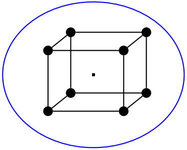 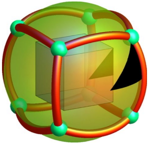 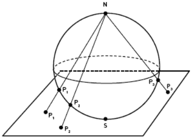 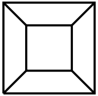

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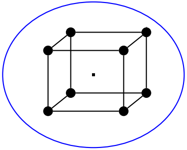 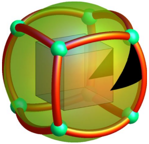 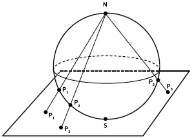 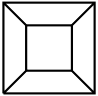

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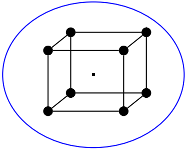
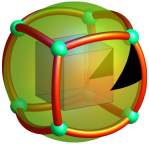
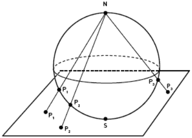

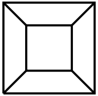

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Project Sphere-N

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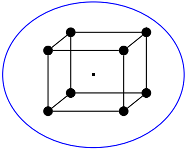
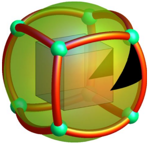
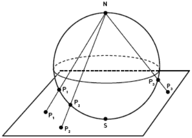

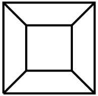

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Project Sphere-N onto Plane:

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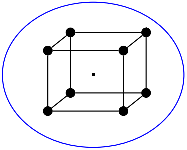
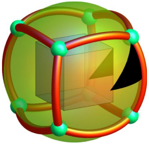
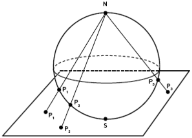
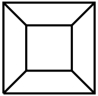

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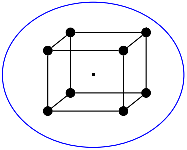
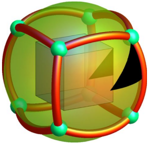
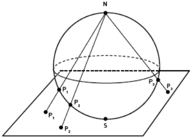

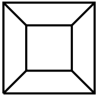

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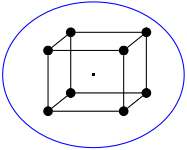

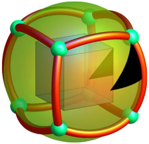

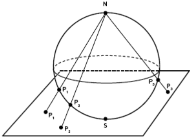

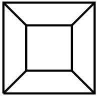

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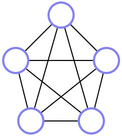
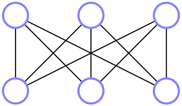
Project Sphere-N onto Plane: drawing on plane.

Euler proved formula thousands of years later!

Euler and non-planarity of K_5 and $K_{3,3}$





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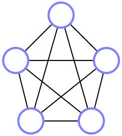

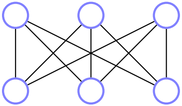

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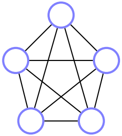

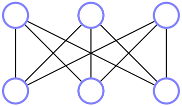

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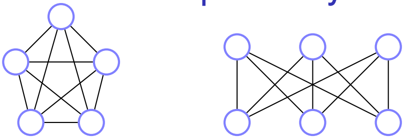

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Consider Face edge Adjacencies with multiplicities

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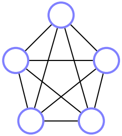

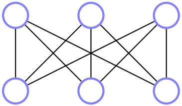

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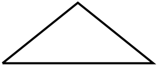

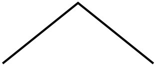

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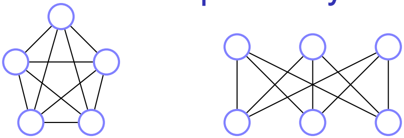

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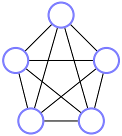
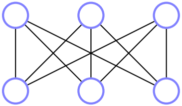
Consider Face edge Adjacencies with multiplicities

Each face is adjacent to at least three edges($v > 2$).

Euler and non-planarity of K_5 and $K_{3,3}$





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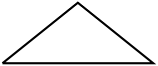

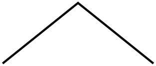

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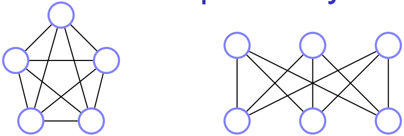
Euler and non-planarity of K_5 and $K_{3,3}$



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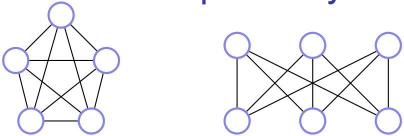
Each edge is adjacent to two faces.

Euler and non-planarity of K_5 and $K_{3,3}$



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Consider Face edge Adjacencies with multiplicities

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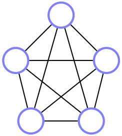
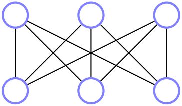
$=2e$ face-edge adjacencies.

Euler and non-planarity of K_5 and $K_{3,3}$





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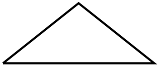
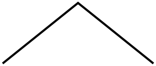
Consider Face edge Adjacencies with multiplicities





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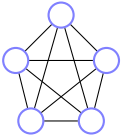

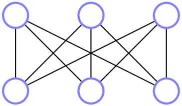

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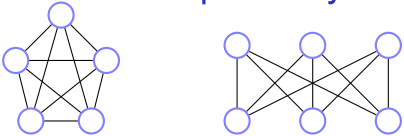

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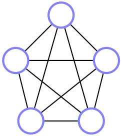

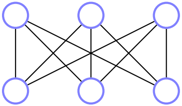

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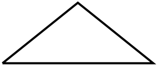

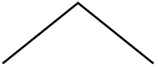

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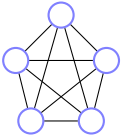

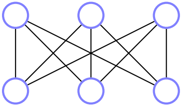

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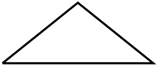

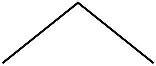

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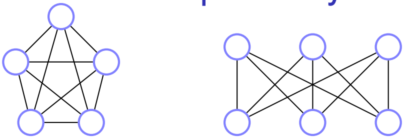

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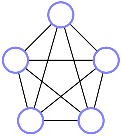
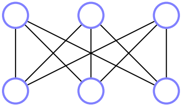

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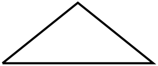
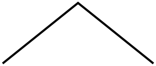

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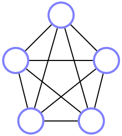
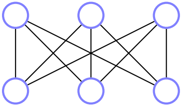
Plug into Euler:

Euler and non-planarity of K_5 and $K_{3,3}$





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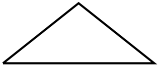

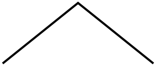

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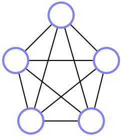
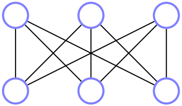
Euler and non-planarity of K_5 and $K_{3,3}$





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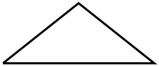

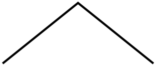

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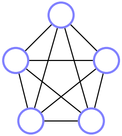
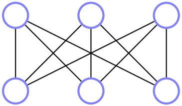
Euler and non-planarity of K_5 and $K_{3,3}$





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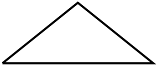

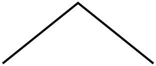

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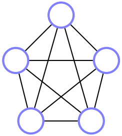
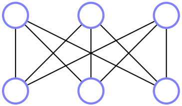
K_5

Euler and non-planarity of K_5 and $K_{3,3}$





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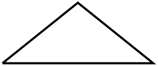

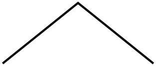

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K_5 Edges?

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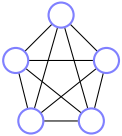

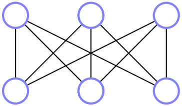

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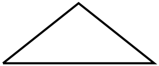

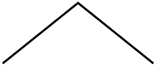

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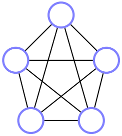
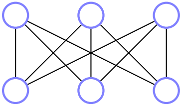
Euler and non-planarity of K_5 and $K_{3,3}$





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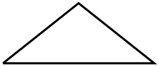

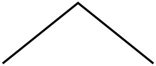

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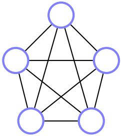

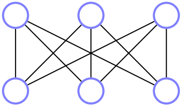

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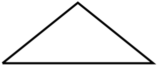

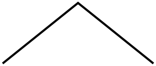

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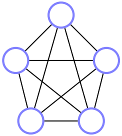
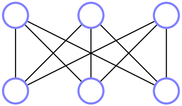
Euler and non-planarity of K_5 and $K_{3,3}$

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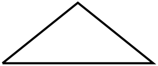
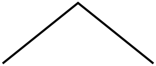

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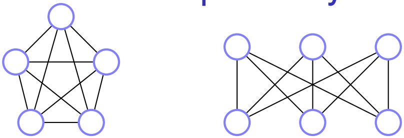
Euler and non-planarity of K_5 and $K_{3,3}$



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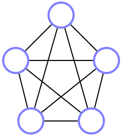
We consider simple graphs where $v \geq 3$.
Consider Face edge Adjacencies with multiplicities

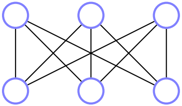
Each face is adjacent to at least three edges($v>2$).
$\geq 3f$ face-edge adjacencies.
Each edge is adjacent to two faces.
$=2e$ face-edge adjacencies.
$\Longrightarrow$ $3f \leq 2e$ for any planar graph with $v>2$. Or $f \leq \frac{2}{3}e$.

Plug into Euler: $v+\frac{2}{3}e \geq e+2 \implies e \leq 3v-6$

K_5 Edges? $e=4+3+2+1=10$. Vertices? $v=5$.
$10 \not\leq 3(5)-6=9$.

Euler and non-planarity of K_5 and $K_{3,3}$





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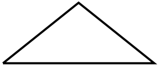

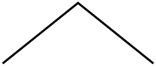

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Plug into Euler: $v + \frac{2}{3}e \geq e + 2 \implies e \leq 3v - 6$

K_5 Edges? $e = 4 + 3 + 2 + 1 = 10$. Vertices? $v = 5$.

$10 \not\leq 3(5) - 6 = 9$. $\implies K_5$ is not planar.

Planar $\implies e \leq 3v - 6$. Flow Poll.

Euler's formula: $v + f = e + 2$

Consider graph with > 2 vertices. Understand the following.

(A) Every face is incident to ≥ 3 edges.

(B) Face-edge incidences $\geq 3f$

(C) Every edge is incident (with multiplicity) to 2 faces.

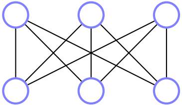
(D) Face edge incidences $= 2e$

(E) $3f \leq$ Face-ege-incidence $= 2e$

(F) $3(e + 2 - v) <= 2e$

Conclusion: $e <= 3v - 6$

Proving non-planarity for $K_{3,3}$



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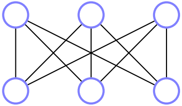

$K_{3,3}$?

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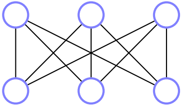

$K_{3,3}$? Edges?

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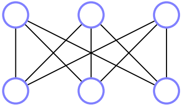

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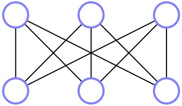

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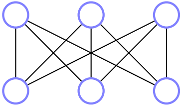

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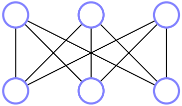
$e \leq 3(v) - 6$ for planar graphs.

Proving non-planarity for $K_{3,3}$



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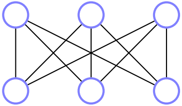

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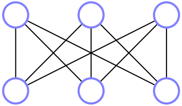

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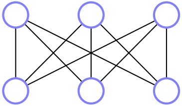
Step in proof of K_5: faces are adjacent to ≥ 3 edges.

Proving non-planarity for $K_{3,3}$



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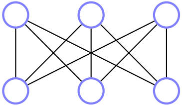
For $K_{3,3}$ every cycle is of even length or incident ≥ 4 faces.

Proving non-planarity for $K_{3,3}$



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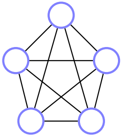
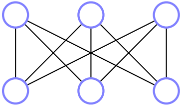
Finish in homework!

Planarity and Euler





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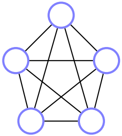
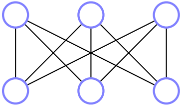
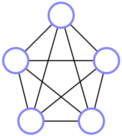
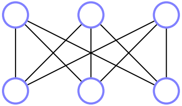




These graphs **cannot** be drawn in the plane without edge crossings.

Planarity and Euler





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Euler's Formula: $v + f = e + 2$ for any planar drawing.

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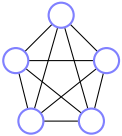

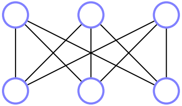

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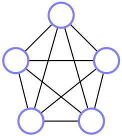

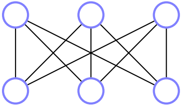

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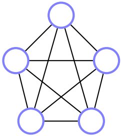

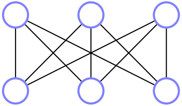

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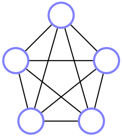

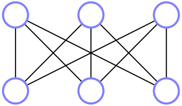

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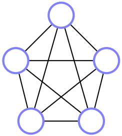

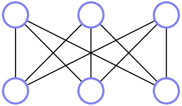

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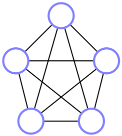

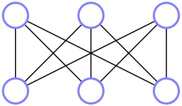

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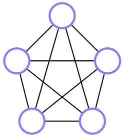

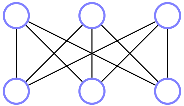

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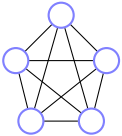
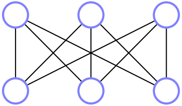
Planarity and Euler





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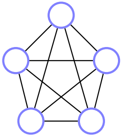

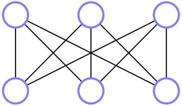

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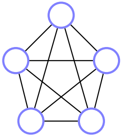
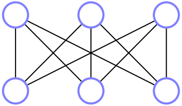
Proved absolutely no drawing can work for these graphs.

Planarity and Euler





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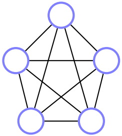
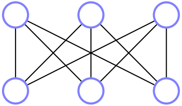
So...

Planarity and Euler





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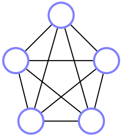
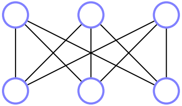
Planarity and Euler

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So......so ...Cool!

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Euler's formula.

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Proof:

Euler's formula.

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Proof: Induction on e.

Euler's formula.

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Proof: Induction on e.
Base:

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$,

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:

Euler's formula.

Euler: Connected planar graph has $v + f = e + 2$.

Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
If it is a tree.

Euler's formula.

Euler: Connected planar graph has $v+f=e+2$.

Proof: Induction on e.
Base: $e=0$, $v=f=1$.
Induction Step:
If it is a tree. $e=v-1$, $f=1$, $v+1=(v-1)+2$. Yes.

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Proof: Induction on e.
Base: $e = 0$, $v = f = 1$.
Induction Step:
If it is a tree. $e = v - 1$, $f = 1$, $v + 1 = (v - 1) + 2$. Yes.
If not a tree.

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Proof: Induction on e.
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Induction Step:
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If not a tree.
Find a cycle.

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Base: $e=0$, $v=f=1$.
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If not a tree.
Find a cycle. Remove edge.

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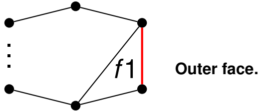

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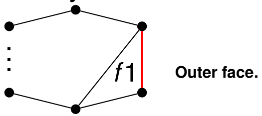

Joins two faces.
New graph: v-vertices.

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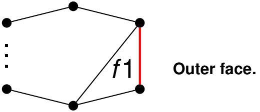

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New graph: v-vertices. $e-1$ edges.

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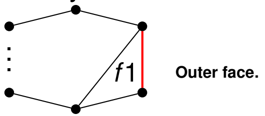

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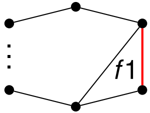

Outer face.

Joins two faces.
New graph: v-vertices. $e-1$ edges. $f-1$ faces. Planar.

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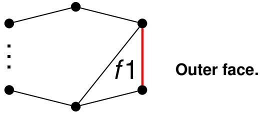

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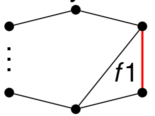

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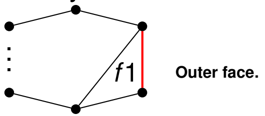

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Therefore $v + f = e + 2$. □

Quick:
$v + 1 = (v - 1) + 2$, add edge: $f \to f + 1$, $e \to e + 1$.

Euler's Proof.Poll.

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Steps/concepts in proof of euler's formula.

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(A) Planar drawing of tree has 1 face.
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All are true and relevant to proof.

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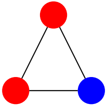

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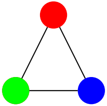

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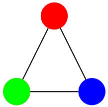

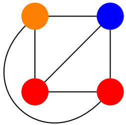

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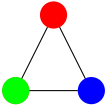

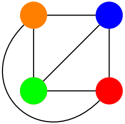

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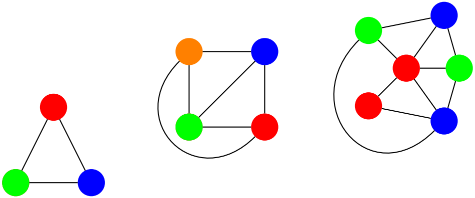

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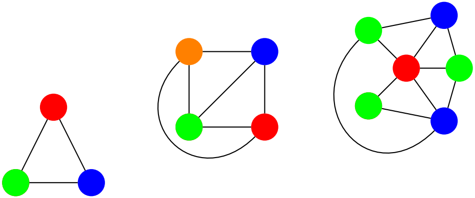

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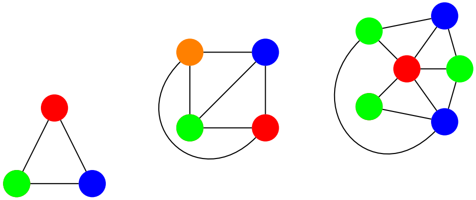

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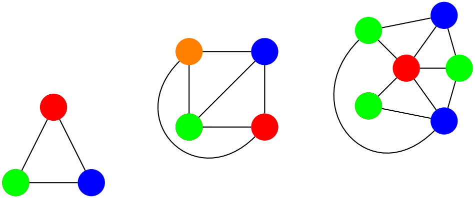

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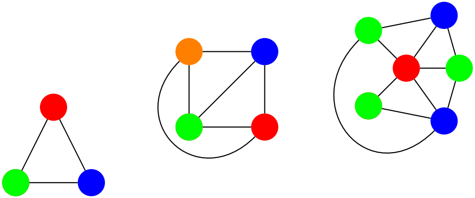

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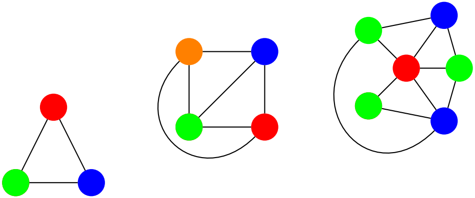

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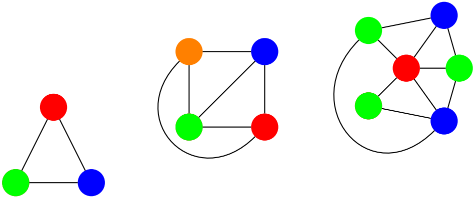

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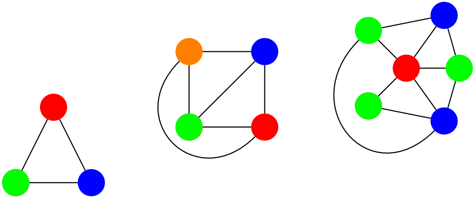

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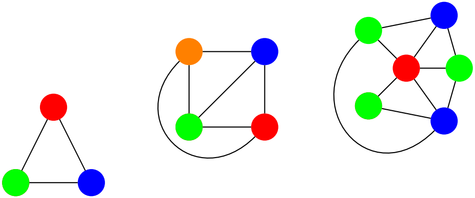

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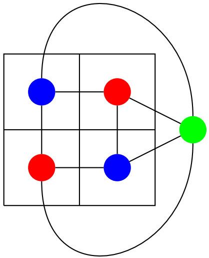
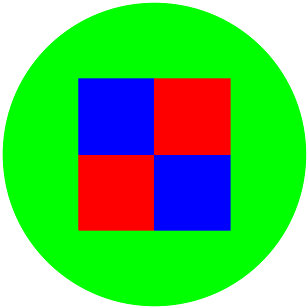
Interesting things to do. Algorithm!

Planar graphs and maps.

Planar graph coloring $\equiv$ map coloring.





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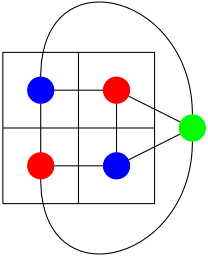

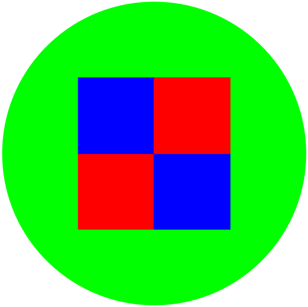

Four color theorem is about planar graphs!

Six color theorem.

Theorem: Every planar graph can be colored with six colors.

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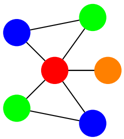
Five color theorem: prelimnary.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.



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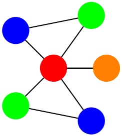

Look at only green and blue.

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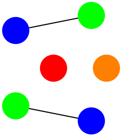

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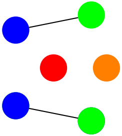

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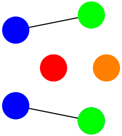

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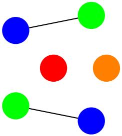

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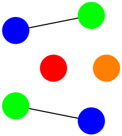

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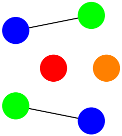

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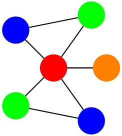

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Proof: Again with the degree 5 vertex.

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Proof: Again with the degree 5 vertex. Again recurse.

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Theorem: Every planar graph can be colored with five colors.

Preliminary Observation: Connected components of vertices with two colors in a legal coloring can switch colors.

Proof: Again with the degree 5 vertex. Again recurse.

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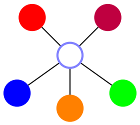

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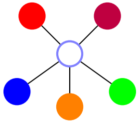

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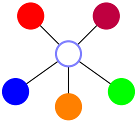

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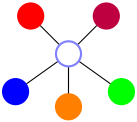

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Switch green and blue in green's component.

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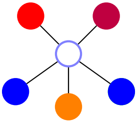

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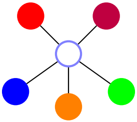

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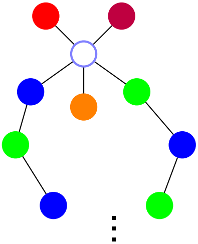

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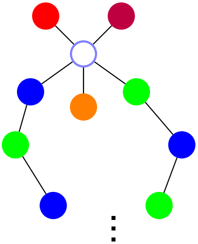

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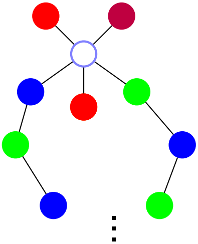

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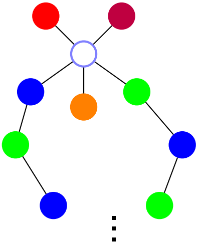

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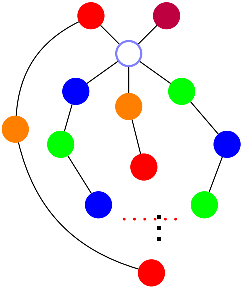

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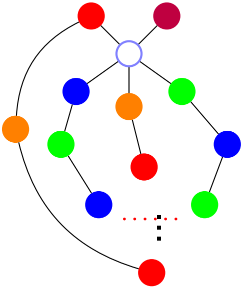

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Planar.

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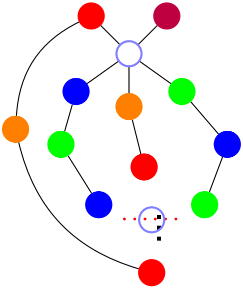

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Planar. $\implies$ paths intersect at a vertex!

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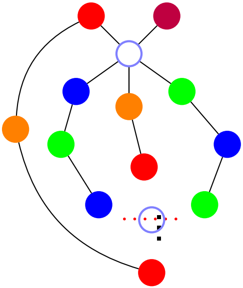

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What color is it?

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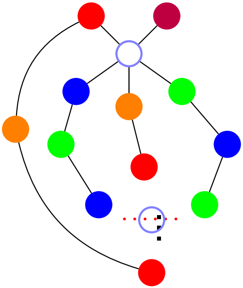

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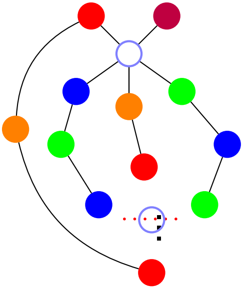

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What color is it?

Must be blue or green to be on that path.

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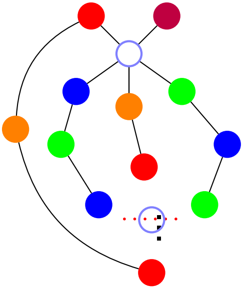

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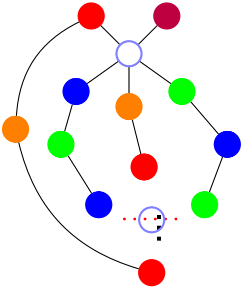

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What color is it?
Must be blue or green to be on that path.
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Contradiction.

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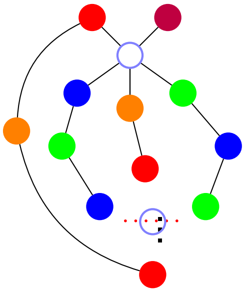

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Planar. $\implies$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.

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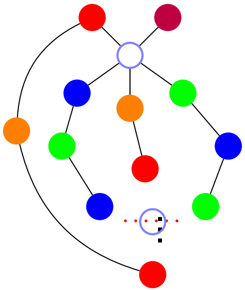

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Planar. $\Longrightarrow$ paths intersect at a vertex!

What color is it?
Must be blue or green to be on that path.
Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex!

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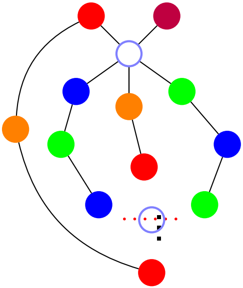

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Must be red or orange to be on that path.

Contradiction. Can recolor one of the neighbors.
Gives an available color for center vertex! $\square$

5 color theorem. Flow poll.

Steps/ideas in 5-color theorem.
(A) There is a degree 5 vertex cuz Euler.
(B) Take subgraph of first and third colors, recolor first components.
(C) If a third's component is different, switched coloring is good.
(D) Subgraph of second and fourth colors, can recolor, recolor second component.
(G) At least one separate component cuz planarity.
(F) Shared color of five neighbors, done.

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All steps in proof!

Four Color Theorem

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

Proof:

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

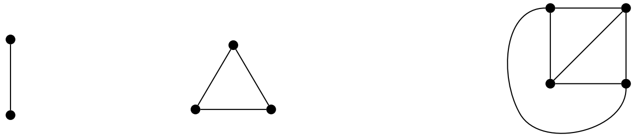
Proof: Not Today!

Four Color Theorem

Theorem: Any planar graph can be colored with four colors.

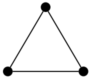
Proof: Not Today!

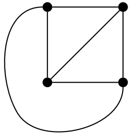
Complete Graph.



K_n complete graph on n vertices.

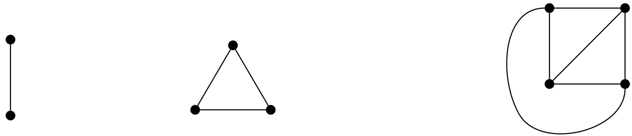
Complete Graph.





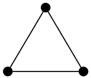
K_n complete graph on n vertices.
All edges are present.

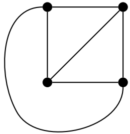
Complete Graph.



K_n complete graph on n vertices.
  All edges are present.
  Everyone is my neighbor.

Complete Graph.





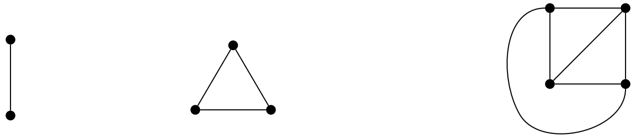
K_n complete graph on n vertices.

All edges are present.

Everyone is my neighbor.

Each vertex is adjacent to every other vertex.

Complete Graph.



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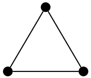

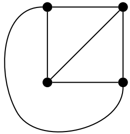

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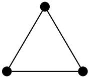
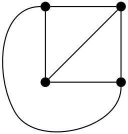
How many edges?

Complete Graph.





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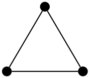
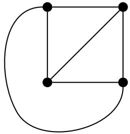
Each vertex is adjacent to every other vertex.

How many edges?

Each vertex is incident to $n-1$ edges.

Complete Graph.





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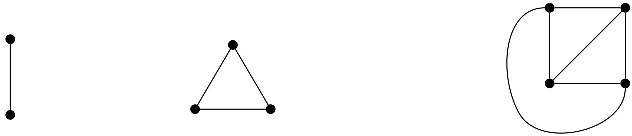
Sum of degrees is $n(n-1)$

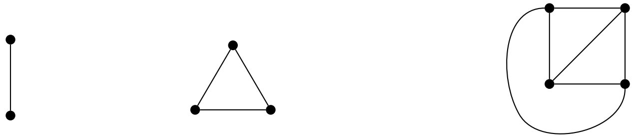
Complete Graph.



K_n complete graph on n vertices.
All edges are present.
Everyone is my neighbor.
Each vertex is adjacent to every other vertex.

How many edges?
Each vertex is incident to $n-1$ edges.
Sum of degrees is $n(n-1) = 2|E|$

Complete Graph.



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Each vertex is adjacent to every other vertex.

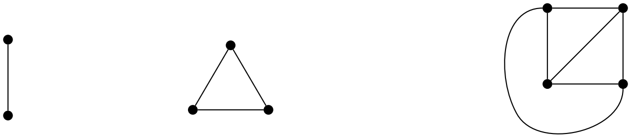
How many edges?

Each vertex is incident to $n-1$ edges.

Sum of degrees is $n(n-1) = 2|E|$

$\implies$ Number of edges is $n(n-1)/2$.

Complete Graph.



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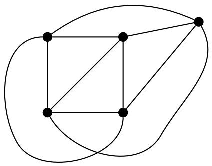
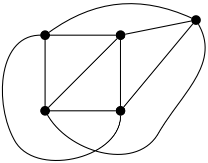
$\implies$ Number of edges is $n(n-1)/2$.

K_4 and K_5



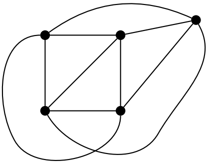
K_5 is not planar.

K_4 and K_5



K_5 is not planar.
Cannot be drawn in the plane without an edge crossing!

K_4 and K_5

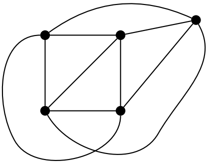


K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it!

K_4 and K_5



K_5 is not planar.

Cannot be drawn in the plane without an edge crossing!

Prove it! We did!

Hypercubes.

Complete graphs, really connected!

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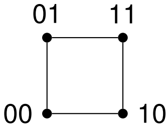

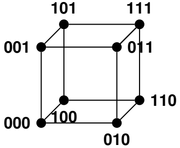

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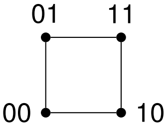

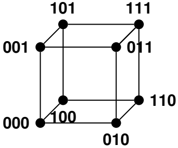

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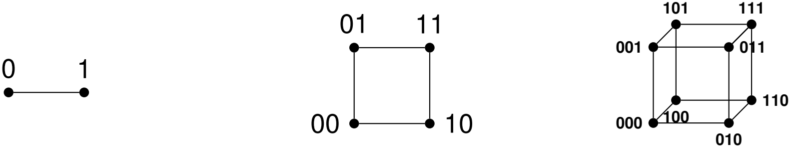

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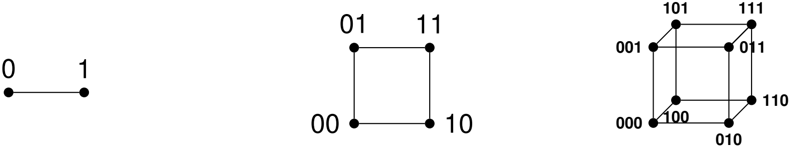

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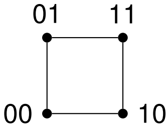

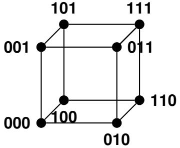

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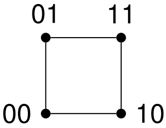

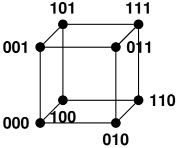

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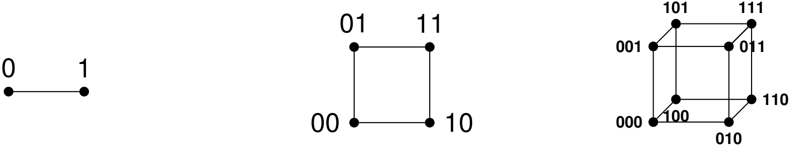

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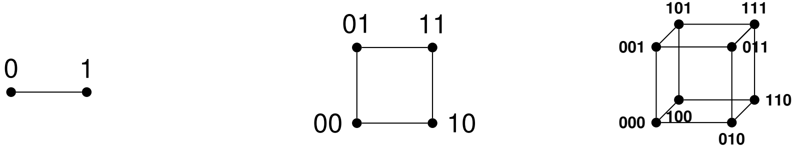

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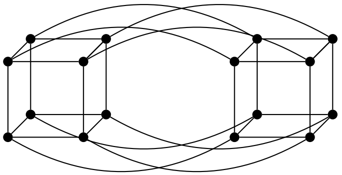

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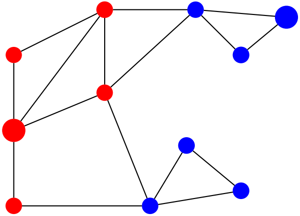
Restatement: for any cut in the hypercube, the number of cut edges is at least the size of the small side.

Cuts in graphs.



S is red, $V - S$ is blue.

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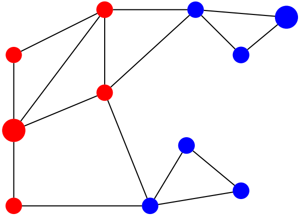

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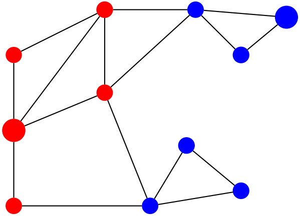

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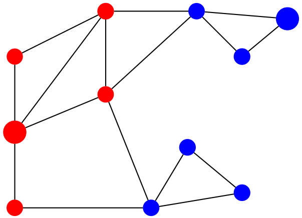

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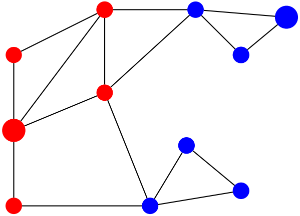

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Hypercube: any cut that cuts off x nodes has $\geq x$ edges.

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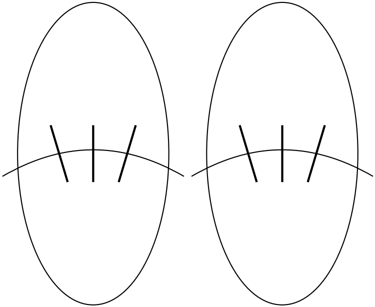

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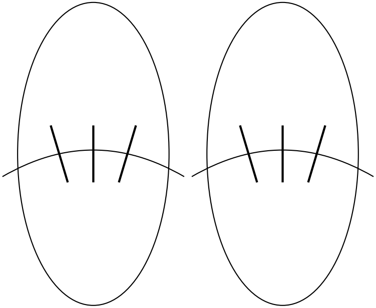

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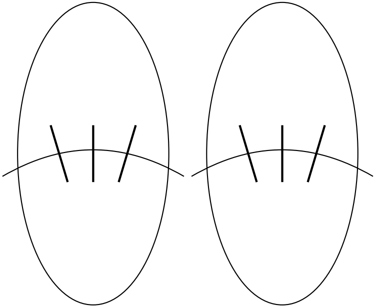

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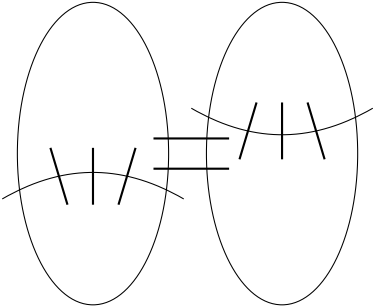

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Thm: For any cut $(S, V - S)$ in the hypercube, the number of cut edges is at least the size of the small side, $|S|$.

Proof: Induction Step.
Recursive definition:
$H_0 = (V_0, E_0)$,$H_1 = (V_1, E_1)$, edges E_x that connect them.
$H = (V_0 \cup V_1, E_0 \cup E_1 \cup E_x)$
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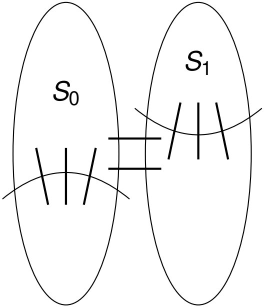

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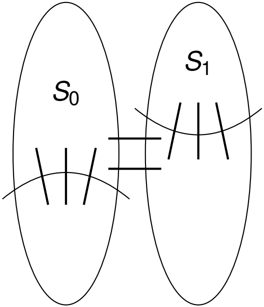

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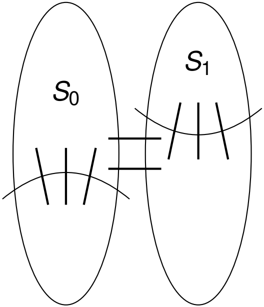

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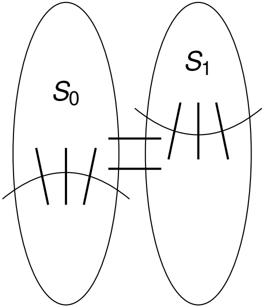

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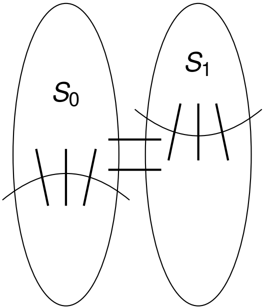

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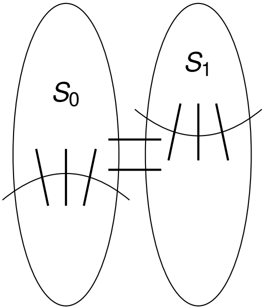

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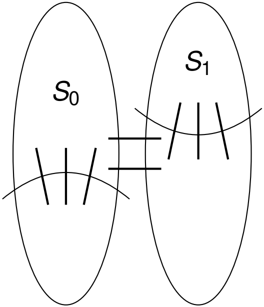

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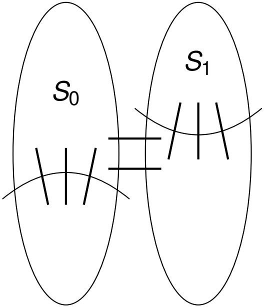

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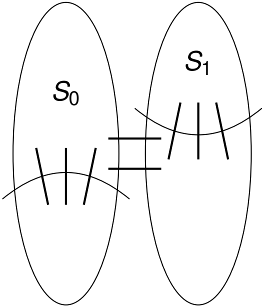

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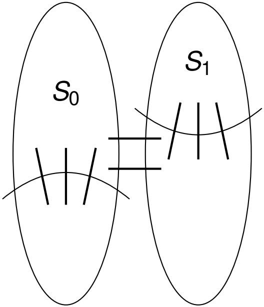

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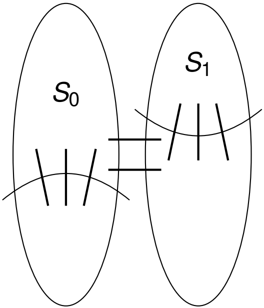

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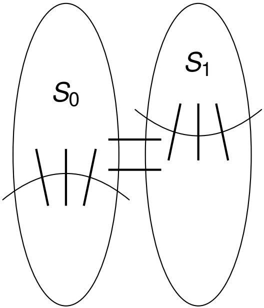

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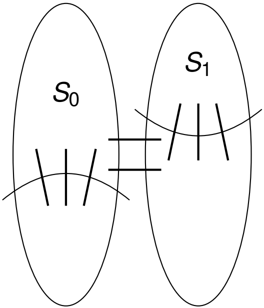

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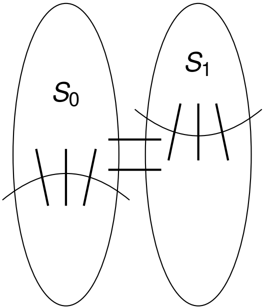

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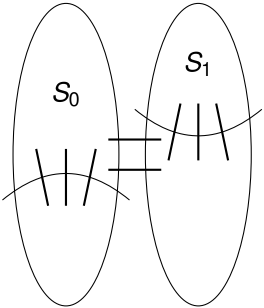

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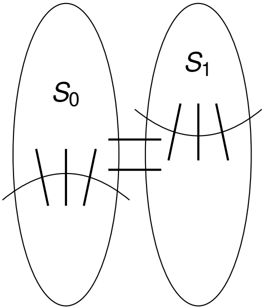

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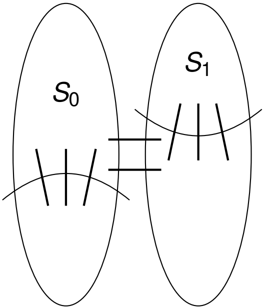

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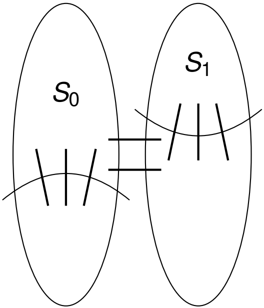

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Also, case 3 where $|S_1| \geq |V|/2$ is symmetric.

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Have a nice weekend!