

1 Beast Arcade

One day you find yourself inside the Mr. Beast Arcade, which is full of games that pay YOU to play them!

- (a) In the first game, Chandler hands you a crisp \$20 bill up front. Then, he flips a coin that shows heads with probability p repeatedly, stopping when a heads comes up for the first time. You receive an additional dollar for each flip. How much money will you get in expectation?
- (b) In the next game, Karl rolls a fair 6-sided die. He then calculates 2^x , where x is the result of that die and hands you that much money. What is the expected amount of money you'll receive?
- (c) For the last game, Jimmy makes your friend flip a fair coin 10,000 times in a row, keeping track of the number of heads that show up. He then hands you a briefcase filled with \$1,000 and says he will also pay you \$5 for each head that comes up. Let X be a random variable representing the number of heads your friend flips. Use it to come up with an expression for Y , a random variable representing the total amount of money you'll receive.
- (d) What is $E[Y]$? What about $P[Y = 26,000]$?

Solution:

- (a) Let X be a random variable representing the total number of flips needed for a heads. Notice that X is geometric with parameter p , and thus its expected value is $\frac{1}{p}$. Then, we see that Y , the total amount of money we'll make, can be written as $20 + X$, so $E[Y] = E[X] + E[20] = 20 + \frac{1}{p}$.
- (b) In game 2, we can use the formula $E[g(X)] = \sum_x g(x)p_X(x)$. In english, we'll sum how much money we'll get in each possible outcome multiplied by its likelihood. Since all 6 dice roll outcomes are equally likely, we know that $\forall x, p_X(x) = \frac{1}{6}$. This gives us $\sum_{i=1}^6 \frac{1}{6} 2^i = 21$.
- (c) We once again have a linear expression for Y in terms of X . This means we can write $Y = 5X + 1000$.
- (d) $E[Y] = E[5X + 1000] = E[5x] + E[1000] = 5E[X] + 1000 = 5 * 5000 + 1000 = 26000$. To calculate $P[Y = 26000]$, we note that you receive exactly \$26000 if and only if your friend flips exactly 5000 heads. This happens with probability $\binom{10000}{5000} \frac{1}{2}^{5000} \frac{1}{2}^{5000} = \binom{10000}{5000} \frac{1}{2}^{10000}$.

2 Linearity

Solve each of the following problems using linearity of expectation. Explain your methods clearly.

- (a) In an arcade, you play game A 10 times and game B 20 times. Each time you play game A , you win with probability $1/3$ (independently of the other times), and if you win you get 3 tickets (redeemable for prizes), and if you lose you get 0 tickets. Game B is similar, but you win with probability $1/5$, and if you win you get 4 tickets. What is the expected total number of tickets you receive?
- (b) A monkey types at a 26-letter keyboard with one key corresponding to each of the lower-case English letters. Each keystroke is chosen independently and uniformly at random from the 26 possibilities. If the monkey types 1 million letters, what is the expected number of times the sequence “book” appears? (*Hint*: Consider where the sequence “book” can appear in the string.)

Solution:

- (a) Let A_i be the indicator you win the i th time you play game A and B_i be the same for game B . The expected value of A_i and B_i are

$$\begin{aligned}\mathbb{E}[A_i] &= 1 \cdot \frac{1}{3} + 0 \cdot \frac{2}{3} = \frac{1}{3}, \\ \mathbb{E}[B_i] &= 1 \cdot \frac{1}{5} + 0 \cdot \frac{4}{5} = \frac{1}{5}.\end{aligned}$$

Then the expected total number of tickets you receive, by linearity of expectation, is

$$3\mathbb{E}[A_1] + \cdots + 3\mathbb{E}[A_{10}] + 4\mathbb{E}[B_1] + \cdots + 4\mathbb{E}[B_{20}] = 10\left(3 \cdot \frac{1}{3}\right) + 20\left(4 \cdot \frac{1}{5}\right) = 26.$$

Note that $10\left(3 \cdot \frac{1}{3}\right)$ and $20\left(4 \cdot \frac{1}{5}\right)$ matches the expression directly gotten using the expected value of a binomial random variable.

- (b) There are $1,000,000 - 4 + 1 = 999,997$ places where “book” can appear, each with a (non-independent) probability of $1/26^4$ of happening. If A is the random variable that tells how many times “book” appears, and A_i is the indicator variable that is 1 if “book” appears starting at the i th letter, then

$$\begin{aligned}\mathbb{E}[A] &= \mathbb{E}[A_1 + \cdots + A_{999,997}] \\ &= \mathbb{E}[A_1] + \cdots + \mathbb{E}[A_{999,997}] \\ &= \frac{999,997}{26^4} \approx 2.19.\end{aligned}$$

3 Family Planning

Mr. and Mrs. Brown decide to continue having children until they either have their first girl or until they have three children. Assume that each child is equally likely to be a boy or a girl, independent of all other children, and that there are no multiple births. Let G denote the numbers of girls that the Browns have. Let C be the total number of children they have.

(a) Determine the sample space, along with the probability of each sample point.

(b) Compute the joint distribution of G and C . Fill in the table below.

	$C = 1$	$C = 2$	$C = 3$
$G = 0$			
$G = 1$			

(c) Use the joint distribution to compute the marginal distributions of G and C and confirm that the values are as you'd expect. Fill in the tables below.

$\mathbb{P}(G = 0)$		$\mathbb{P}(C = 1)$	$\mathbb{P}(C = 2)$	$\mathbb{P}(C = 3)$
$\mathbb{P}(G = 1)$				

(d) Are G and C independent?

(e) What is the expected number of girls the Browns will have? What is the expected number of children that the Browns will have?

Solution:

(a) The sample space is the set of all possible sequences of children that the Browns can have: $\Omega = \{g, bg, bbg, bbb\}$. The probabilities of these sample points are:

$$\begin{aligned}\mathbb{P}[g] &= \frac{1}{2} \\ \mathbb{P}[bg] &= \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \\ \mathbb{P}[bbg] &= \left(\frac{1}{2}\right)^3 = \frac{1}{8} \\ \mathbb{P}[bbb] &= \left(\frac{1}{2}\right)^3 = \frac{1}{8}\end{aligned}$$

(b)

	$C = 1$	$C = 2$	$C = 3$
$G = 0$	0	0	$\mathbb{P}[bbb] = 1/8$
$G = 1$	$\mathbb{P}[g] = 1/2$	$\mathbb{P}[bg] = 1/4$	$\mathbb{P}[bbg] = 1/8$

(c) Marginal distribution for G :

$$\begin{aligned}\mathbb{P}[G = 0] &= 0 + 0 + \frac{1}{8} = \frac{1}{8} \\ \mathbb{P}[G = 1] &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}\end{aligned}$$

Marginal distribution for C :

$$\begin{aligned}\mathbb{P}[C = 1] &= 0 + \frac{1}{2} = \frac{1}{2} \\ \mathbb{P}[C = 2] &= 0 + \frac{1}{4} = \frac{1}{4} \\ \mathbb{P}[C = 3] &= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}\end{aligned}$$

(d) No, G and C are not independent. If two random variables are independent, then

$$\mathbb{P}[X = x, Y = y] = \mathbb{P}[X = x]\mathbb{P}[Y = y].$$

To show this dependence, consider an entry in the joint distribution table, such as $\mathbb{P}[G = 0, C = 3] = 1/8$. This is not equal to $\mathbb{P}[G = 0]\mathbb{P}[C = 3] = (1/8) \cdot (1/4) = 1/32$, so the random variables are not independent.

(e) We can apply the definition of expectation directly for this problem, since we've computed the marginal distribution for both random variables.

$$\begin{aligned}\mathbb{E}[G] &= 0 \cdot \mathbb{P}[G = 0] + 1 \cdot \mathbb{P}[G = 1] = 1 \cdot \frac{7}{8} = \frac{7}{8} \\ \mathbb{E}[C] &= 1 \cdot \mathbb{P}[C = 1] + 2 \cdot \mathbb{P}[C = 2] + 3 \cdot \mathbb{P}[C = 3] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{7}{4}\end{aligned}$$