CS70

Coupon Collecting: Fun with harmonic numbers!

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Variance/ Covariance.

Time to collect coupons

X-time to get n coupons.

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Pr[“get second coupon”|“got milk ”]

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Review: Harmonic sum

$$H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x} dx = \ln(n).$$

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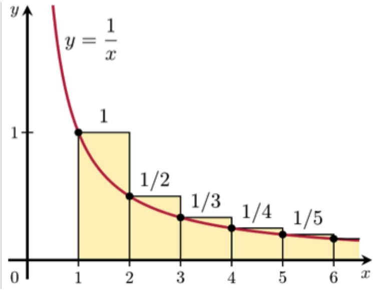

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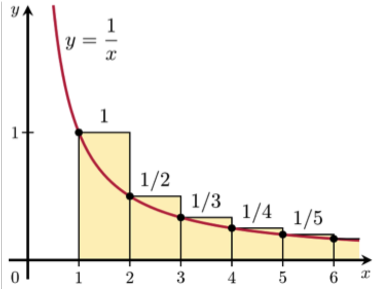

A good approximation is

$$H(n) \approx \ln(n) + \gamma \text{ where } \gamma \approx 0.58 \text{ (Euler-Mascheroni constant).}$$

Harmonic sum: Paradox

Consider this stack of cards (no glue!):

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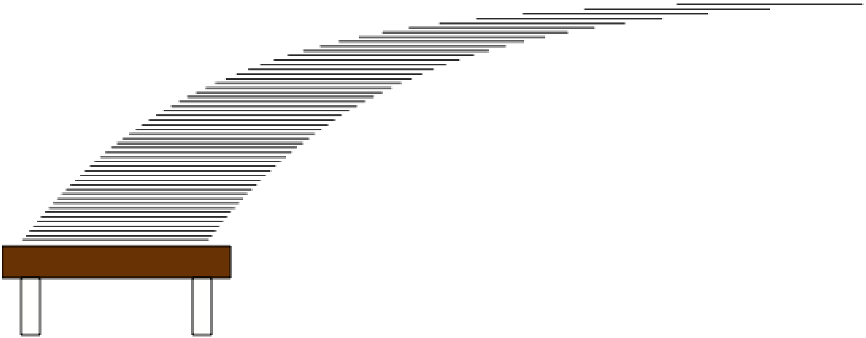

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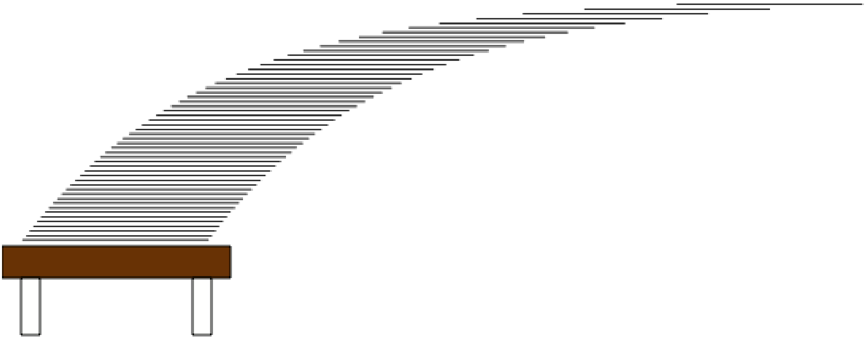

If each card has length 2, the stack can extend $H(n)$ to the right of the table.

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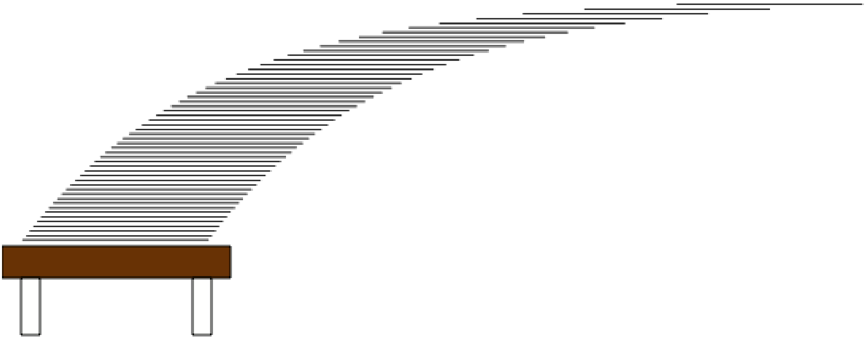


If each card has length 2, the stack can extend $H(n)$ to the right of the table. As n increases, you can go as far as you want!

Paradox

par·a·dox

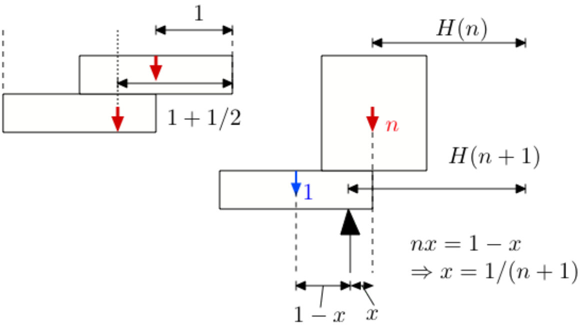
/ˈperəˌdäks/

noun

a statement or proposition that, despite sound (or apparently sound) reasoning from acceptable premises, leads to a conclusion that seems senseless, logically unacceptable, or self-contradictory.
"a potentially serious conflict between quantum mechanics and the general theory of relativity known as the information paradox"

- a seemingly absurd or self-contradictory statement or proposition that when investigated or explained may prove to be well founded or true.
  "in a paradox, he has discovered that stepping back from his job has increased the rewards he gleans from it"
  synonyms: contradiction, contradiction in terms, self-contradiction, inconsistency, incongruity; More
- a situation, person, or thing that combines contradictory features or qualities.
  "the mingling of deciduous trees with elements of desert flora forms a fascinating ecological paradox"

Stacking



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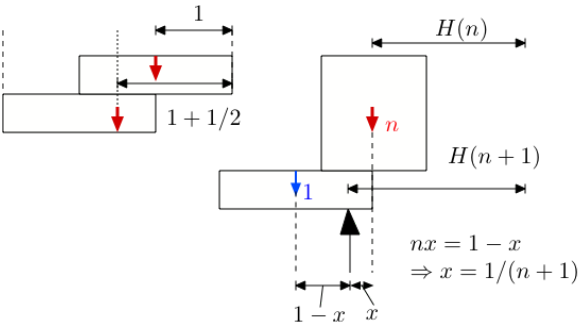

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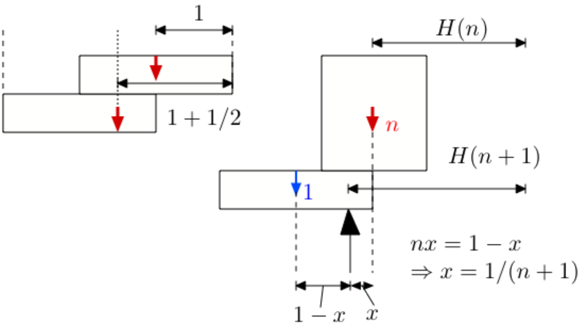

The cards have width 2. Induction shows that the center of gravity after n cards is $H(n)$ away from the right-most edge.

Video.

Calculating $E[g(X)]$: LOTUS

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$\square$

Poll.

Which is LOTUS?

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(A) $E[X] = \sum_{x \in \text{Range}(X)} g(x) Pr[g(X) = g(x)]$
(B) $E[X] = \sum_{x \in \text{Range}(X)} g(x) Pr[X = x]$
(C) $E[X] = \sum_{x \in \text{Range}(g)} x Pr[g(X) = x]$

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Experiment: flip a coin with heads prob. p. until Heads.
Random Variable X: number of flips.

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(A) Distribution of $X \sim G(p)$: $Pr[X = i] = (1-p)^{i-1}p$.

Geometric Distribution: Memoryless - Interpretation

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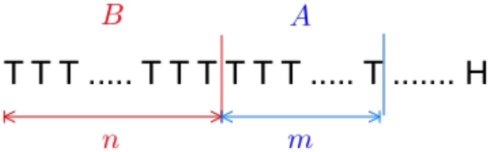

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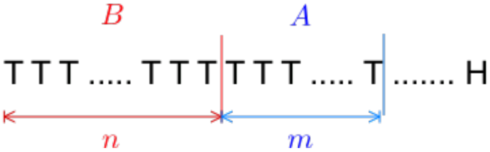

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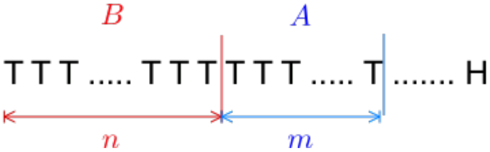

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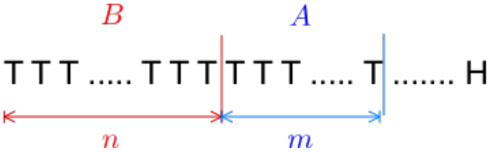

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The coin is memoryless , therefore, so is X.
Independent coin: $Pr[H|'anyprevioussetofcointosses'] = p$

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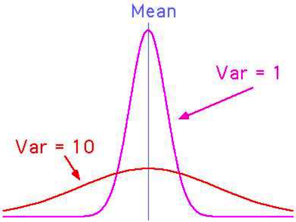

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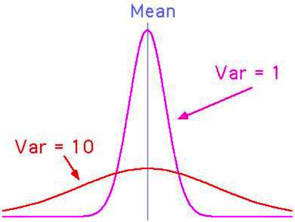

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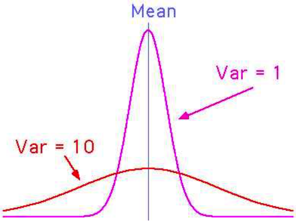

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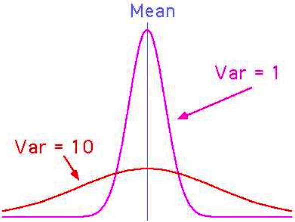

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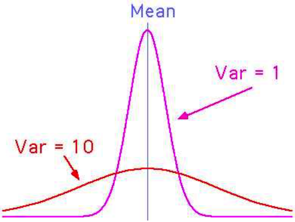

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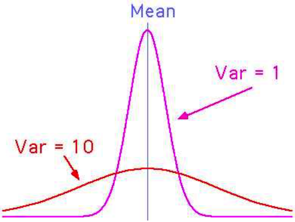

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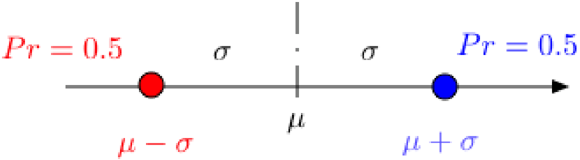

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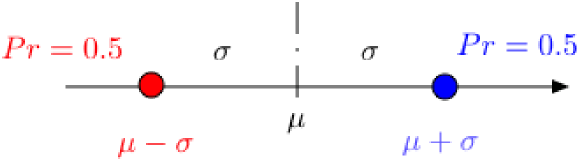

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$$X = \begin{cases} \mu - \sigma, & \text{w.p. } 1/2 \\ \mu + \sigma, & \text{w.p. } 1/2. \end{cases}$$

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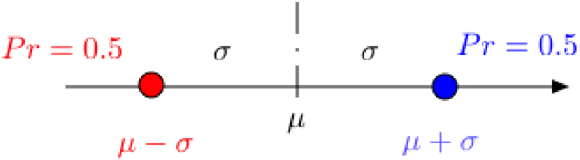

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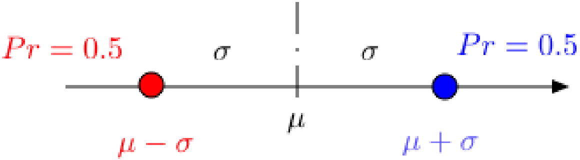

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$$var(X) = \sigma^2 \text{ and } \sigma(X) = \sigma.$$

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Exercise: How big can you make $\frac{\sigma(X)}{E[|X-E[X]|]}$?

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This gives

$$var(X) = \frac{1+3n+2n^2}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}.$$

(Sort of $\int_0^{1/2} x^2 dx = \frac{x^3}{3}$.)

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Poll: fixed points.

What's true?

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(A) X_i and X_j are independent.
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(C) $Pr[X_iX_j] = \frac{(n-2)!}{n!}$
(D) $X_i^2 = X_i$.

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Hence,

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Since shifting the random variables does not change their variance, let us subtract their means.

That is, we assume that $E(X) = 0$ and $E(Y) = 0$.

Then, by independence,

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If X and Y are independent, then

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If $X, Y, Z, \ldots$ are pairwise independent, then

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$\square$

Variance of Binomial Distribution.

Flip coin with heads probability p.

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$$\begin{aligned}
E[(X - E[X])(Y - E[Y])] &= E[XY - E[X]Y - XE[Y] + E[X]E[Y]] \\
&= E[XY] - E[X]E[Y] - E[X]E[Y] + E[X]E[Y] \\
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Correlation

Definition The correlation of X, Y, $Cor(X,Y)$ is

$$corr(X,Y) : \frac{cov(X,Y)}{\sigma(X)\sigma(Y)}.$$

Theorem: $-1 \leq corr(X,Y) \leq 1$.
Proof: Idea: $(a-b)^2 > 0 \rightarrow a^2 + b^2 \geq 2ab$.

Simple case: $E[X] = E[Y] = 0$ and $E[X^2] = E[Y^2] = 1$.

$Cor(X,Y) = E[XY]$.

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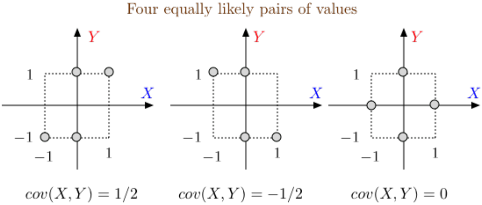
$\rightarrow E[XY] \geq -1$.

$\square$

Shifting and scaling doesn't change correlation.

Examples of Covariance



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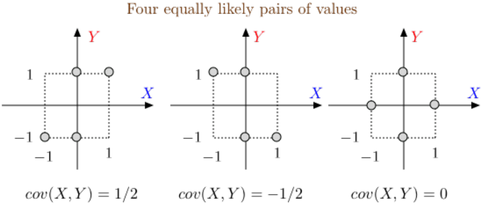

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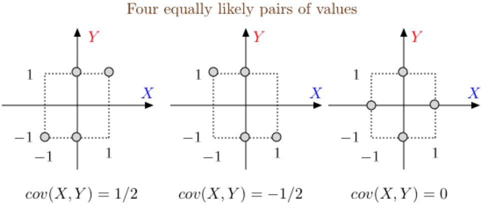

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When $cov(X, Y) > 0$, the RVs X and Y tend to be large or small together.

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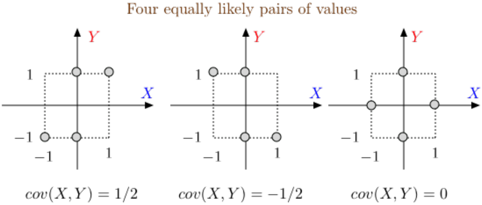

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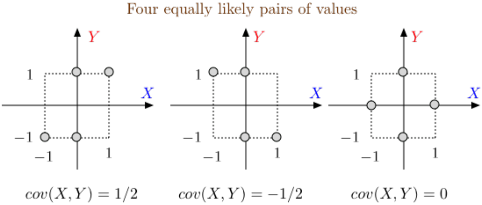

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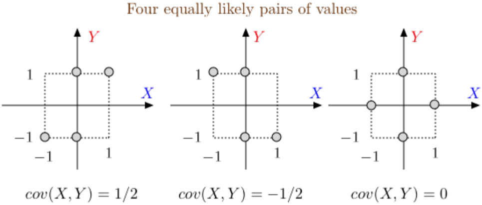

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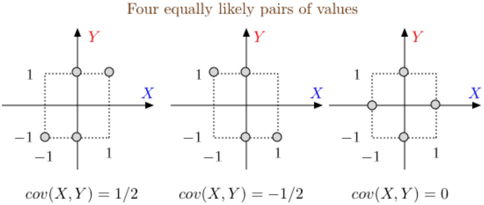

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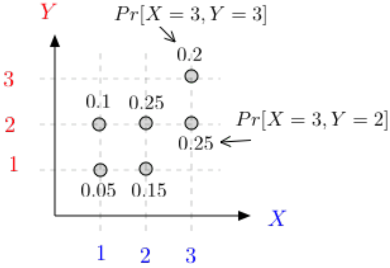

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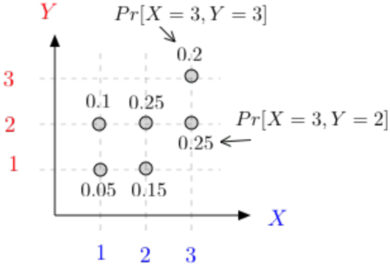

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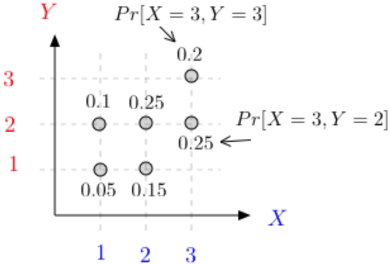

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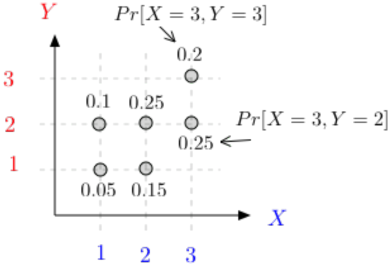

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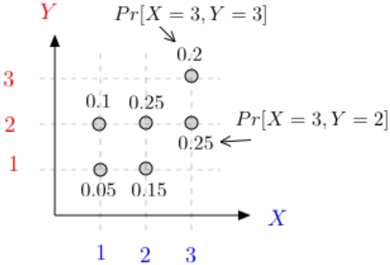

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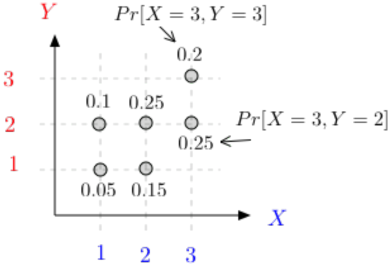

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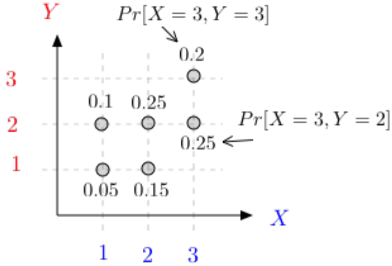

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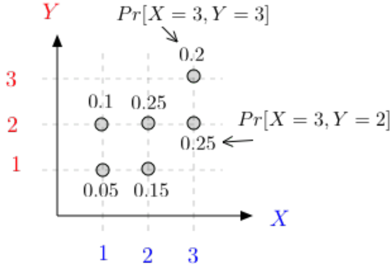

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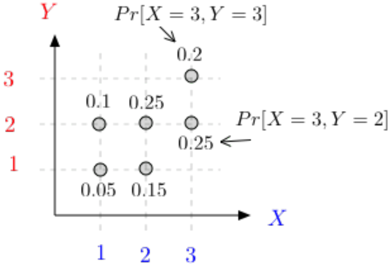

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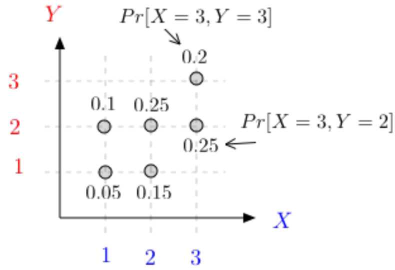

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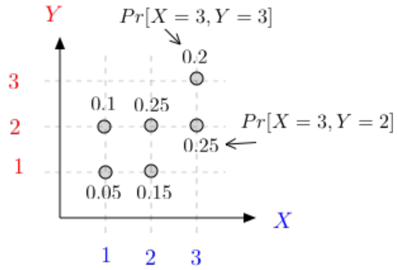

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