1 Counting on Graphs + Symmetry

(a) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

(b) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

(d) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

Solution:

(a) Without considering symmetries there are 6! ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.

(b) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.

(c) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$. (Recall that $2^m = \sum_{k=0}^{m} \binom{m}{k}$, where $m = n(n-1)/2$ in this case.)

(d) The number k of vertices in a cycle is at least 3 and at most n. Without accounting for duplicates, the number of cycles of length k can be counted by choosing any ordered sequence of k vertices from the graph. Hence, there are $n!/(n-k)!$ k-length cycles. We count cycles inverted (abc = cba) and rotated (abc = bca = cab) to be non-distinct cycles. Since every k-length cycle can be inverted in one way and rotated in k-1 ways, we divide $n!/(n-k)!$ by 2 to account for inversions, and by k to account for rotations. Hence the total number of distinct cycles is

$$\sum_{k=3}^{n} \frac{n!}{(n-k)! \cdot 2k}.$$

2 The Count

(a) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

(b) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he have now?

(c) The Count now wants to make a password to secure his phone. His password must be exactly 10 digits long and can only contain the digits 0 and 1. On top of that, he also wants it to contain at least five consecutive 0's. How many possible passwords can he make?

Solution:

(a) This is actually a stars and bars problem in disguise! We have seven positions for digits, and nine dividers to partition these positions into places for nines, places for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. That means there are a total of 16 objects and dividers, and we are looking for where to put the nine dividers, so our answer is $\binom{16}{9}$.

(b) This can be found from just combinations. For any choice of 7 digits, there is exactly one arrangement of them that is strictly decreasing. Thus, the total number of strictly decreasing strings is exactly $\binom{10}{7}$.

(c) One counting strategy is strategic casework - we will split up the problem into exhaustive cases based on where the run of 0's begins. It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the i^{th} digit, then the $i-1^{th}$ digit must be a 1, and the other $(10-5-1=4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$ possibilities. Thus the total number of valid passwords is $2^5 + 5 \cdot 2^4 = 112$.

3 Captain Combinatorial

Please provide combinatorial proofs for the following identities.

(a) $\binom{n}{i} = \binom{n}{n-i}$.

(b) $\sum_{i=1}^{n} i\binom{n}{i}^2 = n\binom{2n-1}{n-1}$. (Hint: Part (a) might be useful.)

(c) $\sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j} = 3^n$. (Hint: consider the number of ways of splitting n elements into 3 groups.)

Solution:

(a) Choosing i players out of n to play on a team is the same as choosing $n-i$ players to not play on the team, i.e. $\binom{n}{i} = \binom{n}{n-i}$.

(b) Assume we have n women and n men. Using part (a) we can rewrite the LHS as $\sum_{i=1}^{n} i\binom{n}{i}\binom{n}{n-i}$, which we can interpret as selecting a team of n players by choosing i women and $n-i$ men, and then choosing one of the women to serve as captain. Again, the RHS first chooses a captain, and then selects a remaining $n-1$ players from all remaining men and women to form the team.

(c) We count the number of ways to split n elements into 3 labeled groups by two different methods.

RHS: There are 3 different choices for each element, so 3^n for all of them.

LHS: For every i from 0 to n, choose i elements to go in group A, then for every j from 0 to $n-i$ choose j elements to go in group B, the remaining go in group C. This gives:

$$\sum_{i=0}^{n} \binom{n}{i} \sum_{j=0}^{n-i} \binom{n-i}{j}$$

4 Inclusion and Exclusion

What is the total number of positive integers strictly less than 100 that are also coprime to 100?

Solution: It is sufficient to count the opposite: what is the total number of positive integers strictly less than 100 and *not* coprime to 100?

If a number is not coprime to 100, this means that the number is either a multiple of 2 or a multiple of 5. In this case, we have:

- 49 multiples of 2
- 19 multiples of 5
- 9 multiples of both 2 and 5

By inclusion-exclusion, the total number of positive integers not coprime to 100 is $49+19-9=59$, and there are 99 positive integers strictly less than 100.

As such, in total there are $99-59=40$ different positive integers strictly less than 100 that are coprime to 100.