CS 70 Discrete Mathematics and Probability Theory
Summer 2022 Jingjia Chen, Michael Psenka and Tarang Srivastava DIS 2B

1 Short Answers

(a) A connected planar simple graph has 5 more edges than it has vertices. How many faces does it have?

(b) How many edges need to be removed from a 3-dimensional hypercube to get a tree?

Solution:

(a) **7.**
Use Euler's formula $v + f = e + 2$.

(b) **5.**
The 3-dimensional hypercube has $3(2^3)/2 = 12$ edges and $2^3 = 8$ vertices. A tree on 8 vertices has 7 edges, so one needs to remove 5 edges.

2 Hamiltonian Tour in a Hypercube

An alternative type of tour to an Eulerian Tour in graph is a Hamiltonian Tour: a tour that visits every vertex exactly once. Prove or disprove that the hypercube contains a Hamiltonian cycle, for hypercubes of dimension $n \geq 2$.
Hint: When proceeding by induction, a good place to start is writing out what this tour would look like in a 3-dimensional hypercube when starting from the 000 vertex, and using the recursive definition of an n-dimensional hypercube.

Solution:
Going off the hint, we get the following Hamiltonian tours:

- $n = 2 : 00, 01, 11, 10.$
- $n = 3 : 000, 001, 011, 010, 110, 111, 101, 100$ [Take the $n = 2$ tour in the 0-subcube (vertices with a 0 in front), move to the 1-subcube (vertices with 1 in front), then take the tour backwards. We know 100 connects to 000 to complete the tour.]

What we've done here is essentially take the tour in the 0-subcube (except for the last edge), transition into the 1-subcube, take the exact same tour in the 1-subcube but backwards, and end at the starting vertex. We can use analogous reasoning to prove this claim with induction on a strengthened inductive hypothesis:

Stronger Inductive Claim: There exists a tour in an n-dimensional hypercube that uses the edge: $(0^n, 10^{n-1})$.
Base Case: $n = 2$, the hypercube is just a four cycle, which is a cycle that contains the edge $(00, 10)$ as required.

Inductive Hypothesis: We assume the claim holds for dimension n.

Inductive Step: The recursive definition of an $n+1$ dimensional hypercube is to take two n dimensional hypercubes, relabel each vertex x in one "subcube" as $0x$ and relabel each vertex in the other "subcube" as $1x$ and add edges $(0x, 1x)$ for each $x \in \{0,1\}^n$.

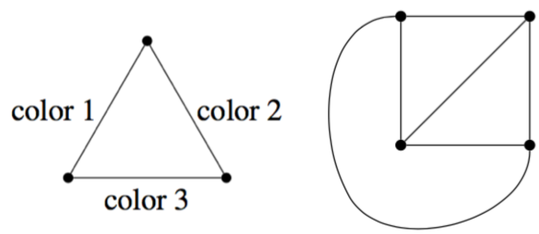
Use the inductive hypothesis to form seperate tours of each subcube which in the 0th subcube contains the edge $(00^{n-1}, 010^{n-2})$ and the 1th subcube contains $(10^{n-1}, 110^{n-1})$. We remove these edges then add the edges between the subcubes; $(00^{n-1}, 10^{n-1})$ and $(010^{n-2}, 110^{n-2})$.

Notice we do not change the degrees of any node in this swap thus the degree of all the nodes is two.

Moreover, the tour is connected as one can reach every node from all zeros in the first cube using the inductive tour, and in the second cube using the edge to the second cube and the rest of the inductive tour.

3 Edge Colorings

An edge coloring of a graph is an assignment of colors to edges in a graph where any two edges incident to the same vertex have different colors. An example is shown on the left.



(a) Show that the 4 vertex complete graph above can be 3 edge colored. (Use the numbers $1,2,3$ for colors. A figure is shown on the right.)

(b) Prove that any graph with maximum degree $d \geq 1$ can be edge colored with $2d-1$ colors.

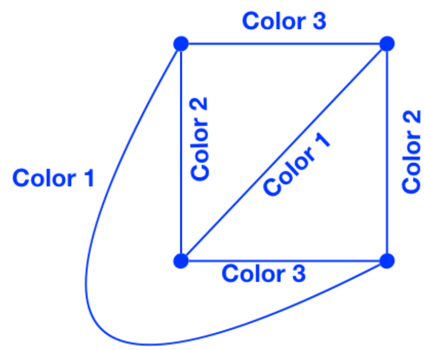
(c) Show that a tree can be edge colored with d colors where d is the maximum degree of any vertex.

Solution:

(a) Three color a triangle u_1, u_2, u_3 where (u_1, u_2) is colored 1, (u_2, u_3) is colored 2, and (u_3, u_1) is colored 3. This is a valid 3 coloring as the edges are all colored differently.

Consider adding a fourth vertex v, the incident edges must be colored differently and each incident edge (v, u_i) needs to be colored differently from the edges incident to u_i. That is, one can color (v, u_1) with 2 as it is not incident to the edge colored 2 and that color is avalaible. Similarly one can color edge (v, u_2) with color 3 and (v, u_3) with color 1.

Another proof is simply provide a coloring which is below.



(b) We will use induction on the number of edges n in the graph to prove the statement: If a graph G has $n \geq 0$ edges and the maximum degree of any vertex is d, then G can be colored with $2d - 1$ colors.
Base case (n = 0). If there are no edges in the graph, then there is nothing to be colored and the statement holds trivially.
Inductive hypothesis. Suppose for $n = k \geq 0$, the statement holds.
Inductive step. Consider a graph G with $n = k + 1$ edges. Remove an edge of your choice, say e from G. Note that in the resulting graph the maximum degree of any vertes is $d' \leq d$. By the inductive hypothesis, we can color this graph using $2d' - 1$ colors and hence with $2d - 1$ colors too. The removed edge is incident to two vertices each of which is incident to at most $d - 1$ other edges, and thus at most $2(d - 1) = 2d - 2$ colors are unavailable for edge e. Thus, we can color edge e without any conflicts. This proves the statement for $n = k + 1$ and hence by induction we get that the statement holds for all $n \geq 0$.

(c) We will use induction on the number of vertices n in the tree to prove the statement: For a tree with $n \geq 1$ vertices, if the maximum degree of any vertex is d, then the tree can be colored with d colors.
Base case (n=1). If there is only one vertex, then there are no edges to color, and thus can be colored with 0 colors.
Inductive hypothesis. Suppose the statement holds for $n = k \geq 1$.
Inductive Step. Remove any leaf v of your choice from the tree. We can then color the remaining tree with d colors by the inductive hypothesis. For any neighboring vertex u of vertex v, the degree of u is at most $d - 1$ since we removed the edge $\{u, v\}$ along with the vertex v. Thus its incident edges use at most $d - 1$ colors and there is a color available for coloring the edge {u,v}. This completes the inductive step and by induction we have that the statement holds for all $n \geq 1$.