1 Pullout Balls

Suppose you have a bag containing four balls numbered $1,2,3,4$.

(a) You perform the following experiment: pull out a single ball and record its number. What is the expected value of the number that you record?

(b) You repeat the experiment from part (a), except this time you pull out two balls together and record the product of their numbers. What is the expected value of the total that you record?

Solution:

(a) Let X be the number that you record. Each ball is equally likely to be chosen, so

$$\mathbb{E}[X] = \sum_x x \cdot \mathbb{P}[X = x] = 1 \times \frac{1}{4} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} + 4 \times \frac{1}{4} = 2.5$$

As demonstrated here, the expected value of a random variable need not, and often is not, a feasible value of that random variable (there is no outcome ω for which $X(\omega) = 2.5$).

(b) Let Y be the product of two numbers that you pull out. Then

$$\mathbb{E}[Y] = \frac{1}{\binom{4}{2}}(1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 3 + 2\cdot 4 + 3\cdot 4) = \frac{2+3+4+6+8+12}{6} = \frac{35}{6}$$

2 Head Count

Consider a coin with $\mathbb{P}[\text{Heads}] = 2/5$. Suppose you flip the coin 20 times, and define X to be the number of heads.

(a) What is $\mathbb{P}[X = k]$, for some $0 \leq k \leq 20$?

(b) Name the distribution of X and what its parameters are.

(c) What is $\mathbb{P}[X \geq 1]$? Hint: You should be able to do this without a summation.

(d) What is $\mathbb{P}[12 \leq X \leq 14]$?

Solution:

(a) There are a total of $\binom{20}{k}$ ways to select k coins to be heads. The probability that the selected k coins to be heads is $(\frac{2}{5})^k$, and the probability that the rest are tails is $(\frac{3}{5})^{20-k}$. Putting this together, we have

$$\mathbb{P}[X = k] = \binom{20}{k} \left(\frac{2}{5}\right)^k \left(\frac{3}{5}\right)^{20-k}.$$

(b) Since we have 20 independent trials, with each trial having a probability $2/5$ of success, $X \sim \text{Binomial}(20, 2/5)$.

(c)

$$\mathbb{P}[X \geq 1] = 1 - \mathbb{P}[X = 0] = 1 - \left(\frac{3}{5}\right)^{20}.$$

(d)

$$\begin{aligned}
\mathbb{P}[12 \leq X \leq 14] &= \mathbb{P}[X = 12] + \mathbb{P}[X = 13] + \mathbb{P}[X = 14] \\
&= \binom{20}{12} \left(\frac{2}{5}\right)^{12} \left(\frac{3}{5}\right)^{8} + \binom{20}{13} \left(\frac{2}{5}\right)^{13} \left(\frac{3}{5}\right)^{7} + \binom{20}{14} \left(\frac{2}{5}\right)^{14} \left(\frac{3}{5}\right)^{6}.
\end{aligned}$$

3 Head Count II

Consider a coin with $\mathbb{P}[\text{Heads}] = 3/4$. Suppose you flip the coin until you see heads for the first time, and define X to be the number of times you flipped the coin.

(a) What is $\mathbb{P}[X = k]$, for some $k \geq 1$?

(b) Name the distribution of X and what its parameters are.

(c) What is the expected number of flips we need before flipping heads for the first time?

(d) What is $\mathbb{P}[X \geq k]$, for some $k \geq 1$?

(e) What is $\mathbb{P}[X \leq k]$, for some $k \geq 1$?

Solution:

(a) If we flipped k times, then we had $k-1$ tails and 1 head, in that order, giving us

$$\mathbb{P}[X = k] = \frac{3}{4}\left(1 - \frac{3}{4}\right)^{k-1} = \frac{3}{4}\left(\frac{1}{4}\right)^{k-1}.$$

(b) $X \sim \text{Geometric}(\frac{3}{4})$

(c) We have $\mathbb{E}[X] = \frac{1}{p} = \frac{4}{3}$, as the parameter of the geometric distribution is $p = \frac{3}{4}$.

(d) If we had to flip *at least* k times before seeing our first heads, then our first $k-1$ flips must have been tails, giving us

$$\mathbb{P}[X \geq k] = \left(1 - \frac{3}{4}\right)^{k-1} = \left(\frac{1}{4}\right)^{k-1}.$$

(e) Notice $\mathbb{P}[X \leq k] = 1 - \mathbb{P}[X > k] = 1 - \mathbb{P}[X \geq k+1]$ since X can only take on integer values. Along similar lines to the previous part, we then have

$$\mathbb{P}[X \leq k] = 1 - \mathbb{P}[X \geq k+1] = 1 - \left(1 - \frac{3}{4}\right)^{k} = 1 - \left(\frac{1}{4}\right)^{k}.$$

4 The Memoryless Property

Let X be a discrete random variable which takes on values in $\mathbb{Z}_+$ (the positive integers). Suppose that for all $m, n \in \mathbb{N}$, we have $\mathbb{P}[X > m+n \mid X > n] = \mathbb{P}[X > m]$. Prove that X is a geometric distribution. Hint: In order to prove that X is geometric, it suffices to prove that there exists a $p \in [0,1]$ such that $\mathbb{P}[X > i] = (1-p)^i$ for all $i > 0$.

Solution:

Notice that

$$\mathbb{P}[X > m] = \mathbb{P}[X > m+n \mid X > n] = \frac{\mathbb{P}[X > m+n]}{\mathbb{P}[X > n]},$$

where that the first equality comes from the given in the question, and the second equality holds from definition of conditional. So, this gives $\mathbb{P}[X > m+n] = \mathbb{P}[X > m]\mathbb{P}[X > n]$.

By repeatedly applying this property, we can deduce $\mathbb{P}[X > n+1] = \mathbb{P}[X > n]\mathbb{P}[X > 1]$, which inductively, gives $\mathbb{P}[X > n+1] = \mathbb{P}[X > 1]^{n+1}$.

Let $p := 1 - \mathbb{P}[X > 1]$. We see that $\mathbb{P}[X > k] = (1-p)^k$, which is the tail probability of the geometric distribution, and hence $X \sim \text{Geometric}(p)$.