CS 70 Spring 2022 | Discrete Mathematics and Probability Theory Satish Rao and Koushik Sen | HW 13

Due: Saturday 4/23, 4:00 PM
Grace period until Saturday 4/23, 6:00 PM

Sundry

Before you start writing your final homework submission, state briefly how you worked on it. Who else did you work with? List names and email addresses. (In case of homework party, you can just describe the group.)

1 Estimating π

In this problem, we discuss some interesting ways that you could probabilistically estimate π, and see how good these techniques are at estimating π.

Technique 1: Buffon's needle is a method that can be used to estimate the value of π. There is a table with infinitely many parallel lines spaced a distance 1 apart, and a needle of length 1. It turns out that if the needle is dropped uniformly at random onto the table, the probability of the needle intersecting a line is $\frac{2}{\pi}$. We have seen a proof of this in the notes.

Technique 2: Consider a square dartboard, and a circular target drawn inscribed in the square dartboard. A dart is thrown uniformly at random in the square. The probability the dart lies in the circle is $\frac{\pi}{4}$.

Technique 3: Pick two integers x and y independently and uniformly at random from 1 to M, inclusive. Let p_M be the probability that x and y are relatively prime. Then

$$\lim_{M\to\infty} p_M = \frac{6}{\pi^2}.$$

Let $p_1 = \frac{2}{\pi}, p_2 = \frac{\pi}{4}$, and $p_3 = \frac{6}{\pi^2}$ be the probabilities of the desired events of **Technique 1**, **Technique 2**, and **Technique 3**, respectively. For each technique, we apply each technique N times, then compute the proportion of the times each technique occurred, getting estimates $\hat{p}_1, \hat{p}_2$, and $\hat{p}_3$,, respectively.

(a) For each $\hat{p}_i$, compute an expression X_i in terms of $\hat{p}_i$ that would be an estimate of π.

(b) Using Chebyshev's Inequality, compute the minimum value of N such that X_2 is within ε of π with $1-\delta$ confidence. Your answer should be in terms of ε and δ.

For X_1 and X_3, computing the minimum value of N will be more tricky, as the expressions for X_1 and X_3 are not as nice as X_2.

(c) For $i = 1$ and 3, compute a constant c_i such that

$$|X_i - \pi| < \varepsilon \implies |\hat{p}_i - p_i| < c_i\varepsilon + o(\varepsilon^2),$$

where the $o(\varepsilon^2)$ represents terms containing powers of ε that are 2 or higher (i.e. $\varepsilon^2, \varepsilon^3$, etc.).

(Hint: You may find the following Taylor series helpful: For x close to 0,

$$\frac{1}{a-x} = \frac{1}{a} + \frac{x}{a^2} + o(x^2)$$
$$\frac{1}{(a-x)^2} = \frac{1}{a^2} + \frac{2x}{a^3} + o(x^2).$$

The $o(x^2)$ represents terms that have x^2 powers or higher.)

In this problem, we assume ε is close enough to 0 such that $o(\varepsilon^2)$ is 0. In other words,

$$\mathbb{P}\left[|\hat{p}_i - p_i| < c_i\varepsilon + o(\varepsilon^2)\right] = \mathbb{P}\left[|\hat{p}_i - p_i| < c_i\varepsilon\right].$$

Combining with part (c) then gives

$$\mathbb{P}\left[|X_i - \pi| < \varepsilon\right] \le \mathbb{P}\left[|\hat{p}_i - p_i| < c_i\varepsilon\right].$$

(d) For $i = 1$ and 3, use Chebyshev's Inequality and the above work to compute the minimum value of N such that X_i is within ε of π with $1 - \delta$ confidence. Your answer should be in terms of ε and δ.

(e) Which technique required the lowest value for N? Which technique required the highest?

2 Random Cuckoo Hashing

Cuckoo birds are parasitic beasts. They are known for hijacking the nests of other bird species and evicting the eggs already inside. Cuckoo hashing is inspired by this behavior. In cuckoo hashing, when we get a collision, the element that was already there gets evicted and rehashed.

We study a simple (but ineffective, as we'll see) version of cuckoo hashing, where all hashes are random. Let's say we want to hash n pieces of data $d_1, d_2, \ldots, d_n$ into n possible hash buckets labeled $1, \ldots, n$. We hash the $d_1, \ldots, d_n$ in that order. When hashing d_i, we assign it a random bucket chosen uniformly from $1, \ldots, n$. If there is no collision, then we place d_i into that bucket. If there is a collision with some other d_j, we evict d_j and assign it another random bucket uniformly from $1, \ldots, n$. (It is possible that d_j gets assigned back to the bucket it was just evicted from!) We again perform the eviction step if we get another collision. We keep doing this until there is no more collision, and we then introduce the next piece of data, d_{i+1} to the hash table.

(a) What is the probability that there are no collisions over the entire process of hashing $d_1, \ldots, d_n$ to buckets $1, \ldots, n$? What value does the probability tend towards as n grows very large?

(b) Assume we have already hashed $d_1, \ldots, d_{n-1}$, and they each occupy their own bucket. We now introduce d_n into our hash table. What is the expected number of collisions that we'll see while hashing d_n? (*Hint*: What happens when we hash d_n and get a collision, so we evict some other d_i and have to hash d_i? Are we at a situation that we've seen before?)

(c) Generalize the previous part: Assume we have already hashed $d_1, \ldots, d_{k-1}$ successfully, where $1 \le k \le n$. Let C_k be the number of collisions that we'll see while hashing d_k. What is $\mathbb{E}[C_k]$?

(d) Let C be the total number of collisions over the entire process of hashing $d_1, \ldots, d_n$. What is $\mathbb{E}[C]$? You may leave your answer as a summation.

3 Coupon Collector Variance

It's that time of the year again—Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

(a) Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\mathrm{Var}(X) = n^2 \left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}[X]$.

(b) The series $\sum_{i=1}^{\infty} i^{-2}$ converges to the constant value $\pi^2/6$. Using this fact and Chebyshev's Inequality, find a lower bound on β for which the probability you need to make more than $\mathbb{E}[X] + \beta n$ visits is less than $1/100$, for large n. *[Hint: Use the approximation $\sum_{i=1}^{n} i^{-1} \approx \ln n$ as n grows large.]*

4 Short Answer

(a) Let X be uniform on the interval $[0,2]$, and define $Y = 2X+1$. Find the PDF, CDF, expectation, and variance of Y.

(b) Let X and Y have joint distribution

$$f(x,y) = \begin{cases} cxy + \frac{1}{4} & x \in [1,2] \text{ and } y \in [0,2] \\ 0 & \text{otherwise.} \end{cases}$$

Find the constant c. Are X and Y independent?

(c) Let $X \sim \mathrm{Exp}(3)$.

(i) Find probability that $X \in [0, 1]$.

(ii) Let $Y = \lfloor X \rfloor$. For each $k \in \mathbb{N}$, what is the probability that $Y = k$? Write the distribution of Y in terms of one of the famous distributions; provide that distribution's name and parameters.

(d) Let $X_i \sim \text{Exp}(\lambda_i)$ for $i = 1, \ldots, n$ be mutually independent. It is a (very nice) fact that $\min(X_1, \ldots, X_n) \sim \text{Exp}(\mu)$. Find μ.

5 Useful Uniforms

Let X be a continuous random variable whose image is all of $\mathbb{R}$; that is, $\mathbb{P}[X \in (a,b)] > 0$ for all $a, b \in \mathbb{R}$ and $a \neq b$.

(a) Give an example of a distribution that X could have, and one that it could not.

(b) Show that the CDF F of X is strictly increasing. That is, $F(x+\varepsilon) > F(x)$ for any $\varepsilon > 0$. Argue why this implies that $F : \mathbb{R} \to (0,1)$ must be invertible.

(c) Let U be a uniform random variable on $(0,1)$. What is the distribution of $F^{-1}(U)$?

(d) Your work in part (c) shows that in order to sample X, it is enough to be able to sample U. If X was a discrete random variable instead, taking finitely many values, can we still use U to sample X?

6 It's Raining Fish

A hurricane just blew across the coast and flung a school of fish onto the road nearby the beach. The road starts at your house and is infinitely long. We will label a point on the road by its distance from your house (in miles). For each $n \in \mathbb{N}$, the number of fish that land on the segment of the road $[n, n+1]$ is independently Poisson(λ) and each fish that is flung into that segment of the road lands uniformly at random within the segment. Keep in mind that you can cite any result from lecture or discussion without proof.

(a) What is the distribution of the number of fish arriving in segment $[0,n]$ of the road, for some $n \in \mathbb{N}$?

(b) Let $[a,b]$ be an interval in $[0,1]$. What is the distribution of the number of fish that lands in the segment $[a,b]$ of the road?

(c) Let $[a,b]$ be any interval such that $a \geq 0$. What is the distribution of the number of fish that land in $[a,b]$?

(d) Suppose you take a stroll down the road. What is the distribution of the distance you walk (in miles) until you encounter the first fish?

(e) Suppose you encounter a fish at distance x. What is the distribution of the distance you walk until you encounter the next fish?

7 Waiting For the Bus

Edward and Jerry are waiting at the bus stop outside of Soda Hall.

Like many bus systems, buses arrive in periodic intervals. However, the Berkeley bus system is unreliable, so the length of these intervals are random, and follow Exponential distributions.

Edward is waiting for the 51B, which arrives according to an Exponential distribution with parameter λ. That is, if we let the random variable X_i correspond to the difference between the arrival time ith and $(i-1)$st bus (also known as the inter-arrival time) of the 51B, $X_i \sim \text{Expo}(\lambda)$.

Jerry is waiting for the 79, whose inter-arrival times also follows Exponential distributions with parameter μ. That is, if we let Y_i denote the inter-arrival time of the 79, $Y_i \sim \text{Expo}(\mu)$. Assume that all inter-arrival times are independent.

(a) What is the probability that Jerry's bus arrives before Edward's bus?

(b) After 20 minutes, the 79 arrives, and Jerry rides the bus. However, the 51B still hasn't arrived yet. Let D be the additional amount of time Edward needs to wait for the 51B to arrive. What is the distribution of D?

(c) Lavanya isn't picky, so she will wait until either the 51B or the 79 bus arrives. Find the distribution of Z, the amount of time Lavanya will wait before catching her bus.

(d) Khalil doesn't feel like riding the bus with Edward. He decides that he will wait for the second arrival of the 51B to ride the bus. Find the distribution of $T = X_1 + X_2$, the amount of time that Khalil will wait to ride the bus.