1 Natural Induction on Inequality

Prove that if $n \in \mathbb{N}$ and $x > 0$, then $(1+x)^n \geq 1+nx$.

Solution:

- *Base Case:* When $n = 0$, the claim holds since $(1+x)^0 \geq 1+0x$.
- *Inductive Hypothesis:* Assume that $(1+x)^k \geq 1+kx$ for some value of $n = k$ where $k \in \mathbb{N}$.
- *Inductive Step:* For $n = k+1$, we can show the following:

$$\begin{aligned}(1+x)^{k+1} = (1+x)^k(1+x) &\geq (1+kx)(1+x) \\ &\geq 1+kx+x+kx^2 \\ &\geq 1+(k+1)x+kx^2 \geq 1+(k+1)x\end{aligned}$$

By induction, we have shown that $\forall n \in \mathbb{N}, (1+x)^n \geq 1+nx$.

2 Make It Stronger

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$ for $n \geq 1$. We want to prove that

$$a_n \leq 3^{(2^n)}$$

for every positive integer n.

(a) Suppose that we want to prove this statement using induction. Can we let our inductive hypothesis be simply $a_n \leq 3^{(2^n)}$? Attempt an induction proof with this hypothesis to show why this does not work.

(b) Try to instead prove the statement $a_n \leq 3^{(2^n-1)}$ using induction.

(c) Why does the hypothesis in part (b) imply the overall claim?

Solution:

(a) Let's try to prove that for every $n \geq 1$, we have $a_n \leq 3^{2^n}$ by induction.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2^1} = 9$.
Inductive Step: For some $n \geq 1$, we assume $a_n \leq 3^{2^n}$. Now, consider $n+1$. We can write:

$$a_{n+1} = 3a_n^2 \leq 3(3^{2^n})^2 = 3 \times 3^{2\times 2^n} = 3 \times 3^{2^{n+1}} = 3^{2^{n+1}+1}.$$

However, what we wanted was to get an inequality of the form: $a_{n+1} \leq 3^{2^{n+1}}$. There is an extra $+1$ in the exponent of what we derived.

(b) This time the induction works.

Base Case: For $n = 1$ we have $a_1 = 1 \leq 3^{2-1} = 3$.
Inductive Step: For some $n \geq 1$ we assume $a_n \leq 3^{2^n-1}$. Now, consider $n+1$. We can write:

$$a_{n+1} = 3a_n^2 \leq 3 \times (3^{2^n-1})^2 = 3 \times 3^{2\times(2^n-1)} = 3 \times 3^{2^{n+1}-2} = 3^{2^{n+1}-1}.$$

This is exactly the induction hypothesis for $n+1$.

(c) For every $n \geq 1$, we have $2^n - 1 \leq 2^n$ and therefore $3^{2^n-1} \leq 3^{2^n}$. This means that our modified hypothesis which we proved in part (b) does indeed imply what we wanted to prove in part (a).

3 Binary Numbers

Prove that every positive integer n can be written in binary. In other words, prove that we can write

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

where $k \in \mathbb{N}$ and $c_i \in \{0,1\}$ for all $i \leq k$.

Solution:

Prove by strong induction on n.

The key insight here is that if n is divisible by 2, then it is easy to get a bit string representation of $(n+1)$ from that of n. However, if n is not divisible by 2, then $(n+1)$ will be, and its binary representation will be more easily derived from that of $(n+1)/2$. More formally:

- Base Case: $n = 1$ can be written as 1×2^0.
- Inductive Step: Assume that the statement is true for all $1 \leq m \leq n$, where n is arbitrary. Now, we need to consider $n+1$. If $n+1$ is divisible by 2, then we can apply our inductive hypothesis to $(n+1)/2$ and use its representation to express $n+1$ in the desired form.

$$(n+1)/2 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0$$
$$n+1 = 2 \cdot (n+1)/2 = c_k \cdot 2^{k+1} + c_{k-1} \cdot 2^k + \cdots + c_1 \cdot 2^2 + c_0 \cdot 2^1 + 0 \cdot 2^0.$$

Otherwise, n must be divisible by 2 and thus have $c_0 = 0$. We can obtain the representation of $n+1$ from n as follows:

$$n = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 0 \cdot 2^0$$
$$n+1 = c_k \cdot 2^k + c_{k-1} \cdot 2^{k-1} + \cdots + c_1 \cdot 2^1 + 1 \cdot 2^0$$

Therefore, the statement is true.

Here is another alternate solution emulating the algorithm of converting a decimal number to a binary number.

- Base Case: $n = 1$ can be written as 1×2^0.
- Inductive Step: Assume that the statement is true for all $1 \leq m \leq n$, for arbitrary n. We show that the statement holds for $n+1$. Let 2^m be the largest power of 2 such that $n+1 \geq 2^m$. Thus, $n+1 < 2^{m+1}$. We examine the number $(n+1) - 2^m$. Since $(n+1) - 2^m < n+1$, the inductive hypothesis holds, so we have a binary representation for $(n+1) - 2^m$. Also, since $n+1 < 2^{m+1}$, $(n+1) - 2^m < 2^m$, so the largest power of 2 in the representation of $(n+1) - 2^m$ is 2^{m-1}. Thus, by the inductive hypothesis,

  $$(n+1) - 2^m = c_{m-1} \cdot 2^{m-1} + c_{m-2} \cdot 2^{m-2} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

  and adding 2^m to both sides gives

  $$n+1 = 2^m + c_{m-1} \cdot 2^{m-1} + c_{m-2} \cdot 2^{m-2} + \cdots + c_1 \cdot 2^1 + c_0 \cdot 2^0,$$

  which is a binary representation for $n+1$. Thus, the induction is complete.

Another intuition is that if x has a binary representation, $2x$ and $2x+1$ do as well: shift the bits and possibly place 1 in the last bit. The above induction could then have proceeded from n and used the binary representation of $\lfloor n/2 \rfloor$, shifting and possibly setting the first bit depending on whether n is odd or even.

Note: In proofs using simple induction, we only use $P(n)$ in order to prove $P(n+1)$. Simple induction gets stuck here because in order to prove $P(n+1)$ in the inductive step, we need to assume more than just $P(n)$. This is because it is not immediately clear how to get a representation for $P(n+1)$ using just $P(n)$, particularly in the case that $n+1$ is divisible by 2. As a result, we assume the statement to be true for all of $1, 2, \ldots, n$ in order to prove it for $P(n+1)$.

4 Fibonacci for Home

Recall, the Fibonacci numbers, defined recursively as

$F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-2} + F_{n-1}$.

Prove that every third Fibonacci number is even. For example, $F_3 = 2$ is even and $F_6 = 8$ is even.

Solution:

First, we should prove that all the Fibonacci numbers are integer by induction: $P(k)$ is "F_k is an integer". This follows from the fact that F_1 and F_2 are integer, and the induction step follows from $F_k = F_{k-1} + F_{k-2}$, the (strong) induction hypothesis that F_{k-1} and F_{k-2} are integers and the fact that the integers are closed under addition.

Now we prove that for all natural numbers $k \geq 1$, F_{3k} is even. The base case, $k = 1$, is that $F_3 = 2$ is even, which is clear.

For the induction step, we have that $F_{3k+3} = F_{3k+2} + F_{3k+1} = 2F_{3k+1} + F_{3k}$.

By the induction hypothesis $F_{3k} = 2q$ for some q, and we have that $F_{3k+3} = 2(F_{3k+1} + q)$, which implies that it is even. Thus, by induction we have that all F_{3k} are even.