Counting - and Lots of It

Computer Science Mentors 70

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Reminder: This worksheet is not meant to be covered in two hours.

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Order? Replacement? Who knows!

1. Say I have a standard 6-sided die. I generate a sequence of numbers by tossing this die 5 times.

   (a) How many distinct sequences of 5 numbers can I generate this way?

   Solution: $6^5 = 7775$

   (b) True or False: The sequence: $(6, 6, 6, 6, 6)$ is less likely than the sequence $(6, 5, 6, 6, 5)$.

   Solution: False. All sequences are equally likely.

   (c) True or False: The sequence: $(6, 6, 6, 6, 6)$ is less likely than a sequence $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{5, 6\}$.

   Solution: True. There are more sequences of the second form.

   (d) How many sequences have the form $(6, 6, 6, 6, 6)$? How many have the form $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{5, 6\}$?

   Solution: 1 and 32. For the first one, there is one choice for each of the digits five times, $1^5 = 5$. In the second one, there are two choices for each of the digits, $2^5 = 32$.

   (e) What is the probability of generating a sequence of the form $(6, 6, 6, 6, 6)$? What about of the form $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{5, 6\}$?

   Solution: $\frac{1}{6^5} = \frac{1}{7776}$ and $\frac{32}{7776} = \frac{1}{243}$. Since the probability space is uniform, the probability of each sequence is the ratio of the number of possibilities to the size of the sample space.

2. Suppose that my 5 friends are roommates, and that they also share a sock drawer. In this sock drawer they have 10 pairs of socks. Each pair of socks is a different color, but it is folded together so that no roommate is ever wearing two different colors of socks.

   (a) How many distinct roommate-sock combinations are there?

   Solution: For the socks of the first roommate, we have 10 choices, for the second one, we have 9 choices and so on. The answer is $10 \times 9 \times 8 \times 7 \times 6 = 30240$.

   (b) Each day, the roommates record the combination of sock colors that they are all wearing in a special scrapbook. How many such combinations are possible?

Solution: The answer is $\binom{10}{5} = 252$ which is the number of ways we can pick 5 out of the 10 socks when the order does not matter.

(c) Say that one roommate decides he likes the pink, red, and teal socks more than all of the rest, and decides to jealously hoard them so that no one else can ever wear them (but he can still also wear the other 7 colors of socks). Now, how many distinct roommate-sock combinations are there?

Solution: Let's begin with the other roommates choosing their socks. The first one has 7 choices, the second one has 6, the third one 5, and the fourth one 4. Now the roommate who has hoarded socks can pick any pair from the remaining 3 hoarded socks and 3 not-hoarded ones. So this roommate has 6 choices. The answer is $7 \times 6 \times 5 \times 4 \times 6 = 5040$.

(d) The other roommates grow frustrated with the hoarding roommate and tell him that if he does not return the pink, red, and teal socks, then he is not allowed to wear the other colors of socks. The hoarding roommate does not relent. Now, how many distinct roommate-sock combinations are there? Please enter your answer as an integer.

Solution: The hoarding roommate has 3 choices. The next roommate has 7 (all the not-hoarded socks), the next one 6, the next 5, and the next 4. So the answer is $7 \times 6 \times 4 \times 3 = 2520$.

(e) How many color combinations of socks are now possible in the scrapbook? Please enter your answer as an integer.

Solution: We need to pick 4 colors from the 7 not-hoarded socks and one from the hoarded socks. So the answer is $\binom{7}{4} \times \binom{3}{1} = 105$.

3. How many ways can you give 10 cookies to 4 friends?

Solution: Count the number of ways to give 10 cookies to 4 friends if some can get no cookies. The number of ways is $\binom{13}{3} = 286$.

Superman?

1. How many ways are there to arrange the letters of the word "SUPERMAN"

(a) On a straight line?

Solution: 8!

(b) On a straight line, such that "SUPER" occurs as a substring?

Solution: 4! Treat "SUPER" as one character.

(c) On a circle? Note: If we arrange elements on a circle, all permutations that are "shifts" are equivalent (i.e. SUPERMAN and UPERMANS).

Solution: 7! Anchor one element, arrange the other 7 around in a line.

(d) On a circle, such that "SUPER" occurs as a substring? Reminder: SUPER can occur anywhere on the circle!

> **Solution:** 3! Treat "SUPER" as a single character, anchor one element, and arrange the other 3 around in a line.

2. Now how many ways are there to arrange the letters of the word "SUPERMAN"

   (a) On a straight line, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

   > **Solution:** 3! * $\binom{8}{3}$ This reduces to a stars and bars problem–the S U P E R are bars, and we want to put M A N somewhere in the sequence. Once we do so, there can be any permutation of M A N within the bars. Equivalently, we can arrange the letters of SUPERMAN 8! ways, but divide by 5! because we have arranged SUPER in any of 5! ways, when we only want one way. This gives us 8! / 5!, which is equal to 3! * 8! / (5! 3!) = 3! * $\binom{8}{3}$.

   (b) On a circle, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?

   > **Solution:** $\binom{7}{3} * 3!$ = $\binom{7}{2} * 2! * 5 = 210$. There are two methods. Method 1: anchor one of S, U, P, E, R. Choose which 3 places to put the M, A, and N (7 choose 3) and allow them to be shuffled (3!). Then the U, P, E, R must fill in the remaining slots in order. Method 2: anchor one of M, A, N. Choose which of the 2 remaining 8 spots to place the A and N, allowing shuffles (7 choose 2 * 2!). Then, choose which of the 5 remaining spots to place the S (the other letters must follow in order after the S).

3. How many solutions does $x + y + z = 10$ have, if all variables must be positive integers?

   > **Solution:** We can think of this in terms of stars and bars. We have two bars between the variables x, y, and z, and our stars are the 10 1s we have to distribute among them. Since all variables must be positive integers, x, y, and z will each be at least 1. So, we have 7 1s left to distribute. So $n = 7$ stars, $k = 2$ bars. Answer = $\binom{n+k}{k} = \binom{9}{2} = 36$.

Combinatorial Proofs

1. A combinatorial proof is a proof which shows that two quantities are the same by explaining that each quantity is a different way of counting the same thing. This question is intended to help you see how this technique is applied.

   Which of the following are valid ways of counting the given quantity (allowing you to deduce that the formulae given by each way of counting are equal)?

   (a) In an $n \times n$ grid, there are n rows of squares, each of which has n squares in it. Thus, there are n^2 squares in an $n \times n$ grid.

   (b) We know there are exactly n squares on the diagonal. Now, when we remove the diagonal, we have two equally sized triangles that have $n - 1$ squares on the hypotenuse. When we remove those, we end up with smaller triangles with $n - 2$ squares on the hypothenuse. We continue this until we are left with one square on each side, and we've counted all of the squares in the grid. This gives us a total of $n + 2\sum_{k=1}^{n-1} k$ squares in the grid.

   (c) Take the $(n - 1) \times (n - 1)$ subgrid that is the upper lefthand corner of this grid. This subgrid has $n - 1$ rows, each of which has $n - 1$ squares, so this part contributes $(n - 1)^2$ squares. Now, the squares that we excluded from this subgrid come to a total of $n + n - 1$ squares. Thus, there are $(n - 1)^2 + 2n - 1$ squares in an $n \times n$ grid.

   (d) First, we peel off the leftmost column, and topmost row, removing exactly $2n - 1$ squares. We then peel off the leftmost column and topmost row remaining, removing exactly $2(n - 1) - 1$ squares. We continue this process until we are left with a single square, which we also remove. This gives us a total of $(2n - 1) + (2n - 3) + \cdots + 3 + 1 = \sum_{k=1}^{n} 2k - 1$ squares in the $n \times n$ grid.

More practice building confidence with combinatorial arguments

2. Prove $k\binom{n}{k} = n\binom{n-1}{k-1}$ by a combinatorial proof.

> **Solution:** Choose a team of k players where one of the players is the captain.
> LHS: Pick a team with k players. This is $\binom{n}{k}$. Then make one of the players the captain. There are k options for the captain so we get $k \times \binom{n}{k}$.
> RHS: Pick the captain. There are n choices for the captain. Now pick the last $k-1$ players on the team. There are now $n-1$ people to choose from. So we get $n \times \binom{n-1}{k-1}$.

3. Prove $n! = \binom{n}{k}k!(n-k)!$ by a combinatorial proof.

> **Solution:** Arrange n items.
> LHS: Number of ways to order n items.
> RHS: Choose k items without ordering. Order these k items. Order the remaining $n-k$ items.

4. Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.

> **Solution:** Suppose that you have a group of n players.
> LHS: Number of ways to pick a team of a of these players, designate one member of the team as captain, and then pick one reserve player from the remaining $n-a$ people.
> RHS: The right-hand side is the number of ways to pick the captain, then the reserve player, and then the other $a-1$ members of the team.