Counting - and Lots of It

Computer Science Mentors 70

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Prepared by: Debayan Bandyopadhyay, Catherine Huang, Abinav Routhu, Robert Wang, Sebastien Whetsel

Reminder: This worksheet is not meant to be covered in two hours.

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Order? Replacement? Who knows!

1. Say I have a standard 6-sided die. I generate a sequence of numbers by tossing this die 5 times.

   (a) How many distinct sequences of 5 numbers can I generate this way?

   (b) True or False: The sequence: $(6, 6, 6, 6, 6)$ is less likely than the sequence $(6, 5, 6, 6, 5)$.

   (c) True or False: The sequence: $(6, 6, 6, 6, 6)$ is less likely than a sequence $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{5, 6\}$.

   (d) How many sequences have the form $(6, 6, 6, 6, 6)$? How many have the form $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{5, 6\}$?

   (e) What is the probability of generating a sequence of the form $(6, 6, 6, 6, 6)$? What about of the form $(x_1, x_2, x_3, x_4, x_5)$ where $x_i \in \{5, 6\}$?

2. Suppose that my 5 friends are roommates, and that they also share a sock drawer. In this sock drawer they have 10 pairs of socks. Each pair of socks is a different color, but it is folded together so that no roommate is ever wearing two different colors of socks.

   (a) How many distinct roommate-sock combinations are there?

   (b) Each day, the roommates record the combination of sock colors that they are all wearing in a special scrapbook. How many such combinations are possible?

   (c) Say that one roommate decides he likes the pink, red, and teal socks more than all of the rest, and decides to jealously hoard them so that no one else can ever wear them (but he can still also wear the other 7 colors of socks). Now, how many distinct roommate-sock combinations are there?

(d) The other roommates grow frustrated with the hoarding roommate and tell him that if he does not return the pink, red, and teal socks, then he is not allowed to wear the other colors of socks. The hoarding roommate does not relent. Now, how many distinct roommate-sock combinations are there? Please enter your answer as an integer.

(e) How many color combinations of socks are now possible in the scrapbook? Please enter your answer as an integer.

3. How many ways can you give 10 cookies to 4 friends?

Superman?

1. How many ways are there to arrange the letters of the word "SUPERMAN"
   (a) On a straight line?
   (b) On a straight line, such that "SUPER" occurs as a substring?
   (c) On a circle? Note: If we arrange elements on a circle, all permutations that are “shifts” are equivalent (i.e. SUPERMAN and UPERMANS).
   (d) On a circle, such that "SUPER" occurs as a substring? Reminder: SUPER can occur anywhere on the circle!
2. Now how many ways are there to arrange the letters of the word "SUPERMAN"
   (a) On a straight line, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?
   (b) On a circle, such that "SUPER" occurs as a subsequence (S U P E R appear in that order, but not necessarily next to each other)?
3. How many solutions does $x + y + z = 10$ have, if all variables must be positive integers?

Combinatorial Proofs

1. A combinatorial proof is a proof which shows that two quantities are the same by explaining that each quantity is a different way of counting the same thing. This question is intended to help you see how this technique is applied.

   Which of the following are valid ways of counting the given quantity (allowing you to deduce that the formulae given by each way of counting are equal)?

   (a) In an $n \times n$ grid, there are n rows of squares, each of which has n squares in it. Thus, there are n^2 squares in an $n \times n$ grid.
   (b) We know there are exactly n squares on the diagonal. Now, when we remove the diagonal, we have two equally sized triangles that have $n - 1$ squares on the hypotenuse. When we remove those, we end up with smaller triangles with $n - 2$

squares on the hypothenuse. We continue this until we are left with one square on each side, and we've counted all of the squares in the grid. This gives us a total of $n + 2\sum_{k=1}^{n-1} k$ squares in the grid.

(c) Take the $(n-1) \times (n-1)$ subgrid that is the upper lefthand corner of this grid. This subgrid has $n-1$ rows, each of which has $n-1$ squares, so this part contributes $(n-1)^2$ squares. Now, the squares that we excluded from this subgrid come to a total of $n + n - 1$ squares. Thus, there are $(n-1)^2 + 2n - 1$ squares in an $n \times n$ grid.

(d) First, we peel off the leftmost column, and topmost row, removing exactly $2n-1$ squares. We then peel off the leftmost column and topmost row remaining, removing exactly $2(n-1)-1$ squares. We continue this process until we are left with a single square, which we also remove. This gives us a total of $(2n-1) + (2n-3) + \cdots + 3 + 1 = \sum_{k=1}^{n} 2k - 1$ squares in the $n \times n$ grid.

More practice building confidence with combinatorial arguments

2. Prove $k\binom{n}{k} = n\binom{n-1}{k-1}$ by a combinatorial proof.

3. Prove $n! = \binom{n}{k}k!(n-k)!$ by a combinatorial proof.

4. Prove $a(n-a)\binom{n}{a} = n(n-1)\binom{n-2}{a-1}$ by a combinatorial proof.