Math 110 Homework 6

Tarang Srivastava

1 Exercises 5.A

Problem 1: The argument is as follows.

(a) Let u be an arbitrary vector $u \in U$. If $U \subset \text{null } T$, then $u \in \text{null } T$. So, $Tu = 0$. Since, U is a vector space, it must be that $0 \in U$, so $Tu \in U$. Thus, U is invariant under T given the condition.

(b) By definition we have $Tu \in \text{range } T$ for all $u \in U$. Since, range $T \subset U$ we have that for all u, $Tu \in U$. Thus, U is invariant under T given the condition.

Problem 3: We wish to show that for all $u \in \text{range } S$ we have that $Tu \in \text{range } S$. Let $v \in V$, then $STv \in \text{range } S$ by definition. Given $ST = TS$, we have that $STv = TSv$. So, $TSv \in \text{range } S$. Let $u \in \text{range } S$, then there exists some $v \in V$ such that $Sv = u$. Since,$TSv \in \text{range } S$, we have $Tu \in \text{range } S$.

Problem 6: True!

We have a subspace U of V such that it is invariant for all $T \in \mathcal{L}(V, V)$, assume for contradiction that $U \neq 0$ and $U \neq V$. Then, since V is finite dimensional we have some basis of U

$$u_1, ..., u_m \text{ is a basis of } U$$

Then we can extend the basis of U to a basis of V, and since we know that $U \neq V$ it must be that we must extend it by at least one vector.

$$u_1, ..., u_m, v_1, ..., v_n \text{ is a basis of } V$$

Then, let $T \in \mathcal{L}(V)$ such that for all $i \in 1, ..., n$ we have that

$$Tu_i = v_i$$

and the remaining basis vectors of U, if there are any, are mapped to 0. Let u be an arbitrary vector $u \in U$, then

$$u = a_1u_1 + ... + a_mu_m$$

for some scalars $a_1, ..., a_m$. Then,

$$Tu = a_1Tu_1 + ... + a_mTu_m$$
$$Tu = a_1v_1 + ... + a_mv_m$$

Since we have that U is invariant under all linear maps it must be that $Tu \in U$ so there exists some linear combination of $u_1, ..., u_m$ that is equal to Tu. So for some scalars $b_1, .., b_m$

$$Tu = b_1u_1 + ... + b_mu_m$$

Substituting the two reprsentations of Tu we get

$$b_1u_1 + ... + b_mu_m = a_1v_1 + ... + a_mv_m$$

Then, we have a contradiction since we claimed that $u_1, ..., u_m, v_1, ..., v_n$ is a basis and therefore a linearly independent list of vectors. But since they can be expressed as a linear combination of each other as such they are not linearly independent, by some previous exercises. Thus, it must be that $U = \{0\}$ or $U = V$.