Math 110 Homework 11

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1 Exercises 7.B

Problem 3: Define the linear map T as follows for each $z = (z_1, z_2, z_3) \in \mathbb{C}^3$.

$$Tz = (2z_1, 3z_2, 2z_1)$$

Note the switch up in the last spot. We can check that T is indeed closed under addition and scalar multiplication, thus a linear map. Firstly, $\lambda = 2$ is an eigenvalue with the eigenvector $w = (1, 0, 1)$.

$$T(1, 0, 1) = (2, 0, 2) = 2(1, 0, 1)$$

Similarly, $\lambda = 3$ is an eigenvalue with the eigenvector $w = (0, 1, 0)$.

$$T(0, 1, 0) = (0, 3, 0) = 3(0, 1, 0)$$

Finally, $(T^2 - 5T + 6I)z \neq 0$ if $z_3 \neq 6z_1$. So given we have values that are nonzero after that linear map, $T^2 - 5T + 6I \neq 0$.

Problem 4: Given that $\mathbb{F} = \mathbb{C}$ by the complex spectral theorem T is normal if and only if T has a diagonal matrix with respect to some orthonormal basis. Since T is diagonalizable if and only if V is equal to the sum of the eigenspaces of distinct eigenvalues, we have shown the last case. Also by the complex spectral theorem T is normal if and only if V has an orthonormal basis consisting of eigenvectors of T. That completes the proof. (Ask Alex if there's some part that guarantees that we have distinct eigenvalues, but it is not that difficult to show that if all the basis eigenvectors are orthogonal they must be associated with distinct eigenvalues.)

Problem 6: $\Longrightarrow$ direction follows easily from 7.13, if T is self-adjoint then all the eigenvalues are real.
$\Longleftarrow$ Given T is normal then T^* and T have the same eigenvectors and for the corresponding eigenvalue λ for T, T^* has the associated eigenvalue $\overline{\lambda}$ for the same eigenvector. If all the eigenvalues are real $\lambda = \overline{\lambda}$, so T and T^* have the same eigenvalues and eigenvectors. By the complex spectral theorem V has an orthogonal basis consisting of eigenvectors of T. Denote these basis eigenvectors as $e_1, ..., e_n$. Let $v \in V$, then for some $a_1, ..., a_n \in \mathbb{C}$

$$\begin{aligned} Tv &= a_1 T(e_1) + ... + a_n T(e_n) \\ &= a_1 \lambda_1 e_1 + ... + a_n \lambda_n e_n \\ &= a_1 T^*(e_1) + ... + a_n T^*(e_n) \\ Tv &= T^* v \end{aligned}$$

For all $v \in V$ so $T = T^*$, therefore T is self adjoint.

Problem 8: Let T be the operator associated with the matrix and some basis B for the complex vector space

$$\mathcal{M}(T, B) = \begin{pmatrix} 0 & \dots & 1 \\ & \ddots & \vdots \\ 0 & & 0 \end{pmatrix}$$

Then, clearly $T^2 = 0$ so $T \neq T^2$ and $T^8 = 0 = T^9$. (I'm not sure what the point of this question was, but I definitely missed it.)

Problem 10: This is so hard wtf.

Problem 11: By the real or complex spectral theorem given that T is self-adjoint it is diagonalizable. Suppose the digaonal matrix is

$$T = \begin{pmatrix} a_1 & \dots & 0 \\ & \ddots & \vdots \\ 0 & & a_n \end{pmatrix}$$

Then, we can take the cube root of each value in the diagonal, which is well defined for complex or real numbers. So we get

$$S = \begin{pmatrix} a_1^{1/3} & \dots & 0 \\ & \ddots & \vdots \\ 0 & & a_n^{1/3} \end{pmatrix}$$

Then, clearly it holds that $S^3 = T$.

Problem 12:

Problem 13:

Problem 14: $\Longleftarrow$ is given by the real spectral theorem.
$\Longrightarrow$ We observe that since there is a basis of eigenvectors of T, we have

$$v = a_1 e_1 + ... + a_n e_n$$

For some basis and scalars. Then

$$Tv = \lambda_1 a_1 e_1 + ... + \lambda_n a_n e_n$$

Suppose we state that $w = v$