Math 110 Homework 8

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1 Exercises 5.B

Problem 1: We wish to show that

$$(I-T)^{-1} = I + T + ... + T^{n-1}$$

So, we can multiply $I - T$ to both sides to get

$$I = (I-T)(I + T + ... + T^{n-1})$$

Then, we can distribute and see we get

$$I = I - T + T - T^2 + T^2 + ... - T^{n-1} + T^{n-1} + T^n$$

After cancelling out all the similar terms we are left with

$$I = I + T^n = I$$

Since, $T^n = 0$. So, to prove the statement we do the following operations

$$I = I + T^n = I$$

$$I = I - T + T - T^2 + T^2 + ... - T^{n-1} + T^{n-1} + T^n$$

$$I = (I-T)(I + T + ... + T^{n-1})$$

Then, we multiply both sides by $(I-T)^{-1}$ to get

$$(I-T)^{-1} = I + T + ... + T^{n-1}$$

as desired.

Problem 2: Assume for contradiction that $\lambda \neq 2$ and $\lambda \neq 3$ and $\lambda \neq 4$. Then, $T - 2I$ and $T - 3I$ and $T - 4I$ must all be invertible. Given,

$$(T-2I)(T-3I)(T-4I) = 0$$

for all $v \in V$ such that $v \neq 0$ we have that

$$(T-2I)(T-3I)(T-4I)v = 0v = 0$$

Then, for one of the values $T - 2I$ or $T - 3I$ or $T - 4I$ one of them maps v to 0. Since, they are all invertible their null spaces is just $\{0\}$, but then we have a contradiction since we had that $v \neq 0$. Therefore, it must be the case that λ is equal to 2, 3 or 4.

Problem 3: We proceed directly. Given

$$T^2 = I$$

We get that $T^2 - I = 0$ so,

$$(T-I)(T+I) = 0$$

Since, $\lambda \neq -1$ it must be that $T + I$ is invertible. Thus, for all non zero v in V we have that $(T+I)v$ is non zero. So it must be that for all $w \in V$, we have $(T-I)v = 0$. By definition $T - I$ is equal to the 0 linear map, so from $T - I = 0$ it follows that $T = I$

Problem 4: From $P^2 = P$ we have that $P^2 - P = 0$ so it must be that

$$P(P-I) = 0$$

So for all $v \in V$ we have that

$$P(P-I)v = 0v = 0$$

Thus

$$Pv = 0 \text{ or } Pv = v$$

To show that $V = \text{null } P \oplus \text{range } P$ we first will show that $\text{null } P \cap \text{range } P = \{0\}$. Suppose $v \in \text{null } P \cap \text{range } P$. Then, $Pv = 0$ and $Pv = v$ it follows directly then that $v = 0$, so

$$\text{null } P \cap \text{range } P = \{0\}$$

Since, $P \in \mathcal{L}(V)$ we already have that range $P \subset V$ and null $P \subset V$. Then for $v \in \text{range } P$ and $w \in \text{null } P$ clearly $v + w \in V$ so we have that

$$V \supset \text{range } P \oplus \text{null } P$$

For the other side, let $v \in V$ we have that $Pv = 0$ or $Pv = v$. So, $v \in \text{range } P$ or $v \in \text{null } P$ then it follows that clearly for all $v \in V$ we have that $v = v + 0$ or $v = 0 + v$. So,

$$V \subset \text{range } P \oplus \text{null } P$$

Therfore,

$$V = \text{range } P \oplus \text{null } P$$

Problem 8: I'm just going to express T as a matrix in terms of the standard basis of $\mathbb{R}^2$, I hope that's okay. So, the goal is that $T^4 = -1$ well if we find T such that it represent an eighth clockwise turn, we are done. Since 4 turns of T would be equivalent to -1. We can construct T by simply seeing where the basis vectors would be mapped if they were turned by an eigth. So, $(1, 0)$ would map to $(\sqrt{2}/2, \sqrt{2}/2)$ and then $(0, 1)$ would map to $(\sqrt{2}/2, -\sqrt{2}/2)$. That is basically the definition of T, but in matrix form it is

$$T = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

Problem 9: If $p(T)v = 0$ by properties of polynomials we can then express $p(T)$ as a polynomial where we factor out the zero. That is for some $q \in \mathcal{P}(\mathbb{F})$ and some constant $\lambda \in \mathbb{F}$,

$$p(T) = q(T) \cdot (T - \lambda)$$

Then we have that, $(T - \lambda)v = 0$, so then by rearranging we get that in fact $Tv = \lambda v$. So, λ must be an eigenvector of T.

Problem 10: Given λ is an eigenvalue of T we have that $Tv = \lambda v$ for some eigenvector v. Thus, we know that $T(Tv) = T\lambda v = \lambda T v = \lambda^2 v$. We then use this inductive argument to get that $T^n v = \lambda^n v$. Since, $p(T)$ is of some form as $p(T)v = a_m T^m v + ...$ we can simply replace each term of $T^m v$ with $\lambda^m v$, then we get it is equivalent that $p(T)v = a_m \lambda^m v + ...$ which is equivalent to expressing the polynomial for λ That is, $p(\lambda)v = a_m \lambda v + ...$ so we get

$$p(T)v = p(\lambda)v$$

Problem 11: $\implies$ direction. Sice we are working with $\mathbb{C}$ we have the nice property that

$$p(T) = c(T - \lambda_1 I)...(T - \lambda_m I)$$

$\Longleftarrow$ direction. We know that if $\alpha = p(\lambda)$ for some eigenvalue λ of T then from Problem 10 we know that

$$p(T)v = p(\lambda)v$$

where v is the corresponding eigenvector. Then, we directly have that

$$p(T)v = \alpha v$$

Thus, by definition α is an eigenvalue of $p(T)$.

2 Exercises 5.C

Problem 1: We use the fact that since T is diagonalizable then

$$V = E(\lambda_1, T) \oplus ... \oplus E(\lambda_m, T)$$

So, now we just wish to show that null T and range T are some combination of these eigenspaces. Observe that null T is equivalent to when $Tv = 0 = 0v$ that is $\lambda = 0$. So, if there is an eigenvalue equal to zero then null $T = E(0, T)$. Then, observe that V has a basis consiting of eigenvectors of T, so it suffices to show that if there is a basis of eigenvectors for all $v \in V$ we have that $v = a_1u_1 + ... + a_mu_m$ where u_i is an eigenvector. Well since $E(\lambda, T)$ is a subspace of V it holds that any scalar multiple of u_i is also in its corresponding eigenspace. Therefore, for all $v \in V$ since we can express it as a sum of $v = a_1u_1 + ... + a_mu_m$ and each term is in a corresponding eigenspace then all the $Tv \neq 0$ are expressed as that sum which is equivalent to the range of T. Thus, $V = \text{null } T \oplus \text{range } T$.

Problem 3: Well (b) and (c) are literally the definition of (a) so showing (b) and (c) are equivalent suffices to show (a) is equivalent. For (b) $\Longrightarrow$ (c). We have by nullity-rank that

$$\dim V = nullity + rank$$

We also have that since the null space and range are subspaces of V

$$\dim(\text{null } T + \text{range } T) = nullity + rank - (\text{null } T \cap \text{range } T)$$

But since we have that $V = \text{null } T + \text{range } T$ We get that

$$nullity + rank = nullity + rank - \dim(\text{null } T \cap \text{range } T)$$

So that implies that $\dim(\text{null } T \cap \text{range } T) = 0$ thus it must be that $(\text{null } T \cap \text{range } T) = \{0\}$.
For (b) $\Longleftarrow$ (c). We have that

$$\dim(\text{null } T + \text{range } T) = nullity + rank - (\text{null } T \cap \text{range } T)$$

Since, $(\text{null } T \cap \text{range } T) = \{0\}$ by combining nullity-rank theorem we have

$$\dim(\text{null } T + \text{range } T) = nullity + rank = \dim V$$

Thus it must be in a finite dimensional space that

$$\text{null } T + \text{range } T = V$$

Problem 5: Given that T is diagonal we know that by subtracting λI we still have a matrix that has elements only in the diagonal and zeros everywhere else. So it follows from definition that $T - \lambda I$ is diagonal as well. After that we can directly apply Problem 1 to show $\Longrightarrow$ direction. For the $\Longleftarrow$ direction.

Problem 16: (a) We proceed by induction on n. For the base case $n = 1$ we have that

$$T(0, 1) = (1, 1) = (F_1, F_2)$$

Then assume the claim holds for some $n = k$. That is,

$$T^k(0, 1) = (F_k, F_{k+1})$$

Then we can just compose T again to get

$$T^{k+1}(0, 1) = T(F_k, F_{k+1}) = (F_{k+1}, F_k + F_{k+1}) = (F_{k+1}, F_{k+2})$$

So we showed the inductive step holds and thus the claim holds for all n.
(b) this homework is so fucking long and so fucking boring fuck this shit . I miss discrete math with its fun hw problems.

3 Exercises 6.A

Problem 11: This is a direct application of Cauchy-Schwarz inequality. The statement holds for all positive a, b, c, d because we set it up with the terms $\frac{1}{\sqrt{a}}$ and $\sqrt{a}$ and $\frac{1}{\sqrt{b}}$ and $\sqrt{b}$ and $\frac{1}{\sqrt{c}}$ and $\sqrt{c}$ and $\frac{1}{\sqrt{c}}$ and $\sqrt{c}$. Then we have something of the form

$$|1 + 1 + 1 + 1|^2 = 16 \leq (a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$$

which we get from just squaring all the terms before to fit the Cauchy-Schwarz inequality form.