

## 1 Exercise 2.A

**Problem 8:** The statement holds. Assume for contradiction that  $\lambda v_1, \dots, \lambda v_m$  is linearly dependent. That is, for  $\lambda \neq 0$ ,  $\lambda v_1 + \dots + \lambda v_m = 0$ . Then we have a contradiction, since we claimed that  $v_1, \dots, v_m$  are linearly independent, and we have an example of a linear combination with non-zero coefficients that is equal to zero. Then,  $\lambda v_1, \dots, \lambda v_m$  must be linearly independent.

**Problem 9:** The statement is false. Consider the counterexample where  $w_i = -v_i$  for  $i \in \{1, \dots, m\}$ . We know that  $w_i \in V$ , since the additive inverse must exist in  $V$  for  $v_i$ . Then, for non-zero coefficients the linear combination  $(v_1 + w_1) + \dots + (v_m + w_m) = 0$  which shows that it is linearly dependent.

**Problem 10:** To show that  $w \in \text{span}(v_1, \dots, v_m)$  we just have to show there exists a linear combination such that

$$w = a_1 v_1 + \dots + a_m v_m$$

By linear dependence we have  $a_1(v_1 + w) + \dots + a_m(v_m + w) = 0$  for nonzero coefficients. We can collect that  $w$  terms and bring it to the other side of the equal sign and divide by the coefficients, again because they are not all zero, to get

$$a_1 v_1 + \dots + a_m v_m = (a_1 + \dots + a_m)w$$

$$\frac{a_1}{(a_1 + \dots + a_m)} v_1 + \dots + \frac{a_m}{(a_1 + \dots + a_m)} v_m = w$$

Thus, showing that  $w$  is in the span of  $v_1, \dots, v_m$ .

**Problem 11:** To show the " $\implies$ " direction, assume for contradiction  $w \in \text{span}(v_1, \dots, v_m)$ . Then, we can express  $w$  as a linear combination of  $v_1, \dots, v_m$ .  $w = a_1 v_1 + \dots + a_m v_m$ . Let  $a_1 v_1 + \dots + a_m v_m - w$  be a linear combination, but here we have a contradiction. Since,  $a_1 v_1 + \dots + a_m v_m - w = a_1 v_1 + \dots + a_m v_m - (a_1 v_1 + \dots + a_m v_m) = 0$  for non-zero coefficients  $a_1, \dots, a_m$ . Therefore,  $w \notin \text{span}(v_1, \dots, v_m)$ .

For the " $\impliedby$ " direction, assume for contradiction  $v_1, \dots, v_m, w$  is linearly dependent. Then, there exists a linear combination with nonzero coefficients such that  $a_1 v_1 + \dots + a_m v_m + cw = 0$ . Subtracting by  $cw$  and dividing by  $c$ , for when  $c \neq 0$ , if  $c = 0$  we are done, we get  $\frac{a_1}{c} v_1 + \dots + \frac{a_m}{c} v_m = w$ , which is a contradiction, since it implies that  $w \in \text{span}(v_1, \dots, v_m)$ . Thus, it must be that  $v_1, \dots, v_m, w$  is linearly independent.