Math 110 Homework 10 4/10/2020

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Exercises 6.C

(3) Firstly, since $v_1,\dots,u_m$ is a basis of U and the Gram-Schmidt process for $u_1,\dots,u_m$ only uses those vectors; $e_1,\dots,e_m$ is a valid orthonormal basis for U. We then just wish to show that $U^\perp = \text{span}(f_1,\dots,f_n)$. Let $v \in V$, then

$v = a_1e_1 + \dots + a_me_m + b_1f_1 + \dots + b_nf_n$. So, $U^\perp = \{x : \langle v, u\rangle = 0 \text{ for all } u \in U\}$. Since, $u \in \text{span}(e_1,\dots,e_m)$ if $a_1,\dots,a_m \neq 0$ then there exist some $u \in U$ such that $\langle v, u\rangle \neq 0$. Since, $\langle v, u\rangle = \langle a_1e_1, u\rangle + \dots + \langle a_me_m, u\rangle$ and we can just let $u = e_m$ then the value is nonzero. Thus, $u^\perp$ comprises only of vectors that are a linear combination of $f_1,\dots,f_n$, so $U^\perp = \text{span}(f_1 \dots f_n)$.

(5)
Let $v \in V$, then for any subspace U of V, there exists an $U^\perp$ such that $V = U \oplus U^\perp$. So, $v = u + w$, where $u \in U$ and $w \in U^\perp$. Then

$(P_{U^\perp})v = w = (I - P_U)v = Iv - (P_U)v$
$= v - u = w$

So, $P_{U^\perp} = I - P_U$.

(6) For $\Leftarrow$ direction since $\langle u, w\rangle = 0$ for all $u \in U$ and all $w \in W$ it must be that $U \subset W^\perp$ or $W \subset U^\perp$. Since $P_Wv \in W$ for all $v \in V$, then $P_Wv \in U^\perp$, so clearly $P_U(P_Wv) = 0$, thus $P_UP_W = 0$.

$\Rightarrow$ IF $P_UP_W = 0$, then for all $w \in W$ $(P_UP_W)w = P_Uw = 0$, so $w \in U^\perp$, thus $W \subset U^\perp$. So it follows for all $u \in U$ and $w \in W$, $\langle u, w\rangle = 0$.

(7) The subspaces $U = \text{range } P$ is the subspace U such that $P = P_U$. Since, has all the properties of P_U from 6.55. Since P is a linear operator the range forms a subspace. Sorry I don't know what more to say it seems pretty straight forward.

(8) We wish to show that $\|Pv\| \leq \|v\|$ implies that range P is orthogonal to null P. Suppose $u \in \text{range } P$, then there exists a $v \in V$ such that $u = Pv$. So, $Pu = P^2v = Pv = u$

Now let $v \in V$ such that $v = u + w$ where $u \in \text{range } P$ and $w \in \text{null } P$. So, $Pv = Pu + Pw = u$. Given that $\|Pv\| \leq \|v\|$ we have $\|u\| = \|Pv\| \leq \|u + w\|$ Then from 6.A.6 as the hint suggests $\langle u, v\rangle = 0$, so we can use 6.C.7 that since all the vectors in the null P are orthogonal to the vectors in range P there exists a U such that $P_U = P$.

Exercises 7.A

(2) By definition of an eigenvalue $T - \lambda I = 0$. Then applying 7.6 we get $(T - \lambda I)^* = T^* - \bar{\lambda} I = 0$, so then by definition $\bar{\lambda}$ is an eigenvalue of T^*.

(3) By the same argument as in lecture. Let $u \in U$ and $w \in U^\perp$. Then, $\langle u, w\rangle = 0$. Since U is T-invariant $\langle Tu, w\rangle = 0$ and $\langle u, T^*w\rangle$. Since the orthogonality condition holds for all $w \in U^\perp$, $U^\perp$ must be T^* invariant.

(4) These are all straightforward from 7.7 and rank-nullity theorems.

a) $\Rightarrow$ if T is injective, then null $T = \{0\}$ so $(\text{range } T^*)^\perp = \{0\}$, thus by dimensions of orthogonal compliments range $T^* = V$, so T^* is surjective.

$\Leftarrow$ if T^* is surjective, then range $T^* = V = (\text{null } T)^\perp$ so null $T = \{0\}$, so T is injective

b) $\Rightarrow$ if T is surjective, range $T = W = (\text{null } T^*)^\perp$ so null $T^* = \{0\}$, then T^* is injective

$\Leftarrow$ if T^* is injective, null $T^* = \{0\} = (\text{range } T)^\perp$ so range $T = W$, thus T is surjective.

(5) All these follow from the rank-nullity theorem, various dimension theorems, and 7.7

$$\dim \operatorname{null} T^* = \dim (\operatorname{range} T)^\perp$$
$$= \dim W - \dim \operatorname{range} T$$
$$= \dim W - \dim V + \dim \operatorname{null} T$$

and

$$\dim \operatorname{range} T^* = \dim (\operatorname{null} T)^\perp$$
$$= \dim V - \dim \operatorname{null} T$$
$$= \dim V - \dim V + \dim \operatorname{range} T$$
$$= \dim \operatorname{range} T$$

(6)

a) let $p = x^2$ and $q = x$. Observe $p, q \in \mathcal{P}_2(\mathbb{R})$. Then, $\langle Tp, q \rangle = 0$, since $T(x^2) = 0$. And $\langle p, T^*q \rangle = \langle p, Tq \rangle = \langle p, q \rangle$ which is something not zero. So proof by counterexample.

b) This is not a contradiction since our inner product space is different and not the traditional euclidean inner product associated with $\mathbb{R}^n$.

(9) The self adjoint operators are not a subspace because they are not closed under scalar multiplication, specifically scalar multiplication by complex numbers. Suppose $T \in \mathcal{L}(V)$. T is a self adjoint operator. Then, T must have a real eigenvalue $Tv = \lambda v$, but $iTv = \lambda i v$ is not self adjoint since its eigenvalue is not real.

(10) From lecture. Let $T \in \mathcal{L}(V)$ not be a normal operator, but since it can be expressed as $T = \frac{T+T^*}{2} + \frac{T-T^*}{2}$ where it is a sum of two normal operators, observe $\left(\frac{T+T^*}{2}\right)^* = \frac{T^*+T}{2}$, so clearly the set of normal operators is not closed under vector addition.

(11) $\Longrightarrow$ if $P = P_U$, then let $v_{1,2} \in V$ and $u_{1,2} \in U$ and $w_{1,2} \in U^\perp$ such that $v_1 = u_1 + w_1$ and $v_2 = u_2 + w_2$. Then, $\langle P_U v_1, v_2 \rangle = \langle u_1, v_2 \rangle = \langle u_1, u_2 \rangle + \langle u_1, w_2 \rangle = \langle u_1, u_2 \rangle$ since $w_2 \in U^\perp$ so $\langle u_1, w_2 \rangle = 0$. Similarly we find that $\langle v_1, P_U v_2 \rangle = \langle u_1, u_2 \rangle$. So $\langle P v_1, v_2 \rangle = \langle v_1, P v_2 \rangle$ thus P is self-adjoint.

$\Longleftarrow$ if P is self-adjoint, then we claim that the subspace U is range P. Let $u \in \operatorname{range} P$ and $w \in \operatorname{null} P$. Let u' be in V such that $Pu' = u$. Then $P^2 u' = Pu$. But $P^2 u' = Pu' = u = Pu$. Then by our hypothesis

$\langle Pu, w \rangle = \langle u, w \rangle = \langle u, Pw \rangle = \langle u, 0 \rangle = 0$. So we showed that for all $u \in \operatorname{range} P$ and $w \in \operatorname{null} P$ they are orthogonal, then by 6.A.7 it must be that $P = P_U$.

(12) Let u and w be the eigenvectors associated with the eigenvalues $\lambda = 3$ and $\lambda = 4$. Observe that $u' = \frac{u}{\|u\|}$ is also an eigenvector since $\frac{1}{\|u\|} Tu = \lambda \frac{u}{\|u\|}$. Same argument holds for $w' = \frac{w}{\|w\|}$. Since u' and w' are orthogonal and have length 1, $v = u' + w'$ will have length $\|v\| = \sqrt{2}$, by Pythagorean theorem, (and linear algebra)

Then observe $Tv = Tu' + Tw' = 3u' + 4w'$

So, $\|Tv\| = \|Tu' + Tw'\| = \|3u' + 4w'\| = \sqrt{25} = 5$

Since, they are orthogonal vectors with length scaled by 3 and 4 respectively.

$\|3u' + 4w'\| = \sqrt{9(1) + 16(1)}$

Math 110 HW 10 Scratch Work

⑤ $P_{U^\perp} = I - P_U$

Let $v \in V$ s.t. $v = u + w$, $u \in U$ and $w \in U^\perp$

$P_{U^\perp} v = w = (I - P_u)v = Iv - P_u v$

$w = v - u = w$

⑥ $\Leftarrow$

if $\langle u, w \rangle = 0$ for all $u \in U$ and $w \in W$

$U \subset W^\perp$ and $W \subset U^\perp$

$U^\perp = W \oplus X$

$V = U \oplus U^\perp$

$= U \oplus W \oplus X$

$P_W v \in W$

$P_W v \in U^\perp$

so $P_U(P_W v) = 0$

⑦ Claim that $U =$ range P is a subspace of V such that $P = P_U$

Since it has all the properties that every vector in null P ...

⑧ Let $u \in$ Range P

$\langle Pv, Pv \rangle \leq \langle v, v \rangle$

$v \in$ null P

$0 \leq \langle v, v \rangle - \langle Pv, Pv \rangle$

$u \in$

$u \in$ range P

$v \in$ null P

P_U

$u \in$ range P

$w \in$ null P

$\exists v \in V$ s.t. $Pv = u$

$Pv = u$

$Pv = P^2 v = Pu$

$u = Pu$

For $v = u + w$

$Pv =$

$\|u\| =$

$(T - \lambda I)^* = T^* - (\lambda I)^*$

$T^* - \bar{\lambda} I^*$

$T^* - \bar{\lambda} I = 0$

$\langle Tv, w \rangle = \langle v, T^* w \rangle$

$x \in U^\perp$

$\langle Tv, w \rangle = \langle v, T^* w \rangle$

$0 = \langle v', w \rangle =$

$0 = \langle v, w \rangle$

$0 = \langle Tv, w \rangle = \langle v, T^* w \rangle = \langle v, w' \rangle$

this holds for all w so

⑤ $\dim \operatorname{null} T^* = \dim (\operatorname{range} T)^\perp$

$= \dim W - \dim \operatorname{range} T$

$= \dim W - \dim V + \dim \operatorname{null} T$

$\dim \operatorname{range} T^* = \dim (\operatorname{null} T)^\perp$

$= \dim V - \dim \operatorname{null} T$

$= \quad - \dim V + \dim \operatorname{range} T$

$(T + S)v$

$(T^* + S^*)(T + S)v$

$(T^* T + S^* T + T^* S + S S^*)v$

T

$T = \frac{T + T^*}{2} + \frac{T - T^*}{2}$

$\frac{T + T^*}{2} = \frac{T^* + T}{2}$

SCRATCH WORK

(11) $\cancel{P}$ $v = u + w$

$\cancel{z = x + y}$

$v' = u' + w'$

(v, v')

$(Pv, v') = (v, P^{*} v')$

(u, v') $(v, u'$

$(u, u') + (u, w')^{0}$ $(u, u') + \cancel{(w, u')}^{0}$

$\cancel{PT} =$

$\cancel{Let}$ $u \in$ range P

$Pv = u$ $w \in$ null P

$= Pv = P^2 v = Pu$ w, u

$Pu = u$

u, v

$Tv = \lambda v$

$\langle Pu, v \rangle =$

$\langle Pu, v \rangle = \langle u, 0 \rangle = 0$

$\langle v, v \rangle = 2$

$\|v\| = \sqrt{2}$

$T(v_1 + v_2)$

$\lambda v_1 + v$

$Tv_1 \perp Tv_2$

$v' = \frac{v}{\|v\|}$ $\langle 3v_1 + 4v_2, 3v_1 + 4v_2 \rangle$

$\sqrt{9 + 16} = \sqrt{25}$

$\lambda^2 \langle v, v \rangle$

$\lambda^2 \langle$