

1 Exercises 5.A

Problem 1: The argument is as follows.

- (a) Let u be an arbitrary vector $u \in U$. If $U \subset \text{null } T$, then $u \in \text{null } T$. So, $Tu = 0$. Since, U is a vector space, it must be that $0 \in U$, so $Tu \in U$. Thus, U is invariant under T given the condition.
- (b) By definition we have $Tu \in \text{range } T$ for all $u \in U$. Since, $\text{range } T \subset U$ we have that for all u , $Tu \in U$. Thus, U is invariant under T given the condition.

Problem 3: We wish to show that for all $u \in \text{range } S$ we have that $Tu \in \text{range } S$. Let $v \in V$, then $STv \in \text{range } S$ by definition. Given $ST = TS$, we have that $STv = TSv$. So, $TSv \in \text{range } S$. Let $u \in \text{range } S$, then there exists some $v \in V$ such that $Sv = u$. Since, $TSv \in \text{range } S$, we have $Tu \in \text{range } S$.

Problem 6: True!

We have a subspace U of V such that it is invariant for all $T \in \mathcal{L}(V, V)$, assume for contradiction that $U \neq 0$ and $U \neq V$. Then, since V is finite dimensional we have some basis of U

$$u_1, \dots, u_m \text{ is a basis of } U$$

Then we can extend the basis of U to a basis of V , and since we know that $U \neq V$ it must be that we must extend it by at least one vector.

$$u_1, \dots, u_m, v_1, \dots, v_n \text{ is a basis of } V$$

Then, let $T \in \mathcal{L}(V)$ such that for all $i \in 1, \dots, n$ we have that

$$Tu_i = v_i$$

and the remaining basis vectors of U , if there are any, are mapped to 0. Let u be an arbitrary vector $u \in U$, then

$$u = a_1u_1 + \dots + a_mu_m$$

for some scalars a_1, \dots, a_m . Then,

$$Tu = a_1Tu_1 + \dots + a_mTu_m$$

$$Tu = a_1v_1 + \dots + a_mv_m$$

Since we have that U is invariant under all linear maps it must be that $Tu \in U$ so there exists some linear combination of u_1, \dots, u_m that is equal to Tu . So for some scalars b_1, \dots, b_m

$$Tu = b_1u_1 + \dots + b_mu_m$$

Substituting the two representations of Tu we get

$$b_1u_1 + \dots + b_mu_m = a_1v_1 + \dots + a_mv_m$$

Then, we have a contradiction since we claimed that $u_1, \dots, u_m, v_1, \dots, v_n$ is a basis and therefore a linearly independent list of vectors. But since they can be expressed as a linear combination of each other as such they are not linearly independent, by some previous exercises. Thus, it must be that $U = \{0\}$ or $U = V$.

Problem 8: By definition we wish to find eigenvalues and eigenvectors, $v = (w, z)$ such that

$$T(w, z) = (z, w) = \lambda(w, z) = (\lambda w, \lambda z)$$

Then, we have to find solutions to $\lambda w = z$ and $\lambda z = w$. Following some substitutions we get

$$z(\lambda^2 - 1) = 0$$

Since, $v \neq 0$ we are left with $\lambda = \pm 1$.

So, $\lambda_1 = 1$ with the corresponding eigenvectors some scalar multiple $v_1 = (1, 1)$ and $\lambda_2 = -1$ with the corresponding eigenvectors some scalar multiple $v_2 = (-1, 1)$.

Problem 12: We wish to find eigenvalues and eigenvectors, $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, such that

$$(Tp)(x) = xp'(x) = \lambda p(x)$$

That is, $4a_4x^4 + 3a_3x^3 + 2a_2x^2 + a_1x = \lambda a_4x^4 + \lambda a_3x^3 + \lambda a_2x^2 + \lambda a_1x + \lambda a_0$. So, clearly $4a_4x^4 = \lambda a_4x^4$. Solving for this we get $\lambda = 4$, but then the following terms do not hold so we must have that $a_3 = a_2 = a_1 = a_0 = 0$. So the polynomial for $\lambda = 4$ must be of the form a_4x^4 . We follow with this argument for the remaining terms to get that the eigenvalues are $\lambda = 4, 3, 2, 1$ and that the corresponding eigenvectors are some scalar multiple of x^4, x^3, x^2, x respectively.

Problem 13: We can just show that $\alpha - \lambda \leq |\alpha - \lambda| < \frac{1}{1000}$ is equivalent to showing $\alpha < \frac{1}{1000} + \lambda$ since we are working with elements in our field and performing field operation it must be the case that $\alpha \in \mathbb{F}$. (Clearly this won't hold in ALL fields but in the world of Axler this is just \mathbb{C} or \mathbb{R}). Then since V is finite dimension there are at most $\dim V$ many eigenvalues, that is finite number of eigenvalues. Again, since we are either working with either the reals or complex, which are both infinite just select a number α that satisfies the inequality and is not equal to one of the eigenvalues. Then, it follows by 5.6, since we chose α such that it is not an eigenvalue $T - \alpha I$ is invertible.

Problem 15: (a) Let λ be an eigenvalue of T . Then, $Tv = \lambda v$ for some corresponding eigenvector v . Let, u be a vector in V such that $Su = v$. We know this exists, since S is an invertible operator on V . Therefore,

$$Tv = \lambda v$$

$$TSu = \lambda Su$$

Then composing with S^{-1} we get

$$S^{-1}TSu = S^{-1}\lambda Su$$

$$= \lambda S^{-1}Su$$

$$= \lambda u$$

Thus, the eigenvalues are the same.

(b) From our argument above we had that the corresponding eigenvectors v of T have the eigenvector u for $S^{-1}TS$ such that $v = Su$ or $S^{-1}v = u$. That is the relationship.

Problem 19: We have that for $T(1, 1, \dots, 1) = (n, n, \dots, n) = \lambda(1, 1, \dots, 1)$ if we have $\lambda = n$ we satisfy the inequality. So, the corresponding eigenvector to $\lambda = n$ is any scalar multiple of $(1, \dots, 1)$. These are all the eigenvalues and eigenvectors of T , since by Kubrat's hint the Trace is equal to the sum of the eigenvalues and the trace is equal to n if there are only 1s in the main diagonal. Since, we found an eigenvalue equal to n there can be no more, so we are done.