Section 1.A – $\mathbb{R}^n$ and $\mathbb{C}^n$

$-\alpha$, subtraction, $1/\alpha$, division: Let $\alpha, \beta \in \mathbf{C}$ · Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that

$$\alpha + (-\alpha) = 0$$

· Subtraction on $\mathbf{C}$ is defined by

$$\beta - \alpha = \beta + (-\alpha)$$

· For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that

$$\alpha(1/\alpha) = 1$$

· Division on $\mathbf{C}$ is defined by

$$\beta/\alpha = \beta(1/\alpha)$$

list, length: Suppose n is a nonnegative integer. A list of length n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this:

$$(x_1, \ldots, x_n)$$

Two lists are equal if and only if they have the same length and the same elements in the same order.

$\mathbb{F}^n$: $\mathbf{F}^n$ is the set of all lists of length n of elements of $\mathbf{F}$:

$$\mathbf{F}^n = \left\{(x_1, \ldots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \ldots, n\right\}$$

For $(x_1, \ldots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the j^{th} coordinate of $(x_1, \ldots, x_n)$

addition in $\mathbb{F}^n$: Addition in $\mathbf{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n)+(y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

Commutativity of addition in $\mathbb{F}^n$: If $x, y \in \mathbf{F}^n$, then $x + y = y + x$

0: Let 0 denote the list of length n whose coordinates are all 0 :

$$0 = (0, \ldots, 0)$$

additive inverse in $\mathbb{F}^n$: For $x \in \mathbf{F}^n$, the additive inverse of x, denoted $-x$, is the vector $-x \in \mathbf{F}^n$ such that $x + (-x) = 0$ In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$

scalar multiplication in $\mathbb{F}^n$: The product of a number λ and a vector in $\mathbf{F}^n$ is computed by multiplying each coordinate of the vector by λ :

$$\lambda\,(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

here $\lambda \in \mathbf{F}$ and $(x_1, \ldots, x_n) \in \mathbf{F}^n$

Section 1.B – Definition of Vector Space

addition, scalar multiplication: · An addition on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$ · A scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$

Vector Space: A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold: commutativity

$$u + v = v + u \text{ for all } u, v \in V$$

associativity $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$
and all $a, b \in \mathbf{F}$

additive identity there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$ additive inverse for every $v \in V$, there exists $w \in V$ such that $v + w = 0$ multiplicative identity $1v = v$ for all $v \in V$ distributive properties

$$a(u+v) = au+av \text{ and } (a+b)v = av+bv \text{ for all } a, b \in \mathbf{F} \text{ and}$$

all $u, v \in V$

vector, point: Elements of a vector space are called vectors or points.

real vector space, complex vector space: • A vector space over $\mathbf{R}$ is called a real vector space. · A vector space over $\mathbf{C}$ is called a complex vector space.

$\mathbb{F}^S$: · If S is a set, then $\mathbf{F}^S$ denotes the set of functions from S to $\mathbf{F}$ · For $f, g \in \mathbf{F}^S$, the sum $f + g \in \mathbf{F}^S$ is the function defined by

$$(f + g)(x) = f(x) + g(x)$$

for all $x \in S$ • For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the product $\lambda f \in \mathbf{F}^S$ is the function defined by

$$(\lambda f)(x) = \lambda f(x)$$

for all $x \in S$

Unique Additive Identity: A vector space has a unique additive identity

Unique additive inverse: Every element in a vector space has a unique additive inverse.

The number 0 times a vector: $0v = 0$ for every $v \in V$

A number times the vector 0: $a0 = 0$ for every $a \in \mathbf{F}$

The number -1 times a vector: $(-1)v = -v$ for every $v \in V$

Section 1.C – Subspaces

Subspace: A subset U of V is called a subspace of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Conditions for a subspace: A subset U of V is a subspace of V if and only if U satisfies the following three conditions: additive identity $0 \in U$ closed under addition $u, w \in U$ implies $u + w \in U$ closed under scalar multiplication $a \in \mathbf{F}$ and $u \in U$ implies $au \in U$

sum of subsets: Suppose $U_1, \ldots, U_m$ are subsets of V. The sum of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$ More precisely,

$$U_1+\cdots+U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}$$

Sum of subspaces is the smalles containing subspace: Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$

direct sum: Suppose $U_1, \ldots, U_m$ are subspaces of V · The sum $U_1 + \cdots + U_m$ is called a direct sum if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j · If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

Condition for a direct sum: Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$

Direct sum of two subspaces: Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$