Section 1.A – $\mathbb{R}^n$ and $\mathbb{C}^n$

$-\alpha$, subtraction, $1/\alpha$, division: Let $\alpha, \beta \in \mathbf{C}$ • Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that $\alpha + (-\alpha) = 0$ • Subtraction on $\mathbf{C}$ is defined by $\beta - \alpha = \beta + (-\alpha)$ • For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that $\alpha(1/\alpha) = 1$ • Division on $\mathbf{C}$ is defined by $\beta/\alpha = \beta(1/\alpha)$

list, length: Suppose n is a nonnegative integer. A list of length n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this: $(x_1, \ldots, x_n)$ Two lists are equal if and only if they have the same length and the same elements in the same order.

$\mathbb{F}^n$: $\mathbf{F}^n$ is the set of all lists of length n of elements of $\mathbf{F}$: $\mathbf{F}^n = \left\{(x_1, \ldots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \ldots, n\right\}$ For $(x_1, \ldots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \ldots, n\}$, we say that x_j is the j^{th} coordinate of $(x_1, \ldots, x_n)$

addition in $\mathbb{F}^n$: Addition in $\mathbf{F}^n$ is defined by adding corresponding coordinates: $(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$

Commutativity of addition in $\mathbb{F}^n$: If $x, y \in \mathbf{F}^n$, then $x + y = y + x$

0: Let 0 denote the list of length n whose coordinates are all 0 : $0 = (0, \ldots, 0)$

additive inverse in $\mathbb{F}^n$: For $x \in \mathbf{F}^n$, the additive inverse of x, denoted $-x$, is the vector $-x \in \mathbf{F}^n$ such that $x + (-x) = 0$ In other words, if $x = (x_1, \ldots, x_n)$, then $-x = (-x_1, \ldots, -x_n)$

scalar multiplication in $\mathbb{F}^n$: The product of a number λ and a vector in $\mathbf{F}^n$ is computed by multiplying each coordinate of the vector by λ : $\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$ here $\lambda \in \mathbf{F}$ and $(x_1, \ldots, x_n) \in \mathbf{F}^n$

Section 1.B – Definition of Vector Space

addition, scalar multiplication: • An addition on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$ • A scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$

Vector Space: A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold: commutativity $u + v = v + u$ for all $u, v \in V$ associativity $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$ additive identity there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$ additive inverse for every $v \in V$, there exists $w \in V$ such that $v + w = 0$ multiplicative identity $1v = v$ for all $v \in V$ distributive properties $a(u+v) = au + av$ and $(a+b)v = av + bv$ for all $a, b \in \mathbf{F}$ and all $u, v \in V$

vector, point: Elements of a vector space are called vectors or points.

real vector space, complex vector space: • A vector space over $\mathbf{R}$ is called a real vector space. • A vector space over $\mathbf{C}$ is called a complex vector space.

$\mathbb{F}^S$: • If S is a set, then $\mathbf{F}^S$ denotes the set of functions from S to $\mathbf{F}$ • For $f, g \in \mathbf{F}^S$, the sum $f + g \in \mathbf{F}^S$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$ • For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the product $\lambda f \in \mathbf{F}^S$ is the function defined by $(\lambda f)(x) = \lambda f(x)$ for all $x \in S$

Unique Additive Identity: A vector space has a unique additive identity

Unique additive inverse: Every element in a vector space has a unique additive inverse.

The number 0 times a vector: $0v = 0$ for every $v \in V$

A number times the vector 0: $a0 = 0$ for every $a \in \mathbf{F}$

The number -1 times a vector: $(-1)v = -v$ for every $v \in V$

Section 1.C – Subspaces

Subspace: A subset U of V is called a subspace of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Conditions for a subspace: A subset U of V is a subspace of V if and only if U satisfies the following three conditions: additive identity $0 \in U$ closed under addition $u, w \in U$ implies $u + w \in U$ closed under scalar multiplication $a \in \mathbf{F}$ and $u \in U$ implies $au \in U$

sum of subsets: Suppose $U_1, \ldots, U_m$ are subsets of V. The sum of $U_1, \ldots, U_m$, denoted $U_1 + \cdots + U_m$, is the set of all possible sums of elements of $U_1, \ldots, U_m$ More precisely,

$$U_1 + \cdots + U_m = \{u_1 + \cdots + u_m : u_1 \in U_1, \ldots, u_m \in U_m\}$$

Sum of subspaces is the smalles containing subspace: Suppose $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is the smallest subspace of V containing $U_1, \ldots, U_m$

direct sum: Suppose $U_1, \ldots, U_m$ are subspaces of V • The sum $U_1 + \cdots + U_m$ is called a direct sum if each element of $U_1 + \cdots + U_m$ can be written in only one way as a sum $u_1 + \cdots + u_m$, where each u_j is in U_j • If $U_1 + \cdots + U_m$ is a direct sum, then $U_1 \oplus \cdots \oplus U_m$ denotes $U_1 + \cdots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.

Condition for a direct sum: Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$

Direct sum of two subspaces: Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$

Section 2.A Span and Linear Independence

Span: The set of all linear combinations of a list of vectors $v_1, \ldots, v_m$ in V is called the span of $v_1, \ldots, v_m$, denoted $\operatorname{span}(v_1, \ldots, v_m)$. In other words,

$$\operatorname{span}(v_1, \ldots, v_m) = \{a_1 v_1 + \cdots + a_m v_m : a_1, \ldots, a_m \in \mathbf{F}\}$$

The span of the empty list ( ) is defined to be $\{0\}$.

Span is the smallest containing subspace: The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

spans: If $\operatorname{span}(v_1, \ldots, v_m)$ equals V, we say that $v_1, \ldots, v_m$ spans V

finite-dimensional vector space: A vector space is called finite-dimensional if some list of vectors in it spans the space.

polynomial over a field F: A function $p : \mathbf{F} \to \mathbf{F}$ is called a polynomial with coefficients in $\mathbf{F}$ if there exist $a_0, \ldots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$. $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.

degree of a polynomial: • A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have degree m if there exist scalars $a_0, a_1, \ldots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

for all $z \in \mathbf{F}$. If p has degree m, we write $\deg p = m$
• The polynomial that is identically 0 is said to have degree $-\infty$.

$\mathcal{P}_m(\mathbb{F})$: For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.

infinite-dimensional vector space: A vector space is called infinite-dimensional if it is not finite-dimensional.

linearly independent: • A list $v_1, \ldots, v_m$ of vectors in V is called linearly independent if the only choice of $a_1, \ldots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \cdots + a_m v_m$ equal 0 is $a_1 = \cdots = a_m = 0$
• The empty list ( ) is also declared to be linearly independent.

linearly dependent: • A list of vectors in V is called linearly dependent if it is not linearly independent.
• In other words, a list $v_1, \ldots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \ldots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \cdots + a_m v_m = 0$

Linear Dependence Lemma: Suppose $v_1, \ldots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \ldots, m\}$ such that the following hold: (a) $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$ (b) if the j^{th} term is removed from $v_1, \ldots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \ldots, v_m)$

Length of linearly independent list $\leq$ length of spanning list: In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Finite-dimensional subspaces: Every subspace of a finite-dimensional vector space is finite dimensional.

Section 2.B Bases

basis: A basis of V is a list of vectors in V that is linearly independent and spans V

Criterion for basis: A list $v_1, \ldots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \cdots + a_n v_n$ where $a_1, \ldots, a_n \in \mathbf{F}$

Spanning list contains a basis: Every spanning list in a vector space can be reduced to a basis of the vector space.

Linearly independent list extends to a basis: Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Every subspace V is part of a direct sum equal to V.: Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$

Section 2.C Dimension

dimension, $\dim V$: The dimension of a finite-dimensional vector space is the length of any basis of the vector space. The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

Dimension of subspace: If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$

Linearly independent list of the right length is a basis: Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V

Spanning list of the right length is a basis: Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V

Dimension of a sum: If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Section 3.A The Vector Space of Linear Maps

linear map: A linear map from V to W is a function $T : V \to W$ with the following properties: additivity

$$T(u + v) = Tu + Tv \text{ for all } u, v \in V$$

homogeneity

$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbf{F} \text{ and all } v \in V$$

Notation $\mathcal{L}(V, W)$: The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$

Linear maps and basis of domain: Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \ldots, n$

additiona nd scalar multiplication on linear maps: Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The sum $S + T$ and the product λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$

$\mathcal{L}(V, W)$ is a vector space: With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.

Product of Linear Maps: If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$

Algebraic Properties of products of linear maps: **associativity**

$$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$

whenever T_1, T_2, and T_3 are linear maps such that the products make sense (meaning that T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1). **identity**

$$TI = IT = T$$

whenever $T \in \mathcal{L}(V, W)$ (the first I is the identity map on V, and the second I is the identity map on W). **distributive properties**

$$(S_1 + S_2) T = S_1 T + S_2 T \quad \text{and} \quad S(T_1 + T_2) = ST_1 + ST_2$$

whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$

Linear maps take 0 to 0: Suppose T is a linear map from V to W. Then $T(0) = 0$

Section 3.B Null Spaces and Ranges

null space: For $T \in \mathcal{L}(V, W)$, the null space of T, denoted null T, is the subset of V consisting of those vectors that T maps to 0 :

$$\operatorname{null} T = \{v \in V : Tv = 0\}$$

injective: A function $T : V \to W$ is called injective if $Tu = Tv$ implies $u = v$

Injectivity is equivalent to null space equals $\{0\}$: Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$

range: For T a function from V to W, the range of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv : v \in V\}$$

The range is subspace: If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W

surjective: A function $T : V \to W$ is called surfective if its range equals W

Fundamental Theorem of Linear Maps: Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

A map to a smaller dimensional space is not injective: Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.

A map to a larger dimensional space is not surjective: Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.

Homogeneous system of linear equations: A homogeneous system of linear equations with more variables than equations has nonzero solutions.

Inhomogeneous system of linear equations: An inhomogeneous system of linear equations with more equations thavariables has no solution for some choice of the constant terms.

Section 3.C Matrices

matrix, $A_{j,k}$: Let m and n denote positive integers. An m -by-n matrix A is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A. In other words, the first index refers to the row number and the second index refers to the column number.

matrix of a linear map, $\mathcal{M}(T)$: Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The matrix of T with respect to these bases is the $m - by - n$ matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k} w_1 + \cdots + A_{m,k} w_m$$

If the bases are not clear from the context, then the notation $\mathcal{M}\left(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m)\right)$ is used.

$\mathbb{F}^{m,n}$: For m and n positive integers, the set of all $m - by - n$ matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$

$\dim \mathbb{F}^{m,n} = mn$: Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space with dimension mn

matrix multiplication: Suppose A is an m – by – n matrix and C is an n -by – p matrix. Then AC is defined to be the m -by-p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

In other words, the entry in row j, column k, of AC is computed by taking row j of A and column k of C, multiplying together corresponding entries, and then summing.

$A_{j,\cdot}, A_{\cdot,k}$: Suppose A is an m -by $-n$ matrix. • If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the 1 – by – n matrix consisting of row j of A • If $1 \leq k \leq n$, then $A_{\cdot,k}$ denotes the m – by – 1 matrix consisting of column k of A

Entry of matrix product equals row times column: Suppose A is an m-by-n matrix and C is an n -by- p matrix. Then

$$(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$$

for $1 \leq j \leq m$ and $1 \leq k \leq p$

Column of matrix product equals matrix times column: Suppose A is an m– by-n matrix and C is an n -by-p matrix. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k}$$

for $1 \leq k \leq p$

Linear combination of columns: Suppose A is an $m - by - n$ matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n -by-1 matrix. Then

$$Ac = c_1 A_{\cdot 1} + \cdots + c_n A_{\cdot n}$$

In other words, Ac is a linear combination of the columns of A, with the scalars that multiply the columns coming from c.

Section 3.D Invertibility and Isomorphic Vector Spaces

invertible, inverse: • A linear map $T \in \mathcal{L}(V, W)$ is called invertible if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity map on V and TS equals the identity map on W • A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an inverse of T (note that the first I is the identity map on V and the second I is the identity map on W).

Inverse is unique: An invertible linear map has a unique inverse.

T^{-1}: If T is invertible, then its inverse is denoted by T^{-1}. In other words, if $T \in \mathcal{L}(V, W)$ is invertible, then T^{-1} is the unique element of $\mathcal{L}(W, V)$ such that $T^{-1}T = I$ and $TT^{-1} = I$

Invertibility is equivalent to injectivity and surjectivity: A linear map is invertible if and only if it is injective and surjective.

isomorphism, isomorphic: • An isomorphism is an invertible linear map. • Two vector spaces are called isomorphic if there is an isomorphism from one vector space onto the other one.

Dimension shows whether vector spaces are isomorphic: Two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.

$\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$ are isomorphic: Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1 \ldots w_m$ is a basis of W Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbf{F}^{m,n}$

$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$: Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finitedimensional implies the title.

matrix of a vector, $\mathcal{M}(v)$: Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. The matrix of v with respect to this basis is the n -by- 1 matrix

$$\mathcal{M}(v) = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

where $c_1, \ldots, c_n$ are the scalars such that

$$v = c_1 v_1 + \cdots + c_n v_n$$

$\mathcal{M}(T)_{\cdot,k} = \mathcal{M}(v_k)$: Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Let $1 \leq k \leq n$. Then the k^{th} column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$, equals $\mathcal{M}(v_k)$

Linear maps act like matrix multiplication: Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$$

operator, $\mathcal{L}(V)$: • A linear map from a vector space to itself is called an operator. • The notation $\mathcal{L}(V)$ denotes the set of all operators on V. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$

Injectivity is equivalent to surjectivity in finite dimensions: Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the following are equivalent: (a) T is invertible; (b) T is injective; (c) T is surjective.

Section 3.E Products and Quotients of Vector Spaces

product of vector space: Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbf{F}$ • The product $V_1 \times \cdots \times V_m$ is defined by

$$V_1 \times \cdots \times V_m = \left\{(v_1, \ldots, v_m) : v_1 \in V_1, \ldots, v_m \in V_m\right\}$$

• Addition on $V_1 \times \cdots \times V_m$ is defined by

$$(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, u_m + v_m)$$

• Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by

$$\lambda(v_1, \ldots, v_m) = (\lambda v_1, \ldots, \lambda v_m)$$

Product of vector spaces is a vector space: Suppose $V_1, \ldots, V_m$ are vector spaces over $\mathbf{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbf{F}$

Dimension of a product is the sum of dimensions: Suppose $V_1, \ldots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m$$

Products and direct sums: Suppose that $U_1, \ldots, U_m$ are subspaces of V. Define a linear map

$$\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m \text{ by}$$
$$\Gamma(u_1, \ldots, u_m) = u_1 + \cdots + u_m$$

Then $U_1 + \cdots + U_m$ is a direct sum if and only if Γ is injective.

A sume is a direct sum if and only if dimensions add up: Suppose V is finite-dimensional and $U_1, \ldots, U_m$ are subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if and only if

$$\dim(U_1 + \cdots + U_m) = \dim U_1 + \cdots + \dim U_m$$

$v + U$: Suppose $v \in V$ and U is a subspace of V. Then $v + U$ is the subset of V defined by

$$v + U = \{v + u : u \in U\}$$

affine subset, parrallel: • An affine subset of V is a subset of V of the form $v + U$ for some $v \in V$ and some subspace U of V • For $v \in V$ and U a subspace of V, the affine subset $v + U$ is said to be parallel to U