Math 110 Homework 12
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1 Exercises 7.D

Problem 1: First we verify that the given operator

$$\sqrt{T^*T}v = \frac{\|x\|}{\|u\|}\langle v,u\rangle u$$

is indeed positive. Observe that when we calculate

$$\langle\sqrt{T^*T}v, v\rangle = \frac{\|x\|}{\|u\|}|\langle v,u\rangle|^2$$

it is always a non-negative value, thus the operator satisfies the positivity condition. Now we wish to show that it is self adjoint, so observe thtat

$$\langle\sqrt{T^*T}v, u\rangle = \frac{\|x\|}{\|u\|}\langle v,u\rangle\langle u,u\rangle$$

$$\langle v, \sqrt{T^*T}u\rangle = \frac{\|x\|}{\|u\|}\overline{\langle u,u\rangle}\langle v,u\rangle$$

So the two inner products are the same, thus the given operator is self adjoint. Secondly, we can verify that the square is equal to the following value we will revisit

$$\begin{aligned}\sqrt{T^*T}\frac{\|x\|}{\|u\|}\langle v,u\rangle u &= \frac{\|x\|}{\|u\|}\langle v,u\rangle\sqrt{T^*T}u \\ &= \frac{\|x\|}{\|u\|}\langle v,u\rangle\frac{\|x\|}{\|u\|}\langle u,u\rangle u \\ &= \|x\|^2\langle v,u\rangle u\end{aligned}$$

Third, we find the adjoint of T. Define T^* as follows

$$T^*v = \langle v,x\rangle u$$

with the same fixed x and u as for T. This is a valid adjoint since for $v, w \in V$ we have

$$\begin{aligned}\langle Tv, w\rangle &= \langle v, T^*w\rangle \\ \langle v,u\rangle\langle x,w\rangle = \langle v,u\rangle\overline{\langle w,x\rangle} &= \langle v,u\rangle\langle x,w\rangle\end{aligned}$$

For all $v \in V$ we have that

$$\begin{aligned}T^*Tv &= T^*(\langle v,u\rangle x) \\ &= \langle v,u\rangle\langle x,x\rangle u\end{aligned}$$

Then, we can verify that indeed $\sqrt{T^*T}^2 = T^*T$ that we calculated earlier. Therefore, given that it is positive and self-adjiont it must be the unique positive square root of T^*T.

Problem 2: Let T be the operator represented by this 2×2 matrix with the standard basis.

$$T = \begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix}$$

The eigenvalues for T are exactly $\lambda = 0$. Then we find T^*T using the adjoint of the matrix provided.

$$T^*T = \begin{pmatrix} 25/2 & -25/2 \\ -25/2 & 25/2 \end{pmatrix}$$

The eigenvalues for which are equal to $\lambda = 25, 0$. So the singular values are equal to $\sigma = 5, 0$.

Problem 4: Consider the polar decomposition for T

$$T = S\sqrt{T^*T}$$

Then let $v \in V$ be such that it is an eigenvector for $\sqrt{T^*T}$ with the associated eigenvalue s. So,

$$Tv = S(sv)$$

$$\|Tv\| = \|S(sv)\| = |s|\|Sv\|$$

Given that S is an isometry we then have

$$\|Tv\| = s\|v\| = s$$

Since, s is nonnegative and the norm of v is 1. By polar decomposition there always exists $\sqrt{T^*T}$ thus an eigenvector associated with it.

Problem 10: The singular values are the eigenvalues of $\sqrt{T^*T}$. If T is self adjoint then $\sqrt{T^*T} = \sqrt{T^2}$. From previous exercises we know that the eigenvalues for T^2 are just the eigenvalues λ for T squared, that is λ^2. So we have that for all eigenvectors v of $\sqrt{T^*T}$ $\sqrt{T^2}v = sv$ such that $T^2v = s^2v = \lambda^2 v$. Taking the square root on both sides we get that $s = |s| = |\lambda|$.

Problem 11: First observe that the singular values for T^* are the eigenvalues of $\sqrt{TT^*}$. We know that T and T^* have the same eigenvectors, and the associated eigenvalues are λ and $\overline{\lambda}$ respectively. Then for some eigenvector we have

$$T^*Tv = T^*\lambda v = |\lambda|^2 v$$

also

$$TT^*v = T^*\overline{\lambda}v = |\lambda|^2 v$$

Since, TT^* and T^*T have the same positive eigenvalues. Their, singular values are the same, equal to the positive square root of those same eigenvalues.

Problem 12: Consider the vector space $V = \mathbb{F}^2$ and the operator T represented by the following matrix in the standard basis.

$$T = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Then this operator has the singular values $\sigma = 1, 0$. Observe that, $T^2 = 0$, so it has the singular values $\sigma = 0$ So, the singular values aren't even equal.

Problem 13: $\Longrightarrow$ First note that $\sqrt{0} = 0 = 0^2$. If T is invertible, then

$$\text{null } T^* = (\text{range } T)^\perp = V^\perp = \{0\}$$

So, T^* is invertible, and then T^*T is invertible. Then from a pervious exercise, T^*T does not have zero as an eigenvalue. Since the singular values are the positive square root of the eigenvalues of T^*T and it has non-zero eigenvalues, then the singular values are all non-zero as well.
$\Longleftarrow$ Consider the polar decomposition of T. If all the singular values are nonzero, then $\sqrt{T^*T}$ has all nonzero eigenvalues. Then, null $\sqrt{T^*T} = \{0\}$ so it is inveritble. Since, S is an isometry and we have a composition of invertible operators S and $\sqrt{T^*T}$ which is invertible, Therefore, T is invertible.

2 8.A

Problem 3: wip
We will proceed by induction on j. Consider the subspace null $(T-\lambda I)^j$, and null $(T^{-1}-\frac{1}{\lambda}I)^j$. It holds that each of these subspaces are a subset of $G(\lambda, T)$ and $G(\lambda^{-1}, T^{-1})$ respectively. For the base case let $j = 1$. Then for all λ it holds that if $v \in$ null $(T - \lambda I)$ then

$$Tv = \lambda v$$

So taking the inverse operator to both sides we get

$$T^{-1}Tv = v = \lambda T^{-1}v$$

Which implies that $T^{-1}v = \lambda^{-1}v$, and by the same argument as before this implies $v \in$ null $(T^{-1} - \frac{1}{\lambda}I)$. Since this holds for all λ and all eigenvectors we have shown that null $(T - \lambda I) \subseteq$ null $(T^{-1} - \lambda^{-1}I)$. We proceed with the exact same steps, but for null $(T^{-1} - \lambda^{-1}I)$ and we get the result null $(T - \lambda I) \supseteq$ null $(T^{-1} - \lambda^{-1}I)$. Therefore,

$$\text{null } (T - \lambda I) = \text{null } (T^{-1} - \lambda^{-1}I)$$

Having shown the base case assume the claim holds for some $j = k$. That is,

$$\text{null } (T - \lambda I)^k = \text{null } (T^{-1} - \lambda^{-1}I)^k$$

Now we take an arbitrary $v \in$ null $(T - \lambda I)^{k+1}$. Since,

$$\text{null } (T^{-1} - \lambda^{-1}I)^k \subset \text{null } (T^{-1} - \lambda^{-1}I)^{k+1}$$

if we have that $v \in$ null $(T - \lambda I)^k$, we are done. Suppose the case that $v \notin$ null $(T - \lambda I)^k$, then it follows

$$(T - \lambda I)^k(T - \lambda I)v = 0 = (T^{-1} - \lambda^{-1}I)^k(T - \lambda I)v$$

Problem 4: Assume for contradiction that the intersection is not empty,

$$G(\alpha, T) \cap G(\beta, T) \neq \{0\}$$

Let v be the eigenvector in the intersection

$$v \in G(\alpha, T) \cap G(\beta, T)$$

Then construct the linearly independent list of eigenvectors as described in 8.13, choosing v to represent both α and β. It is trivial now that that the list is not linearly independent, thus a contradiction.

Problem 6: Assume for contradiction that there exists a $S \in \mathcal{L}(()\mathbb{C}^3)$ such that $S^2 = T$. We can see that $T^3 = 0$, so $S^6 = 0$. But since the dimension of the vector space is 3 we have that

$$\text{null } S^3 = ... = \text{null } S^6$$

Specifically this means that null $S^4 =$ null $S^6 = \mathbb{C}^3$. But this is a contradiction since we know that

$$T^2 \neq 0$$

So null $S^4 \neq$ null T^2.

Problem 8: No! Consider the vector space $\mathbb{C}^2$ and the operators S and T given by following matrices respectively

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

in the standard basis for $\mathbb{C}^2$. It's easy to see that $S^2 = 0$ and $T^2 = 0$, so they both are nilpotent operators. Then it easily follows that the set of nilpotent operators is not closed under vector addition, because for $S + T$ given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We have that $(S + T)^2 = I$. From there it holds that for all $n \in \mathbb{N}$,

$$(S + T)^n = I^{n-1} = I$$

so it is never equal to zero and thus not nilpotent.

Problem 9: We know that ST and TS have the same eigenvalues. Since, ST is nilpotent, there is a diagaonl matrix of ST with all zeros on the diagaonl. So the one and only eigenvalue of ST is $\lambda = 0$.

Problem 11: Let $n = \dim V$, we know that $(ST)^n = 0$. Then,

$$\begin{aligned} S &= S \\ (ST)^n S &= 0S = 0 \\ T(ST)^n S &= T0 = 0 \\ T(ST)(ST)...(ST)(ST)S &= 0 \end{aligned}$$

then we regroup all the operators to get

$$(TS)(TS)...(TS) = 0(TS)^{n+1} \qquad = 0$$

Then we know that for any operator raised to a power greater than the dimension it is unchanged, so $(TS)^n = (TS)^{n+1} = 0$. Therefore, TS is nilpotent.

Problem 14: TL;DR: $N \implies 8.19 \implies 6.37$.
By 8.19 N has an upper triangular matrix with all zeros on the diagonal. Then by 6.37, given N has an upper triangular matrix in some basis, N has an upper triangular matrix in an orthonormal basis.