Math 110 Homework 6

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Problem 7: Since, p is in the standard basis of $\mathcal{P}_m(\mathbb{R})$ it has only one term to consider. Namely, a term x^j where $j \in \{1, ..., m\}$. Then, if p has a degree strictly greater than j it will have some x term for $p^(j)$. Thus, evaluated at 0, it will equal 0. If p has a degree strictly less than j then $p^{(j)} = 0$ by the power rule form calculus. So, when p has degree exactly j we have that it will evaluate to some constant by the power rule. This constant will be exactly $j!$, again by the power rule. So, we normalize it by dividing it by $j!$, so $\varphi(j) = 1$. Then, we have a valid φ_j such that it only evaluates to 1 for one of the polynomials in the standard basis and 0 for all other. Thus, a valid dual basis.

Problem 9: Since, $\psi \in V'$ it is equivalent to the following linear combination with scalars $a_1, .., a_n$. That is,

$$\psi = a_1\varphi_1 + ... + a_n\varphi_n$$

Then for some v_j in the basis of V w ehave that

$$\psi(v_j) = a_j\varphi_j(v_j) = a_j$$

Since, all the other ones are 0, since it is evaluated at a basis vector Then, we simply make direct substitutions for each $j \in \{1, ..., n\}$ and get the desired result.

Problem 11: This is pretty simple. For the $\Longleftarrow$ direction. All d_j are some arbitrary costants in $\mathbb{F}$. So, we construct $A_{j,k}$ such that each column is just a scalar multiple of each other. Since, we are multiplying the same $(c_1, ..., c_m$ by arbitrary constants d_j. Then, clearly if all the vectors are scalar multiples of each other we must remove all of them except 1 to get a linearly independent list. Therefore, the dimension of the row space is 1 and thus rank is 1. For the $\Longrightarrow$ direction we argue that since A has rank 1, then the column space has dimesnion 1. So, it follows that the columns are constructed as follows for some vectors $d_1, ..., d_n$. That is they are scalar multiples of each other.

Problem 19: $\Longrightarrow$. If $U = V$, then the φ such that $\varphi(u) = 0 = \varphi(v)$ for all $u \in U$ and all $v \in V$ is just 0, since 0 is unique in a vector space.
$\Longleftarrow$. We know V is finite-dimensional, so

$$\dim U + \dim U^0 = \dim V$$

But, the annhilator has dimension 0 so $\dim U = \dim V$ since, U is a subspace of V it implies that $U = V$

Problem 20: Pick an arbitrary $\varphi \in W^0$. Since, $U \subset W$ for all $u \in U$, $\varphi(u) = 0$, so clearly $\varphi \in U^0$. We showed this for an arbitrary φ so $W^0 \subset U^0$.

Problem 21: Since V is finite dimensional we abuse the dimension of the annhilator formula.

$$\begin{aligned}\dim U + \dim U^0 &= \dim V \\ \dim W + \dim W^0 &= \dim V \\ \dim W + \dim W^0 &= \dim U + \dim U^0\end{aligned}$$

Since, $W^0 \subset U^0$ we have that

$$\dim W^0 \leq \dim U^0$$

Then, to hold the equality it must be the case that

$$\dim W \geq \dim U$$

Therefore,

$$W \supset U$$

Problem 22: For some $v \in V + U$ we have that $v = u + w$ for $u \in U$ and $w \in W$. Then for some $\varphi \in (U + W)^0$, $\varphi(v) = 0 = \varphi(u + w) = \varphi(u) + \varphi(w) = 0$. Since, $u, w \in U + W$ then clearly they must be zero as well. So, $\varphi \in U^0 \cap W^0$. Thus, $U + W^0 \subset U^0 \cap W^0$. We use the exact same argument in the reverse direction to get the other conclusion $U + W^0 \supset U^0 \cap W^0$ So, then it must be that $U + W^0 = U^0 \cap W^0$

Problem 23: