Math 110 Homework 4

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# 1 Exercies 3.B

**Problem 7:** Let $V$ have dimension 2, and $W$ have some dimension greater than or equal to 2. Let $T$ be a non-injective non-zero linear map $T : V \to W$. We know that null $T$ is a subspace of $V$. Also, null $T \neq \{0\}$ since it is not injective, thus $\dim \text{null } T > 0$. Since, we restricted $V$ to have dimension 2 and null $T$ is nonzero it must have exactly dimension 1. There exists a subspace $U$ of $V$ such that $V = \text{null } T \oplus U$. Therefore, $\dim U = 1$. Let $S$ be another non-zero non-injective linear map $S : V \to W$ such that null $S = U$. Now we have the case that for $v \in V$,

$$(S + T)(v) = Sv + Tv$$

But for any non-zero $v \in V$ it is either in null $S$ or null $T$ but not both. So, the null $(S + T) = \{0\}$, which means that $S + T$ is injective. Therefore, it is not a subspace of $\mathcal{L}(U, V)$ since it is not closed under addition.

**Problem 15:** Assume for contradiction that $T$ is a valid linear map. Given that $T$ is a map $T : \mathbb{R}^5 \to \mathbb{R}^2$, we must have that rank $T = 2$ and nullity $T = 3$. By

$$\dim \mathbb{R}^5 = 5 = \text{rank } + \text{nullity } = 2 + 3$$

We then observe that the null space defined as follows has dimension 2.

$$\left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\right\}$$

We claim that the vectors $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$ span the null space. Observe for some arbitrary coefficients $a, b \in \mathbb{F}$ the linear combination of the vectors $a(3, 1, 0, 0, 0)+b(0, 0, 1, 1, 1) = (3a, a, b, b, b)$. This holds the required property that $3x_2 = x_1$ and that $x_3 = x_4 = x_5$. Thus, it spans our null space and we have a contradiction, since we found a spanning list that has length less than a linearly independent list.

**Problem 19:** We know that for a linear map $T \in \mathcal{L}(V, W)$ the dimension of the range is less than or equal to the codomain. That is,

$$\text{rank } T \leq \dim W$$

Suppose, this was not the case then we could find a $v \in V$ such that $Tv \notin W$ and that is not a linear map. Then, we also have that $\dim \text{null } T = \dim U$. It then follows directly that

$$\dim V = \text{rank } T + \dim \text{null } T = \text{rank } T + \dim U$$

$$\dim V - \text{rank } T = \dim U$$

Therefore, from our first inequality

$$\dim U \geq \dim V - \dim W$$

**Problem 26:**

# 2 Exercises 3.C

**Problem 2:** For $\mathcal{P}_3(\mathbb{R})$ let the basis be the following basis $1, x, x^2, x^3$ in this order. And for $\mathcal{P}_2(\mathbb{R})$ let the basis be $3x^2, 2x, 1$ for this order. $1, x, x^2, x^3$ is clearly a basis since it is just the standard basis reorganized, more formally the linear combination will still only be zero for all zero coefficients, since addition is commutative in $\mathbb{R}$. For $3x^2, 2x, 1$ any linear combination is only zero when all the coefficients are zero since each term has a different degree.

**Problem 3:** We want a matrix that basically has 1s in the diagonal up to the rank $T$ and 0s everywhere else. So, we begin with the null $T$. We know that null $T$ is a subspace of $V$. So let $t_1, ..., t_m$ be a basis for null $T$. Then, since $t_1, ..., t_m$ is a linearly independent list and is less than or equal to the length of the basis of $V$ we can extend it to form a basis of $V$. Note that nullity $T = m$ Let $v_1, ..., v_n$ be the addtional vectors we add to extend our list to be a basis of $V$. Note that rank $T = n$. So, $t_1, ..., t_m, v_1, ..., v_n$ is a basis of $V$. Because we collected all the vectors that are equal to zero we know that $Tv_i = w_i$ for some $w_i \in W$ for $i \in \{1, ..., n\}$. Now we can show these $w_1, ..., w_n$ are linearly independent. We begin with $a_1w_1+...+a_nw_n = 0$ for some coefficients $a_i$. We can substitute the $Tv_i$ to get $a_1Tv_1 + ... + a_nTv_n = 0$. Collecting the terms we have $T(a_1v_1 + ... + a_nv_n) = 0$. We know that $v_1, ..., v_n$ are linearly independent since it is formed by dropping vectors from the basis of $V$. Then, $a_1v_1 + ... + a_nv_n = 0$ only when all the coefficients are zero. So, we know all the $a_i$ are zero, thus $w_1, ..., w_n$ is linearly independent. Then we can finally extend $w_1, ..., w_n$ to form a basis for $W$. Now we basically our done since we have shown that we have a basis for which $Tv_i = w_i$ for $i \in \{1, ..., \text{rank } T\}$. So, we can make our desired matrix since everything else is supposed to be zero because it is in the null space.

**Problem 4:** In a very roundabout way this question is asking if there is a vector $v$ in every basis of $V$ such that for every $T \in \mathcal{L}(V, W)$ it is that $Tv = w$ for some $w$ in the basis of $W$. The stright forward case is if for some $v$ in the basis of $V$ if $Tv = 0$ then put that $v$ at the start of the basis list, so that the column can be all zeros and satisfy the condition. The second also straight forward case if if for some $v$ in the basis of $V$ we happen to have a basis for $W$ such that $Tv = w$ for $w$ in the basis, then we are done as well, by moving both $v$ and $w$ to start of their respective basis lists. The last case is we can always construct a basis such that the condition is met, by just adding $Tv = w$ to the basis of $W$ and then removing all the vectors that are multiples of $w$. Then we will be left with a linearly independent list that we can always extend to form a basis of $W$.