Math 110 Homework 4
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1 Exercies 3.B

Problem 7: Let V have dimension 2, and W have some dimension greater than or equal to 2. Let T be a non-injective non-zero linear map $T : V \to W$. We know that null T is a subspace of V. Also, null $T \neq \{0\}$ since it is not injective, thus $\dim$ null $T > 0$. Since, we restricted V to have dimension 2 and null T is nonzero it must have exactly dimension 1. There exists a subspace U of V such that $V = $ null $T \oplus U$. Therefore, $\dim U = 1$. Let S be another non-zero non-injective linear map $S : V \to W$ such that null $S = U$. Now we have the case that for $v \in V$,

$$(S + T)(v) = Sv + Tv$$

But for any non-zero $v \in V$ it is either in null S or null T but not both. So, the null $(S + T) = \{0\}$, which means that $S + T$ is injective. Therefore, it is not a subspace of $\mathcal{L}(U, V)$ since it is not closed under addition.

Problem 15: Assume for contradiction that T is a valid linear map. Given that T is a map $T : \mathbb{R}^5 \to \mathbb{R}^2$, we must have that rank $T = 2$ and nullity $T = 3$. By

$$\dim \mathbb{R}^5 = 5 = \text{rank } + \text{nullity } = 2 + 3$$

We then observe that the null space defined as follows has dimension 2.

$$\left\{ (x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5 \right\}$$

We claim that the vectors $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$ span the null space. Observe for some arbitrary coefficients $a, b \in \mathbb{F}$ the linear combination of the vectors $a(3, 1, 0, 0, 0) + b(0, 0, 1, 1, 1) = (3a, a, b, b, b)$. This holds the required property that $3x_2 = x_1$ and that $x_3 = x_4 = x_5$. Thus, it spans our null space and we have a contradiction, since we found a spanning list that has length less than a linearly independent list.

Problem 19: We know that for a linear map $T \in \mathcal{L}(V, W)$ the dimension of the range is less than or equal to the codomain. That is,

$$\text{rank } T \leq \dim W$$

Suppose, this was not the case then we could find a $v \in V$ such that $Tv \notin W$ and that is not a linear map. Then, we also have that $\dim$ null $T = \dim U$. It then follows directly that

$$\dim V = \text{rank } T + \dim \text{null } T = \text{rank } T + \dim U$$

$$\dim V - \text{rank } T = \dim U$$

Therefore, from our first inequality

$$\dim U \geq \dim V - \dim W$$

Problem 26:

2 Exercises 3.C

Problem 2: For $\mathcal{P}_3(\mathbb{R})$ let the basis be the following basis $1, x, x^2, x^3$ in this order. And for $\mathcal{P}_2(\mathbb{R})$ let the basis be $3x^2, 2x, 1$ for this order. $1, x, x^2, x^3$ is clearly a basis since it is just the standard basis reorganized, more formally the linear

combination will still only be zero for all zero coefficients, since addition is commutative in $\mathbb{R}$. For $3x^2, 2x, 1$ any linear combination is only zero when all the coefficients are zero since each term has a different degree.

Problem 3: We want a matrix that basically has 1s in the diagonal up to the rank T and 0s everywhere else. So, we begin with the null T. We know that null T is a subspace of V. So let $t_1, ..., t_m$ be a basis for null T. Then, since $t_1, ..., t_m$ is a linearly independent list and is less than or equal to the length of the basis of V we can extend it to form a basis of V. Note that nullity $T = m$ Let $v_1, ..., v_n$ be the addtional vectors we add to extend our list to be a basis of V. Note that rank $T = n$. So, $t_1, ..., t_m, v_1, ..., v_n$ is a basis of V. Because we collected all the vectors that are equal to zero we know that $Tv_i = w_i$ for some $w_i \in W$ for $i \in \{1, ..., n\}$. Now we can show these $w_1, ..., w_n$ are linearly independent. We begin with $a_1 w_1 + ... + a_n w_n = 0$ for some coefficients a_i. We can substitute the Tv_i to get $a_1 Tv_1 + ... + a_n Tv_n = 0$. Collecting the terms we have $T(a_1 v_1 + ... + a_n v_n) = 0$. We know that $v_1, ..., v_n$ are linearly independent since it is formed by dropping vectors from the basis of V. Then, $a_1 v_1 + ... + a_n v_n = 0$ only when all the coefficients are zero. So, we know all the a_i are zero, thus $w_1, ..., w_n$ is linearly independent. Then we can finally extend $w_1, ..., w_n$ to form a basis for W. Now we basically our done since we have shown that we have a basis for which $Tv_i = w_i$ for $i \in \{1, ..., \text{rank } T\}$. So, we can make our desired matrix since everything else is supposed to be zero because it is in the null space.

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