Section 1.A – $\mathbb{R}^n$ and $\mathbb{C}^n$

$-\alpha$, subtraction, $1/\alpha$, division: Let $\alpha, \beta \in \mathbf{C}$ · Let $-\alpha$ denote the additive inverse of α. Thus $-\alpha$ is the unique complex number such that $\alpha + (-\alpha) = 0$ · Subtraction on $\mathbf{C}$ is defined by $\beta - \alpha = \beta + (-\alpha)$ · For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α. Thus $1/\alpha$ is the unique complex number such that $\alpha(1/\alpha) = 1$ · Division on $\mathbf{C}$ is defined by $\beta/\alpha = \beta(1/\alpha)$
list, length: Suppose n is a nonnegative integer. A list of length n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this: $(x_1, \dots, x_n)$ Two lists are equal if and only if they have the same length and the same elements in the same order.
$\mathbb{F}^n$: $\mathbf{F}^n$ is the set of all lists of length n of elements of $\mathbf{F}$: $\mathbf{F}^n = \left\{(x_1, \dots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \dots, n\right\}$ For $(x_1, \dots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \dots, n\}$, we say that x_j is the j^{th} coordinate of $(x_1, \dots, x_n)$
addition in $\mathbb{F}^n$: Addition in $\mathbf{F}^n$ is defined by adding corresponding coordinates: $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$
Commutativity of addition in $\mathbb{F}^n$: If $x, y \in \mathbf{F}^n$, then $x + y = y + x$
0: Let 0 denote the list of length n whose coordinates are all 0 : $0 = (0, \dots, 0)$
additive inverse in $\mathbb{F}^n$: For $x \in \mathbf{F}^n$, the additive inverse of x, denoted $-x$, is the vector $-x \in \mathbf{F}^n$ such that $x + (-x) = 0$ In other words, if $x = (x_1, \dots, x_n)$, then $-x = (-x_1, \dots, -x_n)$
scalar multiplication in $\mathbb{F}^n$: The product of a number λ and a vector in $\mathbf{F}^n$ is computed by multiplying each coordinate of the vector by λ : $\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$ here $\lambda \in \mathbf{F}$ and $(x_1, \dots, x_n) \in \mathbf{F}^n$

Section 1.B – Definition of Vector Space

addition, scalar multiplication: · An addition on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$ · A scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$
Vector Space: A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold: commutativity

$$u + v = v + u \text{ for all } u, v \in V$$

associativity $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$ additive identity there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$ additive inverse for every $v \in V$, there exists $w \in V$ such that $v + w = 0$ multiplicative identity $1v = v$ for all $v \in V$ distributive properties

$$a(u + v) = au + av \text{ and } (a + b)v = av + bv \text{ for all } a, b \in \mathbf{F} \text{ and}$$

all $u, v \in V$
vector, point: Elements of a vector space are called vectors or points.
real vector space, complex vector space: • A vector space over $\mathbf{R}$ is called a real vector space. · A vector space over $\mathbf{C}$ is called a complex vector space.
$\mathbb{F}^S$: · If S is a set, then $\mathbf{F}^S$ denotes the set of functions from S to $\mathbf{F}$ · For $f, g \in \mathbf{F}^S$, the sum $f + g \in \mathbf{F}^S$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$ • For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the product $\lambda f \in \mathbf{F}^S$ is the function defined by $(\lambda f)(x) = \lambda f(x)$ for all $x \in S$
Unique Additive Identity: A vector space has a unique additive identity
Unique additive inverse: Every element in a vector space has a unique additive inverse.
The number 0 times a vector: $0v = 0$ for every $v \in V$
A number times the vector 0: $a0 = 0$ for every $a \in \mathbf{F}$
The number -1 times a vector: $(-1)v = -v$ for every $v \in V$

Section 1.C – Subspaces

Subspace: A subset U of V is called a subspace of V if U is also a vector space (using the same addition and scalar multiplication as on V).
Conditions for a subspace: A subset U of V is a subspace of V if and only if U satisfies the following three conditions: additive identity $0 \in U$ closed under addition $u, w \in U$ implies $u + w \in U$ closed under scalar multiplication $a \in \mathbf{F}$ and $u \in U$ implies $au \in U$
sum of subsets: Suppose $U_1, \dots, U_m$ are subsets of V. The sum of $U_1, \dots, U_m$, denoted $U_1 + \dots + U_m$, is the set of all possible sums of elements of $U_1, \dots, U_m$ More precisely, $U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}$
Sum of subspaces is the smalles containing subspace: Suppose $U_1, \dots, U_m$ are subspaces of V. Then $U_1 + \dots + U_m$ is the smallest subspace of V containing $U_1, \dots, U_m$
direct sum: Suppose $U_1, \dots, U_m$ are subspaces of V · The sum $U_1 + \dots + U_m$ is called a direct sum if each element of $U_1 + \dots + U_m$ can be written in only one way as a sum $u_1 + \dots + u_m$, where each u_j is in U_j · If $U_1 + \dots + U_m$ is a direct sum, then $U_1 \oplus \dots \oplus U_m$ denotes $U_1 + \dots + U_m$, with the $\oplus$ notation serving as an indication that this is a direct sum.
Condition for a direct sum: Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$
Direct sum of two subspaces: Suppose U and W are subspaces of V. Then $U + W$ is a direct sum if and only if $U \cap W = \{0\}$

Section 2.A Span and Linear Independence

Span: The set of all linear combinations of a list of vectors $v_1, \dots, v_m$ in V is called the span of $v_1, \dots, v_m$, denoted $\operatorname{span}(v_1, \dots, v_m)$. In other words,

$$\operatorname{span}(v_1, \dots, v_m) = \{a_1 v_1 + \dots + a_m v_m : a_1, \dots, a_m \in \mathbf{F}\}$$

The span of the empty list ( ) is defined to be $\{0\}$.
Span is the smallest containing subspace: The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.
spans: If $\operatorname{span}(v_1, \dots, v_m)$ equals V, we say that $v_1, \dots, v_m$ spans V
finite-dimensional vector space: A vector space is called finite-dimensional if some list of vectors in it spans the space.
polynomial over a field F: A function $p : \mathbf{F} \to \mathbf{F}$ is called a polynomial with coefficients in $\mathbf{F}$ if there exist $a_0, \dots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$$

for all $z \in \mathbf{F}$. $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in $\mathbf{F}$.
degree of a polynomial: · A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have degree m if there exist scalars $a_0, a_1, \dots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \dots + a_m z^m$$

for all $z \in \mathbf{F}$. If p has degree m, we write deg $p = m$
· The polynomial that is identically 0 is said to have degree $-\infty$.
$\mathcal{P}_m(\mathbb{F})$: For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in $\mathbf{F}$ and degree at most m.
infinite-dimensional vector space: A vector space is called infinite-dimensional if it is not finite-dimensional.
linearly independent: · A list $v_1, \dots, v_m$ of vectors in V is called linearly independent if the only choice of $a_1 \dots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \dots + a_m v_m$ equal 0 is $a_1 = \dots = a_m = 0$
· The empty list ( ) is also declared to be linearly independent.
linearly dependent: · A list of vectors in V is called linearly dependent if it is not linearly independent.
· In other words, a list $v_1, \dots, v_m$ of vectors in V is linearly dependent if there exist $a_1, \dots, a_m \in \mathbf{F}$, not all 0, such that $a_1 v_1 + \dots + a_m v_m = 0$
Linear Dependence Lemma: Suppose $v_1, \dots, v_m$ is a linearly dependent list in V. Then there exists $j \in \{1, 2, \dots, m\}$ such that the following hold: (a) $v_j \in \operatorname{span}\left(v_1, \dots, v_{j-1}\right)$ (b) if the j^{th} term is removed from $v_1, \dots, v_m$, the span of the remaining list equals $\operatorname{span}(v_1, \dots, v_m)$
Length of linearly independent list $\leq$ length of spanning list: In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.
Finite-dimensional subspaces: Every subspace of a finite-dimensional vector space is finite dimensional.

Section 2.B Bases

basis: A basis of V is a list of vectors in V that is linearly independent and spans V
Criterion for basis: A list $v_1, \dots, v_n$ of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \dots + a_n v_n$ where $a_1, \dots, a_n \in \mathbf{F}$
Spanning list contains a basis: Every spanning list in a vector space can be reduced to a basis of the vector space.
Linearly independent list extends to a basis: Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.
Every subspace V is part of a direct sum equal to V.: Suppose V is finite-dimensional and U is a subspace of V. Then there is a subspace W of V such that $V = U \oplus W$

Section 2.C Dimension

dimension, $\dim V$: The dimension of a finite-dimensional vector space is the length of any basis of the vector space. The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.
Dimension of subspace: If V is finite-dimensional and U is a subspace of V, then $\dim U \leq \dim V$
Linearly independent list of the right length is a basis: Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V
Spanning list of the right length is a basis: Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V
Dimension of a sum: If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Section 3.A The Vector Space of Linear Maps

linear map: A linear map from V to W is a function $T : V \to W$ with the following properties: additivity

$$T(u + v) = Tu + Tv \text{ for all } u, v \in V$$

homogeneity

$$T(\lambda v) = \lambda(Tv) \text{ for all } \lambda \in \mathbf{F} \text{ and all } v \in V$$

Notation $\mathcal{L}(V, W)$: The set of all linear maps from V to W is denoted $\mathcal{L}(V, W)$
Linear maps and basis of domain: Suppose $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_n \in W$. Then there exists a unique linear map $T : V \to W$ such that

$$Tv_j = w_j$$

for each $j = 1, \dots, n$
additiona nd scalar multiplication on linear maps: Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbf{F}$. The sum $S + T$ and the product λT are the linear maps from V to W defined by

$$(S + T)(v) = Sv + Tv \quad \text{and} \quad (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$
$\mathcal{L}(V, W)$ is a vector space: With the operations of addition and scalar multiplication as defined above, $\mathcal{L}(V, W)$ is a vector space.
Product of Linear Maps: If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then the product $ST \in \mathcal{L}(U, W)$ is defined by

$$(ST)(u) = S(Tu)$$

for $u \in U$
Algebraic Properties of products of linear maps: **associativity**

$$(T_1 T_2) T_3 = T_1 (T_2 T_3)$$

whenever T_1, T_2, and T_3 are linear maps such that the products make sense (meaning that T_3 maps into the domain of T_2, and T_2 maps into the domain of T_1). **identity**

$$TI = IT = T$$

whenever $T \in \mathcal{L}(V, W)$ (the first I is the identity map on V, and the second I is the identity map on W). **distributive properties**

$$(S_1 + S_2) T = S_1 T + S_2 T \quad \text{and} \quad S(T_1 + T_2) = ST_1 + ST_2$$

whenever $T, T_1, T_2 \in \mathcal{L}(U, V)$ and $S, S_1, S_2 \in \mathcal{L}(V, W)$
Linear maps take 0 to 0: Suppose T is a linear map from V to W. Then $T(0) = 0$

Section 3.B Null Spaces and Ranges

null space: For $T \in \mathcal{L}(V, W)$, the null space of T, denoted null T, is the subset of V consisting of those vectors that T maps to 0 :

$$\operatorname{null} T = \{v \in V : Tv = 0\}$$

injective: A function $T : V \to W$ is called injective if $Tu = Tv$ implies $u = v$
Injectivity is equivalent to null space equals $\{0\}$: Let $T \in \mathcal{L}(V, W)$. Then T is injective if and only if null $T = \{0\}$
range: For T a function from V to W, the range of T is the subset of W consisting of those vectors that are of the form Tv for some $v \in V$:

$$\operatorname{range} T = \{Tv : v \in V\}$$

The range is subspace: If $T \in \mathcal{L}(V, W)$, then range T is a subspace of W
surjective: A function $T : V \to W$ is called surfective if its range equals W
Fundamental Theorem of Linear Maps: Suppose V is finite-dimensional and $T \in \mathcal{L}(V, W)$. Then range T is finite-dimensional and

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

A map to a smaller dimensional space is not injective: Suppose V and W are finite-dimensional vector spaces such that $\dim V > \dim W$. Then no linear map from V to W is injective.
A map to a larger dimensional space is not surjective: Suppose V and W are finite-dimensional vector spaces such that $\dim V < \dim W$. Then no linear map from V to W is surjective.
Homogeneous system of linear equations: A homogeneous system of linear equations with more variables than equations has nonzero solutions.
Inhomogeneous system of linear equations: An inhomogeneous system of linear equations with more equations thavariables has no solution for some choice of the constant terms.

Section 3.C Matrices

matrix, $A_{j,k}$: Let m and n denote positive integers. An m -by-n matrix A is a rectangular array of elements of $\mathbf{F}$ with m rows and n columns:

$$A = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & & \vdots \\ A_{m,1} & \dots & A_{m,n} \end{pmatrix}$$

The notation $A_{j,k}$ denotes the entry in row j, column k of A. In other words, the first index refers to the row number and the second index refers to the column number.
matrix of a linear map, $\mathcal{M}(T)$: Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_n$ is a basis of V and $w_1, \dots, w_m$ is a basis of W. The matrix of T with respect to these bases is the $m - by - n$ matrix $\mathcal{M}(T)$ whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k} w_1 + \dots + A_{m,k} w_m$$

If the bases are not clear from the context, then the notation $\mathcal{M}\left(T, (v_1, \dots, v_n), (w_1, \dots, w_m)\right)$ is used.
$\mathbb{F}^{m,n}$: For m and n positive integers, the set of all $m - by - n$ matrices with entries in $\mathbf{F}$ is denoted by $\mathbf{F}^{m,n}$
$\dim \mathbb{F}^{m,n} = mn$: Suppose m and n are positive integers. With addition and scalar multiplication defined as above, $\mathbf{F}^{m,n}$ is a vector space with dimension mn
matrix multiplication: Suppose A is an $m -$ by $- n$ matrix and C is an n -by $- p$ matrix. Then AC is defined to be the m -by-p matrix whose entry in row j, column k, is given by the following equation:

$$(AC)_{j,k} = \sum_{r=1}^{n} A_{j,r} C_{r,k}$$

In other words, the entry in row j, column k, of AC is computed by taking row j of A and column k of C, multiplying together corresponding entries, and then summing.
$A_{j,\cdot}$, $A_{\cdot,k}$: Suppose A is an m -by $- n$ matrix. · If $1 \leq j \leq m$, then $A_{j,\cdot}$ denotes the $1 -$ by $- n$ matrix consisting of row j of A · If $1 \leq k \leq n$, then $A_{\cdot}, k$ denotes the $m -$ by $- 1$ matrix consisting of column k of A
Entry of matrix product equals row times column: Suppose A is an m-by-n matrix and C is an n -by- p matrix. Then

$$(AC)_{j,k} = A_{j,\cdot} C_{\cdot,k}$$

for $1 \leq j \leq m$ and $1 \leq k \leq p$
Column of matrix product equals matrix times column: Suppose A is an $m -$ by-n matrix and C is an n -by-p matrix. Then

$$(AC)_{\cdot,k} = AC_{\cdot,k}$$

for $1 \leq k \leq p$
Linear combination of columns: Suppose A is an $m - by - n$ matrix and $c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is an n -by-1 matrix. Then

$$Ac = c_1 A_{\cdot 1} + \dots + c_n A_{\cdot n}$$

In other words, Ac is a linear combination of the columns of A, with the scalars that multiply the columns coming from c.