

1 Exercises 7.B

Problem 3: Define the linear map T as follows for each $z = (z_1, z_2, z_3) \in \mathbb{C}^3$.

$$Tz = (2z_1, 3z_2, 2z_1)$$

Note the switch up in the last spot. We can check that T is indeed closed under addition and scalar multiplication, thus a linear map. Firstly, $\lambda = 2$ is an eigenvalue with the eigenvector $w = (1, 0, 1)$.

$$T(1, 0, 1) = (2, 0, 2) = 2(1, 0, 1)$$

Similarly, $\lambda = 3$ is an eigenvalue with the eigenvector $w = (0, 1, 0)$.

$$T(0, 1, 0) = (0, 3, 0) = 3(0, 1, 0)$$

Finally, $(T^2 - 5T + 6I)z \neq 0$ if $z_3 \neq 6z_1$. So given we have values that are nonzero after that linear map, $T^2 - 5T + 6I \neq 0$.

Problem 4: Given that $\mathbb{F} = \mathbb{C}$ by the complex spectral theorem T is normal if and only if T has a diagonal matrix with respect to some orthonormal basis. Since T is diagonalizable if and only if V is equal to the sum of the eigenspaces of distinct eigenvalues, we have shown the last case. Also by the complex spectral theorem T is normal if and only if V has an orthonormal basis consisting of eigenvectors of T . That completes the proof. (Ask Alex if there's some part that guarantees that we have distinct eigenvalues, but it is not that difficult to show that if all the basis eigenvectors are orthogonal they must be associated with distinct eigenvalues.)

Problem 6: \Rightarrow direction follows easily from 7.13, if T is self-adjoint then all the eigenvalues are real.

\Leftarrow Given T is normal then T^* and T have the same eigenvectors and for the corresponding eigenvalue λ for T , T^* has the associated eigenvalue $\bar{\lambda}$ for the same eigenvector. If all the eigenvalues are real $\lambda = \bar{\lambda}$, so T and T^* have the same eigenvalues and eigenvectors. By the complex spectral theorem V has an orthogonal basis consisting of eigenvectors of T . Denote these basis eigenvectors as e_1, \dots, e_n . Let $v \in V$, then for some $a_1, \dots, a_n \in \mathbb{C}$

$$\begin{aligned} Tv &= a_1 T(e_1) + \dots + a_n T(e_n) \\ &= a_1 \lambda_1 e_1 + \dots + a_n \lambda_n e_n \\ &= a_1 T^*(e_1) + \dots + a_n T^*(e_n) \\ Tv &= T^*v \end{aligned}$$

For all $v \in V$ so $T = T^*$, therefore T is self adjoint.

Problem 8: Let T be the operator associated with the matrix and some basis B for the complex vector space

$$\mathcal{M}(T, B) = \begin{pmatrix} 0 & \dots & 1 \\ & \ddots & \vdots \\ 0 & & 0 \end{pmatrix}$$

Then, clearly $T^2 = 0$ so $T \neq T^2$ and $T^8 = 0 = T^9$. (I'm not sure what the point of this question was, but I definitely missed it.)

Problem 10: Let $V = \mathbb{R}^2$ and T be the operator represented by this matrix for the standard basis of \mathbb{R}^2 .

$$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

For $b = 0$ and $c = 1$

$$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} + (1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

Since we can compute that $T^2 = -I$. The zero operator is non invertible so we are done.

Problem 11: By the real or complex spectral theorem given that T is self-adjoint it is diagonalizable. Suppose the diagonal matrix is

$$T = \begin{pmatrix} a_1 & \dots & 0 \\ & \ddots & \vdots \\ 0 & & a_n \end{pmatrix}$$

Then, we can take the cube root of each value in the diagonal, which is well defined for complex or real numbers. So we get

$$S = \begin{pmatrix} a_1^{1/3} & \dots & 0 \\ & \ddots & \vdots \\ 0 & & a_n^{1/3} \end{pmatrix}$$

Then, clearly it holds that $S^3 = T$.

Problem 12: By the spectral theorem there exists an orthonormal basis of V consisting of eigenvectors of T . So v is equal to

$$v = a_1 e_1 + \dots + a_n e_n$$

Where, the a_1, \dots, a_n are some scalars and the e_1, \dots, e_n are the orthonormal basis eigenvectors. Then,

$$Tv = a_1 \lambda_1 e_1 + \dots + a_n \lambda_n e_n$$

So we can restate the inequality as

$$\begin{aligned} \|Tv - \lambda v\| &= \|a_1 \lambda_1 e_1 + \dots + a_n \lambda_n e_n - \lambda v\| \\ &= \| |\lambda_1 - \lambda| a_1 e_1 + \dots + |\lambda_1 - \lambda| a_n e_n \| \end{aligned}$$

Then since the basis vectors are orthogonal we can express them as

$$= \| |\lambda_1 - \lambda| + \dots + |\lambda_1 - \lambda| \| \|a_1 e_1 + \dots + a_n e_n\|$$

Following that $\|v\| = 1$

$$= \| |\lambda_1 - \lambda| + \dots + |\lambda_1 - \lambda| \|$$

So we are given that

$$\| |\lambda_1 - \lambda| + \dots + |\lambda_1 - \lambda| \| < \epsilon$$

Assume for contradiction that $|\lambda - \lambda_i| \geq \epsilon$ for all $i \in 1, \dots, n$. Then, we can make the following substitutions

$$\|\epsilon + \dots + \epsilon\| = n\epsilon < \epsilon$$

Which then implies that $n < 1$ which is a contradiction since it would imply that V has dimension 0, but it clearly doesn't since $v \in V$ and $\|v\| = 1$. Thus, we have a contradiction and it must be that there exists some eigenvalue λ' the statement holds.

Problem 13: The first part of (c) if and only if (a) we proceed as in the provided proof. Then to show that the last part that (a) implies (b) we begin with each part as used in the proof for the real spectral theorem. First note that the equation of $T^2 + bT + cI$ is not needed in the proof since we have the fact that $\|Tv\| = \|T^*v\|$ which we can use in the proof. The, normal operators in complex vector spaces clearly always have eigenvalues, since in any complex vector space the operator has eigenvectors. Then we wish to show the condition that the subspaces are invariant under the orthogonal complement. Observe that if U is a subspace that is invariant under T . Let U be equal to the span of an eigenvector then that is T invariant and the complement is invariant as well since it is just a sum of eigenvalues and scalars. Then we wish to show that $T|_U$ is normal as well, which holds from the fact that $\langle T|_U u, v \rangle = \langle Tu, v \rangle = \langle u, T^*v \rangle = \langle u, T^*|_U v \rangle$. Then I guess we can make very similar arguments to show that it holds.

Problem 14: \Leftarrow By the real spectral theorem if T is self-adjoint and the inner product makes U an inner product space, then U has a basis consisting of eigenvectors of T .

\Rightarrow If T has a basis consisting of eigenvectors, then it is diagonalizable by 5.41. Now if we just provide an inner product on U such that the basis eigenvectors are orthonormal we can just apply the real spectral theorem and be done.

Let e_1, \dots, e_n be the eigenvector basis. Then, for arbitrary $v, w \in V$

$$v = a_1 e_1 + \dots + a_n e_n$$

and

$$w = b_1 e_1 + \dots + b_n e_n$$

For some real number scalars. we define the inner product $\langle v, w \rangle$ as follows

$$\langle v, w \rangle = a_1 \cdot b_1 + \dots + a_n \cdot b_n$$

We will now confirm that all the properties of an inner product do indeed hold.

1. Positivity: Observe that $\langle v, v \rangle = a_1^2 + \dots + a_n^2 \geq 0$.
2. Definiteness: By the property of linearly independent vectors, $\langle v, v \rangle = a_1^2 + \dots + a_n^2 = 0$ only when $a_1 = \dots = a_n = 0$ which is when $v = 0$.
3. Additivity: pretty straightforward from the commutativity and distributivity in \mathbb{R} .
4. Homogeneity: straightforward from the distributivity of real numbers.
5. Conjugate symmetry: Since we are in a real vector space conjugates are equal to the number itself. So, $\langle v, w \rangle = a_1 b_1 + \dots + a_n b_n = b_1 a_1 + \dots + b_n a_n = \langle w, v \rangle$ form the commutativity of real numbers.

Having done ALL that we can confirm that the basis eigenvectors are indeed normal and orthogonal in this definition of the inner product. This follows simply from the fact that $\langle e_i, e_i \rangle = 1 \cdot 1 = 1$ and we have that $\langle e_i, e_j \rangle = 1 \cdot 0 + 0 \cdot 1 = 0$ for different basis vectors. Therefore, we can apply the real spectral theorem and get that T must be self adjoint, since there is a diagonal matrix of orthonormal basis.

2 Exercise 7.C

Problem 1: Counterexample. Consider the operator T associated with the matrix

$$\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$$

Then observe that the matrix is equal to its conjugate transpose. Then the vectors $b_1 = (1/\sqrt{2}, i/\sqrt{2})$ and $b_2 = (1/\sqrt{2}, -i/\sqrt{2})$ form an orthonormal basis and it holds that $\langle T b_1, b_1 \rangle = 2$ and $\langle T b_2, b_2 \rangle = 2$. With all that in place, for $v = (1, 2)$ we have that

$$\langle Tv, v \rangle = -3$$

Problem 2: We have the following inequality by T being positive

$$\begin{aligned} \langle T(v-w), v-w \rangle &\geq 0 \\ \langle Tv - Tw, v-w \rangle &\geq \\ \langle w-v, v-w \rangle &\geq \end{aligned}$$

Multiplying both sides by -1 we get

$$\begin{aligned} -1 \langle w-v, v-w \rangle &\leq 0 \\ \langle v-w, v-w \rangle &\leq 0 \end{aligned}$$

by the property of inner products we also have that

$$\langle v-w, v-w \rangle \geq 0$$

So then it must be the case that

$$\langle v-w, v-w \rangle = 0$$

So again by definiteness

$$\begin{aligned} v-w &= 0 \\ v &= w \end{aligned}$$

Problem 4: We will abuse the associative property and the definition of adjoints to get the result. Observe, that T^*T is already an operator in $\mathcal{L}(V)$. Let $v, w \in V$.

$$\begin{aligned} \langle v, (T^*T)w \rangle &= \langle v, T^*(Tw) \rangle \\ &= \overline{\langle T^*(Tw), v \rangle} \\ &= \overline{\langle Tw, (T^*)^*v \rangle} \\ &= \overline{\langle Tw, Tv \rangle} \\ &= \overline{\langle w, T^*Tv \rangle} \\ &= \langle T^*Tv, w \rangle \end{aligned}$$

Thus, by definition T^*T is self-adjoint. We construct a similar series of steps for TT^* . (Ask Alex if I can state that since T^*T is positive, then its adjoint is also positive so TT^* is positive.) Observe again that TT^* is an operator on W .

$$\begin{aligned} \langle v, TT^*w \rangle &= \overline{\langle TT^*w, v \rangle} \\ &= \overline{\langle T^*w, T^*v \rangle} \\ &= \overline{\langle w, TT^*v \rangle} \\ &= \langle TT^*v, w \rangle \end{aligned}$$

Then, observe that for some eigenvalue λ the adjoint has the eigenvalue $\bar{\lambda}$. So, for any combination of TT^* or T^*T we have that for some eigenvector e that $T^*Te = \lambda T^*e = \lambda\bar{\lambda}$, which is always positive. So all the eigenvalues are positive as well, and since they are both self adjoint they must be positive.

Problem 7: \implies Given that T is positive there exists a R such that $T = R^*R$. From some past hw exercises we know that since T is invertible so must R^* and R . Then we have that $\langle Tv, v \rangle = \langle R^*Rv, v \rangle = \langle Rv, Rv \rangle$ which is never equal to zero for nonzero values, and is positive.

\Leftarrow Assume for contradiction that T is not invertible, that is null $T \neq \{0\}$. Then, there must exist some nonzero $v \in$ null T . This is a contradiction since that would mean that $\langle Tv, v \rangle = \langle 0, v \rangle = 0$.

Problem 8: \implies If $\langle \cdot, \cdot \rangle_T$ is an inner product then the definiteness property ensures that T is invertible, since $\langle v, v \rangle_T = 0$ if and only if $v = 0$ so that ensures that the null $T = \{0\}$. The Positivity property ensures that T is a positive operator since $\langle v, v \rangle_T \geq 0$ so $\langle Tv, v \rangle \geq 0$. (Note: In a real vectors space conjugate symmetry would also ensure that T is self adjoint :)

\Leftarrow If T is positive then it ensures Positivity in the inner product. Since, T is invertible by the previous example and question 7 it ensures that it is definiteness. Additivity in the first slot follows from T being a linear map and closure under vector addition, followed by addition in the first slot of the original inner product. Homogeneity in the first slot follows from T being a linear map and closure under scalar multiplication, followed by Homogeneity in the first slot of the original inner product. And conjugate symmetry is given by the fact that T is positive and given that it is invertible it has no nonzero eigenvalues, so it must be self-adjoint. Thus $\langle Tu, v \rangle = \langle u, Tv \rangle$ so the inner product follows that as well. Therefore, it is a valid inner product.

Problem 9: We know that I is given to be self adjoint and has the lone eigenvalue of 1, so it is positive. Observe that $I^2 = I$, since each positive operator has a unique square root, this is the square root of I . Then, the square root of that is $I^2I^2 = I^4$, so we continue in this fashion forever and thus it has infinite square roots.

Problem 11: This seems like some change of basis stuff going on. Let v be arbitrary. Then by the spectral theorem on T_1 there exists a basis of orthonormal eigenvectors such that $T_1v = a_12e_1 + a_25e_2 + a_37e_3$ for the eigenvectors e_1, \dots, e_3 and some scalars. Then given e_1, \dots, e_3 define S to be the linear map such that Se_1, \dots, Se_3 be equal to the orthonormal basis of eigenvectors for T_2 guaranteed by the spectral theorem. Since, this is just a definition of a linear map but it maps the orthonormal basis of V to another orthonormal we know that S must be an isometry. Observe, that since the vector space has dimension 3 any other orthonormal basis will be a linear combination of the one described, so S will still map it to an orthonormal basis. With this we have showed that such an isometry exists so then S^{-1} exists as well and then it will just map the linear combination in T_2 's eigenvectors back to T_1 's. Preserving the constants, and the eigenvalues since the eigenvalues are the same.