Math 110 Homework 12

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1 Exercises 7.D

Problem 1: First we verify that the given operator

$$\sqrt{T^*T}v = \frac{\|x\|}{\|u\|}\langle v, u\rangle u$$

is indeed positive. Observe that when we calculate

$$\langle\sqrt{T^*T}v, v\rangle = \frac{\|x\|}{\|u\|}|\langle v, u\rangle|^2$$

it is always a non-negative value, thus the operator is positive. Second we find the adjoint of T. Define T^* as follows

$$T^*v = \langle v, x\rangle u$$

with the same fixed x and u as for T. This is a valid adjoint since for $v, w \in V$ we have

$$\langle Tv, w\rangle = \langle v, T^*w\rangle$$
$$\langle v, u\rangle\langle x, w\rangle = \langle v, u\rangle\overline{\langle w, x\rangle} = \langle v, u\rangle\langle x, w\rangle$$

For all $v \in V$ we have that

$$\begin{aligned} T^*Tv &= T^*(\langle v, u\rangle x) \\ &= \langle v, u\rangle\langle x, x\rangle u \end{aligned}$$

Then, we can verify that indeed $\sqrt{T^*T}^2 = T^*T$ that we calculated. Therefore, given that it is positive it is a valid square root of T^*T.

Problem 2: Let T be the operator represented by this 2×2 matrix with the standard basis.

$$T = \begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix}$$

The eigenvalues for T are exactly $\lambda = 0$. Then we find T^*T using the adjoint of the matrix provided.

$$T^*T = \begin{pmatrix} 25/2 & -25/2 \\ -25/2 & 25/2 \end{pmatrix}$$

The eigenvalues for which are equal to $\lambda = 25, 0$. So the singular values are equal to $\sigma = 5, 0$.

Problem 4: Consider the polar decomposition for T

$$T = S\sqrt{T^*T}$$

Then let $v \in V$ be such that it is an eigenvector for $\sqrt{T^*T}$ with the associated eigenvalue s. So,

$$Tv = S(sv)$$

$$\|Tv\| = \|S(sv)\| = |s|\|Sv\|$$

Given that S is an isometry we then have

$$\|Tv\| = s\|v\| = s$$

Since, s is nonnegative and the norm of v is 1. By polar decomposition there always exists $\sqrt{T^*T}$ thus an eigenvector associated with it.

Problem 10: The singular values are the eigenvalues of $\sqrt{T^*T}$. If T is self adjoint then $\sqrt{T^*T} = \sqrt{T^2}$. From previous exercises we know that the eigenvalues for T^2 are just the eigenvalues λ for T squared, that is λ^2. So we have that for all eigenvectors v of $\sqrt{T^*T}$ $\sqrt{T^2}v = sv$ such that $T^2v = s^2v = \lambda^2 v$. Taking the square root on both sides we get that $s = |s| = |\lambda|$.

Problem 11: First observe that the singular values for T^* are the eigenvalues of $\sqrt{TT^*}$. We know that T and T^* have the same eigenvectors, and the associated eigenvalues are λ and $\overline{\lambda}$ respectively. Then for some eigenvector we have

$$T^*Tv = T^*\lambda v = |\lambda|^2 v$$

also

$$TT^*v = T^*\overline{\lambda}v = |\lambda|^2 v$$

Since, TT^* and T^*T have the same positive eigenvalues. Their, singular values are the same, equal to the positive square root of those same eigenvalues.

Problem 12: Consider the vector space $V = \mathbb{F}^2$ and the operator T represented by the following matrix in the standard basis.

$$T = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

Then this operator has the singular values $\sigma = 1, 0$. Observe that, $T^2 = 0$, so it has the singular values $\sigma = 0$ So, the singular values aren't even equal.

Problem 13: $\Longrightarrow$ First note that $\sqrt{0} = 0 = 0^2$. If T is invertible, then

$$\text{null } T^* = (\text{range } T)^\perp = V^\perp = \{0\}$$

So, T^* is invertible, and then T^*T is invertible. Then from a pervious exercise, T^*T does not have zero as an eigenvalue. Since the singular values are the positive square root of the eigenvalues of T^*T and it has non-zero eigenvalues, then the singular values are all non-zero as well.

$\Longleftarrow$ Consider the polar decomposition of T. If all the singular values are nonzero, then $\sqrt{T^*T}$ has all nonzero eigenvalues. Then, null $\sqrt{T^*T} = \{0\}$ so it is inveritble. Since, S is an isometry and we have a composition of invertible operators S and $\sqrt{T^*T}$ which is invertible, Therefore, T is invertible.

2 8.A

Problem 3:

Problem 4: Assume for contradiction that the intersection is not empty,

$$G(\alpha, T) \cap G(\beta, T) \neq \{0\}$$

Let v be the eigenvector in the intersection

$$v \in G(\alpha, T) \cap G(\beta, T)$$

Then construct the linearly independent list of eigenvectors as described in 8.13, choosing v to represent both α and β. It is trivial now that that the list is not linearly independent, thus a contradiction.

Problem 6:

Problem 8:

Problem 9:

Problem 11:

Problem 14: