Math 110 Homework 8

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1 Exercises 5.B

Problem 1: We wish to show that

$$(I-T)^{-1} = I + T + ... + T^{n-1}$$

So, we can multiply $I - T$ to both sides to get

$$I = (I-T)(I + T + ... + T^{n-1})$$

Then, we can distribute and see we get

$$I = I - T + T - T^2 + T^2 + ... - T^{n-1} + T^{n-1} + T^n$$

After cancelling out all the similar terms we are left with

$$I = I + T^n = I$$

Since, $T^n = 0$. So, to prove the statement we do the following operations

$$I = I + T^n = I$$
$$I = I - T + T - T^2 + T^2 + ... - T^{n-1} + T^{n-1} + T^n$$
$$I = (I-T)(I + T + ... + T^{n-1})$$

Then, we multiply both sides by $(I-T)^{-1}$ to get

$$(I-T)^{-1} = I + T + ... + T^{n-1}$$

as desired.

Problem 2: Assume for contradiction that $\lambda \neq 2$ and $\lambda \neq 3$ and $\lambda \neq 4$. Then, $T - 2I$ and $T - 3I$ and $T - 4I$ must all be invertible. Given,

$$(T-2I)(T-3I)(T-4I) = 0$$

for all $v \in V$ such that $v \neq 0$ we have that

$$(T-2I)(T-3I)(T-4I)v = 0v = 0$$

Then, for one of the values $T - 2I$ or $T - 3I$ or $T - 4I$ one of them maps v to 0. Since, they are all invertible their null spaces is just $\{0\}$, but then we have a contradiction since we had that $v \neq 0$. Therefore, it must be the case that λ is equal to 2, 3 or 4.

Problem 3: We proceed directly. Given

$$T^2 = I$$

We get that $T^2 - I = 0$ so,

$$(T-I)(T+I) = 0$$

Since, $\lambda \neq -1$ it must be that $T + I$ is invertible. Thus, for all non zero v in V we have that $(T+I)v$ is non zero. So it must be that for all $w \in V$, we have $(T-I)v = 0$. By definition $T - I$ is equal to the 0 linear map, so from $T - I = 0$ it follows that $T = I$