Math 110 Homework 4

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1 Exercies 3.B

Problem 7: Let V have dimension 2, and W have some dimension greater than or equal to 2. Let T be a non-injective non-zero linear map $T : V \to W$. We know that null T is a subspace of V. Also, null $T \neq \{0\}$ since it is not injective, thus $\dim \text{null } T > 0$. Since, we restricted V to have dimension 2 and null T is nonzero it must have exactly dimension 1. There exists a subspace U of V such that $V = \text{null } T \oplus U$. Therefore, $\dim U = 1$. Let S be another non-zero non-injective linear map $S : V \to W$ such that null $S = U$. Now we have the case that for $v \in V$,

$$(S + T)(v) = Sv + Tv$$

But for any non-zero $v \in V$ it is either in null S or null T but not both. So, the null $(S + T) = \{0\}$, which means that $S + T$ is injective. Therefore, it is not a subspace of $\mathcal{L}(U, V)$ since it is not closed under addition.

Problem 15: Assume for contradiction that T is a valid linear map. Given that T is a map $T : \mathbb{R}^5 \to \mathbb{R}^2$, we must have that rank $T = 2$ and nullity $T = 3$. By

$$\dim \mathbb{R}^5 = 5 = \text{rank } + \text{nullity } = 2 + 3$$

We then observe that the null space defined as follows has dimension 2.

$$\Big\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\Big\}$$

We claim that the vectors $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$ span the null space. Observe for some arbitrary coefficients $a, b \in \mathbb{F}$ the linear combination of the vectors $a(3, 1, 0, 0, 0)+b(0, 0, 1, 1, 1) = (3a, a, b, b, b)$. This holds the required property that $3x_2 = x_1$ and that $x_3 = x_4 = x_5$. Thus, it spans our null space and we have a contradiction, since we found a spanning list that has length less than a linearly independent list.

Problem 19: We know that for a linear map $T \in \mathcal{L}(V, W)$ the dimension of the range is less than or equal to the codomain. That is,

$$\text{rank } T \leq \dim W$$

Suppose, this was not the case then we could find a $v \in V$ such that $Tv \notin W$ and that is not a linear map. Then, we also have that $\dim \text{null } T = \dim U$. It then follows directly that

$$\dim V = \text{rank } T + \dim \text{null } T = \text{rank } T + \dim U$$

$$\dim V - \text{rank } T = \dim U$$

Therefore, from our first inequality

$$\dim U \geq \dim V - \dim W$$

Problem 26: Firstly, we are given that p is a nonconstant polynomial, therefore that implies that p must have degree greater than 0. So, this satisfies the condition that we will not have negative degrees. Then, we show that for any polynomial p we can find a polynomial of greater degree by simply multiplying every value by x. So, then all the polynomials of degree greater than 0 can be chosen, and if we have an operator that drops the degree by 1, then it must be the case that the range $D = \mathcal{P}(\mathbb{R})$. So, then it follows that it is surjective by defintion of surjectivity for linear maps.

2 Exercises 3.C

Problem 2: For $\mathcal{P}_3(\mathbb{R})$ let the basis be the following basis $1, x, x^2, x^3$ in this order. And for $\mathcal{P}_2(\mathbb{R})$ let the basis be $3x^2, 2x, 1$ for this order. $1, x, x^2, x^3$ is clearly a basis since it is just the standard basis reorganized, more formally the linear combination will still only be zero for all zero coefficients, since addition is commutative in $\mathbb{R}$. For $3x^2, 2x, 1$ any linear combination is only zero when all the coefficients are zero since each term has a different degree.

Problem 3: We want a matrix that basically has 1s in the diagonal up to the rank T and 0s everywhere else. So, we begin with the null T. We know that null T is a subspace of V. So let $t_1, ..., t_m$ be a basis for null T. Then, since $t_1, ..., t_m$ is a linearly independent list and is less than or equal to the length of the basis of V we can extend it to form a basis of V. Note that nullity $T = m$ Let $v_1, ..., v_n$ be the addtional vectors we add to extend our list to be a basis of V. Note that rank $T = n$. So, $t_1, ..., t_m, v_1, ..., v_n$ is a basis of V. Because we collected all the vectors that are equal to zero we know that $Tv_i = w_i$ for some $w_i \in W$ for $i \in \{1, ..., n\}$. Now we can show these $w_1, ..., w_n$ are linearly independent. We begin with $a_1w_1+...+a_nw_n = 0$ for some coefficients a_i. We can substitute the Tv_i to get $a_1Tv_1 + ... + a_nTv_n = 0$. Collecting the terms we have $T(a_1v_1 + ... + a_nv_n) = 0$. We know that $v_1, ..., v_n$ are linearly independent since it is formed by dropping vectors from the basis of V. Then, $a_1v_1 + ... + a_nv_n = 0$ only when all the coefficients are zero. So, we know all the a_i are zero, thus $w_1, ..., w_n$ is linearly independent. Then we can finally extend $w_1, ..., w_n$ to form a basis for W. Now we basically our done since we have shown that we have a basis for which $Tv_i = w_i$ for $i \in \{1, ..., \text{rank } T\}$. So, we can make our desired matrix since everything else is supposed to be zero because it is in the null space.

Problem 4: In a very roundabout way this question is asking if there is a vector v in every basis of V such that for every $T \in \mathcal{L}(V, W)$ it is that $Tv = w$ for some w in the basis of W. The stright forward case is if for some v in the basis of V if $Tv = 0$ then put that v at the start of the basis list, so that the column can be all zeros and satisfy the condition. The second also straight forward case if if for some v in the basis of V we happen to have a basis for W such that $Tv = w$ for w in the basis, then we are done as well, by moving both v and w to start of their respective basis lists. The last case is we can always construct a basis such that the condition is met, by just adding $Tv = w$ to the basis of W and then removing all the vectors that are multiples of w. Then we will be left with a linearly independent list that we can always extend to form a basis of W.

3 Exercises 3.D

Problem 3: For the $\Longrightarrow$ direction if there exists an invertible operator T, such that $Tu = Su$, then S is injective since T is injective. For the $\Longleftarrow$ direction since S is injective we know that for some basis $u_1, ..., u_m$ of U we know $Su_1, ..., Su_m$ is linearly independent. Then, we can extend $u_1, ..., u_m$ to form a basis for V. Then select a $T \in \mathcal{L}(V, V)$ such that $Tu_i = Su_i$ and for the extra vectors $Tv = v$. We make a direct map for these vectors. Basically, $Tv = v$. Then, we are guaranteed that T is injective and therefore also invertible.

Problem 4: For the $\Longleftarrow$ direction observe that if $T_1 = ST_2$, then null $S = \{0\}$. So, essentially all the vectors from V that mapped to 0, will again map to zero for S. So null $T =$ null S. For the $\Longrightarrow$ direction if null $T_1 =$ null T_2 then we know there exists a linear operator $S \in \mathcal{L}(W, W)$. We know that for $v \in$ null T_1, null T_2 it will map $T_1 v = T_2 v = 0$. So, then $T_1 v = ST_2 v$. Since all linear maps map 0 to 0. Suppose, $u \notin$ null T_1, null T_2. Then, define S such that $ST_2 = T_1$. It is straightforward to show that S is a linear operator since by our construction of S, null $S = \{0\}$, since we require that S map all nonzero values to a nonzero value. Then, S is injective and therefore also invertible.

Problem 5: For the $\Longleftarrow$ direction, if we let S be the identity linear map, which we know is invertible, then $T_1 = T_2 S = T_2$. So, clearly they are the same and must have the same range. For the $\Longrightarrow$ direction if range $T_1 =$ range T_2 then we know for some $v, u \in V$ it is the case that $T_1 v = T_2 u$. Then we can define S such that $Sv = u$. Then, $T_1 v = T_2 S v$. We now are left to show that such a S is invertible. Observe, that for all $u \in V$ there exists a v such that $Su = v$. Then, by construction of S it is surjective. So, it is also invertible.

Problem 7: **(a)** For all $u \in V$ such that $u \neq v$ we proceed by our normal addition and scalar multiplication for some maps $T, S \in E$. For v we have that $(T + S)(v) = Tv + Sv = 0 + 0$, so it is closed under addition for all vectors in V. For some $\lambda \in \mathbb{F}$ it holds that $T(\lambda v) = \lambda T v = \lambda 0 = 0$, so it satisfies homogeniety as well. Therefore, it is a subspace.
(b) We can start of with v and construct a basis by extending it for V. But, observe then that the column for v will have to be all zeros. Therefore, the dimension of E will be $E = \dim W(1 - \dim V)$.

Problem 9: For the $\Longleftarrow$ direction it is straightforward that if S and T are both invertible then $TSS^{-1}T^{-1} = TIT^{-1} = I$. For the other side, $T^{-1}S^{-1}ST = T^{-1}IT = I$. Therefore, we have found an inverse and it must be invertible. For the $\Longrightarrow$ direction we can show that null $T = \{0\}$ and then make a similar argument for S. Let $v \in$ null T, Then, $v = (ST)^{-1}(ST)v = 0$, since after Tv all the linear maps will map 0 to 0. Therefore, $v = 0$ and we have shown that T is injective and therefore invertible. We proceed with the exact same argument except we select a v such that $Tv \in$ null S. By the same argument as before S is then injective and thus invertible.

Problem 10: $ST = I$ implies by definiton that $S = T^{-1}$. Then, we can take the liear map of T after each following map and get $TS = TT^{-1} = I$. We proceed with the exact same argument in the opposite direction to show the if and only if.

Problem 16: If T is a scalar multiple then $T = \lambda I$. It directly follows then that $ST = S\lambda I = \lambda IS = TS$. We can make the argument that if there is a v such that that $STv = TSv$. Then it must be that if $u = Sv$. Then, Tu can be considered for STu, but clearly the only linear map that satisfies this is a scalar multiple of the identity map.

Problem 19: We know that we can always select a p such that it has higher degree than Tp sine we are in infinite dimensions therefore, there for all p we can always find a p' such that $p = Tp'$. Observe that for this reason it is always surjectve. Since, range $T = \mathcal{P}(\mathbb{R})$. For the second part observe that it must be the case that if T is injective then it must also be invertible therefore $T^{-1}Tp = p$. Thus, there exists no operation such that you can decrease the degree and still be surjective, so it must be that for all nonzero p the statement holds that $degTp = degp$.

4 Exercises 3.E

Problem 7: We know from the definition that $v + U = x + U$, so if we now given that $v + U = x + U = x + W$ then it clearly follows that $U = W$.

Problem 8: Observe for the $\Longleftarrow$ direction that we begin with $\lambda v + (1 - \lambda)w$. We can then do the following algebraic manipulations to get that $\lambda(v - w) + w \in A$ which is clearly the case and then it must be that it is affine for all $v, w \in A$ since it is closed, and follows from that. For the $\Longrightarrow$ direction consider that for an affine subset of V we have that for all v and w as in the argument above we can just make the same manipulations in the other direction and show the statement holds.

Problem 9: For an affine subsets A_1 and A_2 it is parralel to V then it must be that the intersection of two affine subsets that are both parralel to V are parralel to each other. Therefore, and intersection will also be parralel to V or simply not exist in which case be the empty set. Therefore we have that $A_1 \cap A_2$ is an affine subset as well (or the empty set).