Math 110 Homework 6

Tarang Srivastava

1 Exercises 3.F

Problem 7: Since, p is in the standard basis of $\mathcal{P}_m(\mathbb{R})$ it has only one term to consider. Namely, a term x^j where $j \in \{1, ..., m\}$. Then, if p has a degree strictly greater than j it will have some x term for $p^(j)$. Thus, evaluated at 0, it will equal 0. If p has a degree strictly less than j then $p^{(j)} = 0$ by the power rule form calculus. So, when p has degree exactly j we have that it will evaluate to some constant by the power rule. This constant will be exactly $j!$, again by the power rule. So, we normalize it by dividing it by $j!$, so $\varphi(j) = 1$. Then, we have a valid φ_j such that it only evaluates to 1 for one of the polynomials in the standard basis and 0 for all other. Thus, a valid dual basis.

Problem 9: Since, $\psi \in V'$ it is equivalent to the following linear combination with scalars $a_1, .., a_n$. That is,

$$\psi = a_1\varphi_1 + ... + a_n\varphi_n$$

Then for some v_j in the basis of V w ehave that

$$\psi(v_j) = a_j\varphi_j(v_j) = a_j$$

Since, all the other ones are 0, since it is evaluated at a basis vector Then, we simply make direct substitutions for each $j \in \{1, ..., n\}$ and get the desired result.

Problem 11: This is pretty simple. For the $\Longleftarrow$ direction. All d_j are some arbitrary costants in $\mathbb{F}$. So, we construct $A_{j,k}$ such that each column is just a scalar multiple of each other. Since, we are multiplying the same $(c_1, ..., c_m$ by arbitrary constants d_j. Then, clearly if all the vectors are scalar multiples of each other we must remove all of them except 1 to get a linearly independent list. Therefore, the dimension of the row space is 1 and thus rank is 1. For the $\Longrightarrow$ direction we argue that since A has rank 1, then the column space has dimesnion 1. So, it follows that the columns are constructed as follows for some vectors $d_1, ..., d_n$. That is they are scalar multiples of each other.

Problem 19: $\Longrightarrow$. If $U = V$, then the φ such that $\varphi(u) = 0 = \varphi(v)$ for all $u \in U$ and all $v \in V$ is just 0, since 0 is unique in a vector space.
$\Longleftarrow$. We know V is finite-dimensional, so

$$\dim U + \dim U^0 = \dim V$$

But, the annhilator has dimension 0 so $\dim U = \dim V$ since, U is a subspace of V it implies that $U = V$

Problem 20: Pick an arbitrary $\varphi \in W^0$. Since, $U \subset W$ for all $u \in U$, $\varphi(u) = 0$, so clearly $\varphi \in U^0$. We showed this for an arbitrary φ so $W^0 \subset U^0$.

Problem 21: Since V is finite dimensional we abuse the dimension of the annhilator formula.

$$\begin{aligned} \dim U + \dim U^0 &= \dim V \\ \dim W + \dim W^0 &= \dim V \\ \dim W + \dim W^0 &= \dim U + \dim U^0 \end{aligned}$$

Since, $W^0 \subset U^0$ we have that

$$\dim W^0 \leq \dim U^0$$

Then, to hold the equality it must be the case that

$$\dim W \geq \dim U$$

Therefore,

$$W \supset U$$

Problem 22: For some $v \in V + U$ we have that $v = u + w$ for $u \in U$ and $w \in W$. Then for some $\varphi \in (U + W)^0$, $\varphi(v) = 0 = \varphi(u+w) = \varphi(u) + \varphi(w) = 0$. Since, $u, w \in U + W$ then clearly they must be zero as well. So, $\varphi \in U^0 \cap W^0$. Thus, $U + W^0 \subset U^0 \cap W^0$. We use the exact same argument in the reverse direction to get the other conclusion $U + W^0 \supset U^0 \cap W^0$ So, then it must be that $U + W^0 = U^0 \cap W^0$

Problem 23: Since, V is finite dimensional and U and W are subspaces we know

$$\dim V = \dim U + \dim W - \dim U \cap W$$

by adding extra $\dim V$ on each sides and moving things around we get

$$\dim V - \dim U + \dim V - \dim W = \dim V - \dim U \cap W$$

We also know that $\dim U^0 + \dim U = V$ and $\dim W^0 + \dim W = V$ and $\dim(U \cap W) + \dim(U \cap W)^0 = \dim V$. Then making the right substitutions we get,

$$\dim U^0 + \dim W^0 = \dim(U \cap W)^0$$

To make it all work, we just have to show that one of the sides is a subspace of the other. Suppose we have a $\varphi \in (U \cap W)^0$, then for some $\psi in U^0 + W^0$ we know that $\psi = \varphi_U + \varphi_W$ for $\varphi_U \in U^0$ and $\varphi_W \in W^0$. Then, clealry $\psi(v)$ for some $v \in U \cap W$ it holds that $\psi(v) = (\varphi_U + \varphi_W)(v) = \varphi_U(v) + \varphi_W(v)$ and since v is in U and W it must be that $\varphi_U(v) = 0$ and $\varphi_W(v) = 0$ so $\psi(v) = 0$. So, we showed that

$$(U \cap W)^0 \subset U^0 + W^0$$

Therefore,

$$(U \cap W)^0 = U^0 + W^0$$

Problem 34:

Problem 35: We define a linear map $T : \mathcal{P}(\mathbb{R})' \to \mathbb{R}^\infty$ as follows. By our understanding of the standard dual basis of $\mathcal{P}(\mathbb{R})'$ for some $\varphi \in \mathcal{P}(\mathbb{R})'$ is of the form $\varphi = a_1\varphi_1 + a_2\varphi_2 + ...$ So we define $T(\varphi) = (a_1, a_2, ...)$. We know the following coefficients uniquely, so the mapping is to a unique value in $\mathbb{R}^\infty$, that is T is injective. Now we define the inverse T^{-1} as follows. For some $(a_1, a_2, ...) \in \mathbb{R}^\infty$ we map $T^{-1}(a_1, a_2, ...) = \varphi = a_1\varphi_1 + a_2\varphi_2 + ...$ Then, since each φ is uniquely defined by these polynomials we have shown that T^{-1} is injective. Since, bot T and T^{-1} are injective this completes the proof that T is invertible, so the two vector spaces are isomorphic.

2 Exercises 4

Problem 2: No. Consider the following counterexample for a subspace where $m = 2$. $x^2 + x$ and $-x^2$ are in the subspace. But the vector addition of $x^2 + x - x^2 = x$ is not in the subspace since it has degree 1.

Problem 3: No. Consider the following counterexample for a subspace where $m = 2$. $x^2 + x$ and $-x^2$ are in the subspace, since they both have degree 2 which is even. But the vector addition of $x^2 + x - x^2 = x$ is not in the subspace since it has degree 1, which is odd.

Problem 6: For the $\Longleftarrow$ direction if p and p' don't have distinct zeros then it means that there exists no $z \in \mathbb{C}$ such that $p(z) = 0 = p'(z)$. In other words, a point z such that it is zero and the slope at that point is zero. Then it must be that there is no point $a \in \mathbb{C}$ such that $p(z) = (a - z)^m$ for $m > 1$ since by the chain rule $p'(z)$ will have some form of the term $m(a - z)^{m-1}$. This, can only happen if $m > 2$ and that means that it has repeated zeros. We use the exact same argument to go the other way thus showing the if and only if condiiton.

Problem 7: By 4.15 if a polynomial has a real coefficients and a $z \in \mathbb{C}$ as a root, the roots come in pairs. Since, pairs are well pairs, and the polynomial has odd degree, so odd zeros there is a forever lonely zero left out. This, forever lonely zero must be a real number since if it were complex it would come in a pair.

Problem 8: Tp is well defined in $\mathcal{P}(\mathbb{R})$ since for every polynomial the term $\frac{p-p(3)}{x-3}$ evaluates to a polynomial by our definition of polynomial division. And, when $x =$ it evaluates to $p'(3)$ which again is well defined for all polynomials. T is a linear map by closure under addition and scalar multiplication. Observe for $T(p + q)$ we have

$$T(p+q) = \begin{cases} \frac{p+q-p(3)-q(3)}{x-3} & \text{if } x \neq 3 \\ p'(3) + q'(3) & \text{if } x = 3 \end{cases}$$

Which is exactly equivalent to $Tp + Tq$. Then for sclar multiplication,

$$T\lambda p = \begin{cases} \lambda\frac{p-p(3)}{x-3} & \text{if } x \neq 3 \\ \lambda p'(3) & \text{if } x = 3 \end{cases}$$

By distributivity for the the top term, and by the power rule for the bottom term. Which are both equivalent to λTp.

Problem 10: Assume for contradiction that one of the coefficients is complex. Then there is some term in p as az^n for some $a \in \mathbb{C}$ and $n \leq m$. Then, a real number multiplied by a complex number is complex. This is a contradiction and therefore the polynomial must have all real coefficients.