Math 110 Homework 4

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# 1 Exercies 3.B

**Problem 7:** Given that $T$ is injective. For two unique vectors $v, u \in V$.

**Problem 15:** Assume for contradiction that $T$ is a valid linear map. Given that $T$ is a map $T : \mathbb{R}^5 \to \mathbb{R}^2$, we must have that rank $T = 2$ and nullity $T = 3$. By

$$\dim \mathbb{R}^5 = 5 = \text{rank } + \text{nullity } = 2 + 3$$

We then observe that the null space defined as follows has dimension 2.

$$\left\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\right\}$$

We claim that the vectors $(3, 1, 0, 0, 0), (0, 0, 1, 1, 1)$ span the null space. Observe for some arbitrary coefficients $a, b \in \mathbb{F}$ the linear combination of the vectors $a(3, 1, 0, 0, 0)+b(0, 0, 1, 1, 1) = (3a, a, b, b, b)$. This holds the required property that $3x_2 = x_1$ and that $x_3 = x_4 = x_5$. Thus, it spans our null space and we have a contradiction, since we found a spanning list that has length less than a linearly independent list.

**Problem 19:** We know that for a linear map $T \in \mathcal{L}(V, W)$ the dimension of the range is less than or equal to the codomain. That is,

$$\text{rank } T \leq \dim W$$

Suppose, this was not the case then we could find a $v \in V$ such that $Tv \notin W$ and that is not a linear map. Then, we also have that $\dim \text{null } T = \dim U$. It then follows directly that

$$\dim V = \text{rank } T + \dim \text{null } T = \text{rank } T + \dim U$$

$$\dim V - \text{rank } T = \dim U$$

Therefore, from our first inequality

$$\dim U \geq \dim V - \dim W$$

**Problem 26:**