Math 110 Homework 2

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1 Exercise 2.A

Problem 8: The statement holds. Assume for contradiction that $\lambda v_1, .., \lambda v_m$ is linearly dependent. That is, for $\lambda \neq 0$, $\lambda v_1 + ... + \lambda v_m = 0$. Then we have a contradiction, since we claimed that $v_1, ..., v_m$ are linearly independent, and we have an example of a linear combination with non-zero coefficients that is equal to zero. Then, $\lambda v_1, .., \lambda v_m$ must be linearly independent.

Problem 9: The statement is false. Consider the counterexample where $w_i = -v_i$ for $i \in \{1, ..., m\}$. We know that $w_i \in V$, since the additive inverse must exist in V for v_i. Then, for non-zero coefficients the linear combination $(v_1 + w_1) + ... + (v_m + w_m) = 0$ which shows that it is linearly dependent.

Problem 10: To show that $w \in \text{span}(v_1, ..., v_m)$ we just have to show there exists a linear combination such that

$$w = a_1 v_1 + ... + a_m v_m$$

By linear dependence we have $a_1(v_1 + w) + ... + a_1(v_m + w) = 0$ for nonzero coefficients. We can collect that w terms and bring it to the other side of the equal sign and divide by the coefficients, again because they are not all zero, to get

$$a_1 v_1 + ... + a_m v_m = (a_1 + ... + a_m) w$$

$$\frac{a_1}{(a_1 + ... + a_m)} v_1 + ... + \frac{a_m}{(a_1 + ... + a_m)} v_m = w$$

Thus, showing that w is in the span of $v_1, ..., v_m$.

Problem 11: To show the " $\implies$ " direction, assume for contradiction $w \in \text{span}(v_1, ..., v_m)$. Then, we can express w as a linear combination of $v_1, ..., v_m$. $w = a_1 v_1 + ... + a_m v_m$ Let $a_1 v_1 + ... + a_m v_m - w$ be a linear combination, but here we have a contradiction. Since, $a_1 v_1 + ... + a_m v_m - w = a_1 v_1 + ... + a_m v_m - (a_1 v_1 + ... + a_m v_m) = 0$ for non-zero coefficients $a_1, ..., a_m$. Therefore, $w \notin \text{span}(v_1, ..., v_m)$.
For the " $\impliedby$ " direction, assume for contradiction $v_1, ..., v_m, w$ is linearly dependent. Then, there exists a linear combination with nonzero coefficients such that $a_1 v_1 + ... + a_m v_m + cw = 0$. Subtracting by cw and dividing by c, for when $c \neq 0$, if $c = 0$ we are done, we get $\frac{a_1}{c} v_1 + ... + \frac{a_m}{c} v_m = w$, which is a contradiction, since it implies that $w \in \text{span}(v_1, ..., v_m)$. Thus, it must be that $v_1, ..., v_m, w$ is linearly independent.

2 Exercise 2.B

Problem 1: The zero-Vector space, $V = \{0\}$ is the only vector space with exactly one basis, and that is the empty set. Suppose there is another vector space with a basis b with length > 0. Then, let $v \in b$. By the existence of an additive identity we know that $-v$ is also in the vector space, and thus we can have an equally valid basis with $\{-v\}$, and thus having two bases. Therefore, it must be the case that for a vector space to have exactly one basis, the basis must be of length 0, which only the empty set satisfies.

Problem 3: (a) A possible basis for U is as follows $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ observe that for any linear combination it holds that $x_1 = 3x_2$ and $x_3 = 7x_4$.
(b) We can extend the previous basis by adding the following two vectors to get a basis for $\mathbb{R}^5$. $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$ Notice, that by adding the following vectors we can express any arbirary $x_2 = a(3, 1, 0, 0, 0) - 3a(1, 0, 0, 0, 0)$. The same reasoning follows for x_4.
(c) From part (b) we can just form a subspace that handles the case to express arbitrary x_2 and x_4. Therefore, let $W = \{(x, 0, y, 0, 0) \colon x, y \in \mathbb{R}\}$. That is, $W = \text{span}((1, 0, 00, 0), (0, 0, 1, 0, 0))$. We then also observe that $U \cap W = \{0\}$ as desired.

Problem 5: Consider the following basis

$$1, x, x^3 + x^2, x^3$$

Thus, for any arbitrary polynomial of degree two you can select $a(x^3 + x^2) - a(x^3)$.

Problem 7: