Math 110 Homework 6

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1 Exercises 5.A

Problem 1: The argument is as follows.

(a) Let u be an arbitrary vector $u \in U$. If $U \subset \text{null } T$, then $u \in \text{null } T$. So, $Tu = 0$. Since, U is a vector space, it must be that $0 \in U$, so $Tu \in U$. Thus, U is invariant under T given the condition.

(b) By definition we have $Tu \in \text{range } T$ for all $u \in U$. Since, $\text{range } T \subset U$ we have that for all u, $Tu \in U$. Thus, U is invariant under T given the condition.

Problem 3: We wish to show that for all $u \in \text{range } S$ we have that $Tu \in \text{range } S$. Let $v \in V$, then $STv \in \text{range } S$ by definition. Given $ST = TS$, we have that $STv = TSv$. So, $TSv \in \text{range } S$. Let $u \in \text{range } S$, then there exists some $v \in V$ such that $Sv = u$. Since,$TSv \in \text{range } S$, we have $Tu \in \text{range } S$.

Problem 6: True!
We have a subspace U of V such that it is invariant for all $T \in \mathcal{L}(V, V)$, assume for contradiction that $U \neq 0$ and $U \neq V$. Then, since V is finite dimensional we have some basis of U

$$u_1, ..., u_m \text{ is a basis of } U$$

Then we can extend the basis of U to a basis of V, and since we know that $U \neq V$ it must be that we must extend it by at least one vector.

$$u_1, ..., u_m, v_1, ..., v_n \text{ is a basis of } V$$

Then, let $T \in \mathcal{L}(V)$ such that for all $i \in 1, ..., n$ we have that

$$Tu_i = v_i$$

and the remaining basis vectors of U, if there are any, are mapped to 0. Let u be an arbitrary vector $u \in U$, then

$$u = a_1u_1 + ... + a_mu_m$$

for some scalars $a_1, ..., a_m$. Then,

$$\begin{aligned} Tu &= a_1Tu_1 + ... + a_mTu_m \\ Tu &= a_1v_1 + ... + a_mv_m \end{aligned}$$

Since we have that U is invariant under all linear maps it must be that $Tu \in U$ so there exists some linear combination of $u_1, ..., u_m$ that is equal to Tu. So for some scalars $b_1, .., b_m$

$$Tu = b_1u_1 + ... + b_mu_m$$

Substituting the two reprsentations of Tu we get

$$b_1u_1 + ... + b_mu_m = a_1v_1 + ... + a_mv_m$$

Then, we have a contradiction since we claimed that $u_1, ..., u_m, v_1, ..., v_n$ is a basis and therefore a linearly independent list of vectors. But since they can be expressed as a linear combination of each other as such they are not linearly independent, by some previous exercises. Thus, it must be that $U = \{0\}$ or $U = V$.

Problem 8: By defintion we wish to find eigenvalues and eigenvectors, $v = (w, z)$ such that

$$T(w, z) = (z, w) = \lambda(w, z) = (\lambda w, \lambda z)$$

Then, we have to find solutions to $\lambda w = z$ and $\lambda z = w$. Following some substitutions we get

$$z(\lambda^2 - 1) = 0$$

Since, $v \neq 0$ we are left with $\lambda = \pm 1$.
So, $\lambda_1 = 1$ with the corresponding eigenvectors some scalar multiple $v_1 = (1, 1)$ and $\lambda_2 = -1$ with the corresponding eigenvectors some scalar multiple $v_2 = (-1, 1)$.

Problem 12: We wish to find eigenvalues and eigenvectors, $p(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$, such that

$$(Tp)(x) = xp'(x) = \lambda p(x)$$

That is, $4a_4x^4 + 3a_3x^3 + 2a_2x^2 + a_1x = \lambda a_4x^4 + \lambda a_3x^3 + \lambda a_2x^2 + \lambda a_1x + \lambda a_0$. So, clearly $4a_4x^4 = \lambda a_4x^4$. Solving for this we get $\lambda = 4$, but then the following terms do not hold so we must have that $a_3 = a_2 = a_1 = a_0 = 0$. So the polynomial for $\lambda = 4$ must be of the form a_4x^4. We follow with this argument for the remaining terms to get that the eigenvalues are $\lambda = 4, 3, 2, 1$ and that the corresponding eigenvectors are some scalar multiple of x^4, x^3, x^2, x respectively.

Problem 13: We can just show that $\alpha - \lambda \leq |\alpha - \lambda| < \frac{1}{1000}$ is equivalent to showing $\alpha < \frac{1}{1000} + \lambda$ since we are working with elements in our field and performing field operation it must be the case that $\alpha \in \mathbb{F}$. (Clearly this wont hold in ALL fields but in the world of Axler this is just $\mathbb{C}$ or $\mathbb{R}$). Then since V is finite dimension there are at most $\dim V$ many eigenvalues, that is finite number of eigenvalues. Again, since we are either working with either the reals or complex, which are both infinite just select a number α that satisfies the inequality and is not eqaul to one of the eigenvalues. Then, it follows by 5.6, since we chose α such that it is not an eigenvalue $T - \alpha I$ is invertible.

Problem 15: (a) Let λ be an eigenvalue of T. Then, $Tv = \lambda v$ for some corresponding eigenvector v. Let, u be a vector in V such that $Su = v$. We know this exists, since S is an invertible operator on V. Therefore,

$$\begin{aligned} Tv =& \lambda v \\ TSu =& \lambda Su \end{aligned}$$

Then composing with S^{-1} we get

$$\begin{aligned} S^{-1}TSu &= S^{-1}\lambda Su \\ &= \lambda S^{-1}Su \\ &= \lambda u \end{aligned}$$

Thus, the eigenvalues are the same.
(b) From our argumetn above we had that the corresponding eigenvectors v of T have the eigenvector u for $S^{-1}TS$ such that $v = Su$ or $S^{-1}v = u$. That is the relationship.

Problem 19: We have that for $T(1, 1, ..., 1) = (n, n, ..., n) = \lambda(1, 1, ..., 1)$ if we have $\lambda = n$ we satisfy the inequality. So, the corresponding eigenvector to $\lambda = n$ is any scalar multiple of $(1, ..., 1)$. These are all the eigenvalues and eigenvectors of T, since by Kubrat's hint the Trace is equal to the sum of the eigen values and the trace is eqaul to n if there are only 1s in the main diagonal. Since, we found an eigenvalue equal to n thre can be no more, so we are done.

Problem 23: Suppose λ is an eigenvalue of ST and v is the corresponding eigenvector. Then, $STv = \lambda v$. Let w be the vector $w \in V$ such that $Tv = w$. Then, $Sw = \lambda v$. Composing T for both sides we then get $TSw = T\lambda v = \lambda Tv$, but then we can substitute for Tv and get that $TSw = \lambda w$. Therefore, λ is an eigenvalue for TS as well. This holds for all eigenvalues and therefore they have the same eigenvalues.

Problem 24: (a) Let $x = (1, 1, ..., 1) \in \mathbb{F}^n$ then, Ax is just the vector where each row is the sum of the corresponding row. Since, the rows sum to 1, we have that $Ax = (1, 1, ..., 1)$. So, we have the case that

$$T(1, ..., 1) = A(1, ..., 1) = \lambda(1, ..., 1)$$

Then $\lambda = 1$, and will always be an eigenvalue that exists for all n.
(b) Following from Kubrat's answer on Piazza @230. Suppose 1 is an eigenvalue, then $T - \lambda I$ is a matrix where all the columns add up 0. Since, before they sum to 1 and now we subtracted 1 from some element in each column. Now observe that if we take some vector v_j in the standard basis of $\mathbb{F}^n$ and find some basis for $Tv = w$. We have that it is equal to the sum of the j^{th} column of A, but since it is equal to zero we have found a linear combination that would mean that $a_1 w_1 + ... + a_n w_n = 0$ for some nonzero coefficients. So it is the case that $\dim \operatorname{range}(A - \lambda I) < n$. Therefore, $A - \lambda I$ is not surjective and by 5.6 $\lambda = 1$ is an eigenvalue.