

Section 1.A – \mathbb{R}^n and \mathbb{C}^n

$-\alpha$, subtraction, $1/\alpha$, division: Let $\alpha, \beta \in \mathbf{C}$. Let $-\alpha$ denote the additive inverse of α . Thus $-\alpha$ is the unique complex number such that $\alpha + (-\alpha) = 0$. Subtraction on \mathbf{C} is defined by $\beta - \alpha = \beta + (-\alpha)$. For $\alpha \neq 0$, let $1/\alpha$ denote the multiplicative inverse of α . Thus $1/\alpha$ is the unique complex number such that $\alpha(1/\alpha) = 1$. Division on \mathbf{C} is defined by $\beta/\alpha = \beta(1/\alpha)$

list, length: Suppose n is a nonnegative integer. A list of length n is an ordered collection of n elements (which might be numbers, other lists, or more abstract entities) separated by commas and surrounded by parentheses. A list of length n looks like this: (x_1, \dots, x_n) . Two lists are equal if and only if they have the same length and the same elements in the same order.

\mathbb{F}^n : \mathbf{F}^n is the set of all lists of length n of elements of \mathbf{F} : $\mathbf{F}^n = \left\{ (x_1, \dots, x_n) : x_j \in \mathbf{F} \text{ for } j = 1, \dots, n \right\}$. For $(x_1, \dots, x_n) \in \mathbf{F}^n$ and $j \in \{1, \dots, n\}$, we say that x_j is the j^{th} coordinate of (x_1, \dots, x_n)

addition in \mathbb{F}^n : Addition in \mathbf{F}^n is defined by adding corresponding coordinates: $(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$

Commutativity of addition in \mathbb{F}^n : If $x, y \in \mathbf{F}^n$, then $x + y = y + x$

\mathbf{o} : Let \mathbf{o} denote the list of length n whose coordinates are all \mathbf{o} : $\mathbf{0} = (0, \dots, 0)$

additive inverse in \mathbb{F}^n : For $x \in \mathbf{F}^n$, the additive inverse of x , denoted $-x$, is the vector $-x \in \mathbf{F}^n$ such that $x + (-x) = \mathbf{0}$. In other words, if $x = (x_1, \dots, x_n)$, then

$-x = (-x_1, \dots, -x_n)$

scalar multiplication in \mathbb{F}^n : The product of a number λ and a vector in \mathbf{F}^n is computed by multiplying each coordinate of the vector by λ : $\lambda(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$ here $\lambda \in \mathbf{F}$ and $(x_1, \dots, x_n) \in \mathbf{F}^n$

Section 1.B – Definition of Vector Space

addition, scalar multiplication: An addition on a set V is a function that assigns an element $u + v \in V$ to each pair of elements $u, v \in V$. A scalar multiplication on a set V is a function that assigns an element $\lambda v \in V$ to each $\lambda \in \mathbf{F}$ and each $v \in V$.

Vector Space: A vector space is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold: commutativity

$$u + v = v + u \text{ for all } u, v \in V$$

associativity $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbf{F}$ additive identity there exists an element $\mathbf{0} \in V$ such that $v + \mathbf{0} = v$ for all $v \in V$ additive inverse for every $v \in V$, there exists $w \in V$ such that $v + w = \mathbf{0}$ multiplicative identity $1v = v$ for all $v \in V$ distributive properties

$$a(u + v) = au + av \text{ and } (a + b)v = av + bv \text{ for all } a, b \in \mathbf{F} \text{ and}$$

all $u, v \in V$

vector, point: Elements of a vector space are called vectors or points.

real vector space, complex vector space: A vector space over \mathbf{R} is called a real vector space. A vector space over \mathbf{C} is called a complex vector space.

\mathbb{F}^S : If S is a set, then \mathbf{F}^S denotes the set of functions from S to \mathbf{F} . For $f, g \in \mathbf{F}^S$, the sum $f + g \in \mathbf{F}^S$ is the function defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in S$. For $\lambda \in \mathbf{F}$ and $f \in \mathbf{F}^S$, the product $\lambda f \in \mathbf{F}^S$ is the function defined by $(\lambda f)(x) = \lambda f(x)$ for all $x \in S$

Unique Additive Identity: A vector space has a unique additive identity

Unique additive inverse: Every element in a vector space has a unique additive inverse.

The number \mathbf{o} times a vector: $\mathbf{0}v = \mathbf{0}$ for every $v \in V$

A number times the vector \mathbf{o} : $a\mathbf{0} = \mathbf{0}$ for every $a \in \mathbf{F}$

The number -1 times a vector: $(-1)v = -v$ for every $v \in V$

Section 1.C – Subspaces

Subspace: A subset U of V is called a subspace of V if U is also a vector space (using the same addition and scalar multiplication as on V).

Conditions for a subspace: A subset U of V is a subspace of V if and only if U satisfies the following three conditions: additive identity $\mathbf{0} \in U$ closed under addition $u, w \in U$ implies $u + w \in U$ closed under scalar multiplication $a \in \mathbf{F}$ and $u \in U$ implies $au \in U$

sum of subsets: Suppose U_1, \dots, U_m are subsets of V . The sum of U_1, \dots, U_m , denoted $U_1 + \dots + U_m$, is the set of all possible sums of elements of U_1, \dots, U_m . More precisely, $U_1 + \dots + U_m = \{u_1 + \dots + u_m : u_1 \in U_1, \dots, u_m \in U_m\}$

Sum of subspaces is the smallest containing subspace: Suppose U_1, \dots, U_m are subspaces of V . Then $U_1 + \dots + U_m$ is the smallest subspace of V containing U_1, \dots, U_m

direct sum: Suppose U_1, \dots, U_m are subspaces of V . The sum $U_1 + \dots + U_m$ is called a direct sum if each element of $U_1 + \dots + U_m$ can be written in only one way as a sum $u_1 + \dots + u_m$, where each u_j is in U_j . If $U_1 + \dots + U_m$ is a direct sum, then $U_1 \oplus \dots \oplus U_m$ denotes $U_1 + \dots + U_m$, with the \oplus notation serving as an indication that this is a direct sum.

Condition for a direct sum: Suppose U and W are subspaces of V . Then $U + W$ is a direct sum if and only if $U \cap W = \{\mathbf{0}\}$

Direct sum of two subspaces: Suppose U and W are subspaces of V . Then $U + W$ is a direct sum if and only if $U \cap W = \{\mathbf{0}\}$

Section 2.A Span and Linear Independence

Span: The set of all linear combinations of a list of vectors v_1, \dots, v_m in V is called the span of v_1, \dots, v_m , denoted $\text{span}(v_1, \dots, v_m)$. In other words,

$$\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + \dots + a_m v_m : a_1, \dots, a_m \in \mathbf{F}\}$$

The span of the empty list $()$ is defined to be $\{\mathbf{0}\}$.

Span is the smallest containing subspace: The span of a list of vectors in V is the smallest subspace of V containing all the vectors in the list.

spans: If $\text{span}(v_1, \dots, v_m)$ equals V , we say that v_1, \dots, v_m spans V

finite-dimensional vector space: A vector space is called finite-dimensional if some list of vectors in it spans the space.

polynomial over a field \mathbf{F} : A function $p : \mathbf{F} \rightarrow \mathbf{F}$ is called a polynomial with coefficients in \mathbf{F} if there exist $a_0, \dots, a_m \in \mathbf{F}$ such that

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$$

for all $z \in \mathbf{F}$. $\mathcal{P}(\mathbf{F})$ is the set of all polynomials with coefficients in \mathbf{F} .

degree of a polynomial: A polynomial $p \in \mathcal{P}(\mathbf{F})$ is said to have degree m if there exist scalars $a_0, a_1, \dots, a_m \in \mathbf{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \dots + a_m z^m$$

for all $z \in \mathbf{F}$. If p has degree m , we write $\deg p = m$.

The polynomial that is identically \mathbf{o} is said to have degree $-\infty$.

$\mathcal{P}_m(\mathbf{F})$: For m a nonnegative integer, $\mathcal{P}_m(\mathbf{F})$ denotes the set of all polynomials with coefficients in \mathbf{F} and degree at most m .

infinite-dimensional vector space: A vector space is called infinite-dimensional if it is not finite-dimensional.

linearly independent: A list v_1, \dots, v_m of vectors in V is called linearly independent if the only choice of $a_1, \dots, a_m \in \mathbf{F}$ that makes $a_1 v_1 + \dots + a_m v_m$ equal \mathbf{o} is $a_1 = \dots = a_m = 0$

The empty list $()$ is also declared to be linearly independent.

linearly dependent: A list of vectors in V is called linearly dependent if it is not linearly independent.

In other words, a list v_1, \dots, v_m of vectors in V is linearly dependent if there exist $a_1, \dots, a_m \in \mathbf{F}$, not all $\mathbf{0}$, such that $a_1 v_1 + \dots + a_m v_m = \mathbf{0}$

Linear Dependence Lemma: Suppose v_1, \dots, v_m is a linearly dependent list in V . Then there exists $j \in \{1, 2, \dots, m\}$ such that the following hold: (a) $v_j \in \text{span}(v_1, \dots, v_{j-1})$ (b) if the j^{th} term is removed from v_1, \dots, v_m , the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$

Length of linearly independent list \leq length of spanning list: In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Finite-dimensional subspaces: Every subspace of a finite-dimensional vector space is finite dimensional.

Section 2.B Bases

basis: A basis of V is a list of vectors in V that is linearly independent and spans V

Criterion for basis: A list v_1, \dots, v_n of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form $v = a_1 v_1 + \dots + a_n v_n$ where $a_1, \dots, a_n \in \mathbf{F}$

Spanning list contains a basis: Every spanning list in a vector space can be reduced to a basis of the vector space.

Linearly independent list extends to a basis: Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.

Every subspace V is part of a direct sum equal to V : Suppose V is finite-dimensional and U is a subspace of V . Then there is a subspace W of V such that $V = U \oplus W$

Section 2.C Dimension

dimension, $\dim V$: The dimension of a finite-dimensional vector space is the length of any basis of the vector space. The dimension of V (if V is finite-dimensional) is denoted by $\dim V$.

Dimension of subspace: If V is finite-dimensional and U is a subspace of V , then $\dim U \leq \dim V$

Linearly independent list of the right length is a basis: Suppose V is finite-dimensional. Then every linearly independent list of vectors in V with length $\dim V$ is a basis of V

Spanning list of the right length is a basis: Suppose V is finite-dimensional. Then every spanning list of vectors in V with length $\dim V$ is a basis of V

Dimension of a sum: If U_1 and U_2 are subspaces of a finite-dimensional vector space, then

$$\dim(U_1 + U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$