Math 110 Homework 2

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1 Exercise 2.A

Problem 8: The statement holds. Assume for contradiction that $\lambda v_1, .., \lambda v_m$ is linearly dependent. That is, for $\lambda \neq 0$, $\lambda v_1 + ... + \lambda v_m = 0$. Then we have a contradiction, since we claimed that $v_1, ..., v_m$ are linearly independent, and we have an example of a linear combination with non-zero coefficients that is equal to zero. Then, $\lambda v_1, .., \lambda v_m$ must be linearly independent.

Problem 9: The statement is false. Consider the counterexample where $w_i = -v_i$ for $i \in \{1, ..., m\}$. We know that $w_i \in V$, since the additive inverse must exist in V for v_i. Then, for non-zero coefficients the linear combination $(v_1 + w_1) + ... + (v_m + w_m) = 0$ which shows that it is linearly dependent.

Problem 10: To show that $w \in \text{span}(v_1, ..., v_m)$ we just have to show there exists a linear combination such that

$$w = a_1 v_1 + ... + a_m v_m$$

By linear dependence we have $a_1(v_1 + w) + ... + a_1(v_m + w) = 0$ for nonzero coefficients. We can collect that w terms and bring it to the other side of the equal sign and divide by the coefficients, again because they are not all zero, to get

$$a_1 v_1 + ... + a_m v_m = (a_1 + ... + a_m)w$$

$$\frac{a_1}{(a_1 + ... + a_m)} v_1 + ... + \frac{a_m}{(a_1 + ... + a_m)} v_m = w$$

Thus, showing that w is in the span of $v_1, ..., v_m$.

Problem 11: To show the " $\implies$ " direction, assume for contradiction $w \in \text{span}(v_1, ..., v_m)$. Then, we can express w as a linear combination of $v_1, ..., v_m$. $w = a_1 v_1 + ... + a_m v_m$ Let $a_1 v_1 + ... + a_m v_m - w$ be a linear combination, but here we have a contradiction. Since, $a_1 v_1 + ... + a_m v_m - w = a_1 v_1 + ... + a_m v_m - (a_1 v_1 + ... + a_m v_m) = 0$ for non-zero coefficients $a_1, ..., a_m$. Therefore, $w \notin \text{span}(v_1, ..., v_m)$.
For the " $\impliedby$ " direction, assume for contradiction $v_1, ..., v_m, w$ is linearly dependent. Then, there exists a linear combination with nonzero coefficients such that $a_1 v_1 + ... + a_m v_m + cw = 0$. Subtracting by cw and dividing by c, for when $c \neq 0$, if $c = 0$ we are done, we get $\frac{a_1}{c} v_1 + ... + \frac{a_m}{c} v_m = w$, which is a contradiction, since it implies that $w \in \text{span}(v_1, ..., v_m)$. Thus, it must be that $v_1, ..., v_m, w$ is linearly independent.

Problem 14: $\impliedby$. If $v_1, ..., v_m$ is linearly independent then by 2.23, the length of the spanning list must be greater than or equal to m. That $m \leq$ length of spanning list. Therefore, for every positive integer m the length of the spanning list is greater than that. Thus, the length of the spanning list is greater than every positive integer. Hence, no list spans V, from a similar argument as 2.16.
$\implies$ If V is inifinit-dimensional a finite number of vectors cannot span V. Therefore there must exists a sequence of vectors $v_1, v_2, ... \in V$. We also get that $v_1, ..., v_m$ is linearly independent for all positive integers m, since there exists some $v_{m+1} \notin \text{span}(v_1, ..., v_m)$. Which by inductions shows the statement holds for all positive integers.

Problem 17: Assume for contradiction that $p_0, ..., p_m$ are linearly independent. Firstly, we have that $1, x, ..., x^m$ is a basis for $\mathcal{P}_m(\mathbb{F})$. Thus, every linearly independent list is of size $\leq m + 1$. Construct a functions that is always $q(x) = 2$ and add it to the linearly independent list. Since, now the size is $m + 2$ the list is no longer linearly indpendent. Thus, we can state that $a_0 p_0 + ... + a_m p_m + cq = 0$. Since, the list is no longer linearly independent we can move cq to the othe side and divide by c to get a linear combination for just q. $q = b_0 p_0 + ... + b_m p_m$, But for all $p_i(2) = 0$, but $q(2) = 2$. Thus, we have a contradiction $2 = 0$.

2 Exercise 2.B

Problem 1: The zero-Vector space, $V = \{0\}$ is the only vector space with exactly one basis, and that is the empty set. Suppose there is another vector space with a basis b with length > 0. Then, let $v \in b$. By the existence of an additive identity we know that $-v$ is also in the vector space, and thus we can have an equally valid basis with $\{-v\}$, and thus having two bases. Therefore, it must be the case that for a vector space to have exactly one basis, the basis must be of length 0, which only the empty set satisfies.

Problem 3: (a) A possible basis for U is as follows $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1)$ observe that for any linear combination it holds that $x_1 = 3x_2$ and $x_3 = 7x_4$.
(b) We can extend the previous basis by adding the following two vectors to get a basis for $\mathbb{R}^5$. $(3, 1, 0, 0, 0), (0, 0, 7, 1, 0), (0, 0, 0, 0, 1), (1, 0, 0, 0, 0), (0, 0, 1, 0, 0)$ Notice, that by adding the following vectors we can express any arbirary $x_2 = a(3, 1, 0, 0, 0) - 3a(1, 0, 0, 0, 0)$. The same reasoning follows for x_4.
(c) From part (b) we can just form a subspace that handles the case to express arbitrary x_2 and x_4. Therefore, let $W = \{(x, 0, y, 0, 0) \colon x, y \in \mathbb{R}\}$. That is, $W = \text{span}((1, 0, 00, 0), (0, 0, 1, 0, 0))$. We then also observe that $U \cap W = \{0\}$ as desired.

Problem 5: Consider the following basis

$$1, x + 1, x^3 + x^2, x^3$$

Firstly, it meets the condition that none of the vectors are of degree 2. Now to show it is a basis, first consider indpeendence. For some $a_i \in \mathbb{R}$ we have $a_1 + a_2(x + 1) + a_3(x^3 + x^2) + a_4(x^3) = 0$. Rearraning the terms we get $(a_1 + a_2) + a_2 x + a_3 x^2 + (a_3 + a_4)x^3 = 0$ Clearly, $a_2 = a_3 = 0$. Then, it follows that $a_4 = 0 = a_1$, since $1, x, x^2, x^3$ are linearly independent. Thus, it is only zero when all the coefficients are zero. To show that it indeed spans $P_4(F)$ observe that $a = (a_1 + a_2)$ and $(a_3 + a_4)x^3 = ax^3 \implies \frac{a_3 + a_4}{a} x^3 = x^3$. Thus, the standard basis can be expressed as a linear combination of the new basis, so all $p \in$ span. Showing, that it is a valid basis.

Problem 7: Not true, consider the following counterexample. Consider the basis from Problem 5. Let $v_1 = x^3, v_2 = x^3 + x^2, v_3 = x + 1, v_4 = 1$. We know frmom Problem 5 that this forms a basis for $P_4(\mathbb{R}) = V$. Let U be the subspace of polynomials of degree at most 3, that have no integer terms. That is for some $p \in U$, $p(x) = 0$. We observe that it is indeed the case that only $v_1, v_2 \in U$. But they do not form a basis since no linear combination of x^3 and $x^3 + x^2$ is going to equal $x \in U$. Therefore, it does not form a basis.

Problem 8: To show that

$$u_1, ..., u_m, w_1, ..., w_n$$

is a basis of V. We must show that it spans V and that it is linearly independent. We know it spans V by the fact that $V = U \oplus W$. That is, for $v \in V, u \in U, w \in W$

$$v = u + w = a_1u_1 + ... + a_mu_m + b_1w_1 + ... + b_nw_n$$

Therefore, $v \in \text{span}(u_1, ..., u_m, w_1, ..., w_n)$. Now, we are left to show that $u_1, ..., u_m, w_1, ..., w_n$ is linearly independent. If it is linearly independent it must be the case,

$$a_1u_1 + ... + a_mu_m + b_1w_1 + ... + b_nw_n = 0$$

rearraning terms we get that

$$a_1u_1 + ... + a_mu_m = -(b_1w_1 + ... + b_nw_n)$$

Which means that $u = w$, but $U \cap W = \{0\}$ so it must be that

$$a_1u_1 + ... + a_mu_m = -(b_1w_1 + ... + b_nw_n) = 0$$

But since $u_1, ..., u_m$ and $w_1, ..., w_n$ are linearly independent it must be that $a_1 = ... = a_n = b_1 = ... = b_m = 0$. Therefore, we have show linear independence.