Math 110 Homework 3

Tarang Srivastava

1 Exercises 2.C

Problem 1: Since, U is a subspace there exists a basis B such that $|B| = \dim U = \dim V$. We have that B is a linearly independent list in V and of the right length therefore it must be a basis of V. So, $U = \operatorname{span}(B) = V$.

Problem 7:

Problem 8:

Problem 9:

Problem 10: We begin by showing that $p_0, ..., p_m$ is a linearly independent list. Observe, that any linear combination of $p_0, ..., p_m$ is a polynomial of degree m. So, the only polynomial of degree m that is the 0 polynomial is the one with all zero coefficients. Thus, $p_0, ..., p_m$ is a linearly independent list. Since, it is of the right length $m + 1$, and for all $0 \leq i \leq m$, p_i is in $\mathcal{P}_m(\mathbb{F})$. Then, it must be a basis for $\mathcal{P}_m(\mathbb{F})$.

Problem 12: Assume for contradiction that $U \cap W = \{0\}$. Then, $U + W$ is a direct sum, and $U, W \subset V$. So, it must be that $U \oplus W = V$. Thereofre, $\dim V = 9 = \dim W + \dim U - \dim W \cap U = 5 + 5 - 0$. Here we have a contradiction, thus, $U \cap W \neq \{0\}$.

Problem 13: We know that $\dim \mathbb{C}^6 = \dim U + \dim W - \dim(U \cap W)$. So the dimension of the intersection is $\dim(U \cap W) = 4 + 4 - 6 = 2$. There exists a basis B such that $|B| = 2$. Denote the two vectors in the basis of B as $u, w \in B$. Since, u, w form a linearly independent list, then they are not scalar multiples of each other.

Problem 14: Suppose that all the subspaces of V are disjoint. That is, for all $i, j \in [1, m]$ it is the case that $U_i \cap U_j = \{0\}$. Then, the dimension of $U_1 + ... + U_m$ is just the sum of the individual dimensions. That is, $\dim(U_1 + ... + U_m) = \dim U_1 + ... + \dim U_m$. Now, consider that the subspaces are not disjoint. Then, we will have to subtract the intersectin of the subspaces from our previous result. So, the valuse for the dimension will be less. Thus, we get the inequality $\dim(U_1 + ... + U_m) \leq \dim U_1 + ... + \dim U_m$. It follows, that the sum is finite dimensional, since it is a finite sum of finite dimensions.

2 Exercises 3.A

Problem 3: Observe that that for the standard basis $\mathbb{F}^n$, we can arrive at the following scalars

$$\begin{aligned} T(1, ..., 0) &= (A_{1,1}, A_{2,1}, ..., A_{m,1}) \\ &\vdots \\ T(0, ..., 1) &= (A_{1,n}, A_{2,n}, ..., A_{m,n}) \end{aligned}$$

Then, by theorem 3.5 there exists a linear map $T : V \to W$. Since, we can multiply by some scalar x_i for $i = 1, ..., n$ and then add those such vectors. We can claim that,

$$T(x_1, ..., x_n) = (A_{1,1}x_1 + ... + A_{1,n}x_n, ..., A_{m,1}x_1 + ... + A_{m,n}x_n)$$

by the additivity of linear maps.

Problem 4: To show linear independence consider the aribitrary linear combination equal to zero

$$0 = a_1 v_1 + ... + a_m v_m$$

By the additivity and homogeneity properties we have

$$T(0) = 0 = T(a_1 v_1 + ... + a_m v_m) = a_1 T v_1 + ... + a_m T v_m$$

Then, since we know $Tv_1, ..., Tv_m$ is linearly independent it must be the case that all the coefficients are zero. That is, $a_1 = ... = a_m = 0$. So, we have shown that $v_1, ..., v_m$ are linearly independent.

Problem 8: We will define $\varphi : \mathbb{R}^2 \to \mathbb{R}$ as follows,

$$\varphi(x, y) = \sqrt{x^2 + y^2}$$

Notice that $\varphi(a(x, y)) = \varphi(ax, ay) = \sqrt{a^2x^2 + a^2y^2} = a\sqrt{x^2 + y^2}$ by the definition of scalar multplication on $\mathbb{R}^2$. Also, $\varphi(x, y) = a\sqrt{x^2 + y^2}$ showing that φ satisfies the condition. Consider the vectors $(1, 0)$ and $(0, 1)$. We know that $(1, 0) + (0, 1) = (1, 1)$. But φ does not satisfy additivity since $\varphi(1, 0) = 1 = \varphi(0, 1)$, but $\varphi(1, 1) = \sqrt{2}$ and we know that $2 \neq \sqrt{2}$.

Problem 9: We will define $\varphi : \mathbb{C} \to \mathbb{C}$ as follows,

$$\varphi(a + bi) = a$$

We can show that for some $w = w_1 + w_2 i$ and $z = z_1 + z_2 i$, our function respects additivity.

$$\varphi(w + z) = \varphi(w_1 + z_1 + (w_2 + z_2)i) = w_1 + z_1 = \varphi(w) + \varphi(z)$$

Then, if we select the scalar $i \in \mathbb{C}$ we observe that $\varphi(iz) = -z_2 \neq i\varphi(z) = z_1 i$. Thus, it does not satisfy homogeneity.

Problem 10: Let $u \in U$ be a vector such that it does not map to 0, that is $Su \neq 0$. Now, select a $v \in V$ such that $v \notin U$. We can choose such a v, because $U \neq V$ and U is a subspace of V. We know that $v + u \notin U$, since if it were, then there is a $w \in U$ such that $w = v + u$, and $w - u = v$. Since, U is closed under addition that would mean that $v \in U$ which is a contradiction. Finally, if we apply the linear map we get that

$$T(u + v) = 0 \neq Tu + Tv = Su + 0$$

Since, we initially stated that that $Su \neq 0$, it must be the case that T does not respect additivity and is not a linear map.

Problem 11: Since, V is finite dimensional and U is a subspace of V we know there exists a subspace U' such that $U \oplus U' = V$. Let $R \in \mathcal{L}(U', W)$ be a linear map from U' to W. Suppose for some $u \in U$ and $u' \in U'$ we define the linear map T as $Tu = Su$ and $Tu' = Ru'$. Then for any $v \in V$ there is a sum of $u + u' = v$, so $Tv = Tu + Tu' = Su + Ru'$. Since, every v has a linar map Tv, we have found a T that satsifies our condition.

Problem 12:

Problem 14: Let $V = \mathcal{P}_3(\mathbb{R})$, which we know has dimension

3 Exercises 3.B

Problem 1: Let T be a linear map from $\mathbb{R}^5 \to \mathbb{R}^5$ such that for some $x \in \mathbb{R}^5$, where $x = (x_1, x_2, x_3, x_4, x_5)$.

$$Tx = (0, 0, 0, x_4, x_5)$$

First, we wish to show that T is in fact a linear map. For additivity, we know

$$T(x + y) = T(x_1 + y_1, ..., x_5 + y_5) = (0, 0, 0, x_4 + y_4, x_5 + y_5)$$

For the other part we have that

$$T(x) + T(y) = (0, 0, 0, x_4, x_5) + (0, 0, 0, y_4, y_5)$$

$$= (0, 0, 0, x_4 + y_4, x_5 + y_5)$$

Therefore, additivity holds. For homogeneity we simply show again that

$$T(cx) = T(cx_1, ..., cx^5) = (0, 0, 0, cx_4, cx_5)$$

So, then for the other part

$$cT(x) = c(0, 0, 0, x_4, x_5) = (0, 0, 0, cx_4, cx_5)$$

Therefore, showing homogeneity holds as well. So, T is a valid linear map. It follows quite easily now that the null space is all the elements with zeros in the last two indices. That is

$$\text{null } T = \text{span}\big((1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\big)$$

These, are all elements of the standard basis of $\mathbb{R}^5$ so they are linearly independent and therefore form a basis for null T. So, we have shown that the null space has dimension 3. We know $\mathbb{R}^5$ has dimension 5, so it follows that the range has dimension 2, but again that is not very hard to show, but not necessary in this case by 3.22.

Problem 2: Observe that S, T are both maps from $V \to V$ so performing $(ST)^2 = STST$ is completely valid. For any $v \in V$ if $Tv = 0$ or $Sv = 0$ we are immediately done, since S and T are linear maps and must always map 0 to 0, so after the chain of linear maps we will have 0 at the end. To the more interesting part, we first get that $Ts \neq 0$. We are concerned about when $S(Ts) \in$ range S, because we showed the other case just leads to 0. Then, since range$S \subset$ null T it must be then that $T(S(Ts)) = 0$. So, we have shown that $(ST)^2 = 0$, since it maps all v to 0.

Problem 5:

Problem 6: We know that

$$\dim V = \dim \text{range } T + \dim \text{null } T$$

Assume for contradiction that $\dim \text{range } T = \dim \text{null } T$, so we can make the following substitutions and get

$$5 = 2 \dim \text{range } T$$

Here we have a contradiction since it claims that $\dim \text{range } T = \frac{5}{2}$, but a dimensions is the length of a basis so must always be a natural number. So, it must be that for $\mathbb{R}^5$ the range and null space always have unequal dimensions.