Math 110 Homework 12

Tarang Srivastava

April 20, 2020

1 Exercises 7.D

Problem 1: First we verify that the given operator

$$\sqrt{T^*T}v = \frac{\|x\|}{\|u\|}\langle v, u\rangle u$$

is indeed positive. Observe that when we calculate

$$\langle \sqrt{T^*T}v, v\rangle = \frac{\|x\|}{\|u\|}|\langle v, u\rangle|^2$$

it is always a non-negative value, thus the operator is positive. Second we find the adjoint of T. Define T^* as follows

$$T^*v = \langle v, x\rangle u$$

with the same fixed x and u as for T. This is a valid adjoint since for $v, w \in V$ we have

$$\begin{aligned} \langle Tv, w\rangle &= \langle v, T^*w\rangle \\ \langle v, u\rangle\langle x, w\rangle &= \langle v, u\rangle\overline{\langle w, x\rangle} = \langle v, u\rangle\langle x, w\rangle \end{aligned}$$

For all $v \in V$ we have that

$$\begin{aligned} T^*Tv &= T^*(\langle v, u\rangle x) \\ &= \langle v, u\rangle\langle x, x\rangle u \end{aligned}$$

Then, we can verify that indeed $\sqrt{T^*T}^2 = T^*T$ that we calculated. Therefore, given that it is positive it is a valid square root of T^*T.

Problem 2: Let T be the operator represented by this 2×2 matrix with the standard basis.

$$T = \begin{pmatrix} -5/2 & 5/2 \\ -5/2 & 5/2 \end{pmatrix}$$

The eigenvalues for T are exactly $\lambda = 0$. Then we find T^*T using the adjoint of the matrix provided.

$$T^*T = \begin{pmatrix} 25/2 & -25/2 \\ -25/2 & 25/2 \end{pmatrix}$$

The eigenvalues for which are equal to $\lambda = 25, 0$. So the singular values are equal to $\sigma = 5, 0$.