Math 110 Homework 11
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1 Exercises 7.B

Problem 3: Define the linear map T as follows for each $z = (z_1, z_2, z_3) \in \mathbb{C}^3$.

$$Tz = (2z_1, 3z_2, 2z_1)$$

Note the switch up in the last spot. We can check that T is indeed closed under addition and scalar multiplication, thus a linear map. Firstly, $\lambda = 2$ is an eigenvalue with the eigenvector $w = (1, 0, 1)$.

$$T(1, 0, 1) = (2, 0, 2) = 2(1, 0, 1)$$

Similarly, $\lambda = 3$ is an eigenvalue with the eigenvector $w = (0, 1, 0)$.

$$T(0, 1, 0) = (0, 3, 0) = 3(0, 1, 0)$$

Finally, $(T^2 - 5T + 6I)z \neq 0$ if $z_3 \neq 6z_1$. So given we have values that are nonzero after that linear map, $T^2 - 5T + 6I \neq 0$.

Problem 4: Given that $\mathbb{F} = \mathbb{C}$ by the complex spectral theorem T is normal if and only if T has a diagonal matrix with respect to some orthonormal basis. Since T is diagonalizable if and only if V is equal to the sum of the eigenspaces of distinct eigenvalues, we have shown the last case. Also by the complex spectral theorem T is normal if and only if V has an orthonormal basis consisting of eigenvectors of T. That completes the proof. (Ask Alex if there's some part that guarantees that we have distinct eigenvalues, but it is not that difficult to show that if all the basis eigenvectors are orthogonal they must be associated with distinct eigenvalues.)

Problem 6: $\Longrightarrow$ direction follows easily from 7.13, if T is self-adjoint then all the eigenvalues are real.
$\Longleftarrow$ Given T is normal then T^* and T have the same eigenvectors and for the corresponding eigenvalue λ for T, T^* has the associated eigenvalue $\overline{\lambda}$ for the same eigenvector. If all the eigenvalues are real $\lambda = \overline{\lambda}$, so T and T^* have the same eigenvalues and eigenvectors. By the complex spectral theorem V has an orthogonal basis consisting of eigenvectors of T. Denote these basis eigenvectors as $e_1, ..., e_n$. Let $v \in V$, then for some $a_1, ..., a_n \in \mathbb{C}$

$$\begin{aligned}
Tv &= a_1 T(e_1) + ... + a_n T(e_n) \\
&= a_1 \lambda_1 e_1 + ... + a_n \lambda_n e_n \\
&= a_1 T^*(e_1) + ... + a_n T^*(e_n) \\
Tv &= T^* v
\end{aligned}$$

For all $v \in V$ so $T = T^*$, therefore T is self adjoint.

Problem 8: Let T be the operator associated with the matrix and some basis B for the complex vector space

$$\mathcal{M}(T, B) = \begin{pmatrix} 0 & \dots & 1 \\ & \ddots & \vdots \\ 0 & & 0 \end{pmatrix}$$

Then, clearly $T^2 = 0$ so $T \neq T^2$ and $T^8 = 0 = T^9$. (I'm not sure what the point of this question was, but I definitely missed it.)

Problem 10: Let $V = \mathbb{R}^2$ and T be the operator represented by this matrix for the standard basis of $\mathbb{R}^2$.

$$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$$

For $b = 0$ and $c = 1$

$$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} + (1) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$$

Since we can compute that $T^2 = -I$. The zero operator is non invertible so we are done.

Problem 11: By the real or complex spectral theorem given that T is self-adjoint it is diagonalizable. Suppose the digaonal matrix is

$$T = \begin{pmatrix} a_1 & \dots & 0 \\ & \ddots & \vdots \\ 0 & & a_n \end{pmatrix}$$

Then, we can take the cube root of each value in the diagonal, which is well defined for complex or real numbers. So we get

$$S = \begin{pmatrix} a_1^{1/3} & \dots & 0 \\ & \ddots & \vdots \\ 0 & & a_n^{1/3} \end{pmatrix}$$

Then, clearly it holds that $S^3 = T$.

Problem 12: By the spectral theorem there exists an orthonormal basis of V consisiting of eigenvectors of T. So v is equal to

$$v = a_1 e_1 + ... + a_n e_n$$

Where, the $a_1, ..., a_n$ are some scalars and the $e_1, ..., e_n$ are the orthonormal basis eigenvectors. Then,

$$Tv = a_1 \lambda_1 e_1 + ... + a_n \lambda_n e_n$$

So we can restate the inequality as

$$\begin{aligned}
\|Tv - \lambda v\| &= \|a_1 \lambda_1 e_1 + ... + a_n \lambda_n e_n - \lambda v\| \\
&= \big\| |\lambda_1 - \lambda| \, a_1 e_1 + ... + |\lambda_1 - \lambda| \, a_n e_n \big\|
\end{aligned}$$

Then since the basis vectors are orthogonal we can express them as

$$= \big\| |\lambda_1 - \lambda| + ... + |\lambda_1 - \lambda| \big\| \|a_1 e_1 + ... + a_n e_n\|$$

Following that $\|v\| = 1$

$$= \big\| |\lambda_1 - \lambda| + ... + |\lambda_1 - \lambda| \big\|$$

So we are given that

$$\big\| |\lambda_1 - \lambda| + ... + |\lambda_1 - \lambda| \big\| < \epsilon$$

Assume for contradiction that $|\lambda - \lambda_i| \geq \epsilon$ for all $i \in 1, ..., n$. Then, we can make the following substitutions

$$\|\epsilon + ... + \epsilon\| = n\epsilon < \epsilon$$

Which then implies that $n < 1$ which is a contradiction since it would imply that V has dimension 0, but it clearly doesn't since $v \in V$ and $\|v\| = 1$. Thus, we have a contradiction and it must be that there exists some eigenvalue λ' the statement holds.

Problem 13: The first part of (c) if and only if (a) we proceed as in the provided proof. Then to show that the last part that (a) implies (b) we note that if our operator is normal, then we have to consider a basis eigenvector v such that

Problem 14: $\Longleftarrow$ By the real spectral theorem if T is self-adjoint and the inner product makes U an inner product space, then U has a basis consisting of eigenvectors of T.
$\Longrightarrow$ If T has a basis consisiting of eigenvectors, then it is diagonalizable by 5.41. Now if we just provide an inner product on U such that the basis eigenvectors are orthonormal we can just apply the real spectral theorem and be done.
Let $e_1, .., e_n$ be the eigenvector basis. Then, for arbitrary $v, w \in V$

$$v = a_1e_1 + ... + a_ne_n$$

and

$$w = b_1e_1 + ... + b_ne_n$$

For some real number scalars. we define the inner product $\langle v, w\rangle$ as follows

$$\langle v, w\rangle = a_1 \cdot b_1 + ... + a_n \cdot b_n$$

We will now confirm that all the properties of an inner product do indeed hold.

1. Positivity: Observe that $\langle v, v\rangle = a_1^2 + ... + a_n^2 \geq 0$.
2. Definiteness: By the property of linearly independent vectors, $\langle v, v\rangle = a_1^2 + ... + a_n^2 = 0$ only when $a_1 = ... = a_n = 0$ which is when $v = 0$.
3. Additivity: pretty straightforward from the commutitivity and distributivity in $\mathbb{R}$.
4. Homogeniety: straightforward from the distributivity of real numbers.
5. Conjugate symmetry: Since we are in a real vector space conjugates are equal to the number itself. So, $\langle v, w\rangle = a_1b_1 + ... + a_nb_n = b_1a_1 + ... + b_na_n = \langle w, b\rangle$ form the commutitivity of real numbers.

Having done ALL that we can confirm that the basis eigenvectors are indeed normal and orthogonal in this definition of the inner product. This follows simply from the fact that $< e_i, e_i >= 1 * 1 = 1$ and we have that $< e_i, e_j >= 1 * 0 + 0 * 1 = 0$ for different basis vectors. Therefore, we can apply the real spectral theorem and get that T must be self adjoint, since there is a diagonal matrix of orthonormal basis.

2 Exercise 7.C

Problem 1: Counterexample. Consider the operator T associated with the matrix

$$\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$$

Then observe that the matrix is equal to its conjugate transpose. Then the vectors $b_1 = (1/\sqrt{2}, i/\sqrt{2})$ and $b_2 = (1/\sqrt{2}, -i/\sqrt{2})$ form an orthonormal basis and it holds that $\langle Tb_1, b_1\rangle = 2$ and $\langle Tb_2, b_2\rangle = 2$. With all that in place, for $v = (1, 2)$ we have that

$$\langle Tv, v\rangle = -3$$

Problem 2: We have the following inequalitiy by T being positive

$$\begin{aligned} \langle T(v-w), v-w\rangle &\geq 0 \\ \langle Tv - Tw, v-w\rangle &\geq \\ \langle w-v, v-w\rangle &\geq \end{aligned}$$

Multiplying both sides by -1 we get

$$\begin{aligned} -1\langle w-v, v-w\rangle &\leq 0 \\ \langle v-w, v-w\rangle &\leq 0 \end{aligned}$$

by the property of inner products we also have that

$$\langle v-w, v-w\rangle \geq 0$$

So then it must be the case that

$$\langle v-w, v-w\rangle = 0$$

So again by definiteness

$$\begin{aligned} v - w &= 0 \\ v &= w \end{aligned}$$

Problem 4: We will abuse the associative property and the definition of adjoints to get the result. Observe, that T^*T is already an operator in $\mathcal{L}(V)$. Let $v, w \in V$.

$$\begin{aligned} \langle v, (T^*T)w\rangle &= \langle v, T^*(Tw)\rangle \\ &= \overline{\langle T^*(Tw), v\rangle} \\ &= \overline{\langle Tw, (T^*)^*v\rangle} \\ &= \overline{\langle Tw, Tv\rangle} \\ &= \overline{\langle w, T^*Tv\rangle} \\ &= \langle T^*Tv, w\rangle \end{aligned}$$

Thus, by definition T^*T is self-adjoint. We construct a similar series of steps for TT^*. (Ask Alex if I can state that since T^*T is positive, then its adjoint is also positive so TT^* is positive.) Observe again that TT^* is an operator on W.

$$\begin{aligned} \langle v, TT^*w\rangle &= \overline{\langle TT^*w, v\rangle} \\ &= \overline{\langle T^*w, T^*v\rangle} \\ &= \overline{\langle w, TT^*v\rangle} \\ &= \langle TT^*v, w\rangle \end{aligned}$$

Then, observe that for some eigenvalue λ the adjoint has the eigenvalue $\overline{\lambda}$. So, for any combination of TT^* or T^*T we have that for some eigenvector e that $T * Te = \lambda T^*e = \lambda\overline{\lambda}$, which is always positive. So all the eigenvalues are positive as well, and since they are both self adjoint they must be positive.

Problem 7: $\Longrightarrow$ Given that T is positive there exists a R such that $T = R^*R$. From some past hw exercies we know that since T is invertible so must R^* and R. Then we have that $\langle Tv, v\rangle = \langle R^*Rv, v\rangle = \langle Rv, Rv\rangle$ which is never equal to zero for nonzero values, and is positive.
$\Longleftarrow$ Assume for contradiction that T is not invertible, that is null $T \neq \{0\}$. Then, there must exists some nonzero $v \in$ null T. This is a contradiction since that would mean that $\langle Tv, v\rangle = \langle 0, v\rangle = 0$.

Problem 8: $\Longrightarrow$ If $\langle .,. \rangle_T$ is an inner product then the definiteness property ensures that T is invertible, since $\langle v, v \rangle_T = 0$ if and only if $v = 0$ so that ensures that the null $T = \{0\}$. The Positivity property ensures that T is a positive operator since $\langle v, v \rangle_T \geq 0$ so $\langle Tv, v \rangle \geq 0$. (Note: In a real vectors space conjugate symmetry would also ensure that T is self adjoint :) $\Longleftarrow$ If T is positive then it ensures Positivity in the inner product. Since, T is invertible by the previous example and question 7 it ensures that it is definiteness. Additivity in the first slot follows from T being a linear map and closure under vector addition, followed by addition in the first slot of the original inner proudct. Homogeniety in the first slot follows from T being a linear map and closure under scalar multiplication, followed by Homogeniety in the first slot of the original inner proudct. And conjugate symmetry is given by the fact that T is positive and given that it is invertible it has no nonzero eigenvalues, so it must be self-adjoint. Thus $\langle Tu, v \rangle = \langle u, Tv \rangle$ so the inner product follows that as well. Therefore, it is a valid inner product.

Problem 9: We know that I is given to be self adjoint and has the lone eigenvalue of 1, so it is positive. Observe that $I^2 = I$, since each positive operator has a unique square root, this is the square root of I. Then, the square root of that is $I^2 I^2 = I^4$, so we continue in this fashion forever and thus it has infinite square roots.

Problem 11: