Math 110 Homework 8

Tarang Srivastava

1 Exercises 5.B

Problem 1: We wish to show that

$$(I-T)^{-1} = I + T + ... + T^{n-1}$$

So, we can multiply $I - T$ to both sides to get

$$I = (I-T)(I + T + ... + T^{n-1})$$

Then, we can distribute and see we get

$$I = I - T + T - T^2 + T^2 + ... - T^{n-1} + T^{n-1} + T^n$$

After cancelling out all the similar terms we are left with

$$I = I + T^n = I$$

Since, $T^n = 0$. So, to prove the statement we do the following operations

$$I = I + T^n = I$$

$$I = I - T + T - T^2 + T^2 + ... - T^{n-1} + T^{n-1} + T^n$$

$$I = (I-T)(I + T + ... + T^{n-1})$$

Then, we multiply both sides by $(I-T)^{-1}$ to get

$$(I-T)^{-1} = I + T + ... + T^{n-1}$$

as desired.

Problem 2: Assume for contradiction that $\lambda \neq 2$ and $\lambda \neq 3$ and $\lambda \neq 4$. Then, $T - 2I$ and $T - 3I$ and $T - 4I$ must all be invertible. Given,

$$(T-2I)(T-3I)(T-4I) = 0$$

for all $v \in V$ such that $v \neq 0$ we have that

$$(T-2I)(T-3I)(T-4I)v = 0v = 0$$

Then, for one of the values $T - 2I$ or $T - 3I$ or $T - 4I$ one of them maps v to 0. Since, they are all invertible their null spaces is just $\{0\}$, but then we have a contradiction since we had that $v \neq 0$. Therefore, it must be the case that λ is equal to 2, 3 or 4.

Problem 3: We proceed directly. Given

$$T^2 = I$$

We get that $T^2 - I = 0$ so,

$$(T-I)(T+I) = 0$$

Since, $\lambda \neq -1$ it must be that $T + I$ is invertible. Thus, for all non zero v in V we have that $(T + I)v$ is non zero. So it must be that for all $w \in V$, we have $(T - I)v = 0$. By definition $T - I$ is equal to the 0 linear map, so from $T - I = 0$ it follows that $T = I$

Problem 4: From $P^2 = P$ we have that $P^2 - P = 0$ so it must be that

$$P(P-I) = 0$$

So for all $v \in V$ we have that

$$P(P-I)v = 0v = 0$$

Thus

$$Pv = 0 \text{ or } Pv = v$$

To show that $V = \text{null } P \oplus \text{range } P$ we first will show that $\text{null } P \cap \text{range } P = \{0\}$. Suppose $v \in \text{null } P \cap \text{range } P$. Then, $Pv = 0$ and $Pv = v$ it follows directly then that $v = 0$, so

$$\text{null } P \cap \text{range } P = \{0\}$$

Since, $P \in \mathcal{L}(V)$ we already have that $\text{range } P \subset V$ and $\text{null } P \subset V$. Then for $v \in \text{range } P$ and $w \in \text{null } P$ clearly $v + w \in V$ so we have that

$$V \supset \text{range } P \oplus \text{null } P$$

For the other side, let $v \in V$ we have that $Pv = 0$ or $Pv = v$. So, $v \in \text{range } P$ or $v \in \text{null } P$ then it follows that clearly for all $v \in V$ we have that $v = v + 0$ or $v = 0 + v$. So,

$$V \subset \text{range } P \oplus \text{null } P$$

Therfore,

$$V = \text{range } P \oplus \text{null } P$$