MUSA 74 Homework 3

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a. Homework 1.67
Proof. Proceed by induction on n. For the base case consider when $n = 1$. That is, $p|a_1$ The base case is trvially true, and the statement holds. Assume the statement holds for some $k \in \mathbb{N}$. That is, if $p|a_i...a_k$, then there is some $i \in \{1, ..., k\}$ such that $p|a_i$. In the inductive step, we need to show the statement holds for $k+1$. That is, if $p|a_i...a_{k+1}$, then there is some $i \in \{1, ..., k+1\}$ such that $p|a_i$. We will prove the inductive step by cases. The first case is when i is in some set $\{1, ..., k\}$. From our inductive hypothesis we know there exists an $i \in \{1, ..., k\}$ such that $p|a_i$. The second case is when i is not in some set $\{1, ..., k\}$. Therefore, for all $i \in \{1, ..., k\}$, $p \not| a_i$. Which is equivalent to the $\gcd(p, a_i) = 1$ for all $i \in \{1, ..., k\}$, but since $p|a_i...a_{k+1}$, and the gcd for p and $a_i...a_{k+1}$ is p. There must exist an a_i for $i \in \{1, ..., k+1\}$ such that gcd for p and a_i is p and therefore $p|a_i$. Thus, having shown the inductive step the statement holds for all n.

b. Homework 1.68
Proof. Since we are concerned about a Cartesian product an arbitrary n times we can proceed by induction on n. For the base case of $n = 0$, $\mathbb{N}$ is trivially countable. Assume that for some Cartesian product $k \in \mathbb{N}$ times, the product is countable. We will show that the Cartesian product for $k+1$ times is also countable. Let $g : X_1 \times X_2 \times ... \times X_k \to \mathbb{X}$ from the inductive hypothesis. Then we can define a bijective function f such that for $f(2k) = g(2k)$ and $f(2k+1) = n \in X$. This defintion of f creates a bijection for the $k+1$ case and therefore by induction the statement holds for all n.

c. Homework 1.69
Proof. Proceed by induction on n. For the base case consider when $n = 1$. The statement holds since,

$$1^2 = \frac{1(2)(3)}{6} = 1$$

For the inductice step assume the statement holds for some k. That is,

$$1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$$

We will show the statement holds for $k+1$. That is,

$$1^2 + 2^2 + ... + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

From our inductive hypothesis we can substitute in

$$\frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

Which with some algebra is equivalent.