MUSA 74 Homework 3

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a. Homework 1.67
Proof. Proceed by induction on n. For the base case consider when $n = 1$. That is, $p|a_1$ The base case is trvially true, and the statement holds. Assume the statement holds for some $k \in \mathbb{N}$. That is, if $p|a_i...a_k$, then there is some $i \in \{1, ..., k\}$ such that $p|a_i$. In the inductive step, we need to show the statement holds for $k + 1$. That is, if $p|a_i...a_{k+1}$, then there is some $i \in \{1, ..., k + 1\}$ such that $p|a_i$. We will prove the inductive step by cases. The first case is when i is in some set $\{1, ..., k\}$. From our inductive hypothesis we know there exists an $i \in \{1, ..., k\}$ such that $p|a_i$. The second case is when i is not in some set $\{1, ..., k\}$. Therefore, for all $i \in \{1, ..., k\}$, $p \not| a_i$. Which is equivalent to the $\gcd(p, a_i) = 1$ for all $i \in \{1, ..., k\}$, but since $p|a_i...a_{k+1}$, and the gcd for p and $a_i...a_{k+1}$ is p. There must exist an a_i for $i \in \{1, ..., k + 1\}$ such that gcd for p and a_i is p and therefore $p|a_i$. Thus, having shown the inductive step the statement holds for all n.

b. Homework 1.68
Proof. Since we are concerned about a Cartesian product an arbitrary n times we can proceed by induction on n. For the base case of $n = 0$, $\mathbb{N}$ is trivially countable. Assume that for some Cartesian product $k \in \mathbb{N}$ times, the product is countable. We will show that the Cartesian product for $k+1$ times is also countable. Let $g : X_1 \times X_2 \times ... \times X_k \to \mathbb{X}$ from the inductive hypothesis. Then we can define a bijective function f such that for $f(2k) = g(2k)$ and $f(2k + 1) = n \in X$. This defintion of f creates a bijection for the $k + 1$ case and therefore by induction the statement holds for all n.

c. Homework 1.69
Proof. Proceed by induction on n. For the base case consider when $n = 1$. The statement holds since,

$$1^2 = \frac{1(2)(3)}{6} = 1$$

For the inductice step assume the statement holds for some k. That is,

$$1^2 + 2^2 + ... + k^2 = \frac{k(k + 1)(2k + 1)}{6}$$

We will show the statement holds for $k + 1$. That is,

$$1^2 + 2^2 + ... + (k + 1)^2 = \frac{(k + 1)(k + 2)(2k + 3)}{6}$$

From our inductive hypothesis we can substitute in

$$\frac{k(k + 1)(2k + 1)}{6} + (k + 1)^2 = \frac{(k + 1)(k + 2)(2k + 3)}{6}$$

Which with some algebraic manipulations we can show are equivalent.

d. Homework 5.8
Proof. Suppose for contradiction there exists a $\phi \in \alpha$ such that $P(\phi)$ is false. The first case is that $\phi = 0$ in which case the statement does not hold. The second case is that if $P(k+1)$ is false, then $P(k)$ is false. This does not hold for $\phi = 1$ and by strong induction for all γ less than the limit. Lastly, if ϕ is a limit, then by the second condition it is also true. Therefore, no such phi exists and thats and the statement if false. Thus, the original statement is proved by contradiction.

e. Homework 5.10
Proof. For the Zero stage $\beth_b = |\mathbb{N}|$, in which case $\beth_a = |\mathcal{P}(\mathbb{N})|$. It is known that $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$, in which case the statement holds. For the successor stages we will proceed by Cantor's diagonal argument. Assume that $\beth_a = \beth_b$. Then there exists a bijection between the two ordinals. We can now construct a subset ϕ, that is not in the bijectiion by ordering all the subsets. We simply look at the first value, if it is not in the first set we add it ot ϕ otherwise if it is in the first subset we do not add it to ϕ. By this construction we arrive at a subset that is not in the bijection, and therefore the cardinality is greater. Thus, the statement holds for the successor stages. For the Limit stage.

I couldn't prove this part, and to be honest did not completely understand the defintion for the limit case of ordinals