Discussion 6D

Tarang Srivastava - CS70 Summer 2020

Mini Review

Quick Double Integral Practice

$$\int_0^1 \int_0^2 (x^2 y + e^y)\, dx\, dy$$

dx

$$\int_0^1 \frac{8}{3} y + 2e^y \, dy$$

dy

$$2e - \frac{2}{3}$$

$$2e - \frac{2}{3}$$

Lecture Highlights

Let X and Y be two continuous random variables. Then the joint density function $f_{X,Y} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

$$\left[\mathbb{P}(a \leq X \leq b, c \leq Y \leq d) = \int_a^b \int_c^d f_{X,Y}(x,y)dydx \right]$$

and

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = 1$$

and

$$f_{X,Y}(x,y) \geq 0 \qquad \forall x, y \in \mathbb{R}$$

— pdf Closes still apply

$$F_{X,Y}(x,y) = \mathbb{P}(X \leq x, Y \leq y)$$ ← CDF

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

CDF

$$\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$$

Lecture Highlights (Conditional PDFs)

For any y with $f_y(y) > 0$, the conditional distribution of X given $Y = y$ is defined as:

$f_{x,y}$

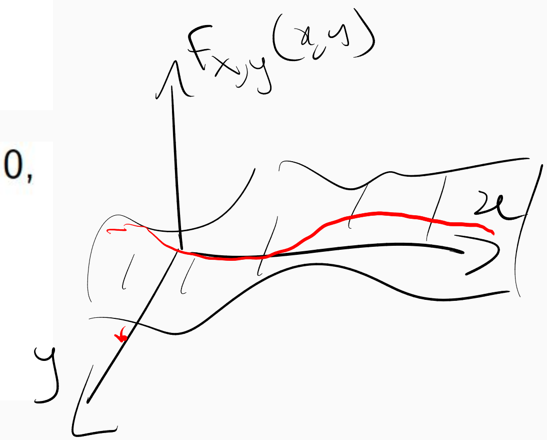
$Y|X$

$$f_{x|y} := \frac{f_{x,y}(x,y)}{f_y(y)}$$

When Y is continuous, even though $P(Y = y) = 0$, if $f_y(y) > 0$, then:

$Y \quad X \quad x$

$$P(a \le X \le b | Y = y) = \int_a^b f_{x|y}(x|y)dx$$



$f_{Y|X}$

Recovering individual PDFs

Marginal for X

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

Independent case

Let X and Y be two continuous random variables. X and Y are independent if:

$$\boxed{f_{x,y}}(x,y) = \underline{f_x(x)}\,\underline{f_y(y)}$$

for all x, y.

Since $f_{x,y}(x,y) = f_{x|y}(x|y)f_y(y)$, this implies $\underline{f_{x|y}(x|y) = f_x(x)}$.

conditional pdf.

Question 1

Question 1

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \leq x \leq 1, 0 \leq y \leq 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

(b) Find $f_X(x)$, the marginal distribution of X.

(c) Find the conditional distribution of Y given $X = x$.

(d) Are X and Y independent?

$f(x,y) = f(x) \cdot f(y)$

What formulas we have

- total probability
- marginal distribution
- conditional " "
- $P(X=x, Y=y) = P(X=x)P(Y=y)$

Question 1

The joint probability density function of two random variables X and Y is given by $f(x,y) = Cxy$ for $0 \leq x \leq 1, 0 \leq y \leq 2$, and 0 otherwise (for a constant C).

(a) Find the constant C that ensures that $f(x,y)$ is indeed a probability density function.

$C = 1 \quad = \quad \int_0^1 \int_0^2 Cxy \, dx \, dy$

(a) Since $f(x,y)$ is a probability density function, it must integrate to 1. Then:

$$1 = \int_0^1 \int_0^2 Cxy\,dy\,dx = \int_0^1 2Cx\,dx = C$$

Therefore, $C = 1$.

Question 1

(b) Find $f_X(x)$, the marginal distribution of X.

$$f_X(x) = \int_0^2 xy\,dy$$

$$= 2x$$

$$0 \leq x \leq 1$$

(b) To get the marginal distribution of X, we integrate the joint distribution with respect to Y. So:

$$f_X(x) = \int_0^2 f(x,y)dy = \int_0^2 xydy = 2x$$

This is the marginal distribution for $0 \leq x \leq 1$.

Question 1

$X =$ r.v.

$x =$ variable (numbers)

(c) Find the conditional distribution of Y given $X = x$.

$$f_{Y|x} = \frac{y}{2}$$

$$\frac{[F_{X,Y}(x,y)]}{[F_X(x)]} = f_{Y|X}(y,x)$$

Bayes' Rule

$$F_X(x) = 2x$$

(c) The conditional distribution of Y given by

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{xy}{2x} = \frac{y}{2}$$

Question 1

(d) Are X and Y independent?

$$f_{Y|X} = F_Y = \int_0^1 xy\,dx = \frac{y}{2}$$

part(c)

$$\frac{y}{2}$$

independence!

(d) The conditional distribution of Y given $X = x$ does not depend on x, so they are independent.

Alternatively, you could find the marginal distribution of Y and see it is the same as the conditional distribution of Y:

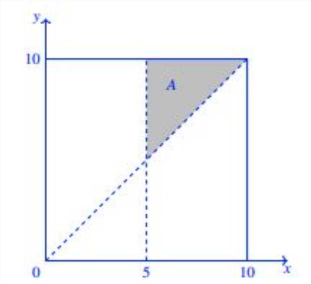
$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 xydx = \frac{y}{2}$$

Notice that since X and Y are independent, $f_X(x)f_Y(y) = xy = f_{X,Y}(x,y)$, i.e. the product of the marginal distributions is the same as the joint distribution.

Question 2

Question 2

You have two spinning wheels, each having a circumference of 10 cm with values in the range $[0, 10)$ marked on the circumference. If you spin both (independently) and let X be the position of the first spinning wheel's mark and Y be the position of the second spinning wheel's mark, what is the probability that $X \geq 5$, given that $Y \geq X$?



hint: easier than it seems

0.13 ?

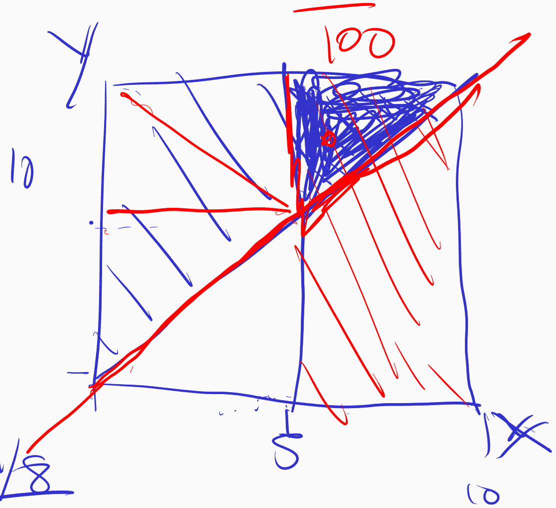
$\frac{1}{2}$

$$P(X \geq 5 \mid Y \geq X)$$

$$P(5 \leq X \leq Y)$$

10

$\frac{12.5}{100}$



$\frac{1}{8}$

$X \in [0, 10)$

$Y \in [0, 10)$ $=$

$$\frac{|A|}{|\Omega|} = \frac{|5 \leq X \leq Y|}{|\Omega|}$$

First we write down what we want and expand out the conditioning:

$$\mathbb{P}[X \geq 5 \mid Y \geq X] = \frac{\mathbb{P}[Y \geq X \cap X \geq 5]}{\mathbb{P}[Y \geq X]}.$$

$\mathbb{P}[Y \geq X] = 1/2$ by symmetry. To find $\mathbb{P}[Y \geq X \cap X \geq 5]$, it helps a lot to just look at the picture of the probability space and use the continuous uniform law $\mathbb{P}[A] = (\text{area of } A)/(\text{area of } \Omega)$. We are interested in the relative area of the region bounded by $x < y < 10$, $5 < x < 10$ to the entire square bounded by $0 < x < 10$, $0 < y < 10$.

$$\mathbb{P}[Y \geq X \cap X \geq 5] = \frac{5 \cdot 5/2}{10 \cdot 10} = \frac{1}{8}.$$

So $\mathbb{P}[X \geq 5 \mid Y \geq X] = (1/8)/(1/2) = 1/4$.

Question 3

Question 3

Let $X \sim \text{Exponential}(\lambda_X)$ and $Y \sim \text{Exponential}(\lambda_Y)$ be independent, where $\lambda_X, \lambda_Y > 0$. Let $U = \min\{X,Y\}$, $V = \max\{X,Y\}$, and $W = V - U$.

(a) Compute $\mathbb{P}(U > t, X \leq Y)$, for $t \geq 0$.

(b) Use the previous part to compute $\mathbb{P}(X \leq Y)$. Conclude that the events $\{U > t\}$ and $\{X \leq Y\}$ are independent.

(c) Compute $\mathbb{P}(W > t \mid X \leq Y)$.

(d) Use the previous part to compute $\mathbb{P}(W > t)$.

(e) Calculate $\mathbb{P}(U > u, W > w)$, for $w > u > 0$. Conclude that U and W are independent. [*Hint*: Think about the approach you used for the previous parts.]

Question 3

Let $X \sim \text{Exponential}(\lambda_X)$ and $Y \sim \text{Exponential}(\lambda_Y)$ be independent, where $\lambda_X, \lambda_Y > 0$. Let $U = \min\{X,Y\}$, $V = \max\{X,Y\}$, and $W = V - U$.

$f_{XY} = f_X \cdot f_Y$

$=$

(a) Compute $\mathbb{P}(U > t, X \leq Y)$, for $t \geq 0$.

X

$$\left[P(t < X < \infty, X < Y < \infty) = \int_0^\infty \int_x^\infty \lambda_x e^{-\lambda_x x} \cdot \lambda_y e^{-\lambda_y y} \, dy \, dx \right]$$

$U > t$

$X = U > t$

$\infty > X > t$

$$= \frac{\lambda_x}{\lambda_x + \lambda_y} \cancel{e^{-(\lambda_x+\lambda_y)0}}$$

$$= \frac{\lambda_x}{\lambda_x + \lambda_Y}$$

$t \in (0, \infty)$

$X < Y$

(a) One has

$$\begin{aligned}
\mathbb{P}(U > t, X \leq Y) = \mathbb{P}(t < X \leq Y) &= \int_t^\infty \int_x^\infty f_{X,Y}(x,y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_t^\infty \int_x^\infty \lambda_X \exp(-\lambda_X x)\lambda_Y \exp(-\lambda_Y y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \lambda_X \lambda_Y \int_t^\infty \exp(-\lambda_X x) \cdot \frac{\exp(-\lambda_Y x)}{\lambda_Y}\,\mathrm{d}x = \lambda_X \int_t^\infty \exp(-(\lambda_X + \lambda_Y)x)\,\mathrm{d}x \\
&= \frac{\lambda_X}{\lambda_X + \lambda_Y} \exp(-(\lambda_X + \lambda_Y)t).
\end{aligned}$$

Question 3

Let $X \sim \text{Exponential}(\lambda_X)$ and $Y \sim \text{Exponential}(\lambda_Y)$ be independent, where $\lambda_X, \lambda_Y > 0$. Let $U = \min\{X,Y\}$, $V = \max\{X,Y\}$, and $W = V - U$.

(b) Use the previous part to compute $\mathbb{P}(X \le Y)$. Conclude that the events $\{U > t\}$ and $\{X \le Y\}$ are independent.

$$\boxed{\frac{\lambda_X}{\lambda_X + \lambda_Y}}$$

part (a)

$$P(X>t) \cdot P(Y>t)$$

$$\int_t^\infty \qquad \int_t^\infty$$

$$e^{-\lambda_X t} \cdot e^{-\lambda_Y t} = \text{part (a)}$$

(b) Take $t = 0$.

$$\mathbb{P}(X \le Y) = \frac{\lambda_X}{\lambda_X + \lambda_Y}.$$

Since X and Y are independent exponentials, $U = \min\{X,Y\} \sim \text{Exponential}(\lambda_X + \lambda_Y)$. So, $\mathbb{P}(U > t) = \exp(-(\lambda_X + \lambda_Y)t)$, and therefore we have $\mathbb{P}(U > t, X \le Y) = \mathbb{P}(X \le Y)\mathbb{P}(U > t)$.

Question 3

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(c) Compute $\mathbb{P}(W > t \mid X \leq Y)$.

Question 3

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(d) Use the previous part to compute $\mathbb{P}(W > t)$.

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(e) Calculate $\mathbb{P}(U > u, W > w)$, for $w > u > 0$. Conclude that U and W are independent. [*Hint*: Think about the approach you used for the previous parts.]