CS70 Discussion 2A

6/30/2026

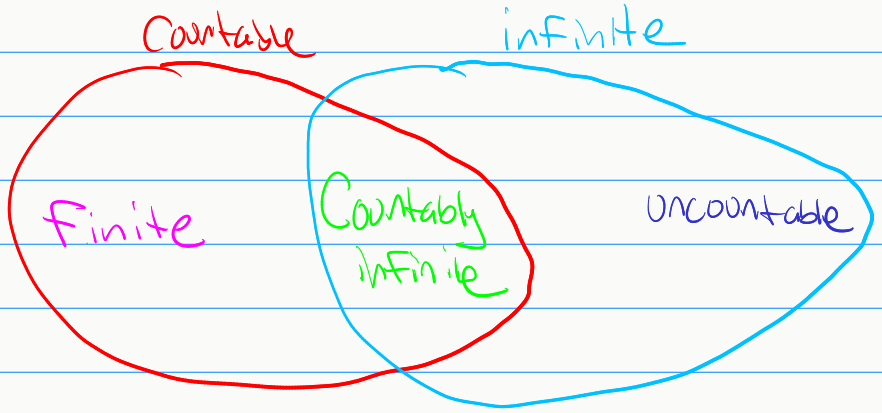
Announcements:

- JoinMe sections have been added for Saturday!
- You can attend up to 5 OH sessions [11 am, 2-3 pm]
- Smaller breakout rooms
- ★ We will work on the problems first in the big group then move to breakout rooms to discuss
- Reminder that participation in class will be used in the event of a "grade bump".
- Please fill out the google form survey, your opinion is valuable! (i.e. Saturday HW Party and 5 OH sessions).

Countability Review

$\mathbb{R}$, $(0,1) \subseteq \mathbb{R}$

$|\mathbb{N}| \leq |(0,1)|$



We care more about stuff being countable, because we can do induction on it.

Definition
S is countably infinite iff ∃ bijection with $\mathbb{N}$

Proof. Assume for contradiction $|\mathbb{N}| = |(0,1)|$

$|\mathbb{N}| \leq |(0,1)|$

0	$0.a_{11}a_{12}a_{13}a_{14}\ldots$
1	$0.a_{21}a_{22}a_{23}$
2	$0.a_{31}a_{32}a_{33}$
3	⋮
4	⋮
5	⋮

$b \in (0,1)$

$b = 0.b_1 b_2 b_3 b_4$

b_i: if $a_{ii} = 1$ then $b_i = 0$; if $a_{ii} \neq 1$ then $b_i = 1$

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1 Unions and Intersections

Given:

- A is a countable, non-empty set. For all $i \in A$, S_i is an uncountable set.
- B is an uncountable set. For all $i \in B$, Q_i is a countable set.

For each of the following, decide if the expression is "Always Countable", "Always Uncountable", "Sometimes Countable, Sometimes Uncountable".

For the "Always" cases, prove your claim. For the "Sometimes" case, provide two examples – one where the expression is countable, and one where the expression is uncountable.

Hints:

① $S_1, S_2, S_3, \ldots$] this sequence has countable terms

BUT each S_i is an uncountable set

② $Q_1, Q_2, Q_3, \ldots$] this sequence has uncountable terms

BUT each Q_i is a countable set

(a) $A \cap B$

a) $A \cap B \subseteq A$

$|A \cap B| \leq |A|$

always countable — countable □

(b) $A \cup B$

b) $B \subseteq A \cup B$

$|B| \leq |A \cup B|$

uncountable — always uncountable

(c) $\bigcup_{i \in A} S_i$

c) countable: $\mathbb{R} \cap \mathbb{N}$ — uncountable: $\mathbb{R} \cap \mathbb{R}$

$S_i = [i, i+1] \subseteq \mathbb{R}$

$\mathbb{N}$

(d) $\bigcap_{i \in A} S_i$

(e) $\bigcup_{i \in B} Q_i$

(f) $\bigcap_{i \in B} Q_i$

2 Counting Cartesian Products

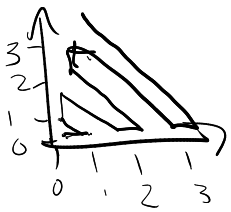
For two sets A and B, define the cartesian product as $A \times B = \{(a,b) : a \in A, b \in B\}$.

(a) Given two countable sets A and B, prove that $A \times B$ is countable.

(b) Given a finite number of countable sets $A_1, A_2, \ldots, A_n$, prove that

$$A_1 \times A_2 \times \cdots \times A_n$$

is countable.



(c) Consider an infinite number of countable sets: $B_1, B_2, \ldots$. Under what condition(s) is $B_1 \times B_2 \times \cdots$ countable? Prove that if this condition is violated, $B_1 \times B_2 \times \cdots$ is uncountable.

HINTS:

(1) $\mathbb{N} \times \mathbb{N}$ is countable

For part (c) first just try to show why it's uncountable and then think of the conditions

a) A and B are countable so have a bijection with $\mathbb{N}$
- $f(n) = a \in A$
- $g(n) = b \in B$

$A \times B$ ↔ $\mathbb{N} \times \mathbb{N}$ countable

$\rightarrow (i,j) \in i,j \in \mathbb{N}$

$\rightarrow (f(i), g(j))$ ($f(i) \in A$, $g(j) \in B$)

Countable

b) Induction

A_i countable ✓ Base Case $n=1$

Inductive Step

Assume $n=k$ $C = A_1 \times A_2 \times \cdots \times A_k$ countable

partition → $C \times A_{k+1}$ countable

$A_1 \times A_2 \times \cdots \times A_{k+1}$ countable

c)

Assume for contradiction it is countable

1	b_{11},	b_{12}, b_{13}
2	b_{21},	b_{22}, b_{23}
3	b_{31},	b_{32}, b_{33}
4		

C_i

$C_i = 1$ if $b_{ii} = c_i$, $C_i = b_{ii}$

$b_{ii} \neq c_i$

3 Hello World!

Determine the computability of the following tasks. If it's not computable, write a reduction or self-reference proof. If it is, write the program.

(a) You want to determine whether a program P on input x prints "Hello World!". Is there a computer program that can perform this task? Justify your answer.

(b) You want to determine whether a program P prints "Hello World!" before running the kth line in the program. Is there a computer program that can perform this task? Justify your answer.

(c) You want to determine whether a program P prints "Hello World!" in the first k steps of its execution. Is there a computer program that can perform this task? Justify your answer.

Hint:

Just try to get to the halting problem somehow