Discussion 4A

Tarang Srivastava - CS 70 Summer 2020

Mini-Review

Lecture Highlights

$$\frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

n choose k

$$\binom{n}{k} = \binom{n}{n-k}$$

part (a)

$$2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$$

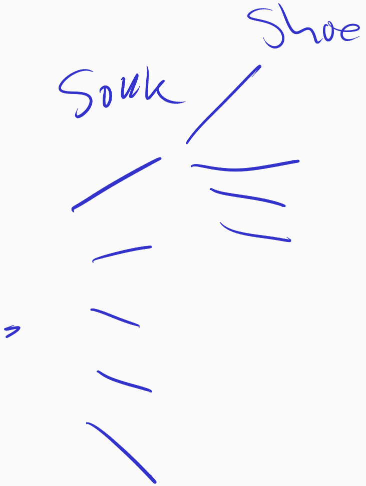
$n! = n(n-1)\cdot(n-2)\cdots$
$2\cdot 1$

Factorial

"n fact"

You have 5 socks and 4 shoes how many combinations of foot attire do you have?

5 · 4



20 in total

How many ways are there to rearrange QWERTY?

WERTQY

$\boxed{6!}$

_ _ _ _ _ _

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

How many ways are there to rearrange TWITTER?

1 2 3 4 5 6 7

$\frac{7!}{3!}$

$\boxed{\frac{7!}{3!}}$

$T_1 W I T_2 T_3 E R$

Overcounting

$\binom{n+k}{n} = \binom{n+k}{k}$

2nd Approach

TTSS

0 0 1 1

$\frac{4!}{2! \cdot 2!}$

You have 6 turtles to race. How many unique head-to-head matchups can you set up?

T1 T2 T3 T4 T5 T6

Ex

T2 vs. T3

$$\binom{6}{2} = \frac{6!}{(6-2)!\,2!} = \frac{6\cdot 5}{2\cdot 1}$$

$$\boxed{\binom{6}{2}} = 15$$

Summary: $S = \{1, 2, \ldots, n\}$

distinct k-element objects constructed from S

$K_n =$

	Sampling	
	with replacement	w/o replacement
ordered	n^k	$n(n-1)\cdots(n-k+1) = \dfrac{n!}{(n-k)!}$
Unordered	$\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$	$\binom{n}{k}$

Announcements

Clobber Policy 50%

Question 1

Clothing Arrangement

Question 1

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

(b) How many outfits are there if we wanted to wear exactly two categories?

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

Clothing Arrangement

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

$$10 \cdot 10 \cdot 10 \cdot 10 = \boxed{10^4}$$

Clothing Arrangement

(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

(a) 10^4 by the first rule of counting.

Clothing Arrangement

(b) How many outfits are there if we wanted to wear exactly two categories?

$$\binom{4}{2} \cdot 10 \cdot 10$$

$$\binom{4}{2} \cdot 10^2$$

Clothing Arrangement

(b) How many outfits are there if we wanted to wear exactly two categories?

(b) $\binom{4}{2} \cdot 10^2$ First we choose the two categories, then we apply first rule of counting.

Clothing Arrangement

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

(1) 2

$$\binom{10}{4} \cdot \cancel{4!} = \frac{10!}{6!}$$

$$\frac{10!}{6! \cdot \cancel{4!}}$$

Clothing Arrangement

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

(c) $\binom{10}{4} \cdot 4! = \dfrac{10!}{6!}$ We choose the 4 hats and then multiply by the number of ways to rearrange them. This is 10 permutation 4 (although that notation was not covered in this class I don't think, so don't worry about that)

<u>First</u> $\binom{10}{4}$

<u>then</u> $4!$

Clothing Arrangement

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

$\binom{16}{4}$

$\frac{c}{4!}$

6

Clothing Arrangement

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

(d) $\binom{10}{4}$ or written as a function of the previous part, $c/4!$.

Question 2

Counting on Graphs + Symmetry

Q2

a, c, b, d

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

K_n $\frac{n \cdot (n-1)}{2}$

$\frac{n \cdot (n-1)}{2} = \binom{n}{2}$

2

(b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

$n = 3$

R G B

G R B

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

Counting on Graphs + Symmetry

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

$\underline{\underline{G}}$ K_n

$$\frac{n \cdot (n-1)}{2} = \binom{n}{2}$$

$$\boxed{2^{\binom{n}{2}}}$$

Counting on Graphs + Symmetry

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

(a) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$. (Recall that $2^m = \sum_{k=0}^{m} \binom{m}{k}$, where $m = n(n-1)/2$ in this case.)

Counting on Graphs + Symmetry

(b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

idea

3 vertices

$$\sum_{k=3}^{n} \binom{n}{k} \cdot \frac{k!}{k \cdot 2}$$

k = # vertices

from part (c)

$V_1 — V_5 — V_6$

V_1, V_5, V_6, V_2

V_5, V_1, V_6, V_2

Counting on Graphs + Symmetry

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(b) The number k of vertices in a cycle is at least 3 and at most n. Without accounting for duplicates, there are $n!/(n-k)!$ cycles. Due to inversions (read from front or end doesn't matter, abc = cba) we divide by 2, and rotations (where we start reading from doesn't matter abc = bca = cab) we divide by k. Hence the total number of distinct cycles is

$$\sum_{k=3}^{n} \frac{n!}{(n-k)! \cdot 2k}. = \binom{n}{k} \cdot \frac{k!}{2k}$$

Counting on Graphs + Symmetry

(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

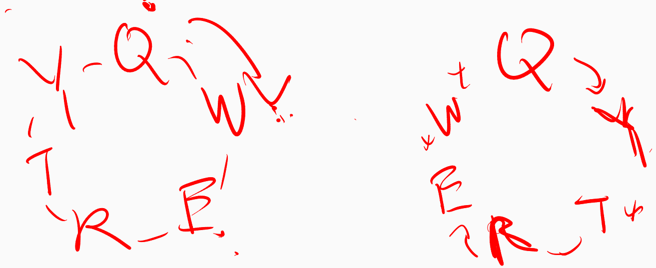
QWERTY

WERTYQ

Rotation → YQWERT

Inversion YTREWQ

$$\frac{n!}{n \cdot 2} = \frac{(n-1)!}{2}$$



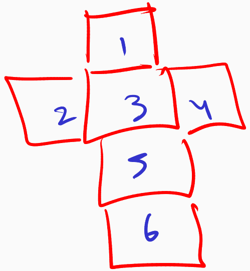
Counting on Graphs + Symmetry

(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.

Counting on Graphs + Symmetry

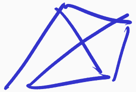
(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.



6 colors

$6!$

$6 \cdot 4$



$4 \cdot 3$

$$\frac{6!}{6 \cdot 4} = \boxed{30}$$

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

(d) Without considering symmetries there are 6! ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.

Question 3

Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.

(a) How many such shortest paths exist?

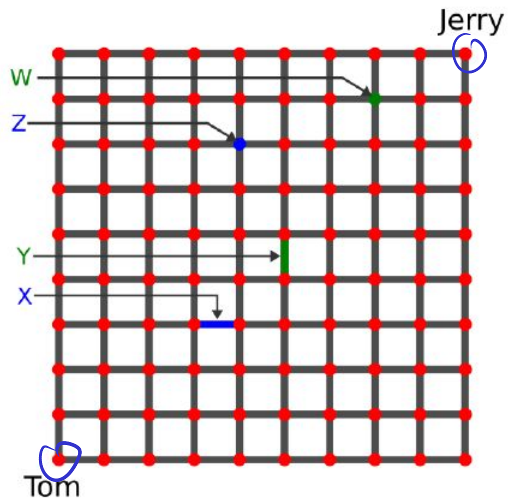
(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

a) $\frac{18!}{9! \cdot 9!} = \binom{18}{9}$

9 right 9 up

$\underbrace{RRRR \ldots R}_{9} \; \underbrace{U \ldots U}_{9}$

b)



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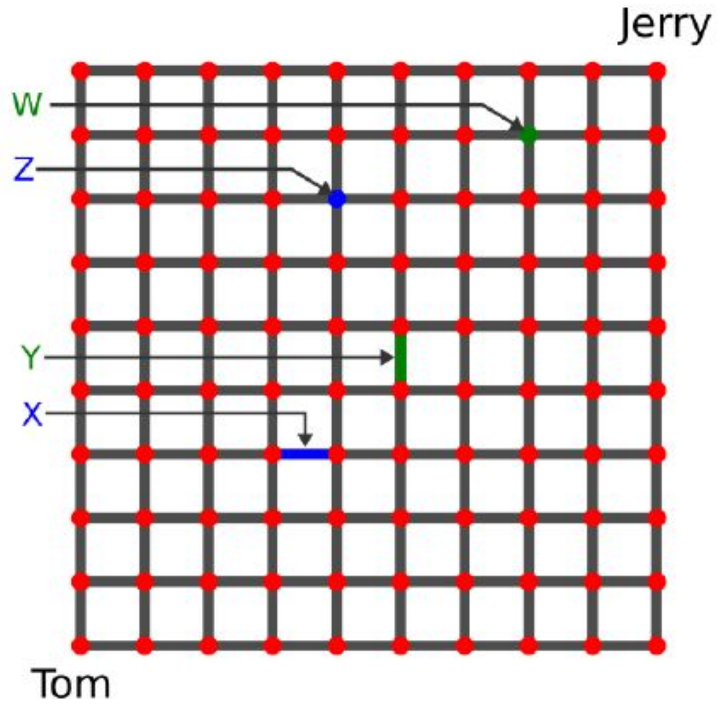


Maze

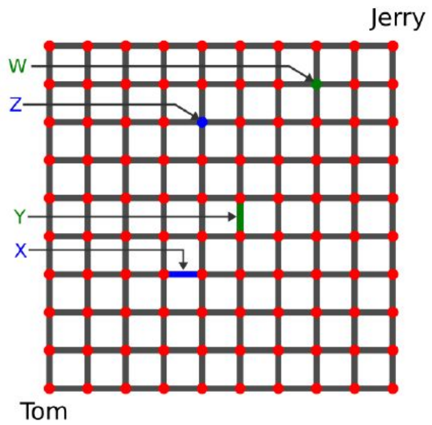
Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.

(a) How many such shortest paths exist?

(a) Each row in the maze has 9 edges, and so does each column. Any shortest path that Tom can take to Jerry will have exactly 9 horizontal edges going right (let's call these "H" edges) and 9 vertical edges going up (let's call these "V" edges).

Observe also that every shortest path from Tom to Jerry can be described by a unique sequence consisting of 9 "H"s and 9 "V"s. For example, one such path is HHHHHHHHHVVVVVVVVV (which represents the path that goes all the way to the right, and then all the way to the top). Conversely, every such sequence of exactly 9 "H"s and 9 "V"s corresponds to a unique shortest path from Tom to Jerry.

Therefore, the number of shortest paths is exactly the same as the number of ways of arranging 9 "H"s and 9 "V"s in a sequence, which is $\binom{18}{9} = 48620$.



Maze

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

(b) 2 legs

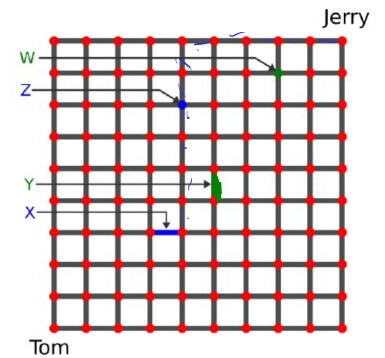
3 rights 3 ups

5 rights 6 ups

$\binom{6}{3}$ $\binom{11}{5} = \frac{11}{6}$

Tom → Y X → Jerry

$\binom{6}{3} \cdot \binom{11}{3} =$



Maze

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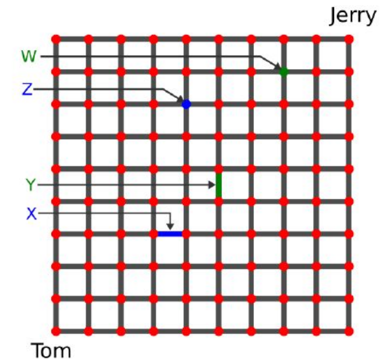


Maze

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

(b) For a shortest path to pass through the edge X, it has to first get to the left vertex of X. So the first portion of the path has to start at the origin, and end at the left vertex of X. Using the same logic as above, there are exactly $\binom{6}{3} = 20$ ways to complete this "first leg" of the path (consisting of 3 "H" edges and 3 "V" edges). Having chosen one of these 20 ways, the path has to then go from the right vertex of X to the top right corner of the maze (the "second leg"). This second leg will consist of 5 "H" edges and 6 "V" edges, and using the same logic, there are exactly $\binom{11}{5} = 462$ possibilities. Therefore, the total number of shortest paths that pass through the edge X is $20 \times 462 = 9240$.

$\frac{11!}{(11-5)!\,5!} = \frac{11!}{6!\,5!} = \frac{11!}{6!\,(11-6)!}$ 3 leg

Jun → X X → Y Y → Jerry

Maze

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

Using similar logic, any shortest path that passes through Y has to consist of 2 legs, the first leg going from the origin to the bottom vertex of Y, and the second leg going from the top vertex of Y to the top right corner of the maze. The first leg will consist of exactly 5 "H"s and 4 "V"s, while the second leg will consist of exactly 4 "H"s and 4 "V"s. So the total number of such shortest paths will be $\binom{9}{5} \times \binom{8}{4} = 8820$.

Maze

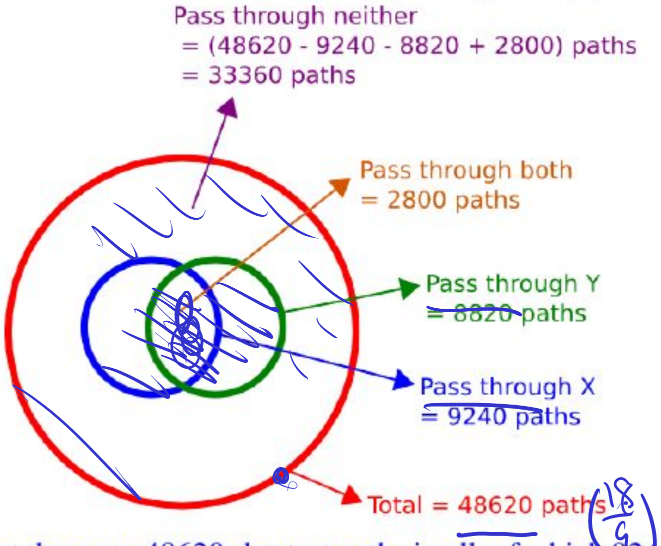
Inclusion–Exclusion

$\binom{18}{9} - Y - X + \binom{\text{go thru}}{Y \text{ and } X} =$

$a \equiv p$

$a^p \equiv a \mod p$

By a similar argument, let's try to figure out how many paths will pass through both X and Y. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 3 "H"s and 3 "V"s (going from the origin to the left edge of X), the second leg consisting of 1 "H" and 1 "V" (going from the right vertex of X to the bottom vertex of Y), and the third leg consisting of 4 "H"s and 4 "V"s (going from the top vertex of Y to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{6}{3} \times \binom{2}{1} \times \binom{8}{4} = 2800$.



Finally, we know that there are 48620 shortest paths in all, of which 9240 pass through X, 8820 pass through Y, and 2800 pass through both. So the number of paths that pass through neither is 33360 (see the figure above for an intuitive explanation).

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part (a)

$$2^n = \binom{n}{n} + \binom{n}{n-1} + \cdots + \binom{n}{0}$$

$n! = n(n-1)\cdot(n-2)\cdots 2\cdot 1$

Factorial

"n fact"

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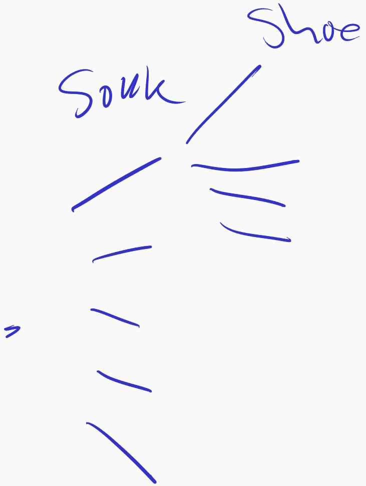

20 in total

How many ways are there to rearrange QWERTY?

WERTQY

$6!$

_ _ _ _ _ _

$6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

How many ways are there to rearrange TWITTER?

1 2 3 4 5 6 7

$$\frac{7!}{3!} -$$

$$\boxed{\frac{7!}{3!}}$$

$T_1WIT_2T_3ER$

Overcounting

You have 6 turtles to race. How many unique head-to-head matchups can you set up?

T1 T2 T3 T4 T5 T6

Ex
T2 vs. T3

$$\binom{6}{2} = \frac{6!}{(6-2)!\,2!} = \frac{6\cdot 5}{2\cdot 1}$$

$$\boxed{\binom{6}{2}} = 15$$

Summary: $S = \{1, 2, \ldots, n\}$

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	Sampling: with replacement	Sampling: w/o replacement
ordered	n^k	$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$
Unordered	$\binom{k+n-1}{n-1} = \binom{k+n-1}{k}$	$\binom{n}{k}$

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(a) There are four categories of clothings (shoes, trousers, shirts, hats) and we have ten distinct items in each category. How many distinct outfits are there if we wear one item of each category?

(a) 10^4 by the first rule of counting.

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(b) How many outfits are there if we wanted to wear exactly two categories?

$$\binom{4}{2} \cdot 10 \cdot 10$$

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Clothing Arrangement

(b) How many outfits are there if we wanted to wear exactly two categories?

(b) $\binom{4}{2} \cdot 10^2$ First we choose the two categories, then we apply first rule of counting.

Clothing Arrangement

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

(1) $\quad$ 2

$$\binom{10}{4} \cdot \cancel{4!} = \frac{10!}{6!}$$

$$\frac{10!}{6! \cdot \cancel{4!}}$$

Clothing Arrangement

(c) How many ways do we have of hanging four of our ten hats in a row on the wall? (Order matters.)

(c) $\binom{10}{4} \cdot 4! = \frac{10!}{6!}$ We choose the 4 hats and then multiply by the number of ways to rearrange them. This is 10 permutation 4 (although that notation was not covered in this class I don't think, so don't worry about that)

First

then

$\binom{10}{4}$

$4!$

Clothing Arrangement

(d) We can pack four hats for travels (order doesn't matter). How many different possibilities for packing four hats are there? Can you express this number in terms of your answer to part (c)?

$\binom{16}{4}$

$\frac{c}{4!}$

6

Clothing Arrangement

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(d) $\binom{10}{4}$ or written as a function of the previous part, $c/4!$.

Question 2

Counting on Graphs + Symmetry

Q2

a, c, b, d

(a) How many distinct undirected graphs are there with n labeled vertices? Assume that there can be at most one edge between any two vertices, and there are no edges from a vertex to itself. The graphs do not have to be connected.

K_n $\quad \frac{n \cdot (n-1)}{2}$ $\quad \frac{n \cdot (n-1)}{2} = \binom{n}{2}$

2

(b) How many distinct cycles are there in a complete graph K_n with n vertices? Assume that cycles cannot have duplicated edges. Two cycles are considered the same if they are rotations or inversions of each other (e.g. (v_1, v_2, v_3, v_1), (v_2, v_3, v_1, v_2) and (v_1, v_3, v_2, v_1) all count as the same cycle).

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$\underline{\underline{G}}$

K_n

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$$\boxed{2^{\binom{n}{2}}}$$

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(a) There are $\binom{n}{2} = n(n-1)/2$ possible edges, and each edge is either present or not. So the answer is $2^{n(n-1)/2}$. (Recall that $2^m = \sum_{k=0}^{m} \binom{m}{k}$, where $m = n(n-1)/2$ in this case.)

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idea

3 vertices

$$\boxed{\sum_{k=3}^{n} \binom{n}{k} \cdot \frac{k!}{k \cdot 2}}$$

k = # vertices

from pnt (c)

$V_1 — V_5 — V_6$

V_1, V_5, V_6, V_2

V_5, V_1, V_6, V_2

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$$\sum_{k=3}^{n} \frac{n!}{(n-k)! \cdot 2k}. = \binom{n}{k} \cdot \frac{k!}{2k}$$

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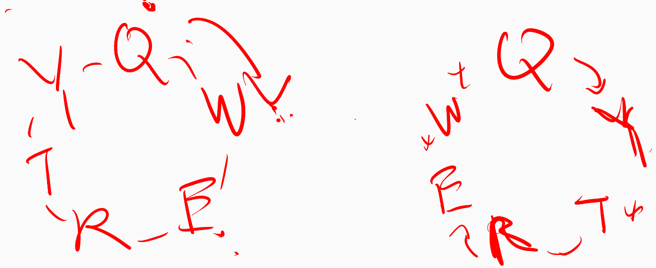
QWERTY

WERTYQ

Rotation → YQWERT

Inversion YTREWQ

$$\frac{n!}{n \cdot 2} = \frac{(n-1)!}{2}$$



Counting on Graphs + Symmetry

(c) How many ways are there to color a bracelet with n beads using n colors, such that each bead has a different color? Note: two colorings are considered the same if one of them can be obtained by rotating the other.

(c) Without considering symmetries there are $n!$ ways to color the beads on the bracelet. Due to rotations, there are n equivalent colorings for any given coloring. Hence taking into account symmetries, there are $(n-1)!$ distinct colorings. Note: if in addition to rotations, we also consider flips/mirror images, then the answer would be $(n-1)!/2$.

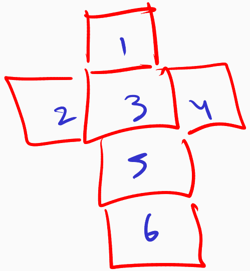
Counting on Graphs + Symmetry

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

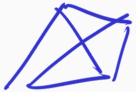
6 colors

6!

$6 \cdot 4$



$$\frac{6!}{6\cdot 4} = \boxed{30}$$



$4 \cdot 3$

Counting on Graphs + Symmetry

(d) How many ways are there to color the faces of a cube using exactly 6 colors, such that each face has a different color? Note: two colorings are considered the same if one can be obtained from the other by rotating the cube in any way.

(d) Without considering symmetries there are $6!$ ways to color the faces of the cube. The number of equivalent colorings, for any given coloring, is $24 = 6 \times 4$: 6 comes from the fact that every given face can be rotated to face any of the six directions. 4 comes from the fact that after we decide the direction of a certain face, we can rotate the cube around this axis in 4 different ways (including no further rotations). Hence there are $6!/24 = 30$ distinct colorings.

Question 3

Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.

(a) How many such shortest paths exist?

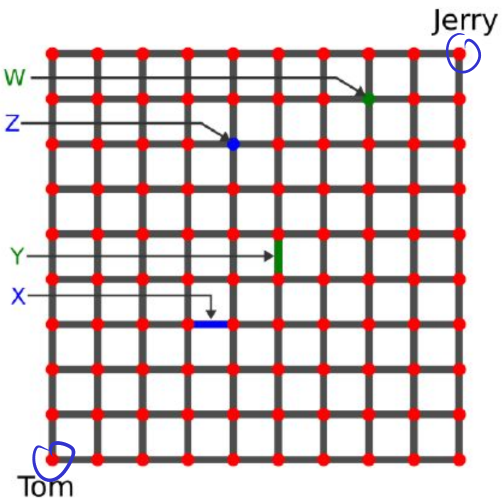
(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

a) $\frac{18!}{9! \cdot 9!} = \binom{18}{9}$

9 right   9 up

$\underbrace{RRRR\ldots R}_{9}\ \underbrace{U\ldots U}_{9}$

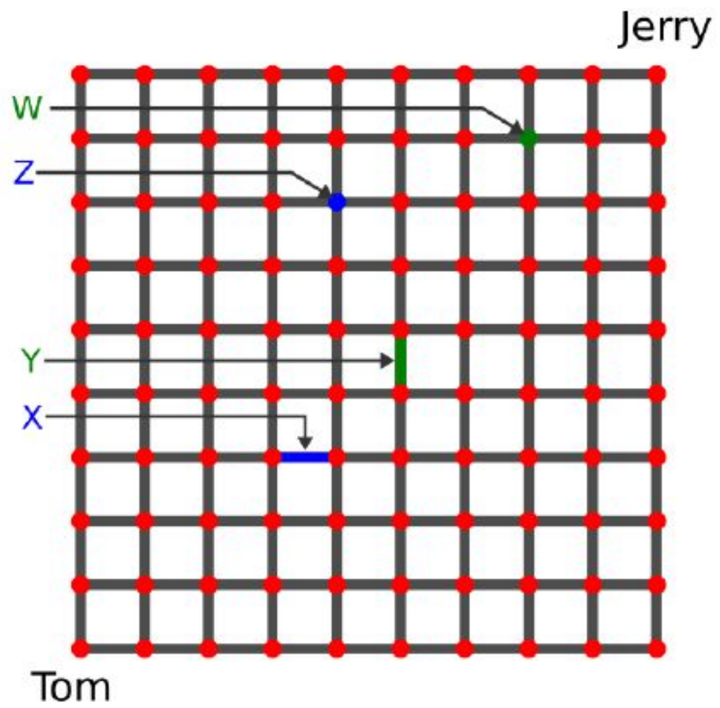
b)



Maze

Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.

(a) How many such shortest paths exist?



Maze

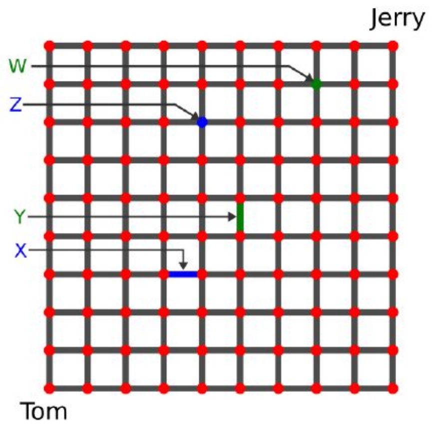
Let's assume that Tom is located at the bottom left corner of the 9×9 maze below, and Jerry is located at the top right corner. Tom of course wants to get to Jerry by the shortest path possible.

(a) How many such shortest paths exist?

(a) Each row in the maze has 9 edges, and so does each column. Any shortest path that Tom can take to Jerry will have exactly 9 horizontal edges going right (let's call these "H" edges) and 9 vertical edges going up (let's call these "V" edges).

Observe also that every shortest path from Tom to Jerry can be described by a unique sequence consisting of 9 "H"s and 9 "V"s. For example, one such path is HHHHHHHHHVVVVVVVVV (which represents the path that goes all the way to the right, and then all the way to the top). Conversely, every such sequence of exactly 9 "H"s and 9 "V"s corresponds to a unique shortest path from Tom to Jerry.

Therefore, the number of shortest paths is exactly the same as the number of ways of arranging 9 "H"s and 9 "V"s in a sequence, which is $\binom{18}{9} = 48620$.



Maze

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

(b) 2 legs

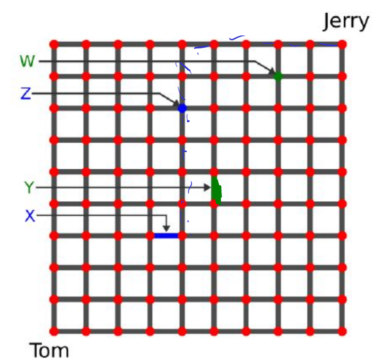
3 rights 3 ups

$\binom{6}{3}$ $\quad \binom{11}{5} = \frac{11}{6}$

5 rights 6 ups

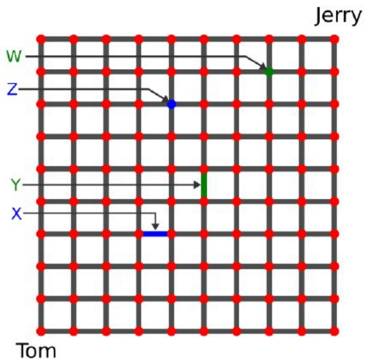
Tom → Y $\quad$ X → Jerry

$\binom{6}{3} \cdot \binom{11}{5} =$



Maze

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?



(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

(b) For a shortest path to pass through the edge X, it has to first get to the left vertex of X. So the first portion of the path has to start at the origin, and end at the left vertex of X. Using the same logic as above, there are exactly $\binom{6}{3} = 20$ ways to complete this "first leg" of the path (consisting of 3 "H" edges and 3 "V" edges). Having chosen one of these 20 ways, the path has to then go from the right vertex of X to the top right corner of the maze (the "second leg"). This second leg will consist of 5 "H" edges and 6 "V" edges, and using the same logic, there are exactly $\binom{11}{5} = 462$ possibilities. Therefore, the total number of shortest paths that pass through the edge X is $20 \times 462 = 9240$.

$\frac{11!}{(11-5)!\,5!} = \frac{11!}{6!\,5!} = \frac{11!}{6!\,(11-6)!}$

3 leg

Tom → X

X → Y

Y → Jerry

Maze

(b) How many shortest paths pass through the edge labeled X? The edge labeled Y? Both the edges X and Y? Neither edge X nor edge Y?

Using similar logic, any shortest path that passes through Y has to consist of 2 legs, the first leg going from the origin to the bottom vertex of Y, and the second leg going from the top vertex of Y to the top right corner of the maze. The first leg will consist of exactly 5 "H"s and 4 "V"s, while the second leg will consist of exactly 4 "H"s and 4 "V"s. So the total number of such shortest paths will be $\binom{9}{5} \times \binom{8}{4} = 8820$.

Maze

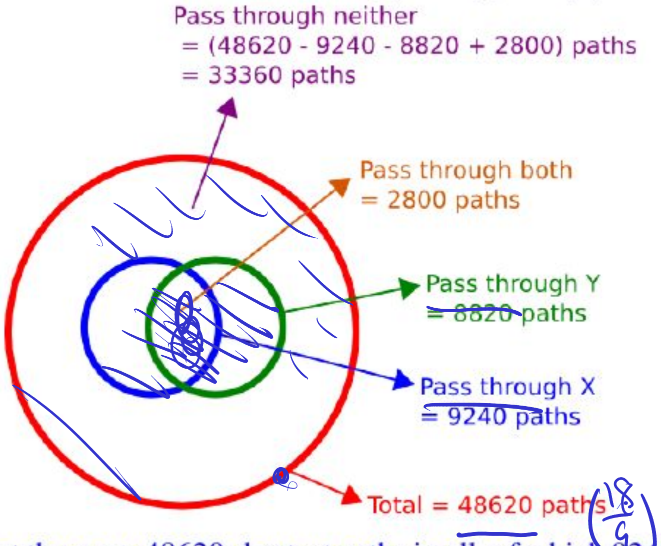
By a similar argument, let's try to figure out how many paths will pass through both X and Y. Clearly, any such path has to consist of 3 legs, with the first leg consisting of 3 "H"s and 3 "V"s (going from the origin to the left edge of X), the second leg consisting of 1 "H" and 1 "V" (going from the right vertex of X to the bottom vertex of Y), and the third leg consisting of 4 "H"s and 4 "V"s (going from the top vertex of Y to the top right corner of the maze). The total number of such shortest paths is therefore $\binom{6}{3} \times \binom{2}{1} \times \binom{8}{4} = 2800$.

Inclusion-Exclusion

$\binom{18}{9} - Y - X + \binom{\text{go through}}{Y \text{ and } X} =$

$a \equiv p$

$a^p \equiv a \mod p$



Finally, we know that there are 48620 shortest paths in all, of which 9240 pass through X, 8820 pass through Y, and 2800 pass through both. So the number of paths that pass through neither is 33360 (see the figure above for an intuitive explanation).