Discussion 6B

Tarang Srivastava - CS70 Summer 2020

Mini Review

$X \sim U[0, 6]$

$f(x) = \frac{1}{6}$

$CDF(Z \leq 2)$

$\left[Z = 20\frac{X}{2} + 4\right]$

Calculus Review

- Chain rule, product, quotient
  - Derivative
  - Integrals
- ☆ Integration by parts
- ☆ $\int_{-\infty}^{\infty} f(x)\,dx$ when it exists
- U-substitution

PDF and CDF

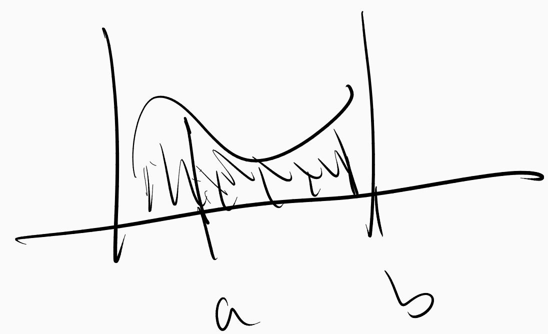
$X = 1, 2, 3 \ldots$

A probability density function (pdf) for a real-valued random variable X is a function $f : \mathbb{R} \to \mathbb{R}$ satisfying:

- $\forall x \in \mathbb{R}, \quad f(x) \geq 0$ ✓
- $\int_{-\infty}^{\infty} f(x)dx = 1$ ✓

$a \leq X \leq b$

2, 3



Cumulate density Function

CDF.

$$\boxed{F(x) = \mathbb{P}[X \leq x] = \int_{-\infty}^{x} f(z)\,dz.}$$

h

Lecture Highlights (sad)

$$\mathbb{P}(a \leq X \leq b) = \int_a^b f(x)dx$$

CDF

$$F(x) = \mathbb{P}(X \leq x)$$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\boxed{f(x)}dx$$

FTC

$$P(a \leq X \leq b) = \int_a^b f(x)dx = F(b) - F(a)$$

$aP(X=a$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$\int_{-\infty}^{\infty}$

issa crossover

Discrete	Continuous
X	X
PMF P(X=x)	PDF $f_x(x)$
CDF	CDF
$E[X] = \Sigma_x x P(X = x)$	$E[X] = \int_{-\infty}^{\infty} x f(x) dx$

Question 1

Question 1

The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2}, & \text{if } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$\frac{c}{x^2}$$

(a) Determine the value of c.

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

(a) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1 = \int_{-\infty}^{\infty} \frac{c}{x^2}\,dx = c\int_1^2 \frac{1}{x^2}\,dx$

$c\left[-\frac{1}{x}\right]_1^2$

$1 = c\,\frac{1}{2}$

$\boxed{c = 2}$

Question 1

The random variable X has the PDF

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(a) Determine the value of c.

(a) Integrate:

$$\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = c\int_1^2 x^{-2}\,\mathrm{d}x = -cx^{-1}\Big|_{x=1}^{2} = -c\left(\frac{1}{2}-1\right) = \frac{c}{2} = 1$$

so $c = 2$.

Question 1

(b) Let A be the event $\{X > 1.5\}$. Calculate $\mathbb{P}(A)$ and the conditional PDF of X given that A has occurred.

$$P(X>1.5) = \int_{1.5}^{2} \frac{2}{x^2}\,dx = 2\int_{3/2}^{2} \frac{1}{x^2}\,d$$

$$\boxed{\int_{1.5}^{2} \frac{c}{x^2}\,dx = 1}$$

$$c = 6$$

$$f_{X|A}(x) = \frac{f_X(x)}{P(A)} = \frac{2x^{-2}}{1/3} = 6x^{-2}$$

$$2\left[-\frac{1}{x}\right]_{3/2}^{2}$$

$$2\left[-\frac{1}{2}+\frac{2}{3}\right]$$

$$2\cdot\frac{1}{6} = \boxed{\frac{1}{3}}$$

(b) To find $\mathbb{P}(A)$,

$$\mathbb{P}(A) = \int_{1.5}^{2} f_X(x)\,dx = 2\int_{1.5}^{2} x^{-2}\,dx = -2x^{-1}\Big|_{x=1.5}^{2} = -2\left(\frac{1}{2}-\frac{2}{3}\right) = \frac{1}{3}.$$

The conditional PDF is thus

$$f_{X|A}(x) = \frac{f_X(x)}{\mathbb{P}(A)} = 6x^{-2}, \qquad x \in [1.5, 2].$$

Question 2

Question 2

Uniform $[0,1]$

at most x

can't be more than x

Let $X_1, \ldots X_n$ be independent $\boxed{U[0,1]}$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

pdf

$F(x) = P[X \le x]$

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

$x \in [0,1]$

$X_1, \ldots, X_n$

a) $P[X \le x] = P[X_1 \le x] \cdot P[X_2 \le x] \ldots P[X_n \le x]$

$x \cdot x \cdot x$

FTC

$= x^n$

b) $n x^{n-1}$

$\int_{-\infty}^{\infty} n x^{n-1} \, dx = 1$?

(a) $Pr[X \le x] = x^n$ since in order for $\max(X_1, \ldots X_n) < x$, we must have $X_i < x$ for all i. Since they are independent, we can multiply together the probabilities of each of them being less than x, which is x itself, as their distributions are uniform.

(b) Taking the derivative of the cdf, we have $f_X(x) = nx^{n-1}$

$(0,1)$

Question 2

Let $X_1, \ldots X_n$ be independent $U[0,1]$ random variables, and let $X = \max(X_1, \ldots X_n)$. Compute each of the following in terms of n.

(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

$$\int_0^1 x \cdot f_X(x)\, dx = \int_0^1 x \cdot n(x)^{n-1}\, dx$$

$$\int_0^1 n x^n\, dx$$

$$n\left[\frac{x^{n+1}}{n+1}\right]_0^1 = n\left[\frac{1}{n+1} - 0\right]$$

$$\boxed{\frac{n}{n+1}}$$

$$\begin{aligned}\mathbb{E}[X] &= \int_0^1 x f_X(x) \\ &= \int_0^1 n x^n dx \\ &= \frac{n}{n+1}\end{aligned}$$

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(a) What is the cdf of X?

(b) What is the pdf of X?

(c) What is $\mathbb{E}[X]$?

(d) What is $Var[X]$?

$$\frac{n^2}{(n+1)^2}$$

$$E[X^2] = \int_0^1 x^2 f_X(x)\, dx = \int_0^1 n x^{n+1}\, dx$$

$$n\left[\frac{1}{n+2}\right]_0^1 = \frac{n}{n+2}$$

$$\frac{n}{n+2} - \frac{n^2}{(n+1)^2}$$

$$\mathbb{E}[X^2] = \int_0^1 x^2 f_X(x) = \int_0^1 n x^{n+1} dx = \frac{n}{n+2}$$

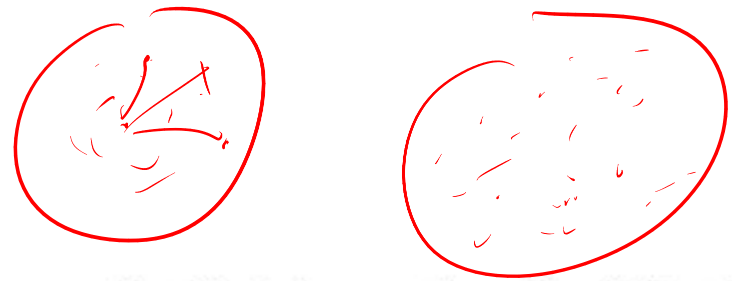
$$Var[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{n}{n+2} - \frac{n^2}{(n+1)^2}$$

Question 3

Question 3

Suppose Alice and Bob are playing darts on a circular board with radius 1. When Alice throws a dart, the distance of the dart from the center is uniform $[0,1]$. When Bob throws the dart, the location of the dart is uniform over the whole board. Let X the a random variable corresponding to the distance of the player's dart from the board.

$X|A$



(a) What is the pdf of X if Alice throws

(b) What is the pdf of X if Bob throws

(c) Suppose we let Alice throw the dart with probability p, and let Bob throw otherwise. What is the pdf of X (your answer should be in terms of p)?

(d) Using the same premise as in part c, suppose you observe a dart on the board but don't know who threw it. Let x be the dart's distance from the center. We would like to come up with a decision rule to determine whether Alice or Bob is more likely to have thrown the dart given your observation, x. Specifically, if we let A be the event that Alice threw the dart and B be the event that Bob threw, we want to guess A if $\mathbb{P}[A|X \in [x, x+dx]] > \mathbb{P}[B|X \in [x, x+dx]]$ (what do these two probabilities have to sum up to?). For what values of x would we guess A? (your answer should be in terms of p)

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(a) What is the pdf of X if Alice throws

$f_{X|A} = 1$

$[0, 1]$

CDF

$x \in 0, 1$

$P(X \leq x)$

x

$\boxed{1}$

(a) If Alice threw, then $X \sim U[0,1]$, so it's pdf is $f_{X|A}(x|A) = 1$. Note, the cdf is $\mathbb{P}[X < x|A] = \int_0^x 1dx = x$, which makes sense because this is exactly the area of a rectangle of length x and height 1.

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(b) What is the pdf of X if Bob throws

$$P[X \leq x \mid B] = \frac{\cancel{\pi} x^2}{\cancel{\pi}}$$

(b) If Bob throws, then the probability that $X < x$ is equaled to the area of the disc of radius x around the center of the dartboard divided by the area of the dartboard. Thus, we have the cdf as:

$$\mathbb{P}[X < x|B] = \frac{\pi x^2}{\pi} = x^2$$

$$f_{X|B}(x|B) = \frac{d}{dx}\mathbb{P}[X < x|B] = 2x$$

CDF $= x^2$

PDF $= 2x$

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(c) Suppose we let Alice throw the dart with probability p, and let Bob throw otherwise. What is the pdf of X (your answer should be in terms of p)?

(c) To find the pdf if X, we can again take the cdf first and take the derivative:

$$\begin{aligned}\mathbb{P}[X < x] &= \mathbb{P}[X < x|A]\mathbb{P}[A] + \mathbb{P}[X < x|B]\mathbb{P}[B] \\ &= px + (1-p)x^2 \\ f_X(x) &= p + 2(1-p)x\end{aligned}$$

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