Discussion 6C

Tarang Srivastava - CS70 Summer 2020

Mini-Review

Lecture Highlights Are Now Notes Highlights - Exponential

Definition 20.6 (Exponential Distribution). *For $\lambda > 0$, a continuous random variable X with p.d.f.*

pdf

$$f(x) = \begin{cases} \boxed{\lambda e^{-\lambda x}} & \text{if } x \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

$$\left[\frac{1}{2} e^{-\frac{1}{2}x} \right]$$

is called an exponential random variable with parameter λ, and we write $X \sim Exp(\lambda)$.

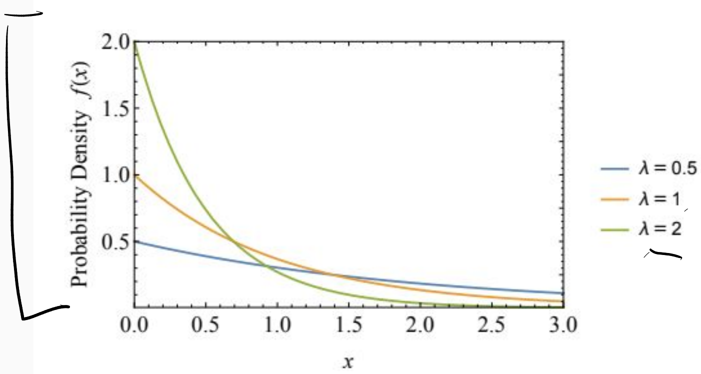
Theorem 20.2. *Let X be an exponential random variable with parameter $\lambda > 0$. Then*

$$\mathbb{E}[X] = \frac{1}{\lambda} \quad \textit{and} \quad \mathrm{Var}(X) = \frac{1}{\lambda^2}.$$

"mean"

$E[X] = \frac{1}{p}$

$Var(X) = \frac{1-p}{p^2}$



Discrete $\Longrightarrow$ Continuous

Geometric $\longrightarrow$ Exponential

how many flips till we get heads? (waiting)

ex how long we wait until a system fails.

$$P(X > x + t \mid X > t) = \frac{P(X > x + t \cap X > t)}{P(X > t)}$$

$$= \frac{P(X > x + t)}{P(X > t)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}}$$

$$= e^{-\lambda x} = P(X > x)$$

20 t 10   10

don't t

20

memoryless property

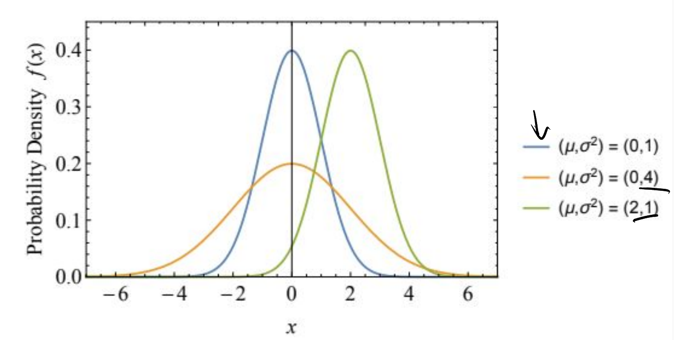
Lecture Highlights - Gaussian/Normal

③ Uniform, Exponential, Gaussian

Definition 20.7 (Normal Distribution). *For any $\mu \in \mathbb{R}$ and $\sigma > 0$, a continuous random variable X with p.d.f.*

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/(2\sigma^2)}}$$

is called a normal random variable with parameters μ and σ^2, and we write $X \sim N(\mu, \sigma^2)$. In the special case $\mu = 0$ and $\sigma = 1$, X is said to have the standard normal distribution.



Theorem 20.3. *For $X \sim N(\mu, \sigma^2)$,*

$$\mathbb{E}[X] = \mu \qquad \textit{and} \qquad \mathrm{Var}(X) = \sigma^2.$$

"mean"

σ = standard deviation

If $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$, then $Z = X + Y$ has distribution $Z \sim \mathcal{N}(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$ — ind. X and Y

Question 1

Question 1

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probability that the new bulb will last at least 30 days?

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

Question 1

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(a) Suppose an electrician is scheduled to check on the lightbulb in 30 days and replace it if it is broken. What is the probability that the electrician will find the bulb broken?

$$E[X] = 50 = \frac{1}{\lambda} \qquad \lambda = \frac{1}{50}$$

$$P(X \leq 30) = P(0 \leq X \leq 30)$$

CDF

$$\int_0^{30} \frac{1}{50} e^{-1/50 \cdot x} \, dx$$

$$-50 \cdot \frac{1}{50} \left[e^{-x/50} \right]_0^{30} = -1 \cdot \left[e^{-30/50} - 1 \right]$$

$$\boxed{1 - e^{-3/5}}$$

(a) Let $X \sim \text{Exponential}(1/50)$ be the time until the bulb is broken. For an exponential random variable with parameter λ, the density function is $f_X(x) = \lambda\, e^{-\lambda x}$ for $x > 0$. So in this case $\lambda = 1/50$. Thus we can integrate the density to find the probability that the lightbulb broke in the first 30 days:

$$\mathbb{P}[X < 30] = \int_0^{30} \left(\frac{1}{50} \cdot e^{-x/50} \right) dx = 1 - e^{-30/50} = 1 - e^{-3/5} \approx 0.451.$$

Question 1

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(b) Suppose the electrician finds the bulb broken and replaces it with a new one. What is the probability that the new bulb will last at least 30 days?

$P(X \geq 30)$

compliment

$1 - P(X \leq 30)$

$1 - (1 - e^{-3/5})$

$\boxed{e^{-3/5}}$ ✓

$P(30 \leq X \leq \infty)$

$\int_{30}^{\infty} \frac{1}{50} e^{-1/50 \cdot x} \, dx$

$-1 \left[e^{-x/50} \right]_{30}^{\infty}$

$-1 \left[0 - e^{-30/50} \right]$

$\boxed{e^{-3/5}}$

(b) The new bulb's waiting time Y is i.i.d. with the old bulb's. So the answer is

$$\mathbb{P}[Y > 30] = 1 - \mathbb{P}[Y < 30] = 1 - (1 - e^{-3/5}) = e^{-3/5} \approx 0.549.$$

Question 1

A brand new lightbulb has just been installed in our classroom, and you know the life span of a lightbulb is exponentially distributed with a mean of 50 days.

(c) Suppose the electrician finds the bulb in working condition and leaves. What is the probability that the bulb will last at least another 30 days?

Memoryless

$$P(X > x + t | X > t) = \frac{P(X > x + t \cap X > t)}{P(X > t)}$$
$$= \frac{P(X > x + t)}{P(X > t)} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}}$$
$$= e^{-\lambda x} = P(X > x)$$

$$P(40 > X > 30 | X > 30) = P(X > 30)$$

$$P(10 > X) \qquad e^{-3/5}$$

[Discrete case]

(c) The bulb is memoryless, so the probability it will last 60 days given that it has lasted 30 days, is just the probability it will last 30 days:

$$\boxed{\mathbb{P}[X > 60 \mid X > 30]} = \mathbb{P}[X - 30 > 30 \mid X > 30] = \mathbb{P}[X > 30] = e^{-3/5} \approx 0.549.$$

Question 2

Question 2

Edward and Khalil are playing darts.

Edward's throws are uniformily distributed over the entire dartboard, which has a radius of 10 inches. Khalil has good aim; the distance of his throws from the center of the dartboard follows an exponential distribution with parameter 1/2.

Say that Edward and Khalil both throw one dart at the dartboard. Let X be the distance of Edward's dart from the center, and Y be the distance of Khalil's dart from the center of the dartboard. What is $\mathbb{P}(X < Y)$, the probability that Edward's throw is closer to the center of the board than Khalil's? Leave your answer in terms of an unevaluated integral.

[*Hint:* X is not uniform over $[0, 10]$. Solve for the distribution of X by first computing the CDF of X, $\mathbb{P}(X < x)$.]

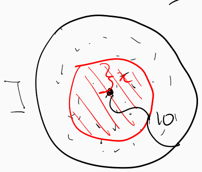
idea!

- We want $P(X<Y)$

$\sum_{d=0}^{10} P(X<d \cap Y>d)$

we can't do this in continuous case

Edward's X uniform over entire board



$P(X \le x)$

① $\boxed{P(X \le d) = \frac{d^2}{100}} = \frac{\pi d^2}{\pi 10^2}$

CDF → Find the PDF

$\frac{2d}{100} = \boxed{\frac{d}{50} = f_X(d)}$

$d \in [0,10]$

exponential ② $P(Y \ge d) = P(d \le Y < \infty)$

$\int_d^\infty \frac{1}{2} e^{-x/2} \, dx = e^{-d/2} = F_Y(d)$

$\underbrace{\int \frac{1}{2} e^{-\frac{x}{2}} \, dx}_{\text{pdf}} = \frac{1}{2}\left[-2 \cdot e^{-\frac{x}{2}}\right]_d^\infty \left[0 - \left(-e^{-\frac{d}{2}}\right)\right] \left[e^{-d/2}\right]$

$\int_0^{10} \boxed{P(X<d \cap d<Y)} \, dx$

X and Y independent

$\int_0^{10} F_X(d) \cdot f_Y(d) \, dx$

Question 2

$$\int_0^{10} \frac{x}{50} \cdot e^{-x/2}\, dx = P(X<Y)$$

Solution: We are given that $Y \sim \text{Exponential}(1/2)$. We now find the distribution of X by solving for the CDF of X, $\mathbb{P}(X < x)$. To get this, we'll consider the ratio of the area where the distance to the center is less than x, compared to the entire available area. This gives us the following expression:

$$\begin{aligned}\mathbb{P}(X < x) &= \frac{\pi x^2}{\pi 10^2} \\ &= \frac{x^2}{100}\end{aligned}$$

Differentiating gives us the PDF of X, which is given by $f_X(x) = \frac{x}{50}$. Now, we solve for $\mathbb{P}(X < Y)$:

$$\begin{aligned}\mathbb{P}(X < Y) &= \int_{x=0}^{10} \mathbb{P}(X \in [x, x+dx])\mathbb{P}(Y > x) \\ &= \int_{x=0}^{10} \frac{x}{50} e^{-0.5x} dx\end{aligned}$$

Evaluating this integral gives us $\mathbb{P}(X < Y) \approx 0.0767$.

Question 3

Question 3

Let X be a normally distributed random variable with mean μ and variance σ^2. Let $Y = aX + b$, where $a > 0$ and b are non-zero real numbers. Show explicitly that Y is normally distributed with mean $a\mu + b$ and variance $a^2\sigma^2$. The PDF for the Gaussian Distribution is $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. One approach is to start with the cumulative distribution function of Y and use it to derive the probability density function of Y.

[1.You can use without proof that the pdf for any gaussian with mean and sd is given by the formula $\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ where μ is the mean value for X and σ^2 is the variance. 2. The drivative of CDF gives PDF.]

Question 3

Let $a > 0$.

We start with the cumulative distribution function (CDF) of Y, F_Y.

$$\begin{aligned}
F_Y(x) &= \mathbb{P}[Y \leq x] && \text{By definition of CDF} \\
&= \mathbb{P}[aX + b \leq x] && \text{Plug in } Y = aX + b \\
&= \mathbb{P}\left[X \leq \frac{x-b}{a}\right] && \text{Because } a > 0 \\
&= F_X\left(\frac{x-b}{a}\right) && \text{By definition of CDF. } F_X \text{ denotes the CDF of } X.
\end{aligned} \tag{1}$$

Let f_Y denote the probability density function (PDF) of Y.

$$\begin{aligned}
f_Y(x) &= \frac{\mathrm{d}}{\mathrm{d}x} F_Y(x) && \text{The PDF is the derivative of the CDF.} \\
&= \frac{\mathrm{d}}{\mathrm{d}x} F_X\left(\frac{x-b}{a}\right) && \text{Plug in the result from (1)} \\
&= \frac{1}{a} \cdot f_X\left(\frac{x-b}{a}\right) && \begin{array}{l}\text{PDF is the derivative of CDF.} \\ \text{Apply chain rule, } \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x-b}{a}\right) = \frac{1}{a}.\end{array} \\
&= \frac{1}{a} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot \mathrm{e}^{-((x-b)/a-\mu)^2/(2\sigma^2)} && X \sim \mathcal{N}(\mu, \sigma^2). \\
&= \frac{1}{a\sigma\sqrt{2\pi}} \cdot \mathrm{e}^{-(x-b-a\mu)^2/(2\sigma^2 a^2)} && \frac{x-b}{a} - \mu = \frac{1}{a}(x - b - a\mu)
\end{aligned} \tag{2}$$

We have shown that f_Y equals the probability density function of a normal random variable with mean $b + a\mu$ and variance $\sigma^2 a^2$. So, Y is normally distributed with mean $b + a\mu$ and variance $\sigma^2 a^2$. The proof is done for $a > 0$. The proof for $a < 0$ is similar.

Come to Join Me Hours right after !!!

Don’t Stay Confused