Discussion 4B

Tarang Srivastava - CS70 Summer 2020

Mini-Review

Summary: $S = \{1, 2, \dots, n\}$

\# distinct k-element objects constructed from S

	Sampling with replacement	Sampling w/o replacement
ordered	n^k ✓	$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$
Unordered "stars & bars"	$\binom{k+n-1}{n-1}$ ✓ $= \binom{k+n-1}{k}$	✓ $\binom{n}{k}$

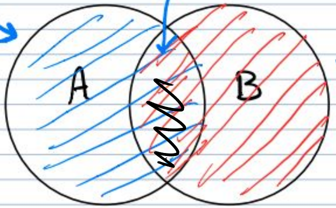
Lecture Highlights

Inclusion-Exclusion Rule:

$$\boxed{|A \cup B| = |A| + |B| - |A \cap B|}$$

$|A \cup B|$

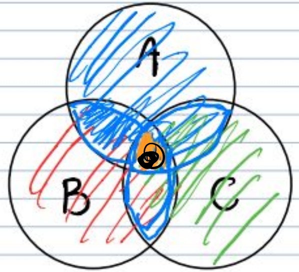
$A \cap B$



$\rightarrow |A| + |B|$ Elements in $|A \cap B|$ are counted twice

$\Rightarrow$ subtract $|A \cap B|$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$



n people = 3

k dollars = 5

P_1 P_2 P_3

\$ \$ \$ \$ \$ | |

TWITTER

$$\frac{7!}{8!}$$

$$\frac{7!}{8!\,2!}$$

Star and Bars:

n = bins

k = balls

$$\binom{n+k-1}{n-1} = \binom{3+5-1}{3-1} = \binom{7}{2}$$

Cardinality of Power Sets

$A = \{ \underset{2}{1}, \underset{2}{2}, \underset{2}{3} \}$ $\quad 3 = |A|$

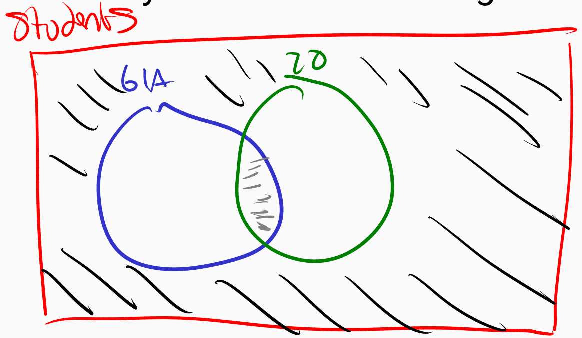
$|P(A)| = \boxed{2^3}$

$2^{|A|}$

Inclusion Exclusion

There are 30,000 students signed up for courses in the fall. 2000 of them have signed up for CS 61A and 900 have signed up for CS 70. Additionally, 150 have signed up for both CS 61A and CS 70.

How many students are not signed up for 61A or 70 this fall?



$$30{,}000 - \underbrace{2000}_{61A} - \underbrace{900}_{70} + 150$$

$$= \underline{27{,}250}$$

Stars and Bars

Days

twitter

We have strawberries, mangoes, and pineapple at the store. We need to select 8 fruits for our salad.

How many different ways can I select the fruit for my salad

Balls . Bins

$$\frac{10!}{8! \cdot 2!} = \binom{10}{2} = \binom{n+k-1}{n-1} = \binom{n+k-1}{k}$$

$$\frac{7!}{\boxed{3!}}$$

S M P

1) Collisions

2) $\boxed{\text{Splitting}}$

$$x_1 + x_2 + \dots + x_6 = 70$$

Question 1

$|A| = 2 \quad w_1, w_2 \in A$

$P(A) = P(w_1) + P(w_2)$

$P(w_1)$

$\frac{P_2(w_1)}{P_2(A)} = \frac{P_1(w_1)}{P_1(A)}$

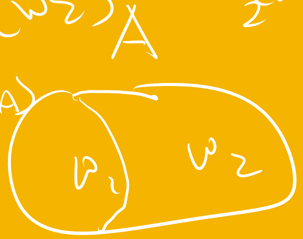
$P_1(w_1 | A) = P_2(w_1 | A)$

$\frac{P_1(w_1 \cap A)}{P_1(A)} = \frac{P_2(w_1 \cap A)}{P_2(A)} = \frac{P_2(w_1)}{P_2(A)} = \frac{P(w_1)}{P_1(w_2}$

$P(A | w_1) = 1 = \frac{P(w_1 \cap A)}{P(w_1) P(w_2)}$

$P_2(A | w_1) = 1$

$P(w_1) = P(w_1 \cap A)$



Question 1: The Count

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have? 4 9s, 2 2s, 1 8

(c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he nave now?

$D_2 \quad D_3 \quad D_5$

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

$$\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{2\cdot 3} \right\rfloor - \left\lfloor \frac{100}{3\cdot 5} \right\rfloor - \left\lfloor \frac{100}{2\cdot 5} \right\rfloor + \left\lfloor \frac{100}{2\cdot 3\cdot 5} \right\rfloor$$

$$D_2 + D_3 + D_5 - D_2 \cap D_3 \quad D_3 \cap D_5 \quad D_2 \cap D_5 \quad D_2 \cap D_3 \cap D_5$$

Inclusion - Exclusion

$D_2 \cup D_3 \cup D_5 =$

$=$

Question 1: The Count

(a) How many of the first 100 positive integers are divisible by 2, 3, or 5?

(a) We use inclusion-exclusion to calculate the number of numbers that satisfy this property. Let A be the set of numbers divisible by 2, B be the set of numbers divisible by 3, and C be the set of numbers divisible by 5. Then, we calculate

$$\overset{D_2}{|A|} + \overset{D_3}{|B|} + \overset{D_5}{|C|} - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{5} \right\rfloor - \left\lfloor \frac{100}{6} \right\rfloor - \left\lfloor \frac{100}{10} \right\rfloor - \left\lfloor \frac{100}{15} \right\rfloor + \left\lfloor \frac{100}{30} \right\rfloor$$

$$= 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$$

numbers.

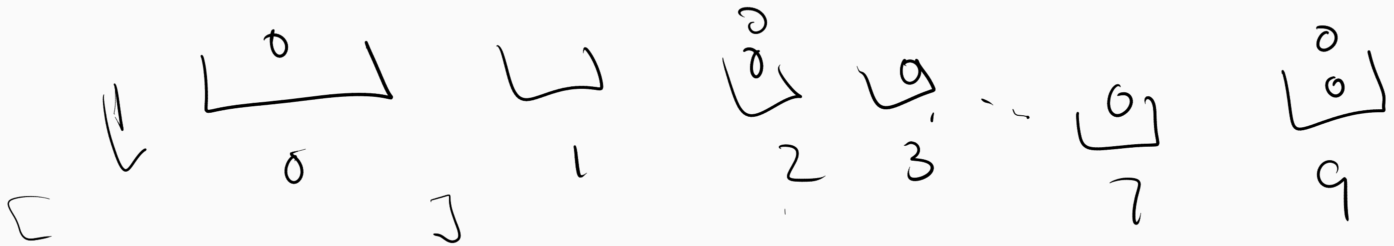
(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

4 9s 3 2s

9999 222

$\downarrow$

$a_1 \; a_2 \; \ldots \; a_7$



Balls = 7 digits

Bins = 0 – 9

$$\binom{n+k-1}{n-1} = \binom{16}{9}$$

Question 1: The Count

(b) The Count is trying to choose his new 7-digit phone number. Since he is picky about his numbers, he wants it to have the property that the digits are non-increasing when read from left to right. For example, 9973220 is a valid phone number, but 9876545 is not. How many choices for a new phone number does he have?

(b) This is actually a stars and bars problem in disguise! We have seven positions for digits, and nine dividers to partition these positions into places for nines, places for eights, etc. This is because we know that the digits are non-increasing, so all the nines (if any) must come first, then all the eights (if any), and so on. That means there are a total of 16 objects and dividers, and we are looking for where to put the nine dividers, so our answer is $\binom{16}{9}$.

Question 1: The Count

(c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he nave now?

① no digit can appear twice

$$\binom{10}{7}$$

Question 1: The Count

(c) Now instead of non-increasing, they must be strictly decreasing. So 9983220 is no longer valid, while 9753210 is valid. How many choices for a new phone number does he nave now?

(c) This can be found from just combinations. For any choice of 7 digits, there is exactly one arrangement of them that is strictly decreasing. Thus, the total number of strictly decreasing strings is exactly $\binom{10}{7}$.

Question 2

Question 2: CS70 *The Musical*

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called pascal's identity)

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

You have to come up with the story

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.

Question 2: CS70 *The Musical*

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\boxed{\binom{2n}{2} = 2\binom{n}{2} + n^2}$

LHS: Edward 2n directer applicants, he must select 2 of them

$\binom{2n}{2}$

RHS: Group A (first n appl.) Group B (last n applicants)

Cases
1) Both from A $\binom{n}{2}$
2) Both from B $\binom{n}{2}$
3) One from A & One from B $n \cdot n = n^2$

$2\binom{n}{2} + n^2$

Question 2: CS70 *The Musical*

(a) First, Edward would like to select directors for his musical. He has received applications from $2n$ directors. Use this to provide a combinatorial argument that proves the following identity: $\binom{2n}{2} = 2\binom{n}{2} + n^2$

(a) Say that we would like to select 2 directors.

LHS: This is the number of ways to choose 2 directors out of the $2n$ candidates.

RHS: Split the $2n$ directors into two groups of n; one group consisting of experienced directors, or inexperienced directors (you can split arbitrarily). Then, we consider three cases: either we choose:

(a) Both directors from the group of experienced directors,

(b) Both directors from the group of inexperienced directors, or

(c) One experienced director and one inexperienced director.

The number of ways we can do each of these things is $\binom{n}{2}$, $\binom{n}{2}$, and n^2, respectively. Since these cases are mutually exclusive and cover all possibilities, it must also count the total number of ways to choose 2 directors out of the $2n$ candidates. This completes the proof.

(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \underbrace{\binom{n-1}{k-1} + \binom{n-1}{k}}$ (this is called pascal's identity)

LHS: n ppl, k roles

RHS: 2 cases

we give him

$\binom{n-1}{k-1}$

+

we don't give job

$\binom{n-1}{k}$

Question 2: CS70 *The Musical*

(b) Edward would now like to select a crew out of n people, Use this to provide a combinatorial argument that proves the following identity: $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (this is called pascal's identity)

(b) Say that we would like to select k crew members.

LHS: This is simply the number of ways to choose k crew members out of n candidates.

RHS: We select the k crew members in a different way. First, Edward looks at the first candidate he sees and decides whether or not he would like to choose the candidate. If he selects the first candidate, then Edward needs to choose $k-1$ more crew members from the remaining $n-1$ candidates. Otherwise, he needs to select all k crew members from the remaining $n-1$ candidates.

We are not double counting here - since in the first case, Edward takes the first candidate he encounters, and in the other case, we do not.

Question 2: CS70 *The Musical*

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

LHS

c) $k =$ # roles

k options of the lead role

$$\sum_{k=1}^{n} k\binom{n}{k} =$$

RHS:

lead role o o o o ... o

$n \cdot 2^{n-1}$

Question 2: CS70 *The Musical*

(c) There are n actors lined up outside of Edward's office, and they would like a role in the musical (including a lead role). However, he is unsure of how many individuals he would like to cast. Use this to provide a combinatorial argument that proves the following identity: $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$

lead role

Cases

k roles

k = # of roles

lead role

(c) In this part, Edward selects a subset of the n actors to be in his musical. Additionally, assume that he must select one individual as the lead for his musical.

LHS: Edward casts k actors in his musical, and then selects one lead among them (note that $k = \binom{k}{1}$). The summation is over all possible sizes for the cast - thus, the expression accounts for all subsets of the n actors.

RHS: From the n people, Edward selects one lead for his musical. Then, for the remaining $n-1$ actors, he decides whether or not he would like to include them in the cast. 2^{n-1} represents the amount of (possibly empty) subsets of the remaining actors. *(Note that for each actor, Edward has 2 choices: to include them, or to exclude them.)*

Question 2: CS70 *The Musical*

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \underbrace{\binom{n}{k}}_{} \underbrace{\binom{k}{j}}_{} = 2^{n-j} \underbrace{\binom{n}{j}}_{}$.

j lead roles

lead roles

j ppl lead

Question 2: CS70 *The Musical*

(d) Generalizing the previous part, provide a combinatorial argument that proves the following identity: $\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j}$.

(d) In this part, Edward selects a subset of the n actors to be in the musical; additionally he must select j lead actors (instead of only 1 in the previous part).

LHS: Edward casts $k \geq j$ actors in his musical, then selects the j leads among them. Again, the summation is over all possible sizes for the cast (note that any cast that has $< j$ members is invalid, since Edward would be unable to select j lead actors) - thus, the expression accounts for all valid subsets of the n actors.

RHS: From the n people, Edward selects j leads for his musical. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. Then, for the remaining $n-j$ actors, he decides whether or not he would like to include them in the cast. 2^{n-j} represents the amount of ways that Edward can do this.

Question 3

Question 3: Bit String

How many bit strings of length 10 contain at least five consecutive 0's?

0 1 0 0/1 ... 0

Case 1

$\underbrace{0\ 0\ 0\ 0\ 0}\ \underbrace{0/1\ \_\ \_\ \_\ \_}_{5}$ $\quad 1 \quad 2^5 \qquad 5 \cdot 2^5$

Case 2

$\underbrace{0/1\ 0\ 0\ 0\ 0}\ 0\ \_\ \_\ \_\ \_$ $\quad 1 \quad 2^4$

$\_\ 1\ 0\ 0\ 0\ 0\ 0\ \_\ \_\ \_$ $\quad 2^4$

2^4

$2^4 \quad 2^4$

$$\boxed{2^5 + 5 \cdot 2^4}$$

Question 3: Bit String

How many bit strings of length 10 contain at least five consecutive 0's?

One counting strategy is strategic casework - we will split up the problem into exhaustive cases based on where the run of 0's begins. It can begin somewhere between the first digit and the sixth digit, inclusively.

If the run begins with the first digit, the first five digits are 0, and there are $2^5 = 32$ choices for the other 5 digits. If the run begins after the i^{th} digit, then the $i-1^{th}$ digit must be a 1, and the other $(10-5-1=4)$ digits can be chosen arbitrarily. The other four digits can be freely chosen with $2^4 = 16$. possibilities. Thus the total number of 10-bit strings with at least five consecutive 0's is $2^5 + 5 \cdot 2^4 = 112$.