Discussion 6A

Tarang Srivastava - CS70 Summer 2020

Mini Review

Lecture Highlights

$\Pr[X=i] = \boxed{(1-p)^{i-1}\,p}$

geometric

$\boxed{E[X] = \frac{1}{p}}$

$\Pr[X \geq i] = (1-p)^{i-1}$

$\mathrm{Var}(X) = \frac{1-p}{p^2}$.

Flip a coin with $\Pr[H] = p$ until we get H.

Let $\underline{X}$ be the flips until we get heads

$X = 5$

$$\boxed{\Pr[X=i] = \frac{\lambda^i}{i!} e^{-\lambda}} \quad i = 0, 1, \dots, n$$

$$\boxed{Var(X) = \lambda}$$

$E[X]$? λ

Theorem: Let $X \sim Poisson(\lambda)$ and $Y \sim Poisson(\mu)$ be independent Poisson random variables.

Then

$$Z = X + Y \sim Poisson(\lambda + \mu)$$

Lecture Highlights

Suppose when we write an article, we make an average of 1 typo per page. We can model this with a Poisson random variable X with $\lambda = 1$. So the probability that a page has 5 typos is

$$\mathbb{P}[X=5] = \boxed{\frac{1^5}{5!}e^{-1}} = \frac{1}{120\,e} \approx \frac{1}{326}.$$

P(X=5)

Proof. Fix $i \in \{0,1,2,\ldots\}$, and assume $n \geq i$ (because we will let $n \to \infty$). Then, because X has binomial distribution with parameter n and $p = \frac{\lambda}{n}$,

$$\mathbb{P}[X=i] = \binom{n}{i}\boxed{p^i(1-p)^{n-i}} = \frac{n!}{i!(n-i)!}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}.$$

n→∞

Let us collect the factors into

$$\mathbb{P}[X=i] = \boxed{\frac{\lambda^i}{i!}}\left(\frac{n!}{(n-i)!}\cdot\frac{1}{n^i}\right)\cdot\left(1-\frac{\lambda}{n}\right)^n\cdot\left(1-\frac{\lambda}{n}\right)^{-i}. \qquad (6)$$

The first parenthesis above becomes, as $n \to \infty$,

$$\frac{n!}{(n-i)!}\cdot\frac{1}{n^i} = \frac{n\cdot(n-1)\cdots(n-i+1)\cdot(n-i)!}{(n-i)!}\cdot\frac{1}{n^i} = \frac{n}{n}\cdot\frac{(n-1)}{n}\cdots\frac{(n-i+1)}{n} \to 1.$$

From calculus, the second parenthesis in (6) becomes, as $n \to \infty$,

$$\left(1-\frac{\lambda}{n}\right)^n \to e^{-\lambda}.$$

compound interest

And since i is fixed, the third parenthesis in (6) becomes, as $n \to \infty$,

$$\left(1-\frac{\lambda}{n}\right)^{-i} \to (1-0)^{-i} = 1.$$

Substituting these results back to (6) gives us

$$\mathbb{P}[X=i] \to \frac{\lambda^i}{i!}\cdot 1\cdot e^{-\lambda}\cdot 1 = \frac{\lambda^i}{i!}e^{-\lambda},$$

as desired. $\square$

Question 1

Question 1 (For all the parts you may leave your answer as an expression)

For each of the following parts, you may leave your answer as an expression.

4-5 mins

(a) You throw darts at a board until you hit the center area. Assume that the throws are i.i.d and the probability of hitting the center area is $p = 0.17$. What is the probability that you hit the center on your eighth throw?

(b) Let $X \sim \text{Geometric}(0.2)$. Calculate the expectation and variance of X.

identically independent distributed

(c) Suppose the accidents occurring weekly on a particular stretch of a highway is Poisson distributed with average number of accidents equal to 3 cars per week. Calculate the probability that there is at least one accident this week.

(d) Consider an experiment that consists of counting the number of α particles given off in a one-second interval by one gram of radioactive material. If we know from past experience that, on average, 3.2 such α-particles are given off per second, what is a good approximation to the probability that no more that 2 α-particles will appear in a second?

$$P(X=i) = \frac{\lambda^i}{i!} e^{-\lambda}$$

Question 1 (For all the parts you may leave your answer as an expression)

(a) You throw darts at a board until you hit the center area. Assume that the throws are i.i.d. and the probability of hitting the center area is $p = 0.17$. What is the probability that you hit the center on your eighth throw?

eighth hit

miss 7 tres

$$(0.17)(1-0.17)^{8-1}$$

Let N denote the random variable that you hit the center on your X-th turn. Then $X \sim$ Geometric(0.17) and hence,

$$\mathbb{P}(X = 8) = (0.17)(1 - 0.17)^7 \approx 0.0461.$$

Question 1 (For all the parts you may leave your answer as an expression)

(b) Let $X \sim \text{Geometric}(0.2)$. Calculate the expectation and variance of X.

$$E[X] = \frac{1}{p} = 5$$

$$Var(X) = \frac{1-p}{p^2} \qquad \frac{0.8}{0.04} = 20$$

$\mathbb{E}(X) = 5$ **and** $\text{Var}(X) = 20$

This follows from $\mathbb{E}(X) = 1/p$ and $\text{Var}(X) = (1-p)/(p^2)$ for $X \sim \text{Geometric}(p)$ as seen in lecture.

Question 1 (For all the parts you may leave your answer as an expression)

(c) Suppose the accidents occurring weekly on a particular stretch of a highway is Poisson distributed with average number of accidents equal to 3 cars per week. Calculate the probability that there is at least one accident this week.

$\lambda = 3$

$P(X \geq 1) =$

$1 - P(X = 0)$

$1 - \frac{3^0}{0!} e^{-3}$

$\boxed{1 - e^{-3}}$

Let X denote the number of accidents occurring on the stretch of highway in question during this week. We have $X \sim \text{Poisson}(3)$ and hence,

$$
\begin{aligned}
\mathbb{P}(X \geq 1) &= 1 - \mathbb{P}(X = 0), \\
&= 1 - e^{-3} \frac{3^0}{0!} \\
&= 1 - e^{-3} \approx \boxed{0.9502.}
\end{aligned}
$$

Question 1 (For all the parts you may leave your answer as an expression)

(d) Consider an experiment that consists of counting the number of α particles given off in a one-second interval by one gram of radioactive material. If we know from past experience that, on average, 3.2 such α-particles are given off per second, what is a good approximation to the probability that no more that 2 α-particles will appear in a second?

$\lambda = 3.2$

$P(X \leq 2) =$

$P(X=0) + P(X=1) + P(X=2)$

We model the number of α-particles given off during the second considered as a Poisson random variable with parameter $\lambda = 3.2$. Hence,

$$\mathbb{P}(X \leq 2) = \boxed{e^{-3.2} + 3.2e^{-3.2} + \frac{(3.2)^2}{2}e^{-3.2}} = 0.382.$$

Question 2

Question 2

It's that time of the year again - Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\mathrm{Var}(X) = n^2 \left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}(X)$. *[Hint: Try to break this problem down using indicators as with the coupon collector's problem. Are the indicators independent?]*

Question 2 (Extra Space)

It's that time of the year again - Safeway is offering its Monopoly Card promotion. Each time you visit Safeway, you are given one of n different Monopoly Cards with equal probability. You need to collect them all to redeem the grand prize.

Let X be the number of visits you have to make before you can redeem the grand prize. Show that $\text{Var}(X) = n^2\left(\sum_{i=1}^{n} i^{-2}\right) - \mathbb{E}(X)$. *[Hint: Try to break this problem down using indicators as with the coupon collector's problem. Are the indicators independent?]*

Question 3

Question 3

Consider a boutique store in a busy shopping mall. Every hour, a large number of people visit the mall, and each independently enters the boutique store with some small probability. The store owner decides to model X, the number of customers that enter her store during a particular hour, as a Poisson random variable with mean λ. — Poisson

Suppose that whenever a customer enters the boutique store, they leave the shop without buying anything with probability p. Assume that customers act independently, i.e. you can assume that they each flip a biased coin to decide whether to buy anything at all. Let us denote the number of customers that buy something as Y and the number of them that do not buy anything as Z (so $X = Y + Z$).

Don't buy anything

(a) What is the probability that $Y = k$ for a given k? How about $\mathbb{P}[Z = j]$? *Hint*: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

hint: $X = \underbrace{k}_{\text{Buy}} + \underbrace{j}_{\text{Don't}}$

$X = k + j$

$P(X = k+j)$

$P(Y = k)$

$P(Y=k) = P(Y = k \mid X = k+j) \cdot P(X = k+j)$

kind of summation

(b) State the name and parameters of the distribution of Y and Z.

(c) Prove that Y and Z are independent. In particular, prove that for every pair of values y, z, we have $\mathbb{P}[Y = y, Z = z] = \mathbb{P}[Y = y]\mathbb{P}[Z = z]$.

Question 3

(a) What is the probability that $Y = k$ for a given k? How about $\mathbb{P}[Z = k]$? *Hint*: You can use the identity

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

X = # ppl

fix k

$X = k + j$

$x = k + j$

$x = k$

$$P(Y = k) = \sum_{j=0}^{\infty} \left[P(X = k + j) \cdot P(Y = k \mid X = k + j) \right]$$

All the possible ways

$$\frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \cdot \binom{k+j}{k} p^j (1-p)^k$$

(a) We consider all possible ways that the event $Y = k$ might happen: namely, $k + j$ people enter the store ($X = k + j$) and then exactly k of them choose to buy something. That is,

$$\begin{aligned}
\mathbb{P}[Y = k] &= \sum_{j=0}^{\infty} \mathbb{P}[X = k + j] \cdot \mathbb{P}[Y = k \mid X = k + j] \\
&= \sum_{j=0}^{\infty} \left(\frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \right) \cdot \left(\binom{k+j}{k} p^j (1-p)^k \right) \\
&= \sum_{j=0}^{\infty} \frac{\lambda^{k+j}}{(k+j)!} e^{-\lambda} \cdot \frac{(k+j)!}{k!\,j!} p^j (1-p)^k \\
&= \frac{(\lambda(1-p))^k e^{-\lambda}}{k!} \cdot \sum_{j=0}^{\infty} \frac{(\lambda p)^j}{j!} \\
&= \frac{(\lambda(1-p))^k e^{-\lambda}}{k!} \cdot e^{\lambda p} \\
&= \left[\frac{(\lambda(1-p))^k e^{-\lambda(1-p)}}{k!} \right].
\end{aligned}$$

$$P[Z = j] = \sum_{k=0}^{\infty} P[X = k + j] \cdot P[Z = j \mid X = k + j]$$

The case for Z is completely analogous:

$$\mathbb{P}[Z = k] = \left[\frac{(\lambda p)^k e^{-\lambda p}}{k!} \right]$$

Question 3

(b) State the name and parameters of the distribution of Y and Z.

$Y \sim \text{Poisson}((1-p)\lambda)$ $\quad \lambda(1-p)$

$Z \sim \text{Poisson}(p\lambda)$ $\quad \lambda p$

(b) Y follows the Poisson distribution with parameter $\lambda(1-p)$ and Z follows the Poisson distribution with parameter λp.

Question 3

(c) Prove that Y and Z are independent. In particular, prove that for every pair of values y, z, we have $\mathbb{P}[Y = y, Z = z] = \mathbb{P}[Y = y]\mathbb{P}[Z = z]$.

$P(X = y + z)$

(c) The joint distribution of Y and Z is given by

$$
\begin{aligned}
\mathbb{P}(Y = y, Z = z) &= \cancel{\sum_{x=0}^{\infty} \mathbb{P}(X = x, Y = y, Z = z)} \\
&= \cancel{\sum_{x=0}^{\infty} \mathbb{P}(Y = y, Z = z \mid X = x)\mathbb{P}(X = x)} \\
&= \mathbb{P}(Y = y, Z = z \mid X = y + z)\mathbb{P}(X = y + z) \\
&= \frac{\cancel{(y+z)!}}{y!z!} p^z (1-p)^y \frac{e^{-\lambda} \lambda^{y+z}}{\cancel{(y+z)!}} \\
&= \frac{e^{-\lambda(1-p)}(\lambda(1-p))^y}{y!} \cdot \frac{e^{-\lambda p}(\lambda p)^z}{z!} \\
&= \mathbb{P}(Y = y) \cdot \mathbb{P}(Z = z).
\end{aligned}
$$

Since $\mathbb{P}(Y = y, Z = z) = \mathbb{P}(Y = y) \cdot \mathbb{P}(Z = z)$ for all $y, z \in \mathbb{N}$, we get that Y and Z are independent.

Ask questions! Don’t stay confused.