DIVERGENCE-FREE TENSOR DENSITIES IN TWO DIMENSIONS

by

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ABSTRACT

Divergence-free Tensor Densities in Two Dimensions

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The study of divergence-free tensors has a rich history, with origins dating back to the advent of general relativity and the Einstein tensor. Currently, research in this topic is focused towards the (non-)existence of divergence-free tensors which are not variational. Should any such tensor exist, it could immediately be used to create a modified version of general relativity. Working in two dimensions for simplicity, we derive a formula which explicitly characterizes all variational, second rank tensor densities dependent on a metric and its derivatives to arbitrary order. Additionally, we create a partial classification of all symmetric, second rank, divergence-free tensor densities dependent on a metric, a scalar field, and derivatives of the scalar field to arbitrary order. Using this partial result, a classification of all symmetric, second rank, divergence-free tensor densities which depend on up to five derivatives of the metric is produced. In particular, such tensor densities with highest derivative of order one, two, three, or five do not exist, while the densities of order zero and order four are variational.

(247 pages)

PUBLIC ABSTRACT

Divergence-free Tensor Densities in Two Dimensions

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In physics, a common method for exploring the way a physical system changes over time is to look at the system's energy. Roughly speaking, the energy in these systems are either motion-based (kinetic energy, a bullet in flight) or position-based (potential energy, a rock sitting at the top of a hill). The difference between the system's total kinetic and potential energies is quantified by an expression called the Lagrangian. Using a special procedure, this Lagrangian is massaged to produce a group of equations called the Euler-Lagrange equations; if the initial configuration of the system is provided, the solution to these equations fully predict the evolution of the system through time. Generally speaking, the system itself is the primary object of interest, with the Lagrangian and Euler-Lagrange equations found afterwards.

However, we may take an alternative route and start with a collection of equations instead. For this "inverse problem", we ask if this collection of equations represent a physical system, that is, does there exist a Lagrangian whose Euler-Lagrange equations are the starting equations? This thesis is dedicated to investigating the "inverse problem" for a certain group of equations related to general relativity, the theory which governs gravity at planetary scales and beyond. The equations tested do not cover all systems in general relativity; for simplicity, we work in a world consisting of just two dimensions, instead of the four dimensions (three of space and one of time) encountered in our daily lives. Even with this restriction, the calculations involved are quite complicated, though we manage to solve the "inverse problem" for the simplest cases.

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CHAPTER 1

INTRODUCTION

The purpose of this thesis is to classify symmetric, divergence-free, natural tensor densities A^{ij} on a two-dimensional manifold M which are functions of a metric and its partial derivatives to fifth order. Our main result asserts that A^{ij} is necessarily of order 4 in the metric and is the Euler-Lagrange expression of a natural Lagrangian which is a function of the metric and its derivatives to order 2. The study of divergence-free tensors has a long history, both in differential geometry and general relativity. However, virtually all of the literature in this area is restricted to tensor densities of second order in the metric, with the few exceptions typically focusing on the third order case, for example, Lovelock [1] and Anderson and Pohjanpelto [2]. As we shall see, the classification of divergence-free tensor densities is surprisingly difficult once one reaches order 5 in the metric, even in two dimensions.

1.1 Notation and Literature Review

Before reviewing the relevant literature, we fix our notation. Let M be a smooth manifold of dimension n with local coordinates $x^i = (x^1, x^2, \ldots, x^n)$. A pseudo-Riemannian metric on M is a nondegenerate, symmetric, covariant, rank two tensor field. The components of a metric tensor in local coordinates are denoted by g_{ij} and the metric signature is $(+,-) = (p,q)$, where $p+q = n$. We define an inverse metric tensor with components g^{ij} via $g_{jk}g^{ik} = \delta^i_j$, where δ^i_j is the Kronecker delta, and denote $g = |\det g_{ij}|$ as (the absolute value of) the metric determinant. The Christoffel symbols of the metric $\Gamma^i{}_{jk}$ define the unique torsion-free connection for which the metric is covariantly constant. In local coordinates, they are given by

$$\Gamma^i{}_{jk} = \frac{1}{2} g^{il} \left(g_{jl,k} + g_{kl,j} - g_{jk,l} \right), \tag{1.1}$$

where commas denote partial derivatives and the summation convention is used throughout this thesis. The Riemann curvature tensor on M is defined by the commutation relation

$$X^i{}_{|jk} - X^i{}_{|kj} = R_l{}^i{}_{jk} X^l, \tag{1.2}$$

where the vertical bar denotes covariant differentiation with respect to the connection coefficients (1.1) and X^i are the components of a (contravariant) vector field on M. In local coordinates, the curvature tensor's components are given by

$$R_l{}^i{}_{jk} = \Gamma^i{}_{lj,k} - \Gamma^i{}_{lk,j} + \Gamma^m{}_{lj}\Gamma^i{}_{km} - \Gamma^m{}_{lk}\Gamma^i{}_{jm}. \tag{1.3}$$

The Ricci tensor, scalar curvature, and Einstein tensor are then defined as

$$R_{ij} = R_i{}^k{}_{jk}, \; R = g^{ij}R_{ij}, \text{ and } G^{ij} = R_{ij} - \frac{1}{2}Rg^{ij}, \tag{1.4}$$

respectively. The permutation symbol $\epsilon^{i_1\cdots i_n}$, permutation tensor $\varepsilon^{i_1\cdots i_n} = g^{-1/2}\epsilon^{i_1\cdots i_n}$, and generalized Kronecker delta

$$\delta^{i_1\cdots i_k}_{j_1\cdots j_k} = \det \begin{bmatrix} \delta^{i_1}_{j_1} & \cdots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \cdots & \delta^{i_k}_{j_k} \end{bmatrix} \tag{1.5}$$

will be used to write expressions in more compact form.

Fundamental to the subject matter of this thesis is the notion of a tensor concomitant or natural tensor T of the metric. This is a tensor (or tensor density) T dependent on the metric and its partial derivatives up to some finite order that obeys the naturality condition

$$T(\phi^* \cdots) = \phi^*[T(\cdots)], \tag{1.6}$$

where $\cdots$ denotes the arguments of T and ϕ^* is the map induced on tensor fields, and their derivatives, by the local diffeomorphism ϕ. A natural tensor dependent on partial derivatives to order r is said to be a natural tensor of order r. The general theory of tensor concomitants is studied in Thomas [3] and Epstein [4], with the following theorem as the key result.

Theorem 1. *If T is a tensor concomitant which is polynomial in the metric, its inverse, and derivatives thereof, then it may be expressed in terms of the metric, the inverse metric, the permutation tensor, the curvature tensor and its covariant derivatives, and contractions of these tensors.*

Since the metric tensor and its inverse are natural tensors of order zero in the metric and the curvature tensor is a natural tensor of metric order 2, an immediate consequence of Theorem 1 is that no natural tensors of metric order 1 exist.

With our conventions set, we begin our literature review with the most physically relevant case: the uniqueness of the Einstein tensor. Shortly after Einstein's foundational paper on general relativity, Hilbert [5] showed that the Einstein equations arise as the Euler-Lagrange equations of what is now known as the Einstein-Hilbert action. The following theorem, established in Cartan [6], Vermeil [7], and Weyl [8], proves the uniqueness of the vacuum Einstein equations.

Theorem 2. *Let $n = 4$ and suppose A^{ij} satisfies the following conditions.*

(A1) *A^{ij} is a contravariant, rank 2 tensor concomitant of the metric tensor and its derivatives up to second order, i.e., $A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd})$.*

(A2) *A^{ij} is divergence-free, i.e., ${A^{ij}}_{|j} = 0$.*

(A3) *A^{ij} is symmetric, i.e., $A^{ij} = A^{ji}$.*

(A4) *A^{ij} is linear in second derivatives of the metric.*

Then

$$A^{ij} = a\, G^{ij} + b\, g^{ij},$$

where a and b are constants.

Moreover, $\sqrt{g}A^{ij} = \dfrac{\delta\lambda}{\delta g_{ij}}$ *is the Euler-Lagrange expression of the Lagrangian density* $\lambda = \sqrt{g}(aR + 2b)$.

Many years later, Lovelock [9] proved a substantial generalization of Theorem 2.

Theorem 3. *If A^{ij} satisfies* (A1)*,* (A2)*, and* (A3) *of Theorem 2, then*

$$A^{ij} = \sum_{p=1}^{2p<n} a_p g^{jk} \delta^{ih_1\cdots h_{2p}}_{kj_1\cdots j_{2p}} R_{h_1h_2}{}^{j_1j_2} \cdots R_{h_{2p-1}h_{2p}}{}^{j_{2p-1}j_{2p}} + ag^{ij}, \tag{1.7}$$

where a and a_p are constants.

Additionally, $\sqrt{g}A^{ij} = \dfrac{\delta\lambda}{\delta g_{ij}}$ *is the Euler-Lagrange expression of the Lagrangian density*

$$\lambda = \sqrt{g} \sum_{p=1}^{2p<n} 2a_p \delta^{h_1\cdots h_{2p}}_{j_1\cdots j_{2p}} R_{h_1h_2}{}^{j_1j_2} \cdots R_{h_{2p-1}h_{2p}}{}^{j_{2p-1}j_{2p}} + 2a\sqrt{g}. \tag{1.8}$$

When $n = 4$, (1.7) gives $A^{ij} = aG^{ij} + bg^{ij}$ and (1.8) simplifies to $\lambda = \sqrt{g}(aR + 2b)$, reproducing Theorem 2 without the linearity condition (A4). We note that the divergence-free condition implies that A^{ij} is a polynomial natural tensor. Expanding on this result, Lovelock [10] showed the symmetry condition (A3) to also be superfluous in 4 dimensions, reducing the necessary conditions for the uniqueness of the Einstein tensor to (A1) and (A2).

Theorem 3 can be generalized in two directions. First, one can introduce auxiliary fields coupled to gravity and replace the divergence-free condition with a divergence identity involving the field equations of the auxiliary fields. Second, one can let A^{ij} depend on higher order derivatives of the metric. We discuss each of these options in turn.

Motivated by attempts to extend general relativity using metric-scalar field theories, e.g., Bergmann [11], Horndeski [12] proved the following theorem.

Theorem 4. *Let $n = 4$ and suppose A^{ij} satisfies the following conditions.*

(A1) *A^{ij} is a contravariant, rank 2 tensor density concomitant of the metric tensor, a scalar field φ, and both of their derivatives up to second order, i.e.,*
$A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd}; \varphi; \varphi_{,a}; \varphi_{,ab})$.

(A2) *$A^{ij}{}_{|j} = \varphi^{|i} B$, where $B = B(g_{ab}; g_{ab,c}; g_{ab,cd}; \varphi; \varphi_{,a}; \varphi_{,ab})$ is a scalar density.*

(A3) *A^{ij} is symmetric, i.e., $A^{ij} = A^{ji}$.*

Then there exists a Lagrangian density $\lambda = \lambda(g_{ab}; g_{ab,c}; g_{ab,cd}; \varphi; \varphi_{,a}; \varphi_{,ab})$ such that A^{ij} is variational.

The expressions for the tensor density A^{ij} and Lagrangian density λ are rather complicated; see Horndeski [12] for the explicit forms.

In a similar vein to Theorem 4, the experimental success of the Einstein-Maxwell equations and the requirement of magnetic monopoles in "grand unifying theories", e.g., 't Hooft [13] and Polyakov [14], motivated generalizations of Theorem 3 to metric-vector and metric-bivector field theories, respectively. For the first, Lovelock [15] produced a modified version of Theorem 3.

Theorem 5. *Suppose A^{ij} satisfies the following conditions.*

(A1) *A^{ij} is a contravariant, rank 2 tensor concomitant of the metric tensor, its derivatives up to second order, and the first derivative of a covector field $\psi_{a,b}$, i.e., $A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd}; \psi_{a,b})$.*

(A2) *$A^{ij}{}_{|j} = \alpha^{ih} F_h{}^j{}_{|j}$, where $\alpha^{ih} = \alpha^{ih}(g_{ab}; \psi_{a,b})$ is a tensor and $F_h{}^j = g^{jk} F_{hk}$ is the field strength tensor $F_{ab} = \psi_{a,b} - \psi_{b,a}$.*

(A3) *A^{ij} is symmetric, i.e., $A^{ij} = A^{ji}$.*

Then A^{ij} is given by

$$A^{ij} = \sum_{p=1}^{2p<n} a_p g^{jk} \delta^{ih_1 \cdots h_{2p}}_{kj_1 \cdots j_{2p}} R_{h_1 h_2}{}^{j_1 j_2} \cdots R_{h_{2p-1} h_{2p}}{}^{j_{2p-1} j_{2p}} + a g^{ij} + b \left(F^{ik} F^j{}_k - \frac{1}{4} g^{ij} F^{kl} F_{kl} \right), \tag{1.9}$$

where $a_p, a,$ and b are constants.

Furthermore, $\sqrt{g}A^{ij} = \dfrac{\delta\lambda}{\delta g_{ij}}$ is the Euler-Lagrange expression of the Lagrangian density

$$\lambda = \sqrt{g}\sum_{p=1}^{2p<n} 2a_p\delta^{h_1\cdots h_{2p}}_{j_1\cdots j_{2p}} R_{h_1h_2}{}^{j_1j_2}\cdots R_{h_{2p-1}h_{2p}}{}^{j_{2p-1}j_{2p}} + 2a\sqrt{g} + \lambda_{EM},$$

where

$$\lambda_{EM} = \begin{cases} -\frac{1}{2}b\sqrt{g}F^{ij}F_{ij} & \text{for } n \text{ odd} \\ -\frac{1}{2}b\sqrt{g}F^{ij}F_{ij} + c\epsilon^{i_1\cdots i_n}F_{i_1i_2}\cdots F_{i_{n-1}i_n} & \text{for } n \text{ even} \end{cases} \tag{1.10}$$

and c is a constant.

We note that the permutation symbol term in (1.10) does not contribute to the Euler-Lagrange expression (1.9) due to a lack of metric tensors.

It is well known, e.g., Lovelock and Rund [16, pp. 146-147], that the definition of F_{ab} in (A2) of Theorem 5 is equivalent to the Bianchi identity $\epsilon^{iabc}F_{bc|a} = 0$. However, if magnetic monopoles exist, the right side of the Bianchi identity gains a magnetic source term and therefore no longer implies the existence of the vector potential ψ_a. Following this line of thought, Lovelock [17] switched from the covectors of Theorem 5 to an antisymmetric tensor F_{ab} and produced the following theorem.

Theorem 6. *Let $n = 4$ and suppose A^{ij} satisfies the following conditions.*

(A1) *A^{ij} is a contravariant, rank 2 tensor density concomitant of the metric tensor, its derivatives up to second order, and a second rank skew-symmetric tensor $F_{ab} = -F_{ba}$, i.e., $A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd}; F_{ab})$.*

(A1b) *B^i and C^i are vector density concomitants of the metric tensor, its partial derivatives up to second order, and the first covariant derivative of F_{ab}.*

(A1c) *If the manifold M is flat, i.e., $R_{abcd} = 0$, then $B^i = \sqrt{g}F^{ij}{}_{|j}$ and $C^i = \epsilon^{ijkl}F_{jk|l}$.*

(A2) *$A^{ij}{}_{|j} = \alpha^i{}_k B^k + \beta^i{}_k C^k$, where $\alpha^i{}_k = \alpha^i{}_k(g_{ab}; F_{ab})$ and $\beta^i{}_k = \beta^i{}_k(g_{ab}; F_{ab})$ are nonzero tensors.*

(A3) *A^{ij} is symmetric, i.e., $A^{ij} = A^{ji}$.*

Then A^{ij}, B^i, and C^i are given by

$$A^{ij} = a\sqrt{g}G^{ij} - b\sqrt{g}\left(F^{ik}F^{j}{}_{k} - \frac{1}{4}g^{ij}F^{kl}F_{kl}\right) + c\sqrt{g}g^{ij}, \tag{1.11a}$$

$$B^i = \sqrt{g}F^{ij}{}_{|j}, \tag{1.11b}$$

$$C^i = \epsilon^{ijkl}F_{kl|j}, \tag{1.11c}$$

where a, b, and c are scalars.

Additionally, $A^{ij} = \dfrac{\delta\lambda}{\delta g_{ij}}$ is the Euler-Lagrange expression of the Lagrangian density

$$\lambda = \sqrt{g}(aR + 2c) + \frac{1}{2}b\sqrt{g}F^{ij}F_{ij} + \frac{1}{2}d\epsilon^{ijkl}F_{ij}F_{kl}, \tag{1.12}$$

where d is another constant.

The expressions in (1.11) codify the Einstein-Maxwell equations upon setting each equal to zero (source-free) or the appropriate source terms. As with Theorem 5, the permutation symbol term in (1.12) does not contribute to the Euler-Lagrange expression (1.11a).

The final generalization of theories which couple auxiliary fields to gravity was given by Anderson [18], who considered a collection of tensor fields ρ_Q, independent of the metric, of arbitrary weight and rank.

Theorem 7. *Suppose A^{ij} satisfies the following conditions.*

(A1) *A^{ij} is a contravariant, rank 2 tensor concomitant of the metric tensor, independent tensor fields ρ_Q, and both of their derivatives up to second order, i.e., $A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd}; \rho_Q; \rho_{Q,a}; \rho_{Q,ab})$.*

(A2) *A^{ij} is divergence-free, i.e., $A^{ij}{}_{|j} = 0$.*

Then

$$A^{ij} = B^{ij} + D^{ij} + E^{ijk}{}_{|k},$$

where B^{ij} is a tensor satisfying conditions (A1), (A2), *and* (A3) *of Theorem 2*, $D^{ij} = -D^{ji}$ *is a tensor satisfying* (A1) *and* (A2), *and E^{ijk} is a totally antisymmetric tensor with the same functional dependency as* (A1).

The form of A^{ij} above places strong restrictions on extensions to general relativity, as exemplified by the following three corollaries to Theorem 7.

Corollary 1. *A^{ij} obeys* (A2) *if and only if* ${A^{ij}}_{|i} = 0$.

Corollary 2. *If A^{ij} is symmetric, then A^{ij} is independent of the fields ρ_Q and their derivatives.*

Corollary 3. *If $n = 4$, then $B^{ij} = aG^{ij} + bg^{ij}$, $D^{ij} = 0$, and, if A^{ij} is not dependent on $\rho_{Q,ab}$, $E^{ijk} = \epsilon^{ijkl}(V^Q \rho_{Q|l} + W_l)$, where V^Q and W_l are arbitrary natural tensors dependent on the metric and ρ_Q.*

A symmetric stress-energy tensor is typically required in general relativity, e.g., Misner *et al.* [19], and so Corollaries 1, 2, and 3 show that auxiliary fields cannot be used to modify the dynamics of general relativity (beyond the standard stress-energy contribution).

The alternative option for extending general relativity is to consider tensors which depend on more than two derivatives of the metric. The following theorem by Lovelock [1] describes all divergence-free third order tensor densities in three dimensions. In this three-dimensional context, the Cotton tensor

$$C_{ijk} = R_{ij|k} - R_{ik|j} + \frac{1}{2(n-1)}\left(R_{|j}g_{ik} - R_{|k}g_{ij}\right)$$

can be used to define the well known Cotton-York tensor [20, 21]

$$C^{ij} = \epsilon^{ikl}{R^j}_{k|l} + \epsilon^{jkl}{R^i}_{k|l}. \tag{1.13}$$

Theorem 8. *Let $n = 3$ and suppose A^{ij} satisfies the following conditions.*

(A1) *A^{ij} is a contravariant, rank 2 tensor density concomitant of the metric tensor and its derivatives up to third order, i.e., $A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd}; g_{ab,cde})$.*

(A2) *A^{ij} is divergence-free, i.e., ${A^{ij}}_{|j} = 0$.*

(A3) *A^{ij} is symmetric, i.e., $A^{ij} = A^{ji}$.*

Then A^{ij} is given by

$$A^{ij} = a\,\sqrt{g}G^{ij} + b\,\sqrt{g}g^{ij} + c\,C^{ij},$$

where a, b, and c are constants.

Additionally, $A^{ij} = \dfrac{\delta\lambda}{\delta g_{ij}}$ is the Euler-Lagrange expression of the Lagrangian density

$$\lambda = \sqrt{g}(aR - 2b) + c\epsilon^{ijk}\left(\frac{1}{2}\Gamma^m_{il}\Gamma^l_{mj,k} + \frac{1}{3}\Gamma^m_{il}\Gamma^n_{mj}\Gamma^l_{nk}\right).$$

The Cotton-York tensor C^{ij} (1.13) is both natural and variational [22]; it is the Euler-Lagrange expression of the Lagrangian density given by the term with coefficient c in Theorem 8. However, this Lagrangian density is not natural and the Cotton-York tensor is not the Euler-Lagrange expression of any natural Lagrangian [23].

The generalization of Theorem 3 to third order has proven to be considerably more challenging than the original works, with no explicit solution in the literature for any dimension other than three. However, Anderson and Pohjanpelto [2] gives a partial generalization of Theorem 3.

Theorem 9. *Suppose A^{ij} satisfies the following conditions.*

(A1) *A^{ij} is a contravariant, rank 2 tensor density concomitant of the metric tensor and its derivatives up to third order, i.e., $A^{ij} = A^{ij}(g_{ab}; g_{ab,c}; g_{ab,cd}; g_{ab,cde})$.*

(A2) *A^{ij} is divergence-free, i.e., ${A^{ij}}_{|j} = 0$.*

(A3) *A^{ij} is symmetric, i.e., $A^{ij} = A^{ji}$.*

Then A^{ij} is variational and can be expressed as

$$A^{ij} = \begin{cases} E^{ij}(\lambda), & \text{if } n = 0, 1, 2 \mod 4, \\ E^{ij}(\lambda) + C_P^{ij}, & \text{if } n = 3 \mod 4, \end{cases}$$

where the Lagrangian λ is a natural scalar density, $E^{ij}(\lambda) = \dfrac{\delta\lambda}{\delta g_{ij}}$ is the associated Euler-Lagrange expression, and C_P^{ij} is the generalized Cotton tensor associated with an invariant polynomial P on $\mathfrak{so}(p,q)$ of degree $(n+1)/2$.

The generalized Cotton tensors, like their namesake and simplest member the Cotton-York tensor, are natural and variational but not the Euler-Lagrange expression of any natural Lagrangian. Further details about the generalized Cotton tensors can be found in Anderson [22].

Recently, a trio of papers on the topic have been published [24–26]. However, each paper is lacking in sufficient rigor and clarity. We now discuss each paper in turn, including our objections to the arguments presented therein. (In light of the issues present within these papers, we believe that the question of natural, divergence-free tensors which are not variational to still be open for tensors dependent on four or more derivatives of the metric.)

1.1.1 S. Deser and Y. Pang, "Are all identically conserved geometric tensors metric variations of actions? A status report" 2019

In this paper [24], the authors claim to derive the most general symmetric, divergence-free tensor A^{ij} dependent on up to six derivatives of the metric in two dimensions and show it is variational. Their analysis depends on a heuristically motivated decomposition [labeled as (7) in the paper]

$$A^{ij} = (\nabla^i\nabla^j - g^{ij}\Box)T - \frac{1}{2}g^{ij}RT + X^{ij}, \tag{1.14}$$

where, if A^{ij} is of order k in the metric, then the scalar T (labeled as S in the paper) and tensor X^{ij} are of order $k-2$. If A^{ij} is variational, the scalar T takes the form $T = \dfrac{\delta\lambda}{\delta R}$ for a Lagrangian λ. Notably, this decomposition fixes the position of the metric order k and $k-1$ terms in A^{ij}.

However, this decomposition is incorrect. The analysis of the divergence-free condition $A^{ij}_{\ |j}$ performed in Chapter 3 of this thesis allows for the following decomposition of a symmetric, divergence-free tensor density A^{ij} of metric order k in two dimensions

$$A^{ij} = (\nabla^i \nabla^j - g^{ij}\Box)T + \tilde{D}^{ija_1\cdots a_{k-3}} R_{|(a_1\cdots a_{k-3})} + \tilde{E}^{ij}, \tag{1.15}$$

where T, $\tilde{D}^{ija_1\cdots a_{k-3}}$, and $\tilde{E}^{ij}$ are of metric order $k-2$. We note that $R_{|(a_1\cdots a_{k-3})}$ is of metric order $k-1$ and so the term $\tilde{D}^{ija_1\cdots a_{k-3}} R_{|(a_1\cdots a_{k-3})}$ is missing from the authors' decomposition.

As an example of this discrepancy, we consider the Lagrangian

$$\lambda^* = \sqrt{g}\varepsilon^{ik}\varepsilon^{jl} R_{|ij} R_{|kl} P, \tag{1.16}$$

where ε^{ij} is the permutation tensor and $P = P(R)$ is a scalar. So defined, λ^* is manifestly of metric order 4. Calculations performed in Section 2.5.4 of this thesis show $T = \dfrac{\delta\lambda^\star}{\delta R}$ and $A^{ij} = \dfrac{\delta\lambda^\star}{\delta g_{ij}}$ are of metric order 4 and 6, respectively. When this Euler-Lagrange expression is organized according to the authors' decomposition, we find that X^{ij} must be of metric order 5, a contradiction. Using the decomposition in (1.15), we see that the discrepancy arises due to a term of the form

$$-\epsilon^{ia}\epsilon^{jb} R^{|c} R_{|(abc)} P = \tilde{D}^{ijabc} R_{|(abc)}. \tag{1.17}$$

Additionally, one of the main results from our thesis is Theorem 10, which contains a decomposition for the Euler-Lagrange expression of a natural Lagrangian λ. Of particular interest is the third term in this decomposition $\dfrac{1}{2}\left(2\,\mathrm{Sym}_{ab}\, S^{abc} - S^{cab}\right)_{|c}$. For an Euler-Lagrange expression $\dfrac{\delta\lambda}{\delta g_{ij}}$ of metric order k, this term is typically of metric order $k-3$ and would appear in the $\tilde{E}^{ij}$ term of the decomposition (1.15) or the X^{ij} term in the authors' proposed decomposition (1.14). However, in the case where λ is degenerate, that is, its Euler-Lagrange expression is lower order than would be expected (e.g., $\lambda^\star$), this term involving the tensor S^{abc} can have metric order as high as (at least) $k-1$. Indeed, as seen in the calculation of this term (2.39), this is the term which provides the discrepancy (1.17).

We note that S^{abc} vanishes identically for Lagrangians below metric order 4 and so $\lambda^\star$ is a minimal example of this phenomenon.

In summary, the Lagrangian $\lambda^\star$ and its Euler-Lagrange expression are counterexamples to the general formula presented in this paper.

1.1.2 S. Deser, "Noether converse: All identically conserved geometric tensors are metric variations of an action in (at least) D=2" 2019

For this paper [25], the author claims to present an analysis which proves that all symmetric, divergence-free tensors are variational in two dimensions. We have three issues with the analysis in this paper.

- First, we fail to understand why terms of the form $\mathrm{Sym}_{ij}\, \varepsilon^{ik} R_{|k\cdots} R^{|j\cdots}$, where the dots denote additional contracted covariant derivative indices, have been excluded from the decomposition ansatz
$$A^{ij} = \frac{1}{2} g^{ij} Q + Y^{ij}, \tag{1.18}$$
where Q is a scalar and Y^{ij} is a tensor whose free indices are attached to derivatives of R.

- Second, we don't fully understand the author's claim that a classification of all divergence-free tensors may be reduced to the question of divergence-free Y-type tensors (those tensors whose free indices are attached to derivatives of R like Y^{ij}). In what follows, we attempt to reproduce Deser's approach to the best of our understanding.

  In particular, Deser appears to subtract the tensor $Z^{ij} = Y^{ij} - \tilde{Y}^{ij}$ from both sides of the ansatz (1.18) to get
$$A^{ij} - Z^{ij} = \frac{1}{2} g^{ij} Q + Y^{ij} - Z^{ij} = \frac{1}{2} g^{ij} Q + \tilde{Y}^{ij}.$$

The author then chooses $\tilde{Y}^{ij} = \dfrac{\delta Q}{\delta g_{ij}}$ so that the previous equation becomes

$$A^{ij} - Z^{ij} = \frac{1}{2} g^{ij} Q + \frac{\delta Q}{\delta g_{ij}} = \frac{1}{\sqrt{g}} \frac{\delta \lambda}{\delta g_{ij}},$$

where $\lambda = \sqrt{g} Q$. Using Noether's second theorem, the divergence of this equation becomes

$$A^{ij}_{\ \ |j} - Z^{ij}_{\ \ |j} = 0 \implies A^{ij}_{\ \ |j} = Z^{ij}_{\ \ |j}.$$

Hence, A^{ij} is divergence-free if and only if Z^{ij} is divergence-free.

At this point, the author claims $Z^{ij} = Y^{ij} - \tilde{Y}^{ij}$ is a Y-type tensor and proceeds with testing the conditions upon which Y-type tensors can be divergence-free. However, because Y^{ij} is a Y-type tensor by definition, this claim implies that $\tilde{Y}^{ij} = \dfrac{\delta Q}{\delta g_{ij}}$ is also a Y-type tensor. (We note that, as the variation of a *scalar* instead of a *scalar density*, Noether's second theorem cannot be invoked to say $\tilde{Y}^{ij}$ is divergence-free. Therefore, the problem cannot be reduced to testing the divergence of Y^{ij} alone.) The requirement that not a single free g^{ij} term may appear in the variation of every possible natural scalar Q seems patently absurd. For example, consider the simplest case $Q = R$, whose variation is (up to sign) the Ricci tensor R^{ij} and hence equal to $\dfrac{1}{2} g^{ij} R$ in two dimensions. As such, in our opinion, the study of divergence-free tensors like Z^{ij} is simply a restatement of the original problem and does not constitute a reduction in complexity.

- Finally, Deser's claim that no divergence-free Y-type tensors exist is not well supported (in addition to the previous arguments about the applicability of this claim to the original problem). As a standalone argument, the analysis presented is insufficient to fully support the claim: only a single case is examined in any detail and it is not general enough to cover all possible Y-type tensors.

1.1.3 S. Deser, "All identically conserved gravitational tensors are metric variations of invariant actions" 2019

In this paper [26], the author claims to present an argument which proves that all symmetric, divergence-free tensors are variational (in all dimensions). We have numerous problems with this paper (especially in terms of rigor and clarity), though we will limit our discussion to the following single issue.

- The author claims to split the problem into two cases, with a single exceptional 1.5 case containing the Cotton-York tensor alone, and is seemingly unaware of the generalized Cotton tensors (this is especially odd considering Anderson and Pohjanpelto [2] was cited in Deser and Pang [24]). Seeing as the rest of the analysis claims to account for all divergence-free tensors but does not present any non-natural actions, this omission casts doubt on the overall claim of the paper.

1.2 Noether's Theorems and Takens' Problem

By virtue of Noether's second theorem, there is a close relationship between divergence-free tensors and Euler-Lagrange expressions, which we now review. Let $\pi : F \to M$ be a fiber bundle with $\dim F = n + m$, and denote local adapted coordinates for π by $(x^i, f^A) \to (x^i)$ where $1 \leq A \leq m$. The r-th jet bundle of F, $J^r(F)$, has coordinates $(x^i; f^A; f^A_{,i}; \cdots ; f^A_{,i_1 \cdots i_r})$. A Lagrangian Λ is a function on $J^r(F)$ taking values in the n-forms of M, that is,

$$\Lambda = \lambda \left(x^i; f^A; f^A_{,i}; \cdots ; f^A_{,i_1 \cdots i_r}\right) \nu,$$

where $\nu = \mathrm{d}^n x = \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$. In general, we will use "Lagrangian" to refer to the coefficient function λ (or the scalar part L of $\lambda = \sqrt{g}L$) instead of the full expression Λ.

We seek local cross sections $f^A = \sigma^A(x^i)$ which yield extremals of the action functional

$$I[\sigma^A] = \int_U \lambda \left(x^i; \sigma^A; \frac{\partial \sigma^A}{\partial x^i}; \ldots ; \frac{\partial^r \sigma^A}{\partial x^{i_1} \cdots \partial x^{i_r}}\right) \nu, \tag{1.19}$$

where U is a suitably chosen subset of the manifold M. Solutions to the variational principle $\delta I = 0$ are found by solving the Euler-Lagrange equations

$$E_A(\lambda) = 0, \tag{1.20}$$

where $E_A(\lambda) = \dfrac{\delta\lambda}{\delta f^A}$ is the variational derivative or Euler-Lagrange expression of λ with respect to the fiber coordinates f^A. To compute E_A, we introduce the total derivative operator

$$D_i = \frac{\mathrm{d}}{\mathrm{d}x^i} = \frac{\partial}{\partial x^i} + f^A_{,i}\frac{\partial}{\partial f^A} + f^A_{,ai}\frac{\partial}{\partial f^A_{,a}} + f^A_{,abi}\frac{\partial}{\partial f^A_{,ab}} + \cdots \tag{1.21}$$

and the Euler-Lagrange expression is

$$E_A(\lambda) = (-D)_J \frac{\partial\lambda}{\partial f^A_{,J}}, \tag{1.22}$$

where $(-D)_J = (-1)^k D_J = (-1)^k D_{j_1} \cdots D_{j_k}$ and the sum on J is over the ordered multi-indices $J = j_1 \cdots j_k$ of all sizes $0 \le k \le r$, with the $k = 0$ case containing no derivatives, $j_i \le j_{i+1}$, and $1 \le j_i \le n$.

In the calculus of variations there are two well-known theorems by Noether (see, e.g., Olver [27]). The first, referred to as Noether's theorem or Noether's first theorem, states that each one parameter symmetry group of an action (1.19) defines a conservation law, that is, a vector field whose total divergence vanishes on solutions to the Euler-Lagrange equations (1.20). Examples of Noether's first theorem include the pairings of time symmetry with conservation of energy and translational (rotational) invariance with conservation of (angular) momentum. Noether's second theorem establishes a relationship between symmetries depending on arbitrary functions (of the base variables x^i) and differential identities involving the Euler-Lagrange expressions. By definition (1.6), natural tensors are preserved by the action of the diffeomorphism group and so Noether's second theorem states that a differential identity should exist for any variational principle involving a natural Lagrangian.

For natural Lagrangians depending solely on the metric and its derivatives, the differential identity is the vanishing covariant divergence of the Euler-Lagrange expression

$$E^{ij}(\lambda)_{|j} = 0. \tag{1.23}$$

Appendix E contains a fairly standard proof of this identity and a more detailed discussion on Noether's theorems. The differential identities of natural Lagrangians which depend on the metric, a scalar field φ or a covector field ψ_i, and both their derivatives to some finite order is given in Horndeski [28], with the results

$$E^{ij}(\lambda)_{|j} = \frac{1}{2}\varphi^{|i}E(\lambda), \tag{1.24}$$

and

$$E^{ij}(\lambda)_{|j} = -\frac{1}{2}F^{i}{}_{j}\,E^{j}(\lambda) - \frac{1}{2}\psi^{i}E^{j}(\lambda)_{|j}, \tag{1.25}$$

respectively, where $E^{ij}(\lambda) = \dfrac{\delta\lambda}{\delta g_{ij}}$, $E(\lambda) = \dfrac{\delta\lambda}{\delta\varphi}$, $E^{i}(\lambda) = \dfrac{\delta\lambda}{\delta\psi_i}$, and $F_{ij} = \psi_{i,j} - \psi_{j,i}$. If the covector field ψ_i is gauge invariant [that is, the transformation $\psi_i \to \psi_i + \phi_{,i}$ leaves the Lagrangian λ invariant for any scalar field $\phi = \phi(x^i)$], then the Euler-Lagrange expression $E^{i}(\lambda)$ is divergence-free and (1.25) reduces to

$$E^{ij}(\lambda)_{|j} = -\frac{1}{2}F^{i}{}_{j}\,E^{j}(\lambda). \tag{1.26}$$

We now observe that the differential identities (1.23), (1.24), and (1.26) are functionally identical to the divergence conditions (A2) of the theorems from the literature review. In particular, Theorems 2, 3, 7, 8, and 9 all follow the first type, while Theorems 4 and 5 correspond to the second and third types, respectively. (The one exception, Theorem 6, is discussed at the end of Appendix E.) Each of these theorems showed A^{ij} to be variational and suggests the following problem.

> Let $A^{ij} = A^{ij}[g, f^A]$ be a natural tensor which satisfies the differential identity associated to the natural variational problem $\lambda = \lambda[g, f^A]$. Is A^{ij} variational?

A more general version of this problem (encompassing both Noether's first and second theorems) was originally put forward by Takens [29], who (in the context of the above problem) showed that natural tensors which are dependent on the metric and its derivatives to order 2 are variational. When compared with the method used to find Theorem 3, Takens' approach is computation-light but does not produce the explicit formulas (1.7) and (1.8).

In the context of Noether's first theorem, the generalization of this problem involves pairs of symmetries and conservation laws instead of a differential identity.

> Let $\Delta = 0$ be a system of differential equations which admits a finite number of paired symmetries and conservation laws. Is Δ variational?

Takens proved that differential equations for a scalar field $\varphi(x^i)$ and its derivatives to order 2 which posses translational invariance and satisfy conservation of momentum are variational. Similarly, differential equations for a scalar field which depend linearly on the vector $(\varphi; \varphi_{,i}; \varphi_{,ij}; \cdots)$ and whose symmetry-conservation law pairs contain at least one pairing which involves the base coordinates x^i are variational.

Generalizations of these two situations were obtained in Anderson and Pohjanpelto [30] and Anderson and Pohjanpelto [31], respectively, with the latter considering polynomial differential equations of any order. Their surprising result is that not all polynomial differential equations with the symmetries and conservation laws of the Euclidean group are variational: there exists a class of non-variational polynomial differential equations with members in every dimension $n \geq 2$. The simplest member of this class is the two-dimensional differential equation defined by the source form (see the reference for details)

$$\Delta_2(1) = \det \begin{bmatrix} 1 & u & u_x & u_y \\ \mathscr{L} & \mathscr{L}u & \mathscr{L}u_x & \mathscr{L}u_y \\ \mathscr{L}^2 & \mathscr{L}^2u & \mathscr{L}^2u_x & \mathscr{L}^2u_y \\ \mathscr{L}^3 & \mathscr{L}^3u & \mathscr{L}^3u_x & \mathscr{L}^3u_y \end{bmatrix} \mathrm{d}u \wedge \mathrm{d}x \wedge \mathrm{d}y,$$

where $u = u(x, y)$, $u_i = D_i u$, $\mathscr{L} = D_{xx} + D_{yy}$ is the Laplacian, and the determinant is expanded along the first column. So defined, $\Delta_2(1)$ represents a tenth order partial differential equation which is symmetric under translations and rotations, satisfies conservation of linear and angular momentum, and is not the Euler-Lagrange expression of any Lagrangian. A similar class of non-variational differential equations may be created by replacing the Euclidean group of translations and rotations with the Poincaré group of translations, rotations, and boosts (hyperbolic rotations). The simplest member of this class is $\Delta_2(1)$ with the obvious modification to the Laplacian.

That $\Delta_2(1)$ and its more complicated generalizations exist at all is quite surprising. Euclidean-invariant polynomial differential equations are a well-studied class of partial differential equations (for example, the wave, heat, and Laplace equations in two or more spatial dimensions) but no non-variational examples exist below tenth order. If we are to classify the complicated equations described by natural tensor densities A^{ij}, then care should be taken to ensure no strange counterexamples analogous to $\Delta_2(1)$ are missed.

1.3 Main Results

This motivates the current paper, which investigates contravariant, rank 2 tensor densities in two dimensions which are dependent on more than two derivatives of the metric. Restricting ourselves to two dimensions significantly reduces the computational difficulty of the problem: the curvature tensor is proportional to the scalar curvature in dimension 2 and so Theorem 1 states that if T is a natural tensor, then the following conditions are equivalent.

(1) $T = T\left(g_{ab}; g_{ab,c}; \ldots ; g_{ab,c_1\cdots c_{r+2}}\right)$

(2) $T = T\left(g_{ab}; R; R_{|a}; R_{|ab}; R_{|(abc)}; \ldots ; R_{|(a_1\cdots a_r)}\right)$

The choice of option (2) allows us to bypass computational difficulties caused by the non-tensorial nature of partial derivatives of the metric and reduces the derivative order of the problem by two. As indicated by matching the indexing variable r with the number of derivatives of R, we will occasionally use the (symmetrized) covariant derivative order of the scalar curvature instead of the metric order for describing various tensorial quantities, e.g., $T = T(g_{ab}; R; R_{|a})$ is a natural tensor of curvature order 1 or a natural tensor of metric order 3.

Our primary results are contained in the following three theorems, with the first of these the previously mentioned general form for the Euler-Lagrange expression (with respect to the metric) of any natural Lagrangian density λ of curvature order r in two dimensions.

Theorem 10. *Let $n = 2$. If λ is a natural scalar density dependent on the metric and its derivatives to order $r + 2$, then $\lambda = \sqrt{g}L$ (where L is a scalar with the same functional dependence as λ) and the Euler-Lagrange expression of λ with respect to the metric is*

$$\begin{aligned}\frac{\delta\lambda}{\delta g_{ab}} = E^{ab}(\lambda) = \sqrt{g}\Bigg[&\left(\nabla^a\nabla^b - g^{ab}\Box\right) E_R(L) - \frac{1}{2}g^{ab}RE_R(L) \\ &+ \frac{1}{2}\left(2S^{(ab)c} - S^{cab}\right)_{|c} + \frac{1}{2}g^{ab}L + \frac{\partial L}{\partial g_{ab}}\Bigg],\end{aligned} \tag{1.27}$$

where $\Box = g^{ab}\nabla_a\nabla_b$ is the d'Alembertian, $\sqrt{g}E_R(L) = E_R(\lambda) = \dfrac{\delta\lambda}{\delta R}$ is the Euler-Lagrange expression of λ with respect to the scalar curvature, and S^{abc} is a particular contravariant, rank 3 tensor which is symmetric in its second and third indices.

The terms involving the Euler-Lagrange expression $E_R(L)$ are due to the variation of λ with respect to R [using the option (2) paradigm mentioned previously], while the final two terms come from the variation of λ with respect to the metric. The tensor S^{abc} is built from derivatives of the Lagrangian with respect to second and higher derivatives of R in a fashion reminiscent of the Euler-Lagrange expression $E_R(L)$; it arises due to the non-trivial commutation of variational and covariant derivatives, with the definition given in equation (2.28) and a full derivation found in Section 2.4. As such, the S^{abc} tensors do not appear in Euler-Lagrange expressions for Lagrangians of metric order less than 4 [e.g., two dimensional $f(R)$ theories in general relativity]. Generally speaking, the first term in (1.27) is of metric order $2r + 4$ (double the Lagrangian metric order of $r + 2$, as expected), the second term is of metric order $2r + 2$, the covariant derivative of S^{abc} is of metric order $2r + 1$, and the remaining two terms are of metric order $r + 2$. However, degenerate Lagrangians reduce the order of certain terms [the Lagrangian $\lambda^\star$ given previously (1.16) is an example, see Section 2.5.4 for details].

The second theorem restricts the highest order terms for all divergence-free, symmetric, contravariant, rank 2 tensor densities dependent on the metric, a scalar field (including the scalar curvature), and derivatives of the scalar field to some finite order.

Theorem 11. *Let $n = 2$. If A^{ij} is a symmetric, divergence-free tensor density dependent on the metric, a scalar field φ, and derivatives of the scalar field to order r, with $r \geq 2$, then A^{ij} takes the form*

$$\begin{aligned} A^{ij} &= \varepsilon^{ia_1}\varepsilon^{ja_2}\varphi_{|(a_1\cdots a_r)}B^{a_3\cdots a_r} + \varepsilon^{ia_1}\varepsilon^{jb_1}\varphi_{|(a_1\cdots a_{r-1})}\varphi_{|(b_1\cdots b_{r-1})}\frac{\partial B^{a_2\cdots a_{r-1}}}{\partial \varphi_{|(b_2\cdots b_{r-1})}} \\ &\quad + D^{ija_1\cdots a_{r-1}}\varphi_{|(a_1\cdots a_{r-1})} + E^{ij}, \end{aligned} \tag{1.28}$$

where $B^{a_3\cdots a_r}$ is a symmetric tensor density of scalar order $r-2$ which obeys the symmetry condition $\dfrac{\partial B^{a_1\cdots a_{r-2}}}{\partial \varphi_{|(b_1\cdots b_{r-2})}} = \dfrac{\partial B^{b_1\cdots b_{r-2}}}{\partial \varphi_{|(a_1\cdots a_{r-2})}}$, $D^{ija_1\cdots a_{r-1}}$ is a tensor density of scalar order $r-2$ which is symmetric in ij and $a_1\cdots a_{r-1}$, and E^{ij} is a symmetric tensor density of scalar order $r-2$.

As this holds for all scalar fields φ, it also holds for the scalar curvature $\varphi = R$, with A^{ij} a natural tensor density of metric order $r+2$.

The (scalar/metric) order of $B^{a_3\cdots a_r}$ cannot be reduced further: in the simplest case $r = 2$, B is of scalar order zero (see Chapter 4 for details). We note the similarity of (1.28) and the double covariant derivative of $E_R(L)$ term in (1.27); a short calculation performed at the end of Chapter 3 (3.17) reveals that the top order term in (1.27) can be cast in the form (1.28), with $B^{a_3\cdots a_r}$ equal to the derivative of $E_R(L)$ with respect to $R_{|(a_3\cdots a_r)}$. A further discussion about this theorem's applications beyond the limited scope of the thesis can be found at the end of Chapter 5.

Finally, we explicitly classify divergence-free, symmetric, contravariant, rank 2 tensor densities dependent on the metric and its derivatives up to order 5 in two dimensions.

Theorem 12. *Let $n = 2$ and suppose A^{ij} is a symmetric, divergence-free, natural tensor density. If A^{ij} is of metric order 0, e.g., $A^{ij} = A^{ij}(g_{ab})$, then A^{ij} is variational and takes the form*

$$A^{ij} = c\sqrt{g}g^{ij},$$

where c is a constant and $\lambda = 2c\sqrt{g}$ is the Lagrangian that has A^{ij} as its Euler-Lagrange expression $A^{ij} = E^{ij}(\lambda)$.

If A^{ij} is of metric order 4, then A^{ij} is variational and takes the form

$$A^{ij} = \sqrt{g}\left(\varepsilon^{ia}\varepsilon^{jb}R_{|ab}\frac{\mathrm{d}^2L}{\mathrm{d}R^2} + \varepsilon^{ia}\varepsilon^{jb}R_{|a}R_{|b}\frac{\mathrm{d}^3L}{\mathrm{d}R^3} + \frac{(-1)^q}{2}g^{ij}R\frac{\mathrm{d}L}{\mathrm{d}R} - \frac{(-1)^q}{2}g^{ij}L\right),$$

where $L = L(R)$ is the scalar part of the Lagrangian density $\lambda = \sqrt{g}L$ that has A^{ij} as its Euler-Lagrange expression $A^{ij} = E^{ij}(\lambda)$.

There are no symmetric, divergence-free, contravariant, rank 2 natural tensor densities of metric order 1, 2, 3, or 5.

The proof of Theorem 12 proceeds by systematically constructing a system of differential and algebraic identities which constitute the most general contravariant, rank 2 natural tensor density which is symmetric, divergence-free, and of the appropriate order. As indicated by the second statement of the theorem, this system frequently has no solution. For example, the fifth metric order natural tensor density

$$\begin{aligned}A^{ij} = \sqrt{g}\Bigg(&2\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)}R^{|c}\frac{\partial P}{\partial S} + 2\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}g^{cd}\frac{\partial P}{\partial S} + 4\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}R^{|c}R^{|d}\frac{\partial^2 P}{\partial S^2}\\ &+ 4\,\mathrm{Sym}_{ij}\,\varepsilon^{ia}\varepsilon^{jb}R_{|b}R^{|c}R_{|ac}\frac{\partial^2 P}{\partial R\partial S} + \varepsilon^{ia}\varepsilon^{jb}R_{|ab}\frac{\partial P}{\partial R} + \varepsilon^{ia}\varepsilon^{jb}R_{|ab}Q\\ &+ \frac{(-1)^q}{3}RR^{|i}R^{|j}\frac{\partial P}{\partial S} + \varepsilon^{ia}\varepsilon^{jb}R_{|a}R_{|b}\frac{\partial^2 P}{\partial R^2} + \varepsilon^{ia}\varepsilon^{jb}R_{|a}R_{|b}\frac{\mathrm{d}Q}{\mathrm{d}R} + \frac{(-1)^q}{2}Rg^{ij}P\Bigg),\end{aligned}$$

where $P = P(R, S)$ and $Q = Q(R)$ are scalar fields and $S = g^{ab}R_{|a}R_{|b}$, was built using all but one identity from the constructed system of equations. However, A^{ij} is not divergence-free (see Appendix D for details), with ${A^{ij}}_{|j}$ proportional to $R^{|i}(P - RQ)$, and there are no divergence-free tensor densities which are fifth order in the metric.

If λ is of metric order 4, then we expect $E^{ij}(\lambda)$ to be of order 8. Hence, no metric order 4 Lagrangians would be expected to produce divergence-free tensors A^{ij} at metric orders 6 or 7. However, the degenerate Lagrangians mentioned previously can produce metric order 6 Euler-Lagrange expressions from a metric order 4 Lagrangian (see Section 2.5.4). This complication makes finding the solution to the metric order 6 case for Theorem 12 problematic and we have failed to complete the proof thereof for this thesis. In the interest of further work in this space, we have placed the partial results in Appendix C. We note that similar, though more tractable, complications were involved in solving the fifth order metric case. Particularly, the presence of S derivatives of the scalar P in the tensor A^{ij} shown above is contrary to the (expected) lack of third metric order Lagrangians until the sixth metric order A^{ij} case. However, since no third metric order degenerate Lagrangians exist (such a Lagrangian would, presumably, be proportional to $\varepsilon^{ij}R_{|i}R_{|j}$ and so vanish identically), we have obtained the null result found in Theorem 12. We expect such complications to continue throughout the higher metric orders, e.g., degenerate Lagrangians of metric order 5 contributing to the metric order 7 analogue of Theorem 12, etc., on top of any contributions from potential non-variational tensors.

1.4 Thesis Contents

We begin in Chapter 2 by exploring the variational principle for a Lagrangian λ dependent on the metric, a scalar field, and symmetrized covariant derivatives of the scalar field. Requiring the action to be invariant under (proper) coordinate transformations forces the Lagrangian to be a scalar density and we derive an "invariance identity" from this restriction which establishes a relation between the partial derivative of λ with respect to the metric and derivatives with respect to the scalar field and its derivatives. We consider a first-order variation in the metric, computing the corresponding variational derivatives of the Christoffel symbols and the scalar curvature. Upon identifying the scalar field as the scalar curvature, these formulas are then used to compute the variational derivative of λ and hence deduce Theorem 10. We test the divergence-free condition of natural Lagrangians (1.23) for low metric order Lagrangians using the formula (1.27). Finally, the chapter concludes by using a degenerate Lagrangian to produce terms with metric order lower than the (theoretically maximum) metric order stated in Theorem 10.

Chapter 3 is devoted to proving Theorem 11, which will be used to simplify the proof of Theorem 12 in Chapter 4. In a fashion similar to that of Chapter 2, we establish an invariance identity for symmetric, rank 2 tensor densities A^{ij} and use it to derive a pair of formulas for computing the covariant divergence of A^{ij}. The first of these formulas has an explicit form (3.6) but most of the explicitly visible covariant derivatives of the scalar field are not fully symmetrized, making it difficult to use in the context of Theorem 12. We use the symmetrization formulas of Appendix A to produce a general form for this fully symmetric formula (3.3) but the combinatorial problem presented by the symmetrization formulas prevents us from explicitly describing this formula beyond a few of the highest order terms. This chapter concludes with a proof of Theorem 11; completing the proof requires some secondary results which also aid the proof of Theorem 12.

The proof of Theorem 12 is performed by cases in Chapter 4. We derive the explicit form of the fully symmetric divergence-free condition (3.3) in each case using the low order symmetrization formulas from Appendix A. The results of Chapter 3 are then used to restrict the form of A^{ij}, with these changes substituted into the divergence-free condition. The simplified equation, once the general theorems of Chapter 3 have been exhausted, is then differentiated with respect to the symmetrized covariant derivatives of the scalar field to produce new identities which are used to further restrict A^{ij}, with the new form substituted back into the divergence-free condition. This process is repeated until the most general tensor density with the desired properties is systematically built; as previously mentioned above, the system of equations produced frequently has no solution.

We conclude in Chapter 5 by discussing possible extensions of the current work. Among the topics covered are the limitations of the current approach to finding a full generalization of Theorem 12, exemplified by the growing algebraic complexity of the problem over the metric order 4, 5, and (incomplete) 6 cases; the potential application of the ideas in this thesis to higher dimensional tensor densities; and some potential for this thesis' primary results to produce examples of non-variational tensors.

CHAPTER 2

EULER-LAGRANGE FORMULAS FOR METRIC VARIATIONS

Chapter 2 starts by tailoring the notation used in the introduction to two dimensions. The rest of the chapter develops a proof of Theorem 10 and explicitly verifies the divergence-free condition (1.23) (see also Theorem 19 and the accompanying discussion) for low order Lagrangians in the final section of this chapter.

2.1 Notation and Two Dimensions

We now restrict our attention to $\dim M = n = 2$. In two dimensions the Riemann and Ricci tensors are proportional to the scalar curvature via the formulas

$$R_l{}^i{}_{jk} = \frac{R}{2}\left(g_{lj}\delta^i_k - g_{lk}\delta^i_j\right) \text{ and } R_{ij} = \frac{R}{2}g_{ij}, \tag{2.1}$$

respectively.

Under a coordinate transformation of M given by the smooth (we note that this is an overly strict restriction), invertible functions

$$\bar{x}^a = \bar{x}^a(x^i), \tag{2.2}$$

the components of a (r, s) tensor field T transform via the equation

$$\bar{T}^{a_1\cdots a_r}{}_{b_1\cdots b_s} = \frac{\partial \bar{x}^{a_1}}{\partial x^{i_1}}\cdots\frac{\partial \bar{x}^{a_r}}{\partial x^{i_r}}\frac{\partial x^{j_1}}{\partial \bar{x}^{b_1}}\cdots\frac{\partial x^{j_s}}{\partial \bar{x}^{b_s}}T^{i_1\cdots i_r}{}_{j_1\cdots j_s}\,, \tag{2.3}$$

and the components of a (r, s) tensor density $\mathcal{T}$ of weight w satisfy the transformation law

$$\bar{\mathcal{T}}^{a_1\cdots a_r}{}_{b_1\cdots b_s} = J^w\frac{\partial \bar{x}^{a_1}}{\partial x^{i_1}}\cdots\frac{\partial \bar{x}^{a_r}}{\partial x^{i_r}}\frac{\partial x^{j_1}}{\partial \bar{x}^{b_1}}\cdots\frac{\partial x^{j_s}}{\partial \bar{x}^{b_s}}\mathcal{T}^{i_1\cdots i_r}{}_{j_1\cdots j_s}\,, \tag{2.4}$$

where $J = \det\frac{\partial x^i}{\partial \bar{x}^a}$ is the Jacobian determinant of the transformation inverse to (2.2). We assume $J > 0$, i.e., the transformation (2.2) is orientation preserving. So defined, the covariant derivative of a contravariant, rank 1 tensor T^i is

$$T^i_{\ |j} = T^i_{\ ,j} + \Gamma^i_{\ jk} T^k, \tag{2.5}$$

and the covariant derivative of a contravariant, rank 1 tensor density $\mathcal{T}^i$ of weight w is

$$\mathcal{T}^i_{\ |j} = \mathcal{T}^i_{\ ,j} + \Gamma^i_{\ kj} \mathcal{T}^k - w \Gamma^k_{\ jk} \mathcal{T}^i, \tag{2.6}$$

with the usual generalizations of these two formulas for tensors and tensor densities of any rank and valence. Hereafter, tensor densities of unit weight will be referred to without their weight and tensorial quantities of rank zero will be referred to as scalars, e.g. a scalar density is a rank zero tensor density of unit weight. Similarly, contravariant and covariant tensors of rank 1 will be called vectors and covectors, respectively, with corresponding terms for rank 1 tensor densities of any weight.

Symmetrization of tensor indices will be denoted by parentheses or $\mathrm{Sym}_{ij\cdots}$, e.g., $T^{(abc)d} = \mathrm{Sym}_{abc} T^{abcd}$, and skew-symmetrized indices via square brackets or $\mathrm{Skew}_{ij\cdots}$, e.g., $T^{[abc]d} = \mathrm{Skew}_{abc} T^{abcd}$. Typically, Sym and Skew are used for (skew-)symmetrization of indices which are not adjacent in a tensorial expression, e.g., $\mathrm{Sym}_{ik} T^{ijk}$. The two-dimensional (contravariant) permutation symbol $\epsilon^{ij} = \epsilon^{[ij]}$ is the rank 2 tensor density defined via $\epsilon^{12} = 1$. The corresponding covariant permutation symbol ϵ_{ij} is a tensor density of weight -1 defined via the identity $\epsilon^{ij}\epsilon_{kl} = \delta^{ij}_{kl}$, where

$$\delta^{ij}_{kl} = \det \begin{bmatrix} \delta^i_k & \delta^i_l \\ \delta^j_k & \delta^j_l \end{bmatrix} = \delta^i_k \delta^j_l - \delta^j_k \delta^i_l$$

is the generalized Kronecker delta. The determinant of the metric g is a scalar density of weight 2 and so the formula

$$\varepsilon^{ij} = \frac{1}{\sqrt{g}} \epsilon^{ij} \tag{2.7}$$

defines a rank 2 tensor field on M called the (contravariant) permutation tensor. So defined, the permutation tensor obeys the following identities

$$\varepsilon^{ij}\varepsilon^{kl} = (-1)^q \left(g^{ik}g^{jl} - g^{il}g^{jk}\right), \tag{2.8a}$$

$$\varepsilon_{ij}\varepsilon_{kl} = (-1)^q \left(g_{ik}g_{jl} - g_{il}g_{jk}\right), \tag{2.8b}$$

$$\varepsilon^{ij}\varepsilon_{ik} = (-1)^q \delta^j_k, \tag{2.8c}$$

$$\varepsilon^{ij}\varepsilon_{ij} = (-1)^q 2, \tag{2.8d}$$

where

$$\varepsilon_{ij} = g_{ik}g_{jl}\varepsilon^{kl} = (-1)^q \sqrt{g}\epsilon_{ij} \tag{2.9}$$

is the covariant permutation tensor. The factor of $(-1)^q$ in (2.8) and (2.9) is noteworthy: unlike the permutation symbols, the covariant and contravariant permutation tensors differ by sign if there are an odd number of negative signs in the signature of the metric. In general, the words "permutation symbol/tensor" will refer to either version of the symbol/tensor, with index placement the determining factor.

2.2 Scalar Field Invariance Identity

We now show that if λ is a natural scalar density dependent on the metric, a scalar field φ, and symmetrized covariant derivatives of φ up to order r, then it takes the form $\lambda = \sqrt{g}L$, where L is a scalar dependent on the same arguments as λ. Additionally, L obeys the invariance identity

$$\frac{\partial L}{\partial g_{ij}} = -\frac{1}{2}\varphi^i \frac{\partial L}{\partial \varphi_j} - g^{ia}\varphi_{ab}\frac{\partial L}{\partial \varphi_{bj}} - \frac{3}{2}g^{ia}\varphi_{abc}\frac{\partial L}{\partial \varphi_{bcj}} - \cdots - \frac{r}{2}g^{ia_1}\varphi_{a_1\cdots a_r}\frac{\partial L}{\partial \varphi_{a_2\cdots a_r j}}, \tag{2.10}$$

where $\varphi_{i_1\cdots i_k} = \varphi_{|(i_1\cdots i_k)}$ is the k-th symmetrized covariant derivative of φ, $\varphi^i = g^{ij}\varphi_j$ and, in general, $\varphi_{a_1\cdots a_{s-1}}{}^{b}{}_{a_{s+1}\cdots a_k} = g^{ba_s}\varphi_{a_1\cdots a_k}$ for $1 \le s \le k$.

We start by letting $\pi : F \to M$ define a fiber bundle with $\dim F = 3$ and local adapted coordinates $(x^i, \varphi) \to (x^i)$. Then, the corresponding action (1.19) is given by

$$I = \int_U \lambda \left(g_{ij}; \varphi; \varphi_i; \varphi_{ij}; \ldots; \varphi_{i_1 \cdots i_r} \right) \nu, \tag{2.11}$$

where U is a suitably chosen subset of M and $\nu = \mathrm{d}x^1 \wedge \mathrm{d}x^2$. The highest order derivative of any fiber coordinate in λ, r, is said to be the order of λ.

If we require (2.11) to be invariant under coordinate transformations (2.2), then λ must be a natural scalar density (as ν is a natural scalar density of weight -1). Under a coordinate transformation, λ obeys an appropriately modified version of (2.4), with explicit form

$$\bar{\lambda} \left(\bar{g}_{ab}; \bar{\varphi}; \bar{\varphi}_a; \bar{\varphi}_{ab}; \ldots; \bar{\varphi}_{a_1 \cdots a_r} \right) = J \lambda \left(g_{ij}; \varphi; \varphi_i; \varphi_{ij}; \ldots; \varphi_{i_1 \cdots i_r} \right). \tag{2.12}$$

The arguments of $\bar{\lambda}$ are covariant tensors (or scalars) and transform using a modified version of (2.3)

$$\bar{g}_{ab} = J^i_a J^j_b g_{ij}, \tag{2.13a}$$

$$\bar{\varphi} = \varphi, \tag{2.13b}$$

$$\bar{\varphi}_a = J^i_a \varphi_i, \tag{2.13c}$$

$$\bar{\varphi}_{ab} = J^i_a J^j_b \varphi_{ij}, \tag{2.13d}$$

$$\bar{\varphi}_{abc} = J^i_a J^j_b J^k_c \varphi_{ijk}, \tag{2.13e}$$

$$\vdots$$

$$\bar{\varphi}_{a_1 \cdots a_r} = J^{i_1}_{a_1} \cdots J^{i_r}_{a_r} \varphi_{i_1 \cdots i_r}, \tag{2.13f}$$

where $J^i_a = \dfrac{\partial x^i}{\partial \bar{x}^a}$. We replace the arguments of (2.12) with these relations, yielding the equation

$$\bar{\lambda} \left(J^i_a J^j_b g_{ij}; \varphi; J^i_a \varphi_i; J^i_a J^j_b \varphi_{ij}; \ldots; J^{i_1}_{a_1} \cdots J^{i_r}_{a_r} \varphi_{i_1 \cdots i_r} \right) = J \lambda \left(g_{ij}; \varphi; \varphi_i; \varphi_{ij}; \ldots; \varphi_{i_1 \cdots i_r} \right). \tag{2.14}$$

Differentiating this equation with respect to g_{ij}, φ, $\varphi_i, \ldots, \varphi_{i_1\cdots i_r}$ reveals the tensorial nature of these derivatives,

$$J_a^i J_b^j \frac{\partial \bar{\lambda}}{\partial g_{ab}} = \frac{\partial \lambda}{\partial g_{ij}},\ \frac{\partial \bar{\lambda}}{\partial \varphi} = \frac{\partial \lambda}{\partial \varphi},\ J_a^i \frac{\partial \bar{\lambda}}{\partial \varphi_a} = \frac{\partial \lambda}{\partial \varphi_i}, \ldots,\ J_{a_1}^{i_1} \cdots J_{a_r}^{i_r} \frac{\partial \bar{\lambda}}{\partial \varphi_{a_1\cdots a_r}} = \frac{\partial \lambda}{\partial \varphi_{i_1\cdots i_r}},$$

while differentiating with respect to J_c^k yields an identity for λ

$$\left(\delta_k^i \delta_a^c J_b^j + J_a^i \delta_k^j \delta_b^c\right) g_{ij} \frac{\partial \bar{\lambda}}{\partial g_{ab}} + \delta_k^i \delta_a^c \varphi_i \frac{\partial \bar{\lambda}}{\partial \varphi_a} + \left(\delta_k^i \delta_a^c J_b^j + J_a^i \delta_k^j \delta_b^c\right) \varphi_{ij} \frac{\partial \bar{\lambda}}{\partial \varphi_{ab}} + \cdots = J K_k^c \lambda, \quad (2.15)$$

where $K_k^c = \dfrac{\partial \bar{x}^c}{\partial x^k}$ is defined via $J_i^j K_j^k = \delta_i^k$ and so $\dfrac{\partial J}{\partial J_c^k} = J K_k^c$. Since (2.15) holds for all transformations (2.2), it holds for the identity transformation $\bar{x}^i = x^i$ with $J_a^i = \delta_a^i = K_a^i$, $J = 1$, and $\bar{\lambda} = \lambda$. Substituting these changes into (2.15), we simplify the resulting equation to get

$$\begin{aligned}
\left(\delta_k^i \delta_a^c \delta_b^j + \delta_a^i \delta_k^j \delta_b^c\right) g_{ij} \frac{\partial \lambda}{\partial g_{ab}} + \delta_k^i \delta_a^c \varphi_i \frac{\partial \lambda}{\partial \varphi_a} + \left(\delta_k^i \delta_a^c \delta_b^j + \delta_a^i \delta_k^j \delta_b^c\right) \varphi_{ij} \frac{\partial \lambda}{\partial \varphi_{ab}} + \cdots &= \delta_k^c \lambda \\
\left(g_{kb} \frac{\partial \lambda}{\partial g_{cb}} + g_{ak} \frac{\partial \lambda}{\partial g_{ac}}\right) + \varphi_k \frac{\partial \lambda}{\partial \varphi_c} + \left(\varphi_{kj} \frac{\partial \lambda}{\partial \varphi_{cj}} + \varphi_{ak} \frac{\partial \lambda}{\partial \varphi_{ac}}\right) + \cdots &= \delta_k^c \lambda \\
2 g_{ki} \frac{\partial \lambda}{\partial g_{ic}} + \varphi_k \frac{\partial \lambda}{\partial \varphi_c} + 2 \varphi_{ki} \frac{\partial \lambda}{\partial \varphi_{ic}} + \cdots + r \varphi_{k i_1 \cdots i_{r-1}} \frac{\partial \lambda}{\partial \varphi_{i_1 \cdots i_{r-1} c}} &= \delta_k^c \lambda.
\end{aligned}$$

Finally, we contract this equation with g^{ak} and solve for the partial derivative of λ with respect to the metric

$$\frac{\partial \lambda}{\partial g_{ij}} = \frac{1}{2} g^{ij} \lambda - \frac{1}{2} \varphi^i \frac{\partial \lambda}{\partial \varphi_j} - g^{ia} \varphi_{ab} \frac{\partial \lambda}{\partial \varphi_{bj}} - \frac{3}{2} g^{ia} \varphi_{abc} \frac{\partial \lambda}{\partial \varphi_{bcj}} - \cdots - \frac{r}{2} g^{ia_1} \varphi_{a_1 \cdots a_r} \frac{\partial \lambda}{\partial \varphi_{a_2 \cdots a_r j}}. \quad (2.16)$$

The equation (2.16) is called the invariance identity of λ, an identity which λ obeys as a scalar density. The invariance identity may be used to extract information about the form of λ. First, we use the matrix calculus identity $\mathrm{d}(\det A) = \det A \operatorname{tr}(A^{-1}\,\mathrm{d}A)$, where tr is the trace, to compute the derivative of $\sqrt{g}$ with respect to the metric (we use the full expression $g = |\det g_{\alpha\beta}|$ to ensure no possible sign problems arise from the signature of the metric and ignore the summation convention for Greek indices in this calculation)

$$
\begin{aligned}
\frac{\partial \sqrt{g}}{\partial g_{ab}} &= \frac{1}{2\sqrt{|\det g_{\alpha\beta}|}} \frac{\partial |\det g_{\alpha\beta}|}{\partial g_{ab}} \\
&= \frac{1}{2\sqrt{|\det g_{\alpha\beta}|}} \frac{|\det g_{\alpha\beta}|}{\det g_{\alpha\beta}} \frac{\partial \det g_{\alpha\beta}}{\partial g_{ab}} \\
&= \frac{1}{2} \frac{\sqrt{|\det g_{\alpha\beta}|}}{\det g_{\alpha\beta}} \left[\det g_{\alpha\beta} \operatorname{tr} \left(g^{ij} \frac{\partial g_{jk}}{\partial g_{ab}} \right) \right] \\
&= \frac{1}{2} \sqrt{|\det g_{\alpha\beta}|} \left[g^{ij} \left(\delta_i^a \delta_j^b \right) \right] \\
&= \frac{1}{2} \sqrt{g} g^{ab},
\end{aligned}
\tag{2.17}
$$

where $\dfrac{\mathrm{d}}{\mathrm{d}x}|x| = \dfrac{x}{|x|} = \dfrac{|x|}{x}$ for $x \neq 0$. If λ has no φ dependence, then (2.16) reduces to

$$\frac{\partial \lambda}{\partial g_{ij}} = \frac{1}{2} g^{ij} \lambda.$$

With the context of (2.17), this equation is clearly satisfied by $\lambda = a\sqrt{g}$, where a is a constant. Hence, $\lambda = \sqrt{g} L$ where L is a scalar dependent on the same arguments as λ. In particular, substituting this form of λ into (2.16) yields the invariance identity for L

$$
\begin{aligned}
\frac{\partial \sqrt{g}}{\partial g_{ij}} L + \sqrt{g} \frac{\partial L}{\partial g_{ij}} &= \frac{1}{2}\sqrt{g} g^{ij} L - \frac{1}{2}\sqrt{g}\varphi^i \frac{\partial L}{\partial \varphi_j} - \sqrt{g} g^{ia} \varphi_{ab} \frac{\partial L}{\partial \varphi_{bj}} - \frac{3}{2}\sqrt{g} g^{ia} \varphi_{abc} \frac{\partial L}{\partial \varphi_{bcj}} \\
&\quad - \cdots - \frac{r}{2}\sqrt{g} g^{ia_1} \varphi_{a_1 \cdots a_r} \frac{\partial L}{\partial \varphi_{a_2 \cdots a_r j}} \\
\frac{1}{2}\sqrt{g} g^{ij} L + \sqrt{g} \frac{\partial L}{\partial g_{ij}} &= \sqrt{g} \left(\frac{1}{2} g^{ij} L - \frac{1}{2}\varphi^i \frac{\partial L}{\partial \varphi_j} - g^{ia} \varphi_{ab} \frac{\partial L}{\partial \varphi_{bj}} - \frac{3}{2} g^{ia} \varphi_{abc} \frac{\partial L}{\partial \varphi_{bcj}} - \cdots \right. \\
&\quad \left. - \frac{r}{2} g^{ia_1} \varphi_{a_1 \cdots a_r} \frac{\partial L}{\partial \varphi_{a_2 \cdots a_r j}} \right) \\
\frac{\partial L}{\partial g_{ij}} &= -\frac{1}{2}\varphi^i \frac{\partial L}{\partial \varphi_j} - g^{ia} \varphi_{ab} \frac{\partial L}{\partial \varphi_{bj}} - \frac{3}{2} g^{ia} \varphi_{abc} \frac{\partial L}{\partial \varphi_{bcj}} - \cdots \\
&\quad - \frac{r}{2} g^{ia_1} \varphi_{a_1 \cdots a_r} \frac{\partial L}{\partial \varphi_{a_2 \cdots a_r j}}.
\end{aligned}
$$

2.3 Metric and Curvature Variation

With the form of λ discovered, we now begin the process of computing the Euler-Lagrange expression in Theorem 10. We start by identifying φ as the scalar curvature R, noting that λ is now a function on the order $r+2$ jet bundle of metrics on M, and let g_{ij} be varied with respect to an infinitesimal variation parameterized by ϵ

$$g_{ab} \to g_{ab} + \epsilon h_{ab} + \mathcal{O}(\epsilon^2), \tag{2.18}$$

where $\left.\dfrac{\partial g_{ab}}{\partial \epsilon}\right|_{\epsilon=0} = h_{ab}$ is a symmetric, non-degenerate, contravariant, rank 2 tensor. To first order, the variation of the inverse metric can be computed by differentiating the identity $\delta^c_a = g_{ab}g^{bc}$ and solving the resulting equation for the variation of the inverse metric

$$0 = \frac{\partial}{\partial \epsilon}\left(g_{ab}g^{bc}\right)\bigg|_{\epsilon=0} = h_{ab}g^{bc} + g_{ab}\frac{\partial g^{bc}}{\partial \epsilon}\bigg|_{\epsilon=0} \implies \frac{\partial g^{cd}}{\partial \epsilon}\bigg|_{\epsilon=0} = -g^{ac}g^{bd}h_{ab}. \tag{2.19}$$

Using the matrix formula that was leveraged to find (2.17), the variation of the metric determinant can be found with a short computation

$$\frac{\partial g}{\partial \epsilon}\bigg|_{\epsilon=0} = \frac{\partial}{\partial \epsilon}|\det g_{\alpha\beta}|\bigg|_{\epsilon=0} = |\det g_{\alpha\beta}|\,\mathrm{tr}\left(g^{ab}\frac{\partial g_{bc}}{\partial \epsilon}\bigg|_{\epsilon=0}\right) = g\ \mathrm{tr}\left(g^{ab}h_{bc}\right) = gg^{ab}h_{ab}. \tag{2.20}$$

With this result in hand, we compute the variation of the square root of the metric determinant

$$\frac{\partial\sqrt{g}}{\partial \epsilon}\bigg|_{\epsilon=0} = \frac{1}{2}\frac{1}{\sqrt{g}}\frac{\partial g}{\partial \epsilon}\bigg|_{\epsilon=0} = \frac{1}{2}\frac{1}{\sqrt{g}}\left(gg^{ab}h_{ab}\right) = \frac{1}{2}\sqrt{g}g^{ab}h_{ab}. \tag{2.21}$$

Next, suppressing the evaluation at $\epsilon = 0$ for brevity, the variation of the Christoffel symbols proceeds in a straightforward manner from the definition (1.1)

$$\begin{aligned}
\frac{\partial \Gamma^a{}_{bc}}{\partial \epsilon} &= \frac{1}{2}\frac{\partial}{\partial \epsilon}\left[g^{ad}\left(g_{bd,c} + g_{cd,b} - g_{bc,d}\right)\right] \\
&= \frac{1}{2}\frac{\partial g^{ad}}{\partial \epsilon}\left(g_{bd,c} + g_{cd,b} - g_{bc,d}\right) + \frac{1}{2}g^{ad}\left[\left(\frac{\partial g_{bd}}{\partial \epsilon}\right)_{,c} + \left(\frac{\partial g_{cd}}{\partial \epsilon}\right)_{,b} - \left(\frac{\partial g_{bc}}{\partial \epsilon}\right)_{,d}\right] \\
&= \frac{1}{2}\left(-g^{fa}g^{ed}h_{ef}\right)\left(g_{bd,c} + g_{cd,b} - g_{bc,d}\right) + \frac{1}{2}g^{ad}\left(h_{bd,c} + h_{cd,b} - h_{bc,d}\right) \\
&= -\left(g^{fa}h_{ef}\right)\Gamma^e{}_{bc} + \frac{1}{2}g^{ad}\left(h_{bd,c} + h_{cd,b} - h_{bc,d}\right) \\
&= \frac{1}{2}g^{ad}\left(h_{bd,c} + h_{cd,b} - h_{bc,d} - 2\Gamma^e{}_{bc}h_{ed}\right) \\
&= \frac{1}{2}g^{ad}\left(h_{bd,c} + h_{cd,b} - h_{bc,d} - 2\Gamma^e{}_{bc}h_{ed} + \Gamma^e{}_{cd}h_{eb} - \Gamma^e{}_{cd}h_{eb} + \Gamma^e{}_{bd}h_{ec} - \Gamma^e{}_{bd}h_{ec}\right)
\end{aligned}$$

$$= \frac{1}{2} g^{ad} \left(h_{bd,c} - \Gamma^e{}_{bc} h_{ed} - \Gamma^e{}_{cd} h_{eb} + h_{cd,b} - \Gamma^e{}_{bc} h_{ed} - \Gamma^e{}_{bd} h_{ec} - h_{bc,d} + \Gamma^e{}_{cd} h_{eb} \right.$$
$$\left. + \Gamma^e{}_{bd} h_{ec} \right)$$
$$= \frac{1}{2} g^{ad} \left(h_{bd|c} + h_{cd|b} - h_{bc|d} \right), \tag{2.22}$$

where we have used the trivial commutation of partial and variational derivatives

$$\frac{\partial T_{,i}}{\partial \epsilon} = \left(\frac{\partial T}{\partial \epsilon} \right)_{,i}$$

to arrive at the second line. Noting that this quantity is manifestly tensorial, we set $\frac{\partial \Gamma^a{}_{bc}}{\partial \epsilon} = \delta \Gamma^a{}_{bc}$ for brevity. With the variation of the Christoffel symbols computed, it is a simple manner to compute the variation of the Riemann tensor using the local definition (1.3)

$$\frac{\partial R_d{}^a{}_{bc}}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \left(\Gamma^a{}_{db,c} - \Gamma^a{}_{dc,b} + \Gamma^e{}_{db} \Gamma^a{}_{ce} - \Gamma^e{}_{dc} \Gamma^a{}_{be} \right)$$
$$= \delta\Gamma^a{}_{db,c} - \delta\Gamma^a{}_{dc,b} + \delta\Gamma^e{}_{db} \Gamma^a{}_{ce} + \Gamma^e{}_{db} \delta\Gamma^a{}_{ce} - \delta\Gamma^e{}_{dc} \Gamma^a{}_{be} - \Gamma^e{}_{dc} \delta\Gamma^a{}_{be}$$
$$= \left(\delta\Gamma^a{}_{db,c} + \Gamma^a{}_{ce} \delta\Gamma^e{}_{db} - \Gamma^e{}_{dc} \delta\Gamma^a{}_{be} \right) - \left(\delta\Gamma^a{}_{dc,b} + \Gamma^a{}_{be} \delta\Gamma^e{}_{dc} - \Gamma^e{}_{db} \delta\Gamma^a{}_{ce} \right)$$
$$= \left(\delta\Gamma^a{}_{db,c} + \Gamma^a{}_{ec} \delta\Gamma^e{}_{db} - \Gamma^e{}_{dc} \delta\Gamma^a{}_{eb} \right) - \left(\delta\Gamma^a{}_{dc,b} + \Gamma^a{}_{eb} \delta\Gamma^e{}_{dc} - \Gamma^e{}_{db} \delta\Gamma^a{}_{ec} \right)$$
$$+ \left(\Gamma^e{}_{bc} \delta\Gamma^a{}_{de} - \Gamma^e{}_{bc} \delta\Gamma^a{}_{de} \right)$$
$$= \left(\delta\Gamma^a{}_{db,c} + \Gamma^a{}_{ec} \delta\Gamma^e{}_{db} - \Gamma^e{}_{dc} \delta\Gamma^a{}_{eb} - \Gamma^e{}_{bc} \delta\Gamma^a{}_{de} \right)$$
$$- \left(\delta\Gamma^a{}_{dc,b} + \Gamma^a{}_{eb} \delta\Gamma^e{}_{dc} - \Gamma^e{}_{db} \delta\Gamma^a{}_{ec} - \Gamma^e{}_{cb} \delta\Gamma^a{}_{de} \right)$$
$$= \delta\Gamma^a{}_{db|c} - \delta\Gamma^a{}_{dc|b}.$$

The variation of the Ricci tensor and scalar curvature (1.4) quickly follow from this result

$$\frac{\partial R_{db}}{\partial \epsilon} = \frac{\partial R_d{}^a{}_{ba}}{\partial \epsilon}$$
$$= \partial\Gamma^a{}_{db|a} - \partial\Gamma^a{}_{da|b},$$
$$\frac{\partial R}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \left(g^{bd} R_{bd} \right)$$
$$= \frac{\partial g^{bd}}{\partial \epsilon} R_{bd} + g^{bd} \left(\delta\Gamma^a{}_{db|a} - \delta\Gamma^a{}_{da|b} \right)$$

$$= -R^{bd}h_{bd} + g^{bd}\left(\delta\Gamma^{a}{}_{db|a} - \delta\Gamma^{a}{}_{da|b}\right).$$

This final expression can be cast in a more convenient form by expressing the variation of the Christoffel symbols in terms of the metric variation h_{ab}

$$
\begin{aligned}
\frac{\partial R}{\partial \epsilon} &= -R^{bd}h_{bd} + \left(g^{bd}\delta\Gamma^{a}{}_{db}\right)_{|a} - \left(g^{bd}\delta\Gamma^{a}{}_{da}\right)_{|b} \\
&= -R^{bd}h_{bd} + \left(g^{bd}\delta\Gamma^{c}{}_{db} - g^{cd}\delta\Gamma^{a}{}_{da}\right)_{|c} \\
&= -R^{bd}h_{bd} + \frac{1}{2}\left[g^{bd}g^{ac}\left(h_{da|b} + h_{ba|d} - h_{db|a}\right) - g^{cd}g^{ab}\left(h_{ab|d} + h_{db|a} - h_{da|b}\right)\right]_{|c} \\
&= -R^{bd}h_{bd} + \frac{1}{2}\left[g^{ac}g^{bd}\left(h_{da|bc} + h_{ba|dc} - h_{db|ac}\right) - g^{ab}g^{cd}\left(h_{ab|dc} + h_{db|ac} - h_{da|bc}\right)\right] \\
&= -R^{ab}h_{ab} + \frac{1}{2}\left(2g^{ad}g^{bc}h_{ab|cd} - 2g^{ab}g^{cd}h_{ab|cd}\right) \\
&= -R^{ab}h_{ab} + \left(g^{ad}g^{bc} - g^{ab}g^{cd}\right)h_{ab|cd} \\
&= -\frac{1}{2}Rg^{ab}h_{ab} + (-1)^{q+1}\varepsilon^{ac}\varepsilon^{bd}h_{ab|cd},
\end{aligned}
\tag{2.23}
$$

where the final line used (2.1) to substitute for the Ricci tensor and (2.8a) to simplify the second term.

If the variation of the scalar curvature is multiplied by a tensor (density) T (with indices suppressed for clarity), the covariant derivatives of h_{ab} in (2.23) can be moved to T using "integration by parts". Explicitly, we have

$$
\begin{aligned}
T\frac{\partial R}{\partial \epsilon} &= T\left(-\frac{1}{2}Rg^{ab}h_{ab} + (-1)^{q+1}\varepsilon^{ac}\varepsilon^{bd}h_{ab|cd}\right) \\
&= -\frac{1}{2}g^{ab}RTh_{ab} + (-1)^{q+1}\varepsilon^{ac}\varepsilon^{bd}Th_{ab|cd} \\
&= -\frac{1}{2}g^{ab}RTh_{ab} + (-1)^{q+1}\varepsilon^{ac}\varepsilon^{bd}\left[\left(Th_{ab|c}\right)_{|d} - T_{|d}h_{ab|c}\right] \\
&= -\frac{1}{2}g^{ab}RTh_{ab} + (-1)^{q+1}\varepsilon^{ac}\varepsilon^{bd}\left[\left(Th_{ab|c}\right)_{|d} - \left(T_{|d}h_{ab}\right)_{|c} + T_{|dc}h_{ab}\right] \\
&= \left[-\frac{1}{2}g^{ab}RT + (-1)^{q+1}\varepsilon^{ac}\varepsilon^{bd}T_{|dc}\right]h_{ab} + (-1)^{q+1}\varepsilon^{ac}\varepsilon^{bd}\left[\left(Th_{ab|c}\right)_{|d} - \left(f_{|d}h_{ab}\right)_{|c}\right] \\
&= \left[-\frac{1}{2}g^{ab}RT - \left(g^{ab}g^{cd} - g^{ad}g^{bc}\right)T_{|dc}\right]h_{ab} + \left[(-1)^{q+1}\varepsilon^{ac}\varepsilon^{be}Th_{ab|c}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (-1)^q \varepsilon^{ae}\varepsilon^{bd} T_{|d}\, h_{ab}\Big]_{|e} \\
&= \left(-\frac{1}{2} g^{ab} R T + g^{ad} g^{bc} T_{|cd} - g^{ab} \Box T\right) h_{ab} + \Big[(-1)^{q+1} \varepsilon^{ac}\varepsilon^{be} T h_{ab|c} \\
&\quad + (-1)^q \varepsilon^{ae}\varepsilon^{bd} T_{|d}\, h_{ab}\Big]_{|e}\,, \qquad (2.24)
\end{aligned}
$$

where $\Box = g^{ab}\nabla_a \nabla_b$ is the d'Alembertian.

Finally, we remark that covariant derivatives do not trivially commute with variational derivatives. To see this, we compute the variational derivative of the covariant derivative of a vector field X^i which varies infinitesimally with the same parameter ϵ as the metric

$$
\begin{aligned}
\frac{\partial}{\partial \epsilon}\left(X^i{}_{|j}\right) &= \frac{\partial}{\partial \epsilon}\left(X^i{}_{,j} + \Gamma^i{}_{jk} X^k\right) \\
&= \left(\frac{\partial X^i}{\partial \epsilon}\right)_{,j} + \left(\frac{\partial \Gamma^i{}_{jk}}{\partial \epsilon}\right) X^k + \Gamma^i{}_{jk}\left(\frac{\partial X^k}{\partial \epsilon}\right) \\
&= \left(\frac{\partial X^i}{\partial \epsilon}\right)_{|j} + \delta\Gamma^i{}_{jk} X^k. \qquad (2.25)
\end{aligned}
$$

The generalization of this formula to tensors and tensor densities of any rank is analogous to the corresponding covariant derivative formulas: replacing the vector X^i with a covector Z_i changes the sign of the Christoffel symbol term (with the appropriate index labeling) and higher rank tensors use multiple (index appropriate) copies of the two configurations.

2.4 Euler-Lagrange Expression for Natural Lagrangians

We now proceed with the proof of Theorem 10. To assist in this venture, we define the tensorial quantities

$$
\mathcal{E}^{a_1 a_2 \cdots a_r} = \frac{\partial L}{\partial R_{|(a_1 a_2 \cdots a_r)}}
$$

and, recursively for $k = r-1, \ldots, 1, 0$,

$$
\mathcal{E}^{a_1 a_2 \cdots a_k} = \frac{\partial L}{\partial R_{|(a_1 a_2 \cdots a_k)}} - \mathcal{E}^{a_1 a_2 \cdots a_{k+1}}{}_{|a_{k+1}}\,,
$$

where $k = 0$ denotes $\mathcal{E} = \frac{\partial L}{\partial R} - \mathcal{E}^a{}_{|a}$. For instance,

$$\mathcal{E}^{a_1 a_2 \cdots a_{r-1}} = \frac{\partial L}{\partial R_{|(a_1 a_2 \cdots a_{r-1})}} - \left(\frac{\partial L}{\partial R_{|(a_1 a_2 \cdots a_r)}} \right)_{|a_r},$$
$$\mathcal{E}^{a_1 a_2 \cdots a_{r-2}} = \frac{\partial L}{\partial R_{|(a_1 a_2 \cdots a_{r-2})}} - \left(\frac{\partial L}{\partial R_{|(a_1 a_2 \cdots a_{r-1})}} \right)_{|a_{r-1}} + \left(\frac{\partial L}{\partial R_{|(a_1 a_2 \cdots a_r)}} \right)_{|a_{r-1} a_r},$$

etc., with $\mathcal{E}$ the covariant Euler-Lagrange expression of L with respect to R

$$\mathcal{E} = E_R(L) = \frac{\partial L}{\partial R} - \left(\frac{\partial L}{\partial R_{|a}} \right)_{|a} + \left(\frac{\partial L}{\partial R_{|ab}} \right)_{|ab} + \cdots + (-1)^n \left(\frac{\partial L}{\partial R_{|(a_1 a_2 \cdots a_r)}} \right)_{|a_1 \cdots a_r}. \tag{2.26}$$

The covariant derivatives in the above definitions are well defined as $\mathcal{E}^{a_1 \cdots a_k}$ is a contravariant, rank k tensor for all $0 \leq k \leq r$. Similarly, $\mathcal{E}^{a_1 \cdots a_k}$ is symmetric in all of its indices. We note that $\mathcal{E}^{i_1 \cdots i_k}$ is typically a natural tensor of metric order $2r + 2 - k$.

Proof of Theorem 10. The functional derivative of the action (2.11) is found by computing the variation of $\lambda = \sqrt{g} L$

$$\begin{aligned}
\delta I &= \int_U \frac{\partial \lambda}{\partial \epsilon} \nu \\
&= \int_U \left(\frac{\partial \lambda}{\partial g_{ab}} \frac{\partial g_{ab}}{\partial \epsilon} + \frac{\partial \lambda}{\partial R} \frac{\partial R}{\partial \epsilon} + \cdots + \frac{\partial \lambda}{\partial R_{|(a_1 \cdots a_r)}} \frac{\partial R_{|(a_1 \cdots i_r)}}{\partial \epsilon} \right) \nu \\
&= \int_U \left[\left(\frac{\partial \sqrt{g}}{\partial g_{ab}} L + \sqrt{g} \frac{\partial L}{\partial g_{ab}} \right) h_{ab} + \sqrt{g} \left(\frac{\partial L}{\partial R} \frac{\partial R}{\partial \epsilon} + \frac{\partial L}{\partial R_{|a}} \frac{\partial R_{|a}}{\partial \epsilon} \right.\right. \\
&\qquad \left.\left. + \cdots + \frac{\partial L}{\partial R_{|(a_1 \cdots a_r)}} \frac{\partial R_{|(a_1 \cdots a_r)}}{\partial \epsilon} \right) \right] \nu \\
&= \int_U \sqrt{g} \left[\left(\frac{1}{2} g^{ab} L + \frac{\partial L}{\partial g_{ab}} \right) h_{ab} + \left(\frac{\partial L}{\partial R} \frac{\partial R}{\partial \epsilon} + \frac{\partial L}{\partial R_{|a}} \frac{\partial R_{|a}}{\partial \epsilon} \right.\right. \\
&\qquad \left.\left. + \cdots + \frac{\partial L}{\partial R_{|(a_1 \cdots a_r)}} \frac{\partial R_{|(a_1 \cdots a_r)}}{\partial \epsilon} \right) \right] \nu.
\end{aligned} \tag{2.27}$$

The variation of the derivatives of R can be found by utilizing the commutation relation (2.25). We start with the top order term alone, noting that $\dfrac{\partial L}{\partial R_{|(a_1 \cdots a_r)}} = \mathcal{E}^{a_1 \cdots a_r}$,

$$\mathcal{E}^{a_1 \cdots a_r} \frac{\partial R_{|(a_1 \cdots a_r)}}{\partial \epsilon} = \mathcal{E}^{a_1 \cdots a_r} \left[\left(\frac{\partial R_{|(a_1 \cdots a_{r-1})}}{\partial \epsilon} \right)_{|a_r} - \delta \Gamma^c{}_{a_r a_1} R_{|(c a_2 \cdots a_{r-1})} - \cdots \right.$$

$$
\begin{aligned}
&\qquad - \delta\Gamma^{c}{}_{a_r a_{r-1}} R_{|(ca_1\cdots a_{r-2})}\Big] \\
&= \mathcal{E}^{a_1\cdots a_r}\left[\left(\frac{\partial R_{|(a_1\cdots a_{r-1})}}{\partial\epsilon}\right)_{|a_r} - (r-1)\delta\Gamma^{c}{}_{a_r a_1} R_{|(ca_2\cdots a_{r-1})}\right] \\
&= \left(\mathcal{E}^{a_1\cdots a_r}\frac{\partial R_{|(a_1\cdots a_{r-1})}}{\partial\epsilon}\right)_{|a_r} - \left(\mathcal{E}^{a_1\cdots a_r}\right)_{|a_r}\frac{\partial R_{|(a_1\cdots a_{r-1})}}{\partial\epsilon} \\
&\qquad - (r-1)R_{|(ca_2\cdots a_{r-1})}\mathcal{E}^{a_1\cdots a_r}\delta\Gamma^{c}{}_{a_r a_1}.
\end{aligned}
$$

The first term of this expression is a total (covariant) divergence and will not contribute to the variation of the action δI. We sum this final line with the $r-1$ curvature order term $\dfrac{\partial L}{\partial R_{|(a_1\cdots a_{r-1})}}\dfrac{\partial R_{|(a_1\cdots a_{r-1})}}{\partial\epsilon}$ and apply the commutation relation (2.25)

$$
\begin{aligned}
&\left[\frac{\partial L}{\partial R_{|(a_1\cdots a_{r-1})}} - \left(\mathcal{E}^{a_1\cdots a_r}\right)_{|a_r}\right]\frac{\partial R_{|(a_1\cdots a_{r-1})}}{\partial\epsilon} - (r-1)R_{|(ca_2\cdots a_{r-1})}\mathcal{E}^{a_1\cdots a_r}\delta\Gamma^{c}{}_{a_r a_1} + \mathrm{Div}(\cdots) \\
&\qquad = \mathcal{E}^{a_1\cdots a_{r-1}}\frac{\partial R_{|(a_1\cdots a_{r-1})}}{\partial\epsilon} - (r-1)R_{|(ca_2\cdots a_{r-1})}\mathcal{E}^{a_1\cdots a_r}\delta\Gamma^{c}{}_{a_r a_1} + \mathrm{Div}(\cdots) \\
&\qquad = -\left(\mathcal{E}^{a_1\cdots a_{r-1}}\right)_{|a_{r-1}}\frac{\partial R_{|(a_1\cdots a_{r-2})}}{\partial\epsilon} - (r-2)R_{|(ca_2\cdots a_{r-2})}\mathcal{E}^{a_1\cdots a_{r-1}}\delta\Gamma^{c}{}_{a_{r-1} a_1} \\
&\qquad\qquad - (r-1)R_{|(ca_2\cdots a_{r-1})}\mathcal{E}^{a_1\cdots a_r}\delta\Gamma^{c}{}_{a_r a_1} + \mathrm{Div}(\cdots),
\end{aligned}
$$

where we have collected the total divergences into the single term $\mathrm{Div}(\cdots)$. This pattern repeats for the remaining scalar curvature terms in (2.27), with each term producing a secondary Christoffel variation term via the commutation relation (2.25). The only exception to this is the penultimate term $\dfrac{\partial L}{\partial R_{|a}}\dfrac{\partial R_{|a}}{\partial\epsilon}$, where the trivial commutation of partial and variational derivatives is used. In total, this procedure produces a final result of

$$
\begin{aligned}
&\frac{\partial L}{\partial R}\frac{\partial R}{\partial\epsilon} + \frac{\partial L}{\partial R_{|a}}\frac{\partial R_{|a}}{\partial\epsilon} + \cdots + \frac{\partial L}{\partial R_{|(a_1\cdots a_r)}}\frac{\partial R_{|(a_1\cdots a_r)}}{\partial\epsilon} \\
&\qquad = \mathcal{E}\frac{\partial R}{\partial\epsilon} - \big[R_{|c}\mathcal{E}^{a_1a_2}\delta\Gamma^{c}{}_{a_2a_1} + 2R_{|ca_2}\mathcal{E}^{a_1a_2a_3}\delta\Gamma^{c}{}_{a_3a_1} + 3R_{|(ca_2a_3)}\mathcal{E}^{a_1a_2a_3a_4}\delta\Gamma^{c}{}_{a_4a_1} \\
&\qquad\qquad + \cdots + (r-1)R_{|(ca_2\cdots a_{r-1})}\mathcal{E}^{a_1\cdots a_r}\delta\Gamma^{c}{}_{a_r a_1}\big] + \mathrm{Div}(\cdots) \\
&\qquad = \mathcal{E}\frac{\partial R}{\partial\epsilon} - \big[R_{|c}\mathcal{E}^{a_1a_2} + 2R_{|ca_3}\mathcal{E}^{a_1a_2a_3} + 3R_{|(ca_3a_4)}\mathcal{E}^{a_1a_2a_3a_4} + \cdots \\
&\qquad\qquad + (r-1)R_{|(ca_3\cdots a_r)}\mathcal{E}^{a_1\cdots a_r}\big]\,\delta\Gamma^{c}{}_{a_1a_2} + \mathrm{Div}(\cdots).
\end{aligned}
\tag{2.28}
$$

Defining the tensor $S_c{}^{a_1a_2} = S_c{}^{a_2a_1}$ as the sum in square brackets, we express the variation of the Christoffel symbols in terms of the metric (2.22) and remove the covariant derivatives from h_{ab}

$$
\begin{aligned}
S_c{}^{ab}\delta\Gamma^c{}_{ab} &= \frac{1}{2}S_c{}^{ab}g^{cd}\left(h_{ad|b}+h_{bd|a}-h_{ab|d}\right)\\
&= \frac{1}{2}S^{cab}\left(h_{ac|b}+h_{bc|a}-h_{ab|c}\right)\\
&= \frac{1}{2}\left(S^{bac}+S^{bca}-S^{cab}\right)h_{ab|c}\\
&= \left[\frac{1}{2}\left(2S^{abc}-S^{cab}\right)h_{ab}\right]_{|c} - \frac{1}{2}\left(2S^{abc}-S^{cab}\right)_{|c}h_{ab}.
\end{aligned}
$$

The first term is a total divergence and we move it to the $\mathrm{Div}(\cdots)$ term. Similarly, the first term of (2.28) can be rewritten using (2.24)

$$
\begin{aligned}
\mathcal{E}\frac{\partial R}{\partial\epsilon} = &\left(-\frac{1}{2}g^{ab}R\mathcal{E} + g^{ad}g^{bc}\mathcal{E}_{|cd} - g^{ab}\Box\mathcal{E}\right)h_{ab} + \Big[(-1)^{q+1}\varepsilon^{ac}\varepsilon^{be}\mathcal{E}h_{ab|c}\\
&+(-1)^{q}\varepsilon^{ae}\varepsilon^{bd}\mathcal{E}_{|d}h_{ab}\Big]_{|e}
\end{aligned}
$$

and the second term of this expression is another total divergence. We use these results to simplify (2.28), yielding an expression of the form

$$
\begin{aligned}
&\frac{\partial L}{\partial R}\frac{\partial R}{\partial\epsilon} + \frac{\partial L}{\partial R_{|a}}\frac{\partial R_{|a}}{\partial\epsilon} + \cdots + \frac{\partial L}{\partial R_{|(a_1\cdots a_r)}}\frac{\partial R_{|(a_1\cdots a_r)}}{\partial\epsilon}\\
&\qquad = \left[-\frac{1}{2}g^{ab}R\mathcal{E} + \mathcal{E}^{|ab} - g^{ab}\Box\mathcal{E} + \frac{1}{2}\left(2S^{abc}-S^{cab}\right)_{|c}\right]h_{ab} + \mathrm{Div}(\cdots).
\end{aligned}
$$

We substitute this expression into (2.27) and use the Divergence theorem to convert the integral over the total divergence $\mathrm{Div}(\cdots)$ into a boundary term

$$
\begin{aligned}
\delta I = &\int_U \left\{\sqrt{g}\left[\frac{1}{2}g^{ab}L + \frac{\partial L}{\partial g_{ab}} - \frac{1}{2}g^{ab}R\mathcal{E} + \mathcal{E}^{|ab} - g^{ab}\Box\mathcal{E} + \frac{1}{2}\left(2S^{abc}-S^{cab}\right)_{|c}\right]h_{ab}\right\}\nu\\
&+\int_{\partial U} X \lrcorner \left(\sqrt{g}\nu\right), \qquad (2.29)
\end{aligned}
$$

where $X = X^i\partial_i$ is the vector inside the final divergence term $\mathrm{Div}(\cdots) = \mathrm{Div}(X)$, ∂U is the boundary of U, and $\lrcorner$ is the interior product. If h_{ab} and its first $r+1$ covariant derivatives vanish on the boundary ∂U, then $X = 0$ and the boundary term vanishes identically.

By definition, the coefficient of h_{ab} in (2.29) is the Euler-Lagrange expression of λ with respect to the metric

$$\begin{aligned}\frac{\delta\lambda}{\delta g_{ab}} &= E^{ab}(\lambda)\\ &= \sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)\mathcal{E} - \frac{1}{2}g^{ab}R\mathcal{E} + \frac{1}{2}\left(2S^{(ab)c} - S^{cab}\right)_{|c} + \frac{1}{2}g^{ab}L + \frac{\partial L}{\partial g_{ab}}\right].\end{aligned} \tag{2.30}$$

Typically, $\sqrt{g}\mathcal{E} = E_R(\lambda) = \dfrac{\delta\lambda}{\delta R}$ is of metric order $2r+2$ (doubling the curvature order from r to $2r$). Hence, the first two terms are identical to their counterparts from (1.27) and are of metric orders $2r+4$ and $2r+2$, respectively. By inspection, the final two terms are of metric order $r+2$. Similarly, S^{abc} is given by the bracketed tensor in (2.28), which is dependent on $\mathcal{E}^{ab}$ (metric order $2r$), and so the the derivative of S^{abc} is of metric order $2r+1$. $\square$

2.5 Special Cases

As mentioned in the opening to this chapter, $E^{ab}(\lambda)$ is the Euler-Lagrange expression of a natural Lagrangian which implies that $E^{ab}(\lambda)_{|b}$ vanishes identically by Noether's second theorem (E.13). We can put heavy restrictions on this identity by computing the divergence of the first two terms in (2.30), which simplifies quite nicely

$$\begin{aligned}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)\mathcal{E} - \frac{1}{2}g^{ab}R\mathcal{E}\right]_{|b} &= g^{ad}g^{bc}\mathcal{E}_{|cdb} - g^{ab}g^{cd}\mathcal{E}_{|cdb} - \frac{1}{2}R^{|a}\mathcal{E} - \frac{1}{2}R\mathcal{E}^{|a}\\ &= g^{ab}g^{cd}\left(\mathcal{E}_{|dbc} - \mathcal{E}_{|cdb}\right) - \frac{1}{2}R^{|a}\mathcal{E} - \frac{1}{2}R\mathcal{E}^{|a}\\ &= g^{ab}g^{cd}\left(\mathcal{E}_{|dbc} - \mathcal{E}_{|dcb}\right) - \frac{1}{2}R^{|a}\mathcal{E} - \frac{1}{2}R\mathcal{E}^{|a}\\ &= -g^{ab}g^{cd}R_{d\ \,bc}^{\ e}\mathcal{E}_{|e} - \frac{1}{2}R^{|a}\mathcal{E} - \frac{1}{2}R\mathcal{E}^{|a}\\ &= -\frac{1}{2}g^{ab}g^{cd}R(g_{bd}\delta^e_c - \delta^e_b g_{cd})\mathcal{E}_{|e} - \frac{1}{2}R^{|a}\mathcal{E} - \frac{1}{2}R\mathcal{E}^{|a}\\ &= -\frac{1}{2}R\left(g^{ae} - 2g^{ae}\right)\mathcal{E}_{|e} - \frac{1}{2}R^{|a}\mathcal{E} - \frac{1}{2}R\mathcal{E}^{|a}\end{aligned}$$

$$= \frac{1}{2}R\mathcal{E}^{|a} - \frac{1}{2}R^{|a}\mathcal{E} - \frac{1}{2}R\mathcal{E}^{|a}$$
$$= -\frac{1}{2}R^{|a}E_R(L). \tag{2.31}$$

With this result, the vanishing of $E^{ab}(\lambda)_{|b}$ is ensured by the following equation

$$\left[\frac{1}{2}\left(2\,\mathrm{Sym}_{ab}\,S^{abc} - S^{cab}\right)_{|c} + \frac{1}{2}g^{ab}L + \frac{\partial L}{\partial g_{ab}}\right]_{|b} = \frac{1}{2}R^{|a}E_R(L). \tag{2.32}$$

We now confirm the vanishing of $E^{ab}(\lambda)_{|b}$ for the metric order 0, 2, and 3 cases.

2.5.1 Order 0

For $\lambda = \sqrt{g}L(g_{ab})$, the tensors S^{abc} and $E_R(L)$ are identically zero. Correspondingly, the variation (2.30) is

$$\delta\lambda = \sqrt{g}\left(\frac{1}{2}g^{ab}L + \frac{\partial L}{\partial g_{ab}}\right)h_{ab}.$$

The invariance identity (2.10) in this case implies L is constant and so the variation is

$$\delta\lambda = \frac{c}{2}\sqrt{g}g^{ab}h_{ab} = E^{ab}h_{ab},$$

where $c = L$ is a constant. The divergence of E^{ab} is identically zero as it is only dependent on the metric.

2.5.2 Order 2

For $\lambda = \sqrt{g}L(g_{ab}; R)$, the tensor S^{abc} is identically zero and the variation (2.30) is

$$\delta\lambda = \sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)E_R(L) - \frac{1}{2}g^{ab}RE_R(L) + \frac{1}{2}g^{ab}L + \frac{\partial L}{\partial g_{ab}}\right]h_{ab} + \mathrm{Div}(\cdots).$$

The invariance identity also vanishes in this case, so $\lambda = \sqrt{g}L(R)$ and the variation of λ is

$$\delta\lambda = \sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)L' - \frac{1}{2}g^{ab}RL' + \frac{1}{2}g^{ab}L\right]h_{ab}, \tag{2.33}$$

where $L' = \dfrac{\mathrm{d}L}{\mathrm{d}R} = E_R(L)$. Using (2.31), we see that the divergence of $E^{ab}(\lambda)$ vanishes

$$\begin{aligned} E^{ab}(\lambda)_{|b} &= \left\{\sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)L' - \frac{1}{2}g^{ab}RL' + \frac{1}{2}g^{ab}L\right]\right\}_{|b} \\ &= \sqrt{g}\left(-\frac{1}{2}R^{|a}E_R(L) + \frac{1}{2}L^{|a}\right) \\ &= \sqrt{g}\left(-\frac{1}{2}R^{|a}L' + \frac{1}{2}R^{|a}L'\right) \\ &= 0, \end{aligned}$$

where we have used the identity

$$g^{ab}L_{|b} = g^{ab}L_{,b} = g^{ab}L'R_{,b} = g^{ab}L'R_{|b} = R^{|a}L'$$

to write the second term of the penultimate line.

2.5.3 Order 3

For $\lambda = \sqrt{g}L(g_{ab}; R; R_{|a})$, the tensor S^{abc} is identically zero and the variation (2.30) is

$$\delta\lambda = \sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)E_R(L) - \frac{1}{2}g^{ab}RE_R(L) + \frac{1}{2}g^{ab}L + \frac{\partial L}{\partial g_{ab}}\right]h_{ab}$$

At first order in R the invariance identity (2.10) is

$$\frac{\partial L}{\partial g_{ab}} = -\frac{1}{2}R^{|a}\frac{\partial L}{\partial R_{|b}} = -\frac{1}{2}R^{|a}\mathcal{E}^b,$$

which implies the symmetry $R^{|a}\mathcal{E}^b = R^{|b}\mathcal{E}^a$. We substitute this expression into the variation, yielding

$$\delta\lambda = \sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)E_R(L) - \frac{1}{2}g^{ab}RE_R(L) + \frac{1}{2}g^{ab}L - \frac{1}{2}R^{|a}\mathcal{E}^b\right]h_{ab} = E^{ab}(\lambda)h_{ab}.$$

We now compute the divergence of the Euler-Lagrange expression

$$\begin{aligned}
E^{ab}(\lambda)_{|b} &= \left\{\sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)E_R(L) - \frac{1}{2}g^{ab}RE_R(L) + \frac{1}{2}g^{ab}L - \frac{1}{2}R^{|a}\mathcal{E}^b\right]\right\}_{|b} \\
&= \sqrt{g}\left[-\frac{1}{2}R^{|a}E_R(L) + \frac{1}{2}L^{|a} - \frac{1}{2}R^{|a}{}_{|b}\mathcal{E}^b - \frac{1}{2}R^{|a}\mathcal{E}^b{}_{|b}\right] \\
&= \sqrt{g}\left[-\frac{1}{2}R^{|a}E_R(L) + \frac{1}{2}g^{ab}\left(\frac{\partial L}{\partial g_{ij}}g_{ij,b} + \frac{\partial L}{\partial R}R_{,b} + \frac{\partial L}{\partial R_{|c}}R_{|c,b}\right) - \frac{1}{2}R^{|a}{}_{|b}\mathcal{E}^b\right. \\
&\qquad \left. - \frac{1}{2}R^{|a}\mathcal{E}^b{}_{|b}\right]
\end{aligned}$$

Before proceeding, we recall that the metric compatibility of the connection can be used to express the first derivative of the metric in terms of the Christoffel symbols

$$0 = g_{ab|j} = g_{ab,j} - \Gamma^e{}_{aj}g_{eb} - \Gamma^e{}_{bj}g_{ae} \implies g_{ab,j} = \Gamma^e{}_{aj}g_{be} + \Gamma^e{}_{bj}g_{ae} = 2\,\mathrm{Sym}_{ab}\,\Gamma^e{}_{bj}g_{ae}. \quad (2.34)$$

Using this expression and the invariance identity, we rewrite the first term in parenthesis and finish the computation

$$\begin{aligned}
E^{ab}(\lambda)_{|b} &= \sqrt{g}\left[-\frac{1}{2}R^{|a}E_R(L) - \frac{1}{4}g^{ab}\left(R^{|i}\mathcal{E}^j\right)\left(2\Gamma^c{}_{jb}g_{ic}\right) + \frac{1}{2}\frac{\partial L}{\partial R}R^{|a}\right. \\
&\qquad \left. + \frac{1}{2}g^{ab}\mathcal{E}^c R_{|c,b} - \frac{1}{2}R^{|a}{}_{|b}\mathcal{E}^b - \frac{1}{2}R^{|a}\mathcal{E}^b{}_{|b}\right] \\
&= \sqrt{g}\left[-\frac{1}{2}R^{|a}E_R(L) - \frac{1}{2}g^{ab}R_{|c}\mathcal{E}^j\Gamma^c{}_{jb} + \frac{1}{2}R^{|a}\left(\frac{\partial L}{\partial R} - \mathcal{E}^b{}_{|b}\right) + \frac{1}{2}g^{ab}\mathcal{E}^c R_{|c,b}\right. \\
&\qquad \left. - \frac{1}{2}R^{|a}{}_{|b}\mathcal{E}^b\right] \\
&= \sqrt{g}\left[-\frac{1}{2}R^{|a}E_R(L) + \frac{1}{2}R^{|a}E_R(L) + \frac{1}{2}g^{ab}\mathcal{E}^c\left(R_{|c,b} - R_{|d}\Gamma^d{}_{cb}\right) - \frac{1}{2}R^{|a}{}_{|b}\mathcal{E}^b\right] \\
&= \sqrt{g}\left(\frac{1}{2}\mathcal{E}^c R_{|c}{}^{|a} - \frac{1}{2}R^{|a}{}_{|b}\mathcal{E}^b\right) \\
&= 0,
\end{aligned}$$

where we used the symmetry of the first two covariant derivatives of a scalar to arrive at the final line.

2.5.4 Degenerate Lagrangians

For a natural Lagrangian λ of metric order $r+2$, the corresponding Euler-Lagrange expression typically has metric order $2r+4$. Lagrangians for which the Euler-Lagrange expression possesses metric order lower than this value are said to be degenerate. As an example of this phenomenon, we consider the natural Lagrangian presented in the introduction (1.16)

$$\lambda^\star = \sqrt{g}L^\star = \sqrt{g}\varepsilon^{ac}\varepsilon^{bd}R_{|ab}R_{|cd}P = \sqrt{g}(-1)^q\left(R^{|a}{}_{|a}R^{|b}{}_{|b} - R^{|ab}R_{|ab}\right)P,$$

where $P = P(R)$ is a scalar. For future reference, the scalar curvature derivatives of $L^\star$ are given by

$$\frac{\partial L^\star}{\partial R} = \varepsilon^{ac}\varepsilon^{bd}R_{|ab}R_{|cd}P', \tag{2.35}$$

where the prime denotes a derivative with respect to R, and

$$\frac{\partial L^\star}{\partial R_{|ij}} = \varepsilon^{ai}\varepsilon^{bj}R_{|ab}P + \varepsilon^{ic}\varepsilon^{jd}R_{|cd}P = 2\varepsilon^{ia}\varepsilon^{jb}R_{|ab}P. \tag{2.36}$$

We note that $\lambda^\star$ is manifestly of metric order 4 and claim that the Euler-Lagrange expression for $\lambda^\star$ is of metric order 6, i.e., $\lambda^\star$ is degenerate. To prove this claim, we check the metric order of each term in the Euler-Lagrange expression (2.30).

By definition, the term $\frac{1}{2}g^{ab}L^\star$ is of metric order 4. Similarly, the derivative with respect to the metric is given by the invariance identity for $L^\star$ (2.10) and is fourth order in the metric

$$\begin{aligned}
\frac{\partial L^\star}{\partial g_{ij}} &= -\frac{1}{2}R^{|i}\frac{\partial L^\star}{\partial R_{|j}} - g^{ik}R_{|kl}\frac{\partial L^\star}{\partial R_{|jl}} \\
&= 0 - 2g^{ik}R_{|kl}\varepsilon^{ja}\varepsilon^{lb}R_{|ab}P \\
&= -2\varepsilon^{ja}\varepsilon^{lb}R_{|ab}R^{|i}{}_{|l}P \\
&= -2(-1)^q g^{ik}R_{kl}(g^{jl}g^{ab} - g^{jb}g^{la})R_{|ab}P \\
&= -2(-1)^q\left(R^{|ij}R^{|l}{}_{|l} - R^{|i}{}_{|l}R^{|jl}\right)P.
\end{aligned} \tag{2.37}$$

Next, the S^{abc} tensors are defined in (2.28) and take the form

$$S^{abc} = R^{|a}\mathcal{E}^{bc} = R^{|a}\frac{\partial L^\star}{\partial R_{|bc}} = R^{|a}\left(2\varepsilon^{bd}\varepsilon^{ce}R_{|de}P\right) = 2(-1)^q R^{|a}\left(g^{bc}R^{|d}{}_{|d} - R^{|bc}\right)P. \quad (2.38)$$

We now show that the term involving these tensors from the Euler-Lagrange expression of $\lambda^\star$ (2.30) is fifth order in the metric by a direct calculation

$$\begin{aligned}
&\frac{1}{2}\left(2\,\mathrm{Sym}_{ab}\,S^{bac} - S^{cab}\right)_{|c} \\
&\quad= \frac{1}{2}\Big[2(-1)^q R^{|b}\left(g^{ac}R^{|d}{}_{|d} - R^{|ac}\right)P + 2(-1)^q R^{|a}\left(g^{bc}R^{|d}{}_{|d} - R^{|bc}\right)P \\
&\qquad - 2(-1)^q R^{|c}\left(g^{ab}R^{|d}{}_{|d} - R^{|ab}\right)P\Big]_{|c} \\
&\quad= (-1)^q\left[g^{bf}\left(g^{ac}g^{de} - g^{ad}g^{ce}\right) + g^{af}\left(g^{bc}g^{de} - g^{bd}g^{ce}\right) - g^{cf}\left(g^{ab}g^{de} - g^{ad}g^{be}\right)\right] \\
&\qquad \times \left(R_{|f}R_{|de}P\right)_{|c} \\
&\quad= (-1)^q\left[g^{bf}\left(g^{ac}g^{de} - g^{ad}g^{ce}\right) + g^{af}\left(g^{bc}g^{de} - g^{bd}g^{ce}\right) - g^{cf}\left(g^{ab}g^{de} - g^{ad}g^{be}\right)\right] \\
&\qquad \times \left\{R_{|fc}R_{|de}P + R_{|f}\left[R_{|(dec)} + \frac{1}{3}\,\mathrm{Sym}_{de}\,R\left(g_{cd}R_{|e} - g_{de}R_{|c}\right)\right]P + R_{|f}R_{|de}R_{|c}P'\right\} \\
&\quad= (-1)^q\Big[\Big(R^{|d}{}_{|d}R^{|ab} - R^{|a}{}_{|e}R^{|be} + R^{|d}{}_{|d}R^{|ab} - R^{|a}{}_{|e}R^{|be} - g^{ab}R^{|d}{}_{|d}R^{|c}{}_{|c} + R^{|c}{}_{|c}R^{|ab}\Big)P \\
&\qquad + \left(0 + 0 - g^{ab}g^{de}R^{|c}R_{|(dec)} + g^{ad}g^{be}R^{|c}R_{|(dec)}\right)P + \frac{1}{6}R\Big(R^{|a}R^{|b} - R^{|a}R^{|b} + R^{|a}R^{|b} \\
&\qquad - R^{|a}R^{|b} - Sg^{ab} + R^{|a}R^{|b} + R^{|a}R^{|b} - 2R^{|a}R^{|b} + R^{|a}R^{|b} - 2R^{|a}R^{|b} - Sg^{ab} \\
&\qquad + R^{|a}R^{|b} - 2R^{|a}R^{|b} + 4R^{|a}R^{|b} - 2R^{|a}R^{|b} + 4R^{|a}R^{|b} + 4Sg^{ab} - 2Sg^{ab}\Big)P \\
&\qquad + \left(R^{|a}R^{|b}R^{|d}{}_{|d} - R^{|b}R_{|c}R^{|ac} + R^{|a}R^{|b}R^{|d}{}_{|d} - R^{|a}R_{|c}R^{|bc} - Sg^{ab}R^{|d}{}_{|d} + SR^{|ab}\right)P'\Big] \\
&\quad= (-1)^q\Big(g^{ac}g^{be}R^{|d}R_{|(ced)}P - g^{ab}g^{ce}R^{|d}R_{|(ced)}P + 3R^{|c}{}_{|c}R^{|ab}P - 2R^{|a}{}_{|e}R^{|be}P \\
&\qquad - g^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P + 2R^{|a}R^{|b}R^{|c}{}_{|c}P' - R^{|b}R_{|c}R^{|ac}P' - R^{|a}R_{|c}R^{|bc}P' - Sg^{ab}R^{|c}{}_{|c}P' \\
&\qquad + SR^{|ab}P' - \frac{2}{3}RR^{|a}R^{|b}P\Big). \qquad (2.39)
\end{aligned}$$

The two leading terms in this expression are of particular interest

$$(-1)^q\left(g^{ac}g^{be}R^{|d}R_{|(ced)}P - g^{ab}g^{ce}R^{|d}R_{|(ced)}P\right) = -\epsilon^{ae}\epsilon^{bc}R^{|d}R_{|(cde)}P$$

and, as a fifth order term not associated with the Euler-Lagrange expression of $L^\star$ through the curvature $E_R(L^\star) = \dfrac{\delta L^\star}{\delta R}$, is the discrepancy mentioned in the introduction (1.17).

For the two remaining terms, we compute the Euler-Lagrange expression of $L^\star$ with respect to R

$$
\begin{aligned}
E_R(L^\star) &= \frac{\partial L^\star}{\partial R} - \left(\frac{\partial L^\star}{\partial R_{|i}}\right)_{|i} + \left(\frac{\partial L^\star}{\partial R_{|ij}}\right)_{|ij} \\
&= \varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|cd} P' + \Big(2\varepsilon^{ia}\varepsilon^{jb} R_{|ab} P\Big)_{|ij} \\
&= \varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|cd} P' + 2\varepsilon^{ia}\varepsilon^{jb} R_{|abij} P + 4\varepsilon^{ia}\varepsilon^{jb} R_{|abi} P_{|j} + 2\varepsilon^{ia}\varepsilon^{jb} R_{|ab} P_{|ij} \\
&= \varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|cd} P' + 2\varepsilon^{ia}\varepsilon^{jb} \left[R_{|(abi)} + \frac{1}{3} R \,\mathrm{Sym}_{ab}\left(g_{ia} R_{|b} - g_{ab} R_{|i}\right)\right]_{|j} P \\
&\quad + 4\varepsilon^{ia}\varepsilon^{jb} \left[R_{|(abi)} + \frac{1}{3} R \,\mathrm{Sym}_{ab}\left(g_{ia} R_{|b} - g_{ab} R_{|i}\right)\right] R_{|j} P' \\
&\quad + 2\varepsilon^{ia}\varepsilon^{jb} R_{|ab}\left(R_{|ij} P' + R_{|i} R_{|j} P''\right) \\
&= 3\varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|cd} P' + 2\varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|c} R_{|d} P'' \\
&\quad + 2\varepsilon^{ia}\varepsilon^{jb} \left[0 + \frac{1}{6} R_{|j}\left(0 + g_{ib} R_{|a} - 2 g_{ab} R_{|i}\right) + \frac{1}{6} R\left(0 + g_{ib} R_{|aj} - 2 g_{ab} R_{|ij}\right)\right] P \\
&\quad + 4\varepsilon^{ia}\varepsilon^{jb} \left[0 + \frac{1}{6} R\left(0 + g_{ib} R_{|a} - 2 g_{ab} R_{|i}\right)\right] R_{|j} P' \\
&= 3\varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|cd} P' + 2\varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|c} R_{|d} P'' \\
&\quad - g_{ab}\varepsilon^{ia}\varepsilon^{jb}\left(R_i R_j + R R_{ij}\right) P - 2 g_{ab}\varepsilon^{ia}\varepsilon^{jb} R R_i R_j P' \\
&= 3\varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|cd} P' + 2\varepsilon^{ac}\varepsilon^{bd} R_{|ab} R_{|c} R_{|d} P'' - g_{ab}\varepsilon^{ac}\varepsilon^{bd}\left(R_{|c} R_{|d} + R R_{|cd}\right) P \\
&\quad - 2 g_{ab}\varepsilon^{ac}\varepsilon^{bd} R R_{|c} R_{|d} P' \\
&= 3(-1)^q (g^{ab} g^{cd} - g^{ad} g^{bc}) R_{|ab} R_{|cd} P' + 2(-1)^q (g^{ab} g^{cd} - g^{ad} g^{bc}) R_{|ab} R_{|c} R_{|d} P'' \\
&\quad - (-1)^q g^{cd}\left(R_{|c} R_{|d} + R R_{|cd}\right) P - 2(-1)^q g^{cd} R R_{|c} R_{|d} P' \\
&= (-1)^q \Big(3 R^{|a}{}_{|a} R^{|b}{}_{|b} P' - 3 R^{|ab} R_{|ab} P' + 2 S R^{|a}{}_{|a} P'' - 2 R^{|a} R^{|b} R_{|ab} P'' - S P \\
&\quad - R R^{|a}{}_{|a} P - 2 R S P'\Big),
\end{aligned}
\tag{2.40}
$$

where, as in the intro, $S = g^{ab} R_{|a} R_{|b}$. Hence, the term $\frac{1}{2} g^{ab} R E_R(L^\star)$ is of metric order 4. We note that $E_R(L^\star)$ is of lower order than the term involving the S^{abc} tensors (2.39), showing a different hierarchy for these terms than the typical orders presented in Theorem 10.

Finally, we find the metric order of the last term involving the double covariant derivative of $E_R(L^\star)$. As we seek the form of the highest order term in this expression, we only consider the fourth order terms in $E_R(L^\star)$ when taking these derivatives and only consider those terms for which both covariant derivatives act on the same $R_{|ab}$ tensor. We find this expression to be of metric order 6 after the short calculation below

$$\begin{aligned}
E_R(L^\star)_{|ij} &= (-1)^q \Big(3R^{|a}{}_{|a}R^{|b}{}_{|b}P' - 3R^{|ab}R_{|ab}P' + 2SR^{|a}{}_{|a}P'' - 2R^{|a}R^{|b}R_{|ab}P'' - SP \\
&\qquad - RR^{|a}{}_{|a}P - 2RSP'\Big)_{|ij} \\
&= (-1)^q \Big(6R^{|a}{}_{|aij}R^{|b}{}_{|b}P' - 6R^{ab}R_{|abij}P' + 2SR^{|a}{}_{|aij}P'' - 2R^aR^bR_{|abij}P'' \\
&\qquad - RR^{|a}{}_{|aij}P + \dots\Big) \\
&= (-1)^q R_{|abij}\Big[6g^{ab}R^{|c}{}_{|c}P' - 6R^{|ab}P' + 2Sg^{ab}P'' - 2R^{|a}R^{|b}P'' - Rg^{ab}P\Big] + \dots \\
&= (-1)^q R_{|(abij)}\Big[6g^{ab}R^{|c}{}_{|c}P' - 6R^{|ab}P' + 2Sg^{ab}P'' - 2R^{|a}R^{|b}P'' - Rg^{ab}P\Big] + \dots,
\end{aligned} \tag{2.41}$$

where the ellipsis denotes those terms which do not contain $R_{|abcd}$ in any fashion. We note that both the third (A.1) and fourth (A.2) scalar order symmetrization formulas have been used to arrive at the final line, for $R_{|abi} \to R_{|abi}$ and $R_{|(abi)j} \to R_{|(abij)}$, respectively. (The additional terms arising from symmetrization are of lower order and so do not appear in this top view.)

$$\begin{aligned}
E_R(L^*)_{|ij} = \varepsilon^{ac}\varepsilon^{bd}\big[&R_{|abij}\left(6R_{|cd}P' + 2R_{|c}R_{|d}P'' - g_{cd}RP\right) \\
&+ 6R_{|abi}R_{|cdj}P' - 2\,\mathrm{Sym}_{ij}\, g_{ab}R_{|i}R_{|cdj}\left(P + RP'\right) \\
&+ 2\,\mathrm{Sym}_{ij}\, R_{|abi}\left(6R_{|cd}R_{|j}P'' + 4R_{|cj}R_{|d}P'' + 2R_{|c}R_{|d}R_{|j}P'''\right) \\
&+ R_{|cij}\left(4R_{|ab}R_{|d}P'' - 2g_{ab}R_{|d}P - 4g_{ab}RR_{|d}P'\right) + \cdots\big]
\end{aligned}$$

Taken together, the preceding calculations show the Euler-Lagrange expression for $\lambda^\star$ to be of metric order 6 (in opposition to the expected metric order of 8 from a fourth order Lagrangian). Using (2.30) and (2.41), we see that, at top order, the Euler-Lagrange expression is given by

$$\begin{aligned}
E^{ab}(\lambda^\star) &= \sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)E_R(L^\star) - \frac{1}{2}g^{ab}RE_R(L^\star) + \frac{1}{2}\left(2S^{(ba)c} - S^{cab}\right)_{|c}\right. \\
&\qquad \left. + \frac{1}{2}g^{ab}L^\star + \frac{\partial L^\star}{\partial g_{ab}}\right] \\
&= \sqrt{g}\left[(g^{ai}g^{bj} - g^{ab}g^{ij})E_R(L^\star)_{|ij} + \dots\right] \\
&= \sqrt{g}\left\{(-1)^q(g^{ai}g^{bj} - g^{ab}g^{ij})R_{|(ijkl)}\left[6g^{kl}R^{|m}{}_{|m}P' - 6R^{|kl}P' + 2Sg^{kl}P''\right.\right. \\
&\qquad \left.\left. - 2R^{|k}R^{|l}P'' - Rg^{kl}P\right] + \dots\right\} \\
&= \sqrt{g}\left[\varepsilon^{ai}\varepsilon^{bj}R_{|(ijkl)}\left(6g^{kl}R^{|m}{}_{|m}P' - 6R^{|kl}P' + 2Sg^{kl}P'' - 2R^{|k}R^{|l}P'' - Rg^{kl}P\right)\right. \\
&\qquad \left. + \dots\right].
\end{aligned}$$

Due to space constraints, complete derivations of $E_R(L^\star)_{|ij}$ and $E^{ab}(\lambda^\star)$ are available in Appendix B.

As mentioned in the introduction, this complicates the work of Chapter 4, where we wish to identify all divergence-free tensor densities for low orders, as degenerate Lagrangians produce extra terms that are of higher order than would be expected from a naive application of Theorem 10. For example, in the case presented above, we see that the coefficient of $R_{|(ijkl)}$ contains a number of terms proportional to $R_{|ij}$ which would not be present in the Euler-Lagrange expression of a Lagrangian of metric order 3. [The highest order term in $E_R(L)$ for a third order Lagrangian is proportional to $R_{|a}R_{|bc}$, which produces a term proportional to $R_{|a}R_{|(bcde)}$ for $E_R(L)_{|ij}$.]

CHAPTER 3

DIVERGENCE-FREE TENSOR DENSITIES: GENERAL CASE

This chapter is devoted to proving Theorem 11. We begin by establishing an invariance identity for symmetric, contravariant, rank 2 tensor densities and use it to compute the covariant divergence. A number of preliminary results, which also aid the proof of Theorem 12 in Chapter 4, are established before completing the proof of Theorem 11. Finally, we explore the similarities between Theorem 11 and Theorem 10 of Chapter 2.

3.1 Tensor Density Invariance Identity

If A^{ij} is a symmetric, natural tensor density dependent on the metric, a scalar field φ, and symmetrized covariant derivatives of the scalar field up to order r, i.e., $A^{ij} = A^{ij}(g_{ab}; \varphi; \varphi_a; \ldots; \varphi_{a_1 \cdots a_r})$, then A^{ij} obeys the invariance identity

$$\begin{aligned}\frac{\partial A^{ij}}{\partial g_{ab}} = \frac{1}{2}\left(g^{ab}A^{ij} - g^{ai}A^{bj} - g^{aj}A^{ib}\right) - \frac{1}{2}\varphi^a A^{ij;b} - g^{al}\varphi_{lc}A^{ij;bc} - \frac{3}{2}g^{al}\varphi_{lcd}A^{ij;bcd} \\ - \cdots - \frac{r}{2}g^{aa_1}\varphi_{a_1\cdots a_r}A^{ij;ba_2\cdots a_r}.\end{aligned} \tag{3.1}$$

While this equation is manifestly symmetric in the indices ij, we note that the symmetry of the metric tensor implies the right side is also symmetric in ab.

We derive this identity in what follows. Under a coordinate transformation (2.2), A^{ij} obeys a version of the transformation law (2.4). We rearrange this equation to

$$J^a_i J^b_j \bar{A}^{ij}\left(\bar{g}_{kl}; \bar{\varphi}; \bar{\varphi}_k; \ldots; \bar{\varphi}_{k_1\cdots k_r}\right) = J A^{ab}(g_{cd}; \varphi; \varphi_c; \ldots; \varphi_{c_1\cdots c_r}), \tag{3.2}$$

where $J^a_i = \frac{\partial x^a}{\partial \bar{x}^i}$. We adopt the shorthand notation

$$A^{ij;ab\cdots} = \frac{\partial A^{ij}}{\partial \varphi_{ab\cdots}}, \qquad A^{ij;ab\cdots;\alpha\beta\cdots} = \frac{\partial A^{ij;ab}}{\partial \varphi_{\alpha\beta\cdots}} = \frac{\partial^2 A^{ij}}{\partial \varphi_{\alpha\beta\cdots}\partial \varphi_{ab\cdots}},$$

etc., for derivatives of A^{ij} with respect to the scalar field and its derivatives.

We suppress the arguments of A^{ij} and, using (2.13), explicitly show the transformed arguments of $\bar{A}^{ab}$ in (3.2)

$$J_i^a J_j^b \bar{A}^{ij}\left(J_k^c J_l^d g_{cd}; \varphi; J_k^c \varphi_c; J_k^c J_l^d \varphi_{cd}; J_k^c J_l^d J_m^e \varphi_{cde}; \ldots; J_{k_1}^{c_1} \cdots J_{k_r}^{c_r} \varphi_{c_1 \cdots c_r}\right) = J A^{ab}.$$

As with the scalar case, if we differentiate this equation with respect to g_{ab}, φ, or $\varphi_{a_1 \cdots a_k}$ for $1 \leq k \leq r$, then we see that the corresponding derivatives of A^{ij} are tensorial. We differentiate this equation with respect to J_n^f, yielding

$$\begin{aligned}
&\left(\delta_i^n \delta_f^a J_j^b + J_i^a \delta_j^n \delta_f^b\right) \bar{A}^{ij} + J_i^a J_j^b \left(\delta_k^n \delta_f^c J_l^d + J_k^c \delta_l^m \delta_f^d\right) g_{cd} \frac{\partial \bar{A}^{ij}}{\partial g_{kl}} \\
&+ J_i^a J_j^b \left(\delta_k^n \delta_f^c\right) \varphi_c \bar{A}^{ij;k} + J_i^a J_j^b \left(\delta_k^n \delta_f^c J_l^d + J_k^c \delta_l^n \delta_f^d\right) \varphi_{cd} \bar{A}^{ij;kl} \\
&+ J_i^a J_j^b \left(\delta_k^n \delta_f^c J_l^d J_m^e + J_k^c \delta_l^n \delta_f^d J_m^e + J_k^c J_l^d \delta_m^n \delta_f^e\right) \varphi_{cde} \bar{A}^{ij;klm} + \cdots = J K_f^n A^{ab},
\end{aligned}$$

where K_f^n is defined via $J_a^b K_b^c = \delta_a^c$ and hence $\frac{\partial J}{\partial J_n^f} = J K_f^n$. Since this equation holds for all transformations (2.2), it holds for the identity transformation $\bar{x}^i = x^i$ with $\bar{A}^{ij} = A^{ij}$, $J = 1$, and $J_a^b = \delta_a^b = K_a^b$. Substituting this transformation into the equation above, we get the identity

$$\begin{aligned}
&\left(\delta_f^a A^{nb} + \delta_f^b A^{an}\right) + \left(g_{fl} \frac{\partial A^{ab}}{\partial g_{nl}} + g_{kf} \frac{\partial A^{ab}}{\partial g_{km}}\right) + \varphi_f A^{ab;n} + \left(\varphi_{fl} A^{ij;nl} + \varphi_{kf} A^{ij;kn}\right) \\
&\qquad + \left(\varphi_{flm} A^{ab;nlm} + \varphi_{kfm} A^{ab;knm} + \varphi_{klf} A^{ab;kln}\right) + \cdots = \delta_f^n A^{ab} \\
&\delta_f^a A^{nb} + \delta_f^b A^{an} + 2 g_{fl} \frac{\partial A^{ab}}{\partial g_{nl}} + \varphi_f A^{ab;n} + 2\varphi_{fl} A^{ij;nl} + 3\varphi_{flm} A^{ab;nlm} + \cdots = \delta_f^n A^{ab}.
\end{aligned}$$

We contract this equation with g^{cf} and solve for $\frac{\partial A^{ij}}{\partial g_{ab}}$, yielding the invariance identity (3.1) given previously.

3.2 Covariant Divergence of Tensor Densities

If A^{ij} is a symmetric, divergence-free, natural tensor density of order r in φ, then it satisfies the (fully symmetrized) divergence-free condition

$$\begin{aligned}0 = A^{ij;a_1\cdots a_r}&\left(\varphi_{a_1\cdots a_r j} + b_{ja_1}\varphi_{a_2\cdots a_r} - c_{a_1a_2}\varphi_{a_3\cdots a_r j} + \cdots + b_{a_1\cdots a_{r-1}}\varphi_{a_r} - c_{a_1\cdots a_r}\varphi_j\right)\\ &+ A^{ij;a_1\cdots a_{r-1}}\left(\varphi_{a_1\cdots a_{r-1}j} + d_{ja_1}\varphi_{a_2\cdots a_{r-1}} - e_{a_1a_2}\varphi_{a_3\cdots a_{r-1}j} + \cdots + d_{a_1\cdots a_{r-2}}\varphi_{a_{r-1}}\right.\\ &\left.- e_{a_1\cdots a_{r-1}}\varphi_j\right) + A^{ij;a_1\cdots a_{r-2}}\left(\varphi_{a_1\cdots a_{r-2}j} + k_{ja_1}\varphi_{a_2\cdots a_{r-2}} - l_{a_1a_2}\varphi_{a_3\cdots a_{r-2}j}\right.\\ &\left.+ \cdots + k_{a_1\cdots a_{r-3}}\varphi_{a_{r-2}} - l_{a_1\cdots a_{r-2}}\varphi_j\right) + \cdots + A^{ij;a_1a_2}\left(\varphi_{a_1a_2j} + y_{ja_1}\varphi_{a_2} - z_{a_1a_2}\varphi_j\right)\\ &+ A^{ij;a_1}\varphi_{a_1j} + \frac{\partial A^{ij}}{\partial\varphi}\varphi_j,\end{aligned} \tag{3.3}$$

where the coefficients $b_{a_1\cdots a_s}$, $c_{a_1\cdots a_t}$, $d_{a_1\cdots a_u}$, etc., are tensors dependent on the metric, scalar curvature, and symmetrized covariant derivatives of the scalar curvature. While (3.3) only contains covariant derivatives of the scalar curvature which are symmetrized, it is not trivial to find the coefficient tensors b, c, d, etc., for higher scalar order A^{ij} (see Appendix A for details).

We derive the divergence-free condition by starting with the covariant derivative of A^{ij}, which is given by an appropriate modification of (2.6)

$$\begin{aligned}A^{ij}{}_{|l} &= A^{ij}{}_{,l} + \Gamma^i{}_{kl}A^{kj} + \Gamma^j{}_{kl}A^{ik} - \Gamma^k{}_{kl}A^{ij}\\ &= \frac{\partial A^{ij}}{\partial g_{ab}}g_{ab,l} + A^{ij;a_1\cdots a_r}\varphi_{a_1\cdots a_r,l} + \cdots + A^{ij;a}\varphi_{a,l} + \frac{\partial A^{ij}}{\partial\varphi}\varphi_{,l} + \Gamma^i{}_{kl}A^{kj} + \Gamma^j{}_{kl}A^{ik}\\ &\quad - \Gamma^k{}_{kl}A^{ij}.\end{aligned} \tag{3.4}$$

We desire a manifestly covariant version of this expression. To this end, we replace coordinate partial derivatives by covariant derivatives with the appropriate Christoffel symbol correction terms

$$\begin{aligned}A^{ij;a_1\cdots a_k}\varphi_{a_1\cdots a_k,j} &= A^{ij;a_1\cdots a_k}\left(\varphi_{a_1\cdots a_k|j} + \Gamma^e{}_{a_1j}\varphi_{ea_2\cdots a_k} + \cdots + \Gamma^e{}_{a_kj}\varphi_{a_1\cdots a_{k-1}e}\right)\\ &= A^{ij;a_1\cdots a_k}\left(\varphi_{a_1\cdots a_k|j} + k\Gamma^e{}_{a_1j}\varphi_{ea_2\cdots a_k}\right)\end{aligned}$$

and replace the partial derivative of the metric with a Christoffel symbol using the formula (2.34). We perform these substitutions in (3.4) to get

$$
\begin{aligned}
A^{ij}{}_{|l} = {} & \frac{\partial A^{ij}}{\partial g_{ab}} \left(2\Gamma^{e}{}_{bl} g_{ae}\right) + A^{ij;a_1\cdots a_r} \left(\varphi_{a_1\cdots a_r|l} + r\Gamma^{e}{}_{a_1 l}\varphi_{ea_2\cdots a_r}\right) + \cdots \\
& + A^{ij;ab}\left(\varphi_{ab|l} + 2\Gamma^{e}{}_{al}\varphi_{eb}\right) + A^{ij;a}\left(\varphi_{a|l} + \Gamma^{e}{}_{al}\varphi_{e}\right) + \frac{\partial A^{ij}}{\partial \varphi}\varphi_l + \Gamma^{i}{}_{kl}A^{kj} \\
& + \Gamma^{j}{}_{kl}A^{ik} - \Gamma^{k}{}_{kl}A^{ij}.
\end{aligned}
$$

Finally, we use the invariance identity (3.1) to replace the partial derivative of A^{ij} with respect to the metric and simplify the resulting expression to arrive at a manifestly tensorial version of (3.4)

$$
\begin{aligned}
A^{ij}{}_{|l} = {} & \left[\frac{1}{2}\left(g^{ab}A^{ij} - g^{ai}A^{bj} - g^{aj}A^{bi}\right) - \frac{1}{2}\varphi^{a}A^{ij;b} - g^{al}\varphi_{lc}A^{ij;bc} - \cdots \right. \\
& \left. - \frac{r}{2} g^{aa_1}\varphi_{a_1\cdots a_r}A^{ij;ba_2\cdots a_r}\right]\left(2\Gamma^{e}{}_{bl}g_{ae}\right) + A^{ij;a_1\cdots a_r}\left(\varphi_{a_1\cdots a_r|l} + r\Gamma^{e}{}_{a_1 l}\varphi_{ea_2\cdots a_r}\right) \\
& + \cdots + A^{ij;ab}\left(\varphi_{ab|l} + 2\Gamma^{e}{}_{al}\varphi_{eb}\right) + A^{ij;a}\left(\varphi_{a|l} + \Gamma^{e}{}_{al}\varphi_{e}\right) + \frac{\partial A^{ij}}{\partial \varphi}\varphi_l + \Gamma^{i}{}_{kl}A^{kj} \\
& + \Gamma^{j}{}_{kl}A^{ik} - \Gamma^{k}{}_{kl}A^{ij} \\
= {} & \Big(\Gamma^{e}{}_{el}A^{ij} - \Gamma^{i}{}_{bl}A^{bj} - \Gamma^{j}{}_{bl}A^{bi} - \Gamma^{e}{}_{bl}\varphi_{e}A^{ij;b} - 2\Gamma^{e}{}_{bl}\varphi_{ec}A^{ij;bc} - \cdots \\
& - r\Gamma^{e}{}_{bl}\varphi_{ea_2\cdots a_r}A^{ij;ba_2\cdots a_r}\Big) + A^{ij;a_1\cdots a_r}\left(\varphi_{a_1\cdots a_r|l} + r\Gamma^{e}{}_{a_1 l}\varphi_{ea_2\cdots a_r}\right) \\
& + \cdots + A^{ij;ab}\left(\varphi_{ab|l} + 2\Gamma^{e}{}_{al}\varphi_{eb}\right) + A^{ij;a}\left(\varphi_{a|l} + \Gamma^{e}{}_{al}\varphi_{e}\right) + \frac{\partial A^{ij}}{\partial \varphi}\varphi_l + \Gamma^{i}{}_{kl}A^{kj} \\
& + \Gamma^{j}{}_{kl}A^{ik} - \Gamma^{k}{}_{kl}A^{ij} \\
= {} & A^{ij;a_1\cdots a_r}\varphi_{a_1\cdots a_r|l} + A^{ij;a_1\cdots a_{r-1}}\varphi_{a_1\cdots a_{r-1}|l} + \cdots + A^{ij;ab}\varphi_{ab|l} + A^{ij;a}\varphi_{al} + \frac{\partial A^{ij}}{\partial \varphi}\varphi_l.
\end{aligned}
\tag{3.5}
$$

If we require A^{ij} to be divergence-free, then changing the index l to j causes the left side of (3.5) to vanish, leaving

$$
0 = A^{ij;a_1\cdots a_r}\varphi_{a_1\cdots a_r|j} + A^{ij;a_1\cdots a_{r-1}}\varphi_{a_1\cdots a_{r-1}|j} + \cdots + A^{ij;ab}\varphi_{ab|j} + A^{ij;a}\varphi_{aj} + \frac{\partial A^{ij}}{\partial \varphi}\varphi_j. \tag{3.6}
$$

Using the symmetrization formulas from Appendix A, each term of the form $\varphi_{a_1\cdots a_k|j}$ in (3.6) may be replaced by a fully symmetrized covariant derivative term $\varphi_{a_1\cdots a_k j}$ and a collection of lower order terms of all scalar orders from 1 to $k-1$ in φ. Additionally, the lower order terms produced by the symmetrization process are linear in derivatives of φ (A.5). We substitute this linear expression into (3.6), yielding the fully symmetrized divergence-free condition given above (3.3).

3.3 Divergence-free Tensor Densities: Preliminary Results

We now build towards the proof of Theorem 11. First, we require the following general result about cyclic tensors and tensor densities in two dimensions.

Lemma 1. *Let $T^{ijI_1I_2}$ be a tensor (density) on a two dimensional manifold M, where I_1 and I_2 are symmetric multi-indices with $|I_1| = |I_2| = k > 0$ and $T^{ijI_1I_2} = T^{jiI_1I_2} = T^{ijI_2I_1}$. If $T^{i(jI_1)I_2} = 0$, then $T^{ijI_1I_2} = 0$.*

Proof. The result follows via a dimensional argument with $\dim M = 2$. Using covectors $X = X_i\,\mathrm{d}x^i$, $Y = Y_j\,\mathrm{d}x^j$, and $Z = Z_k\,\mathrm{d}x^k$ on M, we may express the symmetries of $T^{ijI_1I_2}$ using the notation

$$T\left(X, X; Y^{(k)}; Z^{(k)}\right) = T^{ijI_1I_2} X_i X_j Y_{I_1} Z_{I_2}, \tag{3.7}$$

where $Y^{(k)}$ denotes k copies of Y and Y_{I_1} is a collection of k copies of Y_i contracted against the index I_1 in $T^{ijI_1I_2}$. In this notation, the symmetry condition $T^{i(jI_1)I_2} = 0$ may be represented by

$$T\left(X, Y; Y^{(k)}; Z^{(k)}\right) = T\left(X, Y; Z^{(k)}; Y^{(k)}\right) = T\left(Y, X; Y^{(k)}; Z^{(k)}\right) = 0. \tag{3.8}$$

The symmetry condition allows us to cyclically permute the arguments of T up to a correction factor, e.g.,

$$\begin{aligned} 0 = \frac{1}{k+1}&\left[(k-m+1)T\left(X, X; X^{(k-m)}, Y^{(m)}; Z^{(k)}\right)\right. \\ &\left.\qquad\qquad + mT\left(X, Y; X^{(k-m+1)}, Y^{(m-1)}; Z^{(k)}\right)\right] \end{aligned}$$

$$\Longrightarrow T\left(X, X; X^{(k-m)}, Y^{(m)}; Z^{(k)}\right) = -\frac{m}{k-m+1} T\left(X, Y; X^{(k-m+1)}, Y^{(m-1)}; Z^{(k)}\right).$$

Since $\dim M = 2$ and the covectors of M define a vector space, the span of the covectors X, Y, Z is of dimension 0, 1, or 2. We now show $T\left(X, X; Y^{(k)}; Z^{(k)}\right) = 0$ for all cases, proving the claim. We note that, as a multilinear map, if any of the arguments of T are the zero covector $\mathbf{0}$, then T vanishes identically.

- If $\operatorname{span}(X, Y, Z)$ is of dimension 0, then $X = Y = Z = \mathbf{0}$ and $T(X, X; Y^{(k)}; Z^{(k)})$ vanishes identically.
- If $\operatorname{span}(X, Y, Z)$ is of dimension 1, then X, Y, and Z are linearly dependent. Without loss of generality, we take $Y = \alpha X$ and $Z = \beta X$ for non-zero constants α and β. Hence, T takes the form

$$T\left(X, X; Y^{(k)}; Z^{(k)}\right) = \alpha^k \beta^k T\left(X, X; X^{(k)}; X^{(k)}\right) = 0.$$

- If $\operatorname{span}(X, Y, Z)$ is of dimension 2, then two of X, Y, and Z are linearly independent. Using the symmetries of T, there are two possible cases. First, we assume this to be the case for Y and Z, i.e., we may write $X = \alpha Y + \beta Z$ for constants α and β (with at least one non-zero). We substitute this form for X into the definition (3.7) and simplify the result using the symmetry condition (3.8)

$$\begin{aligned}
T\left(X, X; Y^{(k)}; Z^{(k)}\right) &= T\left(\alpha Y + \beta Z, \alpha Y + \beta Z; Y^{(k)}; Z^{(k)}\right) \\
&= \alpha^2 T\left(Y, Y; Y^{(k)}; Z^{(k)}\right) + \alpha\beta T\left(Y, Z; Y^{(k)}; Z^{(k)}\right) \\
&\qquad + \alpha\beta T\left(Z, Y; Y^{(k)}; Z^{(k)}\right) + \beta^2 T\left(Z, Z; Y^{(k)}; Z^{(k)}\right) \\
&= 0 + 0 + 0 + 0 \\
&= 0.
\end{aligned}$$

For the second case, without loss of generality, we assume X and Z are linearly independent, i.e., $Y = \alpha X + \beta Z$ for constants α and β. Substituting this expression into the definition, we find

$$\begin{aligned} T\left(X, X; Y^{(k)}; Z^{(k)}\right) &= T\left(X, X; (\alpha X + \beta Z)^{(k)}; Z^{(k)}\right) \\ &= \sum_{p=0}^{k} \binom{k-p}{p} \alpha^{k-p} \beta^{p} T\left(X, X; X^{(k-p)}, Z^{(p)}; Z^{(k)}\right), \end{aligned} \tag{3.9}$$

where $\binom{n}{k}$ is the binomial coefficient and the coefficients $\alpha^{k-p}\beta^{p}$ are not identically zero for at least one term in this sum as α and β cannot be zero simultaneously (else $T = 0$ identically by multilinearity). Examining the m-th term in this sum, we cyclically permute the $(\ldots, X; X^{(k-m)}, Z^{(m)}; \ldots)$ arguments, yielding

$$T\left(X, X; X^{(k-m)}, Z^{(m)}; Z^{(k)}\right) = -\frac{m}{k-m+1} T\left(X, Z; X^{(k-m+1)}, Z^{(m-1)}; Z^{(k)}\right).$$

The right side of this equation vanishes for all $m = 0, 1, \ldots, k$ via the symmetry condition and, therefore, each term in the sum (3.9) vanishes identically, leaving

$$T\left(X, X; Y^{(k)}; Z^{(k)}\right) = 0.$$

□

Lemma 1 clearly generalizes to tensors which are covariant in ijI_1I_2 and tensors which posses additional indices (of any valence) beyond ijI_1I_2. An immediate corollary of Lemma 1 is produced by considering multiple partial derivatives of A^{ij} with respect to symmetrized covariant derivatives of φ.

Corollary 4. *If* $A^{i(j;I_1);I_2;\cdots;I_s} = 0$, *where* $s \geq 2$ *is an integer and* $|I_1| = |I_k| > 0$ *for some* $2 \leq k \leq s$, *then* $A^{ij;I_1;I_2;\cdots;I_s} = 0$.

We note that the multi-indices I_2 through I_s (excepting I_k) need not have the same number of indices as I_1 and I_k, i.e., Corollary 4 holds so long as the multi-index contained in the cyclic identity has at least one additional multi-index of the same size (corresponding to two derivatives of the same order). A corollary of Lemma 1 also exists for single partial derivatives, though the conclusion is weaker than the multiple derivative case.

Corollary 5. *If $A^{i(j;I_1)} = 0$ and $|I_1| > 0$, then $A^{ij;I_1;I_2} = 0$ for $|I_1| = |I_2|$.*

We now use Corollary 5 to establish the following proposition.

Proposition 1. *If $A^{ij} = A^{ij}(g_{ab}; \varphi; \varphi_a; \ldots; \varphi_{a_1 \cdots a_r})$ is a symmetric, divergence-free tensor density with $r \geq 1$, then A^{ij} is at most linear in the highest order derivative of φ.*

Proof. Since A^{ij} is of scalar order r, the leading term in the divergence-free condition (3.3) is the only term of scalar order $r+1$. Hence, we differentiate (3.3) with respect to $\varphi_{b_1 \cdots b_{r+1}}$ and obtain a cyclic identity for A^{ij}

$$0 = A^{i(b_{r+1}; b_1 \cdots b_r)}. \tag{3.10}$$

This cyclic identity satisfies Corollary 5, proving the claim. □

We note that the cyclic identity alone suffices to prove the order 2 and 3 cases of Theorem 12, with proof given in Chapter 4.

With the linearity of A^{ij} in the highest order derivative of φ given by Proposition 1, we show the highest scalar order term to be proportional to the product of two permutation tensors by an algebraic argument.

Proposition 2. *If $A^{ij} = A^{ij}(g_{ab}; \varphi; \varphi_a; \ldots; \varphi_{a_1 \cdots a_r})$ is a symmetric, divergence-free tensor density with $r \geq 2$, then A^{ij} takes the form $A^{ij} = \varepsilon^{ia_1}\varepsilon^{ja_2}\varphi_{a_1 \cdots a_r} B^{a_3 \cdots a_r} + C^{ij}$, where $B^{a_3 \cdots a_r}$ and C^{ij} are symmetric tensor densities of (at most) scalar order $r-1$.*

Proof. We define a rank $r+2$ tensor density $U^{ijK} = U^{(ij)(K)} = \mathrm{Sym}_{ij}\, \mathrm{Sym}_K\, \varepsilon^{ik_1}\varepsilon^{jk_2} V^{k_3 \cdots k_r}$, where $K = k_1 \cdots k_r$ is a multi-index and $V^{k_3 \cdots k_r}$ is the symmetric, rank $r-2$ tensor density given by $V^{k_3\cdots k_r} = \frac{r-1}{r+1}\varepsilon_{rt}\varepsilon_{su}A^{rs;tuk_3\cdots k_r}$. Then, using the properties of the permutation tensor (2.7), we have

$$\begin{aligned}
U^{ijK} &= \frac{r-1}{r+1}\operatorname{Sym}_{ij}\operatorname{Sym}_K \varepsilon^{ik_1}\varepsilon^{jk_2}\varepsilon_{rt}\varepsilon_{su}A^{rs;tuk_3\cdots k_r} \\
&= \frac{r-1}{r+1}\operatorname{Sym}_{ij}\operatorname{Sym}_K(-1)^{2q}\left(\delta^i_r\delta^{k_1}_t - \delta^{k_1}_r\delta^i_t\right)\left(\delta^j_s\delta^{k_2}_u - \delta^{k_2}_s\delta^j_u\right)A^{rs;tuk_3\cdots k_r} \\
&= \frac{r-1}{r+1}\operatorname{Sym}_{ij}\operatorname{Sym}_K\left(\delta^i_r\delta^{k_1}_t\delta^j_s\delta^{k_2}_u - \delta^i_r\delta^{k_1}_t\delta^{k_2}_s\delta^j_u - \delta^{k_1}_r\delta^i_t\delta^j_s\delta^{k_2}_u + \delta^{k_1}_r\delta^i_t\delta^{k_2}_s\delta^j_u\right)A^{rs;tuk_3\cdots k_r} \\
&= \frac{r-1}{r+1}\operatorname{Sym}_{ij}\operatorname{Sym}_K\left(A^{ij;k_1k_2k_3\cdots k_r} - A^{ik_2;k_1jk_3\cdots k_r} - A^{k_1j;ik_2k_3\cdots k_r} + A^{k_1k_2;ijk_3\cdots k_r}\right) \\
&= \frac{r-1}{r+1}\operatorname{Sym}_{ij}\operatorname{Sym}_K\left(A^{ij;K} - 2A^{ik_1;jk_2\cdots k_r} + A^{k_1k_2;ijk_3\cdots k_r}\right).
\end{aligned}$$

Since A^{ij} satisfies Proposition 1, it obeys the cyclic identity (3.10). We use the cyclic identity to write

$$\begin{aligned}
0 &= 0 + 0 \\
&= (r+1)\left(\frac{1}{2}A^{i(k_1;jk_2\cdots k_r)} + \frac{1}{2}A^{j(k_1;ik_2\cdots k_r)}\right) \\
&= (r+1)\operatorname{Sym}_{ij}\operatorname{Sym}_K A^{i(k_1;jk_2\cdots k_r)} \\
&= \operatorname{Sym}_{ij}\operatorname{Sym}_K\left(A^{ik_1;jk_2\cdots k_r} + A^{ij;k_2\cdots k_rk_1} + A^{ik_2;k_3\cdots k_rk_1j} + \cdots + A^{ik_r;jk_1\cdots k_{r-1}}\right) \\
&= \operatorname{Sym}_{ij}\operatorname{Sym}_K\left(rA^{ik_1;jk_2\cdots k_r} + A^{ij;K}\right),
\end{aligned}$$

which implies $\operatorname{Sym}_{ij}\operatorname{Sym}_K A^{ik_1;jk_2\cdots k_r} = -\dfrac{1}{r}A^{ij;K}$. Similarly, we write

$$\begin{aligned}
0 &= 0 + 0 + \cdots + 0 \\
&= (r+1)\left(\frac{1}{r}A^{k_1(k_2;ijk_3\cdots k_r)} + \frac{1}{r}A^{k_2(k_3;ijk_4\cdots k_rk_1)} + \cdots + \frac{1}{r}A^{k_r(k_1;ijk_2\cdots k_{r-1})}\right) \\
&= (r+1)\operatorname{Sym}_{ij}\operatorname{Sym}_K A^{k_1(k_2;ijk_3\cdots k_r)} \\
&= \operatorname{Sym}_{ij}\operatorname{Sym}_K\left(A^{k_1k_2;ijk_3\cdots k_r} + A^{k_1i;jk_3\cdots k_rk_2} + A^{k_1j;k_3\cdots k_rk_2i} + \cdots + A^{k_1k_r;ijk_2\cdots k_{r-1}}\right) \\
&= \operatorname{Sym}_{ij}\operatorname{Sym}_K\left[(r-1)A^{k_1k_2;ijk_3\cdots k_r} + 2A^{ik_1;jk_2\cdots k_r}\right],
\end{aligned}$$

which implies $\mathrm{Sym}_{ij}\,\mathrm{Sym}_K\,A^{k_1k_2;ijk_3\cdots k_r} = -\dfrac{2}{r-1}\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_K\,A^{ik_1;jk_2\cdots k_r} = \dfrac{2}{r(r-1)}A^{ij;K}$. We substitute these two results into the equation for U^{ijK} and simplify the resulting expression

$$
\begin{aligned}
U^{ijK} &= \frac{r-1}{r+1}\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_K\left[A^{ij;K} + \frac{2}{r}A^{ij;K} + \frac{2}{r(r-1)}A^{ij;K}\right]\\
&= \frac{r-1}{r+1}\left(\frac{r(r-1)+2(r-1)+2}{r(r-1)}\right)A^{ij;K}\\
&= \frac{r-1}{r+1}\left(\frac{r+1}{r-1}\right)A^{ij;K}\\
&= A^{ij;K}.
\end{aligned}
$$

Integration of this result proves the claim upon stating $V^{k_3\cdots k_r} = B^{k_3\cdots k_r}$. □

Next, we show the tensor $B^{a_3\cdots a_r}$ to be at most of scalar order $r-2$.

Proposition 3. *Let $n=2$. If $A^{ij} = A^{ij}(g_{ab};\varphi;\varphi_a;\ldots;\varphi_{a_1\cdots a_r})$ is a symmetric, divergence-free, natural tensor density with $r \geq 2$, then A^{ij} takes the form*

$$
A^{ij} = \varepsilon^{ia_1}\varepsilon^{ja_2}\varphi_{a_1\cdots a_r}B^{a_3\cdots a_r} + C^{ij}, \tag{3.11}
$$

where $B^{a_3\cdots a_r}$ and C^{ij} are symmetric, natural tensor densities of scalar orders $r-2$ and $r-1$, respectively.

Proof. Since A^{ij} satisfies Proposition 2, we may write

$$
A^{ij} = \varepsilon^{ia_1}\varepsilon^{ja_2}\varphi_{a_1\cdots a_r}B^{a_3\cdots a_r} + C^{ij},
$$

where $B^{a_3\cdots a_r}$ and C^{ij} are tensor densities of (at most) scalar order $r-1$. Taking this result into account, we inspect the divergence-free condition (3.3) and note that only the first term in the second line is quadratic in scalar order r derivatives, taking the form $A^{ij;a_1\cdots a_{r-1}}\varphi_{a_1\cdots a_{r-1}j}$. The first line does not contain any scalar order r derivatives due to the linearity of A^{ij}, while terms in the third and lower lines are at most linear in scalar order r derivatives due to the linearity of A^{ij} (all explicit φ derivatives in the third and lower lines are of scalar order $r-2$ or less).

As with Lemma 1, let U_i, X_j, and Y_k denote covectors on M. We contract the divergence-free condition with U_i and apply the differential operators $X_{j_1}\cdots X_{j_r}\dfrac{\partial}{\partial\varphi_{j_1\cdots j_r}}$ and $Y_{k_1}\cdots Y_{k_r}\dfrac{\partial}{\partial\varphi_{k_1\cdots k_r}}$, yielding the equation

$$\begin{aligned}0 &= U_iX_{j_1}\cdots X_{j_r}Y_{k_1}\cdots Y_{k_r}\left(A^{ik_r;k_1\cdots k_{r-1};j_1\cdots j_r} + A^{ij_r;j_1\cdots j_{r-1};k_1\cdots k_r}\right)\\ &= U_iX_{j_1}\cdots X_{j_r}Y_{k_1}\cdots Y_{k_r}\left(\varepsilon^{ij_1}\varepsilon^{k_rj_2}B^{j_3\cdots j_r;k_1\cdots k_{r-1}} + \varepsilon^{ik_1}\varepsilon^{j_rk_2}B^{k_3\cdots k_r;j_1\cdots j_{r-1}}\right).\end{aligned}$$

Using the function notation introduced for Lemma 1, the previous equation may be written as

$$\begin{aligned}0 &= \det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-1)}\right) + \det(U,Y)\det(X,Y)B\left(Y^{(r-1)};X^{(r-1)}\right)\\ &= \det(Y,X)\left[\det(U,X)B\left(X^{(r-2)};Y^{(r-1)}\right) - \det(U,Y)B\left(Y^{(r-1)};X^{(r-1)}\right)\right],\end{aligned}$$

where $X = X_i\,\mathrm{d}x^i$ and $\det(U,X) = U_iX_j\varepsilon^{ij} = g^{-1/2}U_iX_j\epsilon^{ij} = g^{-1/2}(U_1X_2 - U_2X_1)$. Since the covectors X_j and Y_k were arbitrary, the expression in brackets must vanish for the equation to hold

$$0 = \det(U,X)B\left(X^{(r-2)};Y^{(r-1)}\right) - \det(U,Y)B\left(Y^{(r-2)};X^{(r-1)}\right)$$

or, in coordinates,

$$0 = \mathrm{Sym}_{j_1\cdots j_{r-1}}\,\varepsilon^{ij_1}B^{j_2\cdots j_{r-1};k_1\cdots k_{r-1}} - \mathrm{Sym}_{k_1\cdots k_{r-1}}\,\varepsilon^{ik_1}B^{k_2\cdots k_{r-1};j_1\cdots j_{r-1}}.$$

Expanding the symmetries (and ignoring the resulting coefficients due to the equality with zero), we contract this equation with ε_{ij_1} and use (2.8c) to simplify the resulting equation

$$0 = \varepsilon_{ij_1}\Big(\varepsilon^{ij_1}B^{j_2\cdots j_{r-1};k_1\cdots k_{r-1}} + \varepsilon^{ij_2}B^{j_1j_3\cdots j_{r-1};k_1\cdots k_{r-1}} + \cdots + \varepsilon^{ij_{r-1}}B^{j_1\cdots j_{r-2};k_1\cdots k_{r-1}}$$

$$- \varepsilon^{ik_1} B^{k_2\cdots k_{r-1};j_1\cdots j_{r-1}} - \varepsilon^{ik_2} B^{k_1k_3\cdots k_{r-1};j_1\cdots j_{r-1}} - \cdots - \varepsilon^{ik_{r-1}} B^{k_1\cdots k_{r-2};j_1\cdots j_{r-1}}\Big)$$

$$= (-1)^q \Big(2B^{j_2\cdots j_{r-1};k_1\cdots k_{r-1}} + B^{j_2j_3\cdots j_{r-1};k_1\cdots k_{r-1}} + \cdots + B^{j_{r-1}j_2\cdots j_{r-2};k_1\cdots k_{r-1}}$$

$$- B^{k_2\cdots k_{r-1};k_1j_2\cdots j_{r-1}} - B^{k_1k_3\cdots k_{r-1};k_2j_2\cdots j_{r-1}} - \cdots - B^{k_1\cdots k_{r-2};k_{r-1}j_2\cdots j_{r-1}}\Big)$$

$$= (-1)^q \Big(rB^{j_2\cdots j_{r-1};k_1\cdots k_{r-1}} - B^{k_2\cdots k_{r-1};k_1j_2\cdots j_{r-1}} - B^{k_1k_3\cdots k_{r-1};k_2j_2\cdots j_{r-1}}$$

$$- \cdots - B^{k_1\cdots k_{r-2};k_{r-1}j_2\cdots j_{r-1}}\Big)$$

$$= (-1)^q \Big[rB^{j_2\cdots j_{r-1};k_1\cdots k_{r-1}} - (r-1)\,\mathrm{Sym}_{k_1\cdots k_{r-1}} B^{k_2\cdots k_{r-1};k_1j_2\cdots j_{r-1}}\Big].$$

The coefficient of $(-1)^q$ is never zero and so the above equation is satisfied if and only if the expression in parentheses vanishes

$$0 = rB^{j_2\cdots j_{r-1};k_1\cdots k_{r-1}} - (r-1)\,\mathrm{Sym}_{k_1\cdots k_{r-1}} B^{k_2\cdots k_{r-1};k_1j_2\cdots j_{r-1}}. \tag{3.12}$$

We note that the $r = 2$ version of this equation leads to the claim immediately, with $0 = 2B^{;k_1} - B^{;k_1} = B^{;k_1}$.

As $B^{a_1\cdots a_{r-2};b_1\cdots b_{r-1}} = B^{(a_1\cdots a_{r-2});(b_1\cdots b_{r-1})}$, a choice of $r-1$ (or $r-2$) indices from the collection of $2r-3$ indices $j_2\cdots j_{r-1}k_1\cdots k_{r-1}$ produces a unique version of (3.12), with a total of $\binom{2r-3}{r-1} = \binom{2r-3}{r-2}$ different equations, where $\binom{n}{k}$ is the binomial coefficient. Hence, there exists a $\binom{2r-3}{r-1}$ by $\binom{2r-3}{r-1}$ square matrix A which defines a matrix equation

$$Ax = 0$$

for the vector x of different index combinations $B^{a_1\cdots a_{r-2};b_1\cdots b_{r-1}}$. A row of A, denoted by A_i, defines its version of (3.12) via the "dot product" $A_i \cdot x = 0$.

We choose an index "basis" such that the diagonal entries of A are each r, corresponding with the index combination $B^{j_2\cdots j_{r-1};k_1\cdots k_{r-1}}$ from (3.12). The row A_i represented by this combination will be of the form

$$A_i = (\cdots, -1, \cdots, -1, \cdots, r, \cdots, -1, \cdots, -1, \cdots, -1, \cdots),$$

where each -1 corresponds to an index configuration $B^{k_1\cdots k_{s-1}k_{s+1}\cdots k_{r-2};k_s j_2\cdots j_{r-1}}$ and the remaining entries are zero. There are $r-1$ such index configurations in (3.12) and so A satisfies the Diagonal Dominance Theorem of linear algebra (see, e.g., Strang [32]).

Diagonal Dominance Theorem. *A square matrix A is said to be diagonally dominant if the magnitude of each diagonal entry is greater than the sum of the magnitudes for the remaining (non-diagonal) entries in the corresponding row. If A is a diagonally dominant matrix, then A is non-singular.*

Hence, the only solution to the matrix equation $Ax = 0$ is the trivial solution $x = 0$ and $B^{a_1\cdots a_{r-2};b_1\cdots b_{r-1}} = 0$ for all index choices. □

As examples of the matrix equation $Ax = 0$ defined in the proof of Proposition 3, consider the $r = 3$ equation

$$Ax = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} \begin{pmatrix} B^{a;bc} \\ B^{b;ac} \\ B^{c;ab} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0,$$

which is easily solved by introductory linear algebra techniques, and the substantially more complicated $r = 4$ case

$$Ax = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 4 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 4 & -1 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 4 & 0 & 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 & 0 & 4 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 4 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 4 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} B^{ab;cde} \\ B^{ac;bde} \\ B^{ad;cbe} \\ B^{ae;bcd} \\ B^{bc;ade} \\ B^{bd;ace} \\ B^{be;acd} \\ B^{cd;abe} \\ B^{ce;abd} \\ B^{de;abc} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0.$$

The matrix A rapidly grows in size but becomes dominated by zeros as r increases: for $r = 4$ the matrix equation is a 10 by 10 matrix with 40 non-zero entries (40% nonzero), the $r = 5$ case is a 35 by 35 matrix with 175 non-zero entries ($\sim$14% nonzero), and the $r = 10$ case is a 24310 by 24310 matrix with 243100 non-zero entries ($\sim$0.041% nonzero). We now proceed with the proof of Theorem 11.

Proof. Re-examining the divergence-free condition (3.3) and taking into account Proposition 3, we observe that only the first term of the second line contains a scalar order r derivative and more than one scalar order $r-1$ derivative (it is the only term which combines C^{ij} and $\varphi_{a_1\cdots a_r}$). We apply the differential operators $U_i X_{m_1}\cdots X_{m_r}\dfrac{\partial}{\partial\varphi_{m_1\cdots m_r}}$, $Y_{n_1}\cdots Y_{n_{r-1}}\dfrac{\partial}{\partial\varphi_{n_1\cdots n_{r-1}}}$, and $Z_{o_1}\cdots Z_{o_{r-1}}\dfrac{\partial}{\partial\varphi_{o_1\cdots o_{r-1}}}$ to this equation, yielding

$$\begin{aligned}
0 &= U_i X_{m_1}\cdots X_{m_r} Y_{n_1}\cdots Y_{n_{r-1}} Z_{o_1}\cdots Z_{o_{r-1}} A^{im_1;m_2\cdots m_r;n_1\cdots n_{r-1};o_1\cdots o_{r-1}}\\
&= U_i X_{m_1}\cdots X_{m_r} Y_{n_1}\cdots Y_{n_{r-1}} Z_{o_1}\cdots Z_{o_{r-1}} C^{im_1;m_2\cdots m_r;n_1\cdots n_{r-1};o_1\cdots o_{r-1}}\\
&= C\left(U, X; X^{(r-1)}; Y^{(r-1)}; Z^{(r-1)}\right).
\end{aligned}$$

By Corollary 4, this implies that $C\left(U, U; X^{(r-1)}; Y^{(r-1)}; Z^{(r-1)}\right)$ vanishes identically and so C^{ij} is at most quadratic in order $r-1$ derivatives of φ. Inspecting the divergence-free condition again, we note that only two terms depend on a single scalar order r derivative and at least one order $r-1$ derivative: the first term contracted with $A^{ij;a_1\cdots a_{r-2}}$ and, again, the first term contracted with $A^{ij;a_1\cdots a_{r-1}}$. We apply the differential operators $U_i X_{m_1}\cdots X_{m_r}\dfrac{\partial}{\partial\varphi_{m_1\cdots m_r}}$ and $Y_{n_1}\cdots Y_{n_{r-1}}\dfrac{\partial}{\partial\varphi_{n_1\cdots n_{r-1}}}$ to (3.3), yielding

$$\begin{aligned}
0 &= U_i X_{m_1}\cdots X_{m_r} Y_{n_1}\cdots Y_{n_{r-1}}\left(A^{im_1;m_2\cdots m_r;n_1\cdots n_{r-1}} + A^{in_1;n_2\cdots n_{r-1};m_1\cdots m_r}\right)\\
&= U_i X_{m_1}\cdots X_{m_r} Y_{n_1}\cdots Y_{n_{r-1}}\left(C^{im_1;m_2\cdots m_r;n_1\cdots n_{r-1}} + \varepsilon^{im_1}\varepsilon^{n_1m_2}B^{m_3\cdots m_r;n_2\cdots n_{r-1}}\right)\\
&= C\left(U, X; X^{(r-1)}; Y^{(r-1)}\right) + \det(U, X)\det(Y, X) B\left(X^{(r-2)}; Y^{(r-2)}\right).
\end{aligned}$$

Solving this equation for C, we get

$$C\left(U,X;X^{(r-1)};Y^{(r-1)}\right)=-\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right). \tag{3.13}$$

We change U to Y, yielding the equation

$$C\left(Y,X;X^{(r-1)};Y^{(r-1)}\right)=-\det(Y,X)^2B\left(X^{(r-2)};Y^{(r-2)}\right),$$

which, via the symmetries of C, is equivalent to the $X\leftrightarrow Y$ swapped equation

$$\begin{aligned}C\left(X,Y;Y^{(r-1)};X^{(r-1)}\right)&=-\det(X,Y)^2B\left(Y^{(r-2)};X^{(r-2)}\right)\\&=-\det(Y,X)^2B\left(Y^{(r-2)};X^{(r-2)}\right).\end{aligned}$$

Hence, $B\left(X^{(r-2)};Y^{(r-2)}\right)=B\left(Y^{(r-2)};X^{(r-2)}\right)$ and $B^{a_3\cdots a_r;b_3\cdots b_r}$ is symmetric under the interchange of all a and b indices.

Returning to (3.13), we observe that if $B\left(X^{(r-2)};Y^{(r-2)}\right)=0$, then $C\left(U,U;X^{(r-1)};Y^{(r-1)}\right)=0$ by Corollary 4. Similarly, if the left side of (3.13) is zero, then $B^{a_3\cdots a_r;b_3\cdots a_r}$ vanishes identically. To prove this second statement, we switch to index notation and set $C=0$ in (3.13), contract the equation with a pair of permutation tensors, and expand the symmetries

$$\begin{aligned}0&=-\varepsilon_{ia_1}\varepsilon_{b_1a_2}\left(\operatorname{Sym}_{a_1\cdots a_r}\operatorname{Sym}_{b_1\cdots b_{r-1}}\varepsilon^{ia_1}\varepsilon^{b_1a_2}B^{a_3\cdots a_r;b_2\cdots b_{r-1}}\right)\\&=-(-1)^q(r+1)\varepsilon_{b_1a_2}\left(\operatorname{Sym}_{a_2\cdots a_r}\operatorname{Sym}_{b_1\cdots b_{r-1}}\varepsilon^{b_1a_2}B^{a_3\cdots a_r;b_2\cdots b_{r-1}}\right)\\&=(-1)^{q+1}(r+1)\varepsilon_{b_1a_1}\left(\operatorname{Sym}_{a_1\cdots a_{r-1}}\operatorname{Sym}_{b_1\cdots b_{r-1}}\varepsilon^{b_1a_1}B^{a_2\cdots a_{r-1};b_2\cdots b_{r-1}}\right)\\&=(-1)^{q+1}\frac{r+1}{(r-1)^2}\varepsilon_{b_1a_1}\left(\varepsilon^{b_1a_1}B^{a_2\cdots a_{r-1};b_2\cdots b_{r-1}}+\varepsilon^{b_1a_2}B^{a_1a_3\cdots a_{r-1};b_2\cdots b_{r-1}}\right.\\&\qquad\left.+\cdots+\varepsilon^{b_{r-1}a_{r-1}}B^{a_1\cdots a_{r-2};b_1\cdots b_{r-2}}\right).\end{aligned}\tag{3.14}$$

For $1 \leq l \leq r-1$, we define the multi-indices $\hat{a}^l_j = a_2 \cdots a_{l-1} j a_{l+1} \cdots a_{r-1}$, with $\hat{a}^1_j = a_2 \cdots a_{r-1}$. Next, we consider the $\varepsilon^{b_k a_l}$ term of (3.14)

$$\varepsilon_{b_1 a_1} \varepsilon^{b_k a_l} B^{\hat{a}^l_{a_1};\hat{b}^k_{b_1}} = (-1)^q \left(\delta^{b_k}_{b_1} \delta^{a_l}_{a_1} - \delta^{a_l}_{b_1} \delta^{b_k}_{a_1} \right) B^{\hat{a}^l_{a_1};\hat{b}^k_{b_1}}.$$

If $l = 1 = k$, then this expression simplifies to $2(-1)^q B^{a_2 \cdots a_{r-1};b_2 \cdots b_{2-1}}$. Similarly, if either $l = 1$ or $k = 1$, then the expression simplifies to $(-1)^q B^{a_2 \cdots a_{r-1};b_2 \cdots b_{2-1}}$. There is one term of the first kind and $2(r-2)$ terms of the second kind in (3.14). Together, these terms sum to

$$2(r-1)(-1)^q B^{a_2 \cdots a_{r-1};b_2 \cdots b_{2-1}}.$$

For $l \neq 1 \neq k$, we have $(r-2)^2$ terms of the form

$$\begin{aligned} \varepsilon_{b_1 a_1} \varepsilon^{b_k a_l} B^{\hat{a}^l_{a_1};\hat{b}^k_{b_1}} &= (-1)^q \left(B^{\hat{a}^l_{a_l};\hat{b}^k_{b_k}} - B^{\hat{a}^l_{b_k};\hat{b}^k_{a_l}} \right) \\ &= (-1)^q \left(B^{a_2 \cdots a_r;b_2 \cdots b_r} - B^{\hat{a}^l_{b_k};\hat{b}^k_{a_l}} \right). \end{aligned}$$

The $(r-2)^2$ copies of the first term in parentheses combine with the $2(r-1)$ copies given earlier, totaling $(r-1)^2+1$ copies, while the $(r-2)^2$ versions of the second term can be expressed compactly by exploiting the symmetry of the indices $a_2 \cdots a_{r-1}$ and $b_2 \cdots b_{r-1}$. We substitute this result into (3.14), yielding the equation

$$\begin{aligned} 0 = (-1)^{2q+1} \frac{(r+1)}{(r-1)^2} \Big\{ &\left[(r-1)^2 + 1 \right] B^{a_2 \cdots a_{r-1};b_2 \cdots b_{r-1}} \\ &- (r-2)^2 \operatorname{Sym}_{a_2 \cdots a_{r-1}} \operatorname{Sym}_{b_2 \cdots b_{r-1}} B^{b_2 a_3 \cdots a_{r-1};a_2 b_3 \cdots b_{r-1}} \Big\}. \end{aligned}$$

As the numeric coefficient outside the curly brackets is nonzero, the term inside must vanish

$$0 = \left[(r-1)^2 + 1 \right] B^{a_2 \cdots a_{r-1};b_2 \cdots b_{r-1}} - (r-2)^2 \operatorname{Sym}_{a_2 \cdots a_{r-1}} \operatorname{Sym}_{b_2 \cdots b_{r-1}} B^{b_2 a_3 \cdots a_{r-1};a_2 b_3 \cdots b_{r-1}}. \tag{3.15}$$

In a similar fashion to (3.12), we permute the indices of this equation to create a matrix equation $Ax = b$. The square matrix A produced from (3.15) is diagonally dominant, with diagonal entries $(r-1)^2+1$, each row containing $(r-2)^2$ copies of -1, and all other entries zero. Hence, $B^{a_1\cdots a_{r-2};b_1\cdots b_{r-2}}$ vanishes identically by the Diagonal Dominance Theorem when the left side of (3.13) is zero. Therefore, the kernel of the function $C\left(U,U;X^{(r-1)};Y^{(r-1)}\right)$ is zero and we may express this function directly in terms of the derivative of B.

Lemma 2. *If C^{ij} obeys (3.13), then*

$$C\left(U,U;X^{(r-1)};Y^{(r-1)}\right)=2\det(U,X)\det(U,Y)B(X^{(r-2)};Y^{(r-2)}).$$

Proof. We begin with the ansatz

$$\begin{aligned}\tilde{C}\left(U,U;X^{(r-1)};Y^{(r-1)}\right)=&\,\alpha\det(U,X)\det(Y,U)B\left(X^{(r-2)};Y^{(r-2)}\right)\\&+\beta\det(U,X)\det(Y,X)B\left(U,X^{(r-3)};Y^{(r-2)}\right)\\&+\gamma\det(U,Y)\det(X,Y)B\left(U,Y^{(r-3)};X^{(r-2)}\right),\end{aligned}$$

where α,β, and γ are numeric constants to be determined. We swap the X and Y entries to get

$$\begin{aligned}\tilde{C}\left(U,U;Y^{(r-1)};X^{(r-1)}\right)=&\,\alpha\det(U,Y)\det(X,U)B\left(Y^{(r-2)};X^{(r-2)}\right)\\&+\beta\det(U,Y)\det(X,Y)B\left(U,Y^{(r-3)};X^{(r-2)}\right)\\&+\gamma\det(U,X)\det(Y,X)B\left(U,X^{(r-3)};Y^{(r-2)}\right),\end{aligned}$$

so $\gamma=\beta$ upon comparison with the starting ansatz. Expanding the symmetries, we replace one U with an X to get

$$\begin{aligned}\tilde{C}\left(U,X;X^{(r-1)};Y^{(r-1)}\right)=&\,\frac{\alpha}{2}\left[0+\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right)\right]\\&+\frac{\beta}{2}\left[0+\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right)\right.\\&+\det(X,Y)\det(X,Y)B\left(U,Y^{(r-3)};X^{(r-2)}\right)\end{aligned}$$

$$+ \det(U,Y)\det(X,Y)B\left(X, Y^{(r-3)}; X^{(r-2)}\right)\Big].$$

Using the cyclic identity

$$\det(U,Y)\tau\left(X,\cdots\right) + \det(Y,X)\tau\left(U,\cdots\right) + \det(X,U)\tau\left(Y,\cdots\right) = 0, \tag{3.16}$$

where τ is a tensor (density), we replace the final term of the previous expression

$$\begin{aligned}
\tilde{C}\left(U,X;X^{(r-1)};Y^{(r-1)}\right) &= \frac{\alpha}{2}\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right) \\
&\quad + \frac{\beta}{2}\Big[\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right) \\
&\quad + \det(X,Y)\det(X,Y)B\left(U,Y^{(r-3)};X^{(r-2)}\right) \\
&\quad - \det(Y,X)\det(X,Y)B\left(U,Y^{(r-3)};X^{(r-2)}\right) \\
&\quad - \det(X,U)\det(X,Y)B\left(Y^{(r-2)};X^{(r-2)}\right)\Big] \\
&= \left(\frac{\alpha-\beta}{2}\right)\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right) \\
&\quad + \frac{\beta}{2}\Big[\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right) \\
&\quad + 2\det(X,Y)\det(X,Y)B\left(U,Y^{(r-3)};X^{(r-2)}\right)\Big].
\end{aligned}$$

Checking (3.13), we see $\beta = 0$ and $\alpha = -2$. To prove the uniqueness of $\tilde{C}$, we consider the difference between C and the ansatz

$$\begin{aligned}
\Delta\left(U,U,X^{(r-1)},Y^{(r-1)}\right) &= C\left(U,U;X^{(r-1)};Y^{(r-1)}\right) - \tilde{C}\left(U,U;X^{(r-1)};Y^{(r-1)}\right) \\
&= C\left(U,U;X^{(r-1)};Y^{(r-1)}\right) \\
&\quad + 2\det(U,X)\det(Y,U)B\left(X^{(r-2)};Y^{(r-2)}\right).
\end{aligned}$$

By replacing a U with an X, we get the equation

$$\Delta\left(U,X,X^{(r-1)},Y^{(r-1)}\right) = C\left(U,X;X^{(r-1)};Y^{(r-1)}\right)$$

$$
\begin{aligned}
&+\frac{1}{2}\left[0+2\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right)\right]\\
=&-\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right)\\
&+\det(U,X)\det(Y,X)B\left(X^{(r-2)};Y^{(r-2)}\right)\\
=&\,0.
\end{aligned}
$$

Hence, $\Delta\left(U,U,X^{(r-1)},Y^{(r-1)}\right)$ vanishes identically by Lemma 1 and $C=\tilde{C}$. □

In index notation, Lemma 2 states

$$
C^{ij;a_1\cdots a_{r-1};b_1\cdots b_{r-1}} = 2\,\mathrm{Sym}_{a_1\cdots a_{r-1}}\,\mathrm{Sym}_{b_1\cdots b_{r-1}}\,\varepsilon^{ia_1}\varepsilon^{jb_1}B^{a_2\cdots a_{r-1};b_2\cdots b_{r-1}}
$$

and integration of this expression proves the claim. □

3.4 Euler-Lagrange Expressions and Divergence-free Tensor Densities

The highest scalar order term from Theorem 10 can be seen to correspond with the leading term of (1.28). Let $E_R(L)=\mathcal{E}$ be of curvature order k. The first term of (1.27) is given by

$$
\begin{aligned}
\varepsilon^{ia}\varepsilon^{jb}\mathcal{E}_{|ab} &= \varepsilon^{ia}\varepsilon^{jb}\left(\mathcal{E}_{,ab}-\Gamma^c{}_{ab}\mathcal{E}_{,c}\right)\\
&=\varepsilon^{ia}\varepsilon^{jb}\left[\left(\frac{\partial\mathcal{E}}{\partial R_{|(a_1\cdots a_k)}}R_{|(a_1\cdots a_k),a}+\cdots\right)_{,b}-\Gamma^c{}_{ab}\left(\frac{\partial\mathcal{E}}{\partial R_{|(a_1\cdots a_k)}}R_{|(a_1\cdots a_k),c}+\cdots\right)\right]\\
&=\varepsilon^{ia}\varepsilon^{jb}\left\{\left[\frac{\partial\mathcal{E}}{\partial R_{|(a_1\cdots a_k)}}\left(R_{|(a_1\cdots a_k)a}+k\Gamma^c{}_{a_1a}R_{|(ca_2\cdots a_k)}\right)+\cdots\right]_{,b}\right.\\
&\qquad\left.-\Gamma^c{}_{ab}\left[\frac{\partial\mathcal{E}}{\partial R_{|(a_1\cdots a_k)}}\left(R_{|(a_1\cdots a_k)c}+k\Gamma^d{}_{a_1c}R_{|(da_2\cdots a_k)}\right)+\cdots\right]\right\}\\
&=\varepsilon^{ia}\varepsilon^{jb}\left(\frac{\partial\mathcal{E}}{\partial R_{|(a_1\cdots a_k)}}R_{|(a_1\cdots a_k)a}+\cdots\right)_{|b}\\
&=\varepsilon^{ia}\varepsilon^{jb}\left(\frac{\partial\mathcal{E}}{\partial R_{|(a_1\cdots a_k)}}R_{|(a_1\cdots a_ka)b}+\cdots\right)\\
&=\varepsilon^{ia}\varepsilon^{jb}\left(\frac{\partial\mathcal{E}}{\partial R_{|(a_1\cdots a_k)}}R_{|(a_1\cdots a_kab)}+\cdots\right),
\end{aligned}
$$

where we have used the symmetrization formula (A.4) to arrive at the penultimate and final lines. Comparing this final line with (1.28), we see

$$B^{a_3 \cdots a_r} = \frac{\partial \mathcal{E}}{\partial R_{|(a_3 \cdots a_r)}}. \tag{3.17}$$

CHAPTER 4

DIVERGENCE-FREE TENSOR DENSITIES: SPECIAL CASES

We now prove Theorem 12 by cases.

4.1 Order 0

It is well known, e.g., Lovelock and Rund [16, pp. 319], that the only symmetric, divergence-free, contravariant, rank 2 tensor density dependent solely on the metric is $A^{ij} = c\sqrt{g}g^{ij}$, where c is a constant, and arises as the Euler-Lagrange expression of the Lagrangian $\lambda = 2c\sqrt{g}$.

4.2 Order 1

As mentioned in the introduction, no natural tensor densities of metric order 1 exist by virtue of Theorem 1 (in any dimension $n \geq 2$).

4.3 Order 2

We assume $A^{ij} = A^{ij}(g_{ab}; \varphi)$. The divergence-free condition (3.3) for this case is

$$0 = \frac{\partial A^{ij}}{\partial \varphi}\varphi_j.$$

We differentiate this equation with respect to φ_j, yielding the identity $\frac{\partial A^{ij}}{\partial \varphi} = 0$. Hence, there are no symmetric, divergence-free, contravariant, rank 2 tensor densities dependent on the metric and a scalar field in two dimensions.

When φ is identified with the scalar curvature R (so A^{ij} is second order in derivatives of the metric), this result is not surprising: the summation in (1.7) vanishes identically for $\dim M = 2$, leaving only the inverse metric term. In effect, the vanishing of the generalized Einstein tensor (1.7) in two dimensions is because all information about the second derivative of the metric is encoded in a scalar field, R.

4.4 Order 3

We assume $A^{ij} = A^{ij}(g_{ab}; \varphi; \varphi_a)$. The divergence-free condition (3.3) for this case is

$$0 = A^{ij;k}\varphi_{jk} + \frac{\partial A^{ij}}{\partial \varphi}\varphi_j.$$

We differentiate this equation with respect to φ_{ab} and get the cyclic identity $A^{i(a;b)} = 0$. Creating three versions of this equation by cyclically permuting the indices

$$0 = \frac{1}{2}\left(A^{ia;b} + A^{ib;a}\right), \qquad 0 = \frac{1}{2}\left(A^{ab;i} + A^{ai;b}\right), \qquad 0 = \frac{1}{2}\left(A^{bi;a} + A^{ba;i}\right),$$

we sum the first two equations and subtract the third to get

$$0 = \frac{1}{2}\left(A^{ia;b} + A^{ib;a} + A^{ab;i} + A^{ai;b} - A^{bi;a} - A^{ba;i}\right) = \frac{1}{2}\left(2A^{ia;b}\right) = A^{ia;b}.$$

This implies that no symmetric, divergence-free, contravariant, rank 2 tensor densities dependent on the metric, a scalar field, and the first derivative of the scalar field exist in two dimensions. Equivalently, if we let $\varphi = R$, then there are no tensor densities of the given type dependent on the metric and its first three derivatives.

4.5 Order 4

We assume $A^{ij} = A^{ij}(g_{ab}; \varphi; \varphi_a; \varphi_{ab})$. The divergence-free condition is given by (3.3) and we find the explicit form of this equation using (3.6) and the third scalar order symmetrization formula (A.1)

$$\begin{aligned}
0 &= A^{ij;kl}\varphi_{kl|j} + A^{ij;k}\varphi_{kj} + \frac{\partial A^{ij}}{\partial \varphi}\varphi_j \\
&= A^{ij;kl}\left[\varphi_{klj} + \frac{1}{3}R(g_{jk}\varphi_l - g_{lk}\varphi_j)\right] + A^{ij;k}\varphi_{kj} + \frac{\partial A^{ij}}{\partial \varphi}\varphi_j \\
&= A^{ij;kl}\varphi_{klj} + A^{ij;k}\varphi_{kj} + \left(\frac{\partial A^{ij}}{\partial \varphi} - \frac{1}{3}Rg_{kl}A^{ij;kl} + \frac{1}{3}Rg_{kl}A^{il;kj}\right)\varphi_j. \qquad (4.1)
\end{aligned}$$

As A^{ij} satisfies the requirements of Theorem 11, A^{ij} takes the form

$$A^{ij} = \varepsilon^{ia}\varepsilon^{jb}\varphi_{ab}B + \varepsilon^{ia}\varepsilon^{jb}\varphi_a\varphi_b\frac{\partial B}{\partial \varphi} + D^{ija}\varphi_a + E^{ij}, \tag{4.2}$$

where B, $D^{ija} = D^{jia}$, and $E^{ij} = E^{ji}$ are scalar/tensor densities dependent on the metric and φ. Using the invariance identity for a scalar density (2.16), we write $B = \sqrt{g}P$, where $P = P(\varphi)$ is a scalar. Similarly, D^{ija} vanishes identically (in two dimensions there are no rank 3 tensors which are dependent on the metric and a scalar field). Using these results, we substitute (4.2) into the divergence-free condition and simplify the resulting expression

$$\begin{aligned}
0 &= \left(\sqrt{g}\varepsilon^{ik}\varepsilon^{jb}\varphi_b P' + D^{ijk}\right)\varphi_{kj} + \left(\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{ab}P' + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_a\varphi_b P'' + \frac{\partial D^{ija}}{\partial \varphi}\varphi_a\right.\\
&\quad \left. + \frac{\partial E^{ij}}{\partial \varphi} - \frac{1}{3}\sqrt{g}Rg_{kl}\varepsilon^{ik}\varepsilon^{jl}P + \frac{1}{3}\sqrt{g}Rg_{kl}\,\mathrm{Sym}_{kj}\,\varepsilon^{ik}\varepsilon^{lj}P\right)\varphi_j \\
&= \sqrt{g}\varepsilon^{ik}\varepsilon^{jb}\varphi_b\varphi_{jk}P' - \sqrt{g}\varepsilon^{ia}\varepsilon^{bj}\varphi_j\varphi_{ab}P' + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_a\varphi_b\varphi_j P'' + \frac{\partial E^{ij}}{\partial \varphi}\varphi_j \\
&\quad - \frac{(-1)^q}{3}\sqrt{g}RP\varphi^i + \frac{(-1)^q}{6}\sqrt{g}Rg_{kl}\left(2g^{il}g^{jk} - g^{ij}g^{kl} - g^{ik}g^{jl}\right)P\varphi_j \\
&= \frac{\partial E^{ij}}{\partial \varphi}\varphi_j - \frac{(-1)^q}{3}\sqrt{g}RP\varphi^i - \frac{(-1)^q}{6}\sqrt{g}RP\varphi^i \\
&= \frac{\partial E^{ij}}{\partial \varphi}\varphi_j - \frac{(-1)^q}{2}\sqrt{g}RP\varphi^i,
\end{aligned} \tag{4.3}$$

Differentiating this equation with respect to φ_j, we solve the resulting equation for the partial derivative of E^{ij}

$$\frac{\partial E^{ij}}{\partial \varphi} = \frac{(-1)^q}{2}\sqrt{g}g^{ij}RP.$$

Now, let the scalar φ be the scalar curvature R. We solve this equation using integration by parts, with a final result of

$$E^{ij} = \frac{(-1)^q}{2}\sqrt{g}g^{ij}RL' - \frac{(-1)^q}{2}\sqrt{g}g^{ij}L, \tag{4.4}$$

where $L = L(R)$ is a scalar such that $P = L'' = \frac{\mathrm{d}^2 L}{\mathrm{d}R^2}$ (the integration constant has been absorbed into the definition of L). Combining (4.2) and (4.4), we see that A^{ij} is given by

$$\begin{aligned} A^{ij} &= \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ab}L'' + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|a}R_{|b}L''' + \frac{(-1)^q}{2}\sqrt{g}g^{ij}RL' - \frac{(-1)^q}{2}\sqrt{g}g^{ij}L \\ &= \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}(L')_{|ab} + \frac{(-1)^q}{2}\sqrt{g}g^{ij}RL' - \frac{(-1)^q}{2}\sqrt{g}g^{ij}L. \end{aligned} \tag{4.5}$$

As L is a function of R alone, A^{ij} can be identified as the Euler-Lagrange expression of a two dimensional $f(R)$ field theory, a generalization of general relativity which uses Lagrangians of the form $\lambda = \sqrt{g}f(R)$. The metric signature factor of $(-1)^q$ is necessary in (4.5) to get the correct sign for each term upon expanding the permutation tensors using (2.8). We note that (4.5) has (depending on the signature of the metric) opposite sign from the $\lambda = \sqrt{g}L(R)$ Euler-Lagrange expression found in (2.33) and is due to the convention used for the product of two permutation tensors (2.8). As expected, we reproduce the metric order 0 case when $L = -2(-1)^q c$, where c is a constant.

4.6 Order 5

We assume $A^{ij} = A^{ij}(g_{ab}; \varphi; \varphi_a; \varphi_{ab}; \varphi_{abc})$. As with the metric order 4 case, we find the explicit form of the divergence-free condition (3.3) by starting with the covariant divergence formula (3.6), substituting the fourth scalar order symmetrization formula (A.2) and using the right side of equation (4.1) from the previous case to simplify the remaining terms

$$\begin{aligned} 0 &= A^{ij;klm}\varphi_{klm|j} + A^{ij;kl}\varphi_{kl|j} + A^{ij;k}\varphi_{kj} + \frac{\partial A^{ij}}{\partial\varphi}\varphi_j \\ &= A^{ij;klm}\left\{\varphi_{klmj} + \left[R\left(g_{jk}\varphi_{ml} - g_{mk}\varphi_{jl}\right) + \frac{1}{4}R_{|m}\left(g_{jl}\varphi_k - g_{kl}\varphi_j\right)\right]\right\} \\ &\qquad + A^{ij;kl}\varphi_{klj} + A^{ij;k}\varphi_{kj} + \left(\frac{\partial A^{ij}}{\partial\varphi} - \frac{1}{3}Rg_{kl}A^{ij;kl} + \frac{1}{3}Rg_{kl}A^{il;kj}\right)\varphi_j \\ &= A^{ij;klm}\varphi_{klmj} + A^{ij;kl}\varphi_{klj} + \left(A^{ij;k} - Rg_{ml}A^{ij;klm} + Rg_{ml}A^{im;ljk}\right)\varphi_{kj} \\ &\qquad + \frac{1}{4}\left(g_{kl}A^{ik;jlm} - g_{kl}A^{ij;klm}\right)R_{|m}\varphi_j + \left(\frac{\partial A^{ij}}{\partial\varphi} - \frac{1}{3}Rg_{kl}A^{ij;kl} + \frac{1}{3}Rg_{kl}A^{il;kj}\right)\varphi_j. \end{aligned} \tag{4.6}$$

As A^{ij} satisfies the requirements of Theorem 11, A^{ij} takes the form

$$A^{ij} = \varepsilon^{ia}\varepsilon^{jb}\varphi_{abc}B^c + \varepsilon^{ia}\varepsilon^{jb}\varphi_{ac}\varphi_{bd}B^{c;d} + D^{ijab}\varphi_{ab} + E^{ij}, \tag{4.7}$$

where B^c, $D^{ijab} = D^{jiab} = D^{ijba}$, and $E^{ij} = E^{ji}$ are tensor densities of scalar order 1 and $B^{c;d} = B^{d;c}$. Using the invariance identity for a vector density, we may use this last property to write $B^c = \sqrt{g}P^{;c}$, where $P = P(g_{ij}, \varphi, \varphi_i) = P(\varphi, S)$ is a scalar and $S = g^{ab}\varphi_a\varphi_b$. Hence, we rewrite (4.7) to

$$A^{ij} = \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{abc}P^{;c} + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{ac}\varphi_{bd}P^{;c;d} + D^{ijab}\varphi_{ab} + E^{ij} \tag{4.8}$$

and substitute this result into the divergence-free condition (4.6), simplifying the resulting expression

$$\begin{aligned}
0 &= \left(\sqrt{g}\varepsilon^{ik}\varepsilon^{jl}P^{;m}\right)\varphi_{klmj} + \left(2\sqrt{g}\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{kl}\,\varepsilon^{ik}\varepsilon^{jb}\varphi_{bd}P^{;l;d} + D^{ijkl}\right)\varphi_{klj} \\
&\quad + \Big(\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{abc}P^{;c;k} + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{ac}\varphi_{bd}P^{;c;d;k} + D^{ijab;k}\varphi_{ab} + E^{ij;k} \\
&\quad - \sqrt{g}Rg_{ml}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{jl}P^{;m} + \sqrt{g}Rg_{ml}\,\mathrm{Sym}_{ljk}\,\varepsilon^{il}\varepsilon^{mj}P^{;k}\Big)\varphi_{kj} \\
&\quad + \frac{1}{4}\sqrt{g}\left(g_{kl}\,\mathrm{Sym}_{jlm}\,\varepsilon^{ij}\varepsilon^{kl}P^{;m} - g_{kl}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{jl}P^{;m}\right)R_{|m}\varphi_j \\
&\quad + \Bigg[\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{abc}\frac{\partial P^{;c}}{\partial\varphi} + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{ac}\varphi_{bd}\frac{\partial P^{;c;d}}{\partial\varphi} + \frac{\partial D^{ijab}}{\partial\varphi}\varphi_{ab} + \frac{\partial E^{ij}}{\partial\varphi} \\
&\quad - \frac{1}{3}Rg_{kl}\left(2\sqrt{g}\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{kl}\,\varepsilon^{ik}\varepsilon^{jb}\varphi_{bd}P^{;l;d} + D^{ijkl}\right) \\
&\quad + \frac{1}{3}Rg_{kl}\left(2\sqrt{g}\,\mathrm{Sym}_{il}\,\mathrm{Sym}_{kj}\,\varepsilon^{ik}\varepsilon^{lb}\varphi_{bd}P^{;j;d} + D^{ilkj}\right)\Bigg]\varphi_j \\
&= \left(\sqrt{g}\varepsilon^{ik}\varepsilon^{jb}\varphi_{bd}P^{;l;d} + D^{ijkl}\right)\varphi_{klj} + \Big(\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{abc}P^{;c;k} + D^{ijab;k}\varphi_{ab} + E^{ij;k} \\
&\quad - \sqrt{g}Rg_{ml}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{jl}P^{;m} + \sqrt{g}Rg_{ml}\,\mathrm{Sym}_{ljk}\,\varepsilon^{il}\varepsilon^{mj}P^{;k}\Big)\varphi_{kj} \\
&\quad + \frac{1}{4}\sqrt{g}\left(g_{kl}\,\mathrm{Sym}_{jlm}\,\varepsilon^{ij}\varepsilon^{kl}P^{;m} - g_{kl}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{jl}P^{;m}\right)R_{|m}\varphi_j \\
&\quad + \Bigg[\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{abc}\frac{\partial P^{;c}}{\partial\varphi} + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\varphi_{ac}\varphi_{bd}\frac{\partial P^{;c;d}}{\partial\varphi} + \frac{\partial D^{ijab}}{\partial\varphi}\varphi_{ab} + \frac{\partial E^{ij}}{\partial\varphi} \\
&\quad - \frac{1}{3}Rg_{kl}\left(2\sqrt{g}\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{kl}\,\varepsilon^{ik}\varepsilon^{jb}\varphi_{bd}P^{;l;d} + D^{ijkl}\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{3} R g_{kl} \left(2\sqrt{g} \operatorname{Sym}_{il} \operatorname{Sym}_{kj} \varepsilon^{ik} \varepsilon^{lb} \varphi_{bd} P^{;j;d} + D^{ilkj} \right) \Big] \varphi_j \\
= {} & D^{ijkl} \varphi_{klj} + \Big(D^{ijab;k} \varphi_{ab} + E^{ij;k} - \sqrt{g} R g_{ml} \operatorname{Sym}_{klm} \varepsilon^{ik} \varepsilon^{jl} P^{;m} \\
&+ \sqrt{g} R g_{ml} \operatorname{Sym}_{ljk} \varepsilon^{il} \varepsilon^{mj} P^{;k} \Big) \varphi_{kj} + \frac{1}{4} \sqrt{g} \Big(g_{kl} \operatorname{Sym}_{jlm} \varepsilon^{ij} \varepsilon^{kl} P^{;m} \\
&- g_{kl} \operatorname{Sym}_{klm} \varepsilon^{ik} \varepsilon^{jl} P^{;m} \Big) R_{|m} \varphi_j + \Big[\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} \varphi_{abc} \frac{\partial P^{;c}}{\partial \varphi} + \sqrt{g} \varepsilon^{ia} \varepsilon^{jb} \varphi_{ac} \varphi_{bd} \frac{\partial P^{;c;d}}{\partial \varphi} \\
&+ \frac{\partial D^{ijab}}{\partial \varphi} \varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} - \frac{1}{3} R g_{kl} \left(2\sqrt{g} \operatorname{Sym}_{ij} \operatorname{Sym}_{kl} \varepsilon^{ik} \varepsilon^{jb} \varphi_{bd} P^{;l;d} + D^{ijkl} \right) \\
&+ \frac{1}{3} R g_{kl} \left(2\sqrt{g} \operatorname{Sym}_{il} \operatorname{Sym}_{kj} \varepsilon^{ik} \varepsilon^{lb} \varphi_{bd} P^{;j;d} + D^{ilkj} \right) \Big] \varphi_j .
\end{aligned} \tag{4.9}
$$

We differentiate this equation with respect to φ_{klj} to get

$$
0 = \operatorname{Sym}_{abc} D^{iabc} + \sqrt{g} \operatorname{Sym}_{abc} \varepsilon^{ia} \varepsilon^{jb} \frac{\partial P^{;c}}{\partial \varphi} \varphi_j , \tag{4.10}
$$

which can be rearranged to the covector equation

$$
D(X, Y, Y, Y) = -\sqrt{g} \det(X, Y) \det(\nabla \varphi, Y) \frac{\partial P}{\partial \varphi}(; Y). \tag{4.11}
$$

As the kernel of this equation is non-trivial, we return to the divergence-free condition to extract more information about D^{ijkl}. Using (4.10) to remove the third order φ terms from the divergence-free condition, we get the following equation

$$
\begin{aligned}
0 = {} & \Big(D^{ijab;k} \varphi_{ab} + E^{ij;k} - \sqrt{g} R g_{ml} \operatorname{Sym}_{klm} \varepsilon^{ik} \varepsilon^{jl} P^{;m} + \sqrt{g} R g_{ml} \operatorname{Sym}_{ljk} \varepsilon^{il} \varepsilon^{mj} P^{;k} \Big) \varphi_{kj} \\
&+ \frac{1}{4} \sqrt{g} \Big(g_{kl} \operatorname{Sym}_{jlm} \varepsilon^{ij} \varepsilon^{kl} P^{;m} - g_{kl} \operatorname{Sym}_{klm} \varepsilon^{ik} \varepsilon^{jl} P^{;m} \Big) R_{|m} \varphi_j \\
&+ \Big[\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} \varphi_{ac} \varphi_{bd} \frac{\partial P^{;c;d}}{\partial \varphi} + \frac{\partial D^{ijab}}{\partial \varphi} \varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} \\
&- \frac{1}{3} R g_{kl} \left(2\sqrt{g} \operatorname{Sym}_{ij} \operatorname{Sym}_{kl} \varepsilon^{ik} \varepsilon^{jb} \varphi_{bd} P^{;l;d} + D^{ijkl} \right) \\
&+ \frac{1}{3} R g_{kl} \left(2\sqrt{g} \operatorname{Sym}_{il} \operatorname{Sym}_{kj} \varepsilon^{ik} \varepsilon^{lb} \varphi_{bd} P^{;j;d} + D^{ilkj} \right) \Big] \varphi_j .
\end{aligned} \tag{4.12}
$$

We contract this equation with U_i and apply the differential operators $X_a X_b \dfrac{\partial}{\partial \varphi_{ab}}$ and $Y_c Y_d \dfrac{\partial}{\partial \varphi_{cd}}$ to get

$$\begin{aligned} 0 &= D(U,Y,X,X;Y) + D(U,X,Y,Y;X) + \sqrt{g}\,[\det(U,X)\det(\nabla\varphi,Y) \\ &\quad + \det(U,Y)\det(\nabla\varphi,X)]\,\frac{\partial P}{\partial \varphi}(;Y;X). \end{aligned} \tag{4.13}$$

Setting $Y = X$, this equation simplifies to

$$0 = D(U,X,X,X;X) + \sqrt{g}\det(U,X)\det(\nabla\varphi,X)\frac{\partial P}{\partial \varphi}(;X;X).$$

We note that this equation does not contain new information, as it is identical to the equation produced by applying the differential operator $X^a \frac{\partial}{\partial \varphi_a}$ to (4.11). Equations (4.11) and (4.13) give enough information to fully determine D^{ijab}.

Lemma 3. *If D^{ijab} obeys (4.11) and (4.13), then*

$$D(X,X,Y,Y) = \sqrt{g}\left[2\det(X,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial \varphi}(;Y) + \det(X,Y)^2\frac{\partial P}{\partial \varphi} + \det(X,Y)^2 Q\right],$$

where $Q = Q(\varphi)$ is a scalar.

Proof. We consider the ansatz

$$\tilde{D}(X,X,Y,Y) = 2\sqrt{g}\det(X,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial \varphi}(;Y)$$

which has the same symmetries as D^{ijab}. We define a difference function Δ

$$\begin{aligned} \Delta(X,X,Y,Y) &= D(X,X,Y,Y) - \tilde{D}(X,X,Y,Y) \\ &= D(X,X,Y,Y) - 2\sqrt{g}\det(X,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial \varphi}(;Y), \end{aligned}$$

also possessing the symmetries of D and $\tilde{D}$. Utilizing these symmetries, we replace one X with a Y and use (4.11) to write

$$\begin{aligned}\Delta(X,Y,Y,Y) &= D(X,Y,Y,Y) - \left[0 + \sqrt{g}\det(X,Y)\det(Y,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y)\right]\\ &= -\sqrt{g}\det(X,Y)\det(\nabla\varphi,Y)\frac{\partial P}{\partial\varphi}(;Y) - \sqrt{g}\det(X,Y)\det(Y,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y)\\ &= 0.\end{aligned}$$

This equation has a non-trivial kernel, with the solution given by

$$\Delta(X,X,Y,Y) = \sqrt{g}\det(X,Y)^2\tilde{Q},$$

where $\tilde{Q} = \tilde{Q}(\varphi, S)$ is a scalar. Hence, we may write

$$\begin{aligned}D(X,X,Y,Y) &= \tilde{D}(X,X,Y,Y) + \sqrt{g}\det(X,Y)^2\tilde{Q}\\ &= 2\sqrt{g}\det(X,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y) + \sqrt{g}\det(X,Y)^2\tilde{Q}.\end{aligned}$$

To identify $\tilde{Q}$, we use the symmetry of D to replace an X with a U

$$\begin{aligned}D(U,X,Y,Y) = {}&\sqrt{g}\det(U,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y) + \sqrt{g}\det(X,Y)\det(U,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y)\\ &+ \sqrt{g}\det(U,Y)\det(X,Y)\tilde{Q}\end{aligned}$$

and then apply the differential operator $X^i\dfrac{\partial}{\partial\varphi_i}$ to this equation

$$\begin{aligned}&D(U,X,Y,Y;X)\\ &\quad= \sqrt{g}\det(U,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y;X) + \sqrt{g}\det(X,Y)\det(U,X)\frac{\partial P}{\partial\varphi}(;Y)\\ &\qquad+ \sqrt{g}\det(X,Y)\det(U,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y;X) + \sqrt{g}\det(U,Y)\det(X,Y)\tilde{Q}(;X).\end{aligned}$$

This equation is summed with an $X \leftrightarrow Y$ swapped copy of the same equation

$$
\begin{aligned}
&D(U,X,Y,Y;X) + D(U,Y,X,X;Y) \\
&\quad = \sqrt{g}\det(U,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y;X) + \sqrt{g}\det(X,Y)\det(U,X)\frac{\partial P}{\partial\varphi}(;Y) \\
&\qquad + \sqrt{g}\det(X,Y)\det(U,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y;X) + \sqrt{g}\det(U,Y)\det(X,Y)\tilde{Q}(;X) \\
&\qquad + \sqrt{g}\det(U,X)\det(Y,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;X;Y) + \sqrt{g}\det(Y,X)\det(U,Y)\frac{\partial P}{\partial\varphi}(;X) \\
&\qquad + \sqrt{g}\det(Y,X)\det(U,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;X;Y) + \sqrt{g}\det(U,X)\det(Y,X)\tilde{Q}(;Y) \\
&\quad = \sqrt{g}\det(U,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y;X) + \sqrt{g}\det(U,X)\det(Y,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;X;Y) \\
&\qquad + \sqrt{g}\det(X,Y)\left[\det(U,X)\frac{\partial P}{\partial\varphi}(;Y) - \det(U,Y)\frac{\partial P}{\partial\varphi}(;X)\right] \\
&\qquad + \sqrt{g}\det(X,Y)\left[\det(U,Y)\tilde{Q}(;X) - \det(U,X)\tilde{Q}(;Y)\right].
\end{aligned}
$$

We simplify the terms in brackets using the Jacobi identity (3.16)

$$
\begin{aligned}
&D(U,X,Y,Y;X) + D(U,Y,X,X;Y) \\
&\quad = \sqrt{g}\det(U,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y;X) + \sqrt{g}\det(U,X)\det(Y,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;X;Y) \\
&\qquad + \sqrt{g}\det(X,Y)\left[-\det(X,Y)\frac{\partial P}{\partial\varphi}(;U)\right] + \sqrt{g}\det(X,Y)\left[\det(X,Y)\tilde{Q}(;U)\right] \\
&\quad = \sqrt{g}\det(U,Y)\det(X,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;Y;X) + \sqrt{g}\det(U,X)\det(Y,\nabla\varphi)\frac{\partial P}{\partial\varphi}(;X;Y) \\
&\qquad + \sqrt{g}\det(X,Y)^2\left[\tilde{Q} - \frac{\partial P}{\partial\varphi}\right](;U).
\end{aligned}
$$

Comparing this result with (4.13), we see

$$
\tilde{Q} - \frac{\partial P}{\partial\varphi} = Q,
$$

where $Q = Q(\varphi)$ is a scalar. □

In index notation, Lemma 3 states that D^{ijab} takes the form

$$D^{ijab} = 2\sqrt{g}\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{ab}\,\varepsilon^{ia}\varphi_\star^j \frac{\partial P^{;b}}{\partial\varphi} + \sqrt{g}\,\mathrm{Sym}_{ab}\,\varepsilon^{ia}\varepsilon^{jb}\frac{\partial P}{\partial\varphi} + \sqrt{g}\,\mathrm{Sym}_{ab}\,\varepsilon^{ia}\varepsilon^{jb}Q, \qquad (4.14)$$

where $\varphi_\star^i = \varepsilon^{ij}\varphi_j$. The vector $\varphi_\star^i$ obeys the following identities

$$\begin{aligned}
\varphi_\star^i\varphi_i &= \varepsilon^{ij}\varphi_i\varphi_j = 0,\\
\varphi_\star^i\varepsilon_{ij} &= \varepsilon^{ik}\varphi_k\varepsilon_{ij} = (-1)^q\varphi_j\\
\varphi_\star^i\varphi_{\star i} &= \varepsilon^{ij}\varphi_j g_{ik}\varepsilon^{km}\varphi_m = (-1)^q g^{jm}\varphi_j\varphi_m = (-1)^q S.
\end{aligned} \qquad (4.15)$$

Returning to the divergence-free condition (4.12), we remove the terms quadratic in φ_{ab} using (4.13) and get (leaving D^{ijab} unchanged for brevity)

$$\begin{aligned}
0 = &\left(E^{ij;k} - \sqrt{g}Rg_{ml}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{jl}P^{;m} + \sqrt{g}Rg_{ml}\,\mathrm{Sym}_{ljk}\,\varepsilon^{il}\varepsilon^{mj}P^{;k}\right)\varphi_{kj}\\
&+ \frac{1}{4}\sqrt{g}\left(g_{kl}\,\mathrm{Sym}_{jlm}\,\varepsilon^{ij}\varepsilon^{kl}P^{;m} - g_{kl}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{jl}P^{;m}\right)R_{|m}\varphi_j\\
&+ \left[\frac{\partial D^{ijab}}{\partial\varphi}\varphi_{ab} + \frac{\partial E^{ij}}{\partial\varphi} - \frac{1}{3}Rg_{kl}\left(2\sqrt{g}\,\mathrm{Sym}_{ij}\,\varepsilon^{ia}\varepsilon^{jk}P^{;c;l}\varphi_{ac} + D^{ijkl}\right)\right.\\
&\left.+ \frac{1}{3}Rg_{kl}\left(2\sqrt{g}\,\mathrm{Sym}_{il}\,\mathrm{Sym}_{kj}\,\varepsilon^{ia}\varepsilon^{lk}P^{;c;j}\varphi_{ac} + D^{ilkj}\right)\right]\varphi_j.
\end{aligned} \qquad (4.16)$$

We expand the symmetries and simplify the first term in (4.16) to get

$$\begin{aligned}
&\left(E^{ij;k} - \sqrt{g}Rg_{ml}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{jl}P^{;m} + \sqrt{g}Rg_{ml}\,\mathrm{Sym}_{ljk}\,\varepsilon^{il}\varepsilon^{mj}P^{;k}\right)\varphi_{jk}\\
&\quad= \left[E^{ij;k} - \frac{1}{6}\sqrt{g}Rg_{ml}\left(\varepsilon^{ik}\varepsilon^{jl}P^{;m} + \varepsilon^{ik}\varepsilon^{jm}P^{;l} + \varepsilon^{il}\varepsilon^{jk}P^{;m} + \varepsilon^{il}\varepsilon^{jm}P^{;k} + \varepsilon^{im}\varepsilon^{jk}P^{;l}\right.\right.\\
&\qquad\left. + \varepsilon^{im}\varepsilon^{jl}P^{;k}\right) + \frac{1}{6}\sqrt{g}Rg_{ml}\left(\varepsilon^{il}\varepsilon^{mj}P^{;k} + \varepsilon^{il}\varepsilon^{mk}P^{;j} + \varepsilon^{ij}\varepsilon^{ml}P^{;k} + \varepsilon^{ij}\varepsilon^{mk}P^{;l}\right.\\
&\qquad\left.\left. + \varepsilon^{ik}\varepsilon^{mj}P^{;l} + \varepsilon^{ik}\varepsilon^{ml}P^{;j}\right)\right]\varphi_{jk}\\
&\quad= \left\{E^{ij;k} - \frac{1}{3}\sqrt{g}R\left[g_{ml}\varepsilon^{ik}\varepsilon^{jl}P^{;m} + (-1)^q g^{ij}P^{;k}\right] + \frac{1}{3}\sqrt{g}R\left(-g^{ij}P^{;k}\right.\right.\\
&\qquad\left.\left. + g_{ml}\varepsilon^{ij}\varepsilon^{mk}P^{;l}\right)\right\}\varphi_{jk}\\
&\quad= \left\{E^{ij;k} - \frac{2}{3}\sqrt{g}R\left[g_{ml}\varepsilon^{ik}\varepsilon^{jl}P^{;m} + (-1)^q g^{ij}P^{;k}\right]\right\}\varphi_{jk}
\end{aligned}$$

$$= \left[E^{ij;k} - \frac{2}{3}(-1)^q \sqrt{g} R \left(g^{ij} P^{;k} - g^{jk} P^{;i} + g^{ij} P^{;k} \right) \right] \varphi_{jk}$$
$$= \left[E^{ij;k} - \frac{2}{3}(-1)^q \sqrt{g} R \left(2g^{ij} P^{;k} - g^{jk} P^{;i} \right) \right] \varphi_{jk},$$

while similar procedures simplify the second

$$\begin{aligned}
&\frac{1}{4}\sqrt{g} \left(g_{kl} \operatorname{Sym}_{jlm} \varepsilon^{ij} \varepsilon^{kl} P^{;m} - g_{kl} \operatorname{Sym}_{klm} \varepsilon^{ik} \varepsilon^{jl} P^{;m} \right) R_{|m} \varphi_j \\
&\quad = \frac{1}{24}\sqrt{g} \left[g_{kl} \left(\varepsilon^{ij} \varepsilon^{kl} P^{;m} + \varepsilon^{ij} \varepsilon^{km} P^{;l} + \varepsilon^{il} \varepsilon^{kj} P^{;m} + \varepsilon^{il} \varepsilon^{km} P^{;j} + \varepsilon^{im} \varepsilon^{kl} P^{;j} + \varepsilon^{im} \varepsilon^{kj} P^{;l} \right) \right. \\
&\qquad \left. - g_{kl} \left(\varepsilon^{ik} \varepsilon^{jl} P^{;m} + \varepsilon^{ik} \varepsilon^{jm} P^{;l} + \varepsilon^{il} \varepsilon^{jk} P^{;m} + \varepsilon^{il} \varepsilon^{jm} P^{;k} + \varepsilon^{im} \varepsilon^{jl} P^{;k} + \varepsilon^{im} \varepsilon^{jk} P^{;l} \right) \right] \\
&\qquad \times R_{|m} \varphi_j \\
&\quad = \frac{1}{24}\sqrt{g} \left\{ \left[g_{kl} \varepsilon^{ij} \varepsilon^{km} P^{;l} - (-1)^q g^{ij} P^{;m} - (-1)^q g^{im} P^{;j} + g_{kl} \varepsilon^{im} \varepsilon^{kj} P^{;l} \right] \right. \\
&\qquad \left. - \left[2(-1)^q g^{ij} P^{;m} + 2 g_{kl} \varepsilon^{ik} \varepsilon^{jm} P^{;l} + 2 g_{kl} \varepsilon^{im} \varepsilon^{jl} P^{;k} \right] \right\} R_{|m} \varphi_j \\
&\quad = \frac{1}{24}\sqrt{g} \left[g_{kl} \varepsilon^{ij} \varepsilon^{km} P^{;l} - 3(-1)^q g^{ij} P^{;m} - (-1)^q g^{im} P^{;j} + 3(-1)^q g_{kl} \varepsilon^{im} \varepsilon^{kj} P^{;l} \right. \\
&\qquad \left. - 2 g_{kl} \varepsilon^{ik} \varepsilon^{jm} P^{;l} \right] R_{|m} \varphi_j \\
&\quad = \frac{(-1)^q}{24}\sqrt{g} \left[\left(g^{jm} P^{;i} - g^{im} P^{;j} \right) - 3 g^{ij} P^{;m} - g^{im} P^{;j} + 3 \left(g^{jm} P^{;i} - g^{ij} P^{;m} \right) \right. \\
&\qquad \left. - 2 \left(g^{ij} P^{;m} - g^{im} P^{;j} \right) \right] R_{|m} \varphi_j \\
&\quad = \frac{(-1)^q}{24}\sqrt{g} \left(4 g^{jm} P^{;i} - 8 g^{ij} P^{;m} \right) R_{|m} \varphi_j \\
&\quad = \frac{(-1)^q}{6}\sqrt{g} \left(g^{jm} P^{;i} - 2 g^{ij} P^{;m} \right) R_{|m} \varphi_j
\end{aligned}$$

and third terms

$$\begin{aligned}
&\left[\frac{\partial D^{ijab}}{\partial \varphi} \varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} - \frac{1}{3} R g_{kl} \left(2\sqrt{g} \operatorname{Sym}_{ij} \varepsilon^{ia} \varepsilon^{jk} P^{;c;l} \varphi_{ac} + D^{ijkl} \right) \right. \\
&\qquad \left. + \frac{1}{3} R g_{kl} \left(2\sqrt{g} \operatorname{Sym}_{il} \operatorname{Sym}_{kj} \varepsilon^{ia} \varepsilon^{lk} P^{;c;j} \varphi_{ac} + D^{ilkj} \right) \right] \varphi_j \\
&\quad = \left\{ \frac{\partial D^{ijab}}{\partial \varphi} \varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} - \frac{1}{3} R g_{kl} \left[\sqrt{g} \left(\varepsilon^{ia} \varepsilon^{jk} + \varepsilon^{ja} \varepsilon^{ik} \right) P^{;c;l} \varphi_{ac} + D^{ijkl} \right] \right. \\
&\qquad \left. + \frac{1}{3} R g_{kl} \left[\frac{1}{2}\sqrt{g} \left(\varepsilon^{ia} \varepsilon^{lk} P^{;c;j} + \varepsilon^{ia} \varepsilon^{lj} P^{;c;k} + \varepsilon^{la} \varepsilon^{ik} P^{;c;j} + \varepsilon^{la} \varepsilon^{ij} P^{;c;k} \right) \varphi_{ac} + D^{ilkj} \right] \right\} \varphi_j
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{\partial D^{ijab}}{\partial \varphi}\varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} - \frac{(-1)^q}{3}\sqrt{g}R\left(g_{kl}(g^{ij}g^{ak} - g^{ik}g^{ja}) + g_{kl}(g^{ij}g^{ak} - g^{jk}g^{ia})\right)P^{;c;l}\varphi_{ac} \right.\\
&\quad - \frac{1}{3}Rg_{kl}D^{ijkl} + \frac{(-1)^q}{6}\sqrt{g}R\left(0 + g_{kl}(g^{il}g^{aj} - g^{ij}g^{al})P^{;c;k} - g^{ia}P^{;c;j}\right.\\
&\quad \left.\left. + g_{kl}(g^{il}g^{ja} - g^{jl}g^{ia})P^{;c;k}\right)\varphi_{ac} + \frac{1}{3}Rg_{kl}D^{ilkj}\right]\varphi_j \\
&= \left[\frac{\partial D^{ijab}}{\partial \varphi}\varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} - \frac{(-1)^q}{3}\sqrt{g}R\left(g^{ij}P^{;c;a} - g^{ja}P^{;c;i} + g^{ij}P^{;c;a} - g^{ia}P^{;c;j}\right)\varphi_{ac} \right.\\
&\quad + \frac{(-1)^q}{6}\sqrt{g}R\left(g^{aj}P^{;c;i} - g^{ij}P^{;c;a} - g^{ia}P^{;c;j} + g^{ja}P^{;c;i} - g^{ia}P^{;c;j}\right)\varphi_{ac}\\
&\quad \left. + \frac{1}{3}Rg_{kl}\left(-D^{ijkl} + D^{ilkj}\right)\right]\varphi_j \\
&= \left[\frac{\partial D^{ijab}}{\partial \varphi}\varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} - \frac{(-1)^q}{3}\sqrt{g}R\left(2g^{ij}P^{;c;a} - g^{ja}P^{;c;i} - g^{ia}P^{;c;j}\right)\varphi_{ac} \right.\\
&\quad \left. + \frac{(-1)^q}{6}\sqrt{g}R\left(2g^{aj}P^{;c;i} - g^{ij}P^{;c;a} - 2g^{ia}P^{;c;j}\right)\varphi_{ac} + \frac{1}{3}Rg_{kl}(-D^{ijkl} + D^{ilkj})\right]\varphi_j \\
&= \left(\frac{\partial D^{ijab}}{\partial \varphi}\varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} - \frac{5}{6}(-1)^q\sqrt{g}Rg^{ij}P^{;c;a}\varphi_{ac} + \frac{2}{3}(-1)^q\sqrt{g}Rg^{ja}P^{;c;i}\varphi_{ac} \right.\\
&\quad \left. - \frac{1}{3}Rg_{kl}D^{ijkl} + \frac{1}{3}Rg_{kl}D^{ilkj}\right)\varphi_j.
\end{aligned}
$$

Using these simplified expressions, the divergence-free condition (4.16) is now given by

$$
\begin{aligned}
0 &= \left[E^{ij;k} - \frac{2}{3}(-1)^q\sqrt{g}R\left(2g^{ij}P^{;k} - g^{jk}P^{;i}\right)\right]\varphi_{jk} + \frac{(-1)^q}{6}\sqrt{g}\left(g^{jm}P^{;i} - 2g^{ij}P^{;m}\right)R_{|m}\varphi_j \\
&\quad + \left(\frac{\partial D^{ijab}}{\partial \varphi}\varphi_{ab} + \frac{\partial E^{ij}}{\partial \varphi} - \frac{5}{6}(-1)^q\sqrt{g}Rg^{ij}P^{;c;a}\varphi_{ac} + \frac{2}{3}(-1)^q\sqrt{g}Rg^{aj}P^{;c;i}\varphi_{ac} \right.\\
&\quad \left. - \frac{1}{3}Rg_{kl}D^{ijkl} + \frac{1}{3}Rg_{kl}D^{ilkj}\right)\varphi_j.
\end{aligned} \tag{4.17}
$$

We apply the differential operator $U_i X_j X_k \frac{\partial}{\partial \varphi_{jk}}$ to this equation and solve for the derivative of E^{ij} with respect to φ_k

$$
\begin{aligned}
E(U, X; X) &= \frac{2}{3}(-1)^q\sqrt{g}R\left[2g(U,X)P(;X) - g(X,X)P(;U)\right] - \frac{\partial D}{\partial \varphi}(U, \nabla\varphi, X, X) \\
&\quad + \frac{5}{6}(-1)^q\sqrt{g}Rg(U,\nabla\varphi)P(;X;X) - \frac{2}{3}(-1)^q\sqrt{g}Rg(X,\nabla\varphi)P(;U;X).
\end{aligned} \tag{4.18}
$$

We expand the symmetry of this equation by setting one X to Y and sum the resulting equation with a version which has X and Y swapped

$$\begin{aligned}E(U,X;Y)+E(U,Y;X)=&\left\{\frac{2}{3}\sqrt{g}R\left[2g(U,X)P(;Y)+2g(U,Y)P(;X)-2g(X,Y)P(;U)\right]\right.\\&+\frac{5}{3}\sqrt{g}Rg(U,\nabla\varphi)P(;X;Y)-\frac{2}{3}\sqrt{g}Rg(Y,\nabla\varphi)P(;U;X)\\&\left.-\frac{2}{3}\sqrt{g}Rg(X,\nabla\varphi)P(;U;Y)\right\}(-1)^q-2\frac{\partial D}{\partial\varphi}(U,\nabla\varphi,X,Y).\end{aligned}$$

We create two additional copies of this equation by cyclically permuting U, X, and Y

$$\begin{aligned}E(X,Y;U)+E(X,U;Y)=&\left\{\frac{2}{3}\sqrt{g}R\left[2g(X,Y)P(;U)+2g(X,U)P(;Y)-2g(Y,U)P(;X)\right]\right.\\&+\frac{5}{3}\sqrt{g}Rg(X,\nabla\varphi)P(;Y;U)-\frac{2}{3}\sqrt{g}Rg(U,\nabla\varphi)P(;X;Y)\\&\left.-\frac{2}{3}\sqrt{g}Rg(Y,\nabla\varphi)P(;X;U)\right\}(-1)^q-2\frac{\partial D}{\partial\varphi}(X,\nabla\varphi,Y,U)\\E(Y,U;X)+E(Y,X;U)=&\left\{\frac{2}{3}\sqrt{g}R\left[2g(Y,U)P(;X)+2g(Y,X)P(;U)-2g(U,X)P(;Y)\right]\right.\\&+\frac{5}{3}\sqrt{g}Rg(Y,\nabla\varphi)P(;U;X)-\frac{2}{3}\sqrt{g}Rg(X,\nabla\varphi)P(;Y;U)\\&\left.-\frac{2}{3}\sqrt{g}Rg(U,\nabla\varphi)P(;Y;X)\right\}(-1)^q-2\frac{\partial D}{\partial\varphi}(Y,\nabla\varphi,U,X).\end{aligned}$$

We sum the first two equations and subtract the third, with the left side of the resulting equation given by $2E(U,X;Y)$. Dividing both sides of this equation by 2, we get

$$\begin{aligned}&E(U,X;Y)\\&\quad=\frac{(-1)^q}{2}\left\{\frac{2}{3}\sqrt{g}R\left[2g(U,X)P(;Y)+2g(U,Y)P(;X)-2g(X,Y)P(;U)\right]\right.\\&\quad+\frac{5}{3}\sqrt{g}Rg(U,\nabla\varphi)P(;X;Y)-\frac{2}{3}\sqrt{g}Rg(Y,\nabla\varphi)P(;U;X)-\frac{2}{3}\sqrt{g}Rg(X,\nabla\varphi)P(;U;Y)\\&\quad+\frac{2}{3}\sqrt{g}R\left[2g(X,Y)P(;U)+2g(X,U)P(;Y)-2g(Y,U)P(;X)\right]\\&\quad+\frac{5}{3}\sqrt{g}Rg(X,\nabla\varphi)P(;Y;U)-\frac{2}{3}\sqrt{g}Rg(U,\nabla\varphi)P(;X;Y)-\frac{2}{3}\sqrt{g}Rg(Y,\nabla\varphi)P(;X;U)\\&\quad-\frac{2}{3}\sqrt{g}R\left[2g(Y,U)P(;X)+2g(Y,X)P(;U)-2g(U,X)P(;Y)\right]\\&\quad\left.-\frac{5}{3}\sqrt{g}Rg(Y,\nabla\varphi)P(;U;X)+\frac{2}{3}\sqrt{g}Rg(X,\nabla\varphi)P(;Y;U)+\frac{2}{3}\sqrt{g}Rg(U,\nabla\varphi)P(;Y;X)\right\}\end{aligned}$$

$$+\frac{1}{2}\left(-2\frac{\partial D}{\partial\varphi}(U,\nabla\varphi,X,Y)-2\frac{\partial D}{\partial\varphi}(X,\nabla\varphi,Y,U)+2\frac{\partial D}{\partial\varphi}(Y,\nabla\varphi,U,X)\right)$$
$$=(-1)^q\left\{\frac{2}{3}\sqrt{g}R\left[3g(U,X)P(;Y)-g(Y,U)P(;X)-g(X,Y)P(;U)\right]\right.$$
$$\left.+\frac{5}{6}\sqrt{g}Rg(U,\nabla\varphi)P(;X;Y)+\frac{5}{6}\sqrt{g}Rg(X,\nabla\varphi)P(;Y;U)-\frac{3}{2}\sqrt{g}Rg(Y,\nabla\varphi)P(;X;U)\right\}$$
$$-\frac{\partial D}{\partial\varphi}(U,\nabla\varphi,X,Y)-\frac{\partial D}{\partial\varphi}(X,\nabla\varphi,Y,U)+\frac{\partial D}{\partial\varphi}(Y,\nabla\varphi,U,X).$$

Switching back to index notation, we substitute for D^{ijab} using (4.14) and expand the symmetries

$$\begin{aligned}
E^{ij;k} &= (-1)^q\left(2\sqrt{g}Rg^{ij}P^{;k}-\frac{4}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{ik}P^{;j}+\frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,\varphi^iP^{;j;k}-\frac{3}{2}\sqrt{g}R\varphi^kP^{;i;j}\right)\\
&\quad-2\,\mathrm{Sym}_{ij}\,\frac{\partial D^{iajk}}{\partial\varphi}\varphi_a+\frac{\partial D^{kaij}}{\partial\varphi}\varphi_a\\
&= (-1)^q\left(2\sqrt{g}Rg^{ij}P^{;k}-\frac{4}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{ik}P^{;j}+\frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,\varphi^iP^{;j;k}-\frac{3}{2}\sqrt{g}R\varphi^kP^{;i;j}\right)\\
&\quad-2\,\mathrm{Sym}_{ij}\,\sqrt{g}\left[\frac{1}{2}\left(\varepsilon^{ij}\varphi_\star^a\frac{\partial^2P^{;k}}{\partial\varphi^2}+\varepsilon^{aj}\varphi_\star^i\frac{\partial^2P^{;k}}{\partial\varphi^2}+\varepsilon^{ik}\varphi_\star^a\frac{\partial^2P^{;j}}{\partial\varphi^2}+\varepsilon^{ak}\varphi_\star^i\frac{\partial^2P^{;j}}{\partial\varphi^2}\right)\right.\\
&\quad\left.+\frac{1}{2}\left(\varepsilon^{ij}\varepsilon^{ak}+\varepsilon^{aj}\varepsilon^{ik}\right)\left(\frac{\partial^2P}{\partial\varphi^2}+Q'\right)\right]\varphi_a+\sqrt{g}\left[\frac{1}{2}\left(\varepsilon^{ki}\varphi_\star^a\frac{\partial^2P^{;j}}{\partial\varphi^2}+\varepsilon^{ai}\varphi_\star^k\frac{\partial^2P^{;j}}{\partial\varphi^2}\right.\right.\\
&\quad\left.\left.+\varepsilon^{kj}\varphi_\star^a\frac{\partial^2P^{;i}}{\partial\varphi^2}+\varepsilon^{aj}\varphi_\star^k\frac{\partial^2P^{;i}}{\partial\varphi^2}\right)+\frac{1}{2}\left(\varepsilon^{ki}\varepsilon^{aj}+\varepsilon^{ai}\varepsilon^{kj}\right)\left(\frac{\partial^2P}{\partial\varphi^2}+Q'\right)\right]\varphi_a\\
&= (-1)^q\left(2\sqrt{g}Rg^{ij}P^{;k}-\frac{4}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{ik}P^{;j}+\frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,\varphi^iP^{;j;k}-\frac{3}{2}\sqrt{g}R\varphi^kP^{;i;j}\right)\\
&\quad+2\,\mathrm{Sym}_{ij}\,\sqrt{g}\left[\frac{1}{2}\left(\varphi_\star^i\varphi_\star^j\frac{\partial^2P^{;k}}{\partial\varphi^2}+\varphi_\star^i\varphi_\star^k\frac{\partial^2P^{;j}}{\partial\varphi^2}\right)+\frac{1}{2}\left(\varepsilon^{ij}\varphi_\star^k+\varepsilon^{ik}\varphi_\star^j\right)\left(\frac{\partial^2P}{\partial\varphi^2}+Q'\right)\right]\\
&\quad-\sqrt{g}\left[\frac{1}{2}\left(\varphi_\star^i\varphi_\star^k\frac{\partial^2P^{;j}}{\partial\varphi^2}+\varphi_\star^j\varphi_\star^k\frac{\partial^2P^{;i}}{\partial\varphi^2}\right)+\frac{1}{2}\left(\varepsilon^{ki}\varphi_\star^j+\varepsilon^{kj}\varphi_\star^i\right)\left(\frac{\partial^2P}{\partial\varphi^2}+Q'\right)\right]\\
&= (-1)^q\left(2\sqrt{g}Rg^{ij}P^{;k}-\frac{4}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{ik}P^{;j}+\frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,\varphi^iP^{;j;k}-\frac{3}{2}\sqrt{g}R\varphi^kP^{;i;j}\right)\\
&\quad+\mathrm{Sym}_{ij}\,\sqrt{g}\left[\varphi_\star^i\varphi_\star^j\frac{\partial^2P^{;k}}{\partial\varphi^2}+\varphi_\star^i\varphi_\star^k\frac{\partial^2P^{;j}}{\partial\varphi^2}+\varepsilon^{ik}\varphi_\star^j\left(\frac{\partial^2P}{\partial\varphi^2}+Q'\right)\right]\\
&\quad-\mathrm{Sym}_{ij}\,\sqrt{g}\left[\varphi_\star^i\varphi_\star^k\frac{\partial^2P^{;j}}{\partial\varphi^2}+\varepsilon^{ki}\varphi_\star^j\left(\frac{\partial^2P}{\partial\varphi^2}+Q'\right)\right]\\
&= (-1)^q\left(2\sqrt{g}Rg^{ij}P^{;k}-\frac{4}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{ik}P^{;j}+\frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,\varphi^iP^{;j;k}-\frac{3}{2}\sqrt{g}R\varphi^kP^{;i;j}\right)\\
&\quad+\sqrt{g}\varphi_\star^i\varphi_\star^j\frac{\partial^2P^{;k}}{\partial\varphi^2}+2\sqrt{g}\,\mathrm{Sym}_{ij}\,\varepsilon^{ik}\varphi_\star^j\left(\frac{\partial^2P}{\partial\varphi^2}+Q'\right),
\end{aligned}\tag{4.19}$$

where $Q' = \dfrac{\mathrm{d}Q}{\mathrm{d}\varphi}$. Using $P^{;i} = 2\varphi^i \dfrac{\partial P}{\partial S}$, we rewrite the third and fourth terms in parenthesis

$$
\begin{aligned}
&\frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,\varphi^i P^{;j;k} - \frac{3}{2}\sqrt{g}R\varphi^k P^{;i;j} \\
&\quad = \frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,\varphi^i\left(2g^{jk}\frac{\partial P}{\partial S} + 4\varphi^j\varphi^k\frac{\partial^2 P}{\partial S^2}\right) - \frac{3}{2}\sqrt{g}R\varphi^k\left(2g^{ij}\frac{\partial P}{\partial S} + 4\varphi^i\varphi^j\frac{\partial^2 P}{\partial S^2}\right) \\
&\quad = \frac{10}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{jk}\varphi^i\frac{\partial P}{\partial S} + \frac{20}{3}\sqrt{g}R\varphi^i\varphi^j\varphi^k\frac{\partial^2 P}{\partial S^2} - 3\sqrt{g}Rg^{ij}\varphi^k\frac{\partial P}{\partial S} - 6\sqrt{g}R\varphi^i\varphi^j\varphi^k\frac{\partial^2 P}{\partial S^2} \\
&\quad = \frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{jk}P^{;i} + \frac{2}{3}\sqrt{g}R\varphi^i\varphi^j\varphi^k\frac{\partial^2 P}{\partial S^2} - \frac{3}{2}\sqrt{g}Rg^{ij}P^{;k} \\
&\quad = \frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{jk}P^{;i} + \left(\frac{1}{3}\sqrt{g}R\varphi^i\varphi^j\frac{\partial P}{\partial S}\right)^{;k} - \frac{2}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,\varphi^i g^{jk}\frac{\partial P}{\partial S} \\
&\qquad\quad + \left(-\frac{3}{2}\sqrt{g}Rg^{ij}P\right)^{;k} \\
&\quad = \frac{5}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{jk}P^{;i} + \left(\frac{1}{3}\sqrt{g}R\varphi^i\varphi^j\frac{\partial P}{\partial S} - \frac{3}{2}\sqrt{g}Rg^{ij}P\right)^{;k} - \frac{1}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{jk}P^{;i} \\
&\quad = \frac{4}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{jk}P^{;i} + \left(\frac{1}{3}\sqrt{g}R\varphi^i\varphi^j\frac{\partial P}{\partial S} - \frac{3}{2}\sqrt{g}Rg^{ij}P\right)^{;k},
\end{aligned}
$$

as well as the fifth term

$$
\begin{aligned}
\sqrt{g}\varphi_\star^i\varphi_\star^j\frac{\partial^2 P^{;k}}{\partial\varphi^2} &= \left(\sqrt{g}\varphi_\star^i\varphi_\star^j\frac{\partial^2 P}{\partial\varphi^2}\right)^{;k} - \sqrt{g}\varepsilon^{ik}\varphi_\star^j\frac{\partial^2 P}{\partial\varphi^2} - \sqrt{g}\varepsilon^{jk}\varphi_\star^i\frac{\partial^2 P}{\partial\varphi^2} \\
&= \left(\sqrt{g}\varphi_\star^i\varphi_\star^j\frac{\partial^2 P}{\partial\varphi^2}\right)^{;k} - 2\sqrt{g}\,\mathrm{Sym}_{ij}\,\varepsilon^{ik}\varphi_\star^j\frac{\partial^2 P}{\partial\varphi^2}.
\end{aligned}
$$

We substitute these changes into (4.19), yielding the following equation

$$
\begin{aligned}
E^{ij;k} = (-1)^q &\left\{2\sqrt{g}Rg^{ij}P^{;k} - \frac{4}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{ik}P^{;j} + \left[\frac{4}{3}\sqrt{g}R\,\mathrm{Sym}_{ij}\,g^{jk}P^{;i}\right.\right. \\
&\left.\left. + \left(\frac{1}{3}\sqrt{g}R\varphi^i\varphi^j\frac{\partial P}{\partial S} - \frac{3}{2}\sqrt{g}Rg^{ij}P\right)^{;k}\right]\right\} + \left[\left(\sqrt{g}\varphi_\star^i\varphi_\star^j\frac{\partial^2 P}{\partial\varphi^2}\right)^{;k}\right. \\
&\left. - 2\sqrt{g}\,\mathrm{Sym}_{ij}\,\varepsilon^{ik}\varphi_\star^j\frac{\partial^2 P}{\partial\varphi^2}\right] + 2\sqrt{g}\,\mathrm{Sym}_{ij}\,\varepsilon^{ik}\varphi_\star^j\left(\frac{\partial^2 P}{\partial\varphi^2} + Q'\right) \\
= &\left[2(-1)^q\sqrt{g}Rg^{ij}P\right]^{;k} + \left[\frac{(-1)^q}{3}\sqrt{g}R\varphi^i\varphi^j\frac{\partial P}{\partial S} - \frac{3}{2}(-1)^q\sqrt{g}Rg^{ij}P + \sqrt{g}\varphi_\star^i\varphi_\star^j\frac{\partial^2 P}{\partial\varphi^2}\right]^{;k} \\
&+ \left(\sqrt{g}\varphi_\star^i\varphi_\star^j Q'\right)^{;k}
\end{aligned}
$$

$$= \left[\frac{(-1)^q}{3} \sqrt{g} R \varphi^i \varphi^j \frac{\partial P}{\partial S} + \frac{(-1)^q}{2} \sqrt{g} R g^{ij} P + \sqrt{g} \varphi^i_\star \varphi^j_\star \frac{\partial^2 P}{\partial \varphi^2} + \sqrt{g} \varphi^i_\star \varphi^j_\star Q' \right]^{;k},$$

with the expression in parentheses equal to E^{ij} [the integration constant of $F^{ij} = F^{ij}(g_{ij}, \varphi)$ $= \sqrt{g} g^{ij} F$, where $F = F(\varphi)$ is a scalar, is absorbed into the definition of P]

$$E^{ij} = \frac{(-1)^q}{3} \sqrt{g} R \varphi^i \varphi^j \frac{\partial P}{\partial S} + \sqrt{g} \varphi^i_\star \varphi^j_\star \frac{\partial^2 P}{\partial \varphi^2} + \sqrt{g} \varphi^i_\star \varphi^j_\star Q' + \frac{(-1)^q}{2} \sqrt{g} R g^{ij} P. \tag{4.20}$$

Returning to the divergence condition (4.17), we use (4.18) to remove the second order φ terms, yielding

$$0 = \frac{(-1)^q}{6} \sqrt{g} \left(g^{jm} P^{;i} - 2 g^{ij} P^{;m} \right) R_{|m} \varphi_j + \left(\frac{\partial E^{ij}}{\partial \varphi} - \frac{1}{3} R g_{kl} D^{ijkl} + \frac{1}{3} R g_{kl} D^{ilkj} \right) \varphi_j. \tag{4.21}$$

Setting $\varphi = R$, we simplify this equation using (4.14) and (4.20)

$$\begin{aligned}
0 &= \frac{(-1)^q}{6} \sqrt{g} \left(g^{jm} P^{;i} - 2 g^{ij} P^{;m} \right) R_{|m} R_{|j} + \left(\frac{\partial E^{ij}}{\partial R} - \frac{1}{3} R g_{kl} D^{ijkl} + \frac{1}{3} R g_{kl} D^{ilkj} \right) R_{|j} \\
&= \frac{(-1)^q}{6} \sqrt{g} \left(2 S R^{|i} \frac{\partial P}{\partial S} - 4 S R^{|i} \frac{\partial P}{\partial S} \right) + \left[\left(\frac{(-1)^q}{3} \sqrt{g} R^{|i} R^{|j} \frac{\partial P}{\partial S} + \frac{(-1)^q}{3} \sqrt{g} R R^{|i} R^{|j} \frac{\partial^2 P}{\partial R \partial S} \right.\right. \\
&\quad \left. + \sqrt{g} R^{|i}_\star R^{|j}_\star \frac{\partial^3 P}{\partial R^3} + \sqrt{g} R^{|i}_\star R^{|j}_\star Q'' + \frac{(-1)^q}{2} \sqrt{g} g^{ij} P + \frac{(-1)^q}{2} \sqrt{g} R g^{ij} \frac{\partial P}{\partial R} \right) \\
&\quad - \frac{1}{3} R g_{kl} \left(4 \sqrt{g} \, \mathrm{Sym}_{ij} \, \varepsilon^{ik} R^{|j}_\star R^{|l} \frac{\partial^2 P}{\partial R \partial S} + \sqrt{g} \varepsilon^{ik} \varepsilon^{jl} \frac{\partial P}{\partial R} + \sqrt{g} \varepsilon^{ik} \varepsilon^{jl} Q \right) \\
&\quad + \frac{1}{3} R g_{kl} \left(4 \sqrt{g} \, \mathrm{Sym}_{il} \, \mathrm{Sym}_{jk} \, \varepsilon^{ik} R^{|l}_\star R^{|j} \frac{\partial^2 P}{\partial R \partial S} + \sqrt{g} \, \mathrm{Sym}_{jk} \, \varepsilon^{ik} \varepsilon^{lj} \frac{\partial P}{\partial R} \right. \\
&\quad \left.\left. + \sqrt{g} \, \mathrm{Sym}_{jk} \, \varepsilon^{ik} \varepsilon^{lj} Q \right) \right] R_{|j} \\
&= - \frac{(-1)^q}{3} \sqrt{g} S R^{|i} \frac{\partial P}{\partial S} + \frac{(-1)^q}{3} \sqrt{g} S R^{|i} \frac{\partial P}{\partial S} + \frac{(-1)^q}{3} \sqrt{g} S R R^{|i} \frac{\partial^2 P}{\partial R \partial S} + 0 + \frac{(-1)^q}{2} \sqrt{g} R^{|i} P \\
&\quad + \frac{(-1)^q}{2} \sqrt{g} R R^{|i} \frac{\partial P}{\partial R} + \frac{1}{3} R \left\{ - \left[2 \sqrt{g} g_{kl} \left(\varepsilon^{ik} R^{|j}_\star R^{|l} + \varepsilon^{jk} R^{|i}_\star R^{|l} \right) \frac{\partial^2 P}{\partial R \partial S} \right.\right. \\
&\quad \left. + (-1)^q \sqrt{g} g^{ij} \frac{\partial P}{\partial R} + (-1)^q \sqrt{g} g^{ij} Q \right] + \left[\sqrt{g} g_{kl} \left(\varepsilon^{ik} R^{|l}_\star R^{|j} + \varepsilon^{ij} R^{|l}_\star R^{|k} + \varepsilon^{lk} R^{|i}_\star R^{|j} \right.\right. \\
&\quad \left. + \varepsilon^{lj} R^{|i}_\star R^{|k} \right) \frac{\partial^2 P}{\partial R \partial S} + \frac{1}{2} \sqrt{g} g_{kl} \left(\varepsilon^{ik} \varepsilon^{lj} + \varepsilon^{ij} \varepsilon^{lk} \right) \frac{\partial P}{\partial R} + \frac{1}{2} \sqrt{g} g_{kl} \left(\varepsilon^{ik} \varepsilon^{lj} \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \varepsilon^{ij}\varepsilon^{lk}\Big) Q\Big]\Big\} R_{|j} \\
&= \frac{(-1)^q}{3}\sqrt{g} SRR^{|i}\frac{\partial^2 P}{\partial R \partial S} + \frac{(-1)^q}{2}\sqrt{g}R^{|i}P + \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\frac{\partial P}{\partial R} + \frac{1}{3}R\Big(-\Big[2\sqrt{g}\Big(R_\star^{|i}R_\star^{|j} \\
&\quad + R_\star^{|i}R_\star^{|j}\Big)\frac{\partial^2 P}{\partial R \partial S} + (-1)^q\sqrt{g}g^{ij}\frac{\partial P}{\partial R} + (-1)^q\sqrt{g}g^{ij}Q\Big] + \Big\{\sqrt{g}\Big[-(-1)^q R^{|i}R^{|j} + 0 + 0 \\
&\quad - R_\star^{|i}R_\star^{|j}\Big]\frac{\partial^2 P}{\partial R \partial S} + \frac{1}{2}\sqrt{g}\left[-(-1)^q g^{ij} + 0\right]\frac{\partial P}{\partial R} + \frac{1}{2}\sqrt{g}\left[-(-1)^q g^{ij} + 0\right]Q\Big\}\Big) R_{|j} \\
&= \frac{(-1)^q}{3}\sqrt{g} SRR^{|i}\frac{\partial^2 P}{\partial R \partial S} + \frac{(-1)^q}{2}\sqrt{g}R^{|i}P + \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\frac{\partial P}{\partial R} + \frac{(-1)^q}{3}R\Big(0 - \sqrt{g}R^{|i}\frac{\partial P}{\partial R} \\
&\quad - \sqrt{g}R^{|i}Q - \sqrt{g}SR^{|i}\frac{\partial^2 P}{\partial R \partial S} - \frac{1}{2}\sqrt{g}R^{|i}\frac{\partial P}{\partial R} - \frac{1}{2}\sqrt{g}R^{|i}Q\Big) \\
&= \frac{(-1)^q}{3}\sqrt{g} SRR^{|i}\frac{\partial^2 P}{\partial R \partial S} + \frac{(-1)^q}{2}\sqrt{g}R^{|i}P + \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\frac{\partial P}{\partial R} - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\frac{\partial P}{\partial R} \\
&\quad - \frac{(-1)^q}{3}\sqrt{g}SRR^{|i}\frac{\partial^2 P}{\partial R \partial S} - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}Q \\
&= \frac{(-1)^q}{2}\sqrt{g}R^{|i}\left(P - RQ\right).
\end{aligned}
$$

We observe that the divergence of A^{ij} vanishes if $P = RQ$ or $R_{|j} = 0$. In the first case, P is a function of R only and A^{ij} reproduces the metric order 4 case (4.5) (with the Lagrangian L implicitly defined via the equation $R^2Q = RL' - L$), while the second case requires R to be covariantly constant and hence A^{ij} is at most of metric order 2. Therefore, there are no symmetric, divergence-free, contravariant, rank 2 tensor densities which are fifth order in the metric.

CHAPTER 5
CONCLUSION

In this thesis we presented a novel formula for the Euler-Lagrange expression of any natural Lagrangian which depends on a metric and its derivatives to arbitrary order in two dimensions, described the highest order terms for all symmetric, divergence-free tensor densities dependent on a metric, a scalar field, and derivatives of the scalar field to arbitrary order in two dimensions, and classified all symmetric, divergence-free tensor densities dependent on a metric and its derivatives up to fifth order in two dimensions. The formula for Euler-Lagrange expressions was derived using standard calculus of variations techniques, though the use of Theorem 1 to replace derivatives of the metric with the curvature scalar and derivatives thereof is, to our knowledge, new. The highest order terms for symmetric, divergence-free tensor densities were found using techniques from invariant theory, constraints produced by limiting the problem to two dimensions, and the symmetrization formulas of Appendix A. The same procedures are used in the classification results of Chapter 4, with the explicit symmetrization formulas for these lower order cases allowing for a complete solution in each case.

While the Euler-Lagrange expression formula (1.27) is applicable for Lagrangians of any metric order, an explicit verification of the divergence-free condition (2.32) for all metric orders is missing from the current work. The primary obstacles for this calculation are the covariant derivatives of the S^{abc} tensors, as these tensors contain explicit references to the Lagrangian and require multiple applications of the chain rule to evaluate the covariant derivatives, and the non-trivial algebraic manipulation required for the left and right sides of (2.32) to be represented by the same expression. Also, we note that Theorem 10 can be easily modified to work in any dimension for Lagrangians of the form $\lambda(g_{ij}; R; R_{|i}; \ldots)$ by not substituting the two dimensional form for the Ricci tensor in the variation of the curvature scalar (2.23). The remaining calculations which lead to the Euler-Lagrange expression

(2.30) are (using an appropriately defined action) dimension-agnostic and yield the following theorem.

Theorem 13. *Let λ be a Lagrangian defined on a manifold of finite dimension. If λ is dependent on the metric, curvature scalar R, and (symmetrized) covariant derivatives of R to some finite order, then $\lambda = \sqrt{g}L$ and the Euler-Lagrange expression for λ is given by*

$$\frac{\delta\lambda}{\delta g_{ab}} = \sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)E_R(L) - R^{ab}E_R(L) + \frac{1}{2}\left(2S^{(ab)c} - S^{cab}\right)_{|c} + \frac{1}{2}g^{ab}L + \frac{\partial L}{\partial g_{ab}}\right].$$

The main result of Chapter 2, Theorem 10, is a special case of Theorem 13, as only in two dimensions do Lagrangians of the form $\lambda(g_{ij}; R; R_{|i}; \ldots)$ coincide with Lagrangians dependent solely on the metric and its derivatives (via Theorem 1).

The results of Chapters 3 and 4 can, in principle, be extended to any finite order. However, as can be observed in Chapter 4 and Appendix C, the algebraic complexity of the problem rises rapidly as the order of the candidate tensor increases. Ignoring the work performed in Chapter 3, we observe that the metric order 2 case required a single calculation, the order 3 case took two short calculations, the order 4 case took a page and half, the order 5 case required thirteen pages of work, and the (incomplete) order 6 case fills up 78 pages. A reasonable limit for "by-hand" calculations appears to be sixth (or possibly seventh) metric order and the steep trajectory of work involved suggests that even software-based aid would become insufficient shortly thereafter. For tensors of arbitrary scalar order, expanding each of the terms in (1.28) from Theorem 11 to their fullest extent (a full categorization of symmetric, contravariant, second rank divergence-free tensor densities in two dimensions) would require a complete version of (A.5) (see Appendix A for the complications involved) and the algebraic difficulty of the ensuing system would be considerable, if not completely intractable.

On the surface, extending this work to higher dimensions appears to be even more difficult. The method used to derive the Euler-Lagrange formula from Chapter 2 should work in higher dimensions, with the scalar curvature replaced by the Riemann tensor, though the additional indices will complicate the situation. Moving to Chapter 3, it should be possible to find an analogue of the (fully symmetrized) divergence-free condition (3.3) in terms of the metric, Riemann tensor, and covariant derivatives of the Riemann tensor, which would work for any dimension. Even if the covariant derivatives are symmetrized with appropriately modified symmetrization formulas, the Riemann curvature tensor itself is not symmetric, which complicates the production of a condition analogous to the cyclic identity (3.10). This cyclic identity lead to the linearity condition, Proposition 1, which is predicated on Lemma 1 and its two dimensional proof. Absent some higher dimensional analogue to the lemma, this issue appears to kill the current approach entirely. Even if these problems have a resolution, the algebraic complexity of the resulting system would be astronomical, likely precluding any feasible solution to the problem using this method.

Our results do suggest one possible avenue for producing a non-variational, divergence-free tensor density. In particular, the index interchange symmetry for $B^{a_3\cdots a_r}$ given in Theorem 11 is equivalent to stating that $B^{a_3\cdots a_r} = T^{;a_3\cdots a_r}$, that is, $B^{a_3\cdots a_r}$ is the curl of a scalar T. Using this result, equation (1.28) from Theorem 11 may be written in the form

$$A^{ij} = (\nabla^i\nabla^j - g^{ij}\Box)T + \tilde{D}^{ija_1\cdots a_{r-3}}\varphi_{a_1\cdots a_{r-3}} + \tilde{E}^{ij},$$

where T, $\tilde{D}^{ija_1\cdots a_{r-3}}$, and $\tilde{E}^{ij}$ are of scalar order $r-2$. We note that the decomposition (1.15) mentioned in the introduction is this expression with the identification $\varphi = R$. Unlike the leading two terms of (1.28), $D^{ija_1\cdots a_{r-3}}\varphi_{a_1\cdots a_{r-3}}$ and E^{ij} are not necessarily curls of the scalar T and so do not become completely absorbed into the derivatives of T. Additionally, the derivatives of T produce lower order terms which must be taken from $D^{ija_1\cdots a_{r-3}}\varphi_{a_1\cdots a_{r-3}}$ and E^{ij}, producing the modified tilde versions of these tensors. We note that the original form given in Theorem 11 is more useful in the context of Chapter 4 (all of the tensorial quantities are symmetric, which makes computing derivatives of the expression easier) and so have chosen to present that form as the primary version of the theorem.

We can go further and pull an additional term of the form $-\dfrac{1}{2}g^{ij}RT$ from $\tilde{E}^{ij}$, producing the following corollary to Theorem 11.

Corollary 6. *Let A^{ij} be a symmetric, divergence-free tensor density defined on a two dimensional manifold. If A^{ij} is dependent on the metric, a scalar field φ, and (symmetrized) covariant derivatives of the scalar field to some finite order $k > 2$, then A^{ij} admits the decomposition*

$$\begin{aligned} A^{ij} &= (\nabla^i\nabla^j - g^{ij}\Box)T - \frac{1}{2}g^{ij}RT + \tilde{D}^{ija_1\cdots a_{r-3}}\varphi_{a_1\cdots a_{r-3}} + \tilde{\tilde{E}}^{ij} \\ &= (\nabla^i\nabla^j - g^{ij}\Box)T - \frac{1}{2}g^{ij}RT + X^{ij}, \end{aligned} \tag{5.1}$$

where T and X^{ij} are scalar/tensor densities of scalar order $k-2$ and $k-1$, respectively. Additionally, X^{ij} is (at most) linear in $k-1$ derivatives of φ.

This result bears a striking resemblance to the formula given in Theorem 10

$$\frac{\delta\lambda}{\delta g_{ij}} = \sqrt{g}\left[\left(\nabla^i\nabla^j - g^{ij}\Box\right)E_R(L) - \frac{1}{2}g^{ij}RE_R(L) + \frac{1}{2}\left(2S^{(ij)k} - S^{kij}\right)_{|k} + \frac{1}{2}g^{ij}L + \frac{\partial L}{\partial g_{ij}}\right], \tag{5.2}$$

and, unsurprisingly, the two decompositions do coincide when we identify $\varphi = R$ and A^{ij} is variational, e.g., the metric order 4 case (4.5). Unfortunately, this corollary provides no information about T beyond its scalar (or, in the $\varphi = R$ case, metric/curvature) order and it is not immediately obvious if $T = \dfrac{\delta\lambda}{\delta R}$ for some natural Lagrangian $\lambda = \sqrt{g}L$. Indeed, as mentioned earlier in this conclusion, it is quite difficult to prove that A^{ij}, and therefore T, is variational for even relatively low metric orders.

Conversely, Corollary 6 suggests it may be easier to produce a non-variational counterexample by a direct calculation instead of a categorization approach, as was attempted in Chapters 3 and 4. The divergence of (5.1) can be easily adapted to the calculations given in equation (2.31), which leads to the following divergence-free condition for A^{ij}

$$A^{ij}_{\ \ |j} = 0 \iff \frac{1}{2}R^{|i}T = X^{ij}_{\ \ |j}. \tag{5.3}$$

As such, a possible avenue forward would be to start with a known, non-variational scalar (density) T and then construct a compensating tensor X^{ij} such that the above divergence condition on X^{ij} is satisfied. {This venture is made slightly easier by noting that X^{ij} is (at most) of metric/curvature order one higher than T and is linear at this highest order. We note the difference between this situation and the similar, but erroneous, decomposition produced by Deser and Pang [24], which used the same metric order for T and X^{ij}.} The tensor A^{ij} constructed from T and X^{ij} using (5.1) would then be guaranteed to be divergence-free.

We stress that any tensor A^{ij} produced by this method is not guaranteed to be non-variational, as it may be possible to rewrite terms from X^{ij} as the derivatives of a different scalar $\tilde{T}$, producing a variational decomposition for A^{ij} in the form presented in Theorem 10. Candidate tensors produced using this method would likely need to be checked using the Helmholtz conditions, the necessary and sufficient conditions for a differential equation to be variational (see, e.g., section 5.4 of Olver [27]), and this calculation would constitute a considerable undertaking in its own right. Still, if non-variational tensors A^{ij} do exist, a clever use of this approach may produce a counterexample faster than any attempt to categorize every divergence-free tensor by extending the techniques used by the authors in the literature review or the methods given in this thesis.

The divergence-free condition (5.3) also suggests that it may be worth investigating tensors of the type X^{ij} on their own merits; that is, those tensors which are linear in their highest (curvature) scalar derivative and have divergence proportional to $R^{|i}$ (a metric order 5 example of such a tensor is given in Appendix D and was produced using the analysis in Chapter 4).

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APPENDICES

APPENDIX A
SYMMETRIZATION FORMULAS

Many of the computations performed in this thesis require the covariant derivatives of scalar fields to be fully symmetrized, e.g., φ_{abcd} or $R_{|(abcd)}$. Differentiation with respect to and simplification of expressions involving these symmetrized derivatives then proceeds in a relatively straightforward manner. However, any additional covariant derivatives will break the symmetrization (e.g., $\varphi_{abcd|e}$) and terms of this sort must be fully symmetrized before the previously mentioned differentiation and simplification can be performed.

We now derive a series of formulas which allow us to express terms of the form $\varphi_{|(I)j} = \varphi_{I|j}$, where $I = i_1 \cdots i_s$ is a multi-index, as a fully symmetric derivative term $\varphi_{|(Ij)} = \varphi_{Ij}$ and a collection of fully symmetrized, lower scalar order terms that describe the deviation of the original expression from full symmetry. This reduction in scalar order is obtained using the Riemann curvature tensor commutation formula (1.2). As we shall see, as s increases these formulas become increasingly recursive, requiring reference to lower order formulas to arrive at a non-recursive formula (which we have failed to find for the general case).

Since the first two covariant derivatives of a scalar field commute, we start with $s = 2$, or third scalar order. This formula is found by the following computation

$$\begin{aligned}
\varphi_{|klj} = \varphi_{kl|j} &= \varphi_{klj} + \varphi_{kl|j} - \varphi_{klj} \\
&= \varphi_{klj} + \varphi_{kl|j} - \frac{1}{3}\left(\varphi_{kl|j} + \varphi_{kj|l} + \varphi_{jl|k}\right) \\
&= \varphi_{klj} + \varphi_{kl|j} - \frac{1}{3}\left(\varphi_{kl|j} + 2\,\mathrm{Sym}_{kl}\,\varphi_{kj|l}\right) \\
&= \varphi_{klj} + \frac{2}{3}\left(\varphi_{kl|j} - \mathrm{Sym}_{kl}\,\varphi_{kj|l}\right) \\
&= \varphi_{klj} - \frac{2}{3}\,\mathrm{Sym}_{kl}\left(\varphi_{|kjl} - \varphi_{|klj}\right) \\
&= \varphi_{klj} + \frac{2}{3}\,\mathrm{Sym}_{kl}\,R_k{}^e{}_{jl}\varphi_e \\
&= \varphi_{klj} + \frac{1}{3}\,\mathrm{Sym}_{kl}\,R\left(g_{jk}\delta^e_l - g_{lk}\delta^e_j\right)\varphi_e
\end{aligned}$$

$$= \varphi_{klj} + \frac{1}{3}\,\mathrm{Sym}_{kl}\,R\left(g_{jk}\varphi_l - g_{lk}\varphi_j\right). \tag{A.1}$$

The fourth scalar order case proceeds in a similar fashion

$$\begin{aligned}
\varphi_{klm|j} &= \varphi_{klmj} + \varphi_{klm|j} - \varphi_{klmj} \\
&= \varphi_{klmj} + \varphi_{klm|j} - \frac{1}{4}\left(\varphi_{klm|j} + \varphi_{klj|m} + \varphi_{kjm|l} + \varphi_{jlm|k}\right) \\
&= \varphi_{klmj} + \frac{3}{4}\,\mathrm{Sym}_{klm}\left(\varphi_{klm|j} - \varphi_{klj|m}\right) \\
&= \varphi_{klmj} + \frac{3}{4}\,\mathrm{Sym}_{klm}\left[\varphi_{klm|j} - \frac{1}{3}\left(\varphi_{kl|j} + 2\varphi_{kjl}\right)_{|m}\right] \\
&= \varphi_{klmj} + \frac{3}{4}\,\mathrm{Sym}_{klm}\left[\varphi_{klm|j} - \frac{1}{3}\left(\varphi_{kl|j} + 2\varphi_{kl|j} - 2R_k{}^e{}_{jl}\varphi_e\right)_{|m}\right] \\
&= \varphi_{klmj} + \frac{3}{4}\,\mathrm{Sym}_{klm}\left[\varphi_{klm|j} - \frac{1}{3}\left(3\varphi_{kl|j} - 2R_k{}^e{}_{jl}\varphi_e\right)_{|m}\right] \\
&= \varphi_{klmj} + \frac{3}{4}\,\mathrm{Sym}_{klm}\left[\varphi_{kl|mj} - \varphi_{kl|jm} + \frac{2}{3}\left(R_k{}^e{}_{jl}\varphi_e\right)_{|m}\right] \\
&= \varphi_{klmj} + \frac{3}{4}\,\mathrm{Sym}_{klm}\left[-2R_k{}^e{}_{mj}\varphi_{el} + \frac{2}{3}\left(R_k{}^e{}_{jl}\varphi_e\right)_{|m}\right] \\
&= \varphi_{klmj} + \mathrm{Sym}_{klm}\left[\frac{3}{2}R_k{}^e{}_{jm}\varphi_{el} + \frac{1}{2}\left(R_k{}^e{}_{jl}\varphi_e\right)_{|m}\right] \\
&= \varphi_{klmj} + \mathrm{Sym}_{klm}\left\{\frac{3}{4}R\left(g_{jk}\delta^e_m - g_{mk}\delta^e_j\right)\varphi_{el} + \frac{1}{4}\left[R\left(g_{jk}\delta^e_l - g_{lk}\delta^e_j\right)\varphi_e\right]_{|m}\right\} \\
&= \varphi_{klmj} + \mathrm{Sym}_{klm}\left\{\frac{3}{4}R\left(g_{jk}\varphi_{ml} - g_{mk}\varphi_{jl}\right) + \frac{1}{4}\left[R\left(g_{jl}\varphi_k - g_{kl}\varphi_j\right)\right]_{|m}\right\} \\
&= \varphi_{klmj} + \mathrm{Sym}_{klm}\left[\frac{3}{4}R\left(g_{jk}\varphi_{ml} - g_{mk}\varphi_{jl}\right) + \frac{1}{4}R_{|m}\left(g_{jl}\varphi_k - g_{kl}\varphi_j\right)\right. \\
&\qquad \left. + \frac{1}{4}R\left(g_{jl}\varphi_{km} - g_{kl}\varphi_{jm}\right)\right] \\
&= \varphi_{klmj} + \mathrm{Sym}_{klm}\left[R\left(g_{jk}\varphi_{ml} - g_{mk}\varphi_{jl}\right) + \frac{1}{4}R_{|m}\left(g_{jl}\varphi_k - g_{kl}\varphi_j\right)\right],
\end{aligned} \tag{A.2}$$

as does the fifth scalar order case

$$\begin{aligned}
\varphi_{klmn|j} &= \varphi_{klmnj} + \varphi_{klmn|j} - \varphi_{klmnj} \\
&= \varphi_{klmnj} + \frac{4}{5}\,\mathrm{Sym}_{klmn}\left(\varphi_{klmn|j} - \varphi_{klmj|n}\right) \\
&= \varphi_{klmnj} + \frac{4}{5}\,\mathrm{Sym}_{klmn}\left[\varphi_{klmn|j} - \frac{1}{4}\left(\varphi_{klm|j} + \varphi_{kl|jm} + 2\varphi_{|kjlm}\right)_{|n}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \varphi_{klmnj} + \frac{4}{5}\,\mathrm{Sym}_{klmn}\left\{\varphi_{klmn|j} - \frac{1}{4}\left[\varphi_{klm|j} + \varphi_{kl|jm} + 2\left(\varphi_{kl|j} - R_{k\ \,jl}^{\ e}\varphi_e\right)_{|m}\right]_{|n}\right\} \\
&= \varphi_{klmnj} + \frac{4}{5}\,\mathrm{Sym}_{klmn}\left\{\varphi_{klmn|j} - \frac{1}{4}\left[\varphi_{klm|j} + 3\varphi_{kl|jm} - \left(2R_{k\ \,jl}^{\ e}\varphi_e\right)_{|m}\right]_{|n}\right\} \\
&= \varphi_{klmnj} + \frac{4}{5}\,\mathrm{Sym}_{klmn}\left\{\varphi_{klmn|j} - \frac{1}{4}\left[\varphi_{klm|j} + 3\left(\varphi_{klm|j} - 2R_{k\ \,jm}^{\ e}\varphi_{el}\right)\right.\right. \\
&\qquad \left.\left. - \left(2R_{k\ \,jl}^{\ e}\varphi_e\right)_{|m}\right]_{|n}\right\} \\
&= \varphi_{klmnj} + \frac{4}{5}\,\mathrm{Sym}_{klmn}\left\{\varphi_{klmn|j} - \frac{1}{4}\left[4\varphi_{klm|jn} - \left(6R_{k\ \,jm}^{\ e}\varphi_{el}\right)_{|n}\right.\right. \\
&\qquad \left.\left. - \left(2R_{k\ \,jl}^{\ e}\varphi_e\right)_{|mn}\right]\right\} \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn}\left[-\frac{4}{5}\left(\varphi_{klm|jn} - \varphi_{klm|nj}\right) + \frac{6}{5}\left(R_{k\ \,jm}^{\ e}\varphi_{el}\right)_{|n}\right. \\
&\qquad \left. + \frac{2}{5}\left(R_{k\ \,jl}^{\ e}\varphi_e\right)_{|mn}\right] \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn}\left[\frac{12}{5}\left(R_{k\ \,jn}^{\ e}\varphi_{elm}\right) + \frac{6}{5}\left(R_{k\ \,jm}^{\ e}\varphi_{el}\right)_{|n} + \frac{2}{5}\left(R_{k\ \,jl}^{\ e}\varphi_e\right)_{|mn}\right] \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn}\left\{\frac{6}{5}\left[R\left(g_{jk}\delta_n^e - \delta_j^e g_{kn}\right)\varphi_{elm}\right] + \frac{3}{5}\left[R\left(g_{jk}\delta_m^e - \delta_j^e g_{km}\right)\varphi_{el}\right]_{|n}\right. \\
&\qquad \left. + \frac{1}{5}\left[R\left(g_{jk}\delta_l^e - \delta_j^e g_{kl}\right)\varphi_e\right]_{|mn}\right\} \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn}\left\{\frac{6}{5}R\left(g_{jk}\varphi_{nlm} - g_{kn}\varphi_{jlm}\right) + \frac{3}{5}\left[R\left(g_{jk}\varphi_{lm} - g_{km}\varphi_{jl}\right)\right]_{|n}\right. \\
&\qquad \left. + \frac{1}{5}\left[R\left(g_{jk}\varphi_l - g_{kl}\varphi_j\right)\right]_{|mn}\right\} \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn}\left[\frac{6}{5}R\left(g_{jk}\varphi_{nlm} - g_{kn}\varphi_{jlm}\right) + \frac{3}{5}\left(R_{|n}g_{jk}\varphi_{ml} + Rg_{jk}\varphi_{lmn}\right.\right. \\
&\qquad \left. - R_{|n}g_{km}\varphi_{jl} - Rg_{km}\varphi_{|jln}\right) + \frac{1}{5}\left(R_{|mn}g_{jk}\varphi_l + 2R_{|m}g_{jk}\varphi_{ln} + Rg_{jk}\varphi_{lmn}\right. \\
&\qquad \left.\left. - R_{|mn}g_{kl}\varphi_j - 2R_{|m}g_{kl}\varphi_{jn} - Rg_{kl}\varphi_{|jmn}\right)\right] \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn}\left[\frac{6}{5}R\left(g_{jk}\varphi_{nlm} - g_{kn}\varphi_{jlm}\right) + R_{|n}\left(g_{jk}\varphi_{ml} - g_{km}\varphi_{jl}\right)\right. \\
&\qquad \left. + \frac{1}{5}R_{|mn}\left(g_{jk}\varphi_l - g_{kl}\varphi_j\right) + \frac{4}{5}R\left(g_{jk}\varphi_{lmn} - g_{km}\varphi_{|ljn}\right)\right] \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn}\left\{\frac{6}{5}R\left(g_{jk}\varphi_{nlm} - g_{kn}\varphi_{jlm}\right) + R_{|n}\left(g_{jk}\varphi_{ml} - g_{km}\varphi_{jl}\right)\right. \\
&\qquad \left. + \frac{1}{5}R_{|mn}\left(g_{jk}\varphi_l - g_{kl}\varphi_j\right) + \frac{4}{5}R\left[g_{jk}\varphi_{lmn} - g_{km}\left(\varphi_{|ljn} - \varphi_{jln} + \varphi_{jln}\right)\right]\right\} \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn}\left\{2R\left(g_{jk}\varphi_{lmn} - g_{kl}\varphi_{mnj}\right) + R_{|n}\left(g_{jk}\varphi_{lm} - g_{kl}\varphi_{jm}\right)\right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \frac{1}{5} R_{|mn} \left(g_{jk} \varphi_l - g_{kl} \varphi_j \right) - \frac{4}{5} R g_{km} \left[\varphi_{|ljn} - \frac{1}{3} \left(2\varphi_{|ljn} + \varphi_{|lnj} \right) \right] \Bigg\} \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn} \Big[2R \left(g_{jk} \varphi_{lmn} - g_{kl} \varphi_{mnj} \right) + R_{|n} \left(g_{jk} \varphi_{lm} - g_{kl} \varphi_{jm} \right) \\
&\quad + \frac{1}{5} R_{|mn} \left(g_{jk} \varphi_l - g_{kl} \varphi_j \right) - \frac{4}{15} R g_{km} \left(\varphi_{|ljn} - \varphi_{|lnj} \right) \Big] \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn} \Big[2R \left(g_{jk} \varphi_{lmn} - g_{kl} \varphi_{mnj} \right) + R_{|n} \left(g_{jk} \varphi_{lm} - g_{kl} \varphi_{jm} \right) \\
&\quad + \frac{1}{5} R_{|mn} \left(g_{jk} \varphi_l - g_{kl} \varphi_j \right) + \frac{4}{15} R g_{km} \left(R_l{}^e{}_{jn} \varphi_e \right) \Big] \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn} \Big[2R \left(g_{jk} \varphi_{lmn} - g_{kl} \varphi_{mnj} \right) + R_{|n} \left(g_{jk} \varphi_{lm} - g_{kl} \varphi_{jm} \right) \\
&\quad + \frac{1}{5} R_{|mn} \left(g_{jk} \varphi_l - g_{kl} \varphi_j \right) + \frac{2}{15} R^2 g_{km} \left(g_{jl} \delta_n^e - g_{ln} \delta_j^e \right) \varphi_e \Big] \\
&= \varphi_{klmnj} + \mathrm{Sym}_{klmn} \Big[2R \left(g_{jk} \varphi_{lmn} - g_{kl} \varphi_{mnj} \right) + R_{|n} \left(g_{jk} \varphi_{lm} - g_{kl} \varphi_{jm} \right) \\
&\quad + \frac{1}{5} R_{|mn} \left(g_{jk} \varphi_l - g_{kl} \varphi_j \right) + \frac{2}{15} R^2 g_{kl} \left(g_{jm} \varphi_n - g_{mn} \varphi_j \right) \Big] .
\end{aligned} \tag{A.3}
$$

The general $s+1$ case for a multi-index $I = i_1 \ldots i_s$, with $\varphi_{\hat{I}_t} = \varphi_{i_1 \ldots i_{t-1} | j i_t \ldots i_{s-1}}$, yields

$$
\begin{aligned}
\varphi_{I|j} &= \varphi_{Ij} + \varphi_{I|j} - \varphi_{Ij} \\
&= \varphi_{Ij} + \frac{s}{s+1} \mathrm{Sym}_I \left(\varphi_{I|j} - \varphi_{\hat{I}_s | i_s} \right) \\
&= \varphi_{Ij} + \frac{s}{s+1} \mathrm{Sym}_I \left[\varphi_{I|j} - \frac{1}{s} \left(\varphi_{\hat{I}_s} + \varphi_{\hat{I}_{s-1}} + \cdots + \varphi_{\hat{I}_3} + 2\varphi_{\hat{I}_2} \right)_{|i_s} \right] \\
&= \varphi_{Ij} + \frac{s}{s+1} \mathrm{Sym}_I \Bigg\{ \varphi_{I|j} - \frac{1}{s} \Big[\varphi_{\hat{I}_s} + \varphi_{\hat{I}_{s-1}} + \cdots + \varphi_{\hat{I}_3} \\
&\qquad + 2 \left(\varphi_{i_1 i_2 j} - R_{i_1}{}^e{}_{j i_2} \varphi_e \right)_{|i_3 \cdots i_{s-1}} \Big]_{|i_s} \Bigg\} \\
&= \varphi_{Ij} + \frac{s}{s+1} \mathrm{Sym}_I \left\{ \varphi_{I|j} - \frac{1}{s} \left[\varphi_{\hat{I}_s} + \varphi_{\hat{I}_{s-1}} + \cdots + 3\varphi_{\hat{I}_3} - 2 \left(R_{i_1}{}^e{}_{j i_2} \varphi_e \right)_{|i_3 \cdots i_{s-1}} \right]_{|i_s} \right\} \\
&= \varphi_{Ij} + \frac{s}{s+1} \mathrm{Sym}_I \Bigg\{ \varphi_{I|j} - \frac{1}{s} \Big[\varphi_{\hat{I}_s} + \varphi_{\hat{I}_{s-1}} + \cdots + 3 \left(\varphi_{i_1 i_2 i_3 | j} - 2R_{i_1}{}^e{}_{j i_3} \varphi_{e i_2} \right)_{|i_4 \cdots i_{s-1}} \\
&\qquad - 2 \left(R_{i_1}{}^e{}_{j i_2} \varphi_e \right)_{|i_3 \cdots i_{s-1}} \Big]_{|i_s} \Bigg\} \\
&= \varphi_{Ij} + \frac{s}{s+1} \mathrm{Sym}_I \Bigg\{ \varphi_{I|j} - \frac{1}{s} \Big[\varphi_{\hat{I}_s} + \varphi_{\hat{I}_{s-1}} + \cdots + 4\varphi_{\hat{I}_4} - 3 \left(2R_{i_1}{}^e{}_{j i_3} \varphi_{e i_2} \right)_{|i_4 \cdots i_{s-1}} \\
&\qquad - 2 \left(R_{i_1}{}^e{}_{j i_2} \varphi_e \right)_{|i_3 \cdots i_{s-1}} \Big]_{|i_s} \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \varphi_{Ij} + \frac{s}{s+1}\,\mathrm{Sym}_I \Bigg(\varphi_{I|j} - \frac{1}{s}\Big\{ s\varphi_{\hat{I}_s} - (s-1)\Big[(s-2)R_{i_1}{}^{e}{}_{ji_{s-1}}\varphi_{ei_2\cdots i_{s-2}}\Big] \\
&\qquad - (s-2)\Big[(s-3)R_{i_1}{}^{e}{}_{ji_{s-2}}\varphi_{ei_2\cdots i_{s-3}}\Big]_{|i_{s-1}} - \cdots - 3\left(2R_{i_1}{}^{e}{}_{ji_3}\varphi_{ei_2}\right)_{|i_4\cdots i_{s-1}} \\
&\qquad - 2\left(R_{i_1}{}^{e}{}_{ji_2}\varphi_{e}\right)_{|i_3\cdots i_{s-1}}\Big\}_{|i_s}\Bigg) \\
&= \varphi_{Ij} + \frac{1}{s+1}\,\mathrm{Sym}_I \Big[-s\left(\varphi_{i_1\cdots i_{s-1}|ji_s} - \varphi_{i_1\cdots i_{s-1}|i_sj}\right) + (s-1)(s-2) \\
&\qquad \times \Big(R_{i_1}{}^{e}{}_{ji_{s-1}}\varphi_{ei_2\cdots i_{s-2}}\Big)_{|i_s} + (s-2)(s-3)\Big(R_{i_1}{}^{e}{}_{ji_{s-2}}\varphi_{ei_2\cdots i_{s-3}}\Big)_{|i_{s-1}i_s} \\
&\qquad + \cdots + 3(2)\left(R_{i_1}{}^{e}{}_{ji_3}\varphi_{ei_2}\right)_{|i_4\cdots i_s} + 2(1)\left(R_{i_1}{}^{e}{}_{ji_2}\varphi_{e}\right)_{|i_3\cdots i_s}\Big] \\
&= \varphi_{Ij} + \frac{1}{s+1}\,\mathrm{Sym}_I \Big[s(s-1)\left(R_{i_1}{}^{e}{}_{ji_s}\varphi_{ei_2\cdots i_{s-1}}\right) + (s-1)(s-2)\Big(R_{i_1}{}^{e}{}_{ji_{s-1}}\varphi_{ei_2\cdots i_{s-2}}\Big)_{|i_s} \\
&\qquad + \cdots + 3(2)\left(R_{i_1}{}^{e}{}_{ji_3}\varphi_{ei_2}\right)_{|i_4\cdots i_s} + 2(1)\left(R_{i_1}{}^{e}{}_{ji_2}\varphi_{e}\right)_{|i_3\cdots i_s}\Big] \\
&= \varphi_{Ij} + \frac{1}{2(s+1)}\,\mathrm{Sym}_I \Big\{ s(s-1)\left[R\left(g_{ji_1}\varphi_{i_2\cdots i_s} - g_{i_1i_s}\varphi_{ji_2\cdots i_{s-1}}\right)\right] \\
&\qquad + (s-1)(s-2)\left[R\left(g_{ji_1}\varphi_{i_2\cdots i_{s-1}} - g_{i_1i_{s-1}}\varphi_{ji_2\cdots i_{s-2}}\right)\right]_{|i_s} + \cdots \\
&\qquad + 2(1)\left[R\left(g_{ji_1}\varphi_{i_2} - g_{i_1i_2}\varphi_j\right)\right]_{|i_3\cdots i_s}\Big\}. \qquad\qquad (A.4)
\end{aligned}
$$

We observe the full symmetry of the φ derivatives inside the parentheses but note the covariant derivatives outside the square brackets do not lead to fully symmetric terms (the index j is symmetric with respect to only the interior indices, e.g., the expression $\mathrm{Sym}_I\, \varphi_{ji_1i_2|i_3\cdots i_s}$ is only fully symmetric over the first three indices). For any finite s, it is possible to reduce (A.4) to a form analogous to that of the low scalar order solutions (A.1), (A.2), and (A.3), though reaching this solution is difficult. The application of the covariant derivatives outside the bracketed expression to the second term inside produces expressions that must be symmetrized for $s > 3$ as the index j is not automatically symmetrized like the remaining indices (as can be seen in the derivation for the $s = 4$ case, starting with line 13 and the term $\varphi_{|jln}$). On the brighter side, the derivatives of R that arise from differentiating the bracketed terms are automatically symmetrized due to the symmetrization on the index I (remembering that we eventually want to perform the replacement $\varphi = R$ so it is important that all such derivatives are symmetrized).

Inspecting (A.4) and comparing this result with the lower order formulas, we see that the simplification procedure will produce terms which are linear in derivatives of φ, with a general form of

$$\varphi_{i_1\cdots i_r|j} = \varphi_{i_1\cdots i_r j} + \mathrm{Sym}_I\left(P_{ji_1}\varphi_{i_2\cdots i_r} - Q_{i_1 i_2}\varphi_{i_3\cdots i_r j} + \cdots + P_{ji_1\cdots i_{r-1}}\varphi_{i_r} - Q_{i_1\cdots i_r}\varphi_j\right), \tag{A.5}$$

where, for $2 \le k \le r$, the coefficients $P_{a_1\cdots a_k}$ and $Q_{a_1\cdots a_k}$ are tensors dependent on the metric, scalar curvature, and symmetrized covariant derivatives of the scalar curvature. As previously mentioned, finding the explicit form for the coefficient tensors is challenging.

For example, examining the final term in square brackets from (A.4) and noting the symmetry of the i_k indices, we observe that the outer covariant derivatives follow a binomial expansion when applied to this term

$$\begin{aligned}\mathrm{Sym}_I\left[R\left(g_{ji_1}\varphi_{i_2} - g_{i_1 i_2}\varphi_j\right)\right]_{|i_3\cdots i_s} = \mathrm{Sym}_I\Bigg[&R_{|i_3\cdots i_s}\left(g_{ji_1}\varphi_{i_2} - g_{i_1 i_2}\varphi_j\right) + \cdots \\ &+ \binom{s-2}{k} R_{|i_{k+3}\cdots i_s}\left(g_{ji_1}\varphi_{i_2\cdots i_{k+2}} - g_{i_1 i_2}\varphi_{|ji_3\cdots i_{k+2}}\right) \\ &+ \cdots + R\left(g_{ji_1}\varphi_{i_2\cdots i_s} - g_{i_1 i_2}\varphi_{|ji_3\cdots i_s}\right)\Bigg].\end{aligned} \tag{A.6}$$

For $k \ge 2$, the second φ term in each part of this expansion is not fully symmetrized, with the index j applied first to φ. We use the commutation formula (1.2) to move the j index to the end, producing a number of lower order terms reminiscent of (A.4)

$$\begin{aligned}\mathrm{Sym}_I\,\varphi_{|ji_3\cdots i_{k+2}} &= \mathrm{Sym}_I\,\varphi_{|i_3 j i_4\cdots i_{k+2}} \\ &= \mathrm{Sym}_I\left(\varphi_{|i_3 j i_4}\right)_{|i_5\cdots i_{k+2}} \\ &= \mathrm{Sym}_I\left(\varphi_{|i_3 i_4 j} - R_{i_4}{}^{a}{}_{i_3 j}\varphi_a\right)_{|i_5\cdots i_{k+2}} \\ &= \mathrm{Sym}_I\left[\left(\varphi_{|i_3 i_4 j i_5}\right)_{|i_6\cdots i_{k+2}} + \left(R_{i_4}{}^{a}{}_{ji_3}\varphi_a\right)_{|i_5\cdots i_{k+2}}\right] \\ &= \mathrm{Sym}_I\left[\left(\varphi_{|i_3 i_4 i_5 j} - R_{i_5}{}^{a}{}_{i_3 j}\varphi_{a i_4} - R_{i_5}{}^{a}{}_{i_4 j}\varphi_{i_3 a}\right)_{|i_6\cdots i_{k+2}} + \left(R_{i_4}{}^{a}{}_{ji_3}\varphi_a\right)_{|i_5\cdots i_{k+2}}\right] \\ &= \mathrm{Sym}_I\left[\left(\varphi_{|i_3 i_4 i_5 j i_6}\right)_{|i_7\cdots i_{k+2}} - 2\left(R_{i_5}{}^{a}{}_{i_3 j}\varphi_{a i_4}\right)_{|i_6\cdots i_{k+2}} + \left(R_{i_4}{}^{a}{}_{ji_3}\varphi_a\right)_{|i_5\cdots i_{k+2}}\right]\end{aligned}$$

$$
\begin{aligned}
&= \mathrm{Sym}_I \Big[\varphi_{|i_3\cdots i_{k+2}j} + (k-1)\left(R_{i_{k+2}}{}^{a}{}_{ji_3}\varphi_{ai_4\cdots i_{k+1}}\right) \\
&\quad + (k-2)\left(R_{i_{k+1}}{}^{a}{}_{ji_3}\varphi_{ai_4\cdots i_k}\right)_{|i_{k+2}} + \cdots + 2\left(R_{i_5}{}^{a}{}_{ji_3}\varphi_{ai_4}\right)_{|i_6\cdots i_{k+2}} \\
&\quad + \left(R_{i_4}{}^{a}{}_{ji_3}\varphi_a\right)_{|i_5\cdots i_{k+2}}\Big] \\
&= \mathrm{Sym}_I \Big\{\varphi_{|i_3\cdots i_{k+2}j} + \frac{(k-1)}{2}R\left(g_{ji_{k+2}}\varphi_{i_3\cdots i_{k+1}} - g_{i_{k+2}i_3}\varphi_{ji_4\cdots i_{k+1}}\right) \\
&\quad + \frac{(k-2)}{2}\left[R\left(g_{ji_{k+1}}\varphi_{i_3\cdots i_k} - g_{i_3i_{k+1}}\varphi_{ji_4\cdots i_k}\right)\right]_{|i_{k+2}} + \cdots \\
&\quad + \left[R\left(g_{ji_5}\varphi_{i_3i_4} - g_{i_3i_5}\varphi_{ji_4}\right)\right]_{|i_6\cdots i_{k+2}} + \frac{1}{2}\left[R\left(g_{ji_4}\varphi_{i_3} - g_{i_3i_4}\varphi_j\right)\right]_{|i_5\cdots i_{k+2}}\Big\} \\
&= \mathrm{Sym}_I \Big\{\varphi_{|i_3\cdots i_{k+2}j} + \frac{(k-1)}{2}R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+2}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{k+2}}\right) \\
&\quad + \frac{(k-2)}{2}\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+1}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{k+1}}\right)\right]_{|i_{k+2}} + \cdots \\
&\quad + \left[R\left(g_{ji_3}\varphi_{i_4i_5} - g_{i_3i_4}\varphi_{ji_5}\right)\right]_{|i_6\cdots i_{k+2}} + \frac{1}{2}\left[R\left(g_{ji_3}\varphi_{i_4} - g_{i_3i_4}\varphi_j\right)\right]_{|i_5\cdots i_{k+2}}\Big\}.
\end{aligned}
$$

We use (A.4) to fully symmetrize the highest order term in this expression, noting that the remaining terms can be combined

$$
\begin{aligned}
\mathrm{Sym}_I\,\varphi_{|ji_3\cdots i_{k+2}} &= \varphi_{i_3\cdots i_{k+2}j} + \frac{1}{2(k+1)}\mathrm{Sym}_I\Big\{k(k-1)\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+2}} - g_{i_4i_{k+2}}\varphi_{ji_4\cdots i_{k+1}}\right)\right] \\
&\quad + (k-1)(k-2)\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+1}} - g_{i_3i_{k+1}}\varphi_{ji_4\cdots i_k}\right)\right]_{|i_{k+2}} + \cdots \\
&\quad + 2(1)\left[R\left(g_{ji_3}\varphi_{i_4} - g_{i_3i_4}\varphi_j\right)\right]_{|i_5\cdots i_{k+2}}\Big\} \\
&\quad + \mathrm{Sym}_I\Big\{\frac{(k-1)}{2}R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+2}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{k+2}}\right) \\
&\quad + \frac{(k-2)}{2}\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+1}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{k+1}}\right)\right]_{|i_{k+2}} + \cdots \\
&\quad + \left[R\left(g_{ji_3}\varphi_{i_4i_5} - g_{i_3i_4}\varphi_{ji_5}\right)\right]_{|i_6\cdots i_{k+2}} + \frac{1}{2}\left[R\left(g_{ji_3}\varphi_{i_4} - g_{i_3i_4}\varphi_j\right)\right]_{|i_5\cdots i_{k+2}}\Big\} \\
&= \varphi_{i_3\cdots i_{k+2}j} + \mathrm{Sym}_I\Big\{\frac{(2k+1)(k-1)}{2(k+1)}\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+2}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{k+2}}\right)\right] \\
&\quad + \frac{(2k)(k-2)}{2(k+1)}\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+1}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{k+1}}\right)\right]_{|i_{k+2}} + \cdots \\
&\quad + \frac{k+4}{k+1}\left[R\left(g_{ji_3}\varphi_{i_4i_5} - g_{i_3i_4}\varphi_{ji_5}\right)\right]_{|i_6\cdots i_{k+2}} \\
&\quad + \frac{k+3}{2(k+1)}\left[R\left(g_{ji_3}\varphi_{i_4} - g_{i_3i_4}\varphi_j\right)\right]_{|i_5\cdots i_{k+2}}\Big\}.
\end{aligned}
$$

We substitute this result into (A.6), yielding

$$
\begin{aligned}
&\mathrm{Sym}_I \left[R\left(g_{ji_1}\varphi_{i_2} - g_{i_1i_2}\varphi_j\right)\right]_{|i_3\cdots i_s} \\
&\qquad = \mathrm{Sym}_I \Bigg[R_{|i_3\cdots i_s}\left(g_{ji_1}\varphi_{i_2} - g_{i_1i_2}\varphi_j\right) + \cdots \\
&\qquad\quad + \binom{s-2}{k} R_{|i_{k+3}\cdots i_s}\Bigg(g_{ji_1}\varphi_{i_2\cdots i_{k+2}} - g_{i_1i_2}\Bigg\{ \varphi_{i_3\cdots i_{k+2}j} \\
&\qquad\quad + \frac{(2k+1)(k-1)}{2(k+2)}\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+2}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{k+2}}\right)\right] \\
&\qquad\quad + \frac{(2k)(k-2)}{2(k+1)}\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{k+1}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{k+1}}\right)\right]_{|i_{k+2}} \\
&\qquad\quad + \cdots + \frac{k+4}{k+1}\left[R\left(g_{ji_3}\varphi_{i_4i_5} - g_{i_3i_4}\varphi_{ji_5}\right)\right]_{|i_6\cdots i_{k+2}} \\
&\qquad\quad + \frac{k+3}{2(k+1)}\left[R\left(g_{ji_3}\varphi_{i_4} - g_{i_3i_4}\varphi_j\right)\right]_{|i_5\cdots i_{k+2}} \Bigg\}\Bigg) \\
&\qquad\quad + \cdots + R\Bigg(g_{ji_1}\varphi_{i_2\cdots i_s} - g_{i_1i_2}\Bigg\{ \varphi_{i_3\cdots i_s j} \\
&\qquad\quad + \frac{(2s-3)(s-3)}{2(s-1)}\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_s} - g_{i_3i_4}\varphi_{ji_5\cdots i_s}\right)\right] \\
&\qquad\quad + \frac{(2s-4)(s-4)}{2(s-1)}\left[R\left(g_{ji_3}\varphi_{i_4\cdots i_{s-1}} - g_{i_3i_4}\varphi_{ji_5\cdots i_{s-1}}\right)\right]_{|i_s} \\
&\qquad\quad + \cdots + \frac{s+2}{s-1}\left[R\left(g_{ji_3}\varphi_{i_4i_5} - g_{i_3i_4}\varphi_{ji_5}\right)\right]_{|i_6\cdots i_s} \\
&\qquad\quad + \frac{s+1}{2(s-1)}\left[R\left(g_{ji_3}\varphi_{i_4} - g_{i_3i_4}\varphi_j\right)\right]_{|i_5\cdots i_s} \Bigg\}\Bigg)\Bigg].
\end{aligned}
$$

To continue further, we must re-use this procedure for the final term of each part of the binomial expansion (those terms with $i_5\cdots i_{k+2}$ covariant derivatives) and use similar procedures to simplify the terms which do not have j as the first covariant derivative [and these procedures must then be used on the remaining terms in (A.4)]. Owing to the incredibly recursive nature of these calculations, we do not possess the explicit characterization of the coefficient tensors P and Q from (A.5). An analysis of individual higher orders (6+) might lead to a recognizable pattern which could be leveraged to construct an inductive proof, though we have refrained from pursuing this route due to a combination of time constraints and the above results being sufficient to cover the scope of this thesis.

APPENDIX B

DETAILED DEGENERATE LAGRANGIAN CALCULATIONS

In this appendix we perform more thorough calculations for the degenerate Lagrangian $\lambda^\star$ given in Section 2.5.4. The symmetrization formulas (A.1) and (A.2) will be used extensively. We begin by deriving the expression for the double covariant derivative of the Euler-Lagrange expression with respect to the curvature, a full version of (2.41), using the third to last line of (2.40) as a starting point for index convenience

$$
\begin{aligned}
&E_R(L^\star)_{|ij} \\
&\quad= \varepsilon^{ac}\varepsilon^{bd}\Big[\big(6R_{|abi}R_{|cd}P' + 3R_{|ab}R_{|cd}R_{|i}P''\big) \\
&\qquad+ \big(2R_{|abi}R_{|c}R_{|d}P'' + 4R_{|ab}R_{|ci}R_{|d}P'' + 2R_{|ab}R_{|c}R_{|d}R_{|i}P'''\big) \\
&\qquad- g_{ab}\big(2R_{|ci}R_{|d}P + R_{|c}R_{|d}R_{|i}P'\big) - g_{ab}\big(R_{|i}R_{|cd}P + RR_{|cdi}P + RR_{|cd}R_{|i}P'\big) \\
&\qquad- 2g_{ab}\big(R_{|i}R_{|c}R_{|d}P' + 2RR_{|ci}R_{|d}P' + RR_{|c}R_{|d}R_{|i}P''\big)\Big]_{|j} \\
&\quad= \varepsilon^{ac}\varepsilon^{bd}\big(6R_{|abi}R_{|cd}P' + 3R_{|ab}R_{|cd}R_{|i}P'' + 2R_{|abi}R_{|c}R_{|d}P'' + 4R_{|ab}R_{|ci}R_{|d}P'' \\
&\qquad+ 2R_{|ab}R_{|c}R_{|d}R_{|i}P''' - 2g_{ab}R_{|ci}R_{|d}P - 3g_{ab}R_{|c}R_{|d}R_{|i}P' - g_{ab}R_{|i}R_{|cd}P \\
&\qquad- g_{ab}RR_{|cdi}P - g_{ab}RR_{|cd}R_{|i}P' - 4g_{ab}RR_{|ci}R_{|d}P' - 2g_{ab}RR_{|c}R_{|d}R_{|i}P''\big)_{|j} \\
&\quad= \varepsilon^{ac}\varepsilon^{bd}\big(6R_{|abij}R_{|cd}P' + 6R_{|abi}R_{|cdj}P' + 6R_{|abi}R_{|cd}R_{|j}P'' + 6R_{|abj}R_{|cd}R_{|i}P'' \\
&\qquad+ 3R_{|ab}R_{|cd}R_{|ij}P'' + 3R_{|ab}R_{|cd}R_{|i}R_{|j}P''' + 2R_{|abij}R_{|c}R_{|d}P'' + 4R_{|abi}R_{|cj}R_{|d}P'' \\
&\qquad+ 2R_{|abi}R_{|c}R_{|d}R_{|j}P''' + 4R_{|abj}R_{|ci}R_{|d}P'' + 4R_{|ab}R_{|cij}R_{|d}P'' + 4R_{|ab}R_{|ci}R_{|dj}P'' \\
&\qquad+ 4R_{|ab}R_{|ci}R_{|d}R_{|j}P''' + 2R_{|abj}R_{|c}R_{|d}R_{|i}P''' + 4R_{|ab}R_{|cj}R_{|d}R_{|i}P''' \\
&\qquad+ 2R_{|ab}R_{|c}R_{|d}R_{|ij}P''' + 2R_{|ab}R_{|c}R_{|d}R_{|i}R_{|j}P^{(4)} - 2g_{ab}R_{|cij}R_{|d}P \\
&\qquad- 2g_{ab}R_{|ci}R_{|dj}P - 2g_{ab}R_{|ci}R_{|d}R_{|j}P' - 6g_{ab}R_{|cj}R_{|d}R_{|i}P' - 3g_{ab}R_{|c}R_{|d}R_{|ij}P' \\
&\qquad- 3g_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P'' - g_{ab}R_{|ij}R_{|cd}P - g_{ab}R_{|i}R_{|cdj}P - g_{ab}R_{|i}R_{|j}R_{|cd}P' \\
&\qquad- g_{ab}R_{|j}R_{|cdi}P - g_{ab}RR_{|cdij}P - g_{ab}RR_{|cdi}R_{|j}P' - g_{ab}R_{|j}R_{|cd}R_{|i}P'
\end{aligned}
$$

$$
\begin{aligned}
&- g_{ab}RR_{|cdj}R_{|i}P' - g_{ab}RR_{|cd}R_{|ij}P' - g_{ab}RR_{|cd}R_{|i}R_{|j}P'' \\
&- 4g_{ab}R_{|j}R_{|ci}R_{|d}P' - 4g_{ab}RR_{|cij}R_{|d}P' - 4g_{ab}RR_{|ci}R_{|dj}P' \\
&- 4g_{ab}RR_{|ci}R_{|d}R_{|j}P'' - 2g_{ab}R_{|j}R_{|c}R_{|d}R_{|i}P'' - 4g_{ab}RR_{|cj}R_{|d}R_{|i}P'' \\
&- 2g_{ab}RR_{|c}R_{|d}R_{|ij}P'' - 2g_{ab}RR_{|c}R_{|d}R_{|i}R_{|j}P'''\big) \\
= {}&\varepsilon^{ac}\varepsilon^{bd}\big[R_{|abij}\left(6R_{|cd}P' + 2R_{|c}R_{|d}P'' - g_{cd}RP\right) \\
&+ 6R_{|abi}R_{|cdj}P' - 2\,\mathrm{Sym}_{ij}\, g_{ab}R_{|i}R_{|cdj}\left(P + RP'\right) \\
&+ 2\,\mathrm{Sym}_{ij}\, R_{|abi}\left(6R_{|cd}R_{|j}P'' + 4R_{|cj}R_{|d}P'' + 2R_{|c}R_{|d}R_{|j}P'''\right) \\
&+ R_{|cij}\left(4R_{|ab}R_{|d}P'' - 2g_{ab}R_{|d}P - 4g_{ab}RR_{|d}P'\right) \\
&+ 3R_{|ab}R_{|cd}R_{|ij}P'' + 4R_{|ab}R_{|ci}R_{|dj}P'' + 3R_{|ab}R_{|cd}R_{|i}R_{|j}P''' \\
&+ 2R_{|ab}R_{|ij}R_{|c}R_{|d}P''' + 4R_{|ab}R_{|ci}R_{|d}R_{|j}P''' + 4R_{|ab}R_{|cj}R_{|d}R_{|i}P''' \\
&- 2g_{ab}R_{|ci}R_{|dj}P - 4g_{ab}RR_{|ci}R_{|dj}P' - g_{ab}R_{|cd}R_{|ij}P - g_{ab}RR_{|cd}R_{|ij}P' \\
&+ 2R_{|ab}R_{|c}R_{|d}R_{|i}R_{|j}P^{(4)} - 6g_{ab}R_{|ci}R_{|d}R_{|j}P' - 4g_{ab}RR_{|ci}R_{|d}R_{|j}P'' \\
&- 6g_{ab}R_{|cj}R_{|d}R_{|i}P' - 4g_{ab}RR_{|cj}R_{|d}R_{|i}P'' - 3g_{ab}R_{|c}R_{|d}R_{|ij}P' \\
&- 2g_{ab}RR_{|c}R_{|d}R_{|ij}P'' - 2g_{ab}R_{|i}R_{|j}R_{|cd}P' - g_{ab}RR_{|cd}R_{|i}R_{|j}P'' \\
&- 5g_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P'' - 2g_{ab}RR_{|c}R_{|d}R_{|i}R_{|j}P'''\big] \\
= {}&\varepsilon^{ac}\varepsilon^{bd}\Bigg(\Bigg\{R_{|(abij)} + \mathrm{Sym}_{abi}\left[R\left(g_{ja}R_{|bi} - g_{ab}R_{|ij}\right) + \frac{1}{4}R_{|i}\left(g_{ja}R_{|b} - g_{ab}R_{|j}\right)\right] \\
&+ \frac{1}{3}R_{|j}\left(g_{ia}R_{|b} - g_{ab}R_{|i}\right) + \frac{1}{3}R\left(g_{ia}R_{|bj} - g_{ab}R_{|ij}\right)\Bigg\} \\
&\times\left(6R_{|cd}P' + 2R_{|c}R_{|d}P'' - g_{cd}RP\right) + 6\left[R_{|(abi)} + \frac{1}{3}\,\mathrm{Sym}_{ab}\, R\left(g_{ia}R_{|b} - g_{ab}R_{|i}\right)\right] \\
&\times\left[R_{|(cdj)} + \frac{1}{3}\,\mathrm{Sym}_{cd}\, R\left(g_{jc}R_{|d} - g_{cd}R_{|j}\right)\right]P' \\
&- 2\,\mathrm{Sym}_{ij}\, g_{ab}R_{|i}\left[R_{|(cdj)} + \frac{1}{3}R\left(g_{jc}R_{|d} - g_{cd}R_{|j}\right)\right]\left(P + RP'\right) \\
&+ 2\,\mathrm{Sym}_{ij}\left[R_{|(abi)} + \frac{1}{3}\,\mathrm{Sym}_{ab}\, R\left(g_{ia}R_{|b} - g_{ab}R_{|i}\right)\right]\big(6R_{|cd}R_{|j}P'' + 4R_{|cj}R_{|d}P'' \\
&+ 2R_{|c}R_{|d}R_{|j}P'''\big) + \left[R_{|(cij)} + \frac{1}{3}\,\mathrm{Sym}_{ci}\, R\left(g_{jc}R_{|i} - g_{ci}R_{|j}\right)\right] \\
&\times\left(4R_{|ab}R_{|d}P'' - 2g_{ab}R_{|d}P - 4g_{ab}RR_{|d}P'\right) + 3R_{|ab}R_{|cd}R_{|ij}P''
\end{aligned}
$$

$$
\begin{aligned}
&+ 4R_{|ab}R_{|ci}R_{|dj}P'' + 3R_{|ab}R_{|cd}R_{|i}R_{|j}P''' + 2R_{|ab}R_{|ij}R_{|c}R_{|d}P''' \\
&+ 4R_{|ab}R_{|ci}R_{|d}R_{|j}P''' + 4R_{|ab}R_{|cj}R_{|d}R_{|i}P''' - 2g_{ab}R_{|ci}R_{|dj}P \\
&- 4g_{ab}RR_{|ci}R_{|dj}P' - g_{ab}R_{|cd}R_{|ij}P - g_{ab}RR_{|cd}R_{|ij}P' + 2R_{|ab}R_{|c}R_{|d}R_{|i}R_{|j}P^{(4)} \\
&- 6g_{ab}R_{|ci}R_{|d}R_{|j}P' - 4g_{ab}RR_{|ci}R_{|d}R_{|j}P'' - 6g_{ab}R_{|cj}R_{|d}R_{|i}P' \\
&- 4g_{ab}RR_{|cj}R_{|d}R_{|i}P'' - 3g_{ab}R_{|c}R_{|d}R_{|ij}P' - 2g_{ab}RR_{|c}R_{|d}R_{|ij}P'' \\
&- 2g_{ab}R_{|i}R_{|j}R_{|cd}P' - g_{ab}RR_{|cd}R_{|i}R_{|j}P'' - 5g_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P'' \\
&- 2g_{ab}RR_{|c}R_{|d}R_{|i}R_{|j}P'''\Big) \\
= \varepsilon^{ac}\varepsilon^{bd}\Bigg(&\Bigg\{R_{|(abij)} + \frac{1}{3}\Big[R\left(g_{ja}R_{|bi} + g_{ji}R_{|ba} + g_{jb}R_{|ia} - g_{ab}R_{|ij} - g_{ib}R_{|aj}\right. \\
&\left. - g_{ai}R_{|bj}\right) + \frac{1}{4}\left(g_{ja}R_{|b}R_{|i} + g_{ji}R_{|a}R_{|b} + g_{jb}R_{|a}R_{|i} - g_{ab}R_{|i}R_{|j}\right. \\
&\left. - g_{ib}R_{|a}R_{|j} - g_{ai}R_{|b}R_{|j}\right)\Big] + \frac{1}{3}R_{|j}\left(g_{ia}R_{|b} - g_{ab}R_{|i}\right) + \frac{1}{3}R\left(g_{ia}R_{|bj} - g_{ab}R_{|ij}\right)\Bigg\} \\
&\times\left(6R_{|cd}P' + 2R_{|c}R_{|d}P'' - g_{cd}RP\right) \\
&+ 6\Bigg[R_{|(abi)}R_{|(cdj)} + \frac{1}{3}R\left(g_{ia}R_{|b} - g_{ab}R_{|i}\right)R_{|(cdj)} + \frac{1}{3}R\left(g_{jc}R_{|d} - g_{cd}R_{|j}\right)R_{|(abi)} \\
&+ \frac{1}{36}R^2\left(g_{ia}R_{|b} + g_{ib}R_{|a} - 2g_{ab}R_{|i}\right)\left(g_{jc}R_{|d} + g_{jd}R_{|c} - 2g_{cd}R_{|j}\right)\Bigg]P' \\
&- g_{ab}\Bigg\{R_{|i}\left[R_{|(cdj)} + \frac{1}{3}R\left(g_{jc}R_{|d} - g_{cd}R_{|j}\right)\right] \\
&+ R_{|j}\left[R_{|(cdi)} + \frac{1}{3}R\left(g_{ic}R_{|d} - g_{cd}R_{|i}\right)\right]\Bigg\}\left(P + RP'\right) \\
&+ \Bigg\{\left[R_{|(abi)} + \frac{1}{6}R\left(g_{ia}R_{|b} + g_{ib}R_{|a} - 2g_{ab}R_{|i}\right)\right] \\
&\times\left(6R_{|cd}R_{|j}P'' + 4R_{|cj}R_{|d}P'' + 2R_{|c}R_{|d}R_{|j}P'''\right) \\
&+ \left[R_{|(abj)} + \frac{1}{6}R\left(g_{ja}R_{|b} + g_{jb}R_{|a} - 2g_{ab}R_{|j}\right)\right] \\
&\times\left(6R_{|cd}R_{|i}P'' + 4R_{|ci}R_{|d}P'' + 2R_{|c}R_{|d}R_{|i}P'''\right)\Bigg\} \\
&+ \left[R_{|(cij)} + \frac{1}{6}R\left(g_{jc}R_{|i} + g_{ij}R_{|c} - 2g_{ci}R_{|j}\right)\right] \\
&\times\left(4R_{|ab}R_{|d}P'' - 2g_{ab}R_{|d}P - 4g_{ab}RR_{|d}P'\right) \\
&+ 3R_{|ab}R_{|cd}R_{|ij}P'' + 4R_{|ab}R_{|ci}R_{|dj}P'' + 3R_{|ab}R_{|cd}R_{|i}R_{|j}P'''
\end{aligned}
$$

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\begin{aligned}
&+ 2R_{|ab}R_{|ij}R_{|c}R_{|d}P''' + 4R_{|ab}R_{|ci}R_{|d}R_{|j}P''' + 4R_{|ab}R_{|cj}R_{|d}R_{|i}P''' \\
&- 2g_{ab}R_{|ci}R_{|dj}P - 4g_{ab}RR_{|ci}R_{|dj}P' - g_{ab}R_{|cd}R_{|ij}P - g_{ab}RR_{|cd}R_{|ij}P' \\
&+ 2R_{|ab}R_{|c}R_{|d}R_{|i}R_{|j}P^{(4)} - 6g_{ab}R_{|ci}R_{|d}R_{|j}P' - 4g_{ab}RR_{|ci}R_{|d}R_{|j}P'' \\
&- 6g_{ab}R_{|cj}R_{|d}R_{|i}P' - 4g_{ab}RR_{|cj}R_{|d}R_{|i}P'' - 3g_{ab}R_{|c}R_{|d}R_{|ij}P' \\
&- 2g_{ab}RR_{|c}R_{|d}R_{|ij}P'' - 2g_{ab}R_{|i}R_{|j}R_{|cd}P' - g_{ab}RR_{|cd}R_{|i}R_{|j}P'' \\
&- 5g_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P'' - 2g_{ab}RR_{|c}R_{|d}R_{|i}R_{|j}P'''\Big) \\
= \varepsilon^{ac}\varepsilon^{bd}\Bigg[&\bigg(R_{|(abij)} + \frac{2}{3}Rg_{aj}R_{|bi} + \frac{1}{3}Rg_{ij}R_{|ab} - \frac{1}{3}Rg_{ai}R_{|bj} - \frac{2}{3}Rg_{ab}R_{|ij} \\
&+ \frac{1}{6}g_{aj}R_{|b}R_{|i} + \frac{1}{12}g_{ij}R_{|a}R_{|b} - \frac{5}{12}g_{ab}R_{|i}R_{|j} + \frac{1}{6}g_{ai}R_{|b}R_{|j}\bigg) \\
&\times \big(6R_{|cd}P' + 2R_{|c}R_{|d}P'' - g_{cd}RP\big) \\
&+ \bigg(6R_{|(abi)}R_{|(cdj)} + 2Rg_{ia}R_{|b}R_{|(cdj)} - 2Rg_{ab}R_{|i}R_{|(cdj)} + 2Rg_{jc}R_{|d}R_{|(abi)} \\
&- 2Rg_{cd}R_{|j}R_{|(abi)} + \frac{1}{6}R^2g_{ia}g_{jc}R_{|d}R_{|b} + \frac{1}{6}R^2g_{ib}g_{jc}R_{|d}R_{|a} - \frac{1}{3}R^2g_{ab}g_{jc}R_{|d}R_{|i} \\
&+ \frac{1}{6}R^2g_{ia}g_{jd}R_{|c}R_{|b} + \frac{1}{6}R^2g_{ib}g_{jd}R_{|c}R_{|a} - \frac{1}{3}R^2g_{ab}g_{jd}R_{|c}R_{|i} - \frac{1}{3}R^2g_{ia}g_{cd}R_{|j}R_{|b} \\
&- \frac{1}{3}R^2g_{ib}g_{cd}R_{|j}R_{|a} + \frac{2}{3}R^2g_{ab}g_{cd}R_{|j}R_{|i}\bigg)P' \\
&+ \bigg(-g_{ab}R_{|i}R_{|(cdj)} - \frac{1}{3}Rg_{ab}g_{jc}R_{|d}R_{|i} + \frac{1}{3}Rg_{ab}g_{cd}R_{|i}R_{|j} - g_{ab}R_{|j}R_{|(cdi)} \\
&- \frac{1}{3}Rg_{ab}g_{ic}R_{|d}R_{|j} + \frac{1}{3}Rg_{ab}g_{cd}R_{|i}R_{|j}\bigg)\big(P + RP'\big) \\
&+ \bigg(R_{|(abi)} + \frac{1}{6}Rg_{ia}R_{|b} + \frac{1}{6}Rg_{ib}R_{|a} - \frac{1}{3}Rg_{ab}R_{|i}\bigg) \\
&\times \big(6R_{|cd}R_{|j}P'' + 4R_{|cj}R_{|d}P'' + 2R_{|c}R_{|d}R_{|j}P'''\big) \\
&+ \bigg(R_{|(abj)} + \frac{1}{6}Rg_{ja}R_{|b} + \frac{1}{6}Rg_{jb}R_{|a} - \frac{1}{3}Rg_{ab}R_{|j}\bigg) \\
&\times \big(6R_{|cd}R_{|i}P'' + 4R_{|ci}R_{|d}P'' + 2R_{|c}R_{|d}R_{|i}P'''\big) \\
&+ \bigg(R_{|(cij)} + \frac{1}{6}Rg_{jc}R_{|i} + \frac{1}{6}Rg_{ij}R_{|c} - \frac{1}{3}Rg_{ci}R_{|j}\bigg) \\
&\times \big(4R_{|ab}R_{|d}P'' - 2g_{ab}R_{|d}P - 4g_{ab}RR_{|d}P'\big) \\
&+ 3R_{|ab}R_{|cd}R_{|ij}P'' + 4R_{|ab}R_{|ci}R_{|dj}P'' + 3R_{|ab}R_{|cd}R_{|i}R_{|j}P'''
\end{aligned}
$$

$$
\begin{aligned}
&+2R_{|ab}R_{|ij}R_{|c}R_{|d}P''' + 4R_{|ab}R_{|ci}R_{|d}R_{|j}P''' + 4R_{|ab}R_{|cj}R_{|d}R_{|i}P''' \\
&-2g_{ab}R_{|ci}R_{|dj}P - 4g_{ab}RR_{|ci}R_{|dj}P' - g_{ab}R_{|cd}R_{|ij}P - g_{ab}RR_{|cd}R_{|ij}P' \\
&+2R_{|ab}R_{|c}R_{|d}R_{|i}R_{|j}P^{(4)} - 6g_{ab}R_{|ci}R_{|d}R_{|j}P' - 4g_{ab}RR_{|ci}R_{|d}R_{|j}P'' \\
&-6g_{ab}R_{|cj}R_{|d}R_{|i}P' - 4g_{ab}RR_{|cj}R_{|d}R_{|i}P'' - 3g_{ab}R_{|c}R_{|d}R_{|ij}P' \\
&-2g_{ab}RR_{|c}R_{|d}R_{|ij}P'' - 2g_{ab}R_{|i}R_{|j}R_{|cd}P' - g_{ab}RR_{|cd}R_{|i}R_{|j}P'' \\
&-5g_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P'' - 2g_{ab}RR_{|c}R_{|d}R_{|i}R_{|j}P'''\Big] \\
=\ &\varepsilon^{ac}\varepsilon^{bd}\big(6R_{|cd}R_{|(abij)}P' + 4Rg_{aj}R_{|cd}R_{|bi}P' + 2Rg_{ij}R_{|ab}R_{|cd}P' - 2Rg_{ai}R_{|bj}R_{|cd}P' \\
&-4Rg_{ab}R_{|cd}R_{|ij}P' + g_{aj}R_{|b}R_{|i}R_{|cd}P' + \frac{1}{2}g_{ij}R_{|a}R_{|b}R_{|cd}P' - \frac{5}{2}g_{ab}R_{|i}R_{|j}R_{|cd}P' \\
&+g_{ai}R_{|b}R_{|j}R_{|cd}P' + 2R_{|c}R_{|d}R_{|(abij)}P'' + \frac{4}{3}Rg_{aj}R_{|c}R_{|d}R_{|bi}P'' \\
&+\frac{2}{3}Rg_{ij}R_{|c}R_{|d}R_{|ab}P'' - \frac{2}{3}Rg_{ai}R_{|c}R_{|d}R_{|bj}P'' - \frac{4}{3}Rg_{ab}R_{|c}R_{|d}R_{|ij}P'' \\
&+\frac{1}{3}g_{aj}R_{|b}R_{|c}R_{|d}R_{|i}P'' + \frac{1}{6}g_{ij}R_{|a}R_{|b}R_{|c}R_{|d}P'' - \frac{5}{6}g_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P'' \\
&+\frac{1}{3}g_{ai}R_{|b}R_{|c}R_{|d}R_{|j}P'' - Rg_{cd}R_{|(abij)}P - \frac{2}{3}R^2g_{aj}g_{cd}R_{|bi}P - \frac{1}{3}R^2g_{cd}g_{ij}R_{|ab}P \\
&+\frac{1}{3}R^2g_{ai}g_{cd}R_{|bj}P + \frac{2}{3}R^2g_{ab}g_{cd}R_{|ij}P - \frac{1}{6}Rg_{aj}g_{cd}R_{|b}R_{|i}P \\
&-\frac{1}{12}Rg_{cd}g_{ij}R_{|a}R_{|b}P + \frac{5}{12}Rg_{ab}g_{cd}R_{|i}R_{|j}P - \frac{1}{6}Rg_{ai}g_{cd}R_{|b}R_{|j}P \\
&+6R_{|(abi)}R_{|(cdj)}P' + 2Rg_{ia}R_{|b}R_{|(cdj)}P' - 2Rg_{ab}R_{|i}R_{|(cdj)}P' \\
&+2Rg_{jc}R_{|d}R_{|(abi)}P' - 2Rg_{cd}R_{|j}R_{|(abi)}P' + \frac{1}{6}R^2g_{ia}g_{jc}R_{|d}R_{|b}P' \\
&+\frac{1}{6}R^2g_{ib}g_{jc}R_{|d}R_{|a}P' - \frac{1}{3}R^2g_{ab}g_{jc}R_{|d}R_{|i}P' + \frac{1}{6}R^2g_{ia}g_{jd}R_{|c}R_{|b}P' \\
&+\frac{1}{6}R^2g_{ib}g_{jd}R_{|c}R_{|a}P' - \frac{1}{3}R^2g_{ab}g_{jd}R_{|c}R_{|i}P' - \frac{1}{3}R^2g_{ia}g_{cd}R_{|j}R_{|b}P' \\
&-\frac{1}{3}R^2g_{ib}g_{cd}R_{|j}R_{|a}P' + \frac{2}{3}R^2g_{ab}g_{cd}R_{|j}R_{|i}P' - g_{ab}R_{|i}R_{|(cdj)}P \\
&-\frac{1}{3}Rg_{ab}g_{jc}R_{|d}R_{|i}P + \frac{1}{3}Rg_{ab}g_{cd}R_{|i}R_{|j}P - g_{ab}R_{|j}R_{|(cdi)}P \\
&-\frac{1}{3}Rg_{ab}g_{ic}R_{|d}R_{|j}P + \frac{1}{3}Rg_{ab}g_{cd}R_{|i}R_{|j}P - Rg_{ab}R_{|i}R_{|(cdj)}P' \\
&-\frac{1}{3}R^2g_{ab}g_{jc}R_{|d}R_{|i}P' + \frac{1}{3}R^2g_{ab}g_{cd}R_{|i}R_{|j}P' - Rg_{ab}R_{|j}R_{|(cdi)}P' \\
&-\frac{1}{3}R^2g_{ab}g_{ic}R_{|d}R_{|j}P' + \frac{1}{3}R^2g_{ab}g_{cd}R_{|i}R_{|j}P' + 6R_{|j}R_{|cd}R_{|(abi)}P''
\end{aligned}
$$

$$+ Rg_{ia}R_{|b}R_{|j}R_{|cd}P'' + Rg_{ib}R_{|a}R_{|j}R_{|cd}P'' - 2Rg_{ab}R_{|i}R_{|j}R_{|cd}P''$$
$$+ 4R_{|d}R_{|cj}R_{|(abi)}P'' + \frac{2}{3}Rg_{ia}R_{|b}R_{|d}R_{|cj}P'' + \frac{2}{3}Rg_{ib}R_{|a}R_{|d}R_{|cj}P''$$
$$- \frac{4}{3}Rg_{ab}R_{|d}R_{|i}R_{|cj}P'' + 2R_{|c}R_{|d}R_{|j}R_{|(abi)}P''' + \frac{1}{3}Rg_{ia}R_{|b}R_{|c}R_{|d}R_{|j}P'''$$
$$+ \frac{1}{3}Rg_{ib}R_{|a}R_{|c}R_{|d}R_{|j}P''' - \frac{2}{3}Rg_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P''' + 6R_{|i}R_{|cd}R_{|(abj)}P''$$
$$+ Rg_{ja}R_{|b}R_{|i}R_{|cd}P'' + Rg_{jb}R_{|a}R_{|i}R_{|cd}P'' - 2Rg_{ab}R_{|i}R_{|j}R_{|cd}P''$$
$$+ 4R_{|d}R_{|ci}R_{|(abj)}P'' + \frac{2}{3}Rg_{ja}R_{|b}R_{|d}R_{|ci}P'' + \frac{2}{3}Rg_{jb}R_{|a}R_{|d}R_{|ci}P''$$
$$- \frac{4}{3}Rg_{ab}R_{|d}R_{|j}R_{|ci}P'' + 2R_{|c}R_{|d}R_{|i}R_{|(abj)}P''' + \frac{1}{3}Rg_{ja}R_{|b}R_{|c}R_{|d}R_{|i}P'''$$
$$+ \frac{1}{3}Rg_{jb}R_{|a}R_{|c}R_{|d}R_{|i}P''' - \frac{2}{3}Rg_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P''' + 4R_{|d}R_{|ab}R_{|(cij)}P''$$
$$+ \frac{2}{3}Rg_{jc}R_{|i}R_{|d}R_{|ab}P'' + \frac{2}{3}Rg_{ij}R_{|c}R_{|d}R_{|ab}P'' - \frac{4}{3}Rg_{ci}R_{|d}R_{|j}R_{|ab}P''$$
$$- 2g_{ab}R_{|d}R_{|(cij)}P - \frac{1}{3}Rg_{ab}g_{jc}R_{|d}R_{|i}P - \frac{1}{3}Rg_{ab}g_{ij}R_{|d}R_{|c}P$$
$$+ \frac{2}{3}Rg_{ab}g_{ci}R_{|d}R_{|j}P - 4Rg_{ab}R_{|d}R_{|(cij)}P' - \frac{2}{3}R^2g_{ab}g_{jc}R_{|d}R_{|i}P'$$
$$- \frac{2}{3}R^2g_{ab}g_{ij}R_{|c}R_{|d}P' + \frac{4}{3}R^2g_{ab}g_{ci}R_{|d}R_{|j}P' + 3R_{|ab}R_{|cd}R_{|ij}P''$$
$$+ 4R_{|ab}R_{|ci}R_{|dj}P'' + 3R_{|ab}R_{|cd}R_{|i}R_{|j}P''' + 2R_{|ab}R_{|ij}R_{|c}R_{|d}P'''$$
$$+ 4R_{|ab}R_{|ci}R_{|d}R_{|j}P''' + 4R_{|ab}R_{|cj}R_{|d}R_{|i}P''' - 2g_{ab}R_{|ci}R_{|dj}P$$
$$- 4g_{ab}RR_{|ci}R_{|dj}P' - g_{ab}R_{|cd}R_{|ij}P - g_{ab}RR_{|cd}R_{|ij}P' + 2R_{|ab}R_{|c}R_{|d}R_{|i}R_{|j}P^{(4)}$$
$$- 6g_{ab}R_{|ci}R_{|d}R_{|j}P' - 4g_{ab}RR_{|ci}R_{|d}R_{|j}P'' - 6g_{ab}R_{|cj}R_{|d}R_{|i}P'$$
$$- 4g_{ab}RR_{|cj}R_{|d}R_{|i}P'' - 3g_{ab}R_{|c}R_{|d}R_{|ij}P' - 2g_{ab}RR_{|c}R_{|d}R_{|ij}P''$$
$$- 2g_{ab}R_{|i}R_{|j}R_{|cd}P' - g_{ab}RR_{|cd}R_{|i}R_{|j}P'' - 5g_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P''$$
$$- 2g_{ab}RR_{|c}R_{|d}R_{|i}R_{|j}P'''\Big)$$
$$= (-1)^q\left(g^{ab}g^{cd} - g^{ad}g^{bc}\right)\big(6R_{|cd}R_{|(abij)}P' + 2R_{|c}R_{|d}R_{|(abij)}P'' - Rg_{cd}R_{|(abij)}P$$
$$+ 6R_{|(abi)}R_{|(cdj)}P' + 6R_{|j}R_{|cd}R_{|(abi)}P'' + 4R_{|d}R_{|cj}R_{|(abi)}P''$$
$$+ 4R_{|d}R_{|ci}R_{|(abj)}P'' + 6R_{|i}R_{|cd}R_{|(abj)}P'' + 4R_{|d}R_{|ab}R_{|(cij)}P''$$
$$+ 2R_{|c}R_{|d}R_{|j}R_{|(abi)}P''' + 2R_{|c}R_{|d}R_{|i}R_{|(abj)}P''' + 2Rg_{jc}R_{|d}R_{|(abi)}P'$$
$$- 2Rg_{cd}R_{|j}R_{|(abi)}P' + 2Rg_{ia}R_{|b}R_{|(cdj)}P' - 3Rg_{ab}R_{|i}R_{|(cdj)}P' - g_{ab}R_{|i}R_{|(cdj)}P$$

$$
\begin{aligned}
&- 4Rg_{ab}R_{|d}R_{|(cij)}P' - 2g_{ab}R_{|d}R_{|(cij)}P - g_{ab}R_{|j}R_{|(cdi)}P - Rg_{ab}R_{|j}R_{|(cdi)}P' \\
&+ 3R_{|ab}R_{|cd}R_{|ij}P'' + 4R_{|ab}R_{|ci}R_{|dj}P'' + 3R_{|i}R_{|j}R_{|ab}R_{|cd}P''' \\
&+ 2R_{|c}R_{|d}R_{|ab}R_{|ij}P''' + 4R_{|d}R_{|j}R_{|ab}R_{|ci}P''' + 4R_{|d}R_{|i}R_{|ab}R_{|cj}P''' \\
&- 2g_{ab}R_{|ci}R_{|dj}P - g_{ab}R_{|cd}R_{|ij}P + 4Rg_{aj}R_{|bi}R_{|cd}P' \\
&- 2Rg_{ai}R_{|bj}R_{|cd}P' + 2Rg_{ij}R_{|ab}R_{|cd}P' - 4Rg_{ab}R_{|cd}R_{|ij}P' \\
&- 4Rg_{ab}R_{|ci}R_{|dj}P' - Rg_{ab}R_{|cd}R_{|ij}P' + 2R_{|c}R_{|d}R_{|i}R_{|j}R_{|ab}P^{(4)} \\
&+ g_{ai}R_{|b}R_{|j}R_{|cd}P' + g_{aj}R_{|b}R_{|i}R_{|cd}P' + \frac{1}{2}g_{ij}R_{|a}R_{|b}R_{|cd}P' \\
&- \frac{9}{2}g_{ab}R_{|i}R_{|j}R_{|cd}P' - 6g_{ab}R_{|d}R_{|j}R_{|ci}P' - 6g_{ab}R_{|d}R_{|i}R_{|cj}P' \\
&- 3g_{ab}R_{|c}R_{|d}R_{|ij}P' - 5Rg_{ab}R_{|i}R_{|j}R_{|cd}P'' - \frac{10}{3}Rg_{ab}R_{|c}R_{|d}R_{|ij}P'' \\
&- \frac{16}{3}Rg_{ab}R_{|d}R_{|i}R_{|cj}P'' - \frac{16}{3}Rg_{ab}R_{|d}R_{|j}R_{|ci}P'' - \frac{2}{3}Rg_{ai}R_{|c}R_{|d}R_{|bj}P'' \\
&+ \frac{4}{3}Rg_{aj}R_{|c}R_{|d}R_{|bi}P'' + Rg_{ai}R_{|b}R_{|j}R_{|cd}P'' + Rg_{aj}R_{|b}R_{|i}R_{|cd}P'' \\
&+ \frac{2}{3}Rg_{ai}R_{|b}R_{|d}R_{|cj}P'' + \frac{2}{3}Rg_{aj}R_{|b}R_{|d}R_{|ci}P'' + Rg_{bi}R_{|a}R_{|j}R_{|cd}P'' \\
&+ \frac{2}{3}Rg_{bi}R_{|a}R_{|d}R_{|cj}P'' + Rg_{bj}R_{|a}R_{|i}R_{|cd}P'' + \frac{2}{3}Rg_{bj}R_{|a}R_{|d}R_{|ci}P'' \\
&- \frac{4}{3}Rg_{ci}R_{|d}R_{|j}R_{|ab}P'' + \frac{2}{3}Rg_{cj}R_{|i}R_{|d}R_{|ab}P'' + \frac{4}{3}Rg_{ij}R_{|c}R_{|d}R_{|ab}P'' \\
&+ \frac{2}{3}R^2g_{ab}g_{cd}R_{|ij}P + \frac{1}{3}R^2g_{ai}g_{cd}R_{|bj}P - \frac{2}{3}R^2g_{aj}g_{cd}R_{|bi}P \\
&- \frac{1}{3}R^2g_{cd}g_{ij}R_{|ab}P - \frac{35}{6}g_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P'' + \frac{1}{3}g_{ai}R_{|b}R_{|c}R_{|d}R_{|j}P'' \\
&+ \frac{1}{3}g_{aj}R_{|b}R_{|c}R_{|d}R_{|i}P'' + \frac{1}{6}g_{ij}R_{|a}R_{|b}R_{|c}R_{|d}P'' - \frac{10}{3}Rg_{ab}R_{|c}R_{|d}R_{|i}R_{|j}P''' \\
&+ \frac{1}{3}Rg_{ai}R_{|b}R_{|c}R_{|d}R_{|j}P''' + \frac{1}{3}Rg_{aj}R_{|b}R_{|c}R_{|d}R_{|i}P''' + \frac{1}{3}Rg_{bi}R_{|a}R_{|c}R_{|d}R_{|j}P''' \\
&+ \frac{1}{3}Rg_{bj}R_{|a}R_{|c}R_{|d}R_{|i}P''' + \frac{13}{12}Rg_{ab}g_{cd}R_{|i}R_{|j}P + \frac{1}{3}Rg_{ab}g_{ci}R_{|d}R_{|j}P \\
&- \frac{2}{3}Rg_{ab}g_{cj}R_{|d}R_{|i}P - \frac{1}{3}Rg_{ab}g_{ij}R_{|c}R_{|d}P - \frac{1}{6}Rg_{ai}g_{cd}R_{|b}R_{|j}P \\
&- \frac{1}{6}Rg_{aj}g_{cd}R_{|b}R_{|i}P - \frac{1}{12}Rg_{cd}g_{ij}R_{|a}R_{|b}P + \frac{4}{3}R^2g_{ab}g_{cd}R_{|i}R_{|j}P' \\
&- \frac{2}{3}R^2g_{ab}g_{jc}R_{|d}R_{|i}P' + R^2g_{ab}g_{ci}R_{|d}R_{|j}P' - R^2g_{ab}g_{cj}R_{|d}R_{|i}P' \\
&- \frac{2}{3}R^2g_{ab}g_{ij}R_{|c}R_{|d}P' + \frac{1}{6}R^2g_{ai}g_{jc}R_{|b}R_{|d}P' + \frac{1}{6}R^2g_{ai}g_{jd}R_{|b}R_{|c}P' \\
&- \frac{1}{3}R^2g_{ai}g_{cd}R_{|b}R_{|j}P' + \frac{1}{6}R^2g_{bi}g_{jc}R_{|a}R_{|d}P' + \frac{1}{6}R^2g_{bi}g_{jd}R_{|a}R_{|c}P'
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{3}R^2 g_{bi}g_{cd}R_{|a}R_{|j}P'\Big) \\
= (-1)^q \Bigg[&\Big(6g^{cd}R^{|a}{}_{|a}R_{|(abij)}P' - 6R^{|ab}R_{|(abij)}P'\Big) + \Big(2Sg^{ab}R_{|(abij)}P'' \\
&- 2R^{|a}R^{|b}R_{|(abij)}P''\Big) - Rg^{ab}R_{|(abij)}P + \Big(6g^{ab}g^{cd}R_{|(abi)}R_{|(cdj)}P' \\
&- 6g^{ad}g^{bc}R_{|(abi)}R_{|(cdj)}P'\Big) + \Big(6g^{ab}R_{|j}R^{|c}{}_{|c}R_{|(abi)}P'' - 6R_{|j}R^{|ab}R_{|(abi)}P''\Big) \\
&+ \Big(4g^{ab}R^{|c}R_{|cj}R_{|(abi)}P'' - 4R^{|a}R^{|b}{}_{|j}R_{|(abi)}P''\Big) + \Big(4g^{ab}R^{|c}R_{|ci}R_{|(abj)}P'' \\
&- 4R^{|a}R^{|b}{}_{|i}R_{|(abj)}P''\Big) + \Big(6g^{ab}R_{|i}R^{|c}{}_{|c}R_{|(abj)}P'' - 6R_{|i}R^{|ab}R_{|(abj)}P''\Big) \\
&+ \Big(4R^{|c}R^{|a}{}_{|a}R_{|(cij)}P'' - 4R_{|d}R^{|cd}R_{|(cij)}P''\Big) + \Big(2Sg^{ab}R_{|j}R_{|(abi)}P''' \\
&- 2R^{|a}R^{|b}R_{|j}R_{|(abi)}P'''\Big) + \Big(2Sg^{ab}R_{|i}R_{|(abj)}P''' - 2R^{|a}R^{|b}R_{|i}R_{|(abj)}P'''\Big) \\
&+ \Big(2Rg^{ab}R_{|j}R_{|(abi)}P' - 2RR^{|a}R_{|(aji)}P'\Big) - 2Rg^{ab}R_{|j}R_{|(abi)}P' \\
&+ \Big(2Rg^{cd}R_{|i}R_{|(cdj)}P' - 2RR^{|c}R_{|(cij)}P'\Big) - 3Rg^{cd}R_{|i}R_{|(cdj)}P' - g^{cd}R_{|i}R_{|(cdj)}P \\
&- 4RR^{|c}R_{|(cij)}P' - 2R^{|c}R_{|(cij)}P - g^{cd}R_{|j}R_{|(cdi)}P - Rg^{cd}R_{|j}R_{|(cdi)}P' \\
&+ \Big(3R^{|a}{}_{|a}R^{|c}{}_{|c}R_{|ij}P'' - 3R^{|cd}R_{|cd}R_{|ij}P''\Big) + \Big(4R^{|a}{}_{|a}R^{|d}{}_{|i}R_{|dj}P'' \\
&- 4R^{|cd}R_{|ci}R_{|dj}P''\Big) + \Big(3R_{|i}R_{|j}R^{|a}{}_{|a}R^{|c}{}_{|c}P''' - 3R_{|i}R_{|j}R^{|cd}R_{|cd}P'''\Big) \\
&+ \Big(2SR^{|a}{}_{|a}R_{|ij}P''' - 2R^{|a}R^{|b}R_{|ab}R_{|ij}P'''\Big) + \Big(4R^{|c}R_{|j}R^{|a}{}_{|a}R_{|ci}P''' \\
&- 4R_{|d}R_{|j}R^{|cd}R_{|ci}P'''\Big) + \Big(4R^{|c}R_{|i}R^{|a}{}_{|a}R_{|cj}P''' - 4R_{|d}R_{|i}R^{|cd}R_{|cj}P'''\Big) \\
&- 2R^{|d}{}_{|i}R_{|dj}P - R^{|c}{}_{|c}R_{|ij}P + \Big(4RR^{|c}{}_{|c}R_{|ji}P' - 4RR^{|b}{}_{|j}R_{|bi}P'\Big) \\
&+ \Big(-2RR_{|ij}R^{|c}{}_{|c}P' + 2RR^{|b}{}_{|i}R_{|bj}P'\Big) + \Big(2Rg_{ij}R^{|a}{}_{|a}R^{|c}{}_{|c}P' \\
&- 2Rg_{ij}R^{|cd}R_{|cd}P'\Big) - 4RR^{|c}{}_{|c}R_{|ij}P' - 4RR^{|d}{}_{|i}R_{|dj}P' - RR^{|c}{}_{|c}R_{|ij}P' \\
&+ \Big(2SR_{|i}R_{|j}R^{|a}{}_{|a}P^{(4)} - 2R^{|a}R^{|b}R_{|i}R_{|j}R_{|ab}P^{(4)}\Big) + \Big(R_{|i}R_{|j}R^{|c}{}_{|c}P' \\
&- R^{|c}R_{|j}R_{|ci}P'\Big) + \Big(R_{|i}R_{|j}R^{|c}{}_{|c}P' - R^{|c}R_{|i}R_{|cj}P'\Big) + \Big(\frac{1}{2}Sg_{ij}R^{|c}{}_{|c}P' \\
&- \frac{1}{2}g_{ij}R^{|c}R^{|d}R_{|cd}P'\Big) - \frac{9}{2}R_{|i}R_{|j}R^{|c}{}_{|c}P' - 6R^{|c}R_{|j}R_{|ci}P' - 6R^{|c}R_{|i}R_{|cj}P' \\
&- 3SR_{|ij}P' - 5RR_{|i}R_{|j}R^{|c}{}_{|c}P'' - \frac{10}{3}RSR_{|ij}P'' - \frac{16}{3}RR^{|c}R_{|i}R_{|cj}P'' \\
&- \frac{16}{3}RR^{|c}R_{|j}R_{|ci}P'' + \Big(-\frac{2}{3}RSR_{|ij}P'' + \frac{2}{3}RR^{|b}R_{|i}R_{|bj}P''\Big) + \Big(\frac{4}{3}RSR_{|ji}P''
\end{aligned}
$$

$$
\begin{aligned}
& - \frac{4}{3} R R^{|b} R_{|j} R_{|bi} P'' \Big) + \Big(R R_{|i} R_{|j} R^{|c}{}_{|c} P'' - R R^{|c} R_{|j} R_{|ci} P'' \Big) \\
& + \Big(R R_{|i} R_{|j} R^{|c}{}_{|c} P'' - R R^{|c} R_{|i} R_{|cj} P'' \Big) + 0 + 0 \\
& + \Big(R R_{|i} R_{|j} R^{|c}{}_{|c} P'' - R R^{|d} R_{|j} R_{|di} P'' \Big) + \Big(\frac{2}{3} R R^{|c} R_{|i} R_{|cj} P'' - \frac{2}{3} R S R_{|ij} P'' \Big) \\
& + \Big(R R_{|i} R_{|j} R^{|c}{}_{|c} P'' - R R^{|d} R_{|i} R_{|dj} P'' \Big) + \Big(\frac{2}{3} R R^{|c} R_{|j} R_{|ci} P'' - \frac{2}{3} R S R_{|ji} P'' \Big) \\
& + \Big(- \frac{4}{3} R R_{|i} R_{|j} R^{|a}{}_{|a} P'' + \frac{4}{3} R R^{|a} R_{|j} R_{|ai} P'' \Big) + \Big(\frac{2}{3} R R_{|i} R_{|j} R^{|a}{}_{|a} P'' \\
& - \frac{2}{3} R R^{|a} R_{|i} R_{|aj} P'' \Big) + \Big(\frac{4}{3} R S g_{ij} R^{|a}{}_{|a} P'' - \frac{4}{3} R g_{ij} R^{|a} R^{|b} R_{|ab} P'' \Big) \\
& + \frac{4}{3} R^2 R_{|ij} P + \frac{1}{3} R^2 R_{|ij} P - \frac{2}{3} R^2 R_{|ji} P - \frac{1}{3} R^2 g_{ij} R^{|c}{}_{|c} P - \frac{35}{6} S R_{|i} R_{|j} P'' + 0 \\
& + 0 + 0 - \frac{10}{3} R S R_{|i} R_{|j} P''' + 0 + 0 + 0 + 0 + \frac{13}{6} R R_{|i} R_{|j} P + \frac{1}{3} R R_{|i} R_{|j} P \\
& - \frac{2}{3} R R_{|i} R_{|j} P - \frac{1}{3} R S g_{ij} P - \frac{1}{6} R R_{|i} R_{|j} P - \frac{1}{6} R R_{|i} R_{|j} P - \frac{1}{12} R S g_{ij} P \\
& + \frac{8}{3} R^2 R_{|i} R_{|j} P' - \frac{2}{3} R^2 R_{|i} R_{|j} P' + R^2 R_{|i} R_{|j} P' - R^2 R_{|i} R_{|j} P' - \frac{2}{3} R^2 S g_{ij} P' \\
& + 0 + \Big(\frac{1}{6} R^2 R_{|i} R_{|j} P' - \frac{1}{6} R^2 S g_{ij} P' \Big) - \frac{1}{3} R^2 R_{|i} R_{|j} P' + \Big(\frac{1}{6} R^2 R_{|i} R_{|j} P' \\
& - \frac{1}{6} R^2 S g_{ij} P' \Big) + 0 - \frac{1}{3} R^2 R_{|i} R_{|j} P' \Big]
\end{aligned}
$$

$$
\begin{aligned}
= (-1)^q \Big(& 6 g^{ab} R^{|c}{}_{|c} R_{|(abij)} P' - 6 R^{|ab} R_{|(abij)} P' + 2 S g^{ab} R_{|(abij)} P'' \\
& - 2 R^{|a} R^{|b} R_{|(abij)} P'' - R g^{ab} R_{|(abij)} P + 6 g^{ab} g^{cd} R_{|(abi)} R_{|(cdj)} P' \\
& - 6 g^{ad} g^{bc} R_{|(abi)} R_{|(cdj)} P' + 6 g^{ab} R_{|j} R^{|c}{}_{|c} R_{|(abi)} P'' + 6 g^{ab} R_{|i} R^{|c}{}_{|c} R_{|(abj)} P'' \\
& - 6 R_{|j} R^{|ab} R_{|(abi)} P'' - 6 R_{|i} R^{|ab} R_{|(abj)} P'' + 4 g^{ab} R^{|c} R_{|cj} R_{|(abi)} P'' \\
& + 4 g^{ab} R^{|c} R_{|ci} R_{|(abj)} P'' - 4 R^{|a} R^{|b}{}_{|j} R_{|(abi)} P'' - 4 R^{|a} R^{|b}{}_{|i} R_{|(abj)} P'' \\
& + 4 R^{|a} R^{|b}{}_{|b} R_{|(aij)} P'' - 4 R_{|a} R^{|ab} R_{|(bij)} P'' + 2 S g^{ab} R_{|j} R_{|(abi)} P''' \\
& + 2 S g^{ab} R_{|i} R_{|(abj)} P''' - 2 R^{|a} R^{|b} R_{|j} R_{|(abi)} P''' - 2 R^{|a} R^{|b} R_{|i} R_{|(abj)} P''' \\
& - R g^{ab} R_{|j} R_{|(abi)} P' - R g^{ab} R_{|i} R_{|(abj)} P' - 8 R R^{|a} R_{|(aij)} P' - g^{ab} R_{|j} R_{|(abi)} P \\
& - g^{ab} R_{|i} R_{|(abj)} P - 2 R^{|a} R_{|(aij)} P + 3 R^{|a}{}_{|a} R^{|b}{}_{|b} R_{|ij} P'' + 4 R^{|a}{}_{|a} R^{|b}{}_{|i} R_{|bj} P'' \\
& - 3 R^{|ab} R_{|ab} R_{|ij} P'' - 4 R^{|ab} R_{|ai} R_{|bj} P'' + 3 R_{|i} R_{|j} R^{|a}{}_{|a} R^{|b}{}_{|b} P''' \\
& - 3 R_{|i} R_{|j} R^{|ab} R_{|ab} P''' + 4 R^{|a} R_{|j} R^{|b}{}_{|b} R_{|ai} P''' + 4 R^{|a} R_{|i} R^{|b}{}_{|b} R_{|aj} P'''
\end{aligned}
$$

$$
\begin{aligned}
&- 4R_{|a}R_{|j}R^{|ab}R_{|bi}P''' - 4R_{|a}R_{|i}R^{|ab}R_{|bj}P''' + 2SR^{|a}{}_{|a}R_{|ij}P''' \\
&- 2R^{|a}R^{|b}R_{|ab}R_{|ij}P''' - 3RR^{|a}{}_{|a}R_{|ij}P' - 6RR^{|a}{}_{|j}R_{|ai}P' + 2Rg_{ij}R^{|a}{}_{|a}R^{|b}{}_{|b}P' \\
&- 2Rg_{ij}R^{|ab}R_{|ab}P' - 2R^{|a}{}_{|i}R_{|aj}P - R^{|a}{}_{|a}R_{|ij}P - 4RSR_{|ij}P'' \\
&- \frac{20}{3}RR^{|a}R_{|j}R_{|ai}P'' - \frac{20}{3}RR^{|a}R_{|i}R_{|aj}P'' - \frac{5}{3}RR_{|i}R_{|j}R^{|a}{}_{|a}P'' \\
&+ \frac{4}{3}RSg_{ij}R^{|a}{}_{|a}P'' - \frac{4}{3}Rg_{ij}R^{|a}R^{|b}R_{|ab}P'' + 2SR_{|i}R_{|j}R^{|a}{}_{|a}P^{(4)} \\
&- 2R^{|a}R^{|b}R_{|i}R_{|j}R_{|ab}P^{(4)} - 7R^{|c}R_{|j}R_{|ci}P' - 7R^{|c}R_{|i}R_{|cj}P' \\
&+ \frac{1}{2}Sg_{ij}R^{|c}{}_{|c}P' - \frac{1}{2}g_{ij}R^{|c}R^{|d}R_{|cd}P' - \frac{5}{2}R_{|i}R_{|j}R^{|c}{}_{|c}P' - 3SR_{|ij}P' \\
&+ R^2R_{|ij}P - \frac{1}{3}R^2g_{ij}R^{|c}{}_{|c}P - \frac{35}{6}SR_{|i}R_{|j}P'' - \frac{10}{3}RSR_{|i}R_{|j}P''' + \frac{3}{2}RR_{|i}R_{|j}P \\
&- \frac{5}{12}RSg_{ij}P + \frac{5}{3}R^2R_{|i}R_{|j}P' - R^2Sg_{ij}P'\Bigg). \qquad \text{(B.1)}
\end{aligned}
$$

Next, we substitute this result into (2.30), using (1.16), (2.37), (2.40), and (2.39) for the remaining terms

$$
\begin{aligned}
E^{ab}(\lambda) &= \sqrt{g}\left[\left(\nabla^a\nabla^b - g^{ab}\Box\right)\mathcal{E} - \frac{1}{2}g^{ab}R\mathcal{E} + \frac{1}{2}\left(2\,\mathrm{Sym}_{ab}\,S^{bad} - S^{dab}\right)_{|d} + \frac{1}{2}g^{ab}L^\star + \frac{\partial L^\star}{\partial g_{ab}}\right] \\
&= \sqrt{g}\Bigg[(-1)^q\left(g^{ai}g^{bj} - g^{ab}g^{ij}\right)\Big(6g^{cd}R^{|e}{}_{|e}R_{|(cdij)}P' - 6R^{|cd}R_{|(cdij)}P' \\
&\quad + 2Sg^{cd}R_{|(cdij)}P'' - 2R^{|c}R^{|d}R_{|(cdij)}P'' - Rg^{cd}R_{|(cdij)}P + 6g^{cd}g^{ef}R_{|(cdi)}R_{|(efj)}P' \\
&\quad - 6g^{cf}g^{de}R_{|(cdi)}R_{|(efj)}P' + 6g^{cd}R_{|j}R^{|e}{}_{|e}R_{|(cdi)}P'' + 6g^{cd}R_{|i}R^{|e}{}_{|e}R_{|(cdj)}P'' \\
&\quad - 6R_{|j}R^{|cd}R_{|(cdi)}P'' - 6R_{|i}R^{|cd}R_{|(cdj)}P'' + 4g^{cd}R^{|e}R_{|ej}R_{|(cdi)}P'' \\
&\quad + 4g^{cd}R^{|e}R_{|ei}R_{|(cdj)}P'' - 4R^{|c}R^{|d}{}_{|j}R_{|(cdi)}P'' - 4R^{|c}R^{|d}{}_{|i}R_{|(cdj)}P'' \\
&\quad + 4R^{|c}R^{|d}{}_{|d}R_{|(cij)}P'' - 4R_{|c}R^{|cd}R_{|(dij)}P'' + 2Sg^{cd}R_{|j}R_{|(cdi)}P''' \\
&\quad + 2Sg^{cd}R_{|i}R_{|(cdj)}P''' - 2R^{|c}R^{|d}R_{|j}R_{|(cdi)}P''' - 2R^{|c}R^{|d}R_{|i}R_{|(cdj)}P''' \\
&\quad - Rg^{cd}R_{|j}R_{|(cdi)}P' - Rg^{cd}R_{|i}R_{|(cdj)}P' - 8RR^{|c}R_{|(cij)}P' - g^{cd}R_{|j}R_{|(cdi)}P \\
&\quad - g^{cd}R_{|i}R_{|(cdj)}P - 2R^{|c}R_{|(cij)}P + 3R^{|c}{}_{|c}R^{|d}{}_{|d}R_{|ij}P'' + 4R^{|c}{}_{|c}R^{|d}{}_{|i}R_{|dj}P'' \\
&\quad - 3R^{|cd}R_{|cd}R_{|ij}P'' - 4R^{|cd}R_{|ci}R_{|dj}P'' + 3R_{|i}R_{|j}R^{|c}{}_{|c}R^{|d}{}_{|d}P''' \\
&\quad - 3R_{|i}R_{|j}R^{|cd}R_{|cd}P''' + 4R^{|c}R_{|j}R^{|d}{}_{|d}R_{|ci}P''' + 4R^{|c}R_{|i}R^{|d}{}_{|d}R_{|cj}P'''
\end{aligned}
$$

$$
\begin{aligned}
&- 4R_{|c}R_{|j}R^{|cd}R_{|di}P''' - 4R_{|c}R_{|i}R^{|cd}R_{|dj}P''' + 2SR^{|c}{}_{|c}R_{|ij}P''' \\
&- 2R^{|c}R^{|d}R_{|cd}R_{|ij}P''' - 3RR^{|c}{}_{|c}R_{|ij}P' - 6RR^{|c}{}_{|j}R_{|ci}P' + 2Rg_{ij}R^{|c}{}_{|c}R^{|d}{}_{|d}P' \\
&- 2Rg_{ij}R^{|cd}R_{|cd}P' - 2R^{|c}{}_{|i}R_{|cj}P - R^{|c}{}_{|c}R_{|ij}P - 4RSR_{|ij}P'' \\
&- \frac{20}{3}RR^{|c}R_{|j}R_{|ci}P'' - \frac{20}{3}RR^{|c}R_{|i}R_{|cj}P'' - \frac{5}{3}RR_{|i}R_{|j}R^{|c}{}_{|c}P'' \\
&+ \frac{4}{3}RSg_{ij}R^{|c}{}_{|c}P'' - \frac{4}{3}Rg_{ij}R^{|c}R^{|d}R_{|cd}P'' + 2SR_{|i}R_{|j}R^{|c}{}_{|c}P^{(4)} \\
&- 2R^{|c}R^{|d}R_{|i}R_{|j}R_{|cd}P^{(4)} - 7R^{|c}R_{|j}R_{|ci}P' - 7R^{|c}R_{|i}R_{|cj}P' \\
&+ \frac{1}{2}Sg_{ij}R^{|c}{}_{|c}P' - \frac{1}{2}g_{ij}R^{|c}R^{|d}R_{|cd}P' - \frac{5}{2}R_{|i}R_{|j}R^{|c}{}_{|c}P' \\
&- 3SR_{|ij}P' + R^2R_{|ij}P - \frac{1}{3}R^2g_{ij}R^{|c}{}_{|c}P - \frac{35}{6}SR_{|i}R_{|j}P'' - \frac{10}{3}RSR_{|i}R_{|j}P''' \\
&+ \frac{3}{2}RR_{|i}R_{|j}P - \frac{5}{12}RSg_{ij}P + \frac{5}{3}R^2R_{|i}R_{|j}P' - R^2Sg_{ij}P'\bigg) \\
&- \frac{(-1)^q}{2}g^{ab}R\left(3R^{|c}{}_{|c}R^{|d}{}_{|d}P' - 3R^{|cd}R_{|cd}P' + 2SR^{|c}{}_{|c}P'' - 2R^{|c}R^{|d}R_{|cd}P''\right. \\
&\left.- SP - RR^{|c}{}_{|c}P - 2RSP'\right) + (-1)^q\left(g^{ac}g^{be}R^{|d}R_{|(ced)}P - g^{ab}g^{ce}R^{|d}R_{|(ced)}P\right. \\
&+ 3R^{|c}{}_{|c}R^{|ab}P - 2R^{|a}{}_{|c}R^{|bc}P - g^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P + 2R^{|a}R^{|b}R^{|c}{}_{|c}P' \\
&\left.- R^{|b}R_{|c}R^{|ac}P' - R^{|a}R_{|c}R^{|bc}P' - Sg^{ab}R^{|c}{}_{|c}P' + SR^{|ab}P' - \frac{2}{3}RR^{|a}R^{|b}P\right) \\
&+ \frac{(-1)^q}{2}g^{ab}\left(R^{|c}{}_{|c}R^{|d}{}_{|d} - R^{|cd}R_{|cd}\right)P - 2(-1)^q\left(R^{|ab}R^{|c}{}_{|c} - R^{|a}{}_{|c}R^{|bc}\right)P\bigg] \\
= {}&(-1)^q\sqrt{g}\bigg(6g_{cd}R^{|e}{}_{|e}R^{|(abcd)}P' - 6g^{ab}g_{cd}g_{ef}R^{|i}{}_{|i}R^{|(cdef)}P' - 6R_{|cd}R^{|(abcd)}P' \\
&+ 6g^{ab}g_{cd}R_{|ef}R^{|(cdef)}P' + 2Sg^{ae}g^{bf}g^{cd}R_{|(cdef)}P'' - 2Sg^{ab}g^{cd}g^{ef}R_{|(cdef)}P'' \\
&- 2R_{|c}R_{|d}R^{|(abcd)}P'' + 2g^{ab}g^{ef}R^{|c}R^{|d}R_{|(cdef)}P'' - Rg_{cd}R^{|(abcd)}P \\
&+ Rg^{ab}g^{cd}g^{ef}R_{|(cdef)}P + 6g_{cd}g_{ef}R^{|(acd)}R^{|(bef)}P' - 6g^{ab}g_{cd}g^{ef}R^{|(cdi)}R_{|(efi)}P' \\
&- 6g^{be}R^{|(cda)}R_{|(cde)}P' + 6g^{ab}R^{|(cde)}R_{|(cde)}P' + 6g_{cd}R^{|b}R^{|e}{}_{|e}R^{|(acd)}P'' \\
&- 6g^{ab}g^{cd}R^{|e}R^{|f}{}_{|f}R_{|(cde)}P'' + 6g_{cd}R^{|a}R^{|e}{}_{|e}R^{|(bcd)}P'' - 6g^{ab}g^{cd}R^{|e}R^{|f}{}_{|f}R_{|(cde)}P'' \\
&- 6R^{|b}R_{|cd}R^{|(acd)}P'' + 6g^{ab}R^{|e}R^{|cd}R_{|(cde)}P'' - 6R^{|a}R_{|cd}R^{|(bcd)}P'' \\
&+ 6g^{ab}R^{|e}R^{|cd}R_{|(cde)}P'' + 4g_{cd}R_{|e}R^{|be}R^{|(acd)}P'' - 4g^{ab}g^{cd}R_{|f}R^{|ef}R_{|(cde)}P'' \\
&+ 4g_{cd}R_{|e}R^{|ae}R^{|(bcd)}P'' - 4g^{ab}g^{cd}R_{|f}R^{|ef}R_{|(cde)}P'' - 4R_{|c}R^{|b}{}_{|d}R^{|(acd)}P'' \\
&+ 4g^{ab}R^{|c}R^{|de}R_{|(cde)}P'' - 4R_{|c}R^{|a}{}_{|d}R^{|(bcd)}P'' + 4g^{ab}R^{|c}R^{|de}R_{|(cde)}P''
\end{aligned}
$$

$$
\begin{aligned}
&+ 4R_{|c}R^{|d}{}_{|d}R^{|(abc)}P'' - 4g^{ab}g^{ij}R^{|c}R^{|d}{}_{|d}R_{|(cij)}P'' - 4R^{|d}R_{|cd}R^{|(abc)}P'' \\
&+ 4g^{ab}g^{ij}R_{|c}R^{|cd}R_{|(dij)}P'' + 2Sg_{cd}R^{|b}R^{|(acd)}P''' - 2Sg^{ab}g_{cd}R_{|e}R^{|(cde)}P''' \\
&+ 2Sg_{cd}R^{|a}R^{|(bcd)}P''' - 2Sg^{ab}g_{cd}R_{|e}R^{|(cde)}P''' - 2R^{|b}R_{|c}R_{|d}R^{|(acd)}P''' \\
&+ 2g^{ab}R_{|c}R_{|d}R_{|e}R^{|(cde)}P''' - 2R^{|a}R_{|c}R_{|d}R^{|(bcd)}P''' + 2g^{ab}R^{|c}R^{|d}R^{|e}R_{|(cde)}P''' \\
&- Rg_{cd}R^{|b}R^{|(acd)}P' + Rg^{ab}g_{cd}R_{|e}R^{|(cde)}P' - Rg_{cd}R^{|a}R^{|(bcd)}P' \\
&+ Rg^{ab}g_{cd}R_{|e}R^{|(cde)}P' - 8RR_{|c}R^{|(abc)}P' + 8Rg^{ab}g^{ij}R^{|c}R_{|(cij)}P' \\
&- g_{cd}R^{|b}R^{|(acd)}P + g^{ab}g^{cd}R^{|e}R_{|(cde)}P - g_{cd}R^{|a}R^{|(bcd)}P + g^{ab}g^{cd}R^{|e}R_{|(cde)}P \\
&- 2R_{|c}R^{|(abc)}P + 2g^{ab}g^{ij}R^{|c}R_{|(cij)}P + 3R^{|c}{}_{|c}R^{|d}{}_{|d}R^{|ab}P'' \\
&- 3g^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}R^{|e}{}_{|e}P'' + 4R^{|c}{}_{|c}R^{|a}{}_{|d}R^{|bd}P'' - 4g^{ab}R^{|c}{}_{|c}R^{|de}R_{|de}P'' \\
&- 3R^{|cd}R_{|cd}R^{|ab}P'' + 3g^{ab}R^{|cd}R_{|cd}R^{|e}{}_{|e}P'' - 4R_{|cd}R^{|ac}R^{|bd}P'' \\
&+ 4g^{ab}R^{|c}{}_{|d}R^{|d}{}_{|e}R^{|e}{}_{|c}P'' + 3R^{|a}R^{|b}R^{|c}{}_{|c}R^{|d}{}_{|d}P''' - 3Sg^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P''' \\
&- 3R^{|a}R^{|b}R^{|cd}R_{|cd}P''' + 3Sg^{ab}R^{|cd}R_{|cd}P''' + 4R_{|c}R^{|b}R^{|d}{}_{|d}R^{|ac}P''' \\
&- 4g^{ab}R^{|c}R^{|d}R^{|e}{}_{|e}R_{|cd}P''' + 4R_{|c}R^{|a}R^{|d}{}_{|d}R^{|bc}P''' - 4g^{ab}R^{|c}R^{|d}R^{|e}{}_{|e}R_{|cd}P''' \\
&- 4R^{|c}R^{|b}R_{|cd}R^{|ad}P''' + 4g^{ab}R_{|c}R^{|e}R^{|cd}R_{|de}P''' - 4R^{|c}R^{|a}R^{|bd}R_{|cd}P''' \\
&+ 4g^{ab}R_{|c}R^{|e}R^{|cd}R_{|de}P''' + 2SR^{|c}{}_{|c}R^{|ab}P''' - 2Sg^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P''' \\
&- 2R^{|c}R^{|d}R_{|cd}R^{|ab}P''' + 2g^{ab}R^{|c}R^{|d}R_{|cd}R^{|c}{}_{|c}P''' - 3RR^{|c}{}_{|c}R^{|ab}P' \\
&+ 3Rg^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P' - 6RR^{|b}{}_{|c}R^{|ac}P' + 6Rg^{ab}R^{|cd}R_{|cd}P' - 2Rg^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P' \\
&+ 2Rg^{ab}R^{|cd}R_{|cd}P' - 2R^{|a}{}_{|c}R^{|bc}P + 2g^{ab}R^{|cd}R_{|cd}P - R^{|c}{}_{|c}R^{|ab}P \\
&+ g^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P - 4RSR^{|ab}P'' + 4RSg^{ab}R^{|c}{}_{|c}P'' - \frac{20}{3}RR_{|c}R^{|b}R^{|ac}P'' \\
&+ \frac{20}{3}Rg^{ab}R^{|c}R^{|d}R_{|cd}P'' - \frac{20}{3}RR_{|c}R^{|a}R^{|bc}P'' + \frac{20}{3}Rg^{ab}R^{|c}R^{|d}R_{|cd}P'' \\
&- \frac{5}{3}RR^{|a}R^{|b}R^{|c}{}_{|c}P'' + \frac{5}{3}RSg^{ab}R^{|c}{}_{|c}P'' - \frac{4}{3}RSg^{ab}R^{|c}{}_{|c}P'' + \frac{4}{3}Rg^{ab}R^{|c}R^{|d}R_{|cd}P'' \\
&+ 2SR^{|a}R^{|b}R^{|c}{}_{|c}P^{(4)} - 2S^2g^{ab}R^{|c}{}_{|c}P^{(4)} - 2R^{|a}R^{|b}R^{|c}R^{|d}R_{|cd}P^{(4)} \\
&+ 2Sg^{ab}R^{|c}R^{|d}R_{|cd}P^{(4)} - 7R_{|c}R^{|b}R^{|ac}P' + 7g^{ab}R^{|c}R^{|d}R_{|cd}P' \\
&- 7R_{|c}R^{|a}R^{|bc}P' + 7g^{ab}R^{|c}R^{|d}R_{|cd}P' - \frac{1}{2}Sg^{ab}R^{|c}{}_{|c}P' + \frac{1}{2}g^{ab}R^{|c}R^{|d}R_{|cd}P' \\
&- \frac{5}{2}R^{|a}R^{|b}R^{|c}{}_{|c}P' + \frac{5}{2}Sg^{ab}R^{|c}{}_{|c}P' - 3SR^{|ab}P' + 3Sg^{ab}R^{|c}{}_{|c}P' + R^2R^{|ab}P
\end{aligned}
$$

$$
\begin{aligned}
&- R^2 g^{ab} R^{|c}{}_{|c} P + \frac{1}{3} R^2 g^{ab} R^{|c}{}_{|c} P - \frac{35}{6} S R^{|a} R^{|b} P'' + \frac{35}{6} S^2 g^{ab} P'' \\
&- \frac{10}{3} R S R^{|a} R^{|b} P''' + \frac{10}{3} R S^2 g^{ab} P''' + \frac{3}{2} R R^{|a} R^{|b} P - \frac{3}{2} R S g^{ab} P \\
&+ \frac{5}{12} R S g^{ab} P + \frac{5}{3} R^2 R^{|a} R^{|b} P' - \frac{5}{3} R^2 S g^{ab} P' + R^2 S g^{ab} P' - \frac{3}{2} R g^{ab} R^{|c}{}_{|c} R^{|d}{}_{|d} P' \\
&+ \frac{3}{2} R g^{ab} R^{|cd} R_{|cd} P' - R S g^{ab} R^{|c}{}_{|c} P'' + R g^{ab} R^{|c} R^{|d} R_{|cd} P'' + \frac{1}{2} R S g^{ab} P \\
&+ \frac{1}{2} R^2 g^{ab} R^{|c}{}_{|c} P - R^2 S g^{ab} P' + g^{ac} g^{be} R^{|d} R_{|(ced)} P - g^{ab} g^{ce} R^{|d} R_{|(ced)} P \\
&+ 3 R^{|c}{}_{|c} R^{|ab} P - 2 R^{|a}{}_{|e} R^{|be} P - g^{ab} R^{|c}{}_{|c} R^{|d}{}_{|d} P + 2 R^{|a} R^{|b} R^{|c}{}_{|c} P' \\
&- R^{|b} R_{|d} R^{|ad} P' - R^{|a} R_{|d} R^{|bd} P' - S g^{ab} R^{|c}{}_{|c} P' + S R^{|ab} P' - \frac{2}{3} R R^{|a} R^{|b} P \\
&+ \frac{1}{2} g^{ab} R^{|c}{}_{|c} R^{|d}{}_{|d} P - \frac{1}{2} g^{ab} R^{|cd} R_{|cd} P - 2 R^{|ab} R^{|c}{}_{|c} P + 2 R^{|a}{}_{|c} R^{|bc} P \Big) \\
= (-1)^q \sqrt{g} \Big[& R_{|(cdef)} \Big(6 g^{ae} g^{bf} g^{cd} R^{|i}{}_{|i} P' - 6 g^{ab} g^{cd} g^{ef} R^{|i}{}_{|i} P' + 6 g^{ab} g^{cd} R^{|ef} P' \\
&- 6 g^{ae} g^{bf} R^{|cd} P' - 2 g^{ae} g^{bf} R^{|c} R^{|d} P'' + 2 g^{ab} g^{ef} R^{|c} R^{|d} P'' + 2 S g^{ae} g^{bf} g^{cd} P'' \\
&- 2 S g^{ab} g^{cd} g^{ef} P'' + R g^{ab} g^{cd} g^{ef} P - R g^{ae} g^{bf} g^{cd} P \Big) + R_{|(cde)} R_{|(fgh)} \\
&\times \Big(6 g^{ac} g^{bf} g^{de} g^{gh} P' - 6 g^{ab} g_{cd} g^{fg} g^{eh} P' + 6 g^{ab} g^{cf} g^{dg} g^{eh} P' - 6 g^{ac} g^{bf} g^{dg} g^{eh} P' \Big) \\
&+ R_{|(cde)} \Big(6 g^{ac} g^{de} R^{|b} R^{|f}{}_{|f} P'' + 6 g^{bc} g^{de} R^{|a} R^{|f}{}_{|f} P'' - 12 g^{ab} g^{cd} R^{|e} R^{|f}{}_{|f} P'' \\
&- 6 g^{ac} R^{|b} R_{|de} P'' - 6 g^{bc} R^{|a} R^{|de} P'' + 12 g^{ab} R^{|e} R^{|cd} P'' + 4 g^{ac} g^{de} R_{|f} R^{|bf} P'' \\
&+ 4 g^{bc} g^{de} R_{|f} R^{|af} P'' - 8 g^{ab} g^{cd} R_{|f} R^{|ef} P'' - 4 g^{ac} R^{|d} R^{|be} P'' - 4 g^{bc} R^{|d} R^{|ae} P'' \\
&+ 8 g^{ab} R^{|c} R^{|de} P'' + 4 g^{ac} g^{bd} R^{|e} R^{|f}{}_{|f} P'' - 4 g^{ab} g^{cd} R^{|e} R^{|f}{}_{|f} P'' \\
&- 4 g^{ac} g^{bd} R_{|f} R^{|ef} P'' + 4 g^{ab} g^{cd} R_{|f} R^{|ef} P'' - 2 g^{ac} R^{|b} R^{|d} R^{|e} P''' \\
&- 2 g^{bc} R^{|a} R^{|d} R^{|e} P''' + 4 g^{ab} R^{|c} R^{|d} R^{|e} P''' + 2 S g^{bc} g^{de} R^{|a} P''' + 2 S g^{ac} g^{de} R^{|b} P''' \\
&- 4 S g^{ab} g^{cd} R^{|e} P''' - 8 R g^{ac} g^{bd} R^{|e} P' - R g^{bc} g^{de} R^{|a} P' - R g^{ac} g^{de} R^{|b} P' \\
&+ 10 R g^{ab} g^{cd} R^{|e} P' - g^{ac} g^{bd} R^{|e} P - g^{bc} g^{de} R^{|a} P - g^{ac} g^{de} R^{|b} P + 4 g^{ab} g^{cd} R^{|e} P \Big) \\
&+ R_{|cd} R_{|ef} R_{|gh} \Big(3 g^{ac} g^{bd} g^{ef} g^{gh} P'' - 3 g^{ac} g^{bd} g^{eg} g^{fh} P'' + 4 g^{ac} g^{be} g^{df} g^{gh} P'' \\
&- 4 g^{ac} g^{be} g^{dg} g^{fh} P'' - g^{ab} g^{ce} g^{df} g^{gh} P'' - 3 g^{ab} g^{cd} g^{ef} g^{gh} P'' + 4 g^{ab} g^{ch} g^{de} g^{fg} P'' \Big) \\
&+ R_{|cd} R_{|ef} \Big(3 g^{cd} g^{ef} R^{|a} R^{|b} P''' - 3 g^{ce} g^{df} R^{|a} R^{|b} P''' + 4 g^{ac} g^{ef} R^{|b} R^{|d} P'''
\end{aligned}
$$

$$
\begin{aligned}
&- 4g^{ac}g^{de}R^{|b}R^{|f}P''' + 4g^{bc}g^{ef}R^{|a}R^{|d}P''' - 4g^{bc}g^{de}R^{|a}R^{|f}P''' - 2g^{ac}g^{bd}R^{|e}R^{|f}P''' \\
&- 6g^{ab}g^{cd}R^{|e}R^{|f}P''' + 8g^{ab}g^{de}R^{|c}R^{|f}P''' + 2Sg^{ac}g^{bd}g^{ef}P''' + 3Sg^{ab}g^{ce}g^{df}P''' \\
&- 5Sg^{ab}g^{cd}g^{ef}P''' - 6Rg^{ac}g^{bf}g^{de}P' - 3Rg^{ac}g^{bd}g^{ef}P' + \frac{19}{2}Rg^{ab}g^{ce}g^{df}P' \\
&- \frac{1}{2}Rg^{ab}g^{cd}g^{ef}P' - 2g^{ac}g^{bf}g^{de}P + \frac{1}{2}g^{ab}g^{cd}g^{ef}P - \frac{3}{2}g^{ab}g^{ce}g^{df}P\Big) \\
&+ R_{|cd}\Big(-\frac{20}{3}Rg^{ac}R^{|b}R^{|d}P'' - \frac{20}{3}Rg^{bc}R^{|a}R^{|d}P'' + \frac{47}{3}Rg^{ab}R^{|c}R^{|d}P'' \\
&- \frac{5}{3}Rg^{cd}R^{|a}R^{|b}P'' - 4RSg^{ac}g^{bd}P'' + \frac{10}{3}RSg^{ab}g^{cd}P'' + 2SR^{|a}R^{|b}g^{cd}P^{(4)} \\
&- 2R^{|a}R^{|b}R^{|c}R^{|d}P^{(4)} + 2Sg^{ab}R^{|c}R^{|d}P^{(4)} - 2S^2g^{ab}g^{cd}P^{(4)} - 8g^{ac}R^{|b}R^{|d}P' \\
&- 8g^{bc}R^{|a}R^{|d}P' - \frac{1}{2}g^{cd}R^{|a}R^{|b}P' + \frac{29}{2}g^{ab}R^{|c}R^{|d}P' - 2Sg^{ac}g^{bd}P' + 4Sg^{ab}g^{cd}P' \\
&+ R^2g^{ac}g^{bd}P - \frac{1}{6}R^2g^{ab}g^{cd}P\Big) - \frac{35}{6}SR^{|a}R^{|b}P'' + \frac{35}{6}S^2g^{ab}P'' - \frac{10}{3}RSR^{|a}R^{|b}P''' \\
&+ \frac{10}{3}RS^2g^{ab}P''' + \frac{5}{3}R^2R^{|a}R^{|b}P' - \frac{5}{3}R^2Sg^{ab}P' + \frac{5}{6}RR^{|a}R^{|b}P - \frac{7}{12}RSg^{ab}P\Big].
\end{aligned}
$$

We conclude by verifying the divergence-free condition (2.32) directly using (1.16), (2.37), and (2.39)

$$
\begin{aligned}
&\left[\frac{1}{2}\ \left(2\,\mathrm{Sym}_{ab}\,S^{bad} - S^{dab}\right)_{|d} + \frac{1}{2}g^{ab}L^\star + \frac{\partial L^\star}{\partial g_{ab}}\right]_{|b} \\
&\quad= \Big[(-1)^q\Big(g^{ac}g^{be}R^{|d}R_{|(ced)}P - g^{ab}g^{ce}R^{|d}R_{|(ced)}P + 3R^{|c}{}_{|c}R^{|ab}P - 2R^{|a}{}_{|e}R^{|be}P \\
&\qquad - g^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P + 2R^{|a}R^{|b}R^{|c}{}_{|c}P' - R^{|b}R_{|d}R^{|ad}P' - R^{|a}R_{|d}R^{|bd}P' - Sg^{ab}R^{|c}{}_{|c}P' \\
&\qquad + SR^{|ab}P' - \frac{2}{3}RR^{|a}R^{|b}P\Big) + \frac{(-1)^q}{2}g^{ab}\left(R^{|c}{}_{|c}R^{|d}{}_{|d} - R^{|cd}R_{|cd}\right)P \\
&\qquad - 2(-1)^q\left(R^{|ab}R^{|c}{}_{|c} - R^{|a}{}_{|c}R^{|bc}\right)P\Big]_{|b} \\
&\quad= (-1)^q\Big(g^{ac}g^{be}R^{|d}R_{|(ced)}P - g^{ab}g^{ce}R^{|d}R_{|(ced)}P + R^{|ab}R^{|c}{}_{|c}P - \frac{1}{2}g^{ab}R^{|c}{}_{|c}R^{|d}{}_{|d}P \\
&\qquad - \frac{1}{2}g^{ab}R^{|cd}R_{|cd}P + 2R^{|a}R^{|b}R^{|c}{}_{|c}P' - R^{|b}R_{|d}R^{|ad}P' - R^{|a}R_{|d}R^{|bd}P' - Sg^{ab}R^{|c}{}_{|c}P' \\
&\qquad + SR^{|ab}P' - \frac{2}{3}RR^{|a}R^{|b}P\Big)_{|b} \\
&\quad= (-1)^q\Big(g^{ac}g^{be}g^{df}R_{|f}R_{|(ced)}P - g^{ab}g^{ce}g^{df}R_{|f}R_{|(ced)}P + g^{ac}g^{bd}g^{ef}R_{|cd}R_{|ef}P
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} g^{ab} g^{cd} g^{ef} R_{|cd} R_{|ef} P - \frac{1}{2} g^{ab} g^{ce} g^{df} R_{|cd} R_{|ef} P + 2 g^{ac} g^{bd} g^{ef} R_{|c} R_{|d} R_{|ef} P' \\
&- g^{ae} g^{bc} g^{df} R_{|c} R_{|d} R_{|ef} P' - g^{ac} g^{be} g^{df} R_{|c} R_{|d} R_{|ef} P' - g^{ab} g^{cd} g^{ef} R_{|c} R_{|d} R_{|ef} P' \\
&+ g^{ae} g^{bf} g^{cd} R_{|c} R_{|d} R_{|ef} P' - \frac{2}{3} g^{ac} g^{bd} R R_{|c} R_{|d} P \Bigg)_{|b} \\
= (-1)^q \Bigg(& g^{ac} g^{be} g^{df} R_{|fb} R_{|(ced)} P + g^{ac} g^{be} g^{df} R_{|f} R_{|(ced)b} P + g^{ac} g^{be} g^{df} R_{|b} R_{|f} R_{|(ced)} P' \\
&- g^{ab} g^{ce} g^{df} R_{|fb} R_{|(ced)} P - g^{ab} g^{ce} g^{df} R_{|f} R_{|(ced)b} P - g^{ab} g^{ce} g^{df} R_{|b} R_{|f} R_{|(ced)} P' \\
&+ g^{ac} g^{bd} g^{ef} R_{|cdb} R_{|ef} P + g^{ac} g^{bd} g^{ef} R_{|cd} R_{|efb} P + g^{ac} g^{bd} g^{ef} R_{|b} R_{|cd} R_{|ef} P' \\
&- g^{ab} g^{cd} g^{ef} R_{|cdb} R_{|ef} P - \frac{1}{2} g^{ab} g^{cd} g^{ef} R_{|b} R_{|cd} R_{|ef} P' - g^{ab} g^{ce} g^{df} R_{|cdb} R_{|ef} P \\
&- \frac{1}{2} g^{ab} g^{ce} g^{df} R_{|b} R_{|cd} R_{|ef} P' + 2 g^{ac} g^{bd} g^{ef} R_{|cb} R_{|d} R_{|ef} P' + 2 g^{ac} g^{bd} g^{ef} R_{|c} R_{|db} R_{|ef} P' \\
&+ 2 g^{ac} g^{bd} g^{ef} R_{|c} R_{|d} R_{|efb} P' + 2 g^{ac} g^{bd} g^{ef} R_{|b} R_{|c} R_{|d} R_{|ef} P'' - g^{ae} g^{bc} g^{df} R_{|cb} R_{|d} R_{|ef} P' \\
&- g^{ae} g^{bc} g^{df} R_{|c} R_{|db} R_{|ef} P' - g^{ae} g^{bc} g^{df} R_{|c} R_{|d} R_{|efb} P' - g^{ae} g^{bc} g^{df} R_{|b} R_{|c} R_{|d} R_{|ef} P'' \\
&- g^{ac} g^{be} g^{df} R_{|cb} R_{|d} R_{|ef} P' - g^{ac} g^{be} g^{df} R_{|c} R_{|db} R_{|ef} P' - g^{ac} g^{be} g^{df} R_{|c} R_{|d} R_{|efb} P' \\
&- g^{ac} g^{be} g^{df} R_{|b} R_{|c} R_{|d} R_{|ef} P'' - 2 g^{ab} g^{cd} g^{ef} R_{|cb} R_{|d} R_{|ef} P' - g^{ab} g^{cd} g^{ef} R_{|c} R_{|d} R_{|efb} P' \\
&- g^{ab} g^{cd} g^{ef} R_{|b} R_{|c} R_{|d} R_{|ef} P'' + 2 g^{ae} g^{bf} g^{cd} R_{|cb} R_{|d} R_{|ef} P' + g^{ae} g^{bf} g^{cd} R_{|c} R_{|d} R_{|efb} P' \\
&+ g^{ae} g^{bf} g^{cd} R_{|b} R_{|c} R_{|d} R_{|ef} P'' - \frac{2}{3} g^{ac} g^{bd} R_{|b} R_{|c} R_{|d} P - \frac{2}{3} g^{ac} g^{bd} R R_{|cb} R_{|d} P \\
&- \frac{2}{3} g^{ac} g^{bd} R R_{|c} R_{|db} P - \frac{2}{3} g^{ac} g^{bd} R R_{|b} R_{|c} R_{|d} P' \Bigg) \\
= (-1)^q \Bigg(& R^{|(abc)} R_{|bc} P + g^{ac} g^{be} R^{|d} R_{|(ced)b} P + R^{|(abc)} R_{|b} R_{|c} P' - g^{ce} R^{|ad} R_{|(ced)} P \\
&- g^{ab} g^{ce} R^{|d} R_{|(ced)b} P - g^{ce} R^{|a} R^{|d} R_{|(ced)} P' + g^{ac} g^{bd} R_{|cdb} R^{|e}{}_{|e} P + g^{ef} R^{|ab} R_{|efb} P \\
&+ R^{|ab} R_{|b} R^{|c}{}_{|c} P' - g^{ab} g^{cd} R_{|cdb} R^{|e}{}_{|e} P - \frac{1}{2} R^{|a} R^{|b}{}_{|b} R^{|c}{}_{|c} P' - g^{ab} R^{|cd} R_{|cdb} P \\
&- \frac{1}{2} R^{|a} R^{|bc} R_{|bc} P' + 2 R^{|ab} R_{|b} R^{|c}{}_{|c} P' + 2 R^{|a} R^{|b}{}_{|b} R^{|c}{}_{|c} P' + 2 g^{ef} R^{|a} R^{|b} R_{|efb} P' \\
&+ 2 S R^{|a} R^{|b}{}_{|b} P'' - R^{|ab} R_{|b} R^{|c}{}_{|c} P' - R^{|ab} R_{|bc} R^{|c} P' - g^{ae} R^{|b} R^{|f} R_{|efb} P' \\
&- S R^{|ab} R_{|b} P'' - R^{|ab} R_{|bc} R^{|c} P' - R^{|a} R^{|bc} R_{|bc} P' - g^{be} R^{|a} R^{|f} R_{|efb} P' \\
&- R^{|a} R^{|b} R^{|c} R_{|bc} P'' - 2 R^{|ab} R_{|b} R^{|c}{}_{|c} P' - S g^{ab} g^{ef} R_{|efb} P' - S R^{|a} R^{|b}{}_{|b} P'' \\
&+ 2 R^{|ab} R_{|bc} R^{|c} P' + S g^{ae} g^{bf} R_{|efb} P' + S R^{|ab} R_{|b} P'' - \frac{2}{3} S R^{|a} P - \frac{2}{3} R R^{|ab} R_{|b} P
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{3} R R^{|a} R^{|b}{}_{|b} P - \frac{2}{3} R S R^{|a} P' \Big) \\
&= (-1)^q \Bigg\{ \left(g^{ac} g^{be} - g^{ab} g^{ce} \right) R^{|d} \Bigg[R_{|(cedb)} + \frac{1}{3} R \left(g_{bc} R_{|ed} + g_{be} R_{|cd} + g_{bd} R_{|ce} \right. \\
&\quad \left. - g_{ce} R_{|bd} - g_{cd} R_{|be} - g_{de} R_{|bc} \right) + \frac{1}{24} R_{|c} \left(g_{be} R_{|d} + g_{bd} R_{|e} - 2 g_{ed} R_{|b} \right) \\
&\quad + \frac{1}{24} R_{|e} \left(g_{bc} R_{|d} + g_{bd} R_{|c} - 2 g_{cd} R_{|b} \right) + \frac{1}{24} R_{|d} \left(g_{be} R_{|c} + g_{bc} R_{|e} - 2 g_{ec} R_{|b} \right) \Bigg] P \\
&\quad + R^{|(abc)} R_{|bc} P - g^{ce} R^{|ad} R_{|(ced)} P + R^{|(abc)} R_{|b} R_{|c} P' - g^{ce} R^{|a} R^{|d} R_{|(ced)} P' \\
&\quad + g^{ac} g^{bd} \left[R_{|(cdb)} + \frac{1}{6} R \left(g_{bc} R_{|d} + g_{bd} R_{|c} - 2 g_{cd} R_{|b} \right) \right] R^{|e}{}_{|e} P \\
&\quad + g^{ef} R^{|ab} \left[R_{|(efb)} + \frac{1}{3} R \left(g_{be} R_{|f} - g_{ef} R_{|b} \right) \right] P \\
&\quad - g^{ab} g^{cd} \left[R_{|(cdb)} + \frac{1}{3} R \left(g_{bc} R_{|d} - g_{cd} R_{|b} \right) \right] R^{|e}{}_{|e} P \\
&\quad - g^{ab} R^{|cd} \left[R_{|(cdb)} + \frac{1}{3} R \left(g_{bc} R_{|d} - g_{cd} R_{|b} \right) \right] P \\
&\quad + 2 g^{ef} R^{|a} R^{|b} \left[R_{|(efb)} + \frac{1}{3} R \left(g_{be} R_{|f} - g_{ef} R_{|b} \right) \right] P' \\
&\quad - g^{ae} R^{|b} R^{|f} \left[R_{|(efb)} + \frac{1}{6} R \left(g_{be} R_{|f} + g_{bf} R_{|e} - 2 g_{ef} R_{|b} \right) \right] P' \\
&\quad - g^{be} R^{|a} R^{|f} \left[R_{|(efb)} + \frac{1}{6} R \left(g_{be} R_{|f} + g_{bf} R_{|e} - 2 g_{ef} R_{|b} \right) \right] P' \\
&\quad - S g^{ab} g^{ef} \left[R_{|(efb)} + \frac{1}{3} R \left(g_{be} R_{|f} - g_{ef} R_{|b} \right) \right] P' \\
&\quad + S g^{ae} g^{bf} \left[R_{|(efb)} + \frac{1}{6} R \left(g_{be} R_{|f} + g_{bf} R_{|e} - 2 g_{ef} R_{|b} \right) \right] P' \\
&\quad + \frac{3}{2} R^{|a} R^{|b}{}_{|b} R^{|c}{}_{|c} P' - \frac{3}{2} R^{|a} R^{|bc} R_{|bc} P' + S R^{|a} R^{|b}{}_{|b} P'' - R^{|a} R^{|b} R^{|c} R_{|bc} P'' \\
&\quad - \frac{2}{3} R R^{|ab} R_{|b} P - \frac{2}{3} R R^{|a} R^{|b}{}_{|b} P - \frac{2}{3} S R^{|a} P - \frac{2}{3} R S R^{|a} P' \Bigg\} \\
&= (-1)^q \Bigg[\left(g^{ac} g^{be} - g^{ab} g^{ce} \right) R^{|d} \Bigg(0 + \frac{1}{3} R g_{bc} R_{|ed} + \frac{1}{3} R g_{be} R_{|cd} + \frac{1}{3} R g_{bd} R_{|ce} \\
&\quad - \frac{1}{3} R g_{ce} R_{|bd} - \frac{1}{3} R g_{cd} R_{|be} - \frac{1}{3} R g_{de} R_{|bc} + \frac{1}{12} g_{be} R_{|c} R_{|d} + \frac{1}{12} g_{bd} R_{|c} R_{|e} \\
&\quad - \frac{1}{12} g_{ed} R_{|b} R_{|c} + \frac{1}{12} g_{bc} R_{|d} R_{|e} - \frac{1}{12} g_{cd} R_{|b} R_{|e} - \frac{1}{12} g_{ec} R_{|b} R_{|d} \Bigg) P \\
&\quad + R^{|(abc)} R_{|bc} P - g^{cd} R^{|ab} R_{|(cdb)} P + R^{|(abc)} R_{|b} R_{|c} P' - g^{cb} R^{|a} R^{|d} R_{|(cbd)} P' \\
&\quad + \Bigg(g^{ac} g^{bd} R_{|(cdb)} R^{|e}{}_{|e} P + \frac{1}{6} R R^{|a} R^{|e}{}_{|e} P + \frac{1}{3} R R^{|a} R^{|e}{}_{|e} P - \frac{1}{3} R R^{|a} R^{|e}{}_{|e} P \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\left(g^{cd}R^{|ab}R_{|(cdb)}P+\frac{1}{3}RR^{|ab}R_{|b}P-\frac{2}{3}RR^{|ab}R_{|b}P\right)\\
&+\left(-g^{ab}g^{cd}R_{|(cdb)}R^{|e}{}_{|e}P-\frac{1}{3}RR^{|a}R^{|e}{}_{|e}P+\frac{2}{3}RR^{|a}R^{|e}{}_{|e}P\right)\\
&+\left(-R^{|(abc)}R_{|bc}P-\frac{1}{3}RR^{|ab}R_{|b}P+\frac{1}{3}RR^{|a}R^{|e}{}_{|e}P\right)\\
&+\left(2g^{cd}R^{|a}R^{|b}R_{|(cdb)}P'+\frac{2}{3}RSR^{|a}P'-\frac{4}{3}RSR^{|a}P'\right)\\
&+\left(-R^{|(abc)}R_{|b}R_{|c}P'-\frac{1}{6}RSR^{|a}P'-\frac{1}{6}RSR^{|a}P'+\frac{1}{3}RSR^{|a}P'\right)\\
&+\left(-g^{bc}R^{|a}R^{|d}R_{|(cdb)}P'-\frac{1}{3}RSR^{|a}P'-\frac{1}{6}RSR^{|a}P'+\frac{1}{3}RSR^{|a}P'\right)\\
&+\left(-Sg^{ab}g^{cd}R_{|(cdb)}P'-\frac{1}{3}RSR^{|a}P'+\frac{2}{3}RSR^{|a}P'\right)\\
&+\left(Sg^{ac}g^{bd}R_{|(cdb)}P'+\frac{1}{6}RSR^{|a}P'+\frac{1}{3}RSR^{|a}P'-\frac{1}{3}RSR^{|a}P'\right)\\
&+\frac{3}{2}R^{|a}R^{|b}{}_{|b}R^{|c}{}_{|c}P'-\frac{3}{2}R^{|a}R^{|bc}R_{|bc}P'+SR^{|a}R^{|b}{}_{|b}P''-R^{|a}R^{|b}R^{|c}R_{|bc}P''\\
&-\frac{2}{3}RR^{|ab}R_{|b}P-\frac{2}{3}RR^{|a}R^{|b}{}_{|b}P-\frac{2}{3}SR^{|a}P-\frac{2}{3}RSR^{|a}P'\Bigg]\\
={}&(-1)^q\Bigg[\Bigg(\frac{1}{3}RR^{|ab}R_{|b}P-\frac{1}{3}RR^{|ab}R_{|b}P+\frac{2}{3}RR^{|ab}R_{|b}P-\frac{1}{3}RR^{|ab}R_{|b}P+\frac{1}{3}RR^{|ab}R_{|b}P\\
&-\frac{1}{3}RR^{|a}R^{|b}{}_{|b}P-\frac{1}{3}RR^{|ab}R_{|b}P+\frac{2}{3}RR^{|ab}R_{|b}P-\frac{1}{3}RR^{|a}R^{|b}{}_{|b}P+\frac{1}{3}RR^{|ab}R_{|b}P\\
&-\frac{1}{3}RR^{|ab}R_{|b}P+\frac{1}{3}RR^{|ab}R_{|b}P+\frac{1}{6}SR^{|a}P-\frac{1}{12}SR^{|a}P+\frac{1}{12}SR^{|a}P-\frac{1}{12}SR^{|a}P\\
&-\frac{1}{12}SR^{|a}P+\frac{1}{12}SR^{|a}P+\frac{1}{12}SR^{|a}P-\frac{1}{12}SR^{|a}P-\frac{1}{12}SR^{|a}P+\frac{1}{12}R^{|a}P\\
&-\frac{1}{12}SR^{|a}P+\frac{1}{6}SR^{|a}P\Bigg)+\frac{5}{6}RR^{|a}R^{|e}{}_{|e}P-\frac{2}{3}RR^{|ab}R_{|b}P-\frac{1}{3}RSR^{|a}P'\\
&+\frac{3}{2}R^{|a}R^{|b}{}_{|b}R^{|c}{}_{|c}P'-\frac{3}{2}R^{|a}R^{|bc}R_{|bc}P'+SR^{|a}R^{|b}{}_{|b}P''-R^{|a}R^{|b}R^{|c}R_{|bc}P''\\
&-\frac{2}{3}RR^{|ab}R_{|b}P-\frac{2}{3}RR^{|a}R^{|b}{}_{|b}P-\frac{2}{3}SR^{|a}P-\frac{2}{3}RSR^{|a}P'\Bigg]\\
={}&(-1)^q\left(\frac{3}{2}R^{|a}R^{|b}{}_{|b}R^{|c}{}_{|c}P'-\frac{3}{2}R^{|a}R^{|bc}R_{|bc}P'+SR^{|a}R^{|b}{}_{|b}P''-R^{|a}R^{|b}R^{|c}R_{|bc}P''\right.\\
&\left.-\frac{1}{2}SR^{|a}P-\frac{1}{2}RR^{|a}R^{|b}{}_{|b}P-RSR^{|a}P'\right)\\
={}&\frac{1}{2}R^{|a}\Big[(-1)^q\Big(3R^{|b}{}_{|b}R^{|c}{}_{|c}P'-3R^{|bc}R_{|bc}P'+2SR^{|b}{}_{|b}P''-2R^{|b}R^{|c}R_{|bc}P''\\
&-SP-RR^{|b}{}_{|b}P-2RSP'\Big)\Big]
\end{aligned}
$$

$$= \frac{1}{2} R^{|a} E_R(L).$$

APPENDIX C
THE ORDER 6 CASE

We assume $A^{ij} = A^{ij}(g_{ab}; \varphi; \varphi_a; \varphi_{ab}; \varphi_{abc}; \varphi_{abcd})$. As with the previous cases, we produce the divergence-free condition (3.3) by starting with the covariant divergence equation (3.6), using the fifth scalar order symmetrization formula (A.3) and the symmetrized divergence-free condition from the previous case (4.6) to fully symmetrize the equation

$$\begin{aligned}
0 &= A^{ij;klmn}\varphi_{klmn|j} + A^{ij;klm}\varphi_{klm|j} + A^{ij;kl}\varphi_{kl|j} + A^{ij;k}\varphi_{kj} + \frac{\partial A^{ij}}{\partial \varphi}\varphi_j \\
&= A^{ij;klmn}\Big[\varphi_{klmnj} + 2R(g_{jk}\varphi_{lmn} - g_{kl}\varphi_{jmn}) + R_{|n}(g_{jk}\varphi_{ml} - g_{km}\varphi_{jl}) \\
&\quad + \frac{2}{15}R^2(g_{kl}g_{jm}\varphi_n - g_{kl}g_{mn}\varphi_j) + \frac{1}{5}R_{|mn}(g_{jk}\varphi_l - g_{lk}\varphi_j)\Big] \\
&\quad + A^{ij;klm}\Big[\varphi_{klmj} - R(g_{kl}\varphi_{mj} - g_{jk}\varphi_{lm}) - \frac{1}{4}R_{|m}(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] \\
&\quad + A^{ij;kl}\Big[\varphi_{klj} - \frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] + A^{ij;k}\varphi_{jk} + \frac{\partial A^{ij}}{\partial \varphi}\varphi_j.
\end{aligned} \tag{C.1}$$

As A^{ij} satisfies the requirements of Theorem 11, A^{ij} takes the form

$$A^{ij} = \varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}B^{cd} + \varepsilon^{ia}\varepsilon^{jd}\varphi_{abc}\varphi_{def}B^{bc;ef} + D^{ijabc}\varphi_{abc} + E^{ij}, \tag{C.2}$$

where $B^{cd} = B^{dc}$, $D^{ijabc} = D^{(ij)(abc)}$, and $E^{ij} = E^{ji}$ are tensor densities of scalar order 2 and $B^{cd;ef} = B^{ef;cd}$. We substitute this result into the divergence-free condition ($C.1$), yielding

$$\begin{aligned}
0 = \operatorname{Sym}_{klmn} &\varepsilon^{ik}\varepsilon^{jl}B^{mn}\Big[2R(g_{jk}\varphi_{lmn} - g_{kl}\varphi_{jmn}) + R_{|n}(g_{jk}\varphi_{ml} - g_{km}\varphi_{jl}) \\
&+ \frac{2}{15}R^2(g_{kl}g_{jm}\varphi_n - g_{kl}g_{mn}\varphi_j) + \frac{1}{5}R_{|mn}(g_{jk}\varphi_l - g_{lk}\varphi_j)\Big] \\
&+ \Big(2\operatorname{Sym}_{ij}\operatorname{Sym}_{klm}\varepsilon^{ik}\varepsilon^{ja}B^{lm;bc}\varphi_{abc} + D^{ijklm}\Big)\Big[\varphi_{klmj} - R(g_{kl}\varphi_{mj} - g_{jk}\varphi_{lm})
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{4}R_{|m}(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] + \Big(\varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}B^{cd;kl} + \varepsilon^{ia}\varepsilon^{jd}B^{bc;ef;kl}\varphi_{abc}\varphi_{def} \\
&+ D^{ijabc;kl}\varphi_{abc} + E^{ij;kl}\Big)\Big[\varphi_{klj} - \frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] \\
&+ \Big(\varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}B^{cd;k} + \varepsilon^{ia}\varepsilon^{jd}B^{bc;ef;k}\varphi_{abc}\varphi_{def} + D^{ijabc;k}\varphi_{abc} + E^{ij;k}\Big)\varphi_{jk} \\
&+ \Big(\varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}\frac{\partial B^{cd}}{\partial\varphi} + \varepsilon^{ia}\varepsilon^{jd}\frac{\partial B^{bc;ef}}{\partial\varphi}\varphi_{abc}\varphi_{def} + \frac{\partial D^{ijabc}}{\partial\varphi}\varphi_{abc} + \frac{\partial E^{ij}}{\partial\varphi}\Big)\varphi_j \\
= {} & \mathrm{Sym}_{klmn}\,\varepsilon^{ik}\varepsilon^{jl}B^{mn}\Big[2R(g_{jk}\varphi_{lmn} - g_{kl}\varphi_{jmn}) + R_{|n}(g_{jk}\varphi_{ml} - g_{km}\varphi_{jl}) \\
&+ \frac{2}{15}R^2(g_{kl}g_{jm}\varphi_n - g_{kl}g_{mn}\varphi_j) + \frac{1}{5}R_{|mn}(g_{jk}\varphi_l - g_{lk}\varphi_j)\Big] \\
&+ \Big(\varepsilon^{ik}\varepsilon^{ja}B^{lm;bc}\varphi_{abc} + D^{ijklm}\Big)\varphi_{klmj} + \Big(2\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{ja}B^{lm;bc}\varphi_{abc} + D^{ijklm}\Big) \\
&\times \Big[-R(g_{kl}\varphi_{mj} - g_{jk}\varphi_{lm}) - \frac{1}{4}R_{|m}(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] + \varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}B^{cd;kl} \\
&\times \Big[\varphi_{klj} - \frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] + \varepsilon^{ia}\varepsilon^{jd}B^{bc;ef;kl}\varphi_{abc}\varphi_{def}\Big[-\frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] \\
&+ \Big(D^{ijabc;kl}\varphi_{abc} + E^{ij;kl}\Big)\Big[\varphi_{klj} - \frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] \\
&+ \Big(\varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}B^{cd;k} + \varepsilon^{ia}\varepsilon^{jd}B^{bc;ef;k}\varphi_{abc}\varphi_{def} + D^{ijabc;k}\varphi_{abc} + E^{ij;k}\Big)\varphi_{jk} \\
&+ \Big(\varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}\frac{\partial B^{cd}}{\partial\varphi} + \varepsilon^{ia}\varepsilon^{jd}\frac{\partial B^{bc;ef}}{\partial\varphi}\varphi_{abc}\varphi_{def} + \frac{\partial D^{ijabc}}{\partial\varphi}\varphi_{abc} + \frac{\partial E^{ij}}{\partial\varphi}\Big)\varphi_j \\
= {} & \mathrm{Sym}_{klmn}\,\varepsilon^{ik}\varepsilon^{jl}B^{mn}\Big[2R(g_{jk}\varphi_{lmn} - g_{kl}\varphi_{jmn}) + R_{|n}(g_{jk}\varphi_{ml} - g_{km}\varphi_{jl}) \\
&+ \frac{2}{15}R^2(g_{kl}g_{jm}\varphi_n - g_{kl}g_{mn}\varphi_j) + \frac{1}{5}R_{|mn}(g_{jk}\varphi_l - g_{lk}\varphi_j)\Big] \\
&+ D^{ijklm}\varphi_{klmj} + \Big(2\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{ja}B^{lm;bc}\varphi_{abc} + D^{ijklm}\Big) \\
&\times \Big[-R(g_{kl}\varphi_{mj} - g_{jk}\varphi_{lm}) - \frac{1}{4}R_{|m}(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] \\
&+ \Big(\varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}B^{cd;kl} + \varepsilon^{ia}\varepsilon^{jd}B^{bc;ef;kl}\varphi_{abc}\varphi_{def}\Big)\Big[-\frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] \\
&+ \Big(D^{ijabc;kl}\varphi_{abc} + E^{ij;kl}\Big)\Big[\varphi_{klj} - \frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] \\
&+ \Big(\varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}B^{cd;k} + \varepsilon^{ia}\varepsilon^{jd}B^{bc;ef;k}\varphi_{abc}\varphi_{def} + D^{ijabc;k}\varphi_{abc} + E^{ij;k}\Big)\varphi_{jk} \\
&+ \Big(\varepsilon^{ia}\varepsilon^{jb}\varphi_{abcd}\frac{\partial B^{cd}}{\partial\varphi} + \varepsilon^{ia}\varepsilon^{jd}\frac{\partial B^{bc;ef}}{\partial\varphi}\varphi_{abc}\varphi_{def} + \frac{\partial D^{ijabc}}{\partial\varphi}\varphi_{abc} + \frac{\partial E^{ij}}{\partial\varphi}\Big)\varphi_j. \qquad \text{(C.3)}
\end{aligned}
$$

We apply the differential operator $U_i X_a X_b X_c X_d \dfrac{\partial}{\partial \varphi_{abcd}}$ to this condition and solve the resulting equation for D^{ijabc}

$$\begin{aligned} D(U, X^{(4)}) = \det(U, X) \Bigg[&\frac{1}{3} R B(X, X; \square, \square) \det(\nabla\varphi, X) - \frac{1}{3} R B(X, X; \nabla\varphi, \square) \det(\square, X) \\ &- B(X, X; \nabla^2\varphi) \det(\nabla^2\varphi, X) - \frac{\partial B}{\partial \varphi}(X, X) \det(\nabla\varphi, X) \Bigg], \end{aligned} \tag{C.4}$$

where the $\square$ symbol denotes contraction with a metric index and $\nabla^2\varphi$ denotes contraction with an index of a second derivative of φ. Seeking more information, we use this equation to remove the fourth order φ terms from (C.3), leaving

$$\begin{aligned} 0 = \mathrm{Sym}_{klmn}\, &\varepsilon^{ik} \varepsilon^{jl} B^{mn} \Big[2R(g_{jk}\varphi_{lmn} - g_{kl}\varphi_{jmn}) + R_{|n}(g_{jk}\varphi_{ml} - g_{km}\varphi_{jl}) \\ &+ \frac{2}{15} R^2 (g_{kl} g_{jm} \varphi_n - g_{kl} g_{mn} \varphi_j) + \frac{1}{5} R_{|mn}(g_{jk}\varphi_l - g_{lk}\varphi_j) \Big] \\ &+ \Big(2\, \mathrm{Sym}_{ij}\, \mathrm{Sym}_{klm}\, \varepsilon^{ik} \varepsilon^{ja} B^{lm;bc} \varphi_{abc} + D^{ijklm} \Big) \\ &\times \Big[-R(g_{kl}\varphi_{mj} - g_{jk}\varphi_{lm}) - \frac{1}{4} R_{|m}(g_{kl}\varphi_j - g_{jk}\varphi_l) \Big] \\ &+ \varepsilon^{ia} \varepsilon^{jd} B^{bc;ef;kl} \varphi_{abc} \varphi_{def} \Big[-\frac{1}{3} R(g_{kl}\varphi_j - g_{jk}\varphi_l) \Big] \\ &+ \Big(D^{ijabc;kl} \varphi_{abc} + E^{ij;kl} \Big) \Big[\varphi_{klj} - \frac{1}{3} R(g_{kl}\varphi_j - g_{jk}\varphi_l) \Big] \\ &+ \Big(\varepsilon^{ia} \varepsilon^{jd} B^{bc;ef;k} \varphi_{abc} \varphi_{def} + D^{ijabc;k} \varphi_{abc} + E^{ij;k} \Big) \varphi_{jk} \\ &+ \Big(\varepsilon^{ia} \varepsilon^{jd} \frac{\partial B^{bc;ef}}{\partial \varphi} \varphi_{abc} \varphi_{def} + \frac{\partial D^{ijabc}}{\partial \varphi} \varphi_{abc} + \frac{\partial E^{ij}}{\partial \varphi} \Big) \varphi_j. \end{aligned} \tag{C.5}$$

Applying the differential operators $U_i X_a X_b X_c \dfrac{\partial}{\partial \varphi_{abc}}$ and $Y_d Y_e Y_f \dfrac{\partial}{\partial \varphi_{def}}$ to (C.5), we solve the resulting equation for $D^{idabc;ef}$

$$\begin{aligned} &D(U, Y, X, X, X; Y, Y) + D(U, X, Y, Y, Y; X, X) \qquad \text{(C.6)} \\ &= \det(U, X) \Bigg[\frac{1}{3} R \det(\nabla\varphi, Y) B(X, X; Y, Y; \square, \square) - \frac{1}{3} R \det(\square, Y) B(X, X; Y, Y; \nabla\varphi, \square) \\ &\quad - \det(\nabla^2\varphi, Y) B(X, X; Y, Y; \nabla^2\varphi) - \det(\nabla\varphi, Y) \frac{\partial B}{\partial \varphi}(X, X; Y, Y) \Bigg] + (X \leftrightarrow Y), \end{aligned}$$

where $(X \leftrightarrow Y)$ indicates a second copy of the previous expression with X and Y swapped. We set $Y = X$ in (C.6) to get

$$\begin{aligned}
&D(U,X,X,X,X;X,X)\\
&\quad= \det(U,X)\left[\frac{1}{3}R\det(\nabla\varphi,X)B(X,X;X,X;\square,\square) - \frac{1}{3}R\det(\square,X)B(X,X;X,X;\nabla\varphi,\square)\right.\\
&\qquad\left.- \det(\nabla^2\varphi,X)B(X,X;X,X;\nabla^2\varphi) - \det(\nabla\varphi,X)\frac{\partial B}{\partial\varphi}(X,X;X,X)\right],
\end{aligned}$$

noting that this equation is identical to applying the differential operator $X_a X_b \dfrac{\partial}{\partial\varphi_{ab}}$ to (C.4).

Lemma 4. *If D^{ijabc} obeys (C.4) and (C.6), then*

$$\begin{aligned}
D^{ijabc} &= 2\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{abc}\,\varepsilon^{ia}\varepsilon^{jm}\varphi_{mn}B^{bc;n} + 2\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{abc}\,\varepsilon^{ia}\varphi^j_\star\frac{\partial B^{bc}}{\partial\varphi}\\
&\quad- \frac{2}{3}R\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{abc}\,\varepsilon^{ia}\varphi^j_\star g_{mn}B^{bc;mn}\\
&\quad+ \frac{2}{3}R\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{abc}\,\varepsilon^{ia}\varepsilon^{jm}g_{mn}\varphi_k B^{bc;kn} + \mathrm{Sym}_{abc}\,\varepsilon^{ia}\varepsilon^{jb}Q^c,
\end{aligned} \tag{C.7}$$

where Q^i is a vector density of scalar order 2 such that $Q^{i;ab} = B^{ab;i}$ and $Q^{i;j} = Q^{j;i}$.

Proof. Using the fifth metric order case as a blueprint, we consider the ansatz

$$\begin{aligned}
\tilde{D}(U,U,X^{(3)}) = \det(U,X)\Big[&a_0 RB(U,X;\square,\square)\det(\nabla\varphi,X) + a_1 RB(X,X;\square,\square)\det(\nabla\varphi,U)\\
&+ a_2 RB(U,X;\nabla\varphi,\square)\det(\square,X) + a_3 RB(X,X;\nabla\varphi,\square)\det(\square,U)\\
&+ a_4 B(U,X;\nabla^2\varphi)\det(\nabla^2\varphi,X) + a_5 B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,U)\\
&+ a_6\frac{\partial B}{\partial\varphi}(U,X)\det(\nabla\varphi,X) + a_7\frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,U)\Big].
\end{aligned}$$

We define the difference function

$$\Delta(U,U,X,X,X) = D(U,U,X,X,X) - \tilde{D}(U,U,X,X,X)$$

and convert one U to an X, using (C.4) to express $D(X,Y,Y,Y,Y)$

$$
\begin{aligned}
&\Delta(U,X,X,X,X)\\
&\quad = D(U,X,X,X,X) - \tilde{D}(U,X,X,X,X)\\
&\quad = \det(U,X)\left[\frac{1}{3}RB(X,X;\square,\square)\det(\nabla\varphi,X) - \frac{1}{3}RB(X,X;\nabla\varphi,\square)\det(\square,X)\right.\\
&\qquad \left. - B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,X) - \frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,X)\right]\\
&\qquad - \frac{1}{2}\det(U,X)\left[a_0 RB(X,X;\square,\square)\det(\nabla\varphi,X) + a_1 RB(X,X;\square,\square)\det(\nabla\varphi,X)\right.\\
&\qquad + a_2 RB(X,X;\nabla\varphi,\square)\det(\square,X) + a_3 RB(X,X;\nabla\varphi,\square)\det(\square,X)\\
&\qquad + a_4 B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,X) + a_5 B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,X)\\
&\qquad \left. + a_6\frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,X) + a_7\frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,X)\right]\\
&\quad = \det(U,X)\left[\frac{1}{3}RB(X,X;\square,\square)\det(\nabla\varphi,X) - \frac{1}{3}RB(X,X;\nabla\varphi,\square)\det(\square,X)\right.\\
&\qquad - B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,X) - \frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,X)\\
&\qquad - \frac{a_0+a_1}{2}RB(X,X;\square,\square)\det(\nabla\varphi,X) - \frac{a_2+a_3}{2}RB(X,X;\nabla\varphi,\square)\det(\square,X)\\
&\qquad \left. - \frac{a_4+a_5}{2}B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,X) - \frac{a_6+a_7}{2}\frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,X)\right].
\end{aligned}
$$

Comparing the like terms in this expression, we observe that if

$$\frac{a_0+a_1}{2} = \frac{1}{3},\ \frac{a_2+a_3}{2} = -\frac{1}{3},\ \frac{a_4+a_5}{2} = -1,\ \text{and}\ \frac{a_6+a_7}{2} = -1, \tag{C.8}$$

then $\Delta(U,X^{(4)}) = 0$. As with the previous case, this equation has a non-trivial kernel, with solution

$$\Delta(U,U,X,X,X) = \det(U,X)^2 Q(X),$$

where Q^i is a vector density of scalar order 2. To determine Q^i and the a_i constants, we substitute the definition of Δ into the above equation and solve for D

$$D(U,U,X,X,X) = \tilde{D}(U,U,X,X,X) + \det(U,X)^2 Q(X)$$

$$= \Big[a_0 RB(U,X;\square,\square)\det(\nabla\varphi,X) + a_1 RB(X,X;\square,\square)\det(\nabla\varphi,U)$$
$$+ a_2 RB(U,X;\nabla\varphi,\square)\det(\square,X) + a_3 RB(X,X;\nabla\varphi,\square)\det(\square,U)$$
$$+ a_4 B(U,X;\nabla^2\varphi)\det(\nabla^2\varphi,X) + a_5 B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,U)$$
$$+ a_6 \frac{\partial B}{\partial\varphi}(U,X)\det(\nabla\varphi,X) + a_7\frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,U)\Big]\det(U,X)$$
$$+ \det(U,X)^2 Q(X).$$

Next, we change a U to a Y

$$D(U,Y,X,X,X)$$
$$= \frac{1}{2}\det(Y,X)\Big[a_0 RB(U,X;\square,\square)\det(\nabla\varphi,X) + a_1 RB(X,X;\square,\square)\det(\nabla\varphi,U)$$
$$+ a_2 RB(U,X;\nabla\varphi,\square)\det(\square,X) + a_3 RB(X,X;\nabla\varphi,\square)\det(\square,U)$$
$$+ a_4 B(U,X;\nabla^2\varphi)\det(\nabla^2\varphi,X) + a_5 B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,U)$$
$$+ a_6 \frac{\partial B}{\partial\varphi}(U,X)\det(\nabla\varphi,X) + a_7\frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,U)\Big]$$
$$+ \frac{1}{2}\det(U,X)\Big[a_0 RB(Y,X;\square,\square)\det(\nabla\varphi,X) + a_1 RB(X,X;\square,\square)\det(\nabla\varphi,Y)$$
$$+ a_2 RB(Y,X;\nabla\varphi,\square)\det(\square,X) + a_3 RB(X,X;\nabla\varphi,\square)\det(\square,Y)$$
$$+ a_4 B(Y,X;\nabla^2\varphi)\det(\nabla^2\varphi,X) + a_5 B(X,X;\nabla^2\varphi)\det(\nabla^2\varphi,Y)$$
$$+ a_6 \frac{\partial B}{\partial\varphi}(Y,X)\det(\nabla\varphi,X) + a_7\frac{\partial B}{\partial\varphi}(X,X)\det(\nabla\varphi,Y)\Big]$$
$$+ \det(U,X)\det(Y,X)Q(X)$$

and apply the differential operator $Y_a Y_b \dfrac{\partial}{\partial\varphi_{ab}}$ to this equation

$$D(U,Y,X,X,X;Y,Y)$$
$$= \frac{\det(Y,X)}{2}\Big[a_0 RB(U,X;\square,\square;Y,Y)\det(\nabla\varphi,X) + a_1 RB(X,X;\square,\square;Y,Y)\det(\nabla\varphi,U)$$
$$+ a_2 RB(U,X;\nabla\varphi,\square;Y,Y)\det(\square,X) + a_3 RB(X,X;\nabla\varphi,\square;Y,Y)\det(\square,U)$$
$$+ a_4 B(U,X;Y)\det(Y,X) + a_4 B(U,X;\nabla^2\varphi;Y,Y)\det(\nabla^2\varphi,X)$$

$$
\begin{aligned}
&+ a_5 B(X,X;Y)\det(Y,U) + a_5 B(X,X;\nabla^2\varphi;Y,Y)\det(\nabla^2\varphi,U)\\
&+ a_6 \frac{\partial B}{\partial\varphi}(U,X;Y,Y)\det(\nabla\varphi,X) + a_7 \frac{\partial B}{\partial\varphi}(X,X;Y,Y)\det(\nabla\varphi,U)\Big] + \frac{\det(U,X)}{2}\\
&\times \Big[a_0 RB(Y,X;\Box,\Box;Y,Y)\det(\nabla\varphi,X) + a_1 RB(X,X;\Box,\Box;Y,Y)\det(\nabla\varphi,Y)\\
&+ a_2 RB(Y,X;\nabla\varphi,\Box;Y,Y)\det(\Box,X) + a_3 RB(X,X;\nabla\varphi,\Box;Y,Y)\det(\Box,Y)\\
&+ a_4 B(Y,X;Y)\det(Y,X) + a_4 B(Y,X;\nabla^2\varphi;Y,Y)\det(\nabla^2\varphi,X)\\
&+ a_5 B(X,X;Y)\det(Y,Y) + a_5 B(X,X;\nabla^2\varphi;Y,Y)\det(\nabla^2\varphi,Y)\\
&+ a_6 \frac{\partial B}{\partial\varphi}(Y,X;Y,Y)\det(\nabla\varphi,X) + a_7 \frac{\partial B}{\partial\varphi}(X,X;Y,Y)\det(\nabla\varphi,Y)\Big]\\
&+ \det(U,X)\det(Y,X)Q(X;Y,Y)\\
={}& \frac{1}{2}\Big\{a_1 RB(\Box,\Box;X,X;Y,Y)\left[\det(\nabla\varphi,U)\det(Y,X) + \det(\nabla\varphi,Y)\det(U,X)\right]\\
&+ a_3 RB(\nabla\varphi,\Box;X,X;Y,Y)\left[\det(\Box,U)\det(Y,X) + \det(\Box,Y)\det(U,X)\right]\\
&+ a_4 B(U,X;Y)\det(Y,X)^2 + a_4 B(Y,X;Y)\det(Y,X)\det(U,X)\\
&+ a_5 B(X,X;Y)\det(Y,U)\det(Y,X)\\
&+ a_5 B(X,X;Y,Y;\nabla^2\varphi)\left[\det(\nabla^2\varphi,U)\det(Y,X) + \det(\nabla^2\varphi,Y)\det(U,X)\right]\\
&+ a_7 \frac{\partial B}{\partial\varphi}(X,X;Y,Y)\left[\det(\nabla\varphi,U)\det(Y,X) + \det(\nabla\varphi,Y)\det(U,X)\right]\\
&+ a_0 RB(\Box,\Box;U,X;Y,Y)\det(\nabla\varphi,X)\det(Y,X)\\
&+ a_0 RB(\Box,\Box;Y,X;Y,Y)\det(\nabla\varphi,X)\det(U,X)\\
&+ a_2 RB(\nabla\varphi,\Box;U,X;Y,Y)\det(\Box,X)\det(Y,X)\\
&+ a_2 RB(\nabla\varphi,\Box;Y,X;Y,Y)\det(\Box,X)\det(U,X)\\
&+ a_4 B(U,X;Y,Y;\nabla^2\varphi)\det(\nabla^2\varphi,X)\det(Y,X)\\
&+ a_4 B(Y,X;Y,Y;\nabla^2\varphi)\det(\nabla^2\varphi,X)\det(U,X)\\
&+ a_6 \frac{\partial B}{\partial\varphi}(U,X;Y,Y)\det(\nabla\varphi,X)\det(Y,X)\\
&+ a_6 \frac{\partial B}{\partial\varphi}(Y,X;Y,Y)\det(\nabla\varphi,X)\det(U,X)\Big\} + \det(U,X)\det(Y,X)Q(X;Y,Y).
\end{aligned}
$$

We produce a copy of this equation with X and Y swapped and sum the two together, using the symmetries of B to simplify the resulting expression

$$
\begin{aligned}
&D(U,Y,X,X,X;Y,Y)+D(U,X,Y,Y,Y;X,X)\\
&\quad=\frac{1}{2}\Big\{a_1RB(\square,\square;X,X;Y,Y)\left[\det(\nabla\varphi,U)\det(Y,X)+\det(\nabla\varphi,Y)\det(U,X)\right]\\
&\qquad+a_3RB(\nabla\varphi,\square;X,X;Y,Y)\left[\det(\square,U)\det(Y,X)+\det(\square,Y)\det(U,X)\right]\\
&\qquad+a_4B(U,X;Y)\det(Y,X)^2+a_4B(Y,X;Y)\det(Y,X)\det(U,X)\\
&\qquad+a_5B(X,X;Y)\det(Y,U)\det(Y,X)\\
&\qquad+a_5B(X,X;Y,Y;\nabla^2\varphi)\left[\det(\nabla^2\varphi,U)\det(Y,X)+\det(\nabla^2\varphi,Y)\det(U,X)\right]\\
&\qquad+a_7\frac{\partial B}{\partial\varphi}(X,X;Y,Y)\left[\det(\nabla\varphi,U)\det(Y,X)+\det(\nabla\varphi,Y)\det(U,X)\right]\\
&\qquad+a_0RB(\square,\square;U,X;Y,Y)\det(\nabla\varphi,X)\det(Y,X)\\
&\qquad+a_0RB(\square,\square;Y,X;Y,Y)\det(\nabla\varphi,X)\det(U,X)\\
&\qquad+a_2RB(\nabla\varphi,\square;U,X;Y,Y)\det(\square,X)\det(Y,X)\\
&\qquad+a_2RB(\nabla\varphi,\square;Y,X;Y,Y)\det(\square,X)\det(U,X)\\
&\qquad+a_4B(U,X;Y,Y;\nabla^2\varphi)\det(\nabla^2\varphi,X)\det(Y,X)\\
&\qquad+a_4B(Y,X;Y,Y;\nabla^2\varphi)\det(\nabla^2\varphi,X)\det(U,X)\\
&\qquad+a_6\frac{\partial B}{\partial\varphi}(U,X;Y,Y)\det(\nabla\varphi,X)\det(Y,X)\\
&\qquad+a_6\frac{\partial B}{\partial\varphi}(Y,X;Y,Y)\det(\nabla\varphi,X)\det(U,X)\Big\}+\det(U,X)\det(Y,X)Q(X;Y,Y)\\
&\qquad+\frac{1}{2}\Big\{a_1RB(\square,\square;Y,Y;X,X)\left[\det(\nabla\varphi,U)\det(X,Y)+\det(\nabla\varphi,X)\det(U,Y)\right]\\
&\qquad+a_3RB(\nabla\varphi,\square;Y,Y;X,X)\left[\det(\square,U)\det(X,Y)+\det(\square,X)\det(U,Y)\right]\\
&\qquad+a_4B(U,Y;X)\det(X,Y)^2+a_4B(X,Y;X)\det(X,Y)\det(U,Y)\\
&\qquad+a_5B(Y,Y;X)\det(X,U)\det(X,Y)\\
&\qquad+a_5B(Y,Y;X,X;\nabla^2\varphi)\left[\det(\nabla^2\varphi,U)\det(X,Y)+\det(\nabla^2\varphi,X)\det(U,Y)\right]\\
&\qquad+a_7\frac{\partial B}{\partial\varphi}(Y,Y;X,X)\left[\det(\nabla\varphi,U)\det(X,Y)+\det(\nabla\varphi,X)\det(U,Y)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ a_0 RB(\Box,\Box; U,Y; X,X)\det(\nabla\varphi,Y)\det(X,Y)\\
&+ a_0 RB(\Box,\Box; X,Y; X,X)\det(\nabla\varphi,Y)\det(U,Y)\\
&+ a_2 RB(\nabla\varphi,\Box; U,Y; X,X)\det(\Box,Y)\det(X,Y)\\
&+ a_2 RB(\nabla\varphi,\Box; X,Y; X,X)\det(\Box,Y)\det(U,Y)\\
&+ a_4 B(U,Y; X,X; \nabla^2\varphi)\det(\nabla^2\varphi,Y)\det(X,Y)\\
&+ a_4 B(X,Y; X,X; \nabla^2\varphi)\det(\nabla^2\varphi,Y)\det(U,Y)\\
&+ a_6 \frac{\partial B}{\partial\varphi}(U,Y; X,X)\det(\nabla\varphi,Y)\det(X,Y)\\
&+ a_6 \frac{\partial B}{\partial\varphi}(X,Y; X,X)\det(\nabla\varphi,Y)\det(U,Y)\Bigg\} + \det(U,Y)\det(X,Y)Q(Y;X,X)\\
= \frac{1}{2}\Bigg\{ &a_1 RB(\Box,\Box; X,X; Y,Y)\left[\det(\nabla\varphi,Y)\det(U,X) + \det(\nabla\varphi,X)\det(U,Y)\right]\\
&+ a_3 RB(\nabla\varphi,\Box; X,X; Y,Y)\left[\det(\Box,Y)\det(U,X) + \det(\Box,X)\det(U,Y)\right]\\
&+ a_5 B(X,X; Y,Y; \nabla^2\varphi)\left[\det(\nabla^2\varphi,Y)\det(U,X) + \det(\nabla^2\varphi,X)\det(U,Y)\right]\\
&+ a_7 \frac{\partial B}{\partial\varphi}(X,X; Y,Y)\left[\det(\nabla\varphi,Y)\det(U,X) + \det(\nabla\varphi,X)\det(U,Y)\right]\\
&+ a_0 R\det(X,Y)\left[B(\Box,\Box; U,Y; X,X)\det(\nabla\varphi,Y)\right.\\
&\left.- B(\Box,\Box; U,X; Y,Y)\det(\nabla\varphi,X)\right] + a_0 RB(\Box,\Box; Y,X; Y,Y)\det(\nabla\varphi,X)\det(U,X)\\
&+ a_0 RB(\Box,\Box; X,Y; X,X)\det(\nabla\varphi,Y)\det(U,Y)\\
&+ a_2 R\det(X,Y)\left[B(\nabla\varphi,\Box; U,Y; X,X)\det(\Box,Y)\right.\\
&\left.- B(\nabla\varphi,\Box; U,X; Y,Y)\det(\Box,X)\right] + a_2 RB(\nabla\varphi,\Box; Y,X; Y,Y)\det(\Box,X)\det(U,X)\\
&+ a_2 RB(\nabla\varphi,\Box; X,Y; X,X)\det(\Box,Y)\det(U,Y)\\
&+ a_4 \det(X,Y)^2\left[B(U,X;Y) + B(U,Y;X)\right]\\
&+ a_4 \det(X,Y)\left[B(X,Y;X)\det(U,Y) - B(Y,X;Y)\det(U,X)\right]\\
&+ a_4 \det(X,Y)\left[B(U,Y; X,X; \nabla^2\varphi)\det(\nabla^2\varphi,Y)\right.\\
&\left.- B(U,X; Y,Y; \nabla^2\varphi)\det(\nabla^2\varphi,X)\right] + a_4 B(Y,X; Y,Y; \nabla^2\varphi)\det(\nabla^2\varphi,X)\det(U,X)\\
&+ a_4 B(X,Y; X,X; \nabla^2\varphi)\det(\nabla^2\varphi,Y)\det(U,Y)\\
&+ a_5 \det(X,Y)\left[\det(U,Y)B(X,X;Y) - \det(U,X)B(Y,Y;X)\right]
\end{aligned}
$$

$$+ a_6 \det(X,Y)\left[\frac{\partial B}{\partial\varphi}(U,Y;X,X)\det(\nabla\varphi,Y) - \frac{\partial B}{\partial\varphi}(U,X;Y,Y)\det(\nabla\varphi,X)\right]$$
$$+ a_6\frac{\partial B}{\partial\varphi}(Y,X;Y,Y)\det(\nabla\varphi,X)\det(U,X)$$
$$+ a_6\frac{\partial B}{\partial\varphi}(X,Y;X,X)\det(\nabla\varphi,Y)\det(U,Y)\Big\}$$
$$+ \det(X,Y)\left[\det(U,Y)Q(Y;X,X) - \det(U,X)Q(X;Y,Y)\right].$$

We compare this expression with (C.6), noting that $a_1 = \frac{2}{3}$, $a_3 = -\frac{2}{3}$, and $a_5 = a_7 = -2$ to match the corresponding terms. Taking (C.8) into account, we see that $a_0 = a_2 = a_4 = a_6 = 0$. We perform these substitutions in the previous expression, leaving

$$\begin{aligned}
&D(U,Y,X,X,X;Y,Y) + D(U,X,Y,Y,Y;X,X)\\
&\quad= \frac{1}{3}RB(\square,\square;X,X;Y,Y)\left[\det(\nabla\varphi,Y)\det(U,X) + \det(\nabla\varphi,X)\det(U,Y)\right]\\
&\qquad - \frac{1}{3}RB(\nabla\varphi,\square;X,X;Y,Y)\left[\det(\square,Y)\det(U,X) + \det(\square,X)\det(U,Y)\right]\\
&\qquad - B(X,X;Y,Y;\nabla^2\varphi)\left[\det(\nabla^2\varphi,Y)\det(U,X) + \det(\nabla^2\varphi,X)\det(U,Y)\right]\\
&\qquad - \frac{\partial B}{\partial\varphi}(X,X;Y,Y)\left[\det(\nabla\varphi,Y)\det(U,X) + \det(\nabla\varphi,X)\det(U,Y)\right]\\
&\qquad - \det(X,Y)\left[\det(U,Y)B(X,X;Y) - \det(U,X)B(Y,Y;X)\right]\\
&\qquad + \det(X,Y)\left[\det(U,Y)Q(Y;X,X) - \det(U,X)Q(X;Y,Y)\right]\\
&\quad= \frac{1}{3}RB(\square,\square;X,X;Y,Y)\left[\det(\nabla\varphi,Y)\det(U,X) + \det(\nabla\varphi,X)\det(U,Y)\right]\\
&\qquad - \frac{1}{3}RB(\nabla\varphi,\square;X,X;Y,Y)\left[\det(\square,Y)\det(U,X) + \det(\square,X)\det(U,Y)\right]\\
&\qquad - B(X,X;Y,Y;\nabla^2\varphi)\left[\det(\nabla^2\varphi,Y)\det(U,X) + \det(\nabla^2\varphi,X)\det(U,Y)\right]\\
&\qquad - \frac{\partial B}{\partial\varphi}(X,X;Y,Y)\left[\det(\nabla\varphi,Y)\det(U,X) + \det(\nabla\varphi,X)\det(U,Y)\right]\\
&\qquad + \det(X,Y)\det(U,X)\left[B(Y,Y;X) - Q(X;Y,Y)\right]\\
&\qquad + \det(X,Y)\det(U,Y)\left[Q(Y;X,X) - B(X,X;Y)\right].
\end{aligned}$$

The final two terms of this equation must cancel to match (C.6), i.e.,

$$0 = \det(U,X)\left[B(Y,Y;X) - Q(X;Y,Y)\right] + \det(U,Y)\left[Q(Y;X,X) - B(X,X;Y)\right],$$

where we have removed the common term $\det(X,Y)$ from this condition using the arbitrary nature of the X and Y covectors. Letting $U = X$, this equation reduces to

$$0 = \det(X,Y)\,[Q(Y;X,X) - B(X,X;Y)]$$

or, equivalently, $Q(Y;X,X) = B(X,X;Y)$.

Rewriting this expression as $B^{ij;a} = Q^{a;ij}$, we integrate with respect to φ_{ij} to get

$$\int B^{ij;a}\,\mathrm{d}\varphi_{ij} = Q^a + N^a,$$

where N^a is a vector density of scalar order 1. Differentiating this equation with respect to φ_b, we have

$$\int B^{ij;a;b}\,\mathrm{d}\varphi_{ij} = Q^{a;b} + N^{a;b}.$$

The left side is explicitly symmetric in the indices ab, while the term $N^{a;b}$ is symmetric via the invariance identity for a scalar order 1 vector density, i.e., $N^{a;b} = O^{;a;b}$ for some scalar order 1 scalar density O. Therefore, $Q^{a;b} = Q^{b;a}$. □

Returning to the divergence-free condition (C.5), we remove those terms which are quadratic in φ_{abc} using (C.6), yielding

$$\begin{aligned}
0 = \operatorname{Sym}_{klmn} \varepsilon^{ik}\varepsilon^{jl}B^{mn}\Big[&2R(g_{jk}\varphi_{lmn} - g_{kl}\varphi_{jmn}) + R_{|n}(g_{jk}\varphi_{ml} - g_{km}\varphi_{jl}) \\
&+ \frac{2}{15}R^2(g_{kl}g_{jm}\varphi_n - g_{kl}g_{mn}\varphi_j) + \frac{1}{5}R_{|mn}(g_{jk}\varphi_l - g_{lk}\varphi_j)\Big] \\
&+ \left(2\operatorname{Sym}_{ij}\operatorname{Sym}_{klm}\varepsilon^{ik}\varepsilon^{ja}B^{lm;bc}\varphi_{abc} + D^{ijklm}\right) \\
&\times \left[-R(g_{kl}\varphi_{mj} - g_{jk}\varphi_{lm}) - \frac{1}{4}R_{|m}(g_{kl}\varphi_j - g_{jk}\varphi_l)\right] \\
&+ D^{ijabc;kl}\varphi_{abc}\left[-\frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\right] + E^{ij;kl}\left[\varphi_{klj} - \frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\right] \\
&+ \left(D^{ijabc;k}\varphi_{abc} + E^{ij;k}\right)\varphi_{jk} + \left(\frac{\partial D^{ijabc}}{\partial\varphi}\varphi_{abc} + \frac{\partial E^{ij}}{\partial\varphi}\right)\varphi_j.
\end{aligned} \tag{C.9}$$

We simplify this expression by expanding the symmetries and collecting similar terms. We begin by simplifying the first term

$$
\begin{aligned}
&\mathrm{Sym}_{klmn}\varepsilon^{ik}\varepsilon^{jl}B^{mn}\Big[2R(g_{jk}\varphi_{lmn}-g_{kl}\varphi_{jmn})+R_{|n}(g_{jk}\varphi_{ml}-g_{km}\varphi_{jl})\\
&\qquad+\frac{2}{15}R^2(g_{kl}g_{jm}\varphi_n-g_{kl}g_{mn}\varphi_j)+\frac{1}{5}R_{|mn}(g_{jk}\varphi_l-g_{lk}\varphi_j)\Big]\\
&\quad=\frac{1}{12}\Big(\varepsilon^{ik}\varepsilon^{jl}B^{mn}+\varepsilon^{ik}\varepsilon^{jm}B^{ln}+\varepsilon^{ik}\varepsilon^{jn}B^{lm}+\varepsilon^{il}\varepsilon^{jm}B^{kn}+\varepsilon^{il}\varepsilon^{jn}B^{km}+\varepsilon^{im}\varepsilon^{jn}B^{kl}\\
&\qquad+\varepsilon^{il}\varepsilon^{jk}B^{mn}+\varepsilon^{im}\varepsilon^{jk}B^{ln}+\varepsilon^{in}\varepsilon^{jk}B^{lm}+\varepsilon^{im}\varepsilon^{jl}B^{kn}+\varepsilon^{in}\varepsilon^{jl}B^{km}+\varepsilon^{in}\varepsilon^{jm}B^{kl}\Big)\\
&\qquad\times\Big[2R(g_{jk}\varphi_{lmn}-g_{kl}\varphi_{jmn})+R_{|n}(g_{jk}\varphi_{ml}-g_{km}\varphi_{jl})\\
&\qquad+\frac{2}{15}R^2(g_{kl}g_{jm}\varphi_n-g_{kl}g_{mn}\varphi_j)+\frac{1}{5}R_{|mn}(g_{jk}\varphi_l-g_{lk}\varphi_j)\Big]\\
&\quad=\frac{1}{12}\Big\{2R\Big[-(-1)^q g^{il}B^{mn}-(-1)^q g^{im}B^{ln}-(-1)^q g^{in}B^{lm}+g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn}\\
&\qquad+g_{jk}\varepsilon^{il}\varepsilon^{jn}B^{km}+g_{jk}\varepsilon^{im}\varepsilon^{jn}B^{kl}+0+0+0+g_{jk}\varepsilon^{im}\varepsilon^{jl}B^{kn}+g_{jk}\varepsilon^{in}\varepsilon^{jl}B^{km}\\
&\qquad+g_{jk}\varepsilon^{in}\varepsilon^{jm}B^{kl}\Big]\varphi_{lmn}-2R\big[(-1)^q g^{ij}B^{mn}+0+0+0+0+0+(-1)^q g^{ij}B^{mn}\\
&\qquad+g_{kl}\varepsilon^{im}\varepsilon^{jk}B^{ln}+g_{kl}\varepsilon^{in}\varepsilon^{jk}B^{lm}+g_{kl}\varepsilon^{im}\varepsilon^{jl}B^{kn}+g_{kl}\varepsilon^{in}\varepsilon^{jl}B^{km}+0\big]\varphi_{jmn}\\
&\qquad+R_{|n}\Big[-(-1)^q g^{il}B^{mn}-(-1)^q g^{im}B^{ln}-(-1)^q g^{in}B^{lm}+g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn}\\
&\qquad+g_{jk}\varepsilon^{il}\varepsilon^{jn}B^{km}+g_{jk}\varepsilon^{im}\varepsilon^{jn}B^{kl}+0+0+0+g_{jk}\varepsilon^{im}\varepsilon^{jl}B^{kn}+g_{jk}\varepsilon^{in}\varepsilon^{jl}B^{km}\\
&\qquad+g_{jk}\varepsilon^{in}\varepsilon^{jm}B^{kl}\Big]\varphi_{ml}-R_{|n}\Big[0+(-1)^q g^{ij}B^{ln}+g_{km}\varepsilon^{ik}\varepsilon^{jn}B^{lm}+g_{km}\varepsilon^{il}\varepsilon^{jm}B^{kn}\\
&\qquad+g_{km}\varepsilon^{il}\varepsilon^{jn}B^{km}+g_{km}\varepsilon^{im}\varepsilon^{jn}B^{kl}+g_{km}\varepsilon^{il}\varepsilon^{jk}B^{mn}+(-1)^q g^{ij}B^{ln}\\
&\qquad+g_{km}\varepsilon^{in}\varepsilon^{jk}B^{lm}+0+0+g_{km}\varepsilon^{in}\varepsilon^{jm}B^{kl}\Big]\varphi_{jl}+\frac{2}{15}R^2\big[(-1)^q B^{in}+0\\
&\qquad+g_{kl}g_{jm}\varepsilon^{ik}\varepsilon^{jn}B^{lm}+0+g_{kl}g_{jm}\varepsilon^{il}\varepsilon^{jn}B^{km}-(-1)^q g_{kl}g^{in}B^{kl}+(-1)^q B^{in}\\
&\qquad-(-1)^q B^{in}+0-(-1)^q B^{in}+0+0\big]\varphi_n-\frac{2}{15}R^2\big[(-1)^q g^{ij}g_{mn}B^{mn}\\
&\qquad+g_{kl}g_{mn}\varepsilon^{ik}\varepsilon^{jm}B^{ln}+g_{kl}g_{mn}\varepsilon^{ik}\varepsilon^{jn}B^{lm}+g_{kl}g_{mn}\varepsilon^{il}\varepsilon^{jm}B^{kn}+g_{kl}g_{mn}\varepsilon^{il}\varepsilon^{jn}B^{km}\\
&\qquad+(-1)^q g^{ij}g_{kl}B^{kl}+(-1)^q g^{ij}g_{mn}B^{mn}+g_{kl}g_{mn}\varepsilon^{im}\varepsilon^{jk}B^{ln}+g_{kl}g_{mn}\varepsilon^{in}\varepsilon^{jk}B^{lm}\\
&\qquad+g_{kl}g_{mn}\varepsilon^{im}\varepsilon^{jl}B^{kn}+g_{kl}g_{mn}\varepsilon^{in}\varepsilon^{jl}B^{km}+(-1)^q g^{ij}g_{kl}B^{kl}\Big]\varphi_j\\
&\qquad+\frac{1}{5}R_{|mn}\Big[-(-1)^q g^{il}B^{mn}-(-1)^q g^{im}B^{ln}-(-1)^q g^{in}B^{lm}+g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn}
\end{aligned}
$$

$$+ g_{jk}\varepsilon^{il}\varepsilon^{jn}B^{km} + g_{jk}\varepsilon^{im}\varepsilon^{jn}B^{kl} + 0 + 0 + 0 + g_{jk}\varepsilon^{im}\varepsilon^{jl}B^{kn} + g_{jk}\varepsilon^{in}\varepsilon^{jl}B^{km}$$
$$+ g_{jk}\varepsilon^{in}\varepsilon^{jm}B^{kl}\Big]\varphi_l - \frac{1}{5}R_{|mn}\Big[(-1)^q g^{ij}B^{mn} + g_{kl}\varepsilon^{ik}\varepsilon^{jm}B^{ln} + g_{kl}\varepsilon^{ik}\varepsilon^{jn}B^{lm}$$
$$+ g_{kl}\varepsilon^{il}\varepsilon^{jm}B^{kn} + g_{kl}\varepsilon^{il}\varepsilon^{jn}B^{km} + g_{kl}\varepsilon^{im}\varepsilon^{jn}B^{kl} + (-1)^q g^{ij}B^{mn} + g_{kl}\varepsilon^{im}\varepsilon^{jk}B^{ln}$$
$$+ g_{kl}\varepsilon^{in}\varepsilon^{jk}B^{lm} + g_{kl}\varepsilon^{im}\varepsilon^{jl}B^{kn} + g_{kl}\varepsilon^{in}\varepsilon^{jl}B^{km} + g_{kl}\varepsilon^{in}\varepsilon^{jm}B^{kl}\Big]\varphi_j\Big\}$$
$$= \frac{1}{12}\Big\{2R\Big[-3(-1)^q g^{il}B^{mn} + 6g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn}\Big]\varphi_{lmn} - 2R\Big[2(-1)^q g^{ij}B^{mn}$$
$$+ 4g_{kl}\varepsilon^{im}\varepsilon^{jk}B^{ln}\Big]\varphi_{jmn} + R_{|n}\Big[-2(-1)^q g^{il}B^{mn} - (-1)^q g^{in}B^{lm}$$
$$+ 2g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn} + 2g_{jk}\varepsilon^{il}\varepsilon^{jn}B^{km} + 2g_{jk}\varepsilon^{in}\varepsilon^{jl}B^{km}\Big]\varphi_{ml} - R_{|n}\Big[2(-1)^q g^{ij}B^{ln}$$
$$+ 2g_{km}\varepsilon^{ik}\varepsilon^{jn}B^{lm} + 2g_{km}\varepsilon^{il}\varepsilon^{jm}B^{kn} + g_{km}\varepsilon^{il}\varepsilon^{jn}B^{km} + 2g_{km}\varepsilon^{in}\varepsilon^{jk}B^{lm}\Big]\varphi_{jl}$$
$$+ \frac{2}{15}R^2\Big[2g_{kl}g_{jm}\varepsilon^{ik}\varepsilon^{jn}B^{lm} - (-1)^q g_{kl}g^{in}B^{kl}\Big]\varphi_n - \frac{2}{15}R^2\Big[4(-1)^q g^{ij}g_{kl}B^{kl}$$
$$+ 8g_{kl}g_{mn}\varepsilon^{ik}\varepsilon^{jm}B^{ln}\Big]\varphi_j + \frac{1}{5}R_{|mn}\Big[-(-1)^q g^{il}B^{mn} - 2(-1)^q g^{im}B^{ln}$$
$$+ 2g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn} + 2g_{jk}\varepsilon^{im}\varepsilon^{jn}B^{kl} + 2g_{jk}\varepsilon^{im}\varepsilon^{jl}B^{kn}\Big]\varphi_l - \frac{1}{5}R_{|mn}$$
$$\times\Big[2(-1)^q g^{ij}B^{mn} + 4g_{kl}\varepsilon^{ik}\varepsilon^{jm}B^{ln} + 2g_{kl}\varepsilon^{im}\varepsilon^{jn}B^{kl} + 4g_{kl}\varepsilon^{im}\varepsilon^{jk}B^{ln}\Big]\varphi_j\Big\}$$
$$= \frac{1}{12}\Big\{2R\Big[-5(-1)^q g^{il}B^{mn} + 10g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn}\Big]\varphi_{lmn}$$
$$+ R_{|n}\Big[-4(-1)^q g^{il}B^{mn} - (-1)^q g^{in}B^{lm} + 4g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn} + 2g_{jk}\varepsilon^{il}\varepsilon^{jn}B^{km}$$
$$+ 4g_{jk}\varepsilon^{in}\varepsilon^{jl}B^{km} - 2g_{jk}\varepsilon^{ij}\varepsilon^{ln}B^{km} - g_{jk}\varepsilon^{il}\varepsilon^{mn}B^{kj}\Big]\varphi_{ml}$$
$$+ \frac{2}{15}R^2\Big[10g_{kl}g_{jm}\varepsilon^{ik}\varepsilon^{jn}B^{lm} - 5(-1)^q g_{kl}g^{in}B^{kl}\Big]\varphi_n$$
$$+ \frac{1}{5}R_{|mn}\Big[-3(-1)^q g^{il}B^{mn} - 2(-1)^q g^{im}B^{ln} + 2g_{jk}\varepsilon^{il}\varepsilon^{jm}B^{kn} + 2g_{jk}\varepsilon^{im}\varepsilon^{jn}B^{kl}$$
$$+ 6g_{jk}\varepsilon^{im}\varepsilon^{jl}B^{kn} - 4g_{jk}\varepsilon^{ik}\varepsilon^{lm}B^{jn} - 2g_{jk}\varepsilon^{im}\varepsilon^{ln}B^{jk}\Big]\varphi_l\Big\}$$
$$= \frac{(-1)^q}{12}\Big\{2R\Big[-5g^{il}B^{mn} + 10g_{jk}(g^{ij}g^{lm} - g^{im}g^{jl})B^{kn}\Big]\varphi_{lmn}$$
$$+ R_{|n}\Big[-4g^{il}B^{mn} - g^{in}B^{lm} + 4g_{jk}(g^{ij}g^{lm} - g^{im}g^{jl})B^{kn}$$
$$+ 2g_{jk}(g^{ij}g^{ln} - g^{in}g^{jl})B^{km} + 4g_{jk}(g^{ij}g^{nl} - g^{il}g^{jn})B^{km}$$
$$- 2g_{jk}(g^{il}g^{jn} - g^{in}g^{jl})B^{km} - g_{jk}(g^{im}g^{ln} - g^{in}g^{lm})B^{kj}\Big]\varphi_{ml}$$
$$+ \frac{2}{15}R^2\Big[10g_{kl}g_{jm}(g^{ij}g^{kn} - g^{in}g^{jk})B^{lm} - 5g_{kl}g^{in}B^{kl}\Big]\varphi_n$$

$$
\begin{aligned}
&+\frac{1}{5}R_{|mn}\Big[-3g^{il}B^{mn}-2g^{im}B^{ln}+2g_{jk}(g^{ij}g^{lm}-g^{im}g^{jl})B^{kn}\\
&+2g_{jk}(g^{ij}g^{mn}-g^{in}g^{jm})B^{kl}+6g_{jk}(g^{ij}g^{lm}-g^{il}g^{jm})B^{kn}\\
&-4g_{jk}(g^{il}g^{km}-g^{im}g^{kl})B^{jn}-2g_{jk}(g^{il}g^{mn}-g^{in}g^{lm})B^{jk}\Big]\varphi_l\Big\}\\
=&\frac{(-1)^q}{12}\Big[2R\left(-15g^{il}B^{mn}+10g^{lm}B^{in}\right)\varphi_{lmn}\\
&+R_{|n}\Big(-14g^{il}B^{mn}-g^{in}B^{lm}+4g^{lm}B^{in}+6g^{ln}B^{im}-g^{im}g^{ln}g_{jk}B^{kj}\\
&+g^{in}g^{lm}g_{jk}B^{kj}\Big)\varphi_{lm}+\frac{2}{15}R^2\left(10B^{il}-15g_{jk}g^{il}B^{jk}\right)\varphi_l\\
&+\frac{1}{5}R_{|mn}\Big(-13g^{il}B^{mn}-2g^{im}B^{ln}+8g^{lm}B^{in}+2g^{mn}B^{il}\\
&-2g^{il}g^{mn}g_{jk}B^{jk}+2g^{in}g^{lm}g_{jk}B^{jk}\Big)\varphi_l\Big]\\
=&\frac{(-1)^q}{6}\left(-15Rg^{il}B^{mn}+10Rg^{lm}B^{in}\right)\varphi_{lmn}+\frac{(-1)^q}{12}\Big(-14R_{|n}g^{il}B^{mn}-R^{|i}B^{lm}\\
&+4R_{|n}g^{lm}B^{in}+6R^{|l}B^{im}-R^{|l}g^{im}g_{jk}B^{kj}+R^{|i}g^{lm}g_{jk}B^{kj}\Big)\varphi_{lm}\\
&+\frac{(-1)^q}{12}\Big(\frac{4}{3}R^2B^{il}-2R^2g_{jk}g^{il}B^{jk}-\frac{13}{5}R_{|mn}g^{il}B^{mn}-\frac{2}{5}R_{|mn}g^{im}B^{ln}\\
&+\frac{8}{5}R_{|mn}g^{lm}B^{in}+\frac{2}{5}R_{|mn}g^{mn}B^{il}-\frac{2}{5}R_{|mn}g^{il}g^{mn}g_{jk}B^{jk}\\
&+\frac{2}{5}R_{|mn}g^{in}g^{lm}g_{jk}B^{jk}\Big)\varphi_l.
\end{aligned}
$$

Using a similar procedure, we simplify the second term

$$
\begin{aligned}
&\left(2\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{klm}\,\varepsilon^{ik}\varepsilon^{ja}B^{lm;bc}\varphi_{abc}+D^{ijklm}\right)\\
&\quad\times\left[-R(g_{kl}\varphi_{mj}-g_{jk}\varphi_{lm})-\frac{1}{4}R_{|m}(g_{kl}\varphi_j-g_{jk}\varphi_l)\right]\\
&\quad=-\frac{1}{3}\Big(\varepsilon^{ik}\varepsilon^{ja}B^{lm;bc}+\varepsilon^{jk}\varepsilon^{ia}B^{lm;bc}+\varepsilon^{il}\varepsilon^{ja}B^{km;bc}+\varepsilon^{jl}\varepsilon^{ia}B^{km;bc}+\varepsilon^{im}\varepsilon^{ja}B^{kl;bc}\\
&\qquad+\varepsilon^{jm}\varepsilon^{ia}B^{kl;bc}\Big)\varphi_{abc}\left(Rg_{kl}\varphi_{mj}-Rg_{jk}\varphi_{lm}+\frac{1}{4}R_{|m}g_{kl}\varphi_j-\frac{1}{4}R_{|m}g_{jk}\varphi_l\right)\\
&\qquad-Rg_{kl}\varphi_{mj}D^{ijklm}+Rg_{jk}\varphi_{lm}D^{ijklm}-\frac{1}{4}R_{|m}g_{kl}\varphi_jD^{ijklm}+\frac{1}{4}R_{|m}g_{jk}\varphi_lD^{ijklm}\\
&\quad=-\frac{(-1)^q}{3}\varphi_{abc}\Big\{Rg_{kl}\varphi_{mj}\Big[\left(g^{ij}g^{ak}-g^{ia}g^{jk}\right)B^{lm;bc}+\left(g^{ij}g^{ak}-g^{ja}g^{ik}\right)B^{lm;bc}\\
&\qquad+\left(g^{ij}g^{al}-g^{ia}g^{jl}\right)B^{km;bc}+\left(g^{ij}g^{al}-g^{ja}g^{il}\right)B^{km;bc}+\left(g^{ij}g^{am}-g^{ia}g^{jm}\right)B^{kl;bc}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&- R g_{jk}\varphi_{lm}\Big[\Big(g^{ij}g^{ak} - g^{ia}g^{jk}\Big)B^{lm;bc} + \Big(g^{ij}g^{ak} - g^{ik}g^{aj}\Big)B^{lm;bc}\\
&+ \Big(g^{ij}g^{al} - g^{ia}g^{jl}\Big)B^{km;bc} + \Big(g^{ij}g^{al} - g^{il}g^{aj}\Big)B^{km;bc}\\
&+ \left(g^{ij}g^{am} - g^{ia}g^{jm}\right)B^{kl;bc} + \left(g^{ij}g^{am} - g^{im}g^{aj}\right)B^{kl;bc}\Big]\\
&+ \frac{1}{4}R_{|m}g_{kl}\varphi_j\Big[\Big(g^{ij}g^{ak} - g^{ia}g^{jk}\Big)B^{lm;bc} + \Big(g^{ij}g^{ak} - g^{ik}g^{ja}\Big)B^{lm;bc}\\
&+ \Big(g^{ij}g^{al} - g^{ia}g^{jl}\Big)B^{km;bc} + \Big(g^{ij}g^{al} - g^{il}g^{aj}\Big)B^{km;bc}\\
&+ \left(g^{ij}g^{am} - g^{ia}g^{jm}\right)B^{kl;bc} + \left(g^{ij}g^{am} - g^{im}g^{aj}\right)B^{kl;bc}\Big]\\
&- \frac{1}{4}R_{|m}g_{jk}\varphi_l\Big[\Big(g^{ij}g^{ak} - g^{ia}g^{jk}\Big)B^{lm;bc} + \Big(g^{ij}g^{al} - g^{ia}g^{jl}\Big)B^{km;bc}\\
&+ \Big(g^{ij}g^{al} - g^{il}g^{aj}\Big)B^{km;bc} + \left(g^{ij}g^{am} - g^{ia}g^{jm}\right)B^{kl;bc} + \left(g^{ij}g^{am} - g^{im}g^{ja}\right)B^{kl;bc}\Big]\Big\}\\
&- R g_{kl}\varphi_{mj}D^{ijklm} + R g_{jk}\varphi_{lm}D^{ijklm} - \frac{1}{4}R_{|m}g_{kl}\varphi_j D^{ijklm} + \frac{1}{4}R_{|m}g_{jk}\varphi_l D^{ijklm}\\
= -&\frac{(-1)^q}{3}\varphi_{abc}\Big[\Big(R\varphi_{mj}g^{ij}B^{am;bc} - R\varphi_{mj}g^{ia}B^{jm;bc} + R\varphi_{mj}g^{ij}B^{am;bc} - R\varphi_{mj}g^{ja}B^{im;bc}\\
&+ R\varphi_{mj}g^{ij}B^{am;bc} - R\varphi_{mj}g^{ia}B^{jm;bc} + R\varphi_{mj}g^{ij}B^{am;bc} - R\varphi_{mj}g^{ja}B^{im;bc}\\
&+ R g_{kl}\varphi_{mj}g^{ij}g^{am}B^{kl;bc} - R g_{kl}\varphi_{mj}g^{ia}g^{jm}B^{kl;bc}\Big)\\
&- \Big(R\varphi_{lm}g^{ia}B^{lm;bc} - 2R\varphi_{lm}g^{ia}B^{lm;bc} + R\varphi_{lm}g^{ia}B^{lm;bc} - R\varphi_{lm}g^{ia}B^{lm;bc}\\
&+ R\varphi_{lm}g^{al}B^{im;bc} - R\varphi_{lm}g^{ia}B^{lm;bc} + R\varphi_{lm}g^{al}B^{im;bc} - R\varphi_{lm}g^{il}B^{am;bc}\\
&+ R\varphi_{lm}g^{am}B^{il;bc} - R\varphi_{lm}g^{ia}B^{ml;bc} + R\varphi_{lm}g^{am}B^{il;bc} - R\varphi_{lm}g^{im}B^{al;bc}\Big)\\
&+ \frac{1}{4}R_{|m}\Big(\varphi_j g^{ij}B^{am;bc} - \varphi_j g^{ia}B^{jm;bc} + \varphi_j g^{ij}B^{am;bc} - \varphi_j g^{ja}B^{im;bc}\\
&+ \varphi_j g^{ij}B^{am;bc} - \varphi_j g^{ia}B^{jm;bc} + \varphi_j g^{ij}B^{am;bc} - \varphi_j g^{aj}B^{im;bc}\\
&+ g_{kl}\varphi_j g^{ij}g^{am}B^{kl;bc} - g_{kl}\varphi_j g^{ia}g^{jm}B^{kl;bc} + g_{kl}\varphi_j g^{ij}g^{am}B^{kl;bc}\\
&- g_{kl}\varphi_j g^{im}g^{aj}B^{kl;bc}\Big) - \frac{1}{4}R_{|m}\Big(\varphi_l g^{ia}B^{lm;bc} - 2\varphi_l g^{ia}B^{lm;bc}\\
&+ \varphi_l g^{al}B^{im;bc} - \varphi_l g^{ia}B^{lm;bc} + \varphi_l g^{al}B^{im;bc} - \varphi_l g^{il}B^{am;bc}\\
&+ \varphi_l g^{am}B^{il;bc} - \varphi_l g^{ia}B^{ml;bc} + \varphi_l g^{am}B^{il;bc} - \varphi_l g^{im}B^{al;bc}\Big)\Big]\\
&- R g_{kl}\varphi_{mj}D^{ijklm} + R g_{jk}\varphi_{lm}D^{ijklm} - \frac{1}{4}R_{|m}g_{kl}\varphi_j D^{ijklm} + \frac{1}{4}R_{|m}g_{jk}\varphi_l D^{ijklm}\\
= -&\frac{(-1)^q}{3}\varphi_{abc}\Big[\Big(4R\varphi_{mj}g^{ij}B^{am;bc} - 2R\varphi_{mj}g^{ia}B^{jm;bc} - 2R\varphi_{mj}g^{ja}B^{im;bc}
\end{aligned}
$$

$$
\begin{aligned}
&+ Rg_{kl}\varphi_{mj}g^{ij}g^{am}B^{kl;bc} - Rg_{kl}\varphi_{mj}g^{ia}g^{jm}B^{kl;bc}\Big) \\
&- \Big(-3R\varphi_{lm}g^{ia}B^{lm;bc} + 4R\varphi_{lm}g^{al}B^{im;bc} - 2R\varphi_{lm}g^{il}B^{am;bc}\Big) \\
&+ \frac{1}{4}R_{|m}\Big(4\varphi_j g^{ij}B^{am;bc} - 2\varphi_j g^{ia}B^{jm;bc} - 2\varphi_j g^{ja}B^{im;bc} + 2g_{kl}\varphi_j g^{ij}g^{am}B^{kl;bc} \\
&- g_{kl}\varphi_j g^{ia}g^{jm}B^{kl;bc} - g_{kl}\varphi_j g^{ja}g^{im}B^{kl;bc}\Big) - \frac{1}{4}R_{|m}\Big(-3\varphi_l g^{ia}B^{lm;bc} \\
&+ 2\varphi_l g^{al}B^{im;bc} - \varphi_l g^{il}B^{am;bc} + 2\varphi_l g^{am}B^{il;bc} - \varphi_l g^{im}B^{al;bc}\Big)\Big] \\
&- Rg_{kl}\varphi_{mj}D^{ijklm} + Rg_{jk}\varphi_{lm}D^{ijklm} - \frac{1}{4}R_{|m}g_{kl}\varphi_j D^{ijklm} + \frac{1}{4}R_{|m}g_{jk}\varphi_l D^{ijklm} \\
= &-\frac{(-1)^q}{3}\varphi_{abc}\Big[R\varphi_{mj}\Big(6g^{ij}B^{am;bc} + g^{ia}B^{jm;bc} - 6g^{ja}B^{im;bc} + g_{kl}g^{ij}g^{am}B^{kl;bc} \\
&- g_{kl}g^{ia}g^{jm}B^{kl;bc}\Big) + \frac{1}{4}R_{|m}\varphi_j\Big(5g^{ij}B^{am;bc} + g^{ia}B^{jm;bc} - 4g^{ja}B^{im;bc} \\
&+ 2g_{kl}g^{ij}g^{am}B^{kl;bc} - g_{kl}g^{ia}g^{jm}B^{kl;bc} - g_{kl}g^{ja}g^{im}B^{kl;bc} - 2g^{am}B^{ij;bc} + g^{im}B^{aj;bc}\Big)\Big] \\
&- Rg_{kl}\varphi_{mj}D^{ijklm} + Rg_{jk}\varphi_{lm}D^{ijklm} - \frac{1}{4}R_{|m}g_{kl}\varphi_j D^{ijklm} + \frac{1}{4}R_{|m}g_{jk}\varphi_l D^{ijklm}.
\end{aligned}
$$

We substitute these terms back into the divergence-free condition (C.9), yielding

$$
\begin{aligned}
0 = &\frac{(-1)^q}{6}\Big(-15Rg^{il}B^{mn} + 10Rg^{lm}B^{in}\Big)\varphi_{lmn} + \frac{(-1)^q}{12}\Big(-14R_{|n}g^{il}B^{mn} - R^{|i}B^{ml} \\
&+ 4R_{|n}g^{ml}B^{in} + 6R^{|l}B^{im} - R^{|l}g^{im}g_{kj}B^{kj} + R^{|i}g^{ml}g_{kj}B^{kj}\Big)\varphi_{lm} \\
&+ \frac{(-1)^q}{12}\Big(-2R^2 g^{il}g_{km}B^{km} + \frac{4}{3}R^2 B^{il} - \frac{13}{5}R_{|mn}g^{il}B^{mn} - \frac{2}{5}R_{|mn}g^{im}B^{ln} \\
&+ \frac{8}{5}R_{|mn}g^{ml}B^{in} + \frac{2}{5}R_{|mn}g^{mn}B^{il} - \frac{2}{5}R_{|mn}g^{mn}g^{il}g_{jk}B^{jk} + \frac{2}{5}R_{|mn}g^{in}g_{jk}g^{ml}B^{jk}\Big)\varphi_l \\
&- \frac{(-1)^q}{3}\varphi_{abc}\Big[R\varphi_{mj}\Big(6g^{ij}B^{am;bc} + g^{ia}B^{jm;bc} - 6g^{ja}B^{im;bc} + g_{kl}g^{ij}g^{am}B^{kl;bc} \\
&- g_{kl}g^{ia}g^{jm}B^{kl;bc}\Big) + \frac{1}{4}R_{|m}\varphi_j\Big(5g^{ij}B^{am;bc} + g^{ia}B^{jm;bc} - 4g^{ja}B^{im;bc} \\
&+ 2g_{kl}g^{ij}g^{am}B^{kl;bc} - g_{kl}g^{ia}g^{jm}B^{kl;bc} - g_{kl}g^{ja}g^{im}B^{kl;bc} - 2g^{am}B^{ij;bc} + g^{im}B^{aj;bc}\Big)\Big] \\
&- Rg_{kl}\varphi_{mj}D^{ijklm} + Rg_{jk}\varphi_{lm}D^{ijklm} - \frac{1}{4}R_{|m}g_{kl}\varphi_j D^{ijklm} + \frac{1}{4}R_{|m}g_{jk}\varphi_l D^{ijklm} \\
&+ D^{ijabc;kl}\varphi_{abc}\Big[-\frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] + E^{ij;kl}\Big[\varphi_{klj} - \frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\Big] \\
&+ \Big(D^{ijabc;k}\varphi_{abc} + E^{ij;k}\Big)\varphi_{jk} + \Big(\frac{\partial D^{ijabc}}{\partial\varphi}\varphi_{abc} + \frac{\partial E^{ij}}{\partial\varphi}\Big)\varphi_j. \qquad \text{(C.10)}
\end{aligned}
$$

We apply the differential operator $U_i X_a X_b X_c \dfrac{\partial}{\partial \varphi_{abc}}$ to the divergence-free condition and solve the resulting equation for $E^{ij;kl}$

$$
\begin{aligned}
&E(U,X;X,X)\\
&\quad= \frac{(-1)^q}{6}\left[15Rg(U,X)B(X,X) - 10Rg(X,X)B(U,X)\right]\\
&\qquad + \frac{(-1)^q}{3}\Big\{R\big[6g(U,\nabla^2\varphi)B(X,\nabla^2\varphi;X,X) + g(U,X)B(\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&\qquad - 6g(\nabla^2\varphi,X)B(U,\nabla^2\varphi;X,X) + g(U,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\Box,\Box;X,X)\\
&\qquad - g(U,X)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;X,X)\big] + \frac{1}{4}\left[5g(U,\nabla\varphi)B(X,\nabla R;X,X)\right.\\
&\qquad + g(U,X)B(\nabla\varphi,\nabla R;X,X) - 4g(\nabla\varphi,X)B(U,\nabla R;X,X)\\
&\qquad + 2g(U,\nabla\varphi)g(X,\nabla R)B(\Box,\Box;X,X) - g(U,X)g(\nabla\varphi,\nabla R)B(\Box,\Box;X,X)\\
&\qquad - g(U,\nabla R)g(X,\nabla\varphi)B(\Box,\Box;X,X) - 2g(X,\nabla R)B(U,\nabla\varphi;X,X)\\
&\qquad \left. + g(U,\nabla R)B(X,\nabla\varphi;X,X)\right]\Big\} + \frac{1}{3}R\left[D(U,\nabla\varphi,X,X,X;\Box,\Box)\right.\\
&\qquad \left. - D(U,\Box,X,X,X;\Box,\nabla\varphi)\right] - D(U,\nabla^2\varphi,X,X,X;\nabla^2\varphi) - \frac{\partial D}{\partial\varphi}(U,\nabla\varphi,X,X,X)\\
&\quad= \frac{(-1)^q}{6}\Big[15Rg(U,X)B(X,X) - 10Rg(X,X)B(U,X) + 12Rg(U,\nabla^2\varphi)B(\nabla^2\varphi,X;X,X)\\
&\qquad + 2Rg(U,X)B(\nabla^2\varphi,\nabla^2\varphi;X,X) - 12Rg(\nabla^2\varphi,X)B(\nabla^2\varphi,U;X,X)\\
&\qquad + 2Rg(U,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\Box,\Box;X,X) - 2Rg(U,X)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;X,X)\\
&\qquad + g(U,\nabla\varphi)g(X,\nabla R)B(\Box,\Box;X,X) - \frac{1}{2}g(U,\nabla R)g(X,\nabla\varphi)B(\Box,\Box;X,X)\\
&\qquad - \frac{1}{2}g(U,X)g(\nabla\varphi,\nabla R)B(\Box,\Box;X,X) + \frac{1}{2}g(U,X)B(\nabla\varphi,\nabla R;X,X)\\
&\qquad - 2g(X,\nabla\varphi)B(\nabla R,U;X,X) - g(X,\nabla R)B(\nabla\varphi,U;X,X)\\
&\qquad + \frac{5}{2}g(U,\nabla\varphi)B(\nabla R,X;X,X) + \frac{1}{2}g(U,\nabla R)B(\nabla\varphi,X;X,X)\Big]\\
&\qquad + \frac{1}{3}R\left[D(U,\nabla\varphi,X,X,X;\Box,\Box) - D(U,\Box,X,X,X;\Box,\nabla\varphi)\right]\\
&\qquad - D(U,\nabla^2\varphi,X,X,X;\nabla^2\varphi) - \frac{\partial D}{\partial\varphi}(U,\nabla\varphi,X,X,X).
\end{aligned}
\tag{C.11}
$$

Before proceeding further, we express D (C.7) using covector notation and expand the permutation tensors using (2.8)

$$
\begin{aligned}
&D(U,U,X,X,X)\\
&\quad = 2\det(U,X)\det(U,\nabla^2\varphi)B(X,X;\nabla^2\varphi) + 2\det(U,X)\det(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X)\\
&\qquad - \frac{2}{3}R\det(U,X)\det(U,\nabla\varphi)B(X,X;\Box,\Box)\\
&\qquad + \frac{2}{3}R\det(U,X)\det(U,\Box)B(X,X;\nabla\varphi,\Box) + \det(U,X)^2Q(X)\\
&\quad = (-1)^q\left[2g(U,U)g(X,\nabla^2\varphi)B(X,X;\nabla^2\varphi) - 2g(U,\nabla^2\varphi)g(U,X)B(X,X;\nabla^2\varphi)\right.\\
&\qquad + 2g(U,U)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) - 2g(U,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X)\\
&\qquad - \frac{2}{3}Rg(U,U)g(X,\nabla\varphi)B(X,X;\Box,\Box) + \frac{2}{3}Rg(U,\nabla\varphi)g(U,X)B(X,X;\Box,\Box)\\
&\qquad + \frac{2}{3}Rg(U,U)g(X,\Box)B(X,X;\nabla\varphi,\Box) - \frac{2}{3}Rg(U,\Box)g(U,X)B(X,X;\nabla\varphi,\Box)\\
&\qquad \left.+ g(U,U)g(X,X)Q(X) - g(U,X)g(U,X)Q(X)\right]\\
&\quad = (-1)^q\Bigg[2g(U,U)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) - 2g(U,\nabla^2\varphi)g(U,X)Q(\nabla^2\varphi;X,X)\\
&\qquad + 2g(U,U)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) - 2g(U,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X)\\
&\qquad - \frac{2}{3}Rg(U,U)g(X,\nabla\varphi)B(\Box,\Box;X,X) + \frac{2}{3}Rg(U,\nabla\varphi)g(U,X)B(\Box,\Box;X,X)\\
&\qquad + \frac{2}{3}Rg(U,U)B(\nabla\varphi,X;X,X) - \frac{2}{3}Rg(U,X)B(\nabla\varphi,U;X,X)\\
&\qquad + g(U,U)g(X,X)Q(X) - g(U,X)g(U,X)Q(X)\Bigg].
\end{aligned}
\tag{C.12}
$$

With this result, we expand the D terms in (C.11). First, we expand the terms containing derivatives of D with respect to φ_{ij}, namely

$$
\begin{aligned}
&\frac{1}{3}RD(U,\nabla\varphi,X,X,X;\Box,\Box)\\
&\quad = \frac{(-1)^q}{3}R\Bigg[2g(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) - g(\nabla\varphi,\nabla^2\varphi)g(U,X)Q(\nabla^2\varphi;X,X)\\
&\qquad - g(U,\nabla^2\varphi)g(\nabla\varphi,X)Q(\nabla^2\varphi;X,X) + 2g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X)\\
&\qquad - g(\nabla\varphi,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X) - g(U,\nabla\varphi)g(\nabla\varphi,X)\frac{\partial B}{\partial\varphi}(X,X)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{3} Rg(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;X,X) + \frac{1}{3} Rg(\nabla\varphi,\nabla\varphi)g(U,X)B(\Box_a,\Box_a;X,X) \\
&+ \frac{1}{3} Rg(U,\nabla\varphi)g(\nabla\varphi,X)B(\Box_a,\Box_a;X,X) + \frac{2}{3} Rg(U,\nabla\varphi)B(\nabla\varphi,X;X,X) \\
&- \frac{1}{3} Rg(\nabla\varphi,X)B(\nabla\varphi,U;X,X) - \frac{1}{3} Rg(U,X)B(\nabla\varphi,\nabla\varphi;X,X) \\
&+ g(U,\nabla\varphi)g(X,X)Q(X) - g(U,X)g(\nabla\varphi,X)Q(X)\Big] (;\Box,\Box) \\
= &\frac{(-1)^q}{3} R \Big[2g(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) - g(\nabla\varphi,\nabla^2\varphi)g(U,X)Q(\nabla^2\varphi;X,X) \\
&- g(U,\nabla^2\varphi)g(\nabla\varphi,X)Q(\nabla^2\varphi;X,X) + g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) \\
&- Sg(U,X)\frac{\partial B}{\partial\varphi}(X,X) - \frac{1}{3} Rg(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;X,X) \\
&+ \frac{1}{3} RSg(U,X)B(\Box_a,\Box_a;X,X) + \frac{2}{3} Rg(U,\nabla\varphi)B(\nabla\varphi,X;X,X) \\
&- \frac{1}{3} Rg(\nabla\varphi,X)B(\nabla\varphi,U;X,X) - \frac{1}{3} Rg(U,X)B(\nabla\varphi,\nabla\varphi;X,X) \\
&+ g(U,\nabla\varphi)g(X,X)Q(X) - g(U,X)g(\nabla\varphi,X)Q(X)\Big] (;\Box,\Box) \\
= &\frac{(-1)^q}{3} R \Big[2g(U,\nabla\varphi)g(X,\Box)Q(\Box;X,X) + 2g(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&- g(\nabla\varphi,\Box)g(U,X)Q(\Box;X,X) - g(\nabla\varphi,\nabla^2\varphi)g(U,X)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&- g(U,\Box)g(\nabla\varphi,X)Q(\Box;X,X) - g(U,\nabla^2\varphi)g(\nabla\varphi,X)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&+ g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) - Sg(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) \\
&- \frac{1}{3} Rg(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X) \\
&+ \frac{1}{3} RSg(U,X)B(\Box_a,\Box_a;\Box,\Box;X,X) + \frac{2}{3} Rg(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;X,X) \\
&- \frac{1}{3} Rg(\nabla\varphi,X)B(\Box,\Box;\nabla\varphi,U;X,X) - \frac{1}{3} Rg(U,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X) \\
&+ g(U,\nabla\varphi)g(X,X)Q(X;\Box,\Box) - g(U,X)g(\nabla\varphi,X)Q(X;\Box,\Box)\Big] \\
= &\frac{(-1)^q}{3} R \Big[2g(U,\nabla\varphi)Q(X;X,X) + 2g(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&- g(U,X)Q(\nabla\varphi;X,X) - g(\nabla\varphi,\nabla^2\varphi)g(U,X)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&- g(\nabla\varphi,X)Q(U;X,X) - g(U,\nabla^2\varphi)g(\nabla\varphi,X)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&+ g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) - Sg(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{3} Rg(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X) \\
&+ \frac{1}{3} RSg(U,X)B(\Box_a,\Box_a;\Box,\Box;X,X) + \frac{2}{3} Rg(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;X,X) \\
&- \frac{1}{3} Rg(\nabla\varphi,X)B(\Box,\Box;\nabla\varphi,U;X,X) - \frac{1}{3} Rg(U,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X) \\
&+ g(U,\nabla\varphi)g(X,X)Q(X;\Box,\Box) - g(U,X)g(\nabla\varphi,X)Q(X;\Box,\Box)\Big] \\
= (-1)^q &\Big[\frac{1}{3} Rg(U,\nabla\varphi)g(X,X)Q(X;\Box,\Box) - \frac{1}{3} Rg(U,X)g(X,\nabla\varphi)Q(X;\Box,\Box) \\
&+ \frac{2}{3} Rg(U,\nabla\varphi)Q(X;X,X) - \frac{1}{3} Rg(U,X)Q(\nabla\varphi;X,X) \\
&- \frac{1}{3} Rg(X,\nabla\varphi)Q(U;X,X) + \frac{2}{3} Rg(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&- \frac{1}{3} Rg(U,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&- \frac{1}{3} Rg(U,\nabla^2\varphi)g(X,\nabla\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&+ \frac{1}{3} Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) - \frac{1}{3} RSg(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) \\
&- \frac{1}{9} R^2 g(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X) \\
&+ \frac{1}{9} R^2 Sg(U,X)B(\Box_a,\Box_a;\Box,\Box;X,X) + \frac{2}{9} R^2 g(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;X,X) \\
&- \frac{1}{9} R^2 g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,U;X,X) - \frac{1}{9} R^2 g(U,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X)\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{3} RD(U,\Box,X,X,X;\nabla\varphi,\Box) \\
&\quad= \frac{(-1)^q}{3} R\Big[2g(U,\Box)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) - g(\Box,\nabla^2\varphi)g(U,X)Q(\nabla^2\varphi;X,X) \\
&\qquad - g(U,\nabla^2\varphi)g(\Box,X)Q(\nabla^2\varphi;X,X) + 2g(U,\Box)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) \\
&\qquad - g(\Box,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X) - g(U,\nabla\varphi)g(\Box,X)\frac{\partial B}{\partial\varphi}(X,X) \\
&\qquad - \frac{2}{3} Rg(U,\Box)g(X,\nabla\varphi)B(\Box_a,\Box_a;X,X) + \frac{1}{3} Rg(\Box,\nabla\varphi)g(U,X)B(\Box_a,\Box_a;X,X) \\
&\qquad + \frac{1}{3} Rg(U,\nabla\varphi)g(\Box,X)B(\Box_a,\Box_a;X,X) + \frac{2}{3} Rg(U,\Box)B(\nabla\varphi,X;X,X) \\
&\qquad - \frac{1}{3} Rg(\Box,X)B(\nabla\varphi,U;X,X) - \frac{1}{3} Rg(U,X)B(\nabla\varphi,\Box;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&+ g(U,\square)g(X,X)Q(X) - g(U,X)g(\square,X)Q(X)\Big](;\nabla\varphi,\square) \\
&= \frac{(-1)^q}{3}R\Big[g(U,\square)g(X,\nabla\varphi)Q(\square;X,X) + g(U,\square)g(X,\square)Q(\nabla\varphi;X,X) \\
&+ 2g(U,\square)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,\square;X,X) - \frac{1}{2}g(\square,\nabla\varphi)g(U,X)Q(\square;X,X) \\
&- \frac{1}{2}g(\square,\square)g(U,X)Q(\nabla\varphi;X,X) - g(\square,\nabla^2\varphi)g(U,X)Q(\nabla^2\varphi;\nabla\varphi,\square;X,X) \\
&- \frac{1}{2}g(U,\nabla\varphi)g(\square,X)Q(\square;X,X) - \frac{1}{2}g(U,\square)g(\square,X)Q(\nabla\varphi;X,X) \\
&- g(U,\nabla^2\varphi)g(\square,X)Q(\nabla^2\varphi;\nabla\varphi,\square;X,X) + 2g(U,\square)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\square;X,X) \\
&- g(\square,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\square;X,X) - g(U,\nabla\varphi)g(\square,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\square;X,X) \\
&- \frac{2}{3}Rg(U,\square)g(X,\nabla\varphi)B(\square_a,\square_a;\nabla\varphi,\square;X,X) \\
&+ \frac{1}{3}Rg(\square,\nabla\varphi)g(U,X)B(\square_a,\square_a;\nabla\varphi,\square;X,X) \\
&+ \frac{1}{3}Rg(U,\nabla\varphi)g(\square,X)B(\square_a,\square_a;\nabla\varphi,\square;X,X) \\
&+ \frac{2}{3}Rg(U,\square)B(\nabla\varphi,\square;\nabla\varphi,X;X,X) - \frac{1}{3}Rg(\square,X)B(\nabla\varphi,\square;\nabla\varphi,U;X,X) \\
&- \frac{1}{3}Rg(U,X)B(\nabla\varphi,\square;\nabla\varphi,\square;X,X) + g(U,\square)g(X,X)Q(X;\nabla\varphi,\square) \\
&- g(U,X)g(\square,X)Q(X;\nabla\varphi,\square)\Big] \\
&= \frac{(-1)^q}{3}R\Big[g(X,\nabla\varphi)Q(U;X,X) + g(U,X)Q(\nabla\varphi;X,X) \\
&+ 2g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;X,X) - \frac{1}{2}g(U,X)Q(\nabla\varphi;X,X) \\
&- g(U,X)Q(\nabla\varphi;X,X) - g(U,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X) \\
&- \frac{1}{2}g(U,\nabla\varphi)Q(X;X,X) - \frac{1}{2}g(U,X)Q(\nabla\varphi;X,X) \\
&- g(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;X,X) + 2g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X) \\
&- g(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X) - g(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X) \\
&- \frac{2}{3}Rg(X,\nabla\varphi)B(\square_a,\square_a;\nabla\varphi,U;X,X) + \frac{1}{3}Rg(U,X)B(\square_a,\square_a;\nabla\varphi,\nabla\varphi;X,X) \\
&+ \frac{1}{3}Rg(U,\nabla\varphi)B(\square_a,\square_a;\nabla\varphi,X;X,X) + \frac{2}{3}RB(\nabla\varphi,U;\nabla\varphi,X;X,X) \\
&- \frac{1}{3}RB(\nabla\varphi,X;\nabla\varphi,U;X,X) - \frac{1}{3}Rg(U,X)B(\nabla\varphi,\square;\nabla\varphi,\square;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + g(X,X)Q(X;\nabla\varphi,U) - g(U,X)Q(X;\nabla\varphi,X)\Big] \\
&= (-1)^q \Big[\frac{1}{3}Rg(X,X)Q(X;\nabla\varphi,U) - \frac{1}{3}Rg(U,X)Q(X;\nabla\varphi,X) \\
&\quad + \frac{1}{3}Rg(X,\nabla\varphi)Q(U;X,X) - \frac{1}{3}Rg(U,X)Q(\nabla\varphi;X,X) - \frac{1}{6}Rg(U,\nabla\varphi)Q(X;X,X) \\
&\quad + \frac{2}{3}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;X,X) - \frac{1}{3}Rg(U,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X) \\
&\quad - \frac{1}{3}Rg(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;X,X) + \frac{2}{3}Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X) \\
&\quad - \frac{1}{3}Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X) - \frac{1}{3}Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X) \\
&\quad - \frac{2}{9}R^2g(X,\nabla\varphi)B(\Box_a,\Box_a;\nabla\varphi,U;X,X) + \frac{1}{9}R^2g(U,X)B(\Box_a,\Box_a;\nabla\varphi,\nabla\varphi;X,X) \\
&\quad + \frac{1}{9}R^2g(U,\nabla\varphi)B(\Box_a,\Box_a;\nabla\varphi,X;X,X) + \frac{1}{9}R^2B(\nabla\varphi,U;\nabla\varphi,X;X,X) \\
&\quad - \frac{1}{9}R^2g(U,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,X)\Big].
\end{aligned}
$$

Next, we expand the D term which is differentiated with respect to φ_i

$$
\begin{aligned}
&D(U,\nabla^2\varphi,X,X,X;\nabla^2\varphi) \\
&\quad = (-1)^q \Big[2g(U,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;X,X) - g(\nabla^2\varphi,\nabla_a^2\varphi)g(U,X)Q(\nabla_a^2\varphi;X,X) \\
&\qquad - g(U,\nabla_a^2\varphi)g(\nabla^2\varphi,X)Q(\nabla_a^2\varphi;X,X) + 2g(U,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) \\
&\qquad - g(\nabla^2\varphi,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X) - g(\nabla^2\varphi,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X) \\
&\qquad - \frac{2}{3}Rg(U,\nabla^2\varphi)g(X,\nabla\varphi)B(\Box,\Box;X,X) + \frac{1}{3}Rg(\nabla^2\varphi,\nabla\varphi)g(U,X)B(\Box,\Box;X,X) \\
&\qquad + \frac{1}{3}Rg(U,\nabla\varphi)g(\nabla^2\varphi,X)B(\Box,\Box;X,X) + \frac{2}{3}Rg(U,\nabla^2\varphi)B(\nabla\varphi,X;X,X) \\
&\qquad - \frac{1}{3}Rg(\nabla^2\varphi,X)B(\nabla\varphi,U;X,X) - \frac{1}{3}Rg(U,X)B(\nabla\varphi,\nabla^2\varphi;X,X) \\
&\qquad + g(U,\nabla^2\varphi)g(X,X)Q(X) - g(U,X)g(\nabla^2\varphi,X)Q(X)\Big](;\nabla^2\varphi) \\
&\quad = (-1)^q \Big[2g(U,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X) \\
&\qquad - g(\nabla^2\varphi,\nabla_a^2\varphi)g(U,X)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X) \\
&\qquad - g(U,\nabla_a^2\varphi)g(\nabla^2\varphi,X)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&+2g(U,\nabla^2\varphi)g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X)+2g(U,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X;\nabla^2\varphi)\\
&-g(\nabla^2\varphi,\nabla^2\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X)-g(\nabla^2\varphi,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X;\nabla^2\varphi)\\
&-g(\nabla^2\varphi,\nabla^2\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X)-g(\nabla^2\varphi,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(X,X;\nabla^2\varphi)\\
&-\frac{2}{3}Rg(U,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\Box,\Box;X,X)\\
&-\frac{2}{3}Rg(U,\nabla^2\varphi)g(X,\nabla\varphi)B(\Box,\Box;X,X;\nabla^2\varphi)\\
&+\frac{1}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)g(U,X)B(\Box,\Box;X,X)\\
&+\frac{1}{3}Rg(\nabla^2\varphi,\nabla\varphi)g(U,X)B(\Box,\Box;X,X;\nabla^2\varphi)\\
&+\frac{1}{3}Rg(U,\nabla^2\varphi)g(\nabla^2\varphi,X)B(\Box,\Box;X,X)\\
&+\frac{1}{3}Rg(U,\nabla\varphi)g(\nabla^2\varphi,X)B(\Box,\Box;X,X;\nabla^2\varphi)\\
&+\frac{2}{3}Rg(U,\nabla^2\varphi)B(\nabla^2\varphi,X;X,X)+\frac{2}{3}Rg(U,\nabla^2\varphi)B(\nabla\varphi,X;X,X;\nabla^2\varphi)\\
&-\frac{1}{3}Rg(\nabla^2\varphi,X)B(\nabla^2\varphi,U;X,X)-\frac{1}{3}Rg(\nabla^2\varphi,X)B(\nabla\varphi,U;X,X;\nabla^2\varphi)\\
&-\frac{1}{3}Rg(U,X)B(\nabla^2\varphi,\nabla^2\varphi;X,X)-\frac{1}{3}Rg(U,X)B(\nabla\varphi,\nabla^2\varphi;X,X;\nabla^2\varphi)\\
&+g(U,\nabla^2\varphi)g(X,X)Q(X;\nabla^2\varphi)-g(U,X)g(\nabla^2\varphi,X)Q(X;\nabla^2\varphi)\Big]\\
=(-1)^q&\Big[g(U,\nabla^2\varphi)g(X,X)Q(\nabla^2\varphi;X)-g(U,X)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X)\\
&+2g(U,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)-2g(U,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)\\
&+\frac{2}{3}Rg(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;X,X)-\frac{1}{3}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;X,X)\\
&-\frac{1}{3}Rg(U,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X)+g(U,\nabla^2\varphi)g(X,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi;X,X)\\
&-g(U,X)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi;X,X)\\
&-\frac{2}{3}Rg(U,\nabla^2\varphi)g(X,\nabla\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)\\
&+\frac{1}{3}Rg(U,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)\\
&+\frac{1}{3}Rg(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)+2g(U,\nabla^2\varphi)g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X)\\
&-2g(U,X)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X)-\frac{1}{3}Rg(U,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\Box,\Box;X,X)
\end{aligned}
$$

$$+\frac{1}{3}Rg(U,X)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;X,X)+\frac{2}{3}Rg(U,\nabla^2\varphi)B(\nabla^2\varphi,X;X,X)$$
$$\left.-\frac{1}{3}Rg(X,\nabla^2\varphi)B(\nabla^2\varphi,U;X,X)-\frac{1}{3}Rg(U,X)B(\nabla^2\varphi,\nabla^2\varphi;X,X)\right].$$

Finally, we produce the D term differentiated with respect to φ

$$\begin{aligned}
&\frac{\partial D}{\partial\varphi}(U,\nabla\varphi,X,X,X)\\
&\quad=(-1)^q\Big[2g(U,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)-g(\nabla\varphi,\nabla^2\varphi)g(U,X)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)\\
&\qquad-g(U,\nabla^2\varphi)g(\nabla\varphi,X)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)+2g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial^2 B}{\partial\varphi^2}(X,X)\\
&\qquad-g(\nabla\varphi,\nabla\varphi)g(U,X)\frac{\partial^2 B}{\partial\varphi^2}(X,X)-g(U,\nabla\varphi)g(\nabla\varphi,X)\frac{\partial^2 B}{\partial\varphi^2}(X,X)\\
&\qquad-\frac{2}{3}Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)+\frac{1}{3}Rg(\nabla\varphi,\nabla\varphi)g(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)\\
&\qquad+\frac{1}{3}Rg(U,\nabla\varphi)g(\nabla\varphi,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)+\frac{2}{3}Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X)\\
&\qquad-\frac{1}{3}Rg(\nabla\varphi,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X)-\frac{1}{3}Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X)\\
&\qquad+g(U,\nabla\varphi)g(X,X)\frac{\partial Q}{\partial\varphi}(X)-g(U,X)g(\nabla\varphi,X)\frac{\partial Q}{\partial\varphi}(X)\Big]\\
&\quad=(-1)^q\Big[g(U,\nabla\varphi)g(X,X)\frac{\partial Q}{\partial\varphi}(X)-g(U,X)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(X)\\
&\qquad+2g(U,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)-g(U,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)\\
&\qquad-g(U,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)+g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial^2 B}{\partial\varphi^2}(X,X)\\
&\qquad-Sg(U,X)\frac{\partial^2 B}{\partial\varphi^2}(X,X)-\frac{1}{3}Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)\\
&\qquad+\frac{1}{3}RSg(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)+\frac{2}{3}Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X)\\
&\qquad-\frac{1}{3}Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X)-\frac{1}{3}Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X)\Big].
\end{aligned}$$

We sum these four terms together using the coefficients from (C.11)

$$\begin{aligned}
&\frac{1}{3}R\left[D(U,\nabla\varphi,X,X,X;\Box,\Box)-D(U,\Box,X,X,X;\Box,\nabla\varphi)\right]-D(U,\nabla^2\varphi,X,X,X;\nabla^2\varphi)\\
&\quad-\frac{\partial D}{\partial\varphi}(U,\nabla\varphi,X,X,X)
\end{aligned}$$

$$
\begin{aligned}
= (-1)^q \Bigg[&\frac{1}{3} Rg(U,\nabla\varphi) g(X,X) Q(X;\Box,\Box) - \frac{1}{3} Rg(U,X) g(X,\nabla\varphi) Q(X;\Box,\Box) \\
&+ \frac{2}{3} Rg(U,\nabla\varphi) Q(X;X,X) - \frac{1}{3} Rg(U,X) Q(\nabla\varphi;X,X) \\
&- \frac{1}{3} Rg(X,\nabla\varphi) Q(U;X,X) + \frac{2}{3} Rg(U,\nabla\varphi) g(X,\nabla^2\varphi) Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&- \frac{1}{3} Rg(U,X) g(\nabla\varphi,\nabla^2\varphi) Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&- \frac{1}{3} Rg(U,\nabla^2\varphi) g(X,\nabla\varphi) Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&+ \frac{1}{3} Rg(U,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) - \frac{1}{3} RSg(U,X) \frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) \\
&- \frac{1}{9} R^2 g(U,\nabla\varphi) g(X,\nabla\varphi) B(\Box_a,\Box_a;\Box,\Box;X,X) \\
&+ \frac{1}{9} R^2 Sg(U,X) B(\Box_a,\Box_a;\Box,\Box;X,X) + \frac{2}{9} R^2 g(U,\nabla\varphi) B(\Box,\Box;\nabla\varphi,X;X,X) \\
&- \frac{1}{9} R^2 g(X,\nabla\varphi) B(\Box,\Box;\nabla\varphi,U;X,X) - \frac{1}{9} R^2 g(U,X) B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X) \Bigg] \\
&- (-1)^q \Bigg[\frac{1}{3} Rg(X,X) Q(X;\nabla\varphi,U) - \frac{1}{3} Rg(U,X) Q(X;\nabla\varphi,X) \\
&+ \frac{1}{3} Rg(X,\nabla\varphi) Q(U;X,X) - \frac{1}{3} Rg(U,X) Q(\nabla\varphi;X,X) - \frac{1}{6} Rg(U,\nabla\varphi) Q(X;X,X) \\
&+ \frac{2}{3} Rg(X,\nabla^2\varphi) Q(\nabla^2\varphi;\nabla\varphi,U;X,X) - \frac{1}{3} Rg(U,X) Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X) \\
&- \frac{1}{3} Rg(U,\nabla^2\varphi) Q(\nabla^2\varphi;\nabla\varphi,X;X,X) + \frac{2}{3} Rg(X,\nabla\varphi) \frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X) \\
&- \frac{1}{3} Rg(U,X) \frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X) - \frac{1}{3} Rg(U,\nabla\varphi) \frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X) \\
&- \frac{2}{9} R^2 g(X,\nabla\varphi) B(\Box_a,\Box_a;\nabla\varphi,U;X,X) + \frac{1}{9} R^2 g(U,X) B(\Box_a,\Box_a;\nabla\varphi,\nabla\varphi;X,X) \\
&+ \frac{1}{9} R^2 g(U,\nabla\varphi) B(\Box_a,\Box_a;\nabla\varphi,X;X,X) + \frac{1}{9} R^2 B(\nabla\varphi,U;\nabla\varphi,X;X,X) \\
&- \frac{1}{9} R^2 g(U,X) B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,X) \Bigg] \\
&- (-1)^q \Bigg[g(U,\nabla^2\varphi) g(X,X) Q(\nabla^2\varphi;X) - g(U,X) g(X,\nabla^2\varphi) Q(\nabla^2\varphi;X) \\
&+ 2g(U,\nabla^2\varphi) g(X,\nabla\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) - 2g(U,X) g(\nabla\varphi,\nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) \\
&+ \frac{2}{3} Rg(U,\nabla^2\varphi) Q(\nabla^2\varphi;\nabla\varphi,X;X,X) - \frac{1}{3} Rg(X,\nabla^2\varphi) Q(\nabla^2\varphi;\nabla\varphi,U;X,X) \\
&- \frac{1}{3} Rg(U,X) Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X) + g(U,\nabla^2\varphi) g(X,\nabla_a^2\varphi) Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X) \\
&- g(U,X) g(\nabla^2\varphi,\nabla_a^2\varphi) Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{3} R g(U, \nabla^2\varphi) g(X, \nabla\varphi) Q(\nabla^2\varphi; \square, \square; X, X) \\
&+ \frac{1}{3} R g(U, X) g(\nabla\varphi, \nabla^2\varphi) Q(\nabla^2\varphi; \square, \square; X, X) \\
&+ \frac{1}{3} R g(U, \nabla\varphi) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; \square, \square; X, X) + 2g(U, \nabla^2\varphi) g(X, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X, X) \\
&- 2g(U, X) g(\nabla^2\varphi, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X, X) - \frac{1}{3} R g(U, \nabla^2\varphi) g(X, \nabla^2\varphi) B(\square, \square; X, X) \\
&+ \frac{1}{3} R g(U, X) g(\nabla^2\varphi, \nabla^2\varphi) B(\square, \square; X, X) + \frac{2}{3} R g(U, \nabla^2\varphi) B(\nabla^2\varphi, X; X, X) \\
&- \frac{1}{3} R g(X, \nabla^2\varphi) B(\nabla^2\varphi, U; X, X) - \frac{1}{3} R g(U, X) B(\nabla^2\varphi, \nabla^2\varphi; X, X)\Big] \\
&- (-1)^q \Big[g(U, \nabla\varphi) g(X, X) \frac{\partial Q}{\partial\varphi}(X) - g(U, X) g(\nabla\varphi, X) \frac{\partial Q}{\partial\varphi}(X) \\
&+ 2g(U, \nabla\varphi) g(X, \nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) - g(U, X) g(\nabla\varphi, \nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) \\
&- g(U, \nabla^2\varphi) g(X, \nabla\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) + g(U, \nabla\varphi) g(X, \nabla\varphi) \frac{\partial^2 B}{\partial\varphi^2}(X, X) \\
&- S g(U, X) \frac{\partial^2 B}{\partial\varphi^2}(X, X) - \frac{1}{3} R g(U, \nabla\varphi) g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(\square, \square; X, X) \\
&+ \frac{1}{3} R S g(U, X) \frac{\partial B}{\partial\varphi}(\square, \square; X, X) + \frac{2}{3} R g(U, \nabla\varphi) \frac{\partial B}{\partial\varphi}(\nabla\varphi, X; X, X) \\
&- \frac{1}{3} R g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(\nabla\varphi, U; X, X) - \frac{1}{3} R g(U, X) \frac{\partial B}{\partial\varphi}(\nabla\varphi, \nabla\varphi; X, X)\Big] \\
= {}& (-1)^q \Big[g(U, X) g(X, \nabla\varphi) \frac{\partial Q}{\partial\varphi}(X) - g(U, \nabla\varphi) g(X, X) \frac{\partial Q}{\partial\varphi}(X) \\
&+ g(U, X) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; X) - g(U, \nabla^2\varphi) g(X, X) Q(\nabla^2\varphi; X) \\
&- \frac{1}{3} R g(U, X) g(X, \nabla\varphi) Q(X; \square, \square) + \frac{1}{3} R g(U, \nabla\varphi) g(X, X) Q(X; \square, \square) \\
&+ \frac{1}{3} R g(U, X) Q(X; \nabla\varphi, X) - \frac{1}{3} R g(X, X) Q(X; \nabla\varphi, U) \\
&+ \frac{5}{6} R g(U, \nabla\varphi) Q(X; X, X) - \frac{2}{3} R g(X, \nabla\varphi) Q(U; X, X) \\
&+ 3g(U, X) g(\nabla\varphi, \nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) - g(U, \nabla^2\varphi) g(X, \nabla\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) \\
&- 2g(U, \nabla\varphi) g(X, \nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) + g(U, X) g(\nabla^2\varphi, \nabla_a^2\varphi) Q(\nabla_a^2\varphi; \nabla^2\varphi; X, X) \\
&- g(U, \nabla^2\varphi) g(X, \nabla_a^2\varphi) Q(\nabla_a^2\varphi; \nabla^2\varphi; X, X) + \frac{2}{3} R g(U, X) Q(\nabla^2\varphi; \nabla\varphi, \nabla^2\varphi; X, X) \\
&- \frac{1}{3} R g(U, \nabla^2\varphi) Q(\nabla^2\varphi; \nabla\varphi, X; X, X) - \frac{1}{3} R g(X, \nabla^2\varphi) Q(\nabla^2\varphi; \nabla\varphi, U; X, X) \\
&- \frac{2}{3} R g(U, X) g(\nabla\varphi, \nabla^2\varphi) Q(\nabla^2\varphi; \square, \square; X, X)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{3} R g(U, \nabla^2\varphi) g(X, \nabla\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) \\
&+ \frac{1}{3} R g(U, \nabla\varphi) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) + 2g(U, X) g(\nabla^2\varphi, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X, X) \\
&- 2g(U, \nabla^2\varphi) g(X, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X, X) + S g(U, X) \frac{\partial^2 B}{\partial\varphi^2}(X, X) \\
&- g(U, \nabla\varphi) g(X, \nabla\varphi) \frac{\partial^2 B}{\partial\varphi^2}(X, X) - \frac{1}{3} R g(U, X) g(\nabla^2\varphi, \nabla^2\varphi) B(\Box, \Box; X, X) \\
&+ \frac{1}{3} R g(U, \nabla^2\varphi) g(X, \nabla^2\varphi) B(\Box, \Box; X, X) + \frac{1}{3} R g(U, X) B(\nabla^2\varphi, \nabla^2\varphi; X, X) \\
&- \frac{2}{3} R g(U, \nabla^2\varphi) B(\nabla^2\varphi, X; X, X) + \frac{1}{3} R g(X, \nabla^2\varphi) B(\nabla^2\varphi, U; X, X) \\
&- \frac{2}{3} R S g(U, X) \frac{\partial B}{\partial\varphi}(\Box, \Box; X, X) + \frac{2}{3} R g(U, \nabla\varphi) g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(\Box, \Box; X, X) \\
&+ \frac{2}{3} R g(U, X) \frac{\partial B}{\partial\varphi}(\nabla\varphi, \nabla\varphi; X, X) - \frac{1}{3} R g(U, \nabla\varphi) \frac{\partial B}{\partial\varphi}(\nabla\varphi, X; X, X) \\
&- \frac{1}{3} R g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(\nabla\varphi, U; X, X) + \frac{1}{9} R^2 S g(U, X) B(\Box_a, \Box_a; \Box, \Box; X, X) \\
&- \frac{1}{9} R^2 g(U, \nabla\varphi) g(X, \nabla\varphi) B(\Box_a, \Box_a; \Box, \Box; X, X) \\
&+ \frac{1}{9} R^2 g(U, \nabla\varphi) B(\Box, \Box; \nabla\varphi, X; X, X) + \frac{1}{9} R^2 g(X, \nabla\varphi) B(\Box, \Box; \nabla\varphi, U; X, X) \\
&+ \frac{1}{9} R^2 g(U, X) B(\nabla\varphi, \Box; \nabla\varphi, \Box; X, X) - \frac{2}{9} R^2 g(U, X) B(\Box, \Box; \nabla\varphi, \nabla\varphi; X, X) \\
&- \frac{1}{9} R^2 B(\nabla\varphi, U; \nabla\varphi, X; X, X) \Big].
\end{aligned}
$$

We combine this result with the remaining terms in (C.11), yielding

$$
\begin{aligned}
&E(U, X; X, X) \\
&\quad = \frac{(-1)^q}{6} \big[15 R g(U, X) B(X, X) - 10 R g(X, X) B(U, X) + 12 R g(U, \nabla^2\varphi) B(\nabla^2\varphi, X; X, X) \\
&\qquad + 2 R g(U, X) B(\nabla^2\varphi, \nabla^2\varphi; X, X) - 12 R g(\nabla^2\varphi, X) B(\nabla^2\varphi, U; X, X) \\
&\qquad + 2 R g(U, \nabla^2\varphi) g(X, \nabla^2\varphi) B(\Box, \Box; X, X) - 2 R g(U, X) g(\nabla^2\varphi, \nabla^2\varphi) B(\Box, \Box; X, X) \\
&\qquad + g(U, \nabla\varphi) g(X, \nabla R) B(\Box, \Box; X, X) - \frac{1}{2} g(U, \nabla R) g(X, \nabla\varphi) B(\Box, \Box; X, X) \\
&\qquad - \frac{1}{2} g(U, X) g(\nabla\varphi, \nabla R) B(\Box, \Box; X, X) + \frac{1}{2} g(U, X) B(\nabla\varphi, \nabla R; X, X) \\
&\qquad - 2 g(X, \nabla\varphi) B(\nabla R, U; X, X) - g(X, \nabla R) B(\nabla\varphi, U; X, X) \\
&\qquad + \frac{5}{2} g(U, \nabla\varphi) B(\nabla R, X; X, X) + \frac{1}{2} g(U, \nabla R) B(\nabla\varphi, X; X, X) \Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ (-1)^q \Big[g(U,X)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(X) - g(U,\nabla\varphi)g(X,X)\frac{\partial Q}{\partial\varphi}(X) \\
&+ g(U,X)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X) - g(U,\nabla^2\varphi)g(X,X)Q(\nabla^2\varphi;X) \\
&- \frac{1}{3}Rg(U,X)g(X,\nabla\varphi)Q(X;\Box,\Box) + \frac{1}{3}Rg(U,\nabla\varphi)g(X,X)Q(X;\Box,\Box) \\
&+ \frac{1}{3}Rg(U,X)Q(X;\nabla\varphi,X) - \frac{1}{3}Rg(X,X)Q(X;\nabla\varphi,U) \\
&+ \frac{5}{6}Rg(U,\nabla\varphi)Q(X;X,X) - \frac{2}{3}Rg(X,\nabla\varphi)Q(U;X,X) \\
&+ 3g(U,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) - g(U,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) \\
&- 2g(U,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) + g(U,X)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi;X,X) \\
&- g(U,\nabla^2\varphi)g(X,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi;X,X) + \frac{2}{3}Rg(U,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X) \\
&- \frac{1}{3}Rg(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;X,X) - \frac{1}{3}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;X,X) \\
&- \frac{2}{3}Rg(U,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&+ \frac{1}{3}Rg(U,\nabla^2\varphi)g(X,\nabla\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) \\
&+ \frac{1}{3}Rg(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) + 2g(U,X)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X) \\
&- 2g(U,\nabla^2\varphi)g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X) + Sg(U,X)\frac{\partial^2 B}{\partial\varphi^2}(X,X) \\
&- g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial^2 B}{\partial\varphi^2}(X,X) - \frac{1}{3}Rg(U,X)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;X,X) \\
&+ \frac{1}{3}Rg(U,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\Box,\Box;X,X) + \frac{1}{3}Rg(U,X)B(\nabla^2\varphi,\nabla^2\varphi;X,X) \\
&- \frac{2}{3}Rg(U,\nabla^2\varphi)B(\nabla^2\varphi,X;X,X) + \frac{1}{3}Rg(X,\nabla^2\varphi)B(\nabla^2\varphi,U;X,X) \\
&- \frac{2}{3}RSg(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) + \frac{2}{3}Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) \\
&+ \frac{2}{3}Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X) - \frac{1}{3}Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X) \\
&- \frac{1}{3}Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X) + \frac{1}{9}R^2Sg(U,X)B(\Box_a,\Box_a;\Box,\Box;X,X) \\
&- \frac{1}{9}R^2g(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X) \\
&+ \frac{1}{9}R^2g(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;X,X) + \frac{1}{9}R^2g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,U;X,X) \\
&+ \frac{1}{9}R^2g(U,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,X) - \frac{2}{9}R^2g(U,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \frac{1}{9}R^2 B(\nabla\varphi, U; \nabla\varphi, X; X, X)\Big] \\
&= (-1)^q \Big[g(U,X) g(X,\nabla\varphi) \frac{\partial Q}{\partial\varphi}(X) - g(U,\nabla\varphi) g(X,X) \frac{\partial Q}{\partial\varphi}(X) \\
&\quad + g(U,X) g(X,\nabla^2\varphi) Q(\nabla^2\varphi; X) - g(U,\nabla^2\varphi) g(X,X) Q(\nabla^2\varphi; X) \\
&\quad - \frac{1}{3} R g(U,X) g(X,\nabla\varphi) Q(X; \Box, \Box) + \frac{1}{3} R g(U,\nabla\varphi) g(X,X) Q(X; \Box, \Box) \\
&\quad + \frac{1}{3} R g(U,X) Q(X; \nabla\varphi, X) - \frac{1}{3} R g(X,X) Q(X; \nabla\varphi, U) \\
&\quad + \frac{5}{6} R g(U,\nabla\varphi) Q(X; X, X) - \frac{2}{3} R g(X,\nabla\varphi) Q(U; X, X) \\
&\quad + 3 g(U,X) g(\nabla\varphi,\nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) - g(U,\nabla^2\varphi) g(X,\nabla\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) \\
&\quad - 2 g(U,\nabla\varphi) g(X,\nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi; X, X) + g(U,X) g(\nabla^2\varphi, \nabla^2_a\varphi) Q(\nabla^2_a\varphi; \nabla^2\varphi; X, X) \\
&\quad - g(U,\nabla^2\varphi) g(X,\nabla^2_a\varphi) Q(\nabla^2_a\varphi; \nabla^2\varphi; X, X) + \frac{2}{3} R g(U,X) Q(\nabla^2\varphi; \nabla\varphi, \nabla^2\varphi; X, X) \\
&\quad - \frac{1}{3} R g(U,\nabla^2\varphi) Q(\nabla^2\varphi; \nabla\varphi, X; X, X) - \frac{1}{3} R g(X,\nabla^2\varphi) Q(\nabla^2\varphi; \nabla\varphi, U; X, X) \\
&\quad - \frac{2}{3} R g(U,X) g(\nabla\varphi,\nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) \\
&\quad + \frac{1}{3} R g(U,\nabla^2\varphi) g(X,\nabla\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) \\
&\quad + \frac{1}{3} R g(U,\nabla\varphi) g(X,\nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) + \frac{5}{2} R g(U,X) B(X,X) \\
&\quad - \frac{5}{3} R g(X,X) B(U,X) + \frac{1}{3} g(U,X) g(\nabla^2\varphi,\nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X,X) \\
&\quad - 2 g(U,\nabla^2\varphi) g(X,\nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X,X) + S g(U,X) \frac{\partial^2 B}{\partial\varphi^2}(X,X) \\
&\quad - g(U,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial^2 B}{\partial\varphi^2}(X,X) + \frac{2}{3} R g(U,X) B(\nabla^2\varphi, \nabla^2\varphi; X, X) \\
&\quad + \frac{4}{3} R g(U,\nabla^2\varphi) B(\nabla^2\varphi, X; X, X) - \frac{5}{3} R g(X,\nabla^2\varphi) B(\nabla^2\varphi, U; X, X) \\
&\quad - \frac{2}{3} R g(U,X) g(\nabla^2\varphi,\nabla^2\varphi) B(\Box, \Box; X, X) + \frac{2}{3} R g(U,\nabla^2\varphi) g(X,\nabla^2\varphi) B(\Box, \Box; X, X) \\
&\quad - \frac{1}{12} g(U,X) g(\nabla\varphi,\nabla R) B(\Box, \Box; X, X) - \frac{1}{12} g(U,\nabla R) g(X,\nabla\varphi) B(\Box, \Box; X, X) \\
&\quad + \frac{1}{6} g(U,\nabla\varphi) g(X,\nabla R) B(\Box, \Box; X, X) + \frac{1}{12} g(U,X) B(\nabla\varphi, \nabla R; X, X) \\
&\quad - \frac{1}{3} g(X,\nabla\varphi) B(\nabla R, U; X, X) - \frac{1}{6} g(X,\nabla R) B(\nabla\varphi, U; X, X) \\
&\quad + \frac{5}{12} g(U,\nabla\varphi) B(\nabla R, X; X, X) + \frac{1}{12} g(U,\nabla R) B(\nabla\varphi, X; X, X)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{3} RSg(U,X)\frac{\partial B}{\partial \varphi}(\square,\square;X,X) + \frac{2}{3}Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial \varphi}(\square,\square;X,X)\\
&+ \frac{2}{3}Rg(U,X)\frac{\partial B}{\partial \varphi}(\nabla\varphi,\nabla\varphi;X,X) - \frac{1}{3}Rg(U,\nabla\varphi)\frac{\partial B}{\partial \varphi}(\nabla\varphi,X;X,X)\\
&- \frac{1}{3}Rg(X,\nabla\varphi)\frac{\partial B}{\partial \varphi}(\nabla\varphi,U;X,X) + \frac{1}{9}R^2Sg(U,X)B(\square_a,\square_a;\square,\square;X,X)\\
&- \frac{1}{9}R^2g(U,\nabla\varphi)g(X,\nabla\varphi)B(\square_a,\square_a;\square,\square;X,X)\\
&+ \frac{1}{9}R^2g(U,\nabla\varphi)B(\square,\square;\nabla\varphi,X;X,X) + \frac{1}{9}R^2g(X,\nabla\varphi)B(\square,\square;\nabla\varphi,U;X,X)\\
&+ \frac{1}{9}R^2g(U,X)B(\nabla\varphi,\square;\nabla\varphi,\square;X,X) - \frac{2}{9}R^2g(U,X)B(\square,\square;\nabla\varphi,\nabla\varphi;X,X)\\
&- \frac{1}{9}R^2B(\nabla\varphi,U;\nabla\varphi,X;X,X)\Bigg]. \qquad \text{(C.13)}
\end{aligned}
$$

Lemma 5. *If E^{ij} obeys (C.13), then*

$$
E(U,U;X,X) = \tilde{E}(U,U;X,X) - \det(U,X)^2P, \tag{C.14}
$$

where $\tilde{E}^{ij;kl}$ is defined below and P is a scalar density which is of scalar order 2.

Proof. We start with the following ansatz

$$
\begin{aligned}
&\tilde{E}(U,U;X,X)\\
&\quad = (-1)^q\Bigg[a_1g(U,U)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(X) + a_2g(U,X)g(U,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(X)\\
&\qquad + a_3g(U,X)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(U) + a_4g(U,\nabla\varphi)g(X,X)\frac{\partial Q}{\partial\varphi}(U)\\
&\qquad + a_5g(U,U)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X) + a_6g(U,X)g(U,\nabla^2\varphi)Q(\nabla^2\varphi;X)\\
&\qquad + a_7g(U,X)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;U) + a_8g(U,\nabla^2\varphi)g(X,X)Q(\nabla^2\varphi;U)\\
&\qquad + a_9Rg(U,U)g(X,\nabla\varphi)Q(X;\square,\square) + a_{10}Rg(U,X)g(U,\nabla\varphi)Q(X;\square,\square)\\
&\qquad + a_{11}Rg(U,X)g(X,\nabla\varphi)Q(U;\square,\square) + a_{12}Rg(U,\nabla\varphi)g(X,X)Q(U;\square,\square)\\
&\qquad + a_{13}Rg(U,U)Q(X;\nabla\varphi,X) + a_{14}Rg(U,X)Q(U;\nabla\varphi,X)\\
&\qquad + a_{15}Rg(U,X)Q(X;\nabla\varphi,U) + a_{16}Rg(X,X)Q(U;\nabla\varphi,U)\\
&\qquad + a_{17}Rg(U,\nabla\varphi)Q(U;X,X) + a_{18}Rg(U,\nabla\varphi)Q(X;U,X)
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{19}Rg(X,\nabla\varphi)Q(U;U,X) + a_{20}Rg(X,\nabla\varphi)Q(X;U,U)\\
&+ a_{21}g(U,U)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) + a_{22}g(U,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X)\\
&+ a_{23}g(X,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,U) + a_{24}g(U,\nabla\varphi)g(U,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)\\
&+ a_{25}g(U,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X) + a_{26}g(U,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X)\\
&+ a_{27}g(X,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,U)\\
&+ a_{28}g(U,U)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)\\
&+ a_{29}g(U,X)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,X)\\
&+ a_{30}g(X,X)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,U)\\
&+ a_{31}g(U,\nabla^2\varphi)g(U,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)\\
&+ a_{32}g(U,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,X)\\
&+ a_{33}g(X,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,U)\\
&+ a_{34}Rg(U,U)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X) + a_{35}Rg(U,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;U,X)\\
&+ a_{36}Rg(X,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;U,U) + a_{37}Rg(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;X,X)\\
&+ a_{38}Rg(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;U,X) + a_{39}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;U,X)\\
&+ a_{40}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;U,U)\\
&+ a_{41}Rg(U,U)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)\\
&+ a_{42}Rg(U,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;U,X)\\
&+ a_{43}Rg(X,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;U,U)\\
&+ a_{44}Rg(U,\nabla\varphi)g(U,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)\\
&+ a_{45}Rg(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;U,X)\\
&+ a_{46}Rg(U,\nabla^2\varphi)g(X,\nabla\varphi)Q(\nabla^2\varphi;\Box,\Box;U,X)\\
&+ a_{47}Rg(X,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;U,U)\\
&+ a_{48}Rg(U,U)B(X,X) + a_{49}Rg(U,X)B(U,X) + a_{50}Rg(X,X)B(U,U)\\
&+ a_{51}g(U,U)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X) + a_{52}g(U,X)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(U,X)
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{53} g(X,X) g(\nabla^2\varphi, \nabla^2\varphi) \frac{\partial B}{\partial \varphi}(U,U) + a_{54} g(U,\nabla^2\varphi) g(U,\nabla^2\varphi) \frac{\partial B}{\partial \varphi}(X,X) \\
&+ a_{55} g(U,\nabla^2\varphi) g(X,\nabla^2\varphi) \frac{\partial B}{\partial \varphi}(U,X) + a_{56} g(X,\nabla^2\varphi) g(X,\nabla^2\varphi) \frac{\partial B}{\partial \varphi}(U,U) \\
&+ a_{57} S g(U,U) \frac{\partial^2 B}{\partial \varphi^2}(X,X) + a_{58} S g(U,X) \frac{\partial^2 B}{\partial \varphi^2}(U,X) + a_{59} S g(X,X) \frac{\partial^2 B}{\partial \varphi^2}(U,U) \\
&+ a_{60} g(U,\nabla\varphi) g(U,\nabla\varphi) \frac{\partial^2 B}{\partial \varphi^2}(X,X) + a_{61} g(U,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial^2 B}{\partial \varphi^2}(U,X) \\
&+ a_{62} g(X,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial^2 B}{\partial \varphi^2}(U,U) + a_{63} R g(U,U) B(\nabla^2\varphi, \nabla^2\varphi; X,X) \\
&+ a_{64} R g(U,X) B(\nabla^2\varphi, \nabla^2\varphi; U,X) + a_{65} R g(X,X) B(\nabla^2\varphi, \nabla^2\varphi; U,U) \\
&+ a_{66} R g(U,\nabla^2\varphi) B(\nabla^2\varphi, U; X,X) + a_{67} R g(X,\nabla^2\varphi) B(\nabla^2\varphi, U; U,X) \\
&+ a_{68} R g(U,\nabla^2\varphi) B(\nabla^2\varphi, X; U,X) + a_{69} R g(X,\nabla^2\varphi) B(\nabla^2\varphi, X; U,U) \\
&+ a_{70} R g(U,U) g(\nabla^2\varphi, \nabla^2\varphi) B(\Box,\Box; X,X) + a_{71} R g(U,X) g(\nabla^2\varphi, \nabla^2\varphi) B(\Box,\Box; U,X) \\
&+ a_{72} R g(X,X) g(\nabla^2\varphi, \nabla^2\varphi) B(\Box,\Box; U,U) + a_{73} R g(U,\nabla^2\varphi) g(U,\nabla^2\varphi) B(\Box,\Box; X,X) \\
&+ a_{74} R g(U,\nabla^2\varphi) g(X,\nabla^2\varphi) B(\Box,\Box; U,X) + a_{75} R g(X,\nabla^2\varphi) g(X,\nabla^2\varphi) B(\Box,\Box; U,U) \\
&+ a_{76} g(U,U) g(\nabla\varphi, \nabla R) B(\Box,\Box; X,X) + a_{77} g(U,X) g(\nabla\varphi, \nabla R) B(\Box,\Box; U,X) \\
&+ a_{78} g(X,X) g(\nabla\varphi, \nabla R) B(\Box,\Box; U,U) + a_{79} g(U,\nabla R) g(U,\nabla\varphi) B(\Box,\Box; X,X) \\
&+ a_{80} g(U,\nabla R) g(X,\nabla\varphi) B(\Box,\Box; U,X) + a_{81} g(U,\nabla\varphi) g(X,\nabla R) B(\Box,\Box; U,X) \\
&+ a_{82} g(X,\nabla R) g(X,\nabla\varphi) B(\Box,\Box; U,U) + a_{83} g(U,U) B(\nabla\varphi, \nabla R; X,X) \\
&+ a_{84} g(U,X) B(\nabla\varphi, \nabla R; U,X) + a_{85} g(X,X) B(\nabla\varphi, \nabla R; U,U) \\
&+ a_{86} g(U,\nabla\varphi) B(\nabla R, U; X,X) + a_{87} g(X,\nabla\varphi) B(\nabla R, U; U,X) \\
&+ a_{88} g(U,\nabla\varphi) B(\nabla R, X; U,X) + a_{89} g(X,\nabla\varphi) B(\nabla R, X; U,U) \\
&+ a_{90} g(U,\nabla R) B(\nabla\varphi, U; X,X) + a_{91} g(X,\nabla R) B(\nabla\varphi, U; U,X) \\
&+ a_{92} g(U,\nabla R) B(\nabla\varphi, X; U,X) + a_{93} g(X,\nabla R) B(\nabla\varphi, X; U,U) \\
&+ a_{94} R S g(U,U) \frac{\partial B}{\partial \varphi}(\Box,\Box; X,X) + a_{95} R S g(U,X) \frac{\partial B}{\partial \varphi}(\Box,\Box; U,X) \\
&+ a_{96} R S g(X,X) \frac{\partial B}{\partial \varphi}(\Box,\Box; U,U) + a_{97} R g(U,\nabla\varphi) g(U,\nabla\varphi) \frac{\partial B}{\partial \varphi}(\Box,\Box; X,X) \\
&+ a_{98} R g(U,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial B}{\partial \varphi}(\Box,\Box; U,X) + a_{99} R g(X,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial B}{\partial \varphi}(\Box,\Box; U,U)
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{100} Rg(U,U)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X) + a_{101} Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;U,X)\\
&+ a_{102} Rg(X,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;U,U) + a_{103} Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X)\\
&+ a_{104} Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;U,X) + a_{105} Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;U,X)\\
&+ a_{106} Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;U,U) + a_{107} R^2 Sg(U,U)B(\Box_a,\Box_a;\Box,\Box;X,X)\\
&+ a_{108} R^2 Sg(U,X)B(\Box_a,\Box_a;\Box,\Box;U,X) + a_{109} R^2 Sg(X,X)B(\Box_a,\Box_a;\Box,\Box;U,U)\\
&+ a_{110} R^2 g(U,\nabla\varphi)g(U,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X)\\
&+ a_{111} R^2 g(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;U,X)\\
&+ a_{112} R^2 g(X,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;U,U)\\
&+ a_{113} R^2 g(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,U;X,X) + a_{114} R^2 g(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;U,X)\\
&+ a_{115} R^2 g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,U;U,X) + a_{116} R^2 g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;U,U)\\
&+ a_{117} R^2 g(U,U)B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,X) + a_{118} R^2 g(U,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;U,X)\\
&+ a_{119} R^2 g(X,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;U,U) + a_{120} R^2 g(U,U)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X)\\
&+ a_{121} R^2 g(U,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;U,X) + a_{122} R^2 g(X,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;U,U)\\
&+ a_{123} R^2 B(\nabla\varphi,U;\nabla\varphi,U;X,X) + a_{124} R^2 B(\nabla\varphi,U;\nabla\varphi,X;U,X)\\
&+ a_{125} R^2 B(\nabla\varphi,X;\nabla\varphi,X;U,U)\Big].
\end{aligned}
$$

where a_i, b_j, c_k are constants to be determined. As with the proof for D^{ijabc}, we define a difference function

$$\Delta(U,U,X,X) = E(U,U;X,X) - \tilde{E}(U,U;X,X)$$

and replace a U with an X, using (C.13) to express $E(U,X;X,X)$

$$
\begin{aligned}
&\Delta(U,X,X,X)\\
&\quad = E(U,X;X,X) - \tilde{E}(U,X;X,X)\\
&\quad = (-1)^q\Big[g(U,X)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(X) - g(U,\nabla\varphi)g(X,X)\frac{\partial Q}{\partial\varphi}(X)
\end{aligned}
$$

$$
\begin{aligned}
&+ g(U,X)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X) - g(U,\nabla^2\varphi)g(X,X)Q(\nabla^2\varphi;X)\\
&- \frac{1}{3}Rg(U,X)g(X,\nabla\varphi)Q(X;\Box,\Box) + \frac{1}{3}Rg(U,\nabla\varphi)g(X,X)Q(X;\Box,\Box)\\
&+ \frac{1}{3}Rg(U,X)Q(X;\nabla\varphi,X) - \frac{1}{3}Rg(X,X)Q(X;\nabla\varphi,U)\\
&+ \frac{5}{6}Rg(U,\nabla\varphi)Q(X;X,X) - \frac{2}{3}Rg(X,\nabla\varphi)Q(U;X,X)\\
&+ 3g(U,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) - g(U,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)\\
&- 2g(U,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) + g(U,X)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)\\
&- g(U,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X) + \frac{2}{3}Rg(U,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X)\\
&- \frac{1}{3}Rg(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;X,X) - \frac{1}{3}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;X,X)\\
&- \frac{2}{3}Rg(U,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)\\
&+ \frac{1}{3}Rg(U,\nabla^2\varphi)g(X,\nabla\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)\\
&+ \frac{1}{3}Rg(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X) + \frac{5}{2}Rg(U,X)B(X,X)\\
&- \frac{5}{3}Rg(X,X)B(U,X) + \frac{1}{3}g(U,X)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X)\\
&- 2g(U,\nabla^2\varphi)g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X) + Sg(U,X)\frac{\partial^2 B}{\partial\varphi^2}(X,X)\\
&- g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial^2 B}{\partial\varphi^2}(X,X) + \frac{2}{3}Rg(U,X)B(\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&+ \frac{4}{3}Rg(U,\nabla^2\varphi)B(\nabla^2\varphi,X;X,X) - \frac{5}{3}Rg(X,\nabla^2\varphi)B(\nabla^2\varphi,U;X,X)\\
&- \frac{2}{3}Rg(U,X)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;X,X) + \frac{2}{3}Rg(U,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\Box,\Box;X,X)\\
&- \frac{1}{12}g(U,X)g(\nabla\varphi,\nabla R)B(\Box,\Box;X,X) - \frac{1}{12}g(U,\nabla R)g(X,\nabla\varphi)B(\Box,\Box;X,X)\\
&+ \frac{1}{6}g(U,\nabla\varphi)g(X,\nabla R)B(\Box,\Box;X,X) + \frac{1}{12}g(U,X)B(\nabla\varphi,\nabla R;X,X)\\
&- \frac{1}{3}g(X,\nabla\varphi)B(\nabla R,U;X,X) - \frac{1}{6}g(X,\nabla R)B(\nabla\varphi,U;X,X)\\
&+ \frac{5}{12}g(U,\nabla\varphi)B(\nabla R,X;X,X) + \frac{1}{12}g(U,\nabla R)B(\nabla\varphi,X;X,X)\\
&- \frac{2}{3}RSg(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) + \frac{2}{3}Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)\\
&+ \frac{2}{3}Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X) - \frac{1}{3}Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X)
\end{aligned}
$$

$$- \frac{1}{3} R g(X, \nabla\varphi) \frac{\partial B}{\partial \varphi}(\nabla\varphi, U; X, X) + \frac{1}{9} R^2 S g(U, X) B(\Box_a, \Box_a; \Box, \Box; X, X)$$
$$- \frac{1}{9} R^2 g(U, \nabla\varphi) g(X, \nabla\varphi) B(\Box_a, \Box_a; \Box, \Box; X, X)$$
$$+ \frac{1}{9} R^2 g(U, \nabla\varphi) B(\Box, \Box; \nabla\varphi, X; X, X) + \frac{1}{9} R^2 g(X, \nabla\varphi) B(\Box, \Box; \nabla\varphi, U; X, X)$$
$$+ \frac{1}{9} R^2 g(U, X) B(\nabla\varphi, \Box; \nabla\varphi, \Box; X, X) - \frac{2}{9} R^2 g(U, X) B(\Box, \Box; \nabla\varphi, \nabla\varphi; X, X)$$
$$- \frac{1}{9} R^2 B(\nabla\varphi, U; \nabla\varphi, X; X, X) \Big]$$
$$- (-1)^q \Big[a_1 g(U, X) g(X, \nabla\varphi) \frac{\partial Q}{\partial \varphi}(X) + \frac{a_2}{2} g(X, X) g(U, \nabla\varphi) \frac{\partial Q}{\partial \varphi}(X)$$
$$+ \frac{a_2}{2} g(U, X) g(X, \nabla\varphi) \frac{\partial Q}{\partial \varphi}(X) + \frac{a_3}{2} g(X, X) g(X, \nabla\varphi) \frac{\partial Q}{\partial \varphi}(U)$$
$$+ \frac{a_3}{2} g(U, X) g(X, \nabla\varphi) \frac{\partial Q}{\partial \varphi}(X) + \frac{a_4}{2} g(X, \nabla\varphi) g(X, X) \frac{\partial Q}{\partial \varphi}(U)$$
$$+ \frac{a_4}{2} g(U, \nabla\varphi) g(X, X) \frac{\partial Q}{\partial \varphi}(X) + a_5 g(U, X) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; X)$$
$$+ \frac{a_6}{2} g(X, X) g(U, \nabla^2\varphi) Q(\nabla^2\varphi; X) + \frac{a_6}{2} g(U, X) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; X)$$
$$+ \frac{a_7}{2} g(X, X) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; U) + \frac{a_7}{2} g(U, X) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; X)$$
$$+ \frac{a_8}{2} g(X, \nabla^2\varphi) g(X, X) Q(\nabla^2\varphi; U) + \frac{a_8}{2} g(U, \nabla^2\varphi) g(X, X) Q(\nabla^2\varphi; X)$$
$$+ a_9 R g(U, X) g(X, \nabla\varphi) Q(X; \Box, \Box) + \frac{a_{10}}{2} R g(X, X) g(U, \nabla\varphi) Q(X; \Box, \Box)$$
$$+ \frac{a_{10}}{2} R g(U, X) g(X, \nabla\varphi) Q(X; \Box, \Box) + \frac{a_{11}}{2} R g(X, X) g(X, \nabla\varphi) Q(U; \Box, \Box)$$
$$+ \frac{a_{11}}{2} R g(U, X) g(X, \nabla\varphi) Q(X; \Box, \Box) + \frac{a_{12}}{2} R g(X, \nabla\varphi) g(X, X) Q(U; \Box, \Box)$$
$$+ \frac{a_{12}}{2} R g(U, \nabla\varphi) g(X, X) Q(X; \Box, \Box) + a_{13} R g(U, X) Q(X; \nabla\varphi, X)$$
$$+ \frac{a_{14}}{2} R g(X, X) Q(U; \nabla\varphi, X) + \frac{a_{14}}{2} R g(U, X) Q(X; \nabla\varphi, X)$$
$$+ \frac{a_{15}}{2} R g(X, X) Q(X; \nabla\varphi, U) + \frac{a_{15}}{2} R g(U, X) Q(X; \nabla\varphi, X)$$
$$+ \frac{a_{16}}{2} R g(X, X) Q(X; \nabla\varphi, U) + \frac{a_{16}}{2} R g(X, X) Q(U; \nabla\varphi, X)$$
$$+ \frac{a_{17}}{2} R g(X, \nabla\varphi) Q(U; X, X) + \frac{a_{17}}{2} R g(U, \nabla\varphi) Q(X; X, X)$$
$$+ \frac{a_{18}}{2} R g(X, \nabla\varphi) Q(X; U, X) + \frac{a_{18}}{2} R g(U, \nabla\varphi) Q(X; X, X)$$
$$+ \frac{a_{19}}{2} R g(X, \nabla\varphi) Q(X; U, X) + \frac{a_{19}}{2} R g(X, \nabla\varphi) Q(U; X, X)$$

$$+ a_{20}Rg(X,\nabla\varphi)Q(X;U,X) + a_{21}g(U,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)$$

$$+ \frac{a_{22}}{2}g(X,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X) + \frac{a_{22}}{2}g(U,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)$$

$$+ a_{23}g(X,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X) + \frac{a_{24}}{2}g(X,\nabla\varphi)g(U,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)$$

$$+ \frac{a_{24}}{2}g(U,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) + \frac{a_{25}}{2}g(X,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X)$$

$$+ \frac{a_{25}}{2}g(U,\nabla^2\varphi)g(X,\nabla\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) + \frac{a_{26}}{2}g(X,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X)$$

$$+ \frac{a_{26}}{2}g(U,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X) + a_{27}g(X,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X)$$

$$+ a_{28}g(U,X)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)$$

$$+ \frac{a_{29}}{2}g(X,X)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,X)$$

$$+ \frac{a_{29}}{2}g(U,X)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)$$

$$+ a_{30}g(X,X)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,X)$$

$$+ a_{31}g(U,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)$$

$$+ \frac{a_{32}}{2}g(X,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,X)$$

$$+ \frac{a_{32}}{2}g(U,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;X,X)$$

$$+ a_{33}g(X,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,X)$$

$$+ a_{34}Rg(U,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X) + \frac{a_{35}}{2}Rg(X,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;U,X)$$

$$+ \frac{a_{35}}{2}Rg(U,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X) + a_{36}Rg(X,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;U,X)$$

$$+ \frac{a_{37}}{2}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;X,X) + \frac{a_{37}}{2}Rg(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;X,X)$$

$$+ \frac{a_{38}}{2}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;U,X) + \frac{a_{38}}{2}Rg(U,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;X,X)$$

$$+ \frac{a_{39}}{2}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;U,X) + \frac{a_{39}}{2}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,U;X,X)$$

$$+ a_{40}Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;U,X)$$

$$+ a_{41}Rg(U,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)$$

$$+ \frac{a_{42}}{2}Rg(X,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;U,X)$$

$$+ \frac{a_{42}}{2}Rg(U,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,X)$$

$$
\begin{aligned}
&+ a_{43} Rg(X,X) g(\nabla\varphi, \nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; U, X) \\
&+ \frac{a_{44}}{2} Rg(X,\nabla\varphi) g(U, \nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) \\
&+ \frac{a_{44}}{2} Rg(U,\nabla\varphi) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) \\
&+ \frac{a_{45}}{2} Rg(X,\nabla\varphi) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; U, X) \\
&+ \frac{a_{45}}{2} Rg(U,\nabla\varphi) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) \\
&+ \frac{a_{46}}{2} Rg(X,\nabla^2\varphi) g(X, \nabla\varphi) Q(\nabla^2\varphi; \Box, \Box; U, X) \\
&+ \frac{a_{46}}{2} Rg(U,\nabla^2\varphi) g(X, \nabla\varphi) Q(\nabla^2\varphi; \Box, \Box; X, X) \\
&+ a_{47} Rg(X,\nabla\varphi) g(X, \nabla^2\varphi) Q(\nabla^2\varphi; \Box, \Box; U, X) + a_{48} Rg(U,X) B(X,X) \\
&+ \frac{a_{49}}{2} Rg(X,X) B(U,X) + \frac{a_{49}}{2} Rg(U,X) B(X,X) + a_{50} Rg(X,X) B(U,X) \\
&+ a_{51} g(U,X) g(\nabla^2\varphi, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X,X) + \frac{a_{52}}{2} g(X,X) g(\nabla^2\varphi, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(U,X) \\
&+ \frac{a_{52}}{2} g(U,X) g(\nabla^2\varphi, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X,X) + a_{53} g(X,X) g(\nabla^2\varphi, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(U,X) \\
&+ a_{54} g(U,\nabla^2\varphi) g(X, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X,X) + \frac{a_{55}}{2} g(X,\nabla^2\varphi) g(X, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(U,X) \\
&+ \frac{a_{55}}{2} g(U,\nabla^2\varphi) g(X, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(X,X) + a_{56} g(X,\nabla^2\varphi) g(X, \nabla^2\varphi) \frac{\partial B}{\partial\varphi}(U,X) \\
&+ a_{57} Sg(U,X) \frac{\partial^2 B}{\partial\varphi^2}(X,X) + \frac{a_{58}}{2} Sg(X,X) \frac{\partial^2 B}{\partial\varphi^2}(U,X) + \frac{a_{58}}{2} Sg(U,X) \frac{\partial^2 B}{\partial\varphi^2}(X,X) \\
&+ a_{59} Sg(X,X) \frac{\partial^2 B}{\partial\varphi^2}(U,X) + a_{60} g(U,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial^2 B}{\partial\varphi^2}(X,X) \\
&+ \frac{a_{61}}{2} g(X,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial^2 B}{\partial\varphi^2}(U,X) + \frac{a_{61}}{2} g(U,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial^2 B}{\partial\varphi^2}(X,X) \\
&+ a_{62} g(X,\nabla\varphi) g(X,\nabla\varphi) \frac{\partial^2 B}{\partial\varphi^2}(U,X) + a_{63} Rg(U,X) B(\nabla^2\varphi, \nabla^2\varphi; X, X) \\
&+ \frac{a_{64}}{2} Rg(X,X) B(\nabla^2\varphi, \nabla^2\varphi; U, X) + \frac{a_{64}}{2} Rg(U,X) B(\nabla^2\varphi, \nabla^2\varphi; X, X) \\
&+ a_{65} Rg(X,X) B(\nabla^2\varphi, \nabla^2\varphi; U, X) + \frac{a_{66}}{2} Rg(X,\nabla^2\varphi) B(\nabla^2\varphi, U; X, X) \\
&+ \frac{a_{66}}{2} Rg(U,\nabla^2\varphi) B(\nabla^2\varphi, X; X, X) + \frac{a_{67}}{2} Rg(X,\nabla^2\varphi) B(\nabla^2\varphi, X; U, X) \\
&+ \frac{a_{67}}{2} Rg(X,\nabla^2\varphi) B(\nabla^2\varphi, U; X, X) + \frac{a_{68}}{2} Rg(X,\nabla^2\varphi) B(\nabla^2\varphi, X; U, X) \\
&+ \frac{a_{68}}{2} Rg(U,\nabla^2\varphi) B(\nabla^2\varphi, X; X, X) + a_{69} Rg(X,\nabla^2\varphi) B(\nabla^2\varphi, X; U, X)
\end{aligned}
$$

$$+ a_{70} R g(U,X) g(\nabla^2\varphi, \nabla^2\varphi) B(\square,\square; X,X)$$
$$+ \frac{a_{71}}{2} R g(X,X) g(\nabla^2\varphi, \nabla^2\varphi) B(\square,\square; U,X)$$
$$+ \frac{a_{71}}{2} R g(U,X) g(\nabla^2\varphi, \nabla^2\varphi) B(\square,\square; X,X)$$
$$+ a_{72} R g(X,X) g(\nabla^2\varphi, \nabla^2\varphi) B(\square,\square; U,X)$$
$$+ a_{73} R g(U,\nabla^2\varphi) g(X,\nabla^2\varphi) B(\square,\square; X,X)$$
$$+ \frac{a_{74}}{2} R g(X,\nabla^2\varphi) g(X,\nabla^2\varphi) B(\square,\square; U,X)$$
$$+ \frac{a_{74}}{2} R g(U,\nabla^2\varphi) g(X,\nabla^2\varphi) B(\square,\square; X,X)$$
$$+ a_{75} R g(X,\nabla^2\varphi) g(X,\nabla^2\varphi) B(\square,\square; U,X)$$
$$+ a_{76} g(U,X) g(\nabla\varphi,\nabla R) B(\square,\square; X,X) + \frac{a_{77}}{2} g(X,X) g(\nabla\varphi,\nabla R) B(\square,\square; U,X)$$
$$+ \frac{a_{77}}{2} g(U,X) g(\nabla\varphi,\nabla R) B(\square,\square; X,X) + a_{78} g(X,X) g(\nabla\varphi,\nabla R) B(\square,\square; U,X)$$
$$+ \frac{a_{79}}{2} g(X,\nabla R) g(U,\nabla\varphi) B(\square,\square; X,X) + \frac{a_{79}}{2} g(U,\nabla R) g(X,\nabla\varphi) B(\square,\square; X,X)$$
$$+ \frac{a_{80}}{2} g(X,\nabla R) g(X,\nabla\varphi) B(\square,\square; U,X) + \frac{a_{80}}{2} g(U,\nabla R) g(X,\nabla\varphi) B(\square,\square; X,X)$$
$$+ \frac{a_{81}}{2} g(X,\nabla\varphi) g(X,\nabla R) B(\square,\square; U,X) + \frac{a_{81}}{2} g(U,\nabla\varphi) g(X,\nabla R) B(\square,\square; X,X)$$
$$+ a_{82} g(X,\nabla R) g(X,\nabla\varphi) B(\square,\square; U,X) + a_{83} g(U,X) B(\nabla\varphi,\nabla R; X,X)$$
$$+ \frac{a_{84}}{2} g(X,X) B(\nabla\varphi,\nabla R; U,X) + \frac{a_{84}}{2} g(U,X) B(\nabla\varphi,\nabla R; X,X)$$
$$+ a_{85} g(X,X) B(\nabla\varphi,\nabla R; U,X) + \frac{a_{86}}{2} g(X,\nabla\varphi) B(\nabla R, U; X,X)$$
$$+ \frac{a_{86}}{2} g(U,\nabla\varphi) B(\nabla R, X; X,X) + \frac{a_{87}}{2} g(X,\nabla\varphi) B(\nabla R, X; U,X)$$
$$+ \frac{a_{87}}{2} g(X,\nabla\varphi) B(\nabla R, U; X,X) + \frac{a_{88}}{2} g(X,\nabla\varphi) B(\nabla R, X; U,X)$$
$$+ \frac{a_{88}}{2} g(U,\nabla\varphi) B(\nabla R, X; X,X) + a_{89} g(X,\nabla\varphi) B(\nabla R, X; U,X)$$
$$+ \frac{a_{90}}{2} g(X,\nabla R) B(\nabla\varphi, U; X,X) + \frac{a_{90}}{2} g(U,\nabla R) B(\nabla\varphi, X; X,X)$$
$$+ \frac{a_{91}}{2} g(X,\nabla R) B(\nabla\varphi, X; U,X) + \frac{a_{91}}{2} g(X,\nabla R) B(\nabla\varphi, U; X,X)$$
$$+ \frac{a_{92}}{2} g(X,\nabla R) B(\nabla\varphi, X; U,X) + \frac{a_{92}}{2} g(U,\nabla R) B(\nabla\varphi, X; X,X)$$
$$+ a_{93} g(X,\nabla R) B(\nabla\varphi, X; U,X) + a_{94} R S g(U,X) \frac{\partial B}{\partial\varphi}(\square,\square; X,X)$$

$$+\frac{a_{95}}{2}RSg(X,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;U,X)+\frac{a_{95}}{2}RSg(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)$$
$$+a_{96}RSg(X,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;U,X)+a_{97}Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)$$
$$+\frac{a_{98}}{2}Rg(X,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;U,X)$$
$$+\frac{a_{98}}{2}Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)$$
$$+a_{99}Rg(X,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;U,X)+a_{100}Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X)$$
$$+\frac{a_{101}}{2}Rg(X,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;U,X)+\frac{a_{101}}{2}Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X)$$
$$+a_{102}Rg(X,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;U,X)+\frac{a_{103}}{2}Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X)$$
$$+\frac{a_{103}}{2}Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X)+\frac{a_{104}}{2}Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;U,X)$$
$$+\frac{a_{104}}{2}Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X)+\frac{a_{105}}{2}Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;U,X)$$
$$+\frac{a_{105}}{2}Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X)+a_{106}Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;U,X)$$
$$+a_{107}R^2Sg(U,X)B(\Box_a,\Box_a;\Box,\Box;X,X)+\frac{a_{108}}{2}R^2Sg(X,X)B(\Box_a,\Box_a;\Box,\Box;U,X)$$
$$+\frac{a_{108}}{2}R^2Sg(U,X)B(\Box_a,\Box_a;\Box,\Box;X,X)+a_{109}R^2Sg(X,X)B(\Box_a,\Box_a;\Box,\Box;U,X)$$
$$+a_{110}R^2g(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X)$$
$$+\frac{a_{111}}{2}R^2g(X,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;U,X)$$
$$+\frac{a_{111}}{2}R^2g(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X)$$
$$+a_{112}R^2g(X,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;U,X)$$
$$+\frac{a_{113}}{2}R^2g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,U;X,X)+\frac{a_{113}}{2}R^2g(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;X,X)$$
$$+\frac{a_{114}}{2}R^2g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;U,X)+\frac{a_{114}}{2}R^2g(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;X,X)$$
$$+\frac{a_{115}}{2}R^2g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;U,X)+\frac{a_{115}}{2}R^2g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,U;X,X)$$
$$+a_{116}R^2g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;U,X)+a_{117}R^2g(U,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,X)$$
$$+\frac{a_{118}}{2}R^2g(X,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;U,X)+\frac{a_{118}}{2}R^2g(U,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,X)$$
$$+a_{119}R^2g(X,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;U,X)+a_{120}R^2g(U,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X)$$
$$+\frac{a_{121}}{2}R^2g(X,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;U,X)+\frac{a_{121}}{2}R^2g(U,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X)$$

$$
\begin{aligned}
&+ a_{122} R^2 g(X,X) B(\square,\square;\nabla\varphi,\nabla\varphi;U,X) + a_{123} R^2 B(\nabla\varphi,U;\nabla\varphi,X;X,X)\\
&+ \frac{a_{124}}{2} R^2 B(\nabla\varphi,X;\nabla\varphi,X;U,X) + \frac{a_{124}}{2} R^2 B(\nabla\varphi,U;\nabla\varphi,X;X,X)\\
&+ a_{125} R^2 B(\nabla\varphi,X;\nabla\varphi,X;U,X)\Bigg]\\
= (-1)^q \Bigg[&\left(1 - a_1 - \frac{a_2}{2} - \frac{a_3}{2}\right) g(U,X) g(X,\nabla\varphi) \frac{\partial Q}{\partial\varphi}(X)\\
&+ \left(-1 - \frac{a_2}{2} - \frac{a_4}{2}\right) g(U,\nabla\varphi) g(X,X) \frac{\partial Q}{\partial\varphi}(X)\\
&+ \left(1 - a_5 - \frac{a_6}{2} - \frac{a_7}{2}\right) g(U,X) g(X,\nabla^2\varphi) Q(\nabla^2\varphi;X)\\
&+ \left(-1 - \frac{a_6}{2} - \frac{a_8}{2}\right) g(U,\nabla^2\varphi) g(X,X) Q(\nabla^2\varphi;X)\\
&+ \left(-\frac{1}{3} - a_9 - \frac{a_{10}}{2} - \frac{a_{11}}{2}\right) Rg(U,X) g(X,\nabla\varphi) Q(X;\square,\square)\\
&+ \left(\frac{1}{3} - \frac{a_{10}}{2} - \frac{a_{12}}{2}\right) Rg(U,\nabla\varphi) g(X,X) Q(X;\square,\square)\\
&+ \left(\frac{1}{3} - a_{13} - \frac{a_{14}}{2} - \frac{a_{15}}{2}\right) Rg(U,X) Q(X;\nabla\varphi,X)\\
&+ \left(-\frac{1}{3} - \frac{a_{15}}{2} - \frac{a_{16}}{2}\right) Rg(X,X) Q(X;\nabla\varphi,U)\\
&+ \left(\frac{5}{6} - \frac{a_{17}}{2} - \frac{a_{18}}{2}\right) Rg(U,\nabla\varphi) Q(X;X,X)\\
&+ \left(-\frac{2}{3} - \frac{a_{17}}{2} - \frac{a_{19}}{2}\right) Rg(X,\nabla\varphi) Q(U;X,X)\\
&+ \left(3 - a_{21} - \frac{a_{22}}{2}\right) g(U,X) g(\nabla\varphi,\nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)\\
&+ \left(-1 - \frac{a_{24}}{2} - \frac{a_{25}}{2}\right) g(U,\nabla^2\varphi) g(X,\nabla\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)\\
&+ \left(-2 - \frac{a_{24}}{2} - \frac{a_{26}}{2}\right) g(U,\nabla\varphi) g(X,\nabla^2\varphi) \frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;X,X)\\
&+ \left(1 - a_{28} - \frac{a_{29}}{2}\right) g(U,X) g(\nabla^2\varphi,\nabla^2_a\varphi) Q(\nabla^2_a\varphi;\nabla^2\varphi;X,X)\\
&+ \left(-1 - a_{31} - \frac{a_{32}}{2}\right) g(U,\nabla^2\varphi) g(X,\nabla^2_a\varphi) Q(\nabla^2_a\varphi;\nabla^2\varphi;X,X)\\
&+ \left(\frac{2}{3} - a_{34} - \frac{a_{35}}{2}\right) Rg(U,X) Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;X,X)\\
&+ \left(-\frac{1}{3} - \frac{a_{37}}{2} - \frac{a_{38}}{2}\right) Rg(U,\nabla^2\varphi) Q(\nabla^2\varphi;\nabla\varphi,X;X,X)\\
&+ \left(-\frac{1}{3} - \frac{a_{37}}{2} - \frac{a_{39}}{2}\right) Rg(X,\nabla^2\varphi) Q(\nabla^2\varphi;\nabla\varphi,U;X,X)
\end{aligned}
$$

$$+\left(-\frac{2}{3}-a_{41}-\frac{a_{42}}{2}\right)Rg(U,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square;X,X)$$
$$+\left(\frac{1}{3}-\frac{a_{44}}{2}-\frac{a_{46}}{2}\right)Rg(U,\nabla^2\varphi)g(X,\nabla\varphi)Q(\nabla^2\varphi;\square,\square;X,X)$$
$$+\left(\frac{1}{3}-\frac{a_{44}}{2}-\frac{a_{45}}{2}\right)Rg(U,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square;X,X)$$
$$+\left(\frac{5}{2}-a_{48}-\frac{a_{49}}{2}\right)Rg(U,X)B(X,X)$$
$$+\left(-\frac{5}{3}-\frac{a_{49}}{2}-a_{50}\right)Rg(X,X)B(U,X)$$
$$+\left(\frac{1}{3}-a_{51}-\frac{a_{52}}{2}\right)g(U,X)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X)$$
$$+\left(-2-a_{54}-\frac{a_{55}}{2}\right)g(U,\nabla^2\varphi)g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(X,X)$$
$$+\left(1-a_{57}-\frac{a_{58}}{2}\right)Sg(U,X)\frac{\partial^2 B}{\partial\varphi^2}(X,X)$$
$$+\left(-1-a_{60}-\frac{a_{61}}{2}\right)g(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial^2 B}{\partial\varphi^2}(X,X)$$
$$+\left(\frac{2}{3}-a_{63}-\frac{a_{64}}{2}\right)Rg(U,X)B(\nabla^2\varphi,\nabla^2\varphi;X,X)$$
$$+\left(\frac{4}{3}-\frac{a_{66}}{2}-\frac{a_{68}}{2}\right)Rg(U,\nabla^2\varphi)B(\nabla^2\varphi,X;X,X)$$
$$+\left(-\frac{5}{3}-\frac{a_{66}}{2}-\frac{a_{67}}{2}\right)Rg(X,\nabla^2\varphi)B(\nabla^2\varphi,U;X,X)$$
$$+\left(-\frac{2}{3}-a_{70}-\frac{a_{71}}{2}\right)Rg(U,X)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;X,X)$$
$$+\left(\frac{2}{3}-a_{73}-\frac{a_{74}}{2}\right)Rg(U,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\square,\square;X,X)$$
$$+\left(-\frac{1}{12}-a_{76}-\frac{a_{77}}{2}\right)g(U,X)g(\nabla\varphi,\nabla R)B(\square,\square;X,X)$$
$$+\left(-\frac{1}{12}-\frac{a_{79}}{2}-\frac{a_{80}}{2}\right)g(U,\nabla R)g(X,\nabla\varphi)B(\square,\square;X,X)$$
$$+\left(\frac{1}{6}-\frac{a_{79}}{2}-\frac{a_{81}}{2}\right)g(U,\nabla\varphi)g(X,\nabla R)B(\square,\square;X,X)$$
$$+\left(\frac{1}{12}-a_{83}-\frac{a_{84}}{2}\right)g(U,X)B(\nabla\varphi,\nabla R;X,X)$$
$$+\left(-\frac{1}{3}-\frac{a_{86}}{2}-\frac{a_{87}}{2}\right)g(X,\nabla\varphi)B(\nabla R,U;X,X)$$
$$+\left(-\frac{1}{6}-\frac{a_{90}}{2}-\frac{a_{91}}{2}\right)g(X,\nabla R)B(\nabla\varphi,U;X,X)$$
$$+\left(\frac{5}{12}-\frac{a_{86}}{2}-\frac{a_{88}}{2}\right)g(U,\nabla\varphi)B(\nabla R,X;X,X)$$

$$
\begin{aligned}
&+\left(\frac{1}{12}-\frac{a_{90}}{2}-\frac{a_{92}}{2}\right)g(U,\nabla R)B(\nabla\varphi,X;X,X)\\
&+\left(-\frac{2}{3}-a_{94}-\frac{a_{95}}{2}\right)RSg(U,X)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)\\
&+\left(\frac{2}{3}-a_{97}-\frac{a_{98}}{2}\right)Rg(U,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)\\
&+\left(\frac{2}{3}-a_{100}-\frac{a_{101}}{2}\right)Rg(U,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,X)\\
&+\left(-\frac{1}{3}-\frac{a_{103}}{2}-\frac{a_{104}}{2}\right)Rg(U,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;X,X)\\
&+\left(-\frac{1}{3}-\frac{a_{103}}{2}-\frac{a_{105}}{2}\right)Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,U;X,X)\\
&+\left(\frac{1}{9}-a_{107}-\frac{a_{108}}{2}\right)R^2Sg(U,X)B(\Box_a,\Box_a;\Box,\Box;X,X)\\
&+\left(-\frac{1}{9}-a_{110}-\frac{a_{111}}{2}\right)R^2g(U,\nabla\varphi)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X)\\
&+\left(\frac{1}{9}-\frac{a_{113}}{2}-\frac{a_{114}}{2}\right)R^2g(U,\nabla\varphi)B(\Box,\Box;\nabla\varphi,X;X,X)\\
&+\left(\frac{1}{9}-\frac{a_{113}}{2}-\frac{a_{115}}{2}\right)R^2g(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,U;X,X)\\
&+\left(\frac{1}{9}-a_{117}-\frac{a_{118}}{2}\right)R^2g(U,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,X)\\
&+\left(-\frac{2}{9}-a_{120}-\frac{a_{121}}{2}\right)R^2g(U,X)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,X)\\
&+\left(-\frac{1}{9}-a_{123}-\frac{a_{124}}{2}\right)R^2B(\nabla\varphi,U;\nabla\varphi,X;X,X)\\
&-\left(\frac{a_3}{2}+\frac{a_4}{2}\right)g(X,\nabla\varphi)g(X,X)\frac{\partial Q}{\partial\varphi}(U)\\
&-\left(\frac{a_7}{2}+\frac{a_8}{2}\right)g(X,\nabla^2\varphi)g(X,X)Q(\nabla^2\varphi;U)\\
&-\left(\frac{a_{11}}{2}+\frac{a_{12}}{2}\right)Rg(X,\nabla\varphi)g(X,X)Q(U;\Box,\Box)\\
&-\left(\frac{a_{14}}{2}+\frac{a_{16}}{2}\right)Rg(X,X)Q(U;\nabla\varphi,X)\\
&-\left(\frac{a_{18}}{2}+\frac{a_{19}}{2}+a_{20}\right)Rg(X,\nabla\varphi)Q(X;U,X)\\
&-\left(\frac{a_{22}}{2}+a_{23}\right)g(X,X)g(\nabla\varphi,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X)\\
&-\left(\frac{a_{25}}{2}+\frac{a_{26}}{2}+a_{27}\right)g(X,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial Q}{\partial\varphi}(\nabla^2\varphi;U,X)\\
&-\left(\frac{a_{29}}{2}+a_{30}\right)g(X,X)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi;U,X)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{a_{32}}{2}+a_{33}\right) g(X,\nabla^2\varphi)g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi;U,X)\\
&-\left(\frac{a_{35}}{2}+a_{36}\right) Rg(X,X)Q(\nabla^2\varphi;\nabla\varphi,\nabla^2\varphi;U,X)\\
&-\left(\frac{a_{38}}{2}+\frac{a_{39}}{2}+a_{40}\right) Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,X;U,X)\\
&-\left(\frac{a_{42}}{2}+a_{43}\right) Rg(X,X)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square;U,X)\\
&-\left(\frac{a_{45}}{2}+\frac{a_{46}}{2}+a_{47}\right) Rg(X,\nabla\varphi)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square;U,X)\\
&-\left(\frac{a_{52}}{2}+a_{53}\right) g(X,X)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(U,X)\\
&-\left(\frac{a_{55}}{2}+a_{56}\right) g(X,\nabla^2\varphi)g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(U,X)\\
&-\left(\frac{a_{58}}{2}+a_{59}\right) Sg(X,X)\frac{\partial^2 B}{\partial\varphi^2}(U,X)\\
&-\left(\frac{a_{61}}{2}+a_{62}\right) g(X,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial^2 B}{\partial\varphi^2}(U,X)\\
&-\left(\frac{a_{64}}{2}+a_{65}\right) Rg(X,X)B(\nabla^2\varphi,\nabla^2\varphi;U,X)\\
&-\left(\frac{a_{67}}{2}+\frac{a_{68}}{2}+a_{69}\right) Rg(X,\nabla^2\varphi)B(\nabla^2\varphi,X;U,X)\\
&-\left(\frac{a_{71}}{2}+a_{72}\right) Rg(X,X)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;U,X)\\
&-\left(\frac{a_{74}}{2}+a_{75}\right) Rg(X,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\square,\square;U,X)\\
&-\left(\frac{a_{77}}{2}+a_{78}\right) g(X,X)g(\nabla\varphi,\nabla R)B(\square,\square;U,X)\\
&-\left(\frac{a_{80}}{2}+\frac{a_{81}}{2}+a_{82}\right) g(X,\nabla R)g(X,\nabla\varphi)B(\square,\square;U,X)\\
&-\left(\frac{a_{84}}{2}+a_{85}\right) g(X,X)B(\nabla\varphi,\nabla R;U,X)\\
&-\left(\frac{a_{87}}{2}+\frac{a_{88}}{2}+a_{89}\right) g(X,\nabla\varphi)B(\nabla R,X;U,X)\\
&-\left(\frac{a_{91}}{2}+\frac{a_{92}}{2}+a_{93}\right) g(X,\nabla R)B(\nabla\varphi,X;U,X)\\
&-\left(\frac{a_{95}}{2}+a_{96}\right) RSg(X,X)\frac{\partial B}{\partial\varphi}(\square,\square;U,X)\\
&-\left(\frac{a_{98}}{2}+a_{99}\right) Rg(X,\nabla\varphi)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square;U,X)\\
&-\left(\frac{a_{101}}{2}+a_{102}\right) Rg(X,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;U,X)\\
&-\left(\frac{a_{104}}{2}+\frac{a_{105}}{2}+a_{106}\right) Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,X;U,X)
\end{aligned}
$$

$$
\begin{aligned}
&- \left(\frac{a_{108}}{2} + a_{109}\right) R^2 S g(X,X) B(\Box_a, \Box_a; \Box, \Box; U, X) \\
&- \left(\frac{a_{111}}{2} + a_{112}\right) R^2 g(X,\nabla\varphi) g(X,\nabla\varphi) B(\Box_a, \Box_a; \Box, \Box; U, X) \\
&- \left(\frac{a_{114}}{2} + \frac{a_{115}}{2} + a_{116}\right) R^2 g(X,\nabla\varphi) B(\Box, \Box; \nabla\varphi, X; U, X) \\
&- \left(\frac{a_{118}}{2} + a_{119}\right) R^2 g(X,X) B(\nabla\varphi, \Box; \nabla\varphi, \Box; U, X) \\
&- \left(\frac{a_{121}}{2} + a_{122}\right) R^2 g(X,X) B(\Box, \Box; \nabla\varphi, \nabla\varphi; U, X) \\
&- \left(\frac{a_{124}}{2} + a_{125}\right) R^2 B(\nabla\varphi, X; \nabla\varphi, X; U, X) \Bigg].
\end{aligned}
$$

We assume $\Delta(U, X, X, X)$ vanishes identically and solve for the 125 different a variables. The resulting system of equations is heavily underdetermined, with solution

$$
\begin{array}{llll}
a_1 = 2 + a_4 & a_2 = -2 - a_4 & a_3 = -a_4 & a_4 = a_4 \\
a_5 = 2 + a_8 & a_6 = -2 - a_8 & a_7 = -a_8 & a_8 = a_8 \\
a_9 = -\frac{2}{3} + a_{12} & a_{10} = \frac{2}{3} - a_{12} & a_{11} = -a_{12} & a_{12} = a_{12} \\
a_{13} = \frac{2}{3} + a_{16} & a_{14} = -a_{16} & a_{15} = -\frac{2}{3} - a_{16} & a_{16} = a_{16} \\
a_{17} = a_{17} & a_{18} = \frac{5}{3} - a_{17} & a_{19} = -\frac{4}{3} - a_{17} & a_{20} = -\frac{1}{6} + a_{17} \\
a_{21} = 3 - \frac{a_{22}}{2} & a_{22} = a_{22} & a_{23} = -\frac{a_{22}}{2} & a_{24} = a_{24} \\
a_{25} = -2 - a_{24} & a_{26} = -4 - a_{24} & a_{27} = 3 + a_{24} & a_{28} = 1 - \frac{a_{29}}{2} \\
a_{29} = a_{29} & a_{30} = -\frac{a_{29}}{2} & a_{31} = -1 - \frac{a_{32}}{2} & a_{32} = a_{32} \\
a_{33} = a_{33} & a_{34} = \frac{2}{3} - \frac{a_{35}}{2} & a_{35} = a_{35} & a_{36} = -\frac{a_{35}}{2} \\
a_{37} = a_{37} & a_{38} = -\frac{2}{3} - a_{37} & a_{39} = -\frac{2}{3} - a_{37} & a_{40} = \frac{2}{3} + a_{37} \\
a_{41} = -\frac{2}{3} - \frac{a_{42}}{2} & a_{42} = a_{42} & a_{43} = -\frac{a_{42}}{2} & a_{44} = a_{44} \\
a_{45} = \frac{2}{3} - a_{44} & a_{46} = \frac{2}{3} - a_{44} & a_{47} = -\frac{2}{3} + a_{44} & a_{48} = \frac{5}{2} - \frac{a_{49}}{2} \\
a_{49} = a_{49} & a_{50} = -\frac{5}{3} - \frac{a_{49}}{2} & a_{51} = \frac{1}{3} - \frac{a_{52}}{2} & a_{52} = a_{52} \\
a_{53} = -\frac{a_{52}}{2} & a_{54} = -2 - \frac{a_{55}}{2} & a_{55} = a_{55} & a_{56} = -\frac{a_{55}}{2}
\end{array}
$$

$$
\begin{array}{llll}
a_{57} = 1 - \frac{a_{58}}{2} & a_{58} = a_{58} & a_{59} = -\frac{a_{58}}{2} & a_{60} = -1 - \frac{a_{61}}{2} \\
a_{61} = a_{61} & a_{62} = -\frac{a_{61}}{2} & a_{63} = \frac{2}{3} - \frac{a_{64}}{2} & a_{64} = a_{64} \\
a_{65} = -\frac{a_{64}}{2} & a_{66} = a_{66} & a_{67} = -\frac{10}{3} - a_{66} & a_{68} = \frac{8}{3} - a_{66} \\
a_{69} = \frac{1}{3} + a_{66} & a_{70} = -\frac{2}{3} - \frac{a_{71}}{2} & a_{71} = a_{71} & a_{72} = -\frac{a_{71}}{2} \\
a_{73} = \frac{2}{3} - \frac{a_{74}}{2} & a_{74} = a_{74} & a_{75} = -\frac{a_{74}}{2} & a_{76} = -\frac{1}{12} - \frac{a_{77}}{2} \\
a_{77} = a_{77} & a_{78} = -\frac{a_{78}}{2} & a_{79} = a_{79} & a_{80} = -\frac{1}{6} - a_{79} \\
a_{81} = \frac{1}{3} - a_{79} & a_{82} = -\frac{1}{12} + a_{79} & a_{83} = \frac{1}{12} - \frac{a_{84}}{2} & a_{84} = a_{84} \\
a_{85} = -\frac{a_{85}}{2} & a_{86} = a_{86} & a_{87} = -\frac{2}{3} - a_{86} & a_{88} = \frac{5}{6} - a_{86} \\
a_{89} = -\frac{1}{12} + a_{86} & a_{90} = a_{90} & a_{91} = -\frac{1}{3} - a_{90} & a_{92} = \frac{1}{5} - a_{90} \\
a_{93} = \frac{1}{12} + a_{90} & a_{94} = -\frac{2}{3} - \frac{a_{95}}{2} & a_{95} = a_{95} & a_{96} = -\frac{a_{95}}{2} \\
a_{97} = \frac{2}{3} - \frac{a_{98}}{2} & a_{98} = a_{98} & a_{99} = -\frac{a_{98}}{2} & a_{100} = \frac{2}{3} - \frac{a_{101}}{2} \\
a_{101} = a_{101} & a_{102} = -\frac{a_{101}}{2} & a_{103} = a_{103} & a_{104} = -\frac{2}{3} - a_{103} \\
a_{105} = -\frac{2}{3} - a_{103} & a_{106} = \frac{2}{3} + a_{103} & a_{107} = \frac{1}{9} - \frac{a_{108}}{2} & a_{108} = a_{108} \\
a_{109} = -\frac{a_{108}}{2} & a_{110} = -\frac{1}{9} - \frac{a_{111}}{2} & a_{111} = a_{111} & a_{112} = -\frac{a_{111}}{2} \\
a_{113} = a_{113} & a_{114} = \frac{2}{9} - a_{113} & a_{115} = \frac{2}{9} - a_{113} & a_{116} = -\frac{2}{9} + a_{113} \\
a_{117} = \frac{1}{9} - \frac{a_{118}}{2} & a_{118} = a_{118} & a_{119} = -\frac{a_{118}}{2} & a_{120} = -\frac{2}{9} - \frac{a_{121}}{2} \\
a_{121} = a_{121} & a_{122} = -\frac{a_{121}}{2} & a_{123} = -\frac{1}{9} - \frac{a_{124}}{2} & a_{124} = a_{124} \\
a_{125} = -\frac{a_{124}}{2}. & & &
\end{array}
$$

We note the above solution contains 38 free variables. Using the assumption $\Delta(U, X, X, X) = 0$, we have that $\Delta(U, U, X, X) = \det(U, X)^2 P$, where P is a scalar density of scalar order 2. Hence, we may rearrange the definition of $\Delta(U, U, X, X)$ to get

$$E(U, U; X, X) = \tilde{E}(U, U; X, X) - \Delta(U, U, X, X) = \tilde{E}(U, U; X, X) - \det(U, X)^2 P. \quad \text{(C.15)}$$

$\square$

On top of the numerous free variables in the above system of equations, there is no immediate route for integrating (C.15). To find addition information and lock down some of the a variables, we use (C.11) to remove the third order φ terms from the divergence-free condition (C.10), leaving

$$\begin{aligned}
0 = &\frac{(-1)^q}{12}\Big(-14R_{|n}g^{il}B^{mn} - R^{|i}B^{ml} + 4R_{|n}g^{ml}B^{in} + 6R^{|l}B^{im} - R^{|l}g^{im}g_{kj}B^{kj} \\
&+ R^{|i}g^{ml}g_{kj}B^{kj}\Big)\varphi_{lm} + \frac{(-1)^q}{12}\Big(-2R^2g^{il}g_{km}B^{km} + \frac{4}{3}R^2B^{il} - \frac{13}{5}R_{|mn}g^{il}B^{mn} \\
&- \frac{2}{5}R_{|mn}g^{im}B^{ln} + \frac{8}{5}R_{|mn}g^{ml}B^{in} + \frac{2}{5}R_{|mn}g^{mn}B^{il} - \frac{2}{5}R_{|mn}g^{mn}g^{il}g_{jk}B^{jk} \\
&+ \frac{2}{5}R_{|mn}g^{in}g_{jk}g^{ml}B^{jk}\Big)\varphi_l - Rg_{kl}\varphi_{mj}D^{ijklm} + Rg_{jk}\varphi_{lm}D^{ijklm} - \frac{1}{4}R_{|m}g_{kl}\varphi_jD^{ijklm} \\
&+ \frac{1}{4}R_{|m}g_{jk}\varphi_lD^{ijklm} + E^{ij;kl}\left[-\frac{1}{3}R(g_{kl}\varphi_j - g_{jk}\varphi_l)\right] + E^{ij;k}\varphi_{jk} + \frac{\partial E^{ij}}{\partial\varphi}\varphi_j. \qquad \text{(C.16)}
\end{aligned}$$

We apply the differential operator $U_iX_jX_k\dfrac{\partial}{\partial\varphi_{jk}}$ to this equation and solve for $E^{ij;k}$

$$\begin{aligned}
&E(U,X;X) \\
&= \frac{(-1)^q}{12}\Big[14g(U,X)B(X,\nabla R) + g(U,\nabla R)B(X,X) - 4g(X,X)B(U,\nabla R) \\
&\quad - 6g(\nabla R,X)B(U,X) + g(\nabla R,X)g(U,X)B(\square,\square) - g(U,\nabla R)g(X,X)B(\square,\square) \\
&\quad + 14g(U,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;X,X) + g(U,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;X,X) \\
&\quad - 4g(\nabla^2\varphi,\nabla^2\varphi)B(U,\nabla R;X,X) - 6g(\nabla R,\nabla^2\varphi)B(U,\nabla^2\varphi;X,X) \\
&\quad + g(\nabla R,\nabla^2\varphi)g(U,\nabla^2\varphi)B(\square,\square;X,X) - g(U,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;X,X) \\
&\quad + 2R^2g(U,\nabla\varphi)B(\square,\square;X,X) - \frac{4}{3}R^2B(U,\nabla\varphi;X,X) \\
&\quad + \frac{13}{5}g(U,\nabla\varphi)B(\nabla^2R,\nabla^2R;X,X) + \frac{2}{5}g(U,\nabla^2R)B(\nabla\varphi,\nabla^2R;X,X) \\
&\quad - \frac{8}{5}g(\nabla^2R,\nabla\varphi)B(U,\nabla^2R;X,X) - \frac{2}{5}g(\nabla^2R,\nabla^2R)B(U,\nabla\varphi;X,X) \\
&\quad + \frac{2}{5}g(\nabla^2R,\nabla^2R)g(U,\nabla\varphi)B(\square,\square;X,X) - \frac{2}{5}g(U,\nabla^2R)g(\nabla^2R,\nabla\varphi)B(\square,\square;X,X)\Big] \\
&\quad + \frac{1}{4}\left[D(U,\nabla\varphi,\square,\square,\nabla R;X,X) - D(U,\square,\square,\nabla\varphi,\nabla R;X,X)\right]
\end{aligned}$$

$$+ R\left[D(U,X,\Box,\Box,X) - D(U,\Box,\Box,X,X)\right]$$
$$+ R\left[D(U,\nabla^2\varphi,\Box,\Box,\nabla^2\varphi;X,X) - D(U,\Box,\Box,\nabla^2\varphi,\nabla^2\varphi;X,X)\right]$$
$$+ \frac{R}{3}\left[E(U,\nabla\varphi;\Box,\Box;X,X) - E(U,\Box;\Box,\nabla\varphi;X,X)\right]$$
$$- E(U,\nabla^2\varphi;\nabla^2\varphi;X,X) - \frac{\partial E}{\partial\varphi}(U,\nabla\varphi;X,X),$$

We expand the symmetry by replacing an X with a Y, multiplying both sides by 2 to remove the numeric coefficient from the left side

$$
\begin{aligned}
&E(U,X;Y) + E(U,Y;X)\\
&\quad = \frac{(-1)^q}{12}\Big[14g(U,Y)B(X,\nabla R) + 14g(U,X)B(Y,\nabla R) + 2g(U,\nabla R)B(X,Y)\\
&\qquad - 8g(X,Y)B(U,\nabla R) - 6g(\nabla R,Y)B(U,X) - 6g(\nabla R,X)B(U,Y)\\
&\qquad + g(\nabla R,Y)g(U,X)B(\Box,\Box) + g(\nabla R,X)g(U,Y)B(\Box,\Box)\\
&\qquad - 2g(U,\nabla R)g(X,Y)B(\Box,\Box) + 14g(U,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;X,Y)\\
&\qquad + g(U,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;X,Y) - 4g(\nabla^2\varphi,\nabla^2\varphi)B(U,\nabla R;X,Y)\\
&\qquad - 6g(\nabla R,\nabla^2\varphi)B(U,\nabla^2\varphi;X,Y) + 2g(\nabla R,\nabla^2\varphi)g(U,\nabla^2\varphi)B(\Box,\Box;X,Y)\\
&\qquad - 2g(U,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;X,Y) + 4R^2g(U,\nabla\varphi)B(\Box,\Box;X,Y)\\
&\qquad - \frac{8}{3}R^2B(U,\nabla\varphi;X,Y) + \frac{26}{5}g(U,\nabla\varphi)B(\nabla^2R,\nabla^2R;X,Y)\\
&\qquad + \frac{4}{5}g(U,\nabla^2R)B(\nabla\varphi,\nabla^2R;X,Y) - \frac{16}{5}g(\nabla^2R,\nabla\varphi)B(U,\nabla^2R;X,Y)\\
&\qquad - \frac{4}{5}g(\nabla^2R,\nabla^2R)B(U,\nabla\varphi;X,Y) + \frac{4}{5}g(\nabla^2R,\nabla^2R)g(U,\nabla\varphi)B(\Box,\Box;X,Y)\\
&\qquad - \frac{4}{5}g(U,\nabla^2R)g(\nabla^2R,\nabla\varphi)B(\Box,\Box;X,Y)\Big] + R\left[D(U,Y,\Box,\Box,X) + D(U,X,\Box,\Box,Y)\right.\\
&\qquad \left.- 2D(U,\Box,\Box,X,Y)\right] + \frac{1}{2}\left[D(U,\nabla\varphi,\Box,\Box,\nabla R;X,Y) - D(U,\Box,\Box,\nabla\varphi,\nabla R;X,Y)\right]\\
&\qquad + 2R\left[D(U,\nabla^2\varphi,\Box,\Box,\nabla^2\varphi;X,Y) - D(U,\Box,\Box,\nabla^2\varphi,\nabla^2\varphi;X,Y)\right]\\
&\qquad + \frac{2R}{3}\left[E(U,\nabla\varphi;\Box,\Box;X,Y) - E(U,\Box;\Box,\nabla\varphi;X,Y)\right]\\
&\qquad - 2E(U,\nabla^2\varphi;\nabla^2\varphi;X,Y) - 2\frac{\partial E}{\partial\varphi}(U,\nabla\varphi;X,Y).
\end{aligned}
$$

We permute the covectors to produce two additional copies of this equation and sum the first of these with the original version while subtracting the third:

$$E(U,X;Y)+E(U,Y;X)+E(X,Y;U)+E(X,U;Y)-E(Y,U;X)-E(Y,X;U)=\\ 2E(U,X;Y).$$

We divide both sides of the equation by 2 and simplify

$$\begin{aligned}
&E(U,X;Y)\\
&\quad=\frac{1}{2}\left[E(U,X;Y)+E(U,Y;X)+E(X,Y;U)+E(X,U;Y)-E(Y,U;X)-E(Y,X;U)\right]\\
&\quad=\frac{1}{2}\left\{\frac{(-1)^q}{12}\Big[14g(U,Y)B(X,\nabla R)+14g(U,X)B(Y,\nabla R)+2g(U,\nabla R)B(X,Y)\right.\\
&\qquad-8g(X,Y)B(U,\nabla R)-6g(\nabla R,Y)B(U,X)-6g(\nabla R,X)B(U,Y)\\
&\qquad+g(\nabla R,Y)g(U,X)B(\Box,\Box)+g(\nabla R,X)g(U,Y)B(\Box,\Box)\\
&\qquad-2g(U,\nabla R)g(X,Y)B(\Box,\Box)+14g(U,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;X,Y)\\
&\qquad+g(U,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;X,Y)-4g(\nabla^2\varphi,\nabla^2\varphi)B(U,\nabla R;X,Y)\\
&\qquad-6g(\nabla R,\nabla^2\varphi)B(U,\nabla^2\varphi;X,Y)+2g(\nabla R,\nabla^2\varphi)g(U,\nabla^2\varphi)B(\Box,\Box;X,Y)\\
&\qquad-2g(U,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;X,Y)+4R^2g(U,\nabla\varphi)B(\Box,\Box;X,Y)\\
&\qquad-\frac{8}{3}R^2B(U,\nabla\varphi;X,Y)+\frac{26}{5}g(U,\nabla\varphi)B(\nabla^2R,\nabla^2R;X,Y)\\
&\qquad+\frac{4}{5}g(U,\nabla^2R)B(\nabla\varphi,\nabla^2R;X,Y)-\frac{16}{5}g(\nabla^2R,\nabla\varphi)B(U,\nabla^2R;X,Y)\\
&\qquad-\frac{4}{5}g(\nabla^2R,\nabla^2R)B(U,\nabla\varphi;X,Y)+\frac{4}{5}g(\nabla^2R,\nabla^2R)g(U,\nabla\varphi)B(\Box,\Box;X,Y)\\
&\qquad-\frac{4}{5}g(U,\nabla^2R)g(\nabla^2R,\nabla\varphi)B(\Box,\Box;X,Y)\Big]+R\left[D(U,Y,\Box,\Box,X)+D(U,X,\Box,\Box,Y)\right.\\
&\qquad\left.-2D(U,\Box,\Box,X,Y)\right]+\frac{1}{2}\left[D(U,\nabla\varphi,\Box,\Box,\nabla R;X,Y)-D(U,\Box,\Box,\nabla\varphi,\nabla R;X,Y)\right]\\
&\qquad+2R\left[D(U,\nabla^2\varphi,\Box,\Box,\nabla^2\varphi;X,Y)-D(U,\Box,\Box,\nabla^2\varphi,\nabla^2\varphi;X,Y)\right]\\
&\qquad+\frac{2R}{3}\left[E(U,\nabla\varphi;\Box,\Box;X,Y)-E(U,\Box;\Box,\nabla\varphi;X,Y)\right]\\
&\qquad-2E(U,\nabla^2\varphi;\nabla^2\varphi;X,Y)-2\frac{\partial E}{\partial\varphi}(U,\nabla\varphi;X,Y)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(-1)^q}{12}\Big[14g(X,U)B(Y,\nabla R)+14g(X,Y)B(U,\nabla R)+2g(X,\nabla R)B(Y,U)\\
&-8g(Y,U)B(X,\nabla R)-6g(\nabla R,U)B(X,Y)-6g(\nabla R,Y)B(X,U)\\
&+g(\nabla R,U)g(X,Y)B(\square,\square)+g(\nabla R,Y)g(X,U)B(\square,\square)\\
&-2g(X,\nabla R)g(Y,U)B(\square,\square)+14g(X,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;Y,U)\\
&+g(X,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;Y,U)-4g(\nabla^2\varphi,\nabla^2\varphi)B(X,\nabla R;Y,U)\\
&-6g(\nabla R,\nabla^2\varphi)B(X,\nabla^2\varphi;Y,U)+2g(\nabla R,\nabla^2\varphi)g(X,\nabla^2\varphi)B(\square,\square;Y,U)\\
&-2g(X,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;Y,U)+4R^2g(X,\nabla\varphi)B(\square,\square;Y,U)\\
&-\frac{8}{3}R^2B(X,\nabla\varphi;Y,U)+\frac{26}{5}g(X,\nabla\varphi)B(\nabla^2R,\nabla^2R;Y,U)\\
&+\frac{4}{5}g(X,\nabla^2R)B(\nabla\varphi,\nabla^2R;Y,U)-\frac{16}{5}g(\nabla^2R,\nabla\varphi)B(X,\nabla^2R;Y,U)\\
&-\frac{4}{5}g(\nabla^2R,\nabla^2R)B(X,\nabla\varphi;Y,U)+\frac{4}{5}g(\nabla^2R,\nabla^2R)g(X,\nabla\varphi)B(\square,\square;Y,U)\\
&-\frac{4}{5}g(X,\nabla^2R)g(\nabla^2R,\nabla\varphi)B(\square,\square;Y,U)\Big]+R\left[D(X,U,\square,\square,Y)+D(X,Y,\square,\square,U)\right.\\
&\left.-2D(X,\square,\square,Y,U)\right]+\frac{1}{2}\left[D(X,\nabla\varphi,\square,\square,\nabla R;Y,U)-D(X,\square,\square,\nabla\varphi,\nabla R;Y,U)\right]\\
&+2R\left[D(X,\nabla^2\varphi,\square,\square,\nabla^2\varphi;Y,U)-D(X,\square,\square,\nabla^2\varphi,\nabla^2\varphi;Y,U)\right]\\
&+\frac{2R}{3}\left[E(X,\nabla\varphi;\square,\square;Y,U)-E(X,\square;\square,\nabla\varphi;Y,U)\right]\\
&-2E(X,\nabla^2\varphi;\nabla^2\varphi;Y,U)-2\frac{\partial E}{\partial\varphi}(X,\nabla\varphi;Y,U)\\
&-\frac{(-1)^q}{12}\Big[14g(Y,X)B(U,\nabla R)+14g(Y,U)B(X,\nabla R)+2g(Y,\nabla R)B(U,X)\\
&-8g(U,X)B(Y,\nabla R)-6g(\nabla R,X)B(Y,U)-6g(\nabla R,U)B(Y,X)\\
&+g(\nabla R,X)g(Y,U)B(\square,\square)+g(\nabla R,U)g(Y,X)B(\square,\square)\\
&-2g(Y,\nabla R)g(U,X)B(\square,\square)+14g(Y,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;U,X)\\
&+g(Y,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;U,X)-4g(\nabla^2\varphi,\nabla^2\varphi)B(Y,\nabla R;U,X)\\
&-6g(\nabla R,\nabla^2\varphi)B(Y,\nabla^2\varphi;U,X)+2g(\nabla R,\nabla^2\varphi)g(Y,\nabla^2\varphi)B(\square,\square;U,X)\\
&-2g(Y,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;U,X)+4R^2g(Y,\nabla\varphi)B(\square,\square;U,X)\\
&-\frac{8}{3}R^2B(Y,\nabla\varphi;U,X)+\frac{26}{5}g(Y,\nabla\varphi)B(\nabla^2R,\nabla^2R;U,X)\\
&+\frac{4}{5}g(Y,\nabla^2R)B(\nabla\varphi,\nabla^2R;U,X)-\frac{16}{5}g(\nabla^2R,\nabla\varphi)B(Y,\nabla^2R;U,X)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4}{5}g(\nabla^2 R, \nabla^2 R)B(Y, \nabla\varphi; U, X) + \frac{4}{5}g(\nabla^2 R, \nabla^2 R)g(Y, \nabla\varphi)B(\Box, \Box; U, X) \\
&- \frac{4}{5}g(Y, \nabla^2 R)g(\nabla^2 R, \nabla\varphi)B(\Box, \Box; U, X)\Bigg] - R\left[D(Y, X, \Box, \Box, U) + D(Y, U, \Box, \Box, X)\right. \\
&\left.- 2D(Y, \Box, \Box, U, X)\right] - \frac{1}{2}\left[D(Y, \nabla\varphi, \Box, \Box, \nabla R; U, X) - D(Y, \Box, \Box, \nabla\varphi, \nabla R; U, X)\right] \\
&- 2R\left[D(Y, \nabla^2\varphi, \Box, \Box, \nabla^2\varphi; U, X) - D(Y, \Box, \Box, \nabla^2\varphi, \nabla^2\varphi; U, X)\right] \\
&- \frac{2R}{3}\left[E(Y, \nabla\varphi; \Box, \Box; U, X) - E(Y, \Box; \Box, \nabla\varphi; U, X)\right] \\
&+ 2E(Y, \nabla^2\varphi; \nabla^2\varphi; U, X) + 2\frac{\partial E}{\partial\varphi}(Y, \nabla\varphi; U, X)\Bigg\} \\
= \frac{(-1)^q}{24}&\Bigg[36g(U, X)B(Y, \nabla R) - 8g(U, Y)B(X, \nabla R) - 8g(X, Y)B(U, \nabla R) \\
&2g(U, \nabla R)B(X, Y) + 2g(X, \nabla R)B(U, Y) - 14g(Y, \nabla R)B(U, X) \\
&+ 4g(U, X)g(Y, \nabla R)B(\Box, \Box) - 2g(U, Y)g(X, \nabla R)B(\Box, \Box) \\
&- 2g(U, \nabla R)g(X, Y)B(\Box, \Box) + 14g(U, \nabla^2\varphi)B(\nabla^2\varphi, \nabla R; X, Y) \\
&+ g(U, \nabla R)B(\nabla^2\varphi, \nabla^2\varphi; X, Y) - 4g(\nabla^2\varphi, \nabla^2\varphi)B(U, \nabla R; X, Y) \\
&- 6g(\nabla R, \nabla^2\varphi)B(U, \nabla^2\varphi; X, Y) + 2g(U, \nabla^2\varphi)g(\nabla R, \nabla^2\varphi)B(\Box, \Box; X, Y) \\
&- 2g(U, \nabla R)g(\nabla^2\varphi, \nabla^2\varphi)B(\Box, \Box; X, Y) + 4R^2 g(U, \nabla\varphi)B(\Box, \Box; X, Y) \\
&- \frac{8}{3}R^2 B(U, \nabla\varphi; X, Y) + \frac{26}{5}g(U, \nabla\varphi)B(\nabla^2 R, \nabla^2 R; X, Y) \\
&+ \frac{4}{5}g(U, \nabla^2 R)B(\nabla\varphi, \nabla^2 R; X, Y) - \frac{16}{5}g(\nabla^2 R, \nabla\varphi)B(U, \nabla^2 R; X, Y) \\
&- \frac{4}{5}g(\nabla^2 R, \nabla^2 R)B(U, \nabla\varphi; X, Y) + \frac{4}{5}g(\nabla^2 R, \nabla^2 R)g(U, \nabla\varphi)B(\Box, \Box; X, Y) \\
&- \frac{4}{5}g(U, \nabla^2 R)g(\nabla^2 R, \nabla\varphi)B(\Box, \Box; X, Y) + 14g(X, \nabla^2\varphi)B(\nabla^2\varphi, \nabla R; Y, U) \\
&+ g(X, \nabla R)B(\nabla^2\varphi, \nabla^2\varphi; Y, U) - 4g(\nabla^2\varphi, \nabla^2\varphi)B(X, \nabla R; Y, U) \\
&- 6g(\nabla R, \nabla^2\varphi)B(X, \nabla^2\varphi; Y, U) + 2g(X, \nabla^2\varphi)g(\nabla R, \nabla^2\varphi)B(\Box, \Box; Y, U) \\
&- 2g(X, \nabla R)g(\nabla^2\varphi, \nabla^2\varphi)B(\Box, \Box; Y, U) + 4R^2 g(X, \nabla\varphi)B(\Box, \Box; Y, U) \\
&- \frac{8}{3}R^2 B(X, \nabla\varphi; Y, U) + \frac{26}{5}g(X, \nabla\varphi)B(\nabla^2 R, \nabla^2 R; Y, U) \\
&+ \frac{4}{5}g(X, \nabla^2 R)B(\nabla\varphi, \nabla^2 R; Y, U) - \frac{16}{5}g(\nabla^2 R, \nabla\varphi)B(X, \nabla^2 R; Y, U) \\
&- \frac{4}{5}g(\nabla^2 R, \nabla^2 R)B(X, \nabla\varphi; Y, U) + \frac{4}{5}g(\nabla^2 R, \nabla^2 R)g(X, \nabla\varphi)B(\Box, \Box; Y, U)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4}{5} g(X, \nabla^2 R) g(\nabla^2 R, \nabla\varphi) B(\Box, \Box; Y, U) - 14 g(Y, \nabla^2\varphi) B(\nabla^2\varphi, \nabla R; U, X) \\
&- g(Y, \nabla R) B(\nabla^2\varphi, \nabla^2\varphi; U, X) + 4 g(\nabla^2\varphi, \nabla^2\varphi) B(Y, \nabla R; U, X) \\
&+ 6 g(\nabla R, \nabla^2\varphi) B(Y, \nabla^2\varphi; U, X) - 2 g(Y, \nabla^2\varphi) g(\nabla R, \nabla^2\varphi) B(\Box, \Box; U, X) \\
&+ 2 g(Y, \nabla R) g(\nabla^2\varphi, \nabla^2\varphi) B(\Box, \Box; U, X) - 4 R^2 g(Y, \nabla\varphi) B(\Box, \Box; U, X) \\
&+ \frac{8}{3} R^2 B(Y, \nabla\varphi; U, X) - \frac{26}{5} g(Y, \nabla\varphi) B(\nabla^2 R, \nabla^2 R; U, X) \\
&- \frac{4}{5} g(Y, \nabla^2 R) B(\nabla\varphi, \nabla^2 R; U, X) + \frac{16}{5} g(\nabla^2 R, \nabla\varphi) B(Y, \nabla^2 R; U, X) \\
&+ \frac{4}{5} g(\nabla^2 R, \nabla^2 R) B(Y, \nabla\varphi; U, X) - \frac{4}{5} g(\nabla^2 R, \nabla^2 R) g(Y, \nabla\varphi) B(\Box, \Box; U, X) \\
&+ \frac{4}{5} g(Y, \nabla^2 R) g(\nabla^2 R, \nabla\varphi) B(\Box, \Box; U, X) \Bigg] \\
&+ R \left[D(U, X, \Box, \Box, Y) - D(U, \Box, \Box, X, Y) - D(X, \Box, \Box, Y, U) + D(Y, \Box, \Box, U, X) \right] \\
&+ \frac{1}{4} \left[D(U, \nabla\varphi, \Box, \Box, \nabla R; X, Y) - D(U, \Box, \Box, \nabla\varphi, \nabla R; X, Y) \right. \\
&+ D(X, \nabla\varphi, \Box, \Box, \nabla R; Y, U) - D(X, \Box, \Box, \nabla\varphi, \nabla R; Y, U) \\
&\left. - D(Y, \nabla\varphi, \Box, \Box, \nabla R; U, X) + D(Y, \Box, \Box, \nabla\varphi, \nabla R; U, X) \right] \\
&+ R \left[D(U, \nabla^2\varphi, \Box, \Box, \nabla^2\varphi; X, Y) - D(U, \Box, \Box, \nabla^2\varphi, \nabla^2\varphi; X, Y) \right. \\
&+ D(X, \nabla^2\varphi, \Box, \Box, \nabla^2\varphi; Y, U) - D(X, \Box, \Box, \nabla^2\varphi, \nabla^2\varphi; Y, U) \\
&\left. - D(Y, \nabla^2\varphi, \Box, \Box, \nabla^2\varphi; U, X) + D(Y, \Box, \Box, \nabla^2\varphi, \nabla^2\varphi; U, X) \right] \\
&+ \frac{R}{3} \left[E(U, \nabla\varphi; \Box, \Box; X, Y) - E(U, \Box; \Box, \nabla\varphi; X, Y) + E(X, \nabla\varphi; \Box, \Box; Y, U) \right. \\
&\left. - E(X, \Box; \Box, \nabla\varphi; Y, U) - E(Y, \nabla\varphi; \Box, \Box; U, X) + E(Y, \Box; \Box, \nabla\varphi; U, X) \right] \\
&+ 2 \left[-E(U, \nabla^2\varphi; \nabla^2\varphi; X, Y) - E(X, \nabla^2\varphi; \nabla^2\varphi; Y, U) + E(Y, \nabla^2\varphi; \nabla^2\varphi; U, X) \right] \\
&+ 2 \left[-\frac{\partial E}{\partial\varphi}(U, \nabla\varphi; X, Y) - \frac{\partial E}{\partial\varphi}(X, \nabla\varphi; Y, U) + \frac{\partial E}{\partial\varphi}(Y, \nabla\varphi; U, X) \right].
\end{aligned}
$$

To proceed further, we utilize the symmetry of E^{ij} and set $U = X$

$$
\begin{aligned}
&E(X, X; Y) \\
&\quad = \frac{(-1)^q}{24} \Bigg[36 g(X, X) B(Y, \nabla R) - 8 g(X, Y) B(X, \nabla R) - 8 g(X, Y) B(X, \nabla R) \\
&\qquad 2 g(X, \nabla R) B(X, Y) + 2 g(X, \nabla R) B(X, Y) - 14 g(Y, \nabla R) B(X, X)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4g(X,X)g(Y,\nabla R)B(\square,\square) - 2g(X,Y)g(X,\nabla R)B(\square,\square)\\
&- 2g(X,\nabla R)g(X,Y)B(\square,\square) + 14g(X,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;X,Y)\\
&+ g(X,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;X,Y) - 4g(\nabla^2\varphi,\nabla^2\varphi)B(X,\nabla R;X,Y)\\
&- 6g(\nabla R,\nabla^2\varphi)B(X,\nabla^2\varphi;X,Y) + 2g(X,\nabla^2\varphi)g(\nabla R,\nabla^2\varphi)B(\square,\square;X,Y)\\
&- 2g(X,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;X,Y) + 4R^2g(X,\nabla\varphi)B(\square,\square;X,Y)\\
&- \frac{8}{3}R^2B(X,\nabla\varphi;X,Y) + \frac{26}{5}g(X,\nabla\varphi)B(\nabla^2R,\nabla^2R;X,Y)\\
&+ \frac{4}{5}g(X,\nabla^2R)B(\nabla\varphi,\nabla^2R;X,Y) - \frac{16}{5}g(\nabla^2R,\nabla\varphi)B(X,\nabla^2R;X,Y)\\
&- \frac{4}{5}g(\nabla^2R,\nabla^2R)B(X,\nabla\varphi;X,Y) + \frac{4}{5}g(\nabla^2R,\nabla^2R)g(X,\nabla\varphi)B(\square,\square;X,Y)\\
&- \frac{4}{5}g(X,\nabla^2R)g(\nabla^2R,\nabla\varphi)B(\square,\square;X,Y) + 14g(X,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;Y,X)\\
&+ g(X,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;Y,X) - 4g(\nabla^2\varphi,\nabla^2\varphi)B(X,\nabla R;Y,X)\\
&- 6g(\nabla R,\nabla^2\varphi)B(X,\nabla^2\varphi;Y,X) + 2g(X,\nabla^2\varphi)g(\nabla R,\nabla^2\varphi)B(\square,\square;Y,X)\\
&- 2g(X,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;Y,X) + 4R^2g(X,\nabla\varphi)B(\square,\square;Y,X)\\
&- \frac{8}{3}R^2B(X,\nabla\varphi;Y,X) + \frac{26}{5}g(X,\nabla\varphi)B(\nabla^2R,\nabla^2R;Y,X)\\
&+ \frac{4}{5}g(X,\nabla^2R)B(\nabla\varphi,\nabla^2R;Y,X) - \frac{16}{5}g(\nabla^2R,\nabla\varphi)B(X,\nabla^2R;Y,X)\\
&- \frac{4}{5}g(\nabla^2R,\nabla^2R)B(X,\nabla\varphi;Y,X) + \frac{4}{5}g(\nabla^2R,\nabla^2R)g(X,\nabla\varphi)B(\square,\square;Y,X)\\
&- \frac{4}{5}g(X,\nabla^2R)g(\nabla^2R,\nabla\varphi)B(\square,\square;Y,X) - 14g(Y,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;X,X)\\
&- g(Y,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;X,X) + 4g(\nabla^2\varphi,\nabla^2\varphi)B(Y,\nabla R;X,X)\\
&+ 6g(\nabla R,\nabla^2\varphi)B(Y,\nabla^2\varphi;X,X) - 2g(Y,\nabla^2\varphi)g(\nabla R,\nabla^2\varphi)B(\square,\square;X,X)\\
&+ 2g(Y,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;X,X) - 4R^2g(Y,\nabla\varphi)B(\square,\square;X,X)\\
&+ \frac{8}{3}R^2B(Y,\nabla\varphi;X,X) - \frac{26}{5}g(Y,\nabla\varphi)B(\nabla^2R,\nabla^2R;X,X)\\
&- \frac{4}{5}g(Y,\nabla^2R)B(\nabla\varphi,\nabla^2R;X,X) + \frac{16}{5}g(\nabla^2R,\nabla\varphi)B(Y,\nabla^2R;X,X)\\
&+ \frac{4}{5}g(\nabla^2R,\nabla^2R)B(Y,\nabla\varphi;X,X) - \frac{4}{5}g(\nabla^2R,\nabla^2R)g(Y,\nabla\varphi)B(\square,\square;X,X)\\
&+ \frac{4}{5}g(Y,\nabla^2R)g(\nabla^2R,\nabla\varphi)B(\square,\square;X,X)\Big]\\
&+ R\left[D(X,X,\square,\square,Y) - D(X,\square,\square,X,Y) - D(X,\square,\square,Y,X) + D(Y,\square,\square,X,X)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{4}\left[D(X,\nabla\varphi,\square,\square,\nabla R;X,Y) - D(X,\square,\square,\nabla\varphi,\nabla R;X,Y)\right.\\
&+ D(X,\nabla\varphi,\square,\square,\nabla R;Y,X) - D(X,\square,\square,\nabla\varphi,\nabla R;Y,X)\\
&\left.- D(Y,\nabla\varphi,\square,\square,\nabla R;X,X) + D(Y,\square,\square,\nabla\varphi,\nabla R;X,X)\right]\\
&+ R\left[D(X,\nabla^2\varphi,\square,\square,\nabla^2\varphi;X,Y) - D(X,\square,\square,\nabla^2\varphi,\nabla^2\varphi;X,Y)\right.\\
&+ D(X,\nabla^2\varphi,\square,\square,\nabla^2\varphi;Y,X) - D(X,\square,\square,\nabla^2\varphi,\nabla^2\varphi;Y,X)\\
&\left.- D(Y,\nabla^2\varphi,\square,\square,\nabla^2\varphi;X,X) + D(Y,\square,\square,\nabla^2\varphi,\nabla^2\varphi;X,X)\right]\\
&+ \frac{R}{3}\left[E(X,\nabla\varphi;\square,\square;X,Y) - E(X,\square;\square,\nabla\varphi;X,Y) + E(X,\nabla\varphi;\square,\square;Y,X)\right.\\
&\left.- E(X,\square;\square,\nabla\varphi;Y,X) - E(Y,\nabla\varphi;\square,\square;X,X) + E(Y,\square;\square,\nabla\varphi;X,X)\right]\\
&+ 2\left[-E(X,\nabla^2\varphi;\nabla^2\varphi;X,Y) - E(X,\nabla^2\varphi;\nabla^2\varphi;Y,X) + E(Y,\nabla^2\varphi;\nabla^2\varphi;X,X)\right]\\
&+ 2\left[-\frac{\partial E}{\partial\varphi}(X,\nabla\varphi;X,Y) - \frac{\partial E}{\partial\varphi}(X,\nabla\varphi;Y,X) + \frac{\partial E}{\partial\varphi}(Y,\nabla\varphi;X,X)\right]\\
= \frac{(-1)^q}{24}&\left[36g(X,X)B(Y,\nabla R) - 16g(X,Y)B(X,\nabla R) + 4g(X,\nabla R)B(X,Y)\right.\\
&- 14g(Y,\nabla R)B(X,X) + 4g(X,X)g(Y,\nabla R)B(\square,\square)\\
&- 4g(X,Y)g(X,\nabla R)B(\square,\square) + 28g(X,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;X,Y)\\
&+ 2g(X,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;X,Y) - 8g(\nabla^2\varphi,\nabla^2\varphi)B(X,\nabla R;X,Y)\\
&- 12g(\nabla R,\nabla^2\varphi)B(X,\nabla^2\varphi;X,Y) + 4g(X,\nabla^2\varphi)g(\nabla R,\nabla^2\varphi)B(\square,\square;X,Y)\\
&- 4g(X,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;X,Y) + 8R^2g(X,\nabla\varphi)B(\square,\square;X,Y)\\
&- \frac{16}{3}R^2B(X,\nabla\varphi;X,Y) + \frac{52}{5}g(X,\nabla\varphi)B(\nabla^2R,\nabla^2R;X,Y)\\
&+ \frac{8}{5}g(X,\nabla^2R)B(\nabla\varphi,\nabla^2R;X,Y) - \frac{32}{5}g(\nabla^2R,\nabla\varphi)B(X,\nabla^2R;X,Y)\\
&- \frac{8}{5}g(\nabla^2R,\nabla^2R)B(X,\nabla\varphi;X,Y) + \frac{8}{5}g(\nabla^2R,\nabla^2R)g(X,\nabla\varphi)B(\square,\square;X,Y)\\
&- \frac{8}{5}g(X,\nabla^2R)g(\nabla^2R,\nabla\varphi)B(\square,\square;X,Y) - 14g(Y,\nabla^2\varphi)B(\nabla^2\varphi,\nabla R;X,X)\\
&- g(Y,\nabla R)B(\nabla^2\varphi,\nabla^2\varphi;X,X) + 4g(\nabla^2\varphi,\nabla^2\varphi)B(Y,\nabla R;X,X)\\
&+ 6g(\nabla R,\nabla^2\varphi)B(Y,\nabla^2\varphi;X,X) - 2g(Y,\nabla^2\varphi)g(\nabla R,\nabla^2\varphi)B(\square,\square;X,X)\\
&+ 2g(Y,\nabla R)g(\nabla^2\varphi,\nabla^2\varphi)B(\square,\square;X,X) - 4R^2g(Y,\nabla\varphi)B(\square,\square;X,X)\\
&+ \frac{8}{3}R^2B(Y,\nabla\varphi;X,X) - \frac{26}{5}g(Y,\nabla\varphi)B(\nabla^2R,\nabla^2R;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{4}{5} g(Y, \nabla^2 R) B(\nabla\varphi, \nabla^2 R; X, X) + \frac{16}{5} g(\nabla^2 R, \nabla\varphi) B(Y, \nabla^2 R; X, X) \\
&+ \frac{4}{5} g(\nabla^2 R, \nabla^2 R) B(Y, \nabla\varphi; X, X) - \frac{4}{5} g(\nabla^2 R, \nabla^2 R) g(Y, \nabla\varphi) B(\Box, \Box; X, X) \\
&+ \left. \frac{4}{5} g(Y, \nabla^2 R) g(\nabla^2 R, \nabla\varphi) B(\Box, \Box; X, X) \right] \\
&+ R \left[D(X, X, \Box, \Box, Y) - 2D(X, \Box, \Box, X, Y) + D(Y, \Box, \Box, X, X) \right] \\
&+ \frac{1}{4} [2D(X, \nabla\varphi, \Box, \Box, \nabla R; X, Y) - 2D(X, \Box, \Box, \nabla\varphi, \nabla R; X, Y) \\
&- D(Y, \nabla\varphi, \Box, \Box, \nabla R; X, X) + D(Y, \Box, \Box, \nabla\varphi, \nabla R; X, X)] \\
&+ R \left[2D(X, \nabla^2\varphi, \Box, \Box, \nabla^2\varphi; X, Y) - 2D(X, \Box, \Box, \nabla^2\varphi, \nabla^2\varphi; X, Y) \right. \\
&\left. - D(Y, \nabla^2\varphi, \Box, \Box, \nabla^2\varphi; X, X) + D(Y, \Box, \Box, \nabla^2\varphi, \nabla^2\varphi; X, X) \right] \\
&+ \frac{R}{3} [2E(X, \nabla\varphi; \Box, \Box; X, Y) - 2E(X, \Box; \Box, \nabla\varphi; X, Y) \\
&- E(Y, \nabla\varphi; \Box, \Box; X, X) + E(Y, \Box; \Box, \nabla\varphi; X, X)] \\
&+ 2 \left[-2E(X, \nabla^2\varphi; \nabla^2\varphi; X, Y) + E(Y, \nabla^2\varphi; \nabla^2\varphi; X, X) \right] \\
&+ 2 \left[-2 \frac{\partial E}{\partial\varphi}(X, \nabla\varphi; X, Y) + \frac{\partial E}{\partial\varphi}(Y, \nabla\varphi; X, X) \right].
\end{aligned}
\tag{C.17}
$$

Next, we utilize (C.12) to express the D terms with no derivatives

$$
\begin{aligned}
&D(X, X, \Box, \Box, Y) \\
&\quad = \frac{(-1)^q}{3} \left\{ \left[2g(X, X) g(Y, \nabla^2\varphi) Q(\nabla^2\varphi) - 2g(X, \nabla^2\varphi) g(X, Y) Q(\nabla^2\varphi) \right. \right. \\
&\qquad - \frac{2}{3} R g(X, X) g(Y, \nabla\varphi) B(\Box_a, \Box_a) + \frac{2}{3} R g(X, \nabla\varphi) g(X, Y) B(\Box_a, \Box_a) \\
&\qquad \left. + \frac{2}{3} R g(X, X) B(\nabla\varphi, Y) - \frac{2}{3} R g(X, Y) B(\nabla\varphi, X) \right] (; \Box, \Box) \\
&\qquad + 2 \left[2g(X, X) g(\Box, \nabla^2\varphi) Q(\nabla^2\varphi) - 2g(X, \nabla^2\varphi) g(X, \Box) Q(\nabla^2\varphi) \right. \\
&\qquad - \frac{2}{3} R g(X, X) g(\Box, \nabla\varphi) B(\Box_a, \Box_a) + \frac{2}{3} R g(X, \nabla\varphi) g(X, \Box) B(\Box_a, \Box_a) \\
&\qquad \left. + \frac{2}{3} R g(X, X) B(\nabla\varphi, \Box) - \frac{2}{3} R g(X, \Box) B(\nabla\varphi, X) \right] (; Y, \Box) \\
&\qquad + 2g(X, X) g(Y, \nabla\varphi) \frac{\partial B}{\partial\varphi}(\Box, \Box) + 4g(X, X) g(\Box, \nabla\varphi) \frac{\partial B}{\partial\varphi}(Y, \Box) \\
&\qquad - 2g(X, \nabla\varphi) g(X, Y) \frac{\partial B}{\partial\varphi}(\Box, \Box) - 4g(X, \nabla\varphi) g(X, \Box) \frac{\partial B}{\partial\varphi}(Y, \Box)
\end{aligned}
$$

$$
\begin{aligned}
& - 2g(X,X)g(Y,\Box)Q(\Box) - g(X,X)g(\Box,\Box)Q(Y) + 2g(X,\Box)g(X,Y)Q(\Box) \\
& + g(X,\Box)g(X,\Box)Q(Y)\Big\} \\
= {} & \frac{(-1)^q}{3}\Bigg\{\Bigg[2g(X,X)g(Y,\Box)Q(\Box) + 2g(X,X)g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box) \\
& - 2g(X,\Box)g(X,Y)Q(\Box) - 2g(X,\nabla^2\varphi)g(X,Y)Q(\nabla^2\varphi;\Box,\Box) \\
& - \frac{2}{3}Rg(X,X)g(Y,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box) + \frac{2}{3}Rg(X,\nabla\varphi)g(X,Y)B(\Box_a,\Box_a;\Box,\Box) \\
& + \frac{2}{3}Rg(X,X)B(\nabla\varphi,Y;\Box,\Box) - \frac{2}{3}Rg(X,Y)B(\nabla\varphi,X;\Box,\Box)\Bigg] \\
& + 2\Bigg[g(X,X)g(\Box,Y)Q(\Box) + g(X,X)g(\Box,\Box)Q(Y) \\
& + 2g(X,X)g(\Box,\nabla^2\varphi)Q(\nabla^2\varphi;Y,\Box) - g(X,Y)g(X,\Box)Q(\Box) \\
& - g(X,\Box)g(X,\Box)Q(Y) - 2g(X,\nabla^2\varphi)g(X,\Box)Q(\nabla^2\varphi;Y,\Box) \\
& - \frac{2}{3}Rg(X,X)g(\Box,\nabla\varphi)B(\Box_a,\Box_a;Y,\Box) + \frac{2}{3}Rg(X,\nabla\varphi)g(X,\Box)B(\Box_a,\Box_a;Y,\Box) \\
& + \frac{2}{3}Rg(X,X)B(\nabla\varphi,\Box;Y,\Box) - \frac{2}{3}Rg(X,\Box)B(\nabla\varphi,X;Y,\Box)\Bigg] \\
& + 2g(X,X)g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box) + 4g(X,X)\frac{\partial B}{\partial\varphi}(Y,\nabla\varphi) \\
& - 2g(X,\nabla\varphi)g(X,Y)\frac{\partial B}{\partial\varphi}(\Box,\Box) - 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y) \\
& - 2g(X,X)Q(Y) - 2g(X,X)Q(Y) + 2g(X,Y)Q(X) + g(X,X)Q(Y)\Bigg\} \\
= {} & \frac{(-1)^q}{3}\Bigg[2g(X,X)Q(Y) + 2g(X,X)g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box) \\
& - 2g(X,Y)Q(X) - 2g(X,\nabla^2\varphi)g(X,Y)Q(\nabla^2\varphi;\Box,\Box) \\
& - \frac{2}{3}Rg(X,X)g(Y,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box) + \frac{2}{3}Rg(X,\nabla\varphi)g(X,Y)B(\Box_a,\Box_a;\Box,\Box) \\
& + \frac{2}{3}Rg(X,X)B(\nabla\varphi,Y;\Box,\Box) - \frac{2}{3}Rg(X,Y)B(\nabla\varphi,X;\Box,\Box) \\
& + 2g(X,X)Q(Y) + 4g(X,X)Q(Y) + 4g(X,X)Q(\nabla^2\varphi;Y,\nabla^2\varphi) \\
& - 2g(X,Y)Q(X) - 2g(X,X)Q(Y) - 4g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y) \\
& - \frac{4}{3}Rg(X,X)B(\Box_a,\Box_a;Y,\nabla\varphi) + \frac{4}{3}Rg(X,\nabla\varphi)B(\Box_a,\Box_a;X,Y) \\
& + \frac{4}{3}Rg(X,X)B(\nabla\varphi,\Box;Y,\Box) - \frac{4}{3}RB(\nabla\varphi,X;X,Y)
\end{aligned}
$$

$$
\begin{aligned}
&+2g(X,X)g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square)+4g(X,X)\frac{\partial B}{\partial\varphi}(Y,\nabla\varphi)\\
&-2g(X,\nabla\varphi)g(X,Y)\frac{\partial B}{\partial\varphi}(\square,\square)-4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y)\\
&-3g(X,X)Q(Y)+2g(X,Y)Q(X)\Bigg]\\
=\;&\frac{(-1)^q}{3}\Bigg[3g(X,X)Q(Y)-2g(X,Y)Q(X)+2g(X,X)g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square)\\
&-2g(X,\nabla^2\varphi)g(X,Y)Q(\nabla^2\varphi;\square,\square)+4g(X,X)Q(\nabla^2\varphi;Y,\nabla^2\varphi)\\
&-4g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y)+2g(X,X)g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square)\\
&-2g(X,\nabla\varphi)g(X,Y)\frac{\partial B}{\partial\varphi}(\square,\square)+4g(X,X)\frac{\partial B}{\partial\varphi}(Y,\nabla\varphi)-4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y)\\
&-\frac{2}{3}Rg(X,X)g(Y,\nabla\varphi)B(\square_a,\square_a;\square,\square)+\frac{2}{3}Rg(X,\nabla\varphi)g(X,Y)B(\square_a,\square_a;\square,\square)\\
&-\frac{2}{3}Rg(X,X)B(\nabla\varphi,Y;\square,\square)-\frac{2}{3}Rg(X,Y)B(\nabla\varphi,X;\square,\square)\\
&+\frac{4}{3}Rg(X,\nabla\varphi)B(X,Y;\square,\square)+\frac{4}{3}Rg(X,X)B(\nabla\varphi,\square;Y,\square)\\
&-\frac{4}{3}RB(\nabla\varphi,X;X,Y)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
&D(X,\square,\square,X,Y)\\
&=\frac{(-1)^q}{6}\Bigg\{2\Bigg[2g(X,\square)g(X,\nabla^2\varphi)Q(\nabla^2\varphi)-g(\square,\nabla^2\varphi)g(X,X)Q(\nabla^2\varphi)\\
&-g(X,\nabla^2\varphi)g(\square,X)Q(\nabla^2\varphi)-\frac{2}{3}Rg(X,\square)g(X,\nabla\varphi)B(\square_a,\square_a)\\
&+\frac{1}{3}Rg(\square,\nabla\varphi)g(X,X)B(\square_a,\square_a)+\frac{1}{3}Rg(X,\nabla\varphi)g(\square,X)B(\square_a,\square_a)\\
&+\frac{2}{3}Rg(X,\square)B(\nabla\varphi,X)-\frac{1}{3}Rg(\square,X)B(\nabla\varphi,X)-\frac{1}{3}Rg(X,X)B(\nabla\varphi,\square)\Bigg](;\square,Y)\\
&+2\Bigg[2g(X,\square)g(\square,\nabla^2\varphi)Q(\nabla^2\varphi)-g(\square,\nabla^2\varphi)g(X,\square)Q(\nabla^2\varphi)\\
&-g(X,\nabla^2\varphi)g(\square,\square)Q(\nabla^2\varphi)-\frac{2}{3}Rg(X,\square)g(\square,\nabla\varphi)B(\square_a,\square_a)\\
&+\frac{1}{3}Rg(\square,\nabla\varphi)g(X,\square)B(\square_a,\square_a)+\frac{1}{3}Rg(X,\nabla\varphi)g(\square,\square)B(\square_a,\square_a)\\
&+\frac{2}{3}Rg(X,\square)B(\nabla\varphi,\square)-\frac{1}{3}Rg(\square,\square)B(\nabla\varphi,X)-\frac{1}{3}Rg(X,\square)B(\nabla\varphi,\square)\Bigg](;X,Y)\\
&+2\Bigg[2g(X,\square)g(Y,\nabla^2\varphi)Q(\nabla^2\varphi)-g(\square,\nabla^2\varphi)g(X,Y)Q(\nabla^2\varphi)\\
&-g(X,\nabla^2\varphi)g(\square,Y)Q(\nabla^2\varphi)-\frac{2}{3}Rg(X,\square)g(Y,\nabla\varphi)B(\square_a,\square_a)
\end{aligned}
$$

$$+ \frac{1}{3}Rg(\Box, \nabla\varphi)g(X,Y)B(\Box_a,\Box_a) + \frac{1}{3}Rg(X,\nabla\varphi)g(\Box,Y)B(\Box_a,\Box_a)$$

$$+ \frac{2}{3}Rg(X,\Box)B(\nabla\varphi,Y) - \frac{1}{3}Rg(\Box,Y)B(\nabla\varphi,X) - \frac{1}{3}Rg(X,Y)B(\nabla\varphi,\Box)\Big](;X,\Box)$$

$$+ 4g(X,\Box)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,Y) + 4g(X,\Box)g(\Box,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y)$$

$$+ 4g(X,\Box)g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\Box) - 2g(\Box,\nabla\varphi)g(X,X)\frac{\partial B}{\partial\varphi}(\Box,Y)$$

$$- 2g(\Box,\nabla\varphi)g(X,\Box)\frac{\partial B}{\partial\varphi}(X,Y) - 2g(\Box,\nabla\varphi)g(X,Y)\frac{\partial B}{\partial\varphi}(X,\Box)$$

$$- 2g(X,\nabla\varphi)g(\Box,X)\frac{\partial B}{\partial\varphi}(\Box,Y) - 2g(X,\nabla\varphi)g(\Box,\Box)\frac{\partial B}{\partial\varphi}(X,Y)$$

$$- 2g(X,\nabla\varphi)g(\Box,Y)\frac{\partial B}{\partial\varphi}(X,\Box) - 2g(X,\Box)g(X,\Box)Q(Y) - 2g(X,\Box)g(X,Y)Q(\Box)$$

$$- 2g(X,\Box)g(\Box,Y)Q(X) + g(X,X)g(\Box,\Box)Q(Y) + g(X,X)g(\Box,Y)Q(\Box)$$

$$+ g(X,\Box)g(\Box,X)Q(Y) + g(X,Y)g(\Box,X)Q(\Box) + g(X,\Box)g(\Box,Y)Q(X)$$

$$+ g(X,Y)g(\Box,\Box)Q(X)\Big\}$$

$$= \frac{(-1)^q}{6}\Big\{2\Big[g(X,\Box)g(X,\Box)Q(Y) + g(X,\Box)g(X,Y)Q(\Box)$$

$$+ 2g(X,\Box)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,Y) - \frac{1}{2}g(\Box,\Box)g(X,X)Q(Y)$$

$$- \frac{1}{2}g(\Box,Y)g(X,X)Q(\Box) - g(\Box,\nabla^2\varphi)g(X,X)Q(\nabla^2\varphi;\Box,Y)$$

$$- \frac{1}{2}g(X,\Box)g(\Box,X)Q(Y) - \frac{1}{2}g(X,Y)g(\Box,X)Q(\Box)$$

$$- g(X,\nabla^2\varphi)g(\Box,X)Q(\nabla^2\varphi;\Box,Y) - \frac{2}{3}Rg(X,\Box)g(X,\nabla\varphi)B(\Box_a,\Box_a;\Box,Y)$$

$$+ \frac{1}{3}Rg(\Box,\nabla\varphi)g(X,X)B(\Box_a,\Box_a;\Box,Y) + \frac{1}{3}Rg(X,\nabla\varphi)g(\Box,X)B(\Box_a,\Box_a;\Box,Y)$$

$$+ \frac{2}{3}Rg(X,\Box)B(\nabla\varphi,X;\Box,Y) - \frac{1}{3}Rg(\Box,X)B(\nabla\varphi,X;\Box,Y)$$

$$- \frac{1}{3}Rg(X,X)B(\nabla\varphi,\Box;\Box,Y)\Big] + 2\Big[2g(X,\nabla^2\varphi)Q(\nabla^2\varphi) - g(X,\nabla^2\varphi)Q(\nabla^2\varphi)$$

$$- 2g(X,\nabla^2\varphi)Q(\nabla^2\varphi) - \frac{2}{3}Rg(X,\nabla\varphi)B(\Box_a,\Box_a) + \frac{1}{3}Rg(X,\nabla\varphi)B(\Box_a,\Box_a)$$

$$+ \frac{2}{3}Rg(X,\nabla\varphi)B(\Box_a,\Box_a) + \frac{2}{3}RB(\nabla\varphi,X) - \frac{2}{3}RB(\nabla\varphi,X)$$

$$- \frac{1}{3}RB(\nabla\varphi,X)\Big](;X,Y) + 2\Big[g(X,\Box)g(Y,X)Q(\Box) + g(X,\Box)g(Y,\Box)Q(X)$$

$$+ 2g(X,\Box)g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,\Box) - \frac{1}{2}g(\Box,X)g(X,Y)Q(\Box)$$

$$
\begin{aligned}
&- \frac{1}{2} g(\Box, \Box) g(X, Y) Q(X) - g(\Box, \nabla^2\varphi) g(X, Y) Q(\nabla^2\varphi; X, \Box) \\
&- \frac{1}{2} g(X, X) g(\Box, Y) Q(\Box) - \frac{1}{2} g(X, \Box) g(\Box, Y) Q(X) \\
&- g(X, \nabla^2\varphi) g(\Box, Y) Q(\nabla^2\varphi; X, \Box) - \frac{2}{3} Rg(X, \Box) g(Y, \nabla\varphi) B(\Box_a, \Box_a; X, \Box) \\
&+ \frac{1}{3} Rg(\Box, \nabla\varphi) g(X, Y) B(\Box_a, \Box_a; X, \Box) + \frac{1}{3} Rg(X, \nabla\varphi) g(\Box, Y) B(\Box_a, \Box_a; X, \Box) \\
&+ \frac{2}{3} Rg(X, \Box) B(\nabla\varphi, Y; X, \Box) - \frac{1}{3} Rg(\Box, Y) B(\nabla\varphi, X; X, \Box) \\
&- \frac{1}{3} Rg(X, Y) B(\nabla\varphi, \Box; X, \Box) \Bigg] + 4g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(X, Y) + 4g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(X, Y) \\
&+ 4g(Y, \nabla\varphi) \frac{\partial B}{\partial\varphi}(X, X) - 2g(X, X) \frac{\partial B}{\partial\varphi}(\nabla\varphi, Y) - 2g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(X, Y) \\
&- 2g(X, Y) \frac{\partial B}{\partial\varphi}(X, \nabla\varphi) - 2g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(X, Y) - 4g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(X, Y) \\
&- 2g(X, \nabla\varphi) \frac{\partial B}{\partial\varphi}(X, Y) - 2g(X, X) Q(Y) - 2g(X, Y) Q(X) - 2g(X, Y) Q(X) \\
&+ 2g(X, X) Q(Y) + g(X, X) Q(Y) + g(X, X) Q(Y) + g(X, Y) Q(X) + g(X, Y) Q(X) \\
&+ 2g(X, Y) Q(X) \Bigg\} \\
= &\frac{(-1)^q}{6} \Bigg\{ 2 \Bigg[g(X, X) Q(Y) + g(X, Y) Q(X) + 2g(X, \nabla^2\varphi) Q(\nabla^2\varphi; X, Y) \\
&- g(X, X) Q(Y) - \frac{1}{2} g(X, X) Q(Y) - g(X, X) Q(\nabla^2\varphi; \nabla^2\varphi, Y) \\
&- \frac{1}{2} g(X, X) Q(Y) - \frac{1}{2} g(X, Y) Q(X) - g(X, \nabla^2\varphi) Q(\nabla^2\varphi; X, Y) \\
&- \frac{2}{3} Rg(X, \nabla\varphi) B(\Box_a, \Box_a; X, Y) + \frac{1}{3} Rg(X, X) B(\Box_a, \Box_a; \nabla\varphi, Y) \\
&+ \frac{1}{3} Rg(X, \nabla\varphi) B(\Box_a, \Box_a; X, Y) + \frac{2}{3} RB(\nabla\varphi, X; X, Y) - \frac{1}{3} RB(\nabla\varphi, X; X, Y) \\
&- \frac{1}{3} Rg(X, X) B(\nabla\varphi, \Box; \Box, Y) \Bigg] + 2 \Bigg[-g(X, \nabla^2\varphi) Q(\nabla^2\varphi) + \frac{1}{3} Rg(X, \nabla\varphi) B(\Box_a, \Box_a) \\
&- \frac{1}{3} RB(\nabla\varphi, X) \Bigg] (; X, Y) + 2 \Bigg[g(Y, X) Q(X) + g(X, Y) Q(X) \\
&+ 2g(Y, \nabla^2\varphi) Q(\nabla^2\varphi; X, X) - \frac{1}{2} g(X, Y) Q(X) - g(X, Y) Q(X) \\
&- g(X, Y) Q(\nabla^2\varphi; X, \nabla^2\varphi) - \frac{1}{2} g(X, X) Q(Y) - \frac{1}{2} g(X, Y) Q(X) \\
&- g(X, \nabla^2\varphi) Q(\nabla^2\varphi; X, Y) - \frac{2}{3} R(Y, \nabla\varphi) B(\Box_a, \Box_a; X, X) \\
&+ \frac{1}{3} Rg(X, Y) B(\Box_a, \Box_a; X, \nabla\varphi) + \frac{1}{3} Rg(X, \nabla\varphi) B(\Box_a, \Box_a; X, Y)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2}{3}RB(\nabla\varphi, Y; X, X) - \frac{1}{3}RB(\nabla\varphi, X; X, Y) - \frac{1}{3}Rg(X,Y)B(\nabla\varphi, \Box; X, \Box)\Big] \\
&+ 4g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) - 2g(X,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi, Y) - 2g(X,Y)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi) \\
&- 2g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y) + 2g(X,X)Q(Y)\Big\} \\
= \frac{(-1)^q}{6}&\Big\{2\Big[\frac{1}{2}g(X,Y)Q(X) - g(X,X)Q(Y) + g(X,\nabla^2\varphi)Q(\nabla^2\varphi; X, Y) \\
&- g(X,X)Q(\nabla^2\varphi; \nabla^2\varphi, Y) - \frac{1}{3}Rg(X,\nabla\varphi)B(\Box,\Box; X, Y) \\
&+ \frac{1}{3}Rg(X,X)B(\Box,\Box;\nabla\varphi, Y) + \frac{1}{3}RB(\nabla\varphi, X; X, Y) \\
&- \frac{1}{3}Rg(X,X)B(\nabla\varphi,\Box;\Box, Y)\Big] + 2\Big[-\frac{1}{2}g(X,X)Q(Y) - \frac{1}{2}g(X,Y)Q(X) \\
&- g(X,\nabla^2\varphi)Q(\nabla^2\varphi; X, Y) + \frac{1}{3}Rg(X,\nabla\varphi)B(\Box,\Box; X, Y) \\
&- \frac{1}{3}RB(\nabla\varphi, X; X, Y)\Big] + 2\Big[-\frac{1}{2}g(X,X)Q(Y) + 2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi; X, X) \\
&- g(X,Y)Q(\nabla^2\varphi; X, \nabla^2\varphi) - g(X,\nabla^2\varphi)Q(\nabla^2\varphi; X, Y) - \frac{2}{3}R(Y,\nabla\varphi)B(\Box,\Box; X, X) \\
&+ \frac{1}{3}Rg(X,Y)B(\Box,\Box; X, \nabla\varphi) + \frac{1}{3}Rg(X,\nabla\varphi)B(\Box,\Box; X, Y) \\
&+ \frac{2}{3}RB(\nabla\varphi, Y; X, X) - \frac{1}{3}RB(\nabla\varphi, X; X, Y) - \frac{1}{3}Rg(X,Y)B(\nabla\varphi,\Box; X, \Box)\Big] \\
&+ 4g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) - 2g(X,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi, Y) - 2g(X,Y)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi) \\
&- 2g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y) + 2g(X,X)Q(Y)\Big\} \\
= \frac{(-1)^q}{6}&\Big[-2g(X,X)Q(Y) + 4g(Y,\nabla^2\varphi)Q(\nabla^2\varphi; X, X) - 2g(X,\nabla^2\varphi)Q(\nabla^2\varphi; X, Y) \\
&- 2g(X,X)Q(\nabla^2\varphi;\nabla^2\varphi, Y) - 2g(X,Y)Q(\nabla^2\varphi; X, \nabla^2\varphi) \\
&+ \frac{2}{3}Rg(X,X)B(\Box,\Box; Y, \nabla\varphi) - \frac{2}{3}Rg(X,X)B(\nabla\varphi,\Box; Y, \Box) \\
&- \frac{2}{3}Rg(X,Y)B(\nabla\varphi,\Box; X, \Box) + \frac{2}{3}Rg(X,Y)B(\Box,\Box; X, \nabla\varphi) \\
&+ \frac{2}{3}Rg(X,\nabla\varphi)B(\Box,\Box; X, Y) - \frac{4}{3}R(Y,\nabla\varphi)B(\Box,\Box; X, X) \\
&+ \frac{4}{3}RB(\nabla\varphi, Y; X, X) - \frac{2}{3}RB(\nabla\varphi, X; X, Y) + 4g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) \\
&- 2g(X,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi, Y) - 2g(X,Y)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi) - 2g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y)\Big],
\end{aligned}
$$

and

$$
\begin{aligned}
&D(Y,\square,\square,X,X)\\
&\quad=\frac{(-1)^q}{3}\Bigg\{\Bigg[2g(\square,Y)g(\square,\nabla^2\varphi)Q(\nabla^2\varphi)-g(\square,\nabla^2\varphi)g(Y,\square)Q(\nabla^2\varphi)\\
&\qquad-g(Y,\nabla^2\varphi)g(\square,\square)Q(\nabla^2\varphi)-\frac{2}{3}Rg(Y,\square)g(\square,\nabla\varphi)B(\square_a,\square_a)\\
&\qquad+\frac{1}{3}Rg(\square,\nabla\varphi)g(Y,\square)B(\square_a,\square_a)+\frac{1}{3}Rg(Y,\nabla\varphi)g(\square,\square)B(\square_a,\square_a)\\
&\qquad+\frac{2}{3}Rg(Y,\square)B(\nabla\varphi,\square)-\frac{1}{3}Rg(\square,\square)B(\nabla\varphi,Y)\\
&\qquad-\frac{1}{3}Rg(Y,\square)B(\nabla\varphi,\square)\Bigg](;X,X)+2\Bigg[2g(\square,Y)g(X,\nabla^2\varphi)Q(\nabla^2\varphi)\\
&\qquad-g(\square,\nabla^2\varphi)g(Y,X)Q(\nabla^2\varphi)-g(Y,\nabla^2\varphi)g(\square,X)Q(\nabla^2\varphi)\\
&\qquad-\frac{2}{3}Rg(Y,\square)g(X,\nabla\varphi)B(\square_a,\square_a)+\frac{1}{3}Rg(\square,\nabla\varphi)g(Y,X)B(\square_a,\square_a)\\
&\qquad+\frac{1}{3}Rg(Y,\nabla\varphi)g(\square,X)B(\square_a,\square_a)+\frac{2}{3}Rg(Y,\square)B(\nabla\varphi,X)\\
&\qquad-\frac{1}{3}Rg(\square,X)B(\nabla\varphi,Y)-\frac{1}{3}Rg(Y,X)B(\nabla\varphi,\square)\Bigg](;X,\square)\\
&\qquad+2g(Y,\square)g(\square,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X)+4g(Y,\square)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\square)\\
&\qquad-g(\square,\nabla\varphi)g(Y,\square)\frac{\partial B}{\partial\varphi}(X,X)-2g(\square,\nabla\varphi)g(Y,X)\frac{\partial B}{\partial\varphi}(X,\square)\\
&\qquad-g(Y,\nabla\varphi)g(\square,\square)\frac{\partial B}{\partial\varphi}(X,X)-2g(Y,\nabla\varphi)g(\square,X)\frac{\partial B}{\partial\varphi}(X,\square)\\
&\qquad-2g(Y,\square)g(X,\square)Q(X)-g(Y,\square)g(X,X)Q(\square)+g(Y,\square)g(\square,X)Q(X)\\
&\qquad+g(Y,X)g(\square,\square)Q(X)+g(Y,X)g(\square,X)Q(\square)\Bigg\}\\
&\quad=\frac{(-1)^q}{3}\Bigg\{\Bigg[2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi)-g(Y,\nabla^2\varphi)Q(\nabla^2\varphi)-2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi)\\
&\qquad-\frac{2}{3}Rg(Y,\nabla\varphi)B(\square_a,\square_a)+\frac{1}{3}Rg(Y,\nabla\varphi)B(\square_a,\square_a)+\frac{2}{3}Rg(Y,\nabla\varphi)B(\square_a,\square_a)\\
&\qquad+\frac{2}{3}RB(Y,\nabla\varphi)-\frac{2}{3}RB(\nabla\varphi,Y)-\frac{1}{3}RB(Y,\nabla\varphi)\Bigg](;X,X)\\
&\qquad+2\Bigg[g(\square,Y)g(X,X)Q(\square)+g(\square,Y)g(X,\square)Q(X)\\
&\qquad+2g(\square,Y)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,\square)-\frac{1}{2}g(\square,X)g(Y,X)Q(\square)\\
&\qquad-\frac{1}{2}g(\square,\square)g(Y,X)Q(X)-g(\square,\nabla^2\varphi)g(Y,X)Q(\nabla^2\varphi;X,\square)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} g(Y,X)g(\square,X)Q(\square) - \frac{1}{2} g(Y,\square)g(\square,X)Q(X) \\
&- g(Y,\nabla^2\varphi)g(\square,X)Q(\nabla^2\varphi;X,\square) - \frac{2}{3} Rg(Y,\square)g(X,\nabla\varphi)B(\square_a,\square_a;X,\square) \\
&+ \frac{1}{3} Rg(\square,\nabla\varphi)g(Y,X)B(\square_a,\square_a;X,\square) + \frac{1}{3} Rg(Y,\nabla\varphi)g(\square,X)B(\square_a,\square_a;X,\square) \\
&+ \frac{2}{3} Rg(Y,\square)B(\nabla\varphi,X;X,\square) - \frac{1}{3} Rg(\square,X)B(\nabla\varphi,Y;X,\square) \\
&- \frac{1}{3} Rg(Y,X)B(\nabla\varphi,\square;X,\square)\Big] + 2g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) + 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y) \\
&- g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) - 2g(Y,X)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi) - 2g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) \\
&- 2g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) - 2g(X,Y)Q(X) - g(X,X)Q(Y) + g(X,Y)Q(X) \\
&+ 2g(X,Y)Q(X) + g(X,Y)Q(X)\Big\} \\
= {}& \frac{(-1)^q}{3}\Big\{\Big[2g(Y,X)Q(X) + 2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) - g(Y,X)Q(X) \\
&- g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) - 2g(Y,X)Q(X) - 2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) \\
&- \frac{2}{3} Rg(Y,\nabla\varphi)B(\square,\square;X,X) + \frac{1}{3} Rg(Y,\nabla\varphi)B(\square,\square;X,X) \\
&+ \frac{2}{3} Rg(Y,\nabla\varphi)B(\square,\square;X,X) + \frac{2}{3} RB(Y,\nabla\varphi;X,X) - \frac{2}{3} RB(\nabla\varphi,Y;X,X) \\
&- \frac{1}{3} RB(Y,\nabla\varphi;X,X)\Big] + 2\Big[g(X,X)Q(Y) + g(X,Y)Q(X) \\
&+ 2g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y) - \frac{1}{2} g(X,Y)Q(X) - g(X,Y)Q(X) \\
&- g(X,Y)Q(\nabla^2\varphi;X,\nabla^2\varphi) - \frac{1}{2} g(X,Y)Q(X) - \frac{1}{2} g(X,Y)Q(X) \\
&- g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) - \frac{2}{3} Rg(X,\nabla\varphi)B(\square,\square;X,Y) \\
&+ \frac{1}{3} Rg(Y,X)B(\square,\square;X,\nabla\varphi) + \frac{1}{3} Rg(Y,\nabla\varphi)B(\square,\square;X,X) \\
&+ \frac{2}{3} RB(\nabla\varphi,X;X,Y) - \frac{1}{3} RB(\nabla\varphi,Y;X,X) - \frac{1}{3} Rg(X,Y)B(\nabla\varphi,\square;X,\square)\Big] \\
&- 2g(X,Y)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi) - 3g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X) + 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y) \\
&- g(X,X)Q(Y) + 2g(X,Y)Q(X)\Big\} \\
= {}& \frac{(-1)^q}{3}\Big[g(X,X)Q(Y) - g(X,Y)Q(X) - 2g(X,Y)Q(\nabla^2\varphi;X,\nabla^2\varphi) \\
&+ 4g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y) - 3g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X)
\end{aligned}
$$

$$+\frac{2}{3}Rg(X,Y)B(\square,\square;X,\nabla\varphi)-\frac{4}{3}Rg(X,\nabla\varphi)B(\square,\square;X,Y)$$
$$+Rg(Y,\nabla\varphi)B(\square,\square;X,X)+\frac{4}{3}RB(X,\nabla\varphi;X,Y)-RB(Y,\nabla\varphi;X,X)$$
$$-\frac{2}{3}Rg(X,Y)B(\nabla\varphi,\square;X,\square)-2g(X,Y)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi)-3g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X)$$
$$\left.+4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y)\right].$$

We combine these terms using the coefficients from (C.17) and simplify the result

$$\begin{aligned}
&R[D(X,X,\square,\square,Y)-2D(X,\square,\square,X,Y)+D(Y,\square,\square,X,X)]\\
&=\frac{(-1)^q}{3}R\Big[3g(X,X)Q(Y)-2g(X,Y)Q(X)+2g(X,X)g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square)\\
&\quad-2g(X,\nabla^2\varphi)g(X,Y)Q(\nabla^2\varphi;\square,\square)+4g(X,X)Q(\nabla^2\varphi;Y,\nabla^2\varphi)\\
&\quad-4g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y)+2g(X,X)g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square)\\
&\quad-2g(X,\nabla\varphi)g(X,Y)\frac{\partial B}{\partial\varphi}(\square,\square)+4g(X,X)\frac{\partial B}{\partial\varphi}(Y,\nabla\varphi)-4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y)\\
&\quad-\frac{2}{3}Rg(X,X)g(Y,\nabla\varphi)B(\square_a,\square_a;\square,\square)+\frac{2}{3}Rg(X,\nabla\varphi)g(X,Y)B(\square_a,\square_a;\square,\square)\\
&\quad-\frac{2}{3}Rg(X,X)B(\nabla\varphi,Y;\square,\square)-\frac{2}{3}Rg(X,Y)B(\nabla\varphi,X;\square,\square)\\
&\quad+\frac{4}{3}Rg(X,\nabla\varphi)B(X,Y;\square,\square)+\frac{4}{3}Rg(X,X)B(\nabla\varphi,\square;Y,\square)\\
&\quad-\frac{4}{3}RB(\nabla\varphi,X;X,Y)\Big]-\frac{2(-1)^q}{3}R\Big[3g(X,X)Q(Y)-2g(X,Y)Q(X)\\
&\quad+2g(X,X)g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square)-2g(X,\nabla^2\varphi)g(X,Y)Q(\nabla^2\varphi;\square,\square)\\
&\quad+4g(X,X)Q(\nabla^2\varphi;Y,\nabla^2\varphi)-4g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y)\\
&\quad+2g(X,X)g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square)-2g(X,\nabla\varphi)g(X,Y)\frac{\partial B}{\partial\varphi}(\square,\square)\\
&\quad+4g(X,X)\frac{\partial B}{\partial\varphi}(Y,\nabla\varphi)-4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y)\\
&\quad-\frac{2}{3}Rg(X,X)g(Y,\nabla\varphi)B(\square_a,\square_a;\square,\square)+\frac{2}{3}Rg(X,\nabla\varphi)g(X,Y)B(\square_a,\square_a;\square,\square)\\
&\quad-\frac{2}{3}Rg(X,X)B(\nabla\varphi,Y;\square,\square)-\frac{2}{3}Rg(X,Y)B(\nabla\varphi,X;\square,\square)\\
&\quad+\frac{4}{3}Rg(X,\nabla\varphi)B(X,Y;\square,\square)+\frac{4}{3}Rg(X,X)B(\nabla\varphi,\square;Y,\square)\\
&\quad-\frac{4}{3}RB(\nabla\varphi,X;X,Y)\Big]+\frac{(-1)^q}{3}R\Big[g(X,X)Q(Y)-g(X,Y)Q(X)
\end{aligned}$$

$$
\begin{aligned}
&- 2g(X,Y)Q(\nabla^2\varphi; X, \nabla^2\varphi) + 4g(X,\nabla^2\varphi)Q(\nabla^2\varphi; X, Y) - 3g(Y,\nabla^2\varphi)Q(\nabla^2\varphi; X, X)\\
&+ \frac{2}{3}Rg(X,Y)B(\square,\square; X, \nabla\varphi) - \frac{4}{3}Rg(X,\nabla\varphi)B(\square,\square; X, Y)\\
&+ Rg(Y,\nabla\varphi)B(\square,\square; X, X) + \frac{4}{3}RB(X,\nabla\varphi; X, Y) - RB(Y,\nabla\varphi; X, X)\\
&- \frac{2}{3}Rg(X,Y)B(\nabla\varphi,\square; X, \square) - 2g(X,Y)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi) - 3g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X)\\
&+ 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y)\Bigg]\\
= \frac{(-1)^q}{3}&\Bigg[-2Rg(X,X)Q(Y) + Rg(X,Y)Q(X) - 4Rg(X,X)Q(\nabla^2\varphi; Y, \nabla^2\varphi)\\
&- 2Rg(X,Y)Q(\nabla^2\varphi; X, \nabla^2\varphi) + 8Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi; X, Y)\\
&- 3Rg(Y,\nabla^2\varphi)Q(\nabla^2\varphi; X, X) - 2Rg(X,X)g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square)\\
&+ 2Rg(X,Y)g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square) - 2Rg(X,X)g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square)\\
&+ 2Rg(X,Y)g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) - 4Rg(X,X)\frac{\partial B}{\partial\varphi}(Y,\nabla\varphi)\\
&- 2Rg(X,Y)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi) + 8Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,Y) - 3Rg(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,X)\\
&+ \frac{2}{3}R^2g(X,X)g(Y,\nabla\varphi)B(\square_a,\square_a;\square,\square) - \frac{2}{3}R^2g(X,Y)g(X,\nabla\varphi)B(\square_a,\square_a;\square,\square)\\
&+ \frac{2}{3}R^2g(X,X)B(Y,\nabla\varphi;\square,\square) + \frac{4}{3}R^2g(X,Y)B(X,\nabla\varphi;\square,\square)\\
&+ R^2g(Y,\nabla\varphi)B(X,X;\square,\square) - \frac{8}{3}R^2g(X,\nabla\varphi)B(X,Y;\square,\square)\\
&- \frac{4}{3}R^2g(X,X)B(Y,\square;\nabla\varphi,\square) - \frac{2}{3}R^2g(X,Y)B(\nabla\varphi,\square; X,\square) - R^2B(X,X;Y,\nabla\varphi)\\
&+ \frac{8}{3}R^2B(X,Y;X,\nabla\varphi)\Bigg]. \qquad \text{(C.18)}
\end{aligned}
$$

Next, we consider the D terms in (C.17) which are differentiated with respect to φ_{ij} and are contracted against a second derivative of φ. We use $D(X,X,\square,\square,Y)$ from above as a starting point for the first term

$$
\begin{aligned}
&D(X,\nabla^2\varphi,\square,\square,\nabla^2\varphi; X, Y)\\
&\quad= \frac{(-1)^q}{6}\Bigg[6g(X,\nabla^2\varphi)Q(\nabla^2\varphi) - 2g(\nabla^2\varphi,\nabla^2\varphi)Q(X) - 2g(X,\nabla^2\varphi)Q(\nabla^2\varphi)\\
&\qquad+ 4g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\square,\square) - 2g(\nabla^2\varphi,\nabla^2_a\varphi)g(X,\nabla^2\varphi)Q(\nabla^2_a\varphi;\square,\square)
\end{aligned}
$$

$$
\begin{aligned}
&- 2g(X,\nabla_a^2\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla_a^2\varphi;\square,\square) + 8g(X,\nabla^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi,\nabla_a^2\varphi)\\
&- 4g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;X,\nabla^2\varphi) - 4g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi,\nabla^2\varphi)\\
&+ 4g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) - 2g(\nabla^2\varphi,\nabla\varphi)g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\square,\square)\\
&- 2g(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) + 8g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi)\\
&- 4g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla^2\varphi) - 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi)\\
&- \frac{4}{3}Rg(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\square_a,\square_a;\square,\square)\\
&+ \frac{2}{3}Rg(\nabla^2\varphi,\nabla\varphi)g(X,\nabla^2\varphi)B(\square_a,\square_a;\square,\square)\\
&+ \frac{2}{3}Rg(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\square_a,\square_a;\square,\square)\\
&- \frac{4}{3}Rg(X,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\square,\square) - \frac{2}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(\nabla\varphi,X;\square,\square)\\
&- \frac{2}{3}Rg(X,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\square,\square) + \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(X,\nabla^2\varphi;\square,\square)\\
&+ \frac{4}{3}Rg(X,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\square,\square) + \frac{8}{3}Rg(X,\nabla^2\varphi)B(\nabla\varphi,\square;\nabla^2\varphi,\square)\\
&- \frac{4}{3}RB(\nabla\varphi,\nabla^2\varphi;X,\nabla^2\varphi) - \frac{4}{3}RB(\nabla\varphi,X;\nabla^2\varphi,\nabla^2\varphi)\Big](;X,Y)_a\\
= \frac{(-1)^q}{6}\Big[&4g(X,\nabla^2\varphi)Q(\nabla^2\varphi) - 2g(\nabla^2\varphi,\nabla^2\varphi)Q(X)\\
&+ 2g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\square,\square) - 2g(X,\nabla_a^2\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla_a^2\varphi;\square,\square)\\
&+ 8g(X,\nabla^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi,\nabla_a^2\varphi) - 4g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;X,\nabla^2\varphi)\\
&- 4g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi,\nabla^2\varphi) + 2g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square)\\
&- 2g(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) + 8g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi)\\
&- 4g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla^2\varphi) - 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi)\\
&- \frac{2}{3}Rg(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\square_a,\square_a;\square,\square)\\
&+ \frac{2}{3}Rg(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\square_a,\square_a;\square,\square) - 2Rg(X,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\square,\square)\\
&- \frac{2}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(X,\nabla\varphi;\square,\square) + \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(X,\nabla^2\varphi;\square,\square)\\
&+ \frac{4}{3}Rg(X,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\square,\square) + \frac{8}{3}Rg(X,\nabla^2\varphi)B(\nabla\varphi,\square;\nabla^2\varphi,\square)\\
&- \frac{4}{3}RB(\nabla\varphi,\nabla^2\varphi;X,\nabla^2\varphi) - \frac{4}{3}RB(\nabla\varphi,X;\nabla^2\varphi,\nabla^2\varphi)\Big](;X,Y)_a
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(-1)^q}{6} \Big[4g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y) - 2g(\nabla^2\varphi,\nabla^2\varphi)Q(X;X,Y) \\
&\quad + g(X,\nabla^2\varphi)g(\nabla^2\varphi,X)Q(Y;\square,\square) + g(X,\nabla^2\varphi)g(\nabla^2\varphi,Y)Q(X;\square,\square) \\
&\quad + 2g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\square,\square;X,Y) - g(X,X)g(\nabla^2\varphi,\nabla^2\varphi)Q(Y;\square,\square) \\
&\quad - g(X,Y)g(\nabla^2\varphi,\nabla^2\varphi)Q(X;\square,\square) - 2g(X,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\square,\square;X,Y) \\
&\quad + 4g(X,\nabla^2\varphi)Q(X;\nabla^2\varphi,Y) + 4g(X,\nabla^2\varphi)Q(Y;\nabla^2\varphi,X) \\
&\quad + 8g(X,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,Y) - 2g(\nabla^2\varphi,X)Q(Y;X,\nabla^2\varphi) \\
&\quad - 2g(\nabla^2\varphi,Y)Q(X;X,\nabla^2\varphi) - 4g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;X,\nabla^2\varphi;X,Y) \\
&\quad - 2g(X,X)Q(Y;\nabla^2\varphi,\nabla^2\varphi) - 2g(X,Y)Q(X;\nabla^2\varphi,\nabla^2\varphi) \\
&\quad - 4g(X,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,Y) + 2g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square;X,Y) \\
&\quad - 2g(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\square,\square;X,Y) + 8g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,Y) \\
&\quad - 4g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla^2\varphi;X,Y) - 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,Y) \\
&\quad - \frac{2}{3}Rg(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\square_a,\square_a;\square,\square;X,Y) \\
&\quad + \frac{2}{3}Rg(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\square_a,\square_a;\square,\square;X,Y) \\
&\quad - 2Rg(X,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\square,\square;X,Y) - \frac{2}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(X,\nabla\varphi;\square,\square;X,Y) \\
&\quad + \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(X,\nabla^2\varphi;\square,\square;X,Y) + \frac{4}{3}Rg(X,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\square,\square;X,Y) \\
&\quad + \frac{8}{3}Rg(X,\nabla^2\varphi)B(\nabla\varphi,\square;\nabla^2\varphi,\square;X,Y) - \frac{4}{3}RB(\nabla\varphi,\nabla^2\varphi;X,\nabla^2\varphi;X,Y) \\
&\quad - \frac{4}{3}RB(\nabla\varphi,X;\nabla^2\varphi,\nabla^2\varphi;X,Y) \Big] \\
&= \frac{(-1)^q}{6} \Big[4g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y) - 2g(\nabla^2\varphi,\nabla^2\varphi)Q(X;X,Y) \\
&\quad + g(X,\nabla^2\varphi)g(X,\nabla^2\varphi)Q(Y;\square,\square) - g(X,X)g(\nabla^2\varphi,\nabla^2\varphi)Q(Y;\square,\square) \\
&\quad + g(X,\nabla^2\varphi)g(Y,\nabla^2\varphi)Q(X;\square,\square) - g(X,Y)g(\nabla^2\varphi,\nabla^2\varphi)Q(X;\square,\square) \\
&\quad + 4g(X,\nabla^2\varphi)Q(X;Y,\nabla^2\varphi) + 2g(X,\nabla^2\varphi)Q(Y;X,\nabla^2\varphi) - 2g(Y,\nabla^2\varphi)Q(X;X,\nabla^2\varphi) \\
&\quad - 2g(X,X)Q(Y;\nabla^2\varphi,\nabla^2\varphi) - 2g(X,Y)Q(X;\nabla^2\varphi,\nabla^2\varphi) \\
&\quad + 2g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\square,\square;X,Y)
\end{aligned}
$$

$$
\begin{aligned}
&- 2g(X,\nabla_a^2\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla_a^2\varphi;\Box,\Box;X,Y)\\
&+ 8g(X,\nabla^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi,\nabla_a^2\varphi;X,Y) - 4g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;X,\nabla^2\varphi;X,Y)\\
&- 4g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi,\nabla^2\varphi;X,Y) + 2g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,Y)\\
&- 2g(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,Y) + 8g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,Y)\\
&- 4g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla^2\varphi;X,Y) - 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,Y)\\
&- \frac{2}{3}Rg(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,Y)\\
&+ \frac{2}{3}Rg(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box_a,\Box_a;\Box,\Box;X,Y)\\
&- 2Rg(X,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\Box,\Box;X,Y) - \frac{2}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(X,\nabla\varphi;\Box,\Box;X,Y)\\
&+ \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(X,\nabla^2\varphi;\Box,\Box;X,Y) + \frac{4}{3}Rg(X,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\Box,\Box;X,Y)\\
&+ \frac{8}{3}Rg(X,\nabla^2\varphi)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box;X,Y) - \frac{4}{3}RB(\nabla\varphi,\nabla^2\varphi;X,\nabla^2\varphi;X,Y)\\
&- \frac{4}{3}RB(\nabla\varphi,X;\nabla^2\varphi,\nabla^2\varphi;X,Y)\Bigg],
\end{aligned}
$$

where the a subscript on the outer second derivative $(;X,Y)_a$ is a reminder to pass over $\nabla^2\varphi$ terms without the subscript as the corresponding φ_{ij} term is outside the derivative, i.e., $D(X,\nabla^2\varphi,\Box,\Box,\nabla^2\varphi;X,Y) = D^{ijklm;ab}g_{kl}\varphi_{jm}$. We use $D(Y,\Box,\Box,X,X)$ as the starting point for the next term

$$
\begin{aligned}
&D(X,\Box,\Box,\nabla^2\varphi,\nabla^2\varphi;X,Y)\\
&\quad= \frac{(-1)^q}{3}\Bigg[g(\nabla^2\varphi,\nabla^2\varphi)Q(X) - g(\nabla^2\varphi,X)Q(\nabla^2\varphi) - 2g(\nabla^2\varphi,X)Q(\nabla_a^2\varphi;\nabla^2\varphi,\nabla_a^2\varphi)\\
&\qquad+ 4g(\nabla^2\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi,X) - 3g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla^2\varphi,\nabla^2\varphi)\\
&\qquad+ \frac{2}{3}Rg(\nabla^2\varphi,X)B(\Box,\Box;\nabla^2\varphi,\nabla\varphi) - \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,X)\\
&\qquad+ Rg(X,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla^2\varphi) + \frac{4}{3}RB(\nabla^2\varphi,\nabla\varphi;\nabla^2\varphi,X)\\
&\qquad- RB(X,\nabla\varphi;\nabla^2\varphi,\nabla^2\varphi) - \frac{2}{3}Rg(\nabla^2\varphi,X)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box)\\
&\qquad- 2g(\nabla^2\varphi,X)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi) - 3g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi)
\end{aligned}
$$

$$
\begin{aligned}
&+ 4g(\nabla^2\varphi, \nabla\varphi)\frac{\partial B}{\partial \varphi}(\nabla^2\varphi, X)\Big] (; X, Y)_a \\
= \frac{(-1)^q}{3} \Big[&g(\nabla^2\varphi, \nabla^2\varphi)Q(X; X, Y) - g(\nabla^2\varphi, X)Q(\nabla^2\varphi; X, Y) \\
&- g(\nabla^2\varphi, X)Q(X; \nabla^2\varphi, Y) - g(\nabla^2\varphi, X)Q(Y; \nabla^2\varphi, X) \\
&- 2g(\nabla^2\varphi, X)Q(\nabla^2_a\varphi; \nabla^2\varphi, \nabla^2_a\varphi; X, Y) + 2g(\nabla^2\varphi, X)Q(Y; \nabla^2\varphi, X) \\
&+ 2g(\nabla^2\varphi, Y)Q(X; \nabla^2\varphi, X) + 4g(\nabla^2\varphi, \nabla^2_a\varphi)Q(\nabla^2_a\varphi; \nabla^2\varphi, X; X, Y) \\
&- \frac{3}{2}g(X, X)Q(Y; \nabla^2\varphi, \nabla^2\varphi) - \frac{3}{2}g(X, Y)Q(X; \nabla^2\varphi, \nabla^2\varphi) \\
&- 3g(X, \nabla^2_a\varphi)Q(\nabla^2_a\varphi; \nabla^2\varphi, \nabla^2\varphi; X, Y) + \frac{2}{3}Rg(\nabla^2\varphi, X)B(\Box, \Box; \nabla^2\varphi, \nabla\varphi; X, Y) \\
&- \frac{4}{3}Rg(\nabla^2\varphi, \nabla\varphi)B(\Box, \Box; \nabla^2\varphi, X; X, Y) + Rg(X, \nabla\varphi)B(\Box, \Box; \nabla^2\varphi, \nabla^2\varphi; X, Y) \\
&+ \frac{4}{3}RB(\nabla^2\varphi, \nabla\varphi; \nabla^2\varphi, X; X, Y) - RB(X, \nabla\varphi; \nabla^2\varphi, \nabla^2\varphi; X, Y) \\
&- \frac{2}{3}Rg(\nabla^2\varphi, X)B(\nabla\varphi, \Box; \nabla^2\varphi, \Box; X, Y) - 2g(\nabla^2\varphi, X)\frac{\partial B}{\partial \varphi}(\nabla^2\varphi, \nabla\varphi; X, Y) \\
&- 3g(X, \nabla\varphi)\frac{\partial B}{\partial \varphi}(\nabla^2\varphi, \nabla^2\varphi; X, Y) + 4g(\nabla^2\varphi, \nabla\varphi)\frac{\partial B}{\partial \varphi}(\nabla^2\varphi, X; X, Y)\Big] \\
= \frac{(-1)^q}{3} \Big[&g(\nabla^2\varphi, \nabla^2\varphi)Q(X; X, Y) - g(X, \nabla^2\varphi)Q(\nabla^2\varphi; X, Y) \\
&- g(X, \nabla^2\varphi)Q(X; Y, \nabla^2\varphi) + g(X, \nabla^2\varphi)Q(Y; X, \nabla^2\varphi) + 2g(Y, \nabla^2\varphi)Q(X; X, \nabla^2\varphi) \\
&- \frac{3}{2}g(X, X)Q(Y; \nabla^2\varphi, \nabla^2\varphi) - \frac{3}{2}g(X, Y)Q(X; \nabla^2\varphi, \nabla^2\varphi) \\
&- 2g(X, \nabla^2\varphi)Q(\nabla^2_a\varphi; \nabla^2\varphi, \nabla^2_a\varphi; X, Y) + 4g(\nabla^2\varphi, \nabla^2_a\varphi)Q(\nabla^2_a\varphi; X, \nabla^2\varphi; X, Y) \\
&- 3g(X, \nabla^2_a\varphi)Q(\nabla^2_a\varphi; \nabla^2\varphi, \nabla^2\varphi; X, Y) + \frac{2}{3}Rg(X, \nabla^2\varphi)B(\Box, \Box; \nabla^2\varphi, \nabla\varphi; X, Y) \\
&- \frac{4}{3}Rg(\nabla^2\varphi, \nabla\varphi)B(\Box, \Box; X, \nabla^2\varphi; X, Y) + Rg(X, \nabla\varphi)B(\Box, \Box; \nabla^2\varphi, \nabla^2\varphi; X, Y) \\
&+ \frac{4}{3}RB(\nabla^2\varphi, \nabla\varphi; X, \nabla^2\varphi; X, Y) - RB(X, \nabla\varphi; \nabla^2\varphi, \nabla^2\varphi; X, Y) \\
&- \frac{2}{3}Rg(\nabla^2\varphi, X)B(\nabla\varphi, \Box; \nabla^2\varphi, \Box; X, Y) - 2g(\nabla^2\varphi, X)\frac{\partial B}{\partial \varphi}(\nabla^2\varphi, \nabla\varphi; X, Y) \\
&- 3g(X, \nabla\varphi)\frac{\partial B}{\partial \varphi}(\nabla^2\varphi, \nabla^2\varphi; X, Y) + 4g(\nabla^2\varphi, \nabla\varphi)\frac{\partial B}{\partial \varphi}(X, \nabla^2\varphi; X, Y)\Big].
\end{aligned}
$$

The third term uses $D(X, X, \Box, \Box, Y)$ as the base again, yielding

$D(Y, \nabla^2\varphi, \Box, \Box, \nabla^2\varphi; X, X)$

$$
\begin{aligned}
&= \frac{(-1)^q}{3}\Bigg[3g(Y,\nabla^2\varphi)Q(\nabla^2\varphi) - g(\nabla^2\varphi,\nabla^2\varphi)Q(Y) - g(Y,\nabla^2\varphi)Q(\nabla^2\varphi) \\
&\quad + 2g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\square,\square) - g(\nabla^2\varphi,\nabla^2_a\varphi)g(Y,\nabla^2\varphi)Q(\nabla^2_a\varphi;\square,\square) \\
&\quad - g(Y,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\square,\square) + 4g(Y,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi) \\
&\quad - 2g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;Y,\nabla^2\varphi) - 2g(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi) \\
&\quad + 2g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) - g(\nabla^2\varphi,\nabla\varphi)g(Y,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) \\
&\quad - g(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) + 4g(Y,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi) \\
&\quad - 2g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(Y,\nabla^2\varphi) - 2g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi) \\
&\quad - \frac{2}{3}Rg(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\square_a,\square_a;\square,\square) \\
&\quad + \frac{1}{3}Rg(\nabla^2\varphi,\nabla\varphi)g(Y,\nabla^2\varphi)B(\square_a,\square_a;\square,\square) \\
&\quad + \frac{1}{3}Rg(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\square_a,\square_a;\square,\square) \\
&\quad - \frac{2}{3}Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\square,\square) - \frac{1}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(\nabla\varphi,Y;\square,\square) \\
&\quad - \frac{1}{3}Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\square,\square) + \frac{2}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(Y,\nabla^2\varphi;\square,\square) \\
&\quad + \frac{2}{3}Rg(Y,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\square,\square) + \frac{4}{3}Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\square;\nabla^2\varphi,\square) \\
&\quad - \frac{2}{3}RB(\nabla\varphi,\nabla^2\varphi;Y,\nabla^2\varphi) - \frac{2}{3}RB(\nabla\varphi,Y;\nabla^2\varphi,\nabla^2\varphi)\Bigg](;X,X)_a \\
&= \frac{(-1)^q}{3}\Bigg[2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi) - g(\nabla^2\varphi,\nabla^2\varphi)Q(Y) \\
&\quad + g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\square,\square) - g(Y,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\square,\square) \\
&\quad + 4g(Y,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi) - 2g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;Y,\nabla^2\varphi) \\
&\quad - 2g(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi) + g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) \\
&\quad - g(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\square,\square) + 4g(Y,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi) \\
&\quad - 2g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(Y,\nabla^2\varphi) - 2g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi) \\
&\quad - \frac{1}{3}Rg(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\square_a,\square_a;\square,\square) \\
&\quad + \frac{1}{3}Rg(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\square_a,\square_a;\square,\square) - Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\square,\square) \\
&\quad - \frac{1}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(Y,\nabla\varphi;\square,\square) + \frac{2}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(Y,\nabla^2\varphi;\square,\square)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{3}Rg(Y,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\square,\square)+\frac{4}{3}Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\square;\nabla^2\varphi,\square)\\
&-\frac{2}{3}RB(\nabla\varphi,\nabla^2\varphi;Y,\nabla^2\varphi)-\frac{2}{3}RB(Y,\nabla\varphi;\nabla^2\varphi,\nabla^2\varphi)\Big](;X,X)_a\\
=\frac{(-1)^q}{3}&\Big[2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X)-g(\nabla^2\varphi,\nabla^2\varphi)Q(Y;X,X)\\
&+g(Y,\nabla^2\varphi)g(\nabla^2\varphi,X)Q(X;\square,\square)+g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\square,\square;X,X)\\
&-g(Y,X)g(\nabla^2\varphi,\nabla^2\varphi)Q(X;\square,\square)-g(Y,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\square,\square;X,X)\\
&+4g(Y,\nabla^2\varphi)Q(X;\nabla^2\varphi,X)+4g(Y,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,X)\\
&-2g(\nabla^2\varphi,X)Q(X;Y,\nabla^2\varphi)-2g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;Y,\nabla^2\varphi;X,X)\\
&-2g(Y,X)Q(X;\nabla^2\varphi,\nabla^2\varphi)-2g(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&+g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square;X,X)-g(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\square,\square;X,X)\\
&+4g(Y,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,X)-2g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(Y,\nabla^2\varphi;X,X)\\
&-2g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&-\frac{1}{3}Rg(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\square_a,\square_a;\square,\square;X,X)\\
&+\frac{1}{3}Rg(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\square_a,\square_a;\square,\square;X,X)\\
&-Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\square,\square;X,X)-\frac{1}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(Y,\nabla\varphi;\square,\square;X,X)\\
&+\frac{2}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(Y,\nabla^2\varphi;\square,\square;X,X)+\frac{2}{3}Rg(Y,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\square,\square;X,X)\\
&+\frac{4}{3}Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\square;\nabla^2\varphi,\square;X,X)-\frac{2}{3}RB(\nabla\varphi,\nabla^2\varphi;Y,\nabla^2\varphi;X,X)\\
&-\frac{2}{3}RB(Y,\nabla\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X)\Big]\\
=\frac{(-1)^q}{3}&\Big[2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X)-g(\nabla^2\varphi,\nabla^2\varphi)Q(Y;X,X)\\
&+4g(Y,\nabla^2\varphi)Q(X;X,\nabla^2\varphi)-2g(X,\nabla^2\varphi)Q(X;Y,\nabla^2\varphi)-2g(X,Y)Q(X;\nabla^2\varphi,\nabla^2\varphi)\\
&+g(X,\nabla^2\varphi)g(Y,\nabla^2\varphi)Q(X;\square,\square)-g(X,Y)g(\nabla^2\varphi,\nabla^2\varphi)Q(X;\square,\square)\\
&+g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\square,\square;X,X)+4g(Y,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,X)\\
&-g(Y,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\square,\square;X,X)-2g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;Y,\nabla^2\varphi;X,X)\\
&-2g(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X)+g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&- g(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) + 4g(Y,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,X) \\
&- 2g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(Y,\nabla^2\varphi;X,X) - 2g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,X) \\
&- \frac{1}{3}Rg(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X) \\
&+ \frac{1}{3}Rg(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X) \\
&- Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\Box,\Box;X,X) - \frac{1}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(Y,\nabla\varphi;\Box,\Box;X,X) \\
&+ \frac{2}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(Y,\nabla^2\varphi;\Box,\Box;X,X) + \frac{2}{3}Rg(Y,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\Box,\Box;X,X) \\
&+ \frac{4}{3}Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box;X,X) - \frac{2}{3}RB(\nabla\varphi,\nabla^2\varphi;Y,\nabla^2\varphi;X,X) \\
&- \frac{2}{3}RB(Y,\nabla\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X)\Bigg].
\end{aligned}
$$

We use $D(Y,\Box,\Box,X,X)$ as the final base, yielding

$$
\begin{aligned}
&D(Y,\Box,\Box,\nabla^2\varphi,\nabla^2\varphi;X,X) \\
&\quad = \frac{(-1)^q}{3}\Bigg[g(\nabla^2\varphi,\nabla^2\varphi)Q(Y) - g(\nabla^2\varphi,Y)Q(\nabla^2\varphi) - 2g(\nabla^2\varphi,Y)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi) \\
&\qquad + 4g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,Y) - 3g(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi) \\
&\qquad + \frac{2}{3}Rg(\nabla^2\varphi,Y)B(\Box,\Box;\nabla^2\varphi,\nabla\varphi) - \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,Y) \\
&\qquad + Rg(Y,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla^2\varphi) + \frac{4}{3}RB(\nabla^2\varphi,\nabla\varphi;\nabla^2\varphi,Y) \\
&\qquad - RB(Y,\nabla\varphi;\nabla^2\varphi,\nabla^2\varphi) - \frac{2}{3}Rg(\nabla^2\varphi,Y)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box) \\
&\qquad - 2g(\nabla^2\varphi,Y)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi) - 3g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi) \\
&\qquad + 4g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,Y)\Bigg](;X,X)_a \\
&\quad = \frac{(-1)^q}{3}\Bigg[g(\nabla^2\varphi,\nabla^2\varphi)Q(Y;X,X) - g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) \\
&\qquad - 2g(\nabla^2\varphi,Y)Q(X;\nabla^2\varphi,X) - 2g(\nabla^2\varphi,Y)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,X) \\
&\qquad + 4g(\nabla^2\varphi,X)Q(X;\nabla^2\varphi,Y) + 4g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,Y;X,X) \\
&\qquad - 3g(Y,X)Q(X;\nabla^2\varphi,\nabla^2\varphi) - 3g(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X) \\
&\qquad + \frac{2}{3}Rg(\nabla^2\varphi,Y)B(\Box,\Box;\nabla^2\varphi,\nabla\varphi;X,X) - \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,Y;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&+ Rg(Y, \nabla\varphi)B(\square, \square; \nabla^2\varphi, \nabla^2\varphi; X, X) + \frac{4}{3}RB(\nabla^2\varphi, \nabla\varphi; \nabla^2\varphi, Y; X, X) \\
&- RB(Y, \nabla\varphi; \nabla^2\varphi, \nabla^2\varphi; X, X) - \frac{2}{3}Rg(\nabla^2\varphi, Y)B(\nabla\varphi, \square; \nabla^2\varphi, \square; X, X) \\
&- 2g(\nabla^2\varphi, Y)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi, \nabla\varphi; X, X) - 3g(Y, \nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi, \nabla^2\varphi; X, X) \\
&+ 4g(\nabla^2\varphi, \nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi, Y; X, X)\Big] \\
= \frac{(-1)^q}{3} &\Big[g(\nabla^2\varphi, \nabla^2\varphi)Q(Y; X, X) - g(Y, \nabla^2\varphi)Q(\nabla^2\varphi; X, X) \\
&- 2g(Y, \nabla^2\varphi)Q(X; X, \nabla^2\varphi) + 4g(X, \nabla^2\varphi)Q(X; Y, \nabla^2\varphi) \\
&- 3g(X, Y)Q(X; \nabla^2\varphi, \nabla^2\varphi) - 2g(Y, \nabla^2\varphi)Q(\nabla^2_a\varphi; \nabla^2\varphi, \nabla^2_a\varphi; X, X) \\
&+ 4g(\nabla^2\varphi, \nabla^2_a\varphi)Q(\nabla^2_a\varphi; Y, \nabla^2\varphi; X, X) - 3g(Y, \nabla^2_a\varphi)Q(\nabla^2_a\varphi; \nabla^2\varphi, \nabla^2\varphi; X, X) \\
&+ \frac{2}{3}Rg(Y, \nabla^2\varphi)B(\square, \square; \nabla^2\varphi, \nabla\varphi; X, X) - \frac{4}{3}Rg(\nabla^2\varphi, \nabla\varphi)B(\square, \square; Y, \nabla^2\varphi; X, X) \\
&+ Rg(Y, \nabla\varphi)B(\square, \square; \nabla^2\varphi, \nabla^2\varphi; X, X) + \frac{4}{3}RB(\nabla^2\varphi, \nabla\varphi; Y, \nabla^2\varphi; X, X) \\
&- RB(Y, \nabla\varphi; \nabla^2\varphi, \nabla^2\varphi; X, X) - \frac{2}{3}Rg(Y, \nabla^2\varphi)B(\nabla\varphi, \square; \nabla^2\varphi, \square; X, X) \\
&- 2g(Y, \nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi, \nabla\varphi; X, X) - 3g(Y, \nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi, \nabla^2\varphi; X, X) \\
&+ 4g(\nabla^2\varphi, \nabla\varphi)\frac{\partial B}{\partial\varphi}(Y, \nabla^2\varphi; X, X)\Big].
\end{aligned}
$$

We combine these four terms using the coefficients from (C.17) and simplify

$$
\begin{aligned}
&R\left[2D(X, \nabla^2\varphi, \square, \square, \nabla^2\varphi; X, Y) - 2D(X, \square, \square, \nabla^2\varphi, \nabla^2\varphi; X, Y)\right. \\
&\quad \left.- D(Y, \nabla^2\varphi, \square, \square, \nabla^2\varphi; X, X) + D(Y, \square, \square, \nabla^2\varphi, \nabla^2\varphi; X, X)\right] \\
&\quad = \frac{(-1)^q}{3}R\Big[4g(X, \nabla^2\varphi)Q(\nabla^2\varphi; X, Y) - 2g(\nabla^2\varphi, \nabla^2\varphi)Q(X; X, Y) \\
&\qquad + g(X, \nabla^2\varphi)g(X, \nabla^2\varphi)Q(Y; \square, \square) - g(X, X)g(\nabla^2\varphi, \nabla^2\varphi)Q(Y; \square, \square) \\
&\qquad + g(X, \nabla^2\varphi)g(Y, \nabla^2\varphi)Q(X; \square, \square) - g(X, Y)g(\nabla^2\varphi, \nabla^2\varphi)Q(X; \square, \square) \\
&\qquad + 4g(X, \nabla^2\varphi)Q(X; Y, \nabla^2\varphi) + 2g(X, \nabla^2\varphi)Q(Y; X, \nabla^2\varphi) - 2g(Y, \nabla^2\varphi)Q(X; X, \nabla^2\varphi) \\
&\qquad - 2g(X, X)Q(Y; \nabla^2\varphi, \nabla^2\varphi) - 2g(X, Y)Q(X; \nabla^2\varphi, \nabla^2\varphi) \\
&\qquad + 2g(X, \nabla^2\varphi)g(\nabla^2\varphi, \nabla^2_a\varphi)Q(\nabla^2_a\varphi; \square, \square; X, Y)
\end{aligned}
$$

$$
\begin{aligned}
&- 2g(X,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\Box,\Box;X,Y)\\
&+ 8g(X,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,Y) - 4g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;X,\nabla^2\varphi;X,Y)\\
&- 4g(X,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,Y) + 2g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,Y)\\
&- 2g(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,Y) + 8g(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,Y)\\
&- 4g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla^2\varphi;X,Y) - 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,Y)\\
&- \frac{2}{3}Rg(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,Y)\\
&+ \frac{2}{3}Rg(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box_a,\Box_a;\Box,\Box;X,Y)\\
&- 2Rg(X,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\Box,\Box;X,Y) - \frac{2}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(X,\nabla\varphi;\Box,\Box;X,Y)\\
&+ \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(X,\nabla^2\varphi;\Box,\Box;X,Y) + \frac{4}{3}Rg(X,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\Box,\Box;X,Y)\\
&+ \frac{8}{3}Rg(X,\nabla^2\varphi)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box;X,Y) - \frac{4}{3}RB(\nabla\varphi,\nabla^2\varphi;X,\nabla^2\varphi;X,Y)\\
&- \frac{4}{3}RB(\nabla\varphi,X;\nabla^2\varphi,\nabla^2\varphi;X,Y)\Bigg] - \frac{2(-1)^q}{3}R\Bigg[g(\nabla^2\varphi,\nabla^2\varphi)Q(X;X,Y)\\
&- g(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y) - g(X,\nabla^2\varphi)Q(X;Y,\nabla^2\varphi) + g(X,\nabla^2\varphi)Q(Y;X,\nabla^2\varphi)\\
&+ 2g(Y,\nabla^2\varphi)Q(X;X,\nabla^2\varphi) - \frac{3}{2}g(X,X)Q(Y;\nabla^2\varphi,\nabla^2\varphi)\\
&- \frac{3}{2}g(X,Y)Q(X;\nabla^2\varphi,\nabla^2\varphi) - 2g(X,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,Y)\\
&+ 4g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;X,\nabla^2\varphi;X,Y) - 3g(X,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,Y)\\
&+ \frac{2}{3}Rg(X,\nabla^2\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla\varphi;X,Y) - \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(\Box,\Box;X,\nabla^2\varphi;X,Y)\\
&+ Rg(X,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla^2\varphi;X,Y) + \frac{4}{3}RB(\nabla^2\varphi,\nabla\varphi;X,\nabla^2\varphi;X,Y)\\
&- RB(X,\nabla\varphi;\nabla^2\varphi,\nabla^2\varphi;X,Y) - \frac{2}{3}Rg(\nabla^2\varphi,X)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box;X,Y)\\
&- 2g(\nabla^2\varphi,X)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,Y) - 3g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,Y)\\
&+ 4g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla^2\varphi;X,Y)\Bigg] + \frac{(-1)^q}{3}R\Bigg[2g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X)\\
&- g(\nabla^2\varphi,\nabla^2\varphi)Q(Y;X,X) + 4g(Y,\nabla^2\varphi)Q(X;X,\nabla^2\varphi)\\
&- 2g(X,\nabla^2\varphi)Q(X;Y,\nabla^2\varphi) - 2g(X,Y)Q(X;\nabla^2\varphi,\nabla^2\varphi)\\
&+ g(X,\nabla^2\varphi)g(Y,\nabla^2\varphi)Q(X;\Box,\Box) - g(X,Y)g(\nabla^2\varphi,\nabla^2\varphi)Q(X;\Box,\Box)
\end{aligned}
$$

$$
\begin{aligned}
&+ g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\Box,\Box;X,X) + 4g(Y,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,X)\\
&- g(Y,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\Box,\Box;X,X) - 2g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;Y,\nabla^2\varphi;X,X)\\
&- 2g(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X) + g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)\\
&- g(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X) + 4g(Y,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,X)\\
&- 2g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(Y,\nabla^2\varphi;X,X) - 2g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&- \frac{1}{3}Rg(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X)\\
&+ \frac{1}{3}Rg(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X)\\
&- Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\nabla^2\varphi;\Box,\Box;X,X) - \frac{1}{3}Rg(\nabla^2\varphi,\nabla^2\varphi)B(Y,\nabla\varphi;\Box,\Box;X,X)\\
&+ \frac{2}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(Y,\nabla^2\varphi;\Box,\Box;X,X) + \frac{2}{3}Rg(Y,\nabla\varphi)B(\nabla^2\varphi,\nabla^2\varphi;\Box,\Box;X,X)\\
&+ \frac{4}{3}Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box;X,X) - \frac{2}{3}RB(\nabla\varphi,\nabla^2\varphi;Y,\nabla^2\varphi;X,X)\\
&- \frac{2}{3}RB(Y,\nabla\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X)\Bigg] - \frac{(-1)^q}{3}R\Bigg[g(\nabla^2\varphi,\nabla^2\varphi)Q(Y;X,X)\\
&- g(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X) - 2g(Y,\nabla^2\varphi)Q(X;X,\nabla^2\varphi) + 4g(X,\nabla^2\varphi)Q(X;Y,\nabla^2\varphi)\\
&- 3g(X,Y)Q(X;\nabla^2\varphi,\nabla^2\varphi) - 2g(Y,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,X)\\
&+ 4g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;Y,\nabla^2\varphi;X,X) - 3g(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&+ \frac{2}{3}Rg(Y,\nabla^2\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla\varphi;X,X) - \frac{4}{3}Rg(\nabla^2\varphi,\nabla\varphi)B(\Box,\Box;Y,\nabla^2\varphi;X,X)\\
&+ Rg(Y,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla^2\varphi;X,X) + \frac{4}{3}RB(\nabla^2\varphi,\nabla\varphi;Y,\nabla^2\varphi;X,X)\\
&- RB(Y,\nabla\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X) - \frac{2}{3}Rg(Y,\nabla^2\varphi)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box;X,X)\\
&- 2g(Y,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,X) - 3g(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&+ 4g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(Y,\nabla^2\varphi;X,X)\Bigg]\\
= {}& \frac{(-1)^q}{3}\Bigg[-4Rg(\nabla^2\varphi,\nabla^2\varphi)Q(X;X,Y) - 2Rg(\nabla^2\varphi,\nabla^2\varphi)Q(Y;X,X)\\
&- 2Rg(X,Y)g(\nabla^2\varphi,\nabla^2\varphi)Q(X;\Box,\Box) - Rg(X,X)g(\nabla^2\varphi,\nabla^2\varphi)Q(Y;\Box,\Box)\\
&+ 2Rg(X,\nabla^2\varphi)g(Y,\nabla^2\varphi)Q(X;\Box,\Box) + Rg(X,\nabla^2\varphi)g(X,\nabla^2\varphi)Q(Y;\Box,\Box)\\
&+ 2Rg(X,Y)Q(X;\nabla^2\varphi,\nabla^2\varphi) + Rg(X,X)Q(Y;\nabla^2\varphi,\nabla^2\varphi)
\end{aligned}
$$

$$
\begin{aligned}
&+ 6Rg(X,\nabla^2\varphi)Q(\nabla^2\varphi;X,Y) + 3Rg(Y,\nabla^2\varphi)Q(\nabla^2\varphi;X,X)\\
&+ 2Rg(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\Box,\Box;X,Y)\\
&+ Rg(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\Box,\Box;X,X)\\
&- 2Rg(X,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\Box,\Box;X,Y)\\
&- Rg(Y,\nabla^2_a\varphi)g(\nabla^2\varphi,\nabla^2\varphi)Q(\nabla^2_a\varphi;\Box,\Box;X,X)\\
&+ 12Rg(X,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,Y) + 6Rg(Y,\nabla^2\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2_a\varphi;X,X)\\
&- 12Rg(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;X,\nabla^2\varphi;X,Y) - 6Rg(\nabla^2\varphi,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;Y,\nabla^2\varphi;X,X)\\
&+ 2Rg(X,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,Y) + Rg(Y,\nabla^2_a\varphi)Q(\nabla^2_a\varphi;\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&+ 2Rg(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,Y)\\
&+ Rg(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)\\
&- 2Rg(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,Y)\\
&- Rg(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,X)\\
&+ 12Rg(X,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,Y) + 6Rg(Y,\nabla^2\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla\varphi;X,X)\\
&- 12Rg(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla^2\varphi;X,Y) - 6Rg(\nabla^2\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(Y,\nabla^2\varphi;X,X)\\
&+ 2Rg(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,Y) + Rg(Y,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&- \frac{2}{3}R^2g(X,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,Y)\\
&- \frac{1}{3}R^2g(Y,\nabla^2\varphi)g(\nabla^2\varphi,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X)\\
&+ \frac{2}{3}R^2g(X,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box_a,\Box_a;\Box,\Box;X,Y)\\
&+ \frac{1}{3}R^2g(Y,\nabla\varphi)g(\nabla^2\varphi,\nabla^2\varphi)B(\Box_a,\Box_a;\Box,\Box;X,X)\\
&- 2R^2g(X,\nabla^2\varphi)B(\Box,\Box;\nabla\varphi,\nabla^2\varphi;X,Y) - R^2g(Y,\nabla^2\varphi)B(\Box,\Box;\nabla\varphi,\nabla^2\varphi;X,X)\\
&- 2R^2g(X,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla^2\varphi;X,Y) - R^2g(Y,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla^2\varphi;X,X)\\
&+ 4R^2g(\nabla^2\varphi,\nabla\varphi)B(\Box,\Box;X,\nabla^2\varphi;X,Y) + 2R^2g(\nabla^2\varphi,\nabla\varphi)B(\Box,\Box;Y,\nabla^2\varphi;X,X)\\
&- \frac{2}{3}R^2g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;X,\nabla\varphi;X,Y) - \frac{1}{3}R^2g(\nabla^2\varphi,\nabla^2\varphi)B(\Box,\Box;Y,\nabla\varphi;X,X)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{4}{3}R^2 g(X,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla^2\varphi;X,Y) + \frac{2}{3}R^2 g(Y,\nabla\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla^2\varphi;X,X) \\
&- \frac{4}{3}R^2 g(X,\nabla^2\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla\varphi;X,Y) - \frac{2}{3}R^2 g(Y,\nabla^2\varphi)B(\Box,\Box;\nabla^2\varphi,\nabla\varphi;X,X) \\
&+ 4R^2 g(X,\nabla^2\varphi)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box;X,Y) + 2R^2 g(Y,\nabla^2\varphi)B(\nabla\varphi,\Box;\nabla^2\varphi,\Box;X,X) \\
&- \frac{4}{3}R^2 B(\nabla^2\varphi,\nabla^2\varphi;X,\nabla\varphi;X,Y) - \frac{2}{3}R^2 B(\nabla^2\varphi,\nabla^2\varphi;Y,\nabla\varphi;X,X) \\
&- 4R^2 B(\nabla^2\varphi,\nabla\varphi;X,\nabla^2\varphi;X,Y) - 2R^2 B(\nabla^2\varphi,\nabla\varphi;Y,\nabla^2\varphi;X,X) \\
&\left. + 2R^2 B(\nabla^2\varphi,\nabla^2\varphi;X,\nabla\varphi;X,Y) + R^2 B(\nabla^2\varphi,\nabla^2\varphi;Y,\nabla\varphi;X,X)\right].
\end{aligned}
$$

The remaining D terms follow in a similar fashion, with $\nabla^2\varphi$ replaced by a combination of $\nabla\varphi$ and ∇R. We use $D(X,X,\Box,\Box,Y)$ to express the first term

$$
\begin{aligned}
&D(X,\nabla\varphi,\Box,\Box,\nabla R;X,Y) \\
&\quad = \frac{(-1)^q}{3}\Big[3g(X,\nabla\varphi)Q(\nabla R) - g(\nabla\varphi,\nabla R)Q(X) - g(X,\nabla R)Q(\nabla\varphi) \\
&\qquad + 2g(X,\nabla\varphi)g(\nabla R,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box) - g(\nabla\varphi,\nabla^2\varphi)g(X,\nabla R)Q(\nabla^2\varphi;\Box,\Box) \\
&\qquad - g(X,\nabla^2\varphi)g(\nabla\varphi,\nabla R)Q(\nabla^2\varphi;\Box,\Box) + 4g(X,\nabla\varphi)Q(\nabla^2\varphi;\nabla R,\nabla^2\varphi) \\
&\qquad - 2g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;X,\nabla R) - 2g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,\nabla R) \\
&\qquad + 2g(X,\nabla\varphi)g(\nabla R,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box) - g(\nabla\varphi,\nabla\varphi)g(X,\nabla R)\frac{\partial B}{\partial\varphi}(\Box,\Box) \\
&\qquad - g(X,\nabla\varphi)g(\nabla\varphi,\nabla R)\frac{\partial B}{\partial\varphi}(\Box,\Box) + 4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla R,\nabla\varphi) \\
&\qquad - 2g(\nabla\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla R) - 2g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla R) \\
&\qquad - \frac{2}{3}Rg(X,\nabla\varphi)g(\nabla R,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box) \\
&\qquad + \frac{1}{3}Rg(\nabla\varphi,\nabla\varphi)g(X,\nabla R)B(\Box_a,\Box_a;\Box,\Box) \\
&\qquad + \frac{1}{3}Rg(X,\nabla\varphi)g(\nabla\varphi,\nabla R)B(\Box_a,\Box_a;\Box,\Box) - \frac{2}{3}Rg(X,\nabla\varphi)B(\nabla\varphi,\nabla R;\Box,\Box) \\
&\qquad - \frac{1}{3}Rg(\nabla\varphi,\nabla R)B(\nabla\varphi,X;\Box,\Box) - \frac{1}{3}Rg(X,\nabla R)B(\nabla\varphi,\nabla\varphi;\Box,\Box) \\
&\qquad + \frac{2}{3}Rg(\nabla\varphi,\nabla\varphi)B(X,\nabla R;\Box,\Box) + \frac{2}{3}Rg(X,\nabla\varphi)B(\nabla\varphi,\nabla R;\Box,\Box) \\
&\qquad + \frac{4}{3}Rg(X,\nabla\varphi)B(\nabla\varphi,\Box;\nabla R,\Box) - \frac{2}{3}RB(\nabla\varphi,\nabla\varphi;X,\nabla R)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{3} RB(\nabla\varphi, X; \nabla\varphi, \nabla R)\Big] (; X, Y) \\
&= \frac{(-1)^q}{3} \Big[3g(X,\nabla\varphi)Q(\nabla R) - g(\nabla\varphi,\nabla R)Q(X) - g(X,\nabla R)Q(\nabla\varphi) \\
&\quad + 2g(X,\nabla\varphi)g(\nabla R,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box) - g(X,\nabla R)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box) \\
&\quad - g(X,\nabla^2\varphi)g(\nabla\varphi,\nabla R)Q(\nabla^2\varphi;\Box,\Box) + 4g(X,\nabla\varphi)Q(\nabla^2\varphi;\nabla R,\nabla^2\varphi) \\
&\quad - 2g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;X,\nabla R) - 2g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,\nabla R) \\
&\quad + g(X,\nabla\varphi)g(\nabla R,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box) - Sg(X,\nabla R)\frac{\partial B}{\partial\varphi}(\Box,\Box) - 2S\frac{\partial B}{\partial\varphi}(X,\nabla R) \\
&\quad + 2g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla R,\nabla\varphi) + \frac{1}{3}RSg(X,\nabla R)B(\Box_a,\Box_a;\Box,\Box) \\
&\quad - \frac{1}{3}Rg(X,\nabla\varphi)g(\nabla R,\nabla\varphi)B(\Box_a,\Box_a;\Box,\Box) - \frac{1}{3}Rg(\nabla\varphi,\nabla R)B(\nabla\varphi,X;\Box,\Box) \\
&\quad - \frac{1}{3}Rg(X,\nabla R)B(\nabla\varphi,\nabla\varphi;\Box,\Box) + \frac{2}{3}RSB(X,\nabla R;\Box,\Box) \\
&\quad + \frac{4}{3}Rg(X,\nabla\varphi)B(\nabla\varphi,\Box;\nabla R,\Box) - \frac{2}{3}RB(\nabla\varphi,\nabla\varphi;X,\nabla R) \\
&\quad - \frac{2}{3}RB(X,\nabla\varphi;\nabla\varphi,\nabla R)\Big] (; X, Y) \\
&= \frac{(-1)^q}{3} \Big[3g(X,\nabla\varphi)Q(\nabla R;X,Y) - g(\nabla\varphi,\nabla R)Q(X;X,Y) - g(X,\nabla R)Q(\nabla\varphi;X,Y) \\
&\quad + g(X,\nabla\varphi)g(\nabla R,X)Q(Y;\Box,\Box) + g(X,\nabla\varphi)g(\nabla R,Y)Q(X;\Box,\Box) \\
&\quad + 2g(X,\nabla\varphi)g(\nabla R,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,Y) - \frac{1}{2}g(X,\nabla R)g(\nabla\varphi,X)Q(Y;\Box,\Box) \\
&\quad - \frac{1}{2}g(X,\nabla R)g(\nabla\varphi,Y)Q(X;\Box,\Box) - g(X,\nabla R)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\Box,\Box;X,Y) \\
&\quad - \frac{1}{2}g(X,X)g(\nabla\varphi,\nabla R)Q(Y;\Box,\Box) - \frac{1}{2}g(X,Y)g(\nabla\varphi,\nabla R)Q(X;\Box,\Box) \\
&\quad - g(X,\nabla^2\varphi)g(\nabla\varphi,\nabla R)Q(\nabla^2\varphi;\Box,\Box;X,Y) + 2g(X,\nabla\varphi)Q(X;\nabla R,Y) \\
&\quad + 2g(X,\nabla\varphi)Q(Y;\nabla R,X) + 4g(X,\nabla\varphi)Q(\nabla^2\varphi;\nabla R,\nabla^2\varphi;X,Y) \\
&\quad - g(\nabla\varphi,X)Q(Y;X,\nabla R) - g(\nabla\varphi,Y)Q(X;X,\nabla R) \\
&\quad - 2g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;X,\nabla R;X,Y) - g(X,X)Q(Y;\nabla\varphi,\nabla R) \\
&\quad - g(X,Y)Q(X;\nabla\varphi,\nabla R) - 2g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,\nabla R;X,Y) \\
&\quad + g(X,\nabla\varphi)g(\nabla R,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,Y) - Sg(X,\nabla R)\frac{\partial B}{\partial\varphi}(\Box,\Box;X,Y) \\
&\quad - 2S\frac{\partial B}{\partial\varphi}(X,\nabla R;X,Y) + 2g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla R,\nabla\varphi;X,Y)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}RSg(X,\nabla R)B(\square_a,\square_a;\square,\square;X,Y)\\
&-\frac{1}{3}Rg(X,\nabla\varphi)g(\nabla R,\nabla\varphi)B(\square_a,\square_a;\square,\square;X,Y)\\
&-\frac{1}{3}Rg(\nabla\varphi,\nabla R)B(\nabla\varphi,X;\square,\square;X,Y)-\frac{1}{3}Rg(X,\nabla R)B(\nabla\varphi,\nabla\varphi;\square,\square;X,Y)\\
&+\frac{2}{3}RSB(X,\nabla R;\square,\square;X,Y)+\frac{4}{3}Rg(X,\nabla\varphi)B(\nabla\varphi,\square;\nabla R,\square;X,Y)\\
&-\frac{2}{3}RB(\nabla\varphi,\nabla\varphi;X,\nabla R;X,Y)-\frac{2}{3}RB(X,\nabla\varphi;\nabla\varphi,\nabla R;X,Y)\Big]\\
=\frac{(-1)^q}{3}&\Big[3g(X,\nabla\varphi)Q(\nabla R;X,Y)-g(X,\nabla R)Q(\nabla\varphi;X,Y)-g(\nabla\varphi,\nabla R)Q(X;X,Y)\\
&+g(X,\nabla\varphi)g(X,\nabla R)Q(Y;\square,\square)+g(X,\nabla\varphi)g(Y,\nabla R)Q(X;\square,\square)\\
&+2g(X,\nabla\varphi)Q(X;Y,\nabla R)+g(X,\nabla\varphi)Q(Y;X,\nabla R)-g(Y,\nabla\varphi)Q(X;X,\nabla R)\\
&-g(X,X)Q(Y;\nabla\varphi,\nabla R)-g(X,Y)Q(X;\nabla\varphi,\nabla R)\\
&-\frac{1}{2}g(X,\nabla R)g(Y,\nabla\varphi)Q(X;\square,\square)-\frac{1}{2}g(X,X)g(\nabla\varphi,\nabla R)Q(Y;\square,\square)\\
&-\frac{1}{2}g(X,Y)g(\nabla\varphi,\nabla R)Q(X;\square,\square)-\frac{1}{2}g(X,\nabla\varphi)g(X,\nabla R)Q(Y;\square,\square)\\
&+2g(X,\nabla\varphi)g(\nabla R,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square;X,Y)\\
&-g(X,\nabla R)g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;\square,\square;X,Y)\\
&-g(X,\nabla^2\varphi)g(\nabla\varphi,\nabla R)Q(\nabla^2\varphi;\square,\square;X,Y)+4g(X,\nabla\varphi)Q(\nabla^2\varphi;\nabla R,\nabla^2\varphi;X,Y)\\
&-2g(\nabla\varphi,\nabla^2\varphi)Q(\nabla^2\varphi;X,\nabla R;X,Y)-2g(X,\nabla^2\varphi)Q(\nabla^2\varphi;\nabla\varphi,\nabla R;X,Y)\\
&+g(X,\nabla\varphi)g(\nabla R,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\square,\square;X,Y)-Sg(X,\nabla R)\frac{\partial B}{\partial\varphi}(\square,\square;X,Y)\\
&-2S\frac{\partial B}{\partial\varphi}(X,\nabla R;X,Y)+2g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla R,\nabla\varphi;X,Y)\\
&+\frac{1}{3}RSg(X,\nabla R)B(\square_a,\square_a;\square,\square;X,Y)\\
&-\frac{1}{3}Rg(X,\nabla\varphi)g(\nabla R,\nabla\varphi)B(\square_a,\square_a;\square,\square;X,Y)\\
&-\frac{1}{3}Rg(\nabla\varphi,\nabla R)B(\square,\square;X,\nabla\varphi;X,Y)-\frac{1}{3}Rg(X,\nabla R)B(\square,\square;\nabla\varphi,\nabla\varphi;X,Y)\\
&+\frac{2}{3}RSB(\square,\square;X,\nabla R;X,Y)+\frac{4}{3}Rg(X,\nabla\varphi)B(\nabla\varphi,\square;\nabla R,\square;X,Y)\\
&-\frac{2}{3}RB(\nabla\varphi,\nabla\varphi;X,\nabla R;X,Y)-\frac{2}{3}RB(\nabla\varphi,\nabla R;X,\nabla\varphi;X,Y)\Big],
\end{aligned}
$$

while the second term is easily produced from the second term of the previous case

$$
\begin{aligned}
&D(X,\Box,\Box,\nabla\varphi,\nabla R;X,Y)\\
&\quad=\frac{(-1)^q}{3}\Big[g(\nabla\varphi,\nabla R)Q(X;X,Y)-\frac{1}{2}g(X,\nabla\varphi)Q(\nabla R;X,Y)-\frac{1}{2}g(X,\nabla R)Q(\nabla\varphi;X,Y)\\
&\qquad-\frac{1}{2}g(X,\nabla\varphi)Q(X;Y,\nabla R)-\frac{1}{2}g(X,\nabla R)Q(X;Y,\nabla\varphi)+\frac{1}{2}g(X,\nabla\varphi)Q(Y;X,\nabla R)\\
&\qquad+\frac{1}{2}g(X,\nabla R)Q(Y;X,\nabla\varphi)+g(Y,\nabla\varphi)Q(X;X,\nabla R)+g(Y,\nabla R)Q(X;X,\nabla\varphi)\\
&\qquad-\frac{3}{2}g(X,X)Q(Y;\nabla\varphi,\nabla R)-\frac{3}{2}g(X,Y)Q(X;\nabla\varphi,\nabla R)\\
&\qquad-g(X,\nabla\varphi)Q(\nabla_a^2\varphi;\nabla R,\nabla_a^2\varphi;X,Y)-g(X,\nabla R)Q(\nabla_a^2\varphi;\nabla\varphi,\nabla_a^2\varphi;X,Y)\\
&\qquad+2g(\nabla\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;X,\nabla R;X,Y)+2g(\nabla R,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;X,\nabla\varphi;X,Y)\\
&\qquad-3g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla\varphi,\nabla R;X,Y)+\frac{1}{3}Rg(X,\nabla\varphi)B(\Box,\Box;\nabla R,\nabla\varphi;X,Y)\\
&\qquad+\frac{1}{3}Rg(X,\nabla R)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,Y)-\frac{2}{3}Rg(\nabla\varphi,\nabla\varphi)B(\Box,\Box;X,\nabla R;X,Y)\\
&\qquad-\frac{2}{3}Rg(\nabla R,\nabla\varphi)B(\Box,\Box;X,\nabla\varphi;X,Y)+Rg(X,\nabla\varphi)B(\Box,\Box;\nabla\varphi,\nabla R;X,Y)\\
&\qquad+\frac{2}{3}RB(\nabla\varphi,\nabla\varphi;X,\nabla R;X,Y)+\frac{2}{3}RB(\nabla R,\nabla\varphi;X,\nabla\varphi;X,Y)\\
&\qquad-RB(\nabla\varphi,\nabla R;X,\nabla\varphi;X,Y)-\frac{1}{3}Rg(\nabla\varphi,X)B(\nabla\varphi,\Box;\nabla R,\Box;X,Y)\\
&\qquad-\frac{1}{3}Rg(\nabla R,X)B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,Y)-g(\nabla\varphi,X)\frac{\partial B}{\partial\varphi}(\nabla R,\nabla\varphi;X,Y)\\
&\qquad-g(\nabla R,X)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,Y)-3g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla R;X,Y)\\
&\qquad+2g(\nabla\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla R;X,Y)+2g(\nabla R,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi;X,Y)\Big]\\
&\quad=\frac{(-1)^q}{3}\Big[g(\nabla\varphi,\nabla R)Q(X;X,Y)-\frac{1}{2}g(X,\nabla\varphi)Q(\nabla R;X,Y)-\frac{1}{2}g(X,\nabla R)Q(\nabla\varphi;X,Y)\\
&\qquad-\frac{1}{2}g(X,\nabla\varphi)Q(X;Y,\nabla R)-\frac{1}{2}g(X,\nabla R)Q(X;Y,\nabla\varphi)+\frac{1}{2}g(X,\nabla\varphi)Q(Y;X,\nabla R)\\
&\qquad+\frac{1}{2}g(X,\nabla R)Q(Y;X,\nabla\varphi)+g(Y,\nabla\varphi)Q(X;X,\nabla R)+g(Y,\nabla R)Q(X;X,\nabla\varphi)\\
&\qquad-\frac{3}{2}g(X,X)Q(Y;\nabla\varphi,\nabla R)-\frac{3}{2}g(X,Y)Q(X;\nabla\varphi,\nabla R)\\
&\qquad-g(X,\nabla\varphi)Q(\nabla_a^2\varphi;\nabla R,\nabla_a^2\varphi;X,Y)-g(X,\nabla R)Q(\nabla_a^2\varphi;\nabla\varphi,\nabla_a^2\varphi;X,Y)\\
&\qquad+2g(\nabla\varphi,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;X,\nabla R;X,Y)+2g(\nabla R,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;X,\nabla\varphi;X,Y)\\
&\qquad-3g(X,\nabla_a^2\varphi)Q(\nabla_a^2\varphi;\nabla\varphi,\nabla R;X,Y)+\frac{4}{3}Rg(X,\nabla\varphi)B(\Box,\Box;\nabla R,\nabla\varphi;X,Y)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{3}Rg(X,\nabla R)B(\Box,\Box;\nabla\varphi,\nabla\varphi;X,Y)-\frac{2}{3}Rg(\nabla\varphi,\nabla\varphi)B(\Box,\Box;X,\nabla R;X,Y)\\
&-\frac{2}{3}Rg(\nabla R,\nabla\varphi)B(\Box,\Box;X,\nabla\varphi;X,Y)+\frac{2}{3}RB(\nabla\varphi,\nabla\varphi;X,\nabla R;X,Y)\\
&-\frac{1}{3}RB(\nabla R,\nabla\varphi;X,\nabla\varphi;X,Y)-\frac{1}{3}Rg(X,\nabla\varphi)B(\nabla\varphi,\Box;\nabla R,\Box;X,Y)\\
&-\frac{1}{3}Rg(X,\nabla R)B(\nabla\varphi,\Box;\nabla\varphi,\Box;X,Y)-4g(X,\nabla\varphi)\frac{\partial B}{\partial\varphi}(\nabla R,\nabla\varphi;X,Y)\\
&-g(X,\nabla R)\frac{\partial B}{\partial\varphi}(\nabla\varphi,\nabla\varphi;X,Y)+2g(\nabla\varphi,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla R;X,Y)\\
&+2g(\nabla R,\nabla\varphi)\frac{\partial B}{\partial\varphi}(X,\nabla\varphi;X,Y)\Bigg].
\end{aligned}
$$

APPENDIX D
COMPUTATION OF FIFTH ORDER DIVERGENCE

This appendix is dedicated to computing the divergence of a tensor A^{ij} which can be produced using the analysis in Chapter 4.6. We combine (4.8), (4.14), and (4.20) to obtain

$$\begin{aligned}
A^{ij} &= \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)}P^{;c} + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}P^{;c;d} + D^{ijab}R_{|ab} + E^{ij} \\
&= 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)}R^{|c}\frac{\partial P}{\partial S} + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}\left(2g^{cd}\frac{\partial P}{\partial S} + 4R^{|c}R^{|d}\frac{\partial^2 P}{\partial S^2}\right) \\
&\quad + \left[4\sqrt{g}\,\mathrm{Sym}_{ij}\,\mathrm{Sym}_{ab}\,\varepsilon^{ia}R_\star^{|j}R^{|b}\frac{\partial^2 P}{\partial R\partial S} + \sqrt{g}\,\mathrm{Sym}_{ab}\,\varepsilon^{ia}\varepsilon^{jb}\left(\frac{\partial P}{\partial R} + Q\right)\right]R_{|ab} \\
&\quad + \frac{(-1)^q}{3}\sqrt{g}RR^{|i}R^{|j}\frac{\partial P}{\partial S} + \sqrt{g}R_\star^{|i}R_\star^{|j}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) + \frac{(-1)^q}{2}\sqrt{g}Rg^{ij}P \\
&= 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)}R^{|c}\frac{\partial P}{\partial S} + 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}g^{cd}\frac{\partial P}{\partial S} + 4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}R^{|c}R^{|d}\frac{\partial^2 P}{\partial S^2} \\
&\quad + 4\sqrt{g}\,\mathrm{Sym}_{ij}\,\varepsilon^{ia}R_\star^{|j}R^{|b}R_{|ab}\frac{\partial^2 P}{\partial R\partial S} + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ab}\left(\frac{\partial P}{\partial R} + Q\right) \\
&\quad + \frac{(-1)^q}{3}\sqrt{g}RR^{|i}R^{|j}\frac{\partial P}{\partial S} + \sqrt{g}R_\star^{|i}R_\star^{|j}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) + \frac{(-1)^q}{2}\sqrt{g}Rg^{ij}P,
\end{aligned} \tag{D.1}$$

where $R_\star^{|i} = \varepsilon^{ij}R_{|j}$, $P = P(R,S)$ and $Q = Q(R)$ are scalars, $Q' = \dfrac{\mathrm{d}Q}{\mathrm{d}R}$, etc., and $S = g^{ab}R_{|a}R_{|b}$.

First, we find ${A^{ij}}_{|j}$ "by hand", computing the divergence of each term using the third (A.1) and fourth (A.2) scalar order symmetrization formulas and the covariant derivative formulas $S_{|j} = 2R^{|i}R_{|ij}$ and

$$P_{|i} = P_{,i} = R_{,i}\frac{\partial P}{\partial R} + S_{,i}\frac{\partial P}{\partial S} = R_{|i}\frac{\partial P}{\partial R} + S_{|i}\frac{\partial P}{\partial S} = R_{|i}\frac{\partial P}{\partial R} + 2R^{|i}R_{|ij}\frac{\partial P}{\partial S}.$$

Defining $A_1^{ij} = 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)}R^{|c}\frac{\partial P}{\partial S}$, the divergence of this first term is given by

$$A^{ij}_{1\,|j} = \left(2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)|j}R^{|c} + 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)}{R^{|c}}_{|j}\right)\frac{\partial P}{\partial S} + 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)}R^{|c}\left(\frac{\partial P}{\partial S}\right)_{|j}$$

$$
\begin{aligned}
&= (-1)^q \left\{ 2\sqrt{g} \left(g^{ij} g^{ab} - g^{ib} g^{ja} \right) \left[R_{|(abcj)} - \frac{1}{3} R \left(g_{ab} R_{|cj} + g_{bc} R_{|aj} + g_{ac} R_{|bj} - g_{ja} R_{|bc} \right.\right.\right. \\
&\quad \left. - g_{jb} R_{|ac} - g_{jc} R_{|ab} \right) - \frac{1}{12} \left(g_{ab} R_{|c} R_{|j} + g_{bc} R_{|a} R_{|j} + g_{ac} R_{|b} R_{|j} - g_{ja} R_{|b} R_{|c} \right. \\
&\quad \left.\left. - g_{jb} R_{|c} R_{|a} - g_{jc} R_{|a} R_{|b} \right) \right] R^{|c} + 2\sqrt{g} \left(g^{ij} g^{ab} - g^{ib} g^{ja} \right) R_{|(abc)} R^{|c}{}_{|j} \left.\right\} \frac{\partial P}{\partial S} \\
&\quad + 2\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|(abc)} R^{|c} R_{|j} \frac{\partial^2 P}{\partial R \partial S} + 4\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|(abc)} R^{|c} R^{|d} R_{|dj} \frac{\partial^2 P}{\partial S^2} \\
&= \left[-\frac{2}{3} \sqrt{g} R \left(2R^{|ij} R_{|j} + R^{|ij} R_{|j} + R^{|ij} R_{|j} - R^{|ij} R_{|j} - R^{|ij} R_{|j} - R^{|i} R_{|ab} g^{ab} - R^{|ij} R_{|j} \right.\right. \\
&\quad \left. - R^{|i} R_{|ab} g^{ab} - R^{|ij} R_{|j} + 2R^{|ij} R_{|j} + R^{|ij} R_{|j} + R^{|ij} R_{|j} \right) - \frac{1}{6} \sqrt{g} \left(2SR^{|i} + SR^{|i} \right. \\
&\quad \left. + SR^{|i} - SR^{|i} - SR^{|i} - SR^{|i} - SR^{|i} - SR^{|i} - SR^{|i} + 2SR^{|i} + SR^{|i} + SR^{|i} \right) \\
&\quad \left. + 2\sqrt{g} \left(R^{|ic} R_{|(abc)} g^{ab} - R^{|(ibc)} R_{|bc} \right) \right] \frac{\partial P}{\partial S} (-1)^q + 2\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|(abc)} R^{|c} R_{|j} \frac{\partial^2 P}{\partial R \partial S} \\
&\quad + 4\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|(abc)} R^{|c} R^{|d} R_{|dj} \frac{\partial^2 P}{\partial S^2} \\
&= (-1)^q \left[-\frac{2}{3} \sqrt{g} R \left(4R^{|ij} R_{|j} - 2R^{|i} R_{|ab} g^{ab} \right) - \frac{1}{3} \sqrt{g} SR^{|i} + 2(-1)^q \sqrt{g} \left(R^{|ic} R_{|(abc)} g^{ab} \right.\right. \\
&\quad \left.\left. - R^{|(ibc)} R_{|bc} \right) \right] \frac{\partial P}{\partial S} + 2(-1)^q \sqrt{g} \left(g^{ij} g^{ab} - g^{ib} g^{ja} \right) R_{|(abc)} R^{|c} R_{|j} \frac{\partial^2 P}{\partial R \partial S} \\
&\quad + 4\sqrt{g} \left(g^{ij} g^{ab} - g^{ib} g^{ja} \right) R_{|(abc)} R^{|c} R^{|d} R_{|dj} \frac{\partial^2 P}{\partial S^2} \\
&= \left[-\frac{2}{3} \sqrt{g} R \left(4g^{ia} R^{|b} - 2R^{|i} g^{ab} \right) R_{|ab} - \frac{1}{3} \sqrt{g} SR^{|i} + 2\sqrt{g} \left(R^{|ic} g^{ab} - g^{ia} R^{|bc} \right) R_{|(abc)} \right] \\
&\quad \times \frac{\partial P}{\partial S} (-1)^q + 2(-1)^q \sqrt{g} \left(R^{|i} R^{|c} g^{ab} - g^{ia} R^{|b} R^{|c} \right) R_{|(abc)} \frac{\partial^2 P}{\partial R \partial S} \\
&\quad + 4(-1)^q \sqrt{g} \left(g^{ab} R^{|c} R_{|d} R^{|id} - g^{ia} R^{|b} R_{|d} R^{|cd} \right) R_{|(abc)} \frac{\partial^2 P}{\partial S^2}.
\end{aligned} \tag{D.2}
$$

Defining $A_2^{ij} = 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb} R_{|ac} R_{|bd} g^{cd} \frac{\partial P}{\partial S}$, we compute its divergence

$$
\begin{aligned}
A^{ij}_{2\,|j} &= \left(2\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|acj} R_{|bd} g^{cd} + 2\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|ac} R_{|bdj} g^{cd} \right) \frac{\partial P}{\partial S} \\
&\quad + 2\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|ac} R_{|bd} g^{cd} \left(\frac{\partial P}{\partial S} \right)_{|j} \\
&= \left\{ 2\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} \left[R_{|(acj)} - \frac{1}{6} R \left(2g_{ac} R_{|j} - g_{ja} R_{|c} - g_{jc} R_{|a} \right) \right] R_{|bd} g^{cd} \right. \\
&\quad \left. + 2\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|ac} \left[R_{|(bdj)} - \frac{1}{6} R \left(2g_{bd} R_{|j} - g_{jb} R_{|d} - g_{jd} R_{|b} \right) \right] g^{cd} \right\} \frac{\partial P}{\partial S} \\
&\quad + 2\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|ac} R_{|bd} g^{cd} R_{|j} \frac{\partial^2 P}{\partial R \partial S} + 4\sqrt{g} \varepsilon^{ia} \varepsilon^{jb} R_{|ac} R_{|bd} g^{cd} R^{|e} R_{|ej} \frac{\partial^2 P}{\partial S^2}
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^q \left\{ 2\sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right)\left[R_{|(acj)} - \frac{1}{6}R\left(2g_{ac}R_{|j} - g_{ja}R_{|c} - g_{jc}R_{|a}\right)\right] R_{|bd}g^{cd} \right. \\
&\quad \left. - \frac{1}{3}\sqrt{g}R\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right) R_{|ac}\left(2g_{bd}R_{|j} - g_{jb}R_{|d} - g_{jd}R_{|b}\right) g^{cd} \right\} \frac{\partial P}{\partial S} \\
&\quad + 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}g^{cd}R_{|j}\frac{\partial^2 P}{\partial R \partial S} + 4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}g^{cd}R^{|e}R_{|ej}\frac{\partial^2 P}{\partial S^2} \\
&= (-1)^q \left\{ 2\sqrt{g}\left[R^{|(iac)}R_{|ac} - \frac{1}{6}R\left(2R^{i}R_{|ab}g^{ab} - R^{|ij}R_{|j} - R^{|ij}R_{|j}\right) - R^{|ic}R_{|(abc)}g^{ab} \right. \right. \\
&\quad \left. + \frac{1}{6}R\left(2R^{|ij}R_{|j} - 2R^{|ij}R_{|j} - R^{|ij}R_{|j}\right)\right] - \frac{1}{3}\sqrt{g}R\left(2R^{i}R_{|ab}g^{ab} - R^{|ij}R_{|j} \right. \\
&\quad \left. \left. - R^{|ij}R_{|j} - 2R^{|ij}R_{|j} + R^{|ij}R_{|j} + R^{i}R_{|ab}g^{ab}\right) \right\} \frac{\partial P}{\partial S} \\
&\quad + 2(-1)^q\sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right) R_{|ac}R_{|bd}g^{cd}R_{|j}\frac{\partial^2 P}{\partial R \partial S} \\
&\quad + 4(-1)^q\sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right) R_{|ac}R_{|bd}g^{cd}R^{|e}R_{|ej}\frac{\partial^2 P}{\partial S^2} \\
&= (-1)^q \left[2\sqrt{g}\left(g^{ia}R^{|bc} - R^{|ia}g^{bc}\right)R_{|(abc)} + \frac{1}{3}\sqrt{g}R\left(-5R^{|i}g^{ab} + 4g^{ia}R^{|b}\right)R_{|ab}\right]\frac{\partial P}{\partial S} \\
&\quad + 2(-1)^q\sqrt{g}\left(g^{ac}g^{bd}R^{|i} - g^{ic}g^{bd}R^{|a}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R \partial S} + 4(-1)^q\sqrt{g}\left(g^{if}g^{ab}g^{cd}R^{|e} \right. \\
&\quad \left. - g^{ib}g^{af}g^{cd}R^{|e}\right)R_{|ac}R_{|bd}R_{|ef}\frac{\partial^2 P}{\partial S^2}. \qquad \text{(D.3)}
\end{aligned}
$$

We sum these two divergences, yielding

$$
\begin{aligned}
A^{ij}_{1\ |j} + A^{ij}_{2\ |j} &= (-1)^q \left[-\frac{2}{3}\sqrt{g}R\left(4g^{ia}R^{|b} - 2R^{|i}g^{ab}\right)R_{|ab} - \frac{1}{3}\sqrt{g}SR^{|i} + 2\sqrt{g}\left(R^{|ic}g^{ab} \right.\right. \\
&\quad \left.\left. - g^{ia}R^{|bc}\right)R_{|(abc)}\right]\frac{\partial P}{\partial S} + 2(-1)^q\sqrt{g}\left(R^{|i}R^{|c}g^{ab} - g^{ia}R^{|b}R^{|c}\right)R_{|(abc)}\frac{\partial^2 P}{\partial R \partial S} \\
&\quad + 4(-1)^q\sqrt{g}\left(g^{ab}R^{|c}R_{|d}R^{|id} - g^{ia}R^{|b}R_{|d}R^{|cd}\right)R_{|(abc)}\frac{\partial^2 P}{\partial S^2} \\
&\quad + (-1)^q\left[2\sqrt{g}\left(g^{ia}R^{|bc} - R^{|ia}g^{bc}\right)R_{|(abc)} + \frac{1}{3}\sqrt{g}R\left(-5R^{|i}g^{ab} \right.\right. \\
&\quad \left.\left. + 4g^{ia}R^{|b}\right)R_{|ab}\right]\frac{\partial P}{\partial S} + 2(-1)^q\sqrt{g}\left(g^{ac}g^{bd}R^{|i} - g^{ic}g^{bd}R^{|a}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R \partial S} \\
&\quad + 4(-1)^q\sqrt{g}\left(g^{if}g^{ab}g^{cd}R^{|e} - g^{ib}g^{af}g^{cd}R^{|e}\right)R_{|ac}R_{|bd}R_{|ef}\frac{\partial^2 P}{\partial S^2} \\
&= (-1)^q\left[-\frac{1}{3}\sqrt{g}R\left(4g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab} - \frac{1}{3}\sqrt{g}SR^{|i}\right]\frac{\partial P}{\partial S} + 2(-1)^q\sqrt{g} \\
&\quad \times\left(R^{|i}R^{|c}g^{ab} - g^{ia}R^{|b}R^{|c}\right)R_{|(abc)}\frac{\partial^2 P}{\partial R \partial S} + 4(-1)^q\sqrt{g}\left(g^{ab}R^{|c}R_{|d}R^{|id}\right.
\end{aligned}
$$

$$- g^{ia}R^{|b}R_{|d}R^{|cd}\Big) R_{|(abc)}\frac{\partial^2 P}{\partial S^2} + 2(-1)^q\sqrt{g}\left(g^{ac}g^{bd}R^{|i} - g^{ic}g^{bd}R^{|a}\right)R_{|ab}R_{|cd}$$
$$\times \frac{\partial^2 P}{\partial R\partial S} + 4(-1)^q\sqrt{g}\left(g^{if}g^{ab}g^{cd}R^{|e} - g^{ib}g^{af}g^{cd}R^{|e}\right)R_{|ac}R_{|bd}R_{|ef}\frac{\partial^2 P}{\partial S^2}. \tag{D.4}$$

Defining $A_3^{ij} = 4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}R^{|c}R^{|d}\frac{\partial^2 P}{\partial S^2}$, the divergence of A_3^{ij} is

$$\begin{aligned}
A^{ij}_{3\;|j} = &\Big(4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|acj}R_{|bd}R^{|c}R^{|d} + 4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bdj}R^{|c}R^{|d} + 4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}R^{|c}{}_{|j}R^{|d} \\
&+ 4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}R^{|c}R^{|d}{}_{|j}\Big)\frac{\partial^2 P}{\partial S^2} + 4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}R^{|c}R^{|d}\left(\frac{\partial^2 P}{\partial S^2}\right)_{|j} \\
= &(-1)^q\Big\{4\sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right)\left[R_{|(acj)} - \frac{1}{6}R\left(2g_{ac}R_{|j} - g_{ja}R_{|c} - g_{jc}R_{|a}\right)\right]R_{|bd}R^{|c}R^{|d} \\
&+ 4\sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right)\left(0 - \frac{1}{6}R(2g_{bd}R_{|j} - g_{jb}R_{|d} - g_{jd}R_{|b})\right)R_{|ac}R^{|c}R^{|d} \\
&+ 4\sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right)R_{|ac}R_{|bd}R^{|c}{}_{|j}R^{|d} + 0\Big\}\frac{\partial^2 P}{\partial S^2} \\
&+ 4\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}R^{|c}R^{|d}R_{|j}\frac{\partial^3 P}{\partial S^2\partial R} + 8\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}R^{|c}R^{|d}R^{|e}R_{|ej}\frac{\partial^3 P}{\partial S^3} \\
= &(-1)^q\Big\{4\sqrt{g}\Big[R^{|(ibc)}R_{|bd}R_{|c}R^{|d} - \frac{1}{6}R\left(2R^{|i}R^{|j}R^{|k}R_{|jk} - SR^{|ij}R_{|j} - R^{|i}R^{|b}R^{|d}R_{|bd}\right) \\
&- R^{|ij}R_{|j}R^{|c}R_{|(abc)}g^{ab} + \frac{1}{6}R\left(2SR^{|ij}R_{|j} - 2SR^{|ij}R_{|j} - SR^{|ij}R_{|j}\right)\Big] \\
&- \frac{2}{3}\sqrt{g}R\Big(2R^{|i}R^{|b}R^{|d}R_{|bd} - SR^{|ij}R_{|j} - R^{|i}R^{|b}R^{|d}R_{|bd} - 2R^{|i}R^{|b}R^{|d}R_{|bd} \\
&+ SR^{|ij}R_{|j} + R^{|i}R^{|b}R^{|d}R_{|bd}\Big) + 4\sqrt{g}\left(R^{|ia}R_{|ab}R^{|bc}R_{|c} - R^{|ij}R_{|j}R_{|ab}R^{|ab}\right)\Big\}\frac{\partial^2 P}{\partial S^2} \\
&+ 4(-1)^q\sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right)R_{|ac}R_{|bd}R^{|c}R^{|d}R_{|j}\frac{\partial^3 P}{\partial S^2\partial R} + 0 \\
= &(-1)^q\Big[4\sqrt{g}\left(g^{ia}R^{|bd}R^{|c}R_{|d} - R^{|ij}R_{|j}R^{|c}g^{ab}\right)R_{|(abc)} - \frac{2}{3}\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk} \\
&+ 4(-1)^q\sqrt{g}\left(g^{ia}g^{be}g^{cf}R^{|d} - g^{ia}g^{ce}g^{df}R^{|b}\right)R_{|ab}R_{|cd}R_{|ef}\Big]\frac{\partial^2 P}{\partial S^2} \\
&+ 4(-1)^q\sqrt{g}\left(g^{ab}R^{|i}R^{|c}R^{|d} - g^{ib}R^{|a}R^{|c}R^{|d}\right)R_{|ac}R_{|bd}\frac{\partial^3 P}{\partial S^2\partial R}.
\end{aligned} \tag{D.5}$$

The sum of the first three terms is given by

$$A^{ij}_{1\;|j} + A^{ij}_{2\;|j} + A^{ij}_{3\;|j}$$

$$
\begin{aligned}
&= (-1)^q \left[-\frac{1}{3}\sqrt{g}R\left(4g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab} - \frac{1}{3}\sqrt{g}SR^{|i}\right]\frac{\partial P}{\partial S} + 2(-1)^q\sqrt{g} \\
&\quad \times \left(R^{|i}R^{|c}g^{ab} - g^{ia}R^{|b}R^{|c}\right)R_{|(abc)}\frac{\partial^2 P}{\partial R\partial S} + 4(-1)^q\sqrt{g}\left(g^{ab}R^{|c}R_{|d}R^{|id}\right. \\
&\quad \left. - g^{ia}R^{|b}R_{|d}R^{|cd}\right)R_{|(abc)}\frac{\partial^2 P}{\partial S^2} + 2(-1)^q\sqrt{g}\left(g^{ac}g^{bd}R^{|i} - g^{ic}g^{bd}R^{|a}\right)R_{|ab}R_{|cd} \\
&\quad \times \frac{\partial^2 P}{\partial R\partial S} + 4(-1)^q\sqrt{g}\left(g^{if}g^{ab}g^{cd}R^{|e} - g^{ib}g^{af}g^{cd}R^{|e}\right)R_{|ac}R_{|bd}R_{|ef}\frac{\partial^2 P}{\partial S^2} \\
&\quad + (-1)^q\left[4\sqrt{g}\left(g^{ia}R^{|bd}R^{|c}R_{|d} - R^{|ij}R_{|j}R^{|c}g^{ab}\right)R_{|(abc)} - \frac{2}{3}\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\right. \\
&\quad \left. + 4(-1)^q\sqrt{g}\left(g^{ia}g^{be}g^{cf}R^{|d} - g^{ia}g^{ce}g^{df}R^{|b}\right)R_{|ab}R_{|cd}R_{|ef}\right]\frac{\partial^2 P}{\partial S^2} \\
&\quad + 4(-1)^q\sqrt{g}\left(g^{ab}R^{|i}R^{|c}R^{|d} - g^{ib}R^{|a}R^{|c}R^{|d}\right)R_{|ac}R_{|bd}\frac{\partial^3 P}{\partial S^2\partial R} \\
&= -\frac{(-1)^q}{3}\sqrt{g}R\left(4g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab}\frac{\partial P}{\partial S} - \frac{(-1)^q}{3}\sqrt{g}SR^{|i}\frac{\partial P}{\partial S} \\
&\quad - \frac{2}{3}(-1)^q\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\frac{\partial^2 P}{\partial S^2} \\
&\quad + 2(-1)^q\sqrt{g}\left(g^{ac}g^{bd}R^{|i} - g^{ic}g^{bd}R^{|a}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S} \\
&\quad + 4(-1)^q\sqrt{g}\left(g^{ab}R^{|i}R^{|c}R^{|d} - g^{ib}R^{|a}R^{|c}R^{|d}\right)R_{|ac}R_{|bd}\frac{\partial^3 P}{\partial S^2\partial R} \\
&\quad + 2(-1)^q\sqrt{g}\left(R^{|i}R^{|c}g^{ab} - g^{ia}R^{|b}R^{|c}\right)R_{|(abc)}\frac{\partial^2 P}{\partial R\partial S}.
\end{aligned}
\tag{D.6}
$$

Defining $A_4^{ij} = 2\,\mathrm{Sym}_{ij}\,\varepsilon^{ia}\varepsilon^{jc}R_{|c}R_{|ab}\frac{\partial P^{;b}}{\partial R} = 4\sqrt{g}\,\mathrm{Sym}_{ij}\,\varepsilon^{ia}\varepsilon^{jc}R_{|c}R^{|b}R_{|ab}\frac{\partial^2 P}{\partial R\partial S}$, the divergence of this term is

$$
\begin{aligned}
A^{ij}_{4\;|j} &= 2\sqrt{g}\left(\varepsilon^{ia}\varepsilon^{jc} + \varepsilon^{ja}\varepsilon^{ic}\right)\left(R_{|cj}R_{|ab}R^{|b} + R_{|c}R_{|abj}R^{|b} + R_{|c}R_{|ab}R^{|b}{}_{|j}\right)\frac{\partial^2 P}{\partial R\partial S} \\
&\quad + 2\sqrt{g}\left(\varepsilon^{ia}\varepsilon^{jc} + \varepsilon^{ja}\varepsilon^{ic}\right)R_{|c}R_{|ab}R^{|b}\left(\frac{\partial^2 P}{\partial R\partial S}\right)_{|j} \\
&= 2\sqrt{g}\left(\left(g^{ij}g^{ac} - g^{ic}g^{ja}\right)\left[0 + R_{|c}R^{|b}\left(R_{|(abj)} - \frac{1}{6}R(2g_{ab}R_{|j} - g_{ja}R_{|b} - g_{jb}R_{|a})\right)\right.\right. \\
&\quad \left. + R_{|c}R_{|ab}R^{|b}{}_{|j}\right] + \left(g^{ij}g^{ac} - g^{jc}g^{ia}\right)\left\{{}_{|cj}R_{|ab}R^{|b} + R_{|c}R^{|b}\left[R_{|(abj)} - \frac{1}{6}R\left(2g_{ab}R_{|j}\right.\right.\right. \\
&\quad \left.\left.\left.\left. - g_{ja}R_{|b} - g_{jb}R_{|a}\right)\right] + R_{|c}R_{|ab}R^{|b}{}_{|j}\right\}\right)\frac{\partial^2 P}{\partial R\partial S}(-1)^q + 2(-1)^q\sqrt{g}\left[\left(g^{ij}g^{ac} - g^{ic}g^{ja}\right)\right. \\
&\quad \left. \times R_{|c}R_{|ab}R^{|b} + \left(g^{ij}g^{ac} - g^{jc}g^{ia}\right)R_{|c}R_{|ab}R^{|b}\right]\left(R_{|j}\frac{\partial^3 P}{\partial R^2\partial S} + 2R^{|d}R_{|dj}\frac{\partial^3 P}{\partial R\partial S^2}\right) \\
&= 2(-1)^q\sqrt{g}\left(\left\{g^{ij}R^{|a}R^{|b}\left[R_{|(abj)} - \frac{1}{6}R\left(2g_{ab}R_{|j} - g_{ja}R_{|b} - g_{jb}R_{|a}\right)\right] + R^{|ia}R_{|ab}R^{|b}\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&- g^{ja}R^{|i}R^{|b}\left[R_{|(abj)} - \frac{1}{6}R\left(2g_{ab}R_{|j} - g_{ja}R_{|b} - g_{jb}R_{|a}\right)\right] - R^{|i}R_{|ab}R^{|ab}\Big\}\\
&+ \Big\{R^{|ia}R_{|ab}R^{|b} + g^{ij}R^{|a}R^{|b}\left[R_{|(abj)} - \frac{1}{6}R\left(2g_{ab}R_{|j} - g_{ja}R_{|b} - g_{jb}R_{|a}\right)\right]\\
&+ R^{|ia}R_{|ab}R^{|b} - R^{|ia}R_{|a}g^{jc}R_{|cj} - g^{ia}R^{|j}R^{|b}\Big[R_{|(abj)} - \frac{1}{6}R\left(2g_{ab}R_{|j}\right.\\
&\left.- g_{ja}R_{|b} - g_{jb}R_{|a}\right)\Big] - R^{|ia}R_{|ab}R^{|b}\Big\}\Big)\frac{\partial^2 P}{\partial R\partial S} + 2(-1)^q\sqrt{g}\left(g^{ij}R^{|a}R^{|b} - g^{ja}R^{|i}R^{|b}\right.\\
&\left.+ g^{ij}R^{|a}R^{|b} - g^{ia}R^{|j}R^{|b}\right)R_{|ab}\left(R_{|j}\frac{\partial^3 P}{\partial R^2\partial S} + 2R^{|c}R_{|cj}\frac{\partial^3 P}{\partial R\partial S^2}\right)\\
={}& 2(-1)^q\sqrt{g}\Big[g^{ij}R^{|a}R^{|b}R_{|(abj)} - \frac{1}{6}R\left(2SR^{|i} - SR^{|i} - SR^{|i}\right) + 2R^{|ia}R_{|ab}R^{|b}\\
&- g^{ja}R^{|i}R^{|b}R_{|(abj)} + \frac{1}{6}R\left(2SR^{|i} - 2SR^{|i} - SR^{|i}\right) - R^{|i}R_{|ab}R^{|ab}\\
&- \frac{1}{6}R\left(2SR^{|i} - SR^{|i} - SR^{|i}\right) - R^{|ia}R_{|a}g^{jc}R_{|cj} + \frac{1}{6}R\left(2SR^{|i} - SR^{|i} - SR^{|i}\right)\Big]\\
&\times\frac{\partial^2 P}{\partial R\partial S} + 2(-1)^q\sqrt{g}\left(2g^{ij}R^{|a}R^{|b} - g^{ja}R^{|i}R^{|b} - g^{ia}R^{|j}R^{|b}\right)R_{|ab}\Big(R_{|j}\frac{\partial^3 P}{\partial R^2\partial S}\\
&+ 2R^{|c}R_{|cj}\frac{\partial^3 P}{\partial R\partial S^2}\Big)\\
={}& 2(-1)^q\sqrt{g}\left(g^{ia}R^{|b}R^{|c} - R^{|i}R^{|a}g^{bc}\right)R_{|(abc)}\frac{\partial^2 P}{\partial R\partial S} - \frac{(-1)^q}{3}\sqrt{g}RSR^{|i}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(2g^{ia}g^{bc}R^{|d} - g^{ac}g^{bd}R^{|i} - g^{ia}g^{cd}R^{|b}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(R^{|i}R^{|a}R^{|b} - Sg^{ia}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S}\\
&+ 4(-1)^q\sqrt{g}\left(g^{id}R^{|a}R^{|b}R^{|c} - g^{ad}R^{|i}R^{|b}R^{|c}\right)R_{|ab}R_{|cd}\frac{\partial^3 P}{\partial R\partial S^2}
\end{aligned}
\tag{D.7}
$$

and we sum this value with the previous three divergences to get

$$
\begin{aligned}
\sum_{k=1}^{4} A_{k\ |j}^{ij} ={}& -\frac{(-1)^q}{3}\sqrt{g}R\left(4g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab}\frac{\partial P}{\partial S} - \frac{(-1)^q}{3}\sqrt{g}SR^{|i}\frac{\partial P}{\partial S}\\
&- \frac{2}{3}(-1)^q\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\frac{\partial^2 P}{\partial S^2}\\
&+ 2(-1)^q\sqrt{g}\left(g^{ac}g^{bd}R^{|i} - g^{ic}g^{bd}R^{|a}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 4(-1)^q\sqrt{g}\left(g^{ab}R^{|i}R^{|c}R^{|d} - g^{ib}R^{|a}R^{|c}R^{|d}\right)R_{|ac}R_{|bd}\frac{\partial^3 P}{\partial S^2\partial R}\\
&+ 2(-1)^q\sqrt{g}\left(R^{|i}R^{|c}g^{ab} - g^{ia}R^{|b}R^{|c}\right)R_{|(abc)}\frac{\partial^2 P}{\partial R\partial S}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(-1)^q\sqrt{g}\left(g^{ia}R^{|b}R^{|c} - R^{|i}R^{|a}g^{bc}\right)R_{|(abc)}\frac{\partial^2 P}{\partial R\partial S} - \frac{(-1)^q}{3}\sqrt{g}RSR^{|i}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(2g^{ia}g^{bc}R^{|d} - g^{ac}g^{bd}R^{|i} - g^{ia}g^{cd}R^{|b}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(R^{|i}R^{|a}R^{|b} - Sg^{ia}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S}\\
&+ 4(-1)^q\sqrt{g}\left(g^{id}R^{|a}R^{|b}R^{|c} - g^{ad}R^{|i}R^{|b}R^{|c}\right)R_{|ab}R_{|cd}\frac{\partial^3 P}{\partial R\partial S^2}\\
= -&\frac{(-1)^q}{3}\sqrt{g}R\left(4g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab}\frac{\partial P}{\partial S} - \frac{(-1)^q}{3}\sqrt{g}SR^{|i}\frac{\partial P}{\partial S}\\
&- \frac{2}{3}(-1)^q\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\frac{\partial^2 P}{\partial S^2} - \frac{(-1)^q}{3}\sqrt{g}RSR^{|i}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(g^{ia}g^{bc}R^{|d} - g^{ia}g^{cd}R^{|b}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(R^{|i}R^{|a}R^{|b} - Sg^{ia}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S}.
\end{aligned} \tag{D.8}
$$

Defining $A_5^{ij} = \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ab}\left(\frac{\partial P}{\partial R} + Q\right)$, we compute the divergence

$$
\begin{aligned}
A^{ij}_{5\ |j} &= \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|abj}\left(\frac{\partial P}{\partial R} + Q\right) + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ab}\left(\frac{\partial P}{\partial R} + Q\right)_{|j}\\
&= \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}\left[R_{|(abj)} - \frac{1}{6}R\left(2g_{ab}R_{|j} - g_{ja}R_{|b} - g_{jb}R_{|a}\right)\right]\left(\frac{\partial P}{\partial R} + Q\right)\\
&\quad + \sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right)R_{|ab}\left(R_{|j}\frac{\partial^2 P}{\partial R^2} + R^{|c}R_{|cj}\frac{\partial^2 P}{\partial R\partial S} + R_{|j}Q'\right)\\
&= (-1)^q\sqrt{g}\left[0 - \frac{1}{6}R\left(2g^{ij}R_{|j} + g^{ib}R_{|b} + 0\right)\right]\left(\frac{\partial P}{\partial R} + Q\right)\\
&\quad + (-1)^q\sqrt{g}\left(R^{|i}g^{ab} - g^{ib}R^{|a}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right)\\
&\quad + 2(-1)^q\sqrt{g}\left(g^{id}g^{ab}R^{|c} - g^{ib}g^{ad}R^{|c}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S}\\
&= -\frac{(-1)^q}{2}\sqrt{g}RR^{|i}\left(\frac{\partial P}{\partial R} + Q\right) + (-1)^q\sqrt{g}\left(g^{ab}R^{|i} - g^{ib}R^{|a}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right)\\
&\quad + 2(-1)^q\sqrt{g}\left(g^{id}g^{ab}R^{|c} - g^{ib}g^{ad}R^{|c}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S}.
\end{aligned} \tag{D.9}
$$

We sum this result with the previous four terms, yielding

$$
\begin{aligned}
\sum_{k=1}^{5}A^{ij}_{k\ |j} = -&\frac{(-1)^q}{3}\sqrt{g}R\left(4g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab}\frac{\partial P}{\partial S} - \frac{(-1)^q}{3}\sqrt{g}SR^{|i}\frac{\partial P}{\partial S}\\
&- \frac{2}{3}(-1)^q\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\frac{\partial^2 P}{\partial S^2} - \frac{(-1)^q}{3}\sqrt{g}RSR^{|i}\frac{\partial^2 P}{\partial R\partial S}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2(-1)^q\sqrt{g}\left(g^{ia}g^{bc}R^{|d} - g^{ia}g^{cd}R^{|b}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(R^{|i}R^{|a}R^{|b} - Sg^{ia}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S} - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\left(\frac{\partial P}{\partial R} + Q\right)\\
&+ (-1)^q\sqrt{g}\left(g^{ab}R^{|i} - g^{ib}R^{|a}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right)\\
&+ 2(-1)^q\sqrt{g}\left(g^{id}g^{ab}R^{|c} - g^{ib}g^{ad}R^{|c}\right)R_{|ab}R_{|cd}\frac{\partial^2 P}{\partial R\partial S}\\
= &-\frac{(-1)^q}{3}\sqrt{g}R\left(4g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab}\frac{\partial P}{\partial S} - \frac{(-1)^q}{3}\sqrt{g}SR^{|i}\frac{\partial P}{\partial S}\\
&- \frac{2}{3}(-1)^q\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\frac{\partial^2 P}{\partial S^2} - \frac{(-1)^q}{3}\sqrt{g}RSR^{|i}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(R^{|i}R^{|a}R^{|b} - Sg^{ia}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S} - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\left(\frac{\partial P}{\partial R} + Q\right)\\
&+ (-1)^q\sqrt{g}\left(g^{ab}R^{|i} - g^{ib}R^{|a}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right).
\end{aligned} \tag{D.10}
$$

Computing the divergence of $A_6^{ij} = \frac{(-1)^q}{3}\sqrt{g}RR^{|i}R^{|j}\frac{\partial P}{\partial S}$, we have

$$
\begin{aligned}
A^{ij}_{6\;|j} = &\frac{(-1)^q}{3}\sqrt{g}R_{|j}R^{|i}R^{|j}\frac{\partial P}{\partial S} + \frac{(-1)^q}{3}\sqrt{g}RR^{|i}{}_{|j}R^{|j}\frac{\partial P}{\partial S} + \frac{(-1)^q}{3}\sqrt{g}RR^{|i}R^{|j}{}_{|j}\frac{\partial P}{\partial S}\\
&+ \frac{(-1)^q}{3}\sqrt{g}RR^{|i}R^{|j}\left(\frac{\partial P}{\partial S}\right)_{|j}\\
= &\frac{(-1)^q}{3}\sqrt{g}SR^{|i}\frac{\partial P}{\partial S} + \frac{(-1)^q}{3}\sqrt{g}R\left(g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab}\frac{\partial P}{\partial S}\\
&+ \frac{(-1)^q}{3}\sqrt{g}RSR^{|i}\frac{\partial^2 P}{\partial R\partial S} + \frac{2}{3}(-1)^q\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\frac{\partial^2 P}{\partial S^2}.
\end{aligned} \tag{D.11}
$$

Summing with the previous result, we get

$$
\begin{aligned}
\sum_{k=1}^{6} A^{ij}_{k\;|j} = &-\frac{(-1)^q}{3}\sqrt{g}R\left(4g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab}\frac{\partial P}{\partial S} - \frac{(-1)^q}{3}\sqrt{g}SR^{|i}\frac{\partial P}{\partial S}\\
&- \frac{2}{3}(-1)^q\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\frac{\partial^2 P}{\partial S^2} - \frac{(-1)^q}{3}\sqrt{g}RSR^{|i}\frac{\partial^2 P}{\partial R\partial S}\\
&+ 2(-1)^q\sqrt{g}\left(R^{|i}R^{|a}R^{|b} - Sg^{ia}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S} - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\left(\frac{\partial P}{\partial R} + Q\right)\\
&+ (-1)^q\sqrt{g}\left(g^{ab}R^{|i} - g^{ib}R^{|a}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) + \frac{(-1)^q}{3}\sqrt{g}SR^{|i}\frac{\partial P}{\partial S}\\
&+ \frac{(-1)^q}{3}\sqrt{g}RSR^{|i}\frac{\partial^2 P}{\partial R\partial S} + \frac{(-1)^q}{3}\sqrt{g}R\left(g^{ia}R^{|b} + R^{|i}g^{ab}\right)R_{|ab}\frac{\partial P}{\partial S}\\
&+ \frac{2}{3}(-1)^q\sqrt{g}RR^{|i}R^{|j}R^{|k}R_{|jk}\frac{\partial^2 P}{\partial S^2}
\end{aligned}
$$

$$
\begin{aligned}
&= -(-1)^q\sqrt{g}Rg^{ia}R^{|b}R_{|ab}\frac{\partial P}{\partial S} + 2(-1)^q\sqrt{g}\left(R^{|i}R^{|a}R^{|b} - Sg^{ia}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S} \\
&\quad - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\left(\frac{\partial P}{\partial R} + Q\right) + (-1)^q\sqrt{g}\left(g^{ab}R^{|i} - g^{ib}R^{|a}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right).
\end{aligned}
\tag{D.12}
$$

Computing the divergence of $A_7^{ij} = \sqrt{g}R_\star^{|i}R_\star^{|j}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) = \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|a}R_{|b}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right)$, we have

$$
\begin{aligned}
A^{ij}_{7\;|j} &= \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|aj}R_{|b}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|a}R_{|bj}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) \\
&\quad + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|a}R_{|b}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right)_{|j} \\
&= (-1)^q\sqrt{g}\left(g^{ij}g^{ab} - g^{ib}g^{ja}\right)R_{|aj}R_{|b}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) + 0 + \sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|a}R_{|b}\left(R_{|j}\frac{\partial^3 P}{\partial R^3}\right. \\
&\quad \left. + 2R^{|c}R_{|cj}\frac{\partial^3 P}{\partial R^2\partial S} + R_{|j}Q''\right) \\
&= (-1)^q\sqrt{g}\left(g^{ia}R^{|b} - g^{ab}R^{|i}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) + (-1)^q\sqrt{g}\left(Sg^{ij} - R^{|i}R^{|j}\right) \\
&\quad \times\left(0 + 2R^{|c}R_{|cj}\frac{\partial^3 P}{\partial R^2\partial S} + 0\right) \\
&= (-1)^q\sqrt{g}\left(g^{ia}R^{|b} - g^{ab}R^{|i}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) + 2(-1)^q\sqrt{g}\left(Sg^{ia}R^{|b} - R^{|i}R^{|a}R^{|b}\right) \\
&\quad \times R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S}.
\end{aligned}
\tag{D.13}
$$

We sum this term with the previous results to get

$$
\begin{aligned}
\sum_{k=1}^{7} A^{ij}_{k\;|j} &= -(-1)^q\sqrt{g}Rg^{ia}R^{|b}R_{|ab}\frac{\partial P}{\partial S} + 2(-1)^q\sqrt{g}\left(R^{|i}R^{|a}R^{|b} - Sg^{ia}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S} \\
&\quad - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\left(\frac{\partial P}{\partial R} + Q\right) + (-1)^q\sqrt{g}\left(g^{ab}R^{|i} - g^{ib}R^{|a}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) \\
&\quad + (-1)^q\sqrt{g}\left(g^{ia}R^{|b} - g^{ab}R^{|i}\right)R_{|ab}\left(\frac{\partial^2 P}{\partial R^2} + Q'\right) \\
&\quad + 2(-1)^q\sqrt{g}\left(Sg^{ia}R^{|b} - R^{|i}R^{|a}R^{|b}\right)R_{|ab}\frac{\partial^3 P}{\partial R^2\partial S} \\
&= -(-1)^q\sqrt{g}Rg^{ia}R^{|b}R_{|ab}\frac{\partial P}{\partial S} - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\left(\frac{\partial P}{\partial R} + Q\right).
\end{aligned}
\tag{D.14}
$$

Computing the divergence of $A_8^{ij} = \frac{(-1)^q}{2}\sqrt{g}Rg^{ij}P$, we get

$$\begin{aligned}
A^{ij}_{8\ |j} &= \frac{(-1)^q}{2}\sqrt{g}R_{|j}g^{ij}P + \frac{(-1)^q}{2}\sqrt{g}Rg^{ij}P_{|j} \\
&= \frac{(-1)^q}{2}\sqrt{g}R^{|i}P + \frac{(-1)^q}{2}\sqrt{g}Rg^{ij}\left(R_{|j}\frac{\partial P}{\partial R} + R^{|k}R_{|jk}\frac{\partial P}{\partial S}\right) \\
&= \frac{(-1)^q}{2}\sqrt{g}R^{|i}P + \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\frac{\partial P}{\partial R} + (-1)^q\sqrt{g}Rg^{ia}R^{|b}R_{|ab}\frac{\partial P}{\partial S},
\end{aligned} \tag{D.15}$$

and summing this term with the previous seven terms yields the final result of

$$\begin{aligned}
\sum_{k=1}^{8} A^{ij}_{k\ |j} &= -(-1)^q\sqrt{g}Rg^{ia}R^{|b}R_{|ab}\frac{\partial P}{\partial S} - \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\left(\frac{\partial P}{\partial R} + Q\right) + \frac{(-1)^q}{2}\sqrt{g}R^{|i}P \\
&\quad + \frac{(-1)^q}{2}\sqrt{g}RR^{|i}\frac{\partial P}{\partial R} + (-1)^q\sqrt{g}Rg^{ia}R^{|b}R_{|ab}\frac{\partial P}{\partial S} \\
&= \frac{(-1)^q}{2}\sqrt{g}R^{|i}(P - RQ).
\end{aligned} \tag{D.16}$$

We check the previous "by hand" computation with custom Maple code. For coding simplicity, we have expanded the permutation tensors in all terms using (2.7) and absorbed the factor of $(-1)^q$ into the scalars P and Q throughout the Maple code. We begin by loading the **DifferentialGeometry**, **Tensor**, and **ATensor** packages. The first two are standard Maple packages for handling manifolds and tensorial quantities, respectively, while the third is a custom module (with code found at the end of this appendix) designed to compute the tensorial coefficients of the scalar fields P and Q (and their derivatives) in A^{ij}.

```
> with(ATensor);
          [B1, B2, B3, B4, B5, B6, B7, B8, Div, Init, divB1, divB2, divB3, divB4, divB5, divB6, divB7, divB8]          (1)
> with(DifferentialGeometry):
> with(Tensor):
```

Figure D.1: Loading packages.

We initialize a two dimensional manifold M with local coordinates (x, y) using the *DGEnvironment* command from the **DifferentialGeometry** package and define an arbitrary conformal Lorentzian metric g. [As all two dimensional metrics are conformally flat, this is the most generic metric with (+,-) signature. The Riemannian case follows in an identical matter, with the same general results, and has been omitted for brevity.]

```
> DGEnvironment[Manifold]([x,y],M)
```

$$Manifold: M \tag{2}$$

```
M > g := evalDG(F(x,y)*(dx &t dx - dy &t dy))
```

$$g := F(x, y)\ dx \otimes dx - F(x, y)\ dy \otimes dy \tag{3}$$

Figure D.2: Initialization of a Lorentzian manifold.

We produce the curvature scalar R and its first three symmetrized covariant derivatives using the *Init* command of the **ATensor** package and label them accordingly.

```
M > TensorList := Init(g):
M > gij, R, Ra, Rab, Rabc := op(TensorList):
```

Figure D.3: Curvature scalar derivatives.

The output for these commands are suppressed with a colon for brevity; the use of a very general metric produces rather intractable expressions, with a few of the subsequent calculations going beyond Maple's one million character limit. For example, we note that $R_{|a}$ comprises three lines, $R_{|ab}$ takes up fifteen lines, and $R_{|(abc)}$ requires 71 lines to express.

Next, we define a number of useful intermediate quantities: the inverse metric, square root of the determinant, the raised covariant derivative of the metric, the scalar $S = g^{ab}R_{|a}R_{|b}$, and the tensor $R^{|a}R_{|ab}$.

```
M > gIJ := InverseMetric(gij)
```

$$gIJ := \frac{1}{F(x, y)}\ \partial_x \otimes \partial_x - \frac{1}{F(x, y)}\ \partial_y \otimes \partial_y \tag{4}$$

```
M > rootg := MetricDensity(gij,1)
```

$$rootg := \sqrt{-F(x, y)^2}\ \mathbf{1}_{[1]} \tag{5}$$

```
M > RA := RaiseLowerIndices(gIJ,Ra,[1]):
M > S := ContractIndices(Ra &t RA, [[1,2]]):
M > RA_Rab := ContractIndices(RA &t Rab, [[1,2]]):
```

Figure D.4: Intermediate quantities.

Then we define the scalar fields $P(R, S)$ and $Q(R)$, and compute a number of different partial derivatives.

```
M > P := p(r,s)
```

$$P := p(r, s) \quad (6)$$

```
M > Q := q(r)
```

$$Q := q(r) \quad (7)$$

```
M > dP := [P,seq(diff(P,r$i),i=1..3),seq(diff(P,s$i),i=1..3), diff(P,r,s), diff(P,r,r,s), diff(P,r,s,s)]
```

$$dP := \left[p(r,s),\ \frac{\partial}{\partial r}p(r,s),\ \frac{\partial^2}{\partial r^2}p(r,s),\ \frac{\partial^3}{\partial r^3}p(r,s),\ \frac{\partial}{\partial s}p(r,s),\ \frac{\partial^2}{\partial s^2}p(r,s),\ \frac{\partial^3}{\partial s^3}p(r,s),\ \frac{\partial^2}{\partial s\,\partial r}p(r,s),\ \frac{\partial^3}{\partial s\,\partial r^2}p(r,s),\ \frac{\partial^3}{\partial s^2\,\partial r}p(r,s)\right] \quad (8)$$

```
M > dQ := [Q,seq(diff(Q,r$i),i=1..3)]
```

$$dQ := \left[q(r),\ \frac{\mathrm{d}}{\mathrm{d}r}q(r),\ \frac{\mathrm{d}^2}{\mathrm{d}r^2}q(r),\ \frac{\mathrm{d}^3}{\mathrm{d}r^3}q(r)\right] \quad (9)$$

Figure D.5: Scalar fields and their partial derivatives.

Finally, we compute the covariant derivative of the scalar expressions in A^{ij} (D.1): in order, we have $Q_{|a}$, $\left(\frac{\mathrm{d}Q}{\mathrm{d}R}\right)_{|a}$, $P_{|a}$, $\left(\frac{\partial P}{\partial R}\right)_{|a}$, $\left(\frac{\partial P}{\partial S}\right)_{|a}$, $\left(\frac{\partial^2 P}{\partial R^2}\right)_{|a}$, $\left(\frac{\partial^2 P}{\partial S^2}\right)_{|a}$, and $\left(\frac{\partial^2 P}{\partial S \partial R}\right)_{|a}$.

```
M > Diff_Q := evalDG(Ra * dQ[2]):
M > Diff_dQdR := evalDG(Ra * dQ[3]):
M > Diff_P := evalDG(Ra * dP[2] + 2 * RA_Rab * dP[5]):
M > Diff_dPdR := evalDG(Ra * dP[3] + 2 * RA_Rab * dP[8]):
M > Diff_dPdS := evalDG(Ra * dP[8] + 2 * RA_Rab * dP[6]):
M > Diff_d2PdR2 := evalDG(Ra * dP[4] + 2 * RA_Rab * dP[9]):
M > Diff_d2PdS2 := evalDG(Ra * dP[10] + 2 * RA_Rab * dP[7]):
M > Diff_d2PdSdR := evalDG(Ra * dP[9] + 2 * RA_Rab * dP[10]):
```

Figure D.6: Scalar fields and their covariant derivatives.

Before continuing, we define $B_1^{ij} = 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|(abc)}R^{|c}$, $B_2^{ij} = 2\sqrt{g}\varepsilon^{ia}\varepsilon^{jb}R_{|ac}R_{|bd}g^{cd}$, etc., and note that these tensor densities are computed using the *B#* commands of the **ATensor** package. In particular, we use the *Div* command of the **ATensor** package to compute the divergence of each $B_{\#}^{ij}$ term and subtract from this value the "by-hand" calculations for the divergence contained in the *divB#* command of the **ATensor** package, with the difference returning zero in each case.

```
M > evalDG(Div(B1(TensorList),gij) - divB1(TensorList))
                                                    0 1[1] ⊗ ∂x                        (10)
M > evalDG(Div(B2(TensorList),gij) - divB2(TensorList))
                                                    0 1[1] ⊗ ∂x                        (11)
M > evalDG(Div(B3(TensorList),gij) - divB3(TensorList))
                                                    0 1[1] ⊗ ∂x                        (12)
M > evalDG(Div(B4(TensorList),gij) - divB4(TensorList))
                                                    0 1[1] ⊗ ∂x                        (13)
M > evalDG(Div(B5(TensorList),gij) - divB5(TensorList))
                                                    0 1[1] ⊗ ∂x                        (14)
M > evalDG(Div(B6(TensorList),gij) - divB6(TensorList))
                                                    0 1[1] ⊗ ∂x                        (15)
M > evalDG(Div(B7(TensorList),gij) - divB7(TensorList))
                                                    0 1[1] ⊗ ∂x                        (16)
M > evalDG(Div(B8(TensorList),gij) - divB8(TensorList))
                                                    0 1[1] ⊗ ∂x                        (17)
```

Figure D.7: Divergence of $B^{ij}_{\#}$ terms.

We now compute the divergence of the eight $A^{ij}_{\#}$ terms in (D.1), using the chain rule and the $B\#$ (and $divB\#$) commands to express the divergence of each term, and sum the terms together.

```
M > DivA1Part1 := evalDG(divB1(TensorList) * dP[5]):
M > DivA1Part2 := ContractIndices(B1(TensorList) &t (Diff_dPdS),[[2,3]]):
M > DivA1 := evalDG(DivA1Part1 + DivA1Part2):
M > DivA2Part1 := evalDG(divB2(TensorList) * dP[5]):
M > DivA2Part2 := ContractIndices(B2(TensorList) &t (Diff_dPdS),[[2,3]]):
M > DivA2 := evalDG(DivA2Part1 + DivA2Part2):
M > DivA3Part1 := evalDG(divB3(TensorList) * dP[6]):
M > DivA3Part2 := ContractIndices(B3(TensorList) &t (Diff_d2PdS2),[[2,3]]):
M > DivA3 := evalDG(DivA3Part1 + DivA3Part2):
M > DivA4Part1 := evalDG(divB4(TensorList) * dP[8]):
M > DivA4Part2 := ContractIndices(B4(TensorList) &t (Diff_d2PdSdR),[[2,3]]):
M > DivA4 := evalDG(DivA4Part1 + DivA4Part2):
M > DivA5Part1 := evalDG(divB5(TensorList) * (dP[2] + dQ[1])):
M > DivA5Part2 := ContractIndices(B5(TensorList) &t (Diff_dPdR + Diff_Q),[[2,3]]):
M > DivA5 := evalDG(DivA5Part1 + DivA5Part2):
M > DivA6Part1 := evalDG(divB6(TensorList) * dP[5]):
M > DivA6Part2 := ContractIndices(B6(TensorList) &t (Diff_dPdS),[[2,3]]):
M > DivA6 := evalDG(DivA6Part1 + DivA6Part2):
M > DivA7Part1 := evalDG(divB7(TensorList) * (dP[3] + dQ[2])):
M > DivA7Part2 := ContractIndices(B7(TensorList) &t (Diff_d2PdR2 + Diff_dQdR),[[2,3]]):
M > DivA7 := evalDG(DivA7Part1 + DivA7Part2):
M > DivA8Part1 := evalDG(divB8(TensorList) * dP[1]):
M > DivA8Part2 := ContractIndices(B8(TensorList) &t Diff_P,[[2,3]]):
M > DivA8 := evalDG(DivA8Part1 + DivA8Part2):
M > a := [DivA1,DivA2,DivA3,DivA4,DivA5,DivA6,DivA7,DivA8]:
M > SUM := evalDG(add(a[i],i=1..8)):
```

Figure D.8: Divergence of A^{ij}.

We compare this sum with the "by-hand" computation for $A^{ij}_{|j}$ (D.16) by subtracting the two results and find the difference to be zero, confirming the "by-hand" calculations done previously.

```
M > DivA := evalDG(1/2 * rootg &t RA * (dP[1] - R * dQ[1]))
```

$$DivA := -\frac{1}{2F(x,y)^8}\Bigg(\Bigg(p(r,s)\,F(x,y)^3 + F(x,y)\left(\frac{\partial^2}{\partial x^2}F(x,y)\right)q(r) - F(x,y)\left(\frac{\partial^2}{\partial y^2}F(x,y)\right)q(r) - \left(\frac{\partial}{\partial x}F(x,y)\right)^2 q(r) + \left(\frac{\partial}{\partial y}F(x,y)\right)^2 q(r)\Bigg)\sqrt{-F(x,y)^2}\Bigg(\left(\frac{\partial^3}{\partial x^3}F(x,y)\right)F(x,y)^2 - \left(\frac{\partial^3}{\partial x\partial y^2}F(x,y)\right)F(x,y)^2 - 4\left(\frac{\partial^2}{\partial x^2}F(x,y)\right)\left(\frac{\partial}{\partial x}F(x,y)\right)F(x,y) + 2\left(\frac{\partial^2}{\partial y^2}F(x,y)\right)\left(\frac{\partial}{\partial x}F(x,y)\right)F(x,y) + 2\left(\frac{\partial^2}{\partial x\partial y}F(x,y)\right)\left(\frac{\partial}{\partial y}F(x,y)\right)F(x,y) - 3\left(\frac{\partial}{\partial y}F(x,y)\right)^2\left(\frac{\partial}{\partial x}F(x,y)\right) + 3\left(\frac{\partial}{\partial x}F(x,y)\right)^3\Bigg)\Bigg)\mathbf{1}_{[1]}\otimes\partial_x - \frac{1}{2F(x,y)^8}\Bigg(\Bigg(p(r,s)\,F(x,y)^3 + F(x,y)\left(\frac{\partial^2}{\partial x^2}F(x,y)\right)q(r) - F(x,y)\left(\frac{\partial^2}{\partial y^2}F(x,y)\right)q(r) - \left(\frac{\partial}{\partial x}F(x,y)\right)^2 q(r) + \left(\frac{\partial}{\partial y}F(x,y)\right)^2 q(r)\Bigg)\sqrt{-F(x,y)^2}\Bigg(-\left(\frac{\partial^3}{\partial x^2\partial y}F(x,y)\right)F(x,y)^2 + \left(\frac{\partial^3}{\partial y^3}F(x,y)\right)F(x,y)^2 + 2\left(\frac{\partial^2}{\partial x^2}F(x,y)\right)\left(\frac{\partial}{\partial y}F(x,y)\right)F(x,y) - 4\left(\frac{\partial^2}{\partial y^2}F(x,y)\right)\left(\frac{\partial}{\partial y}F(x,y)\right)F(x,y) + 2\left(\frac{\partial^2}{\partial x\partial y}F(x,y)\right)\left(\frac{\partial}{\partial x}F(x,y)\right)F(x,y) + 3\left(\frac{\partial}{\partial y}F(x,y)\right)^3 - 3\left(\frac{\partial}{\partial y}F(x,y)\right)\left(\frac{\partial}{\partial x}F(x,y)\right)^2\Bigg)\Bigg)\mathbf{1}_{[1]}\otimes\partial_y \tag{18}$$

```
M > evalDG(SUM - DivA)
```

$$0\,\mathbf{1}_{[1]}\otimes\partial_x \tag{19}$$

Figure D.9: Confirmation of “by-hand” calculations.

The source code for the **ATensor** package is listed below.

```
ATensor := module()
description "A package containing commands relating to the fifth order
    symmetric (2,0) tensor density that is not divergence-free.";
option package;
export Init, Div, B1, B2, B3, B4, B5, B6, B7, B8, divB1, divB2, divB3, divB4,
    divB5, divB6, divB7, divB8;
# Throughout this module lowercase letters after the initial name (g for the
    metric, R for the Ricci scalar/covariant derviatives) indicate lower
    indices and uppercase letters denote upper indices, with summation
    notation in effect. As an example, RAaD = gAB_Rabc_gCD denotes the trace
    of the first two indices and the raised third index of the symmetrized
    third covariant derivative of the Ricci scalar.

# Command list
Init := proc(g)
local C, R, Ra, Rab, R3, Rabc;
description "Computes and returns the Ricci scalar and first three symmetrized
    covariant derivatives computed from a given metric tensor.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Compute the Levi-Civita connection of the metric.
C := Christoffel(g);
# Compute the Ricci scalar.
```

```
R := RicciScalar(g);
# Compute the first covariant derivative of the Ricci scalar.
Ra := CovariantDerivative(R,C);
# Compute the second covariant derivative of the Ricci scalar.
Rab := CovariantDerivative(Ra,C);
# Compute the (unsymmetrized) third covariant derivative of the Ricci scalar.
R3 := CovariantDerivative(Rab,C);
# Symmetrize the third covariant derivative of the Ricci scalar.
Rabc := SymmetrizeIndices(R3,[1,2,3], "Symmetric");
# Return the metric and 4 computed tensors.
[g, R, Ra, Rab, Rabc]
end proc:

Div := proc(T,g)
local C, delta_T;
description "Computes the divergence of a rank 2 symmetric contravariant
    tensor using the Levi-Civita connection of the given metric.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor, DifferentialGeometry
    :-Tools;
# Compute the Levi-Civita connection of the metric.
C := Christoffel(g);
# Take the covariant derivative of the given tensor.
delta_T := CovariantDerivative(T,C);
# Return zero if delta_T is zero. If delta_T is nonzero, contract one index of
     the given tensor with the covariant derivative index.
if evalb(DGinfo(delta_T,"TensorType") = [["con_bas", "con_bas", "cov_bas"], [[
    "bas", 1]]]) = 'false'
then return 0
else ContractIndices(delta_T, [[2,3]])
end if
end proc:

B1 := proc(tensors)
local gij, R, Ra, Rab, Rabc, rootg, gIJ, RA, RAbc, RAac, gIJ_RAac, gIJ_RAac_RD
     , gIJ_RAac_RC, RIJc, RIJc_RD, RIJc_RC;
```

```
description "Computes the term linear in third order derivatives in the tensor
    A from a list containing the metric, Ricci scalar, and the first three
    symmetrized covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the square root of the determinant of the metric.
rootg := MetricDensity(gij,1);
# Compute the inverse metric for raising indices.
gIJ := InverseMetric(gij);
# Raise the index of the first covariant derivative of the Ricci scalar
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# There are two terms in B1: both terms involve the product of two metrics,
    with the first term tracing two indices of the third covariant derivative
    of the metric and the second term raising the first two indices of the
    metric. Computing the first term, raise the first index of the third
    covariant derivative and contract over the first and second indices.
RAbc := RaiseLowerIndices(gIJ,Rabc,[1]);
RAac := ContractIndices(RAbc,[[1,2]]);
# Multiply the traced third covariant derivative by an inverse metric and a
    raised index first covariant derivative.
gIJ_RAac := evalDG(gIJ &t RAac);
gIJ_RAac_RD := evalDG(gIJ_RAac &t RA);
# Contract the third index of the third covariant derivative with the first
    covariant derivative.
gIJ_RAac_RC := ContractIndices(gIJ_RAac_RD, [[3,4]]);
# Computing the second term, raise both the first and second indices of the
    third covariant derivative.
RIJc := RaiseLowerIndices(gIJ, RAbc, [2]);
# Contract the remaining lower index of the third covariant derivative with a
    raised index first covariant derivative
RIJc_RD := evalDG(RIJc &t RA);
RIJc_RC := ContractIndices(RIJc_RD, [[3,4]]);
```

```
# Take the difference between the two terms and multiply the result by a
    factor of 2*rootg
evalDG(2*rootg &t (gIJ_RAac_RC - RIJc_RC))
end proc:

B2 := proc(tensors)
local gij, R, Ra, Rab, Rabc, rootg, gIJ, Rab_Rcd, Rab_RcD, Rab_RcB, Rab_RCB,
    Rab_RAB, gIJ_Rab_RAB, RIb_RcB, RIb_RJB;
description "Computes the term quadratic in second order derivatives only in
    the tensor A from a list containing the metric, Ricci scalar, and the
    first three symmetrized covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
     covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the square root of the determinant of the metric.
rootg := MetricDensity(gij,1);
# Compute the inverse metric for raising indices.
gIJ := InverseMetric(gij);
# There are two terms in B2: both terms involve the product of two metrics,
    with the first term contracting the two second covariant derivatives and
    the second term raising single indices. In both terms, we compute the
    product of two second covariant derivatives and contract a pair of indices
     between the two.
Rab_Rcd := evalDG(Rab &t Rab);
Rab_RcD := RaiseLowerIndices(gIJ, Rab_Rcd, [4]);
Rab_RcB := ContractIndices(Rab_RcD, [[2,4]]);
# Starting with the first term, contract across the remaining indices and
    multiply by the inverse metric.
Rab_RCB := RaiseLowerIndices(gIJ,Rab_RcB, [2]);
Rab_RAB := ContractIndices(Rab_RCB, [[1,2]]);
gIJ_Rab_RAB := evalDG(gIJ &t Rab_RAB);
# For the second term, raise both remaining lower indices.
RIb_RcB := RaiseLowerIndices(gIJ, Rab_RcB, [1]);
RIb_RJB := RaiseLowerIndices(gIJ, RIb_RcB, [2]);
```

```
# Take the difference between the two terms and multiply the result by a
    factor of 2*rootg
evalDG(2*rootg &t (gIJ_Rab_RAB - RIb_RJB))
end proc:

B3 := proc(tensors)
local gij, R, Ra, Rab, Rabc, rootg, gIJ, RA, Rab_Rcd, Rab_Rcd_RE_RF,
    Rab_Rcd_RB_RD, Rab_RCd_RB_RD, Rab_RAd_RB_RD, gIJ_Rab_RAd_RB_RD,
    RIb_Rcd_RB_RD, RIb_RJd_RB_RD;
description "Computes the term quadratic in second order and first order
    derivatives in the tensor A from a list containing the metric, Ricci
    scalar, and the first three symmetrized covariant derivatives of the Ricci
     scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
     covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the square root of the determinant of the metric.
rootg := MetricDensity(gij,1);
# Compute the inverse metric for raising indices.
gIJ := InverseMetric(gij);
# Raise the index of the first covariant derivative of the Ricci scalar
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# There are two terms in B3: both terms involve the product of two metrics,
    with the first term contracting the two second covariant derivatives and
    the second term raising single indices. In both terms, we compute the
    product of two second covariant derivatives and contract each with a
    raised first covariant derivative.
Rab_Rcd := evalDG(Rab &t Rab);
Rab_Rcd_RE_RF := evalDG(Rab_Rcd &t RA &t RA);
Rab_Rcd_RB_RD := ContractIndices(Rab_Rcd_RE_RF,[[2,5],[3,6]]);
# Starting with the first term, contract across the remaining indices and
    multiply by the inverse metric.
Rab_RCd_RB_RD := RaiseLowerIndices(gIJ, Rab_Rcd_RB_RD, [2]);
Rab_RAd_RB_RD := ContractIndices(Rab_RCd_RB_RD, [[1,2]]);
```

```
gIJ_Rab_RAd_RB_RD := evalDG(gIJ &t Rab_RAd_RB_RD);
# For the second term, raise both remaining lower indices.
RIb_Rcd_RB_RD := RaiseLowerIndices(gIJ, Rab_Rcd_RB_RD, [1]);
RIb_RJd_RB_RD := RaiseLowerIndices(gIJ, RIb_Rcd_RB_RD, [2]);
# Take the difference between the two terms and multiply the result by a
    factor of 4*rootg
evalDG(4*rootg &t (gIJ_Rab_RAd_RB_RD - RIb_RJd_RB_RD))
end proc:

B4 := proc(tensors)
local gij, R, Ra, Rab, Rabc, rootg, gIJ, RA, Rab_Rc_RD, Rab_Rc_RB, Rab_RC_RB,
    Rab_RA_RB, gIJ_Rab_RA_RB, RIb_Rc_RB, RIb_RJ_RB, sym_RIb_RJ_RB;
description "Computes the term quadratic in first order and linear in second
    order derivatives in the tensor A from a list containing the metric, Ricci
    scalar, and the first three symmetrized covariant derivatives of the
    Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the square root of the determinant of the metric.
rootg := MetricDensity(gij,1);
# Compute the inverse metric for raising indices.
gIJ := InverseMetric(gij);
# Raise the index of the first covariant derivative of the Ricci scalar
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# There are two terms in B4: both terms involve the product of two metrics,
    with the first term contracting between a first and second derivative and
    the second term raising the two free indices. In both terms, we compute
    the product of a second covariant derivative and two first covariant
    derivatives (one up, one down), contracting across the second covariant
    derviative and the upper first covariant derivative.
Rab_Rc_RD := evalDG(Rab &t Ra &t RA);
Rab_Rc_RB := ContractIndices(Rab_Rc_RD, [[2,4]]);
```

```
# For the first term, contract across the remaining indices and multiply by
    the inverse metric.
Rab_RC_RB := RaiseLowerIndices(gIJ, Rab_Rc_RB, [2]);
Rab_RA_RB := ContractIndices(Rab_RC_RB, [[1,2]]);
gIJ_Rab_RA_RB := evalDG(gIJ &t Rab_RA_RB);
# For the second term, raise both remaining lower indices then symmetrize.
RIb_Rc_RB := RaiseLowerIndices(gIJ, Rab_Rc_RB, [1]);
RIb_RJ_RB := RaiseLowerIndices(gIJ, RIb_Rc_RB, [2]);
sym_RIb_RJ_RB := SymmetrizeIndices(RIb_RJ_RB, [1,2], "Symmetric");
# Take the difference between the two terms and multiply the result by a
    factor of 4*rootg
evalDG(4*rootg &t (gIJ_Rab_RA_RB - sym_RIb_RJ_RB))
end proc:

B5 := proc(tensors)
local gij, R, Ra, Rab, Rabc, rootg, gIJ, RAb, RAa, gIJ_RAa, RIb, RIJ;
description "Computes the term linear in second order derivatives in the
    tensor A from a list containing the metric, Ricci scalar, and the first
    three symmetrized covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
     covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the square root of the determinant of the metric.
rootg := MetricDensity(gij,1);
# Compute the inverse metric for raising indices.
gIJ := InverseMetric(gij);
# There are two terms in B5: both terms involve the product of two metrics,
    with the first term tracing the second derivative and the second term
    raising both indices of the second derivative.
# For the first term, compute the trace of the second derivative and multiply
    by an inverse metric.
RAb := RaiseLowerIndices(gIJ, Rab, [1]);
RAa := ContractIndices(RAb, [[1,2]]);
gIJ_RAa := evalDG(gIJ &t RAa);
```

```
# For the second term, raise both remaining lower indices then symmetrize.
RIb := RaiseLowerIndices(gIJ, Rab, [1]);
RIJ := RaiseLowerIndices(gIJ, RIb, [2]);
# Take the difference between the two terms and multiply the result by a
    factor of rootg
evalDG(rootg &t (gIJ_RAa - RIJ))
end proc:

B6 := proc(tensors)
local gij, R, Ra, Rab, Rabc, rootg, gIJ, RA, R_RI_RJ;
description "Computes the term quadratic in first order derivatives and linear
     in the Ricci scalar in the tensor A from a list containing the metric,
    Ricci scalar, and the first three symmetrized covariant derivatives of the
     Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
     covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the square root of the determinant of the metric.
rootg := MetricDensity(gij,1);
# Compute the inverse metric for raising indices.
gIJ := InverseMetric(gij);
# Raise the index of the first covariant derivative of the Ricci scalar
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# There is one term in B6, comprised of a Ricci scalar and two raised first
    covariant derivatives, with a coefficicent of 1/3*rootg.
R_RI_RJ := evalDG(R * RA &t RA);
evalDG(1/3*rootg &t R_RI_RJ)
end proc:

B7 := proc(tensors)
local gij, R, Ra, Rab, Rabc, rootg, gIJ, RA, Ra_RB, Ra_RA, gIJ_S, RI_RJ;
description "Computes the term quadratic in first order derivatives in the
    tensor A from a list containing the metric, Ricci scalar, and the first
    three symmetrized covariant derivatives of the Ricci scalar.";
```

```
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the square root of the determinant of the metric.
rootg := MetricDensity(gij,1);
# Compute the inverse metric for raising indices.
gIJ := InverseMetric(gij);
# Raise the index of the first covariant derivative of the Ricci scalar
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# There are two terms in B7: both terms involve the product of two metrics,
   with the first term contracting the two first derivatives and the second
   raising the indices of both.
# For the first term, contract the two first derivatives and multiply by an
   inverse metric.
Ra_RB := evalDG(Ra &t RA);
Ra_RA := ContractIndices(Ra_RB, [[1,2]]);
gIJ_S := evalDG(gIJ*Ra_RA);
# For the second term, raise both first derivatives.
RI_RJ := evalDG(RA &t RA);
# Take the difference between the two terms and multiply the result by a
   factor of rootg
evalDG(rootg &t (gIJ_S - RI_RJ))
end proc:

B8 := proc(tensors)
local gij, R, Ra, Rab, Rabc, rootg, gIJ;
description "Computes the term linear in the Ricci scalar in the tensor A from
    a list containing the metric, Ricci scalar, and the first three
   symmetrized covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the square root of the determinant of the metric.
```

```
rootg := MetricDensity(gij,1);
# Compute the inverse metric for raising indices.
gIJ := InverseMetric(gij);
# There is one term in B8, consisting of the Ricci tensor and an inverse
    metric multiplied by a factor of 1/2*rootg.
evalDG(1/2*R*rootg &t gIJ)
end proc:

divB1 := proc(tensors)
local gij, R, Ra, Rab, Rabc, gIJ, rootg, RA, Ra_RB, S, R_Rab_RC, R_RIb_RC,
    R_RIb_RB, R_RAa_RI, S_RI, Rabc_Rde, RAbc_RDI, RAac_RCI, RIBC_Rde, RIBC_Rbc
    ;
description "Computes the divergence of the B1 term in the tensor A from a
    list containing the metric, Ricci scalar, and the first three symmetrized
    covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
     covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the inverse metric.
gIJ := InverseMetric(gij);
# Compute the square root of the metric determinant.
rootg := MetricDensity(gij,1);
# Compute the raised first covariant derivative of R.
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# Compute S = Ra_RA.
Ra_RB := evalDG(Ra &t RA);
S := ContractIndices(Ra_RB, [[1,2]]);
# Compute all 5 terms of divB1.
# The first term consists of R, as well as a second and first derivative, with
     the second derivative contracted against the first derivative and with
    its free index raised.
R_Rab_RC := evalDG(R * Rab &t RA);
R_RIb_RC := RaiseLowerIndices(gIJ, R_Rab_RC,[1]);
R_RIb_RB := ContractIndices(R_RIb_RC, [[2,3]]);
```

```
# The second term is similar to the first term but with the second derivative
    traced and the first derivative raised.
R_RAa_RI := ContractIndices(R_RIb_RC, [[1,2]]);
# The third index is the product of S and a raised first derivative.
S_RI := evalDG(S*RA);
# The fourth term is the product of one second and one third derivative. with
    two indices traced on the third derivative and the remaining index
    contracted against a second derivative. The free index on the second
    derivative is raised.
Rabc_Rde := evalDG(Rabc &t Rab);
RAbc_RDI := RaiseLowerIndices(gIJ, Rabc_Rde, [1,4,5]);
RAac_RCI := ContractIndices(RAbc_RDI, [[1,2],[3,4]]);
# The fifth term is similar to the fourth but the second derivative is
    contracted against two of the third derivative's indices and the remaining
    index is raised.
RIBC_Rde := RaiseLowerIndices(gIJ, Rabc_Rde, [1,2,3]);
RIBC_Rbc := ContractIndices(RIBC_Rde, [[2,4],[3,5]]);
# The coefficients of each term are -8/3 for term one, +4/3 for term two, -1/3
    for term three, +2 for term four, and -2 for term five. Sum all five and
    multiply by rootg.
evalDG(rootg &t (-8/3*R_RIb_RB + 4/3*R_RAa_RI - 1/3*S_RI + 2*RAac_RCI - 2*
    RIBC_Rbc))
end proc:

divB2 := proc(tensors)
local gij, R, Ra, Rab, Rabc, gIJ, rootg, RA, Rabc_Rde, RIBC_Rde, RIBC_Rbc,
    RAbc_RDI, RAac_RCI, R_Rab_RI, R_RAb_RI, R_RAa_RI, R_RIa_RA;
description "Computes the divergence of the B2 term in the tensor A from a
    list containing the metric, Ricci scalar, and the first three symmetrized
    covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the inverse metric.
```

```
gIJ := InverseMetric(gij);
# Compute the square root of the metric determinant.
rootg := MetricDensity(gij,1);
# Compute the raised first covariant derivative of R.
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# Compute all 4 terms of divB2. The first term is the product of one second
    and one third derivative. The second derivative is contracted against two
    of the third derivative's indices, with the remaining index raised.
Rabc_Rde := evalDG(Rabc &t Rab);
RIBC_Rde := RaiseLowerIndices(gIJ, Rabc_Rde, [1,2,3]);
RIBC_Rbc := ContractIndices(RIBC_Rde, [[2,4],[3,5]]);
# The second term is similar to the first but with two indices traced on the
    third derivative and the remaining index contracted against a second
    derivative. The free index on the second derivative is raised.
RAbc_RDI := RaiseLowerIndices(gIJ, Rabc_Rde, [1,4,5]);
RAac_RCI := ContractIndices(RAbc_RDI, [[1,2],[3,4]]);
# The third term consists of R, as well as a second and first derivative, with
     the second derivative traced and the first derivative raised.
R_Rab_RI := evalDG(R * Rab &t RA);
R_RAb_RI := RaiseLowerIndices(gIJ, R_Rab_RI,[1]);
R_RAa_RI := ContractIndices(R_RAb_RI, [[1,2]]);
# The fourth term is similar to the third term but with the second derivative
    contracted against the first derivative and with its free index raised.
R_RIa_RA := ContractIndices(R_RAb_RI, [[2,3]]);
# The coefficients of each term are +2 for term one, -2 for term two, -5/3 for
     term three, and +4/3 for term four. Sum all four and multiply by rootg.
evalDG(rootg &t (2*RIBC_Rbc - 2*RAac_RCI - 5/3*R_RAa_RI + 4/3*R_RIa_RA))
end proc:

divB3 := proc(tensors)
local gij, R, Ra, Rab, Rabc, gIJ, rootg, RA, Rabc_Rde_RF_RG, RIbc_RDe_RF_RG,
    RIbc_RBe_RC_RE, RBbc_RIe_RC_RE, R_Rab_RC_RD_RE, R_Rab_RA_RB_RI,
    Rab_Rcd_Ref_RG, RIb_RCd_REf_RG, RIb_RBd_RDf_RF, RAB_Rcd_RIf_RG,
    RAB_Rab_RIf_RF;
```

```
description "Computes the divergence of the B3 term in the tensor A from a
    list containing the metric, Ricci scalar, and the first three symmetrized
    covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
     covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the inverse metric.
gIJ := InverseMetric(gij);
# Compute the square root of the metric determinant.
rootg := MetricDensity(gij,1);
# Compute the raised first covariant derivative of R.
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# Compute all 5 terms of divB3. The first term is the product of one third,
    one second, and two first derivatives. The free index is on the third
    derivative, with the remaining indices contracted against a second and
    first derivative. The second index of the second derivative is contracted
    against the final first derivative.
Rabc_Rde_RF_RG := evalDG(Rabc &t Rab &t RA &t RA);
RIbc_RDe_RF_RG := RaiseLowerIndices(gIJ,Rabc_Rde_RF_RG, [1,4]);
RIbc_RBe_RC_RE := ContractIndices(RIbc_RDe_RF_RG, [[2,4],[3,6],[5,7]]);
# The second term is the same product as the first, with the third derivative
    traced over two indices and contracted with a first derivative on the
    remaining index, as well as a first derivative contracted against the
    second derivative. The free index on the second derivative is raised.
RBbc_RIe_RC_RE := ContractIndices(RIbc_RDe_RF_RG, [[1,2],[3,6],[5,7]]);
# The third term consists of R, three first derivatives, and one second
    derivative. The second derivative is contracted against two of the first
    derivatives.
R_Rab_RC_RD_RE := evalDG(R * Rab &t RA &t RA &t RA);
R_Rab_RA_RB_RI := ContractIndices(R_Rab_RC_RD_RE, [[1,3],[2,4]]);
# The fourth term contains one first derivative and three second derivatives.
    One of the second derivatives is contracted over a single index with each
    of the other two second derivatives, with one free index raised and the
    other contracted with the first derivative.
```

```
Rab_Rcd_Ref_RG := evalDG(Rab &t Rab &t Rab &t RA);
RIb_RCd_REf_RG := RaiseLowerIndices(gIJ, Rab_Rcd_Ref_RG, [1,3,5]);
RIb_RBd_RDf_RF := ContractIndices(RIb_RCd_REf_RG, [[2,3],[4,5],[6,7]]);
# The fifth term contains the same product as the fourth, with two of the
   second derivatives contracted against each other and the remaining second
   derivative contracted against the first derivative. The free index on this
    last second derivative is raised.
RAB_Rcd_RIf_RG := RaiseLowerIndices(gIJ, Rab_Rcd_Ref_RG, [1,2,5]);
RAB_Rab_RIf_RF := ContractIndices(RAB_Rcd_RIf_RG, [[1,3],[2,4],[6,7]]);
# The coefficients of each term are +4 for terms one and four, -4 for terms
   two and five, and -2/3 for term three. Sum all five and multiply by rootg.
evalDG(rootg &t (4*RIbc_RBe_RC_RE - 4*RBbc_RIe_RC_RE - 2/3*R_Rab_RA_RB_RI + 4*
   RIb_RBd_RDf_RF - 4*RAB_Rab_RIf_RF))
end proc:

divB4 := proc(tensors)
local gij, R, Ra, Rab, Rabc, gIJ, rootg, RA, Ra_RB, S, RIbc, RIbc_RD_RE,
   RIbc_RB_RC, RAab, RAab_RC_RI, RAab_RB_RI, R_S_RI, Rab_Rcd_RE, RIB_Rcd_RE,
   RIB_Rbe_RE, RAB_Rcd_RI, RAB_Rab_RI, RIb_RCd_RE, RIe_RCc_RE;
description "Computes the divergence of the B4 term in the tensor A from a
   list containing the metric, Ricci scalar, and the first three symmetrized
   covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the inverse metric.
gIJ := InverseMetric(gij);
# Compute the square root of the metric determinant.
rootg := MetricDensity(gij,1);
# Compute the raised first covariant derivative of R.
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# Compute S = Ra_RA.
Ra_RB := evalDG(Ra &t RA);
S := ContractIndices(Ra_RB, [[1,2]]);
```

```
# Compute all 6 terms of divB4. The first term raises the first index of a
    third covariant derivative and contracts the remaining indices against two
     first covariant derivatives.
RIbc := RaiseLowerIndices(gIJ,Rabc,[1]);
RIbc_RD_RE := evalDG(RIbc &t RA &t RA);
RIbc_RB_RC := ContractIndices(RIbc_RD_RE, [[2,4],[3,5]]);
# The second term traces off two indices of a third covariant derivative,
    contracts the remaining index against a first covariant derivative, and
    multiplies by another first covariant derivative.
RAab := ContractIndices(RIbc, [[1,2]]);
RAab_RC_RI := evalDG(RAab &t RA &t RA);
RAab_RB_RI := ContractIndices(RAab_RC_RI, [[1,2]]);
# The third term is R*S multiplied by a raised first covariant derivative.
R_S_RI := evalDG(R * S * RA);
# The fourth term is the product of two second and one first derivatives,
    contracting the two second derivatives against each other, while
    contracting one free index against a first derivative and raising the
    remaining first index.
Rab_Rcd_RE := evalDG(Rab &t Rab &t RA);
RIB_Rcd_RE := RaiseLowerIndices(gIJ, Rab_Rcd_RE, [1,2]);
RIB_Rbe_RE := ContractIndices(RIB_Rcd_RE, [[2,3], [4,5]]);
# The fifth term consists of the same product as the fourth term but with the
    two second derivatives contracted against each other and the first
    derivative has the free index.
RAB_Rcd_RI := RaiseLowerIndices(gIJ, Rab_Rcd_RE, [1,2]);
RAB_Rab_RI := ContractIndices(RAB_Rcd_RI, [[1,3], [2,4]]);
# The sixth term is simlilar to the fourth and fifth, with the trace of one
    second derivative multiplied by a second derivative contracted against a
    first derivative with the free index raised.
RIb_RCd_RE := RaiseLowerIndices(gIJ, Rab_Rcd_RE, [1,3]);
RIe_RCc_RE := ContractIndices(RIb_RCd_RE, [[2,5],[3,4]]);
# The coefficients of each term are +2 for term one, -2 for terms two, five
    and six, -1/3 for term three, and +4 for term four. Sum all six and
    multiply by rootg.
```

```
evalDG(rootg &t (2*RIbc_RB_RC - 2*RAab_RB_RI - 1/3*R_S_RI + 4*RIB_Rbe_RE - 2*
    RAB_Rab_RI - 2*RIe_RCc_RE))
end proc:

divB5 := proc(tensors)
local gij, R, Ra, Rab, Rabc, gIJ, rootg, RA;
description "Computes the divergence of the B5 term in the tensor A from a
    list containing the metric, Ricci scalar, and the first three symmetrized
    covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the inverse metric.
gIJ := InverseMetric(gij);
# Compute the square root of the metric determinant.
rootg := MetricDensity(gij,1);
# Compute the raised first covariant derivative of R.
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# Compute divB5. The only term is -1/2*rootg*R multiplied by a raised first
    covariant derivative.
evalDG(-1/2*R*rootg &t RA)
end proc:

divB6 := proc(tensors)
local gij, R, Ra, Rab, Rabc, gIJ, rootg, RA, Ra_RB, S, S_RI, Rab_RC, Rab_RB,
    RIb_RB, R_RIb_RB, RaB, RaA, RaA_RI, R_RaA_RI;
description "Computes the divergence of the B6 term in the tensor A from a
    list containing the metric, Ricci scalar, and the first three symmetrized
    covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the inverse metric.
```

```
gIJ := InverseMetric(gij);
# Compute the square root of the metric determinant.
rootg := MetricDensity(gij,1);
# Compute the raised first covariant derivative of R.
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# Compute S = Ra_RA.
Ra_RB := evalDG(Ra &t RA);
S := ContractIndices(Ra_RB, [[1,2]]);
# Compute the three terms of divB6. The first term is S multiplied by a raised
     first covariant derivative.
S_RI := evalDG(S * RA);
# The second term contracts one raised first covariant derivative of R with a
    second covariant derivative and raises the remaining index, while
    multiplying by R.
Rab_RC := evalDG(Rab &t RA);
Rab_RB := ContractIndices(Rab_RC, [[2,3]]);
RIb_RB := RaiseLowerIndices(gIJ, Rab_RB, [1]);
R_RIb_RB := evalDG(R * RIb_RB);
# The third term contains a raised first derivative and the trace of a second
    derivative, while multiplying by R.
RaB := RaiseLowerIndices(gIJ, Rab, [2]);
RaA := ContractIndices(RaB,[[1,2]]);
RaA_RI := evalDG(RaA * RA);
R_RaA_RI := evalDG(R * RaA_RI);
# Sum all three terms and multiply by 1/3*rootg.
evalDG(1/3*rootg &t (S_RI + R_RIb_RB + R_RaA_RI))
end proc:

divB7 := proc(tensors)
local gij, R, Ra, Rab, Rabc, gIJ, rootg, RA, Rab_RC, Rab_RB, RIb_RB, RaB, RaA,
     RaA_RI;
description "Computes the divergence of the B7 term in the tensor A from a
    list containing the metric, Ricci scalar, and the first three symmetrized
    covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
```

```
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the inverse metric.
gIJ := InverseMetric(gij);
# Compute the square root of the metric determinant.
rootg := MetricDensity(gij,1);
# Compute the raised first covariant derivative of R.
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# Compute the two terms of divB7. The first term contracts one raised first
    covariant derivative of R with a second covariant derivative and raises
    the remaining index.
Rab_RC := evalDG( RA &t Rab);
Rab_RB := ContractIndices(Rab_RC, [[1,2]]);
RIb_RB := RaiseLowerIndices(gIJ, Rab_RB, [1]);
# The second term contains a raised first derivative and the trace of a second
    derivative.
RaB := RaiseLowerIndices(gIJ, Rab, [2]);
RaA := ContractIndices(RaB,[[1,2]]);
RaA_RI := evalDG(RaA * RA);
# Subtract the second term from the first and multiply by rootg.
evalDG(rootg &t (RIb_RB - RaA_RI))
end proc:

divB8 := proc(tensors)
local gij, R, Ra, Rab, Rabc, gIJ, rootg, RA;
description "Computes the divergence of the B8 term in the tensor A from a
    list containing the metric, Ricci scalar, and the first three symmetrized
    covariant derivatives of the Ricci scalar.";
uses DifferentialGeometry, DifferentialGeometry:-Tensor;
# Name the metric and compute the Ricci scalar and its first three symmetrized
    covariant derivatives using Init.
gij, R, Ra, Rab, Rabc := op(tensors);
# Compute the inverse metric.
gIJ := InverseMetric(gij);
```

```
# Compute the square root of the metric determinant.
rootg := MetricDensity(gij,1);
# Compute the raised first covariant derivative of R.
RA := RaiseLowerIndices(gIJ,Ra,[1]);
# Compute divB8.
evalDG(1/2*rootg &t RA)
end proc:

end module:
```

APPENDIX E

THE CALCULUS OF VARIATIONS AND NOETHER'S THEOREMS

For manifolds M and N, a symmetry of a function $f : M \to N$ is a transformation (2.2) which leaves the function unchanged, i.e., $f(\bar{x}) = f(x)$. A symmetry of a (differential or algebraic) system of equations $\Delta = 0$ is a transformation which maps solutions to other solutions, i.e., if y is a solution to a system of equations $\Delta = 0$, then $\bar{y}$ is also a solution. A variational symmetry of a functional $I[f^A]$ is a transformation of the fiber bundle F (both the base manifold and the dependent variables f^A) which leaves the action (1.19) unchanged.

More formally, let G be a connected local group of transformations, i.e., a collection of transformations of the type (2.2) which form a s-dimensional connected Lie group under composition. We denote the action of an element $g \in G$ on a valid point $x^i \in M$ by $g \cdot x^i$. A function $f : M \to N$ is said to be G-invariant if $f(g \cdot x^i) = f(x^i)$ for all $g \in G$ and for all $x^i \in M$ such that $g \cdot x^i$ is defined. A real-valued G-invariant function $\zeta : M \to \mathbb{R}$ is said to be an invariant of G. The group action G is fully characterized by a collection of s vector fields on M called the infinitesimal generators of G. In particular, the following well-known theorem (see, e.g., Olver [27]) relates these vector fields with the invariants of G.

Theorem 14. *Let G be a connected group of transformations acting on the manifold M. A smooth, real-valued function $\zeta : M \to \mathbb{R}$ is an invariant function for G if and only if*

$$X^i \zeta_{,i} = 0 \tag{E.1}$$

for all $x^i \in M$ and every infinitesimal generator X^i of G.

A symmetry group of a system of equations $\Delta = 0$ is a local group of transformations G acting on the space of independent and dependent variables for the system (the fiber bundle F of the system) with the property that whenever $u = f(x^i)$ is a solution of Δ then $u = g \cdot f(x^i)$ is also a solution.

The r-th prolongation of a vector field

$$X = X^i(x^j, f^B)\partial_i + X^A(x^j, f^B)\frac{\partial}{\partial f^A}, \tag{E.2}$$

is the vector field

$$\operatorname{pr}^{(r)} X = X + X^A_J(x^i, f^A_{(r)})\frac{\partial}{\partial f^A_{,J}} \tag{E.3}$$

defined on the r-th jet space of the fiber bundle $J^r(F)$ and where the sum over J is to be done over all (unordered) multi-indices $J = (j_1, \ldots, j_n)$ of sizes $1 \leq k \leq r$. The coefficient functions X^A_J are dependent on the base coordinates x^i, the dependent variables f^A, and derivatives of f^A to order r, with explicit formula

$$X^A_J(x^i, f^A_{(r)}) = D_J\left(X^A - X^i f^A_{,i}\right) + X^i f^A_{,Ji}. \tag{E.4}$$

The expression in parentheses is also called the characteristic of the vector field X, defined as the m-tuple of functions

$$Q^A = X^A - X^i f^A_{,i}. \tag{E.5}$$

Then the formula (E.4) can be rewritten more compactly as

$$\operatorname{pr}^{(r)} X = X^i D_i + D_J Q^A \frac{\partial}{\partial f^A_{,J}}. \tag{E.6}$$

The variational symmetry group of a functional $I[f^A]$ is the local group of transformations G on the fiber bundle F which leave the value of the action (1.19) unchanged. Locally, the infinitesimal generators of this symmetry group are given by vector fields on F (E.2) and the corresponding version of Theorem 14 is the following.

Theorem 15. *A connected group of transformations G acting on a subset of F is a variational symmetry group of the functional* (1.19) *if and only if*

$$\operatorname{pr}^{(r)} X(\lambda) + \lambda D_i X^i = 0 \tag{E.7}$$

for all $(x^i, f^A) \in F$ and every infinitesimal generator X.

A well-known theorem (e.g., Theorem 4.14 of Olver [27]) of the calculus of variations relates the symmetries of the Lagrangian λ and Euler-Lagrange equations $E(\lambda) = 0$.

Theorem 16. *If X is a symmetry of the Lagrangian λ, then X is also a symmetry of the Euler-Lagrange equations $E(\lambda) = 0$.*

The converse to this theorem is generally not true; consider the two dimensional wave equation $u_{,11} - u_{,22} - u_{,33} = 0$ and its Lagrangian $\lambda = \frac{1}{2}(u_{,1}^2 - u_{,2}^2 - u_{,3}^2)$. If $v(t, x, y)$ is a solution, then the dilation transformation $\bar{x}^i = ax^i$ for $a \neq 0$ transforms v to $v(at, ax, ay)$ which still solves the 2D wave equation. However, the Lagrangian scales by a factor of a^2.

A system of differential equations $\Delta(x^i, f^A_{(r)}) = 0$ is locally solvable at the point $(x^i_0, f^A_{0,(r)}) \in \mathcal{V}_\Delta = \{(x^i, f^A_{(r)}) : \Delta(x, f^A_{(r)}) = 0\}$ if there exists a smooth solution $f^A = u^A(x^i)$ of the system, defined for all x^i in a neighborhood of x^i_0, which has the initial conditions $f^A_{0,(r)} = \operatorname{pr}^{(r)} u^A(x^i)$. The system is locally solvable if it is locally solvable at every point in the subvariety $\mathcal{V}_\Delta$. The system is said to be nondegenerate if at every point in the subvariety it is both locally solvable and of maximum rank. The following theorem gives the infinitesimal criteria for a system of differential equations.

Theorem 17. *Let $\Delta(x^i, f^A_{(r)}) = 0$ be a nondegenerate system of differential equations. A connected local group of transformations G acting on an open subset of the fiber bundle F is a symmetry group of the system if and only if*

$$\operatorname{pr}^{(r)} X[\Delta_x(x^i, f^A_{(r)})] = 0,\ x = 1, \ldots, s,\ \textit{whenever}\ \Delta(x^i, f^A_{(r)}) = 0, \tag{E.8}$$

for every infinitesimal generator X of G.

A conservation law of the functional $I[f^A]$ is an equation of the form

$$\mathrm{Div}\, P = D_i P^i = 0 \tag{E.9}$$

which is satisfied for all solutions $f^A = f^A(x^i)$ of the variational principle $\delta I = 0$. Here Div is the total divergence operator, an appropriate total derivative (1.21) applied to the horizontal part of a generalized vector field P on F

$$P = P^i \partial_i + P^A \frac{\partial}{\partial f^A}, \tag{E.10}$$

where the component functions P^i and P^A are functions of the base coordinates x^i, the fiber coordinates f^A, and derivatives of the fiber coordinates to some finite order. A trivial conservation law of the first kind is a conservation law (E.9) such that P vanishes identically on solutions to the Euler-Lagrange equations. A trivial conservation law of the second kind is a conservation law which holds for all functions f^A, regardless of whether they solve the Euler-Lagrange equations. Two conservation laws are said to be equivalent if they differ by a trivial conservation law (any linear combination of the two types). If $\mathrm{Div}\, P = 0$ is a conservation law of an Euler-Lagrange equation $E(\lambda) = 0$, the characteristic of this conservation law is given by the equation

$$\mathrm{Div}\, P = Q \cdot E = Q^v \Delta_v, \tag{E.11}$$

where the characteristic Q is a s-tuple function of the base manifold, the fiber coordinates f^A, and their derivatives to finite order and Δ_v is the Euler-Lagrange form $E(\lambda) = \Delta_v \wedge \mathrm{d} f^v$.

Conservation laws and variational symmetries are related via the well-known Noether's (first) theorem, e.g., Olver [27].

Theorem 18. *If a transformation T is a variational symmetry of the action I, then there exists a corresponding conservation law. This correspondence is one to one for systems* *which are nondegenerate (up to equivalence classes of variational symmetries and equivalence classes of nontrivial conservation laws).*

In particular, if X is an infinitesimal generator of a one-parameter local group of symmetries of the variational problem $I[f^A]$ and Q_a (E.5) is the corresponding characteristic of X, then Q_a is also the characteristic of a conservation law for the Euler-Lagrange equations $E_A(L) = 0$, i.e., there exists $P = (P_1, \ldots, P_n)$ such that (E.11) is satisfied for $v = 1, \ldots, m$.

If the variational problem $I[f^A]$ is degenerate (in that the Euler-Lagrange equations are under- or over-determined), then the system is governed by Noether's second theorem [27].

Theorem 19. *The action $I[f^A]$ admits an infinite dimensional group of variational symmetries linearly parameterized by an arbitrary function $h(x^i)$ and its derivatives if and only if there exist differential operators $\mathcal{D}^A$ (not all zero) for $1 \leq i \leq m$ such that*

$$\mathcal{D}^A E_A(\lambda) = 0 \tag{E.12}$$

for all x^i and f^A, i.e., the variational principle is under-determined.

The symmetry group parameterized by h in the theorem is also known as a gauge symmetry. As an example of Noether's second theorem, consider the action $I[g]$ where the Lagrangian $\lambda[g]$ is a natural tensor density of metric order r. Then, if ϕ is a diffeomorphism, the action I is invariant under diffeomorphisms as seen by considering the two metrics g and $\phi^* g$:

$$\begin{aligned} I[g] &= \int_U \lambda[g]\nu \\ &= \int_{\phi(U)} \lambda[g]\,\nu \\ &= \int_U \phi^*(\lambda[g]\,\nu) \\ &= \int_U \lambda[\phi^* g]\,\phi^*\nu \\ &= I[\phi^* g], \end{aligned}$$

where we have used the fact that (sub-)manifolds are invariant under diffeomorphisms to arrive at the second line. The diffeomorphism group is of infinite dimension and so Noether's second theorem states there must exist a differential identity of the type (E.12) for this problem. Explicitly, if X^i is an infinitesimal generator of the diffeomorphism ϕ, then the variation of the metric [to first order $g_{ij} \to g_{ij} + \epsilon h_{ij} + \mathcal{O}(\epsilon^2)$] with respect to ϕ is given by the Lie derivative of the metric with respect to the vector field $X = X^i \partial_i$

$$\begin{aligned}
h_{ij} &= \mathcal{L}_X g_{ij} \\
&= X \lrcorner \, \mathrm{d} g_{ij} + \mathrm{d}(X \lrcorner g_{ij}) \\
&= X^k g_{ij,k} + \left(g_{il} X^l\right)_{,j} + \left(g_{jl} X^l\right)_{,i} \\
&= X^k \left(g_{ij|k} + \Gamma^l{}_{ki} g_{lj} + \Gamma^l{}_{kj} g_{il}\right) + g_{il}\left(X^l{}_{|j} - \Gamma^l{}_{jk} X^k\right) + g_{jl}\left(X^l{}_{|i} - \Gamma^l{}_{ik} X^k\right) \\
&= X^k g_{ij|k} + g_{il} X^l{}_{|j} + g_{jl} X^l{}_{|i} \\
&= X_{i|j} + X_{j|i},
\end{aligned}$$

where d is the exterior derivative and $\lrcorner$ is the interior product. The variation of the action is then

$$\begin{aligned}
\delta I &= \int_U \delta\lambda[g]\, \nu \\
&= \int_U \left(\frac{\delta\lambda}{\delta g_{ij}} h_{ij} + \frac{\delta\lambda}{\delta g_{ij,k}} h_{ij,k} + \cdots + \frac{\delta\lambda}{\delta g_{ij,k_1\cdots k_r}} \delta h_{ij,k_1\cdots k_r} \right) \nu \\
&= \int_U E^{ij}(\lambda) h_{ij}\, \nu \\
&= \int_U E^{ij}(\lambda)(X_{i|j} + X_{j|i})\, \nu \\
&= 2\int_U E^{ij}(\lambda) X_{i|j}\, \nu \\
&= 2\int_U \left\{ -X_i \nabla_j E^{ij}(\lambda) + \nabla_j \left[E^{ij}(\lambda) X_i\right] \right\} \nu \\
&= -2\int_U X_i \nabla_j E^{ij}(\lambda)\, \nu + 2\int_{\partial U} E^{ij}(\lambda)\, \left(X_i \lrcorner\, \nu\right),
\end{aligned}$$

where the divergence theorem was used to arrive at the final line and ∂U is the boundary of U. We assume the diffeomorphism is the identity transformation on the boundary ∂U (otherwise the manifold would "tear") and so X^i vanishes there, removing the boundary term. Since the action is diffeomorphism invariant, the variation of the action vanishes and we are left with the equation

$$0 = \int_U X_i \nabla_j E^{ij}(\lambda)\,\nu = \int_U E^{ij}(\lambda)_{|j} X_i\,\nu.$$

Since this equation must hold for any diffeomorphism (and hence any infinitesimal generator X^i), we now have the explicit form for the differential identity implied by Theorem 19

$$E^{ij}(\lambda)_{|j} = 0, \tag{E.13}$$

with the differential operator $\mathcal{D}^A$ given by the covariant derivative. We note that this is a trivial conservation law of the first kind mentioned earlier.

With this result in hand, almost all of the theorems from the literature review in Chapter 1 can be thought of as inverse versions of Noether's second theorem. In particular, Theorems 2, 3, 7, 8, and 9 all have differential identities of the form (E.13), while Theorems 4 and 5 have differential identities of the form

$$E^{ij}(\lambda)_{|j} = \frac{1}{2}\varphi^{|i} E(\lambda) \tag{E.14}$$

and

$$E^{ij}(\lambda)_{|j} = -\frac{1}{2}F^i_{\ j}\,E^j(\lambda) - \frac{1}{2}\psi^i E^j(\lambda)_{|j}, \tag{E.15}$$

respectively, where $E(\lambda) = \dfrac{\delta\lambda}{\delta\varphi}$ and $E^i(\lambda) = \dfrac{\delta\lambda}{\delta\psi_i}$.

Theorem 6 is the only exception: as detailed in Lovelock [17], the differential identity is given explicitly by

$$\begin{aligned} E^{ij}(\lambda)_{|j} &= \alpha^i_{\ b} B^b + \beta^i_{\ b} C^b \\ &= \lambda\left[\sqrt{g}F^i_{\ b}\left(F^{bj}_{\ \ \ |j}\right) - \frac{1}{4}\sqrt{g}g^{ia}\epsilon_{abcd}F^{cd}\left(\epsilon^{bjkl}F_{kl|j}\right)\right] \end{aligned}$$

$$= \lambda\sqrt{g}\left\{F^i_{\ k}F^{kj}_{\ \ |j} + \frac{1}{4}g^{ia}\left[\delta^j_a\left(\delta^k_c\delta^l_d - \delta^l_c\delta^k_d\right) - \delta^k_a\left(\delta^j_c\delta^l_d - \delta^l_c\delta^j_d\right)\right.\right.$$
$$\left.\left. + \delta^l_a\left(\delta^j_c\delta^k_d - \delta^k_c\delta^j_d\right)\right]F^{cd}F_{kl|j}\right\}$$
$$= \lambda\sqrt{g}\left[F^i_{\ k}F^{kj}_{\ \ |j} + \frac{1}{2}g^{ia}\left(\delta^j_a\delta^k_c\delta^l_d - \delta^k_a\delta^j_c\delta^l_d + \delta^l_a\delta^j_c\delta^k_d\right)F^{cd}F_{kl|j}\right]$$
$$= \lambda\sqrt{g}\left[F^i_{\ k}F^{kj}_{\ \ |j} + \frac{1}{2}g^{ia}\left(F^{cd}F_{cd|a} - F^{cd}F_{ad|c} + F^{cd}F_{da|c}\right)\right]$$
$$= \lambda\sqrt{g}\left[F^i_{\ k}F^{kj}_{\ \ |j} + \frac{1}{2}g^{ia}\left(F^{cd}F_{cd|a} - 2F^{cd}F_{ad|c}\right)\right]$$
$$= \lambda\sqrt{g}\left(F^i_{\ k}F^{kj}_{\ \ |j} - F^i_{\ k|j}F^{jk} + \frac{1}{2}F^{jk|i}F_{jk}\right)$$
$$= \lambda\sqrt{g}\left(F^i_{\ k}F^{kj}_{\ \ |j} + F^i_{\ k|j}F^{kj} + \frac{1}{2}F^{jk|i}F_{jk}\right).$$

On the other hand, the Euler-Lagrange expression of λ with respect to the skew-symmetric tensor F_{ij} is

$$\frac{\delta\lambda}{\delta F_{ij}} = \frac{\partial\lambda}{\partial F_{ij}} = b\sqrt{g}F^{ij} + d\,\epsilon^{ijkl}F_{kl},$$

which has a divergence similar to the form above

$$\frac{\delta\lambda}{\delta F_{ij}}_{|j} = b\sqrt{g}F^{ij}_{\ \ |j} + d\,\epsilon^{ijkl}F_{kl|j} = bB^i + dC^i$$

but it is not proportional. This failure is a known property of any electromagnetic field theory: no variational theory of electromagnetism exists which uses the electric and magnetic fields **E** and **B** alone as the dependent variables (or, equivalently, just the field strength tensor F_{ab} as shown). Instead, the use of covector potentials ψ_i as the dynamic variables is required.

CURRICULUM VITAE

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(December 2023)

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EDUCATION

M.S. (expected 12/23)	**Utah State University**, Logan, UT Physics. GPA: 4.0 (4.0 = A) Thesis: *Divergence-free Tensor Densities in Two Dimensions* Major Professor: Ian Anderson
B.S. May 2020	**Utah State University**, Logan, UT Physics. GPA: 3.69 *Cum Laude*
B.S. May 2020	**Utah State University**, Logan, UT Mathematics. GPA: 3.69 *Cum Laude*

RESEARCH SKILLS

As a graduate student in the Department of Physics at Utah State university, I primarily worked under the supervision of the Department of Mathematics' Ian Anderson. As such, my focus is towards mathematical physics, with an emphasis on differential geometry, tensors and tensor calculus, and variational calculus (particularly the inverse problem). In addition to these more focused topics, my extensive mathematics background includes experience with partial differential equations, topology, abstract algebra, probability, and analysis (both real and, to a lesser extent, complex). I also possess knowledge at the master's level in many physics disciplines, including classical mechanics, general relativity, quantum mechanics, electromagnetism, statistical mechanics, and field theory.

My thesis on divergence-free tensor densities required the manipulation of complex tensorial calculations which involved considerable index manipulation. While I performed all of the work by hand, my familiarity with the computer algebra systems Maple and Cadabra allowed me to verify the accuracy of the more complicated expressions in a timely fashion.

As an undergraduate project, I wrote Maple code which finds solutions to the inverse problem of the calculus of variations and implements a number of tools for solving problems in Hamiltonian mechanics using the language of differential geometry. I also designed and took part in the execution of an experiment which recorded the motion of a double pendulum using a phone camera and used the Python package Trackpy to track the motion of the pendulum arms from the camera footage. The resulting data was tested for characteristics of chaos (sensitivity to initial conditions, Lyapunov exponents) and compared with a Python-produced simulation of the pendulum.

TECHNICAL SKILLS

Operating Systems	Windows, Linux	Programming	Python (SciPy, NumPy, Trackpy)
Scientific Computing	Maple, Cadabra, Mathematica	Typography	LaTeX

RELEVANT EMPLOYMENT

Sept 2020 - Apr 2023	**Graduate Student Teaching Assistant**, *Utah State University.* Taught introductory labs and provided tutoring services for the introductory physics class.
Aug 2019 - Sept 2019	**Teaching Assistant and Grader**, *Utah State University.* Reference: Ian Anderson. Assisted in-class presentations and graded assignments for an Introduction to Maple class.
Oct 2016 - Aug 2017	**Instructor**, *AcerPlacer.* Taught and graded preparatory courses for the ACCUPLACER placement exam.

HONORS

Lillywhite Scholarship	Neville & Annie Hunsaker Scholarship
USU Presidential Academic Scholarship	Best Experiment, USU Advanced Physics Lab