$$c = \vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = a_x b_x + a_y b_y + a_z b_z \quad \Leftrightarrow \quad c = \mathbf{a}^T \mathbf{b}$$

$$\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \mapsto \tilde{\mathbf{a}} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = (a_y b_z - a_z b_y)\vec{\mathbf{i}} + (a_z b_x - a_x b_z)\vec{\mathbf{j}} + (a_x b_y - a_y b_x)\vec{\mathbf{k}} \mapsto \tilde{\mathbf{a}} \cdot \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\tilde{\mathbf{a}}^T = -\tilde{\mathbf{a}}$$

$$\mathbf{s} = \mathbf{A}\bar{\mathbf{s}} \qquad \bar{\mathbf{s}} = \mathbf{A}^T \mathbf{s}$$

$$\tilde{\bar{\mathbf{s}}} = (\widetilde{\mathbf{A}^T \mathbf{s}}) = \mathbf{A}^T \tilde{\mathbf{s}} \mathbf{A} \qquad \tilde{\mathbf{s}} = (\widetilde{\mathbf{A}\bar{\mathbf{s}}}) = \mathbf{A} \tilde{\bar{\mathbf{s}}} \mathbf{A}^T$$

$$\tilde{\omega} = \dot{\mathbf{A}}\mathbf{A}^T \qquad \tilde{\bar{\omega}} = \mathbf{A}^T \dot{\mathbf{A}}$$

Angular velocity of L-RF w.r.t. G-RF

$$\dot{\mathbf{A}} = \tilde{\omega}\mathbf{A} \qquad \dot{\mathbf{A}} = \mathbf{A}\tilde{\bar{\omega}}$$

$$\ddot{\mathbf{A}} = \dot{\tilde{\omega}}\mathbf{A} + \tilde{\omega}\dot{\mathbf{A}} = \tilde{\dot{\omega}}\mathbf{A} + \tilde{\omega}\tilde{\omega}\mathbf{A} = (\tilde{\dot{\omega}} + \tilde{\omega}\tilde{\omega})\mathbf{A} \qquad \ddot{\mathbf{A}} = \mathbf{A}\dot{\tilde{\bar{\omega}}} + \dot{\mathbf{A}}\tilde{\bar{\omega}} = \mathbf{A}\tilde{\dot{\bar{\omega}}} + \mathbf{A}\tilde{\bar{\omega}}\tilde{\bar{\omega}} = \mathbf{A}(\tilde{\dot{\bar{\omega}}} + \tilde{\bar{\omega}}\tilde{\bar{\omega}})$$

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}} = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\,\mathbf{I}_{3\times 3} \qquad \widetilde{\tilde{\mathbf{a}}\mathbf{b}} = \mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T = \tilde{\mathbf{a}}\tilde{\mathbf{b}} - \tilde{\mathbf{b}}\tilde{\mathbf{a}}$$

$$\mathbf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} \quad \text{where} \quad e_0 = \cos\frac{\chi}{2} \quad \text{and} \quad \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \mathbf{u}\sin\frac{\chi}{2}$$

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1 \qquad \mathbf{p}^T\mathbf{p} = 1$$

$$\mathbf{A} = [(2e_0^2 - 1)\mathbf{I} + 2(\mathbf{e}\mathbf{e}^T + e_0\tilde{\mathbf{e}})] = 2\begin{bmatrix} e_0^2 + e_1^2 - \frac{1}{2} & e_1e_2 - e_0e_3 & e_1e_3 + e_0e_2 \\ e_1e_2 + e_0e_3 & e_0^2 + e_2^2 - \frac{1}{2} & e_2e_3 - e_0e_1 \\ e_1e_3 - e_0e_2 & e_2e_3 + e_0e_1 & e_0^2 + e_3^2 - \frac{1}{2} \end{bmatrix}$$

$$\boxed{e_0 \neq 0}$$

- Step 1: Compute e_0^2 as
$$e_0^2 = \frac{tr(\mathbf{A}) + 1}{4}$$
- Step 2: Compute e_0 from e_0^2; choose any sign you wish for e_0, maybe should be close to some previous value of it that you just computed
- Step 3: For $i = 1, 2, 3$, compute
$$e_i^2 = \frac{2a_{ii} - tr(\mathbf{A}) + 1}{4}$$
- Step 4: When computing e_i from e_i^2, the sign of e_i should be determined based on the following relations:
$$a_{32} - a_{23} = 4e_0e_1$$
$$a_{13} - a_{31} = 4e_0e_2$$
$$a_{21} - a_{12} = 4e_0e_3$$

$e_0 = 0$

Their computation draws on the following remark: since now $e_1^2 + e_2^2 + e_3^2 = 1$, at least one of e_1, e_2, and e_3 is nonzero. Whichever that one is, you'll use it to solve for the other two using two out of the three following conditions:
$$a_{21} + a_{12} = 4e_1e_2$$
$$a_{31} + a_{13} = 4e_1e_3$$
$$a_{32} + a_{23} = 4e_2e_3$$

$$\mathbf{E} \equiv [-\mathbf{e}, \tilde{\mathbf{e}} + e_0\mathbf{I}] = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} \qquad \mathbf{G} \equiv [-\mathbf{e}, -\tilde{\mathbf{e}} + e_0\mathbf{I}] = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{E}\mathbf{G}^T$$

$$\mathbf{E}\mathbf{p} = \mathbf{0}_3 \qquad \mathbf{G}\mathbf{p} = \mathbf{0}_3 \qquad \mathbf{E}\mathbf{E}^T = \mathbf{G}\mathbf{G}^T = \mathbf{I}_3$$

$$\mathbf{G}^T\mathbf{G} = \mathbf{E}^T\mathbf{E} = \mathbf{I}_4 - \mathbf{p}\mathbf{p}^T \qquad \tilde{\mathbf{e}}\tilde{\mathbf{e}} = \mathbf{e}\mathbf{e}^T + (e_0^2 - 1)\,\mathbf{I}_{3\times 3}$$

$$\mathbf{p}^T\dot{\mathbf{p}} = \dot{\mathbf{p}}^T\mathbf{p} = \mathbf{0} \qquad \mathbf{E}\dot{\mathbf{p}} = -\dot{\mathbf{E}}\mathbf{p} \qquad \mathbf{G}\dot{\mathbf{p}} = -\dot{\mathbf{G}}\mathbf{p} \qquad \mathbf{E}\dot{\mathbf{G}}^T = \dot{\mathbf{E}}\mathbf{G}^T$$

$$\dot{\mathbf{A}} = \dot{\mathbf{E}}\mathbf{G}^T + \mathbf{E}\dot{\mathbf{G}}^T = \mathbf{2}\mathbf{E}\dot{\mathbf{G}}^T \qquad \widetilde{\mathbf{G}\dot{\mathbf{p}}} = \mathbf{G}\dot{\mathbf{G}}^T$$

$$\tilde{\bar{\omega}} = \mathbf{A}^T\dot{\mathbf{A}} = 2\mathbf{G}\mathbf{E}^T\mathbf{E}\dot{\mathbf{G}}^T$$
$$\Downarrow$$
$$\tilde{\bar{\omega}} = 2\mathbf{G}\dot{\mathbf{G}}^T$$
$$\Downarrow$$
$$\tilde{\bar{\omega}} = 2(\widetilde{\mathbf{G}\dot{\mathbf{p}}})$$
$$\Downarrow$$
$$\bar{\omega} = 2(\mathbf{G}\dot{\mathbf{p}})$$

$$\omega = \mathbf{A}\bar{\omega} = 2\mathbf{E}\mathbf{G}^T\mathbf{G}\dot{\mathbf{p}}$$
$$\Downarrow$$
$$\omega = 2\mathbf{E}\dot{\mathbf{p}}$$

$$\bar{\omega} = 2\mathbf{G}\dot{\mathbf{p}}$$
$$\downarrow$$
$$\mathbf{G}^T\bar{\omega} = 2\mathbf{G}^T\mathbf{G}\dot{\mathbf{p}}$$
$$\swarrow$$
$$\dot{\mathbf{p}} = \frac{1}{2}\mathbf{G}^T\bar{\omega} \longrightarrow \dot{\mathbf{p}} = \frac{1}{2}\mathbf{E}^T\omega$$

$$\mathbf{r}^P = \mathbf{r} + \mathbf{s}^P = \mathbf{r} + \mathbf{A}\bar{\mathbf{s}}^P$$

$$\vec{\mathbf{v}}^P = \frac{d\vec{\mathbf{r}}^P}{dt} = \dot{\vec{\mathbf{r}}} + \dot{\vec{\mathbf{s}}}^P = \dot{\vec{\mathbf{r}}} + \vec{\omega} \times \vec{\mathbf{s}}^P \qquad \vec{\mathbf{a}}^P \equiv \frac{d^2\vec{\mathbf{r}}^P}{dt^2} = \ddot{\vec{\mathbf{r}}} + \vec{\omega} \times \vec{\omega} \times \vec{\mathbf{s}}^P + \dot{\vec{\omega}} \times \vec{\mathbf{s}}^P$$

$$\dot{\mathbf{r}}^P = \dot{\mathbf{r}} + \dot{\mathbf{s}}^P = \dot{\mathbf{r}} + \dot{\mathbf{A}}\bar{\mathbf{s}}^P = \dot{\mathbf{r}} + \tilde{\omega}\mathbf{A}\bar{\mathbf{s}}^P = \dot{\mathbf{r}} + \tilde{\omega}\mathbf{s}^P$$

$$\mathbf{a}^P \equiv \ddot{\mathbf{r}}^P = \ddot{\mathbf{r}} + \ddot{\mathbf{s}}^P = \ddot{\mathbf{r}} + \tilde{\omega}\tilde{\omega}\mathbf{A}\bar{\mathbf{s}}^P + \tilde{\dot{\omega}}\mathbf{A}\bar{\mathbf{s}}^P = \ddot{\mathbf{r}} + \tilde{\omega}\tilde{\omega}\mathbf{s}^P + \tilde{\dot{\omega}}\mathbf{s}^P$$

- First, recall that we introduced a matrix $\mathbf{B}$ as follows:

$$\frac{\partial[\mathbf{A}(\mathbf{p}) \cdot \bar{\mathbf{s}}]}{\partial \mathbf{p}} \equiv \mathbf{B}(\mathbf{p}, \bar{\mathbf{s}})$$

- Some helpful identities:

$\dot{\mathbf{B}}(\mathbf{p}, \bar{\mathbf{s}}) = \mathbf{B}(\dot{\mathbf{p}}, \bar{\mathbf{s}}) \quad \longrightarrow \quad$ due to the linearity of the $\mathbf{B}(\mathbf{p}, \bar{\mathbf{s}})$ matrix in relation to the variable $\mathbf{p}$

$$\frac{d[\mathbf{B}(\mathbf{p}, \bar{\mathbf{s}})\dot{\mathbf{p}}]}{d\,t} = \mathbf{B}(\dot{\mathbf{p}}, \bar{\mathbf{s}})\dot{\mathbf{p}} + \mathbf{B}(\mathbf{p}, \bar{\mathbf{s}})\ddot{\mathbf{p}}$$

$$\mathbf{a}_i = \mathbf{A}_i\bar{\mathbf{a}}_i \quad \Rightarrow \quad \dot{\mathbf{a}}_i = \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i \quad \Rightarrow \quad \ddot{\mathbf{a}}_i = \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{a}}_i)\dot{\mathbf{p}}_i + \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{a}}_i)\ddot{\mathbf{p}}_i$$

$$\dot{\mathbf{d}}_{ij} = \dot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j - \dot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i$$

$$\ddot{\mathbf{d}}_{ij} = \ddot{\mathbf{r}}_j + \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)\ddot{\mathbf{p}}_j + \mathbf{B}(\dot{\mathbf{p}}_j, \bar{\mathbf{s}}_j^Q)\dot{\mathbf{p}}_j - \ddot{\mathbf{r}}_i - \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P)\ddot{\mathbf{p}}_i - \mathbf{B}(\dot{\mathbf{p}}_i, \bar{\mathbf{s}}_i^P)\dot{\mathbf{p}}_i$$

$$\mathbf{q}_i = \begin{bmatrix} \mathbf{r}_i \\ \mathbf{p}_i \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \\ z_i \\ e_{0,i} \\ e_{1,i} \\ e_{2,i} \\ e_{3,i} \end{bmatrix} \begin{array}{l} \left.\begin{matrix} \\ \\ \end{matrix}\right\} \text{Tell us where the body is located} \\ \left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} \text{Tell us how the body is oriented} \end{array}$$

Note: $\mathbf{q}_i \in \mathbb{R}^7$

$$\mathbf{q} = \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \\ \cdots \\ \mathbf{q}_{nb} \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ \vdots \\ e_{2,nb} \\ e_{3,nb} \end{bmatrix} \equiv \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ \vdots \\ q_{nc} \end{bmatrix} \in \mathbb{R}^{nc}$$

- NOTE: for a mechanism with *nb* bodies, the number *nc* of Cartesian generalized coordinates is

$$nc = 7 \cdot nb$$

  - "nc" stands for "number of coordinates"

- Recall we have a number of *nb* "Euler Parameter normalization constraints":

$$\mathbf{p}_i^T \cdot \mathbf{p}_i = 1 \,, \qquad i = 1, 2, \ldots, nb$$

- We'll call $\mathbf{\Phi^P}$ the specific set of nb Euler Parameter normalization constraints (note lack of dependency on $\mathbf{r}$):

$$\mathbf{\Phi^P}(\mathbf{p}) \equiv \begin{bmatrix} \mathbf{p}_1^T \mathbf{p}_1 - 1.0 \\ \cdots \\ \mathbf{p}_{nb}^T \, \mathbf{p}_{nb} - 1.0 \end{bmatrix}$$

- Number of degrees of freedom (NDOF, ndof) is equal to total number of generalized coordinates minus the number of constraints that these coordinates must satisfy
  - Sometimes also called "Gruebler Count"

$$NDOF = nc - m_G - m_D - nb$$

- Quick Remarks:
  - NDOF is an attribute of the model, and it is independent of the set of generalized coordinates used to represent the motion of the mechanism
  - Since we're using Euler Parameters for body orientation, one should also include the set of nb normalization constraints

- In general, for carrying out Kinematic Analysis, $NDOF = 0$
  - For Dynamics Analysis, we need $NDOF \geq 0$

- Approach leading to Kinematic Analysis
  - Prescribe motions for various components of the mechanical system until $NDOF = 0$
  - For a well posed problem, you'll be able to uniquely determine **q**(t) as the solution of an algebraic problem

- Approach leading to Dynamics Analysis
  - Apply a set of forces upon the mechanism and specify a number of motions, but when doing the latter make sure you end up with NDOF $\geq$ 0
  - For a well posed problem, **q**(t) found as the solution of a differential problem

Kinematics Analysis – the process of computing the position, velocity, and acceleration of a system of interconnected bodies that make up a mechanical system independent of the forces that produce its motion

$$\boxed{\begin{array}{ll}\boldsymbol{\Phi}^F(\mathbf{q},t) = \begin{bmatrix} \boldsymbol{\Phi}^G(\mathbf{q}) \\ \boldsymbol{\Phi}^D(\mathbf{q},t) \\ \boldsymbol{\Phi}^p(\mathbf{p}) \end{bmatrix}_{nc\times 1} = \mathbf{0} \\ \mathbf{q} \in \mathbb{R}^{nc} \qquad \boldsymbol{\Phi} : \mathbb{R}^{nc+1} \to \mathbb{R}^{nc}\end{array}}$$

IMPORTANT: This is a nonlinear systems with *nc* equations and *nc* unknowns that you must solve to find **q**

- The set of L0 generalized coordinates $\mathbf{q}$ is iteratively improved as:

$$\mathbf{q}^{(k+1)} = \mathbf{q}^{(k)} - \left[\boldsymbol{\Phi}^F_{\mathbf{q}}(\mathbf{q}^{(k)})\right]^{-1} \boldsymbol{\Phi}(\mathbf{q}^{(k)}, t)$$

- In a practical implementation, this is done like:

$$\begin{aligned}\left[\boldsymbol{\Phi}^F_{\mathbf{q}}(\mathbf{q}^{(k)})\right] \Delta\mathbf{q}^{(k+1)} &= \boldsymbol{\Phi}(\mathbf{q}^{(k)}, t) \\ \mathbf{q}^{(k+1)} &= \mathbf{q}^{(k)} - \Delta\mathbf{q}^{(k+1)}\end{aligned}$$

- You will have to provide a good guess $\mathbf{q}^{(0)}$ for starting the iterative process

$$\begin{aligned}\left[\boldsymbol{\Phi}^F_{\mathbf{q}}(\mathbf{q}^{(k)})\right] \Delta\mathbf{q}^{(k+1)} &= \boldsymbol{\Phi}(\mathbf{q}^{(k)}, t) \\ \mathbf{q}^{(k+1)} &= \mathbf{q}^{(k)} - \Delta\mathbf{q}^{(k+1)}\end{aligned}$$

This is why we need the Jacobian; we'll talk about how to compute it in a bit

- The three stages of Kinematics Analysis: <u>position</u> analysis, <u>velocity</u> analysis, and <u>acceleration</u> analysis they each follow *very* similar recipes for finding for each body of the mechanism its position, velocity, and acceleration, respectively

- ALL STAGES RELY ON THE CONCEPT OF JACOBIAN MATRIX:
  - $\boldsymbol{\Phi}_{\mathbf{q}}$ – the partial derivative of the constraints wrt the generalized coordinates

- ALL STAGES REQUIRE THE SOLUTION OF A SYSTEM OF EQUATIONS

$$\boldsymbol{\Phi}_{\mathbf{q}}\, \mathbf{x} = \mathbf{b}$$

- WHAT IS *DIFFERENT* BETWEEN THE THREE STAGES IS THE EXPRESSION OF THE RIGHT-SIDE OF THE LINEAR EQUATION, "**b**"

Paying attention to the fact that $\boldsymbol{\Phi}^{\mathbf{p}}(\mathbf{p}) \equiv \begin{bmatrix} \mathbf{p}_1^T\mathbf{p}_1 - 1.0 \\ \cdots \\ \mathbf{p}_{nb}^T\,\mathbf{p}_{nb} - 1.0 \end{bmatrix} = \mathbf{0}_{nb}.$

$$\Phi^K(\mathbf{q},t) = \begin{bmatrix} \Phi^G(\mathbf{q}) \\ \Phi^D(\mathbf{q},t) \end{bmatrix} = \mathbf{0}_{6nb} \qquad \Phi^F(\mathbf{q},t) = \begin{bmatrix} \Phi^G(\mathbf{q}) \\ \Phi^D(\mathbf{q},t) \\ \Phi^{\mathbf{p}}(\mathbf{p}) \end{bmatrix} = \mathbf{0}_{7nb}$$

- Subsequently, when carrying out Velocity Analysis, if solving for $\dot{\mathbf{r}}$ and $\dot{\mathbf{p}}$, you have to include $\Phi^{\mathbf{p}} = \mathbf{0}_{nb}$ in the set of constraints $\Phi(\mathbf{q},t)$
  - Justification: although not stemming from a GCon, the Euler Parameter normalization constraints are nonetheless constraints induced by the particular choice of generalized coordinates. It's important to make this distinction between ACEs (a) stemming from the geometry of the motion (from GCon's), and (b) induced by the choice of generalized coordinates we've decided to work with (normalization constraints for $\mathbf{p}$, in our case).
- Note that for the Velocity and Acceleration right-hand side of the linear equations, you have:

$$\nu^{\mathbf{p}} = \mathbf{0}_{nb} \qquad \& \qquad \gamma^{\mathbf{p}} = \begin{bmatrix} -2\dot{\mathbf{p}}_1^T\dot{\mathbf{p}}_1 \\ \cdots \\ -2\dot{\mathbf{p}}_{nb}^T\dot{\mathbf{p}}_{nb} \end{bmatrix}$$

$$\Phi_{\mathbf{q}}\ddot{\mathbf{q}} = \gamma_{7nb}$$

$$\Phi_{\mathbf{q}}\dot{\mathbf{q}} = \nu_{7nb}$$

Basic GCon DP1: $\Phi_{\mathbf{r}}^{DP1}$ and $\Phi_{\mathbf{p}}^{DP1}$

- Recall that

$$\Phi^{DP1}(i,\bar{\mathbf{a}}_i,j,\bar{\mathbf{a}}_j,f(t)) = \bar{\mathbf{a}}_i^T\mathbf{A}_i^T\mathbf{A}_j\bar{\mathbf{a}}_j - f(t) = \mathbf{a}_i{}^T\mathbf{a}_j - f(t) = 0$$

- Then, it follows that

$$\frac{\partial\Phi^{DP1}}{\partial\mathbf{r}_i} = \mathbf{0}_{1\times3} \qquad \frac{\partial\Phi^{DP1}}{\partial\mathbf{p}_i} = \mathbf{a}_j^T\mathbf{B}(\mathbf{p}_i,\bar{\mathbf{a}}_i)$$

$$\frac{\partial\Phi^{DP1}}{\partial\mathbf{r}_j} = \mathbf{0}_{1\times3} \qquad \frac{\partial\Phi^{DP1}}{\partial\mathbf{p}_j} = \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_j,\bar{\mathbf{a}}_j)$$

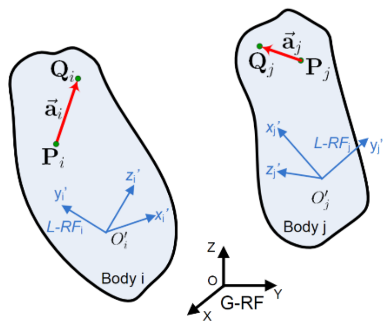
- Putting it all together (note that $\Phi_{\mathbf{q}}^{DP1} \in \mathbb{R}^{1\times7nb}$) ,

$$\Phi_{\mathbf{q}}^{DP1} = \left[\mathbf{0}_{1\times3}\ldots\mathbf{0}_{1\times3}\ldots\mathbf{0}_{1\times3}\ldots\ldots\mathbf{0}_{1\times4}\ \frac{\partial\Phi^{DP1}}{\partial\mathbf{p}_i}\ \mathbf{0}_{1\times4}\ldots\mathbf{0}_{1\times4}\ \frac{\partial\Phi^{DP1}}{\partial\mathbf{p}_j}\ \mathbf{0}_{1\times4}\ldots\mathbf{0}_{1\times4}\right]$$

Partials with respect to $\mathbf{r}$ ← → Partials with respect to $\mathbf{p}$

$$= \left[\mathbf{0}_{1\times3}\ldots\mathbf{0}_{1\times3}\ldots\mathbf{0}_{1\times3}\ldots\ldots\mathbf{0}_{1\times4}\ \mathbf{a}_j^T\mathbf{B}(\mathbf{p}_i,\bar{\mathbf{a}}_i)\ \mathbf{0}_{1\times4}\ldots\mathbf{0}_{1\times4}\ \mathbf{a}_i^T\mathbf{B}(\mathbf{p}_j,\bar{\mathbf{a}}_j)\ \mathbf{0}_{1\times4}\ldots\mathbf{0}_{1\times4}\right]$$

Body 1, **r** | Body i, **r** | Body j, **r** | Body i-1, **p** | Body i, **p** | Body i+1, **p** | Body j-1, **p** | Body j, **p** | Body j+1, **p** | Body nb, **p**



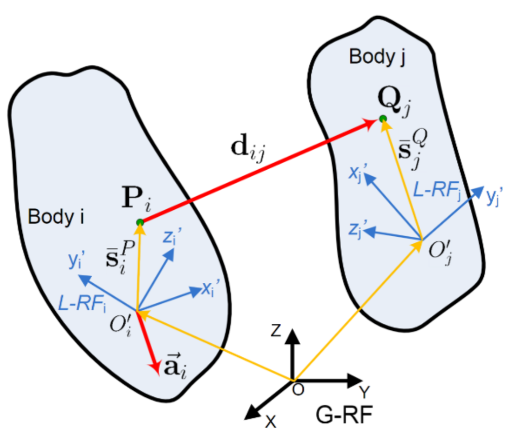
Basic GCon DP2: $\mathbf{\Phi}_{\mathbf{r}}^{DP2}$ and $\mathbf{\Phi}_{\mathbf{p}}^{DP2}$

- Recall that

$$\Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} - f(t) = \mathbf{a}_i{}^T \mathbf{d}_{ij} - f(t) = 0$$

- It follows that

$$\begin{aligned}
\Phi^{DP2}_{\mathbf{q}_i,\mathbf{q}_j}(\mathbf{a}_i, \mathbf{d}_{ij}) &= \mathbf{a}_i^T (\mathbf{d}_{ij})_{\mathbf{q}_i,\mathbf{q}_j} + \mathbf{d}_{ij}^T (\mathbf{a}_i)_{\mathbf{q}_i,\mathbf{q}_j} \\
&= \mathbf{a}_i^T [\ -\mathbf{I}_3 \quad -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad \mathbf{I}_3 \quad \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)\ ] + \mathbf{d}_{ij}^T [\ \mathbf{0} \quad \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{a}}_i) \quad \mathbf{0} \quad \mathbf{0}\ ] \\
&= [\ -\mathbf{a}_i^T \qquad \mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{a}}_i) - \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \qquad \mathbf{a}_i^T \qquad \mathbf{a}_i^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q)\ ]
\end{aligned}$$



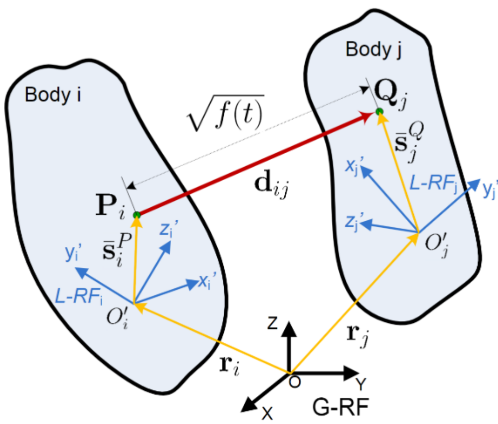
Basic GCon D: $\mathbf{\Phi}_{\mathbf{r}}^{D}$ and $\mathbf{\Phi}_{\mathbf{p}}^{D}$

- Recall that the GCon-D assumes the expression

$$\Phi^D(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) \;=\; \mathbf{d}_{ij}^T \mathbf{d}_{ij} - f(t) = 0$$

- It follows that

$$\begin{aligned}
\Phi^D_{\mathbf{q}_i,\mathbf{q}_j} &= (\mathbf{d}_{ij}^T \mathbf{d}_{ij})_{\mathbf{q}_i,\mathbf{q}_j} = 2\mathbf{d}_{ij}^T [\mathbf{d}_{ij}]_{\mathbf{q}_i,\mathbf{q}_j} \\
&= 2\mathbf{d}_{ij}^T [\; -\mathbf{I}_3 \quad -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad \mathbf{I}_3 \quad \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \;] \\
&= [\; -2\mathbf{d}_{ij}^T \quad -2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) \quad 2\mathbf{d}_{ij}^T \quad 2\mathbf{d}_{ij}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \;]
\end{aligned}$$



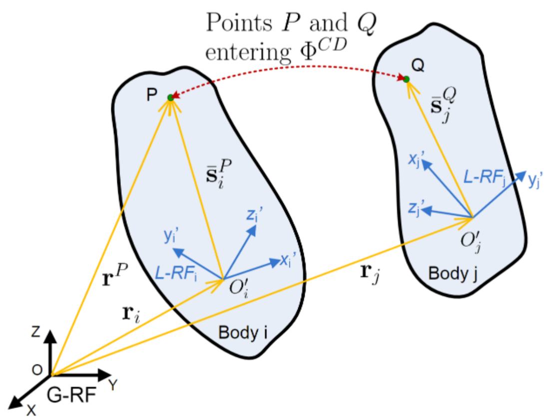
Basic GCon CD: $\mathbf{\Phi}_{\mathbf{r}}^{CD}$ and $\mathbf{\Phi}_{\mathbf{p}}^{CD}$

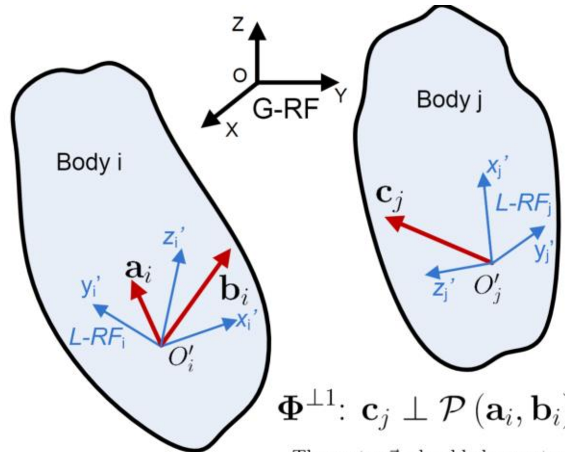
- Recall that the GCon-CD assumes the expression

$$\Phi^{CD}(\mathbf{c}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, f(t)) = \mathbf{c}^T \mathbf{d}_{ij} - f(t) = 0$$

- It follows that

$$\begin{aligned}
\Phi^{CD}_{\mathbf{q}_i,\mathbf{q}_j} &= (\mathbf{c}^T \mathbf{d}_{ij})_{\mathbf{q}_i,\mathbf{q}_j} = \mathbf{c}^T [\mathbf{d}_{ij}]_{\mathbf{q}_i,\mathbf{q}_j} \\
&= \mathbf{c}^T \begin{bmatrix} -\mathbf{I}_3 & -\mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{I}_3 & \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix} \\
&= \begin{bmatrix} -\mathbf{c}^T & -\mathbf{c}^T \mathbf{B}(\mathbf{p}_i, \bar{\mathbf{s}}_i^P) & \mathbf{c}^T & \mathbf{c}^T \mathbf{B}(\mathbf{p}_j, \bar{\mathbf{s}}_j^Q) \end{bmatrix}
\end{aligned}$$



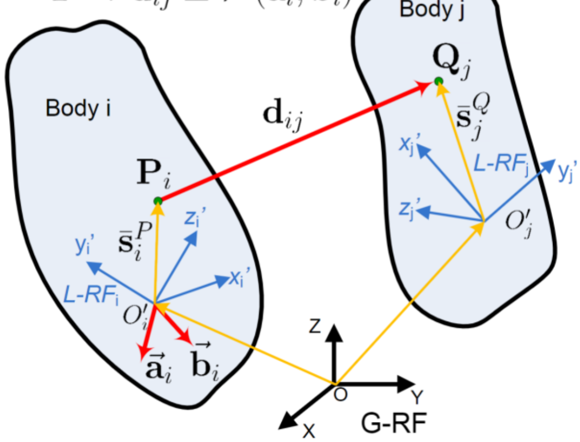


$\mathbf{\Phi}^{\perp 1}$: $\mathbf{c}_j \perp \mathcal{P}\left(\mathbf{a}_i, \mathbf{b}_i\right)$

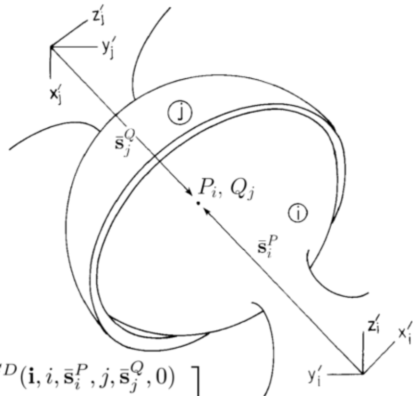
The vector $\vec{\mathbf{c}}_j$ should always stay perpendicular to the plan defined by the vectors $\vec{\mathbf{a}}_i$ and $\vec{\mathbf{b}}_i$

$$\mathbf{\Phi}^{\perp 1}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{c}}_j) = \begin{bmatrix} \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{c}}_j, 0) \\ \Phi^{DP1}(i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{c}}_j, 0) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{c}}_j \\ \bar{\mathbf{b}}_i^T \mathbf{A}_i^T \mathbf{A}_j \bar{\mathbf{c}}_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\mathbf{\Phi}^{\perp 2}$: $\mathbf{d}_{ij} \perp \mathcal{P}\left(\mathbf{a}_i, \mathbf{b}_i\right)$

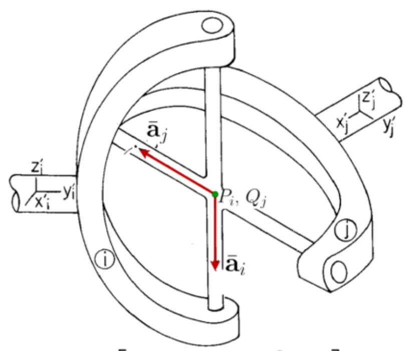


$$\mathbf{\Phi}^{\perp 2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q) = \begin{bmatrix} \Phi^{DP2}(i, \bar{\mathbf{a}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{DP2}(i, \bar{\mathbf{b}}_i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{a}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} \\ \bar{\mathbf{b}}_i^T \mathbf{A}_i^T \mathbf{d}_{ij} \end{bmatrix} = 0$$



$$\mathbf{\Phi}^{SJ} = \begin{bmatrix} \Phi^{CD}(\mathbf{i}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{CD}(\mathbf{j}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{CD}(\mathbf{k}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \end{bmatrix}$$

$$\mathbf{\Phi}^{SJ}(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q) = \begin{bmatrix} \Phi^{CD}(\mathbf{i}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{CD}(\mathbf{j}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{CD}(\mathbf{k}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\mathbf{\Phi}^{UJ} = \begin{bmatrix} \mathbf{\Phi}^{SJ}(i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q) \\ \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, 0) \end{bmatrix}$$

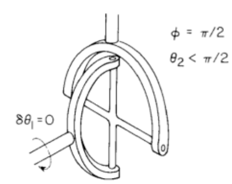
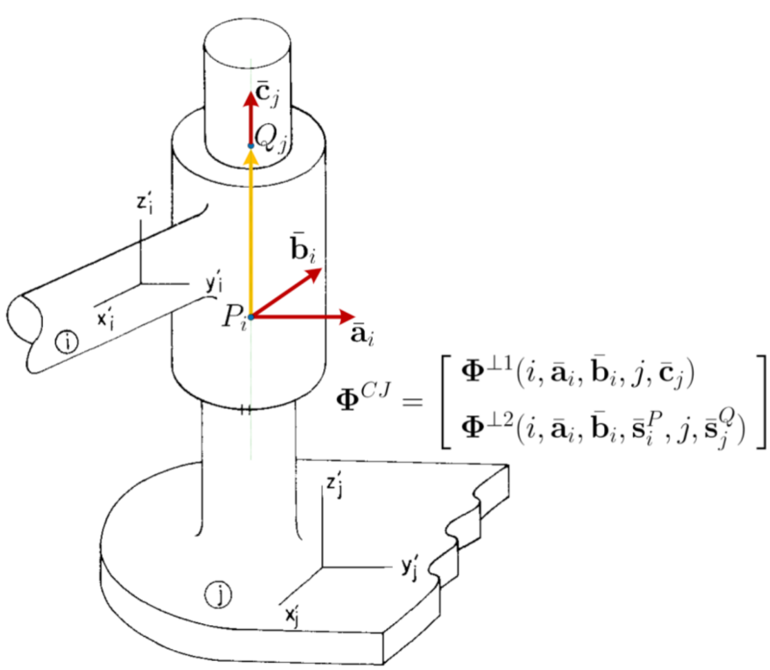
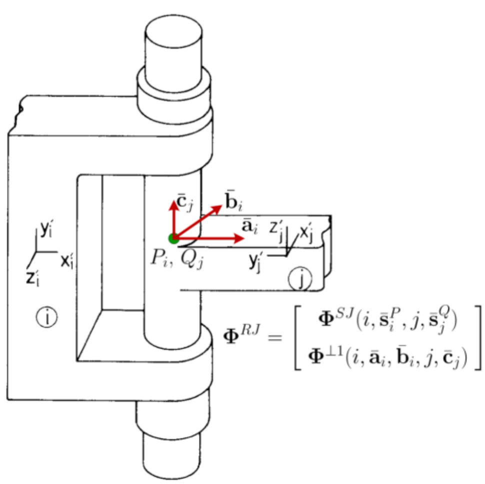


Figure 9.4.15 Singular behavior of universal joint.

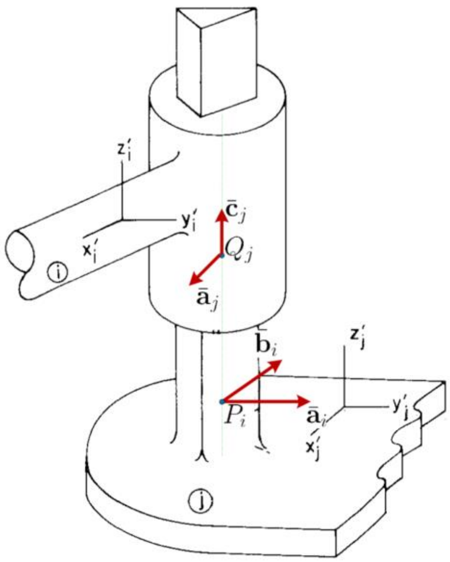
$$\mathbf{\Phi}^{UJ}(i, \bar{\mathbf{s}}_i^P, \bar{\mathbf{a}}_i, j, \bar{\mathbf{s}}_j^Q, \bar{\mathbf{a}}_j) = \begin{bmatrix} \Phi^{CD}(\mathbf{i}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{CD}(\mathbf{j}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{CD}(\mathbf{k}, i, \bar{\mathbf{s}}_i^P, j, \bar{\mathbf{s}}_j^Q, 0) \\ \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$



$$\mathbf{\Phi}^{CJ}(i,\bar{\mathbf{s}}_i^P,\bar{\mathbf{a}}_i,\bar{\mathbf{b}}_i,j,\bar{\mathbf{s}}_j^Q,\bar{\mathbf{c}}_j) = \begin{bmatrix}\mathbf{\Phi}^{\perp 1}(i,\bar{\mathbf{a}}_i,\bar{\mathbf{b}}_i,j,\bar{\mathbf{c}}_j)\\ \mathbf{\Phi}^{\perp 2}(i,\bar{\mathbf{a}}_i,\bar{\mathbf{b}}_i,\bar{\mathbf{s}}_i^P,j,\bar{\mathbf{s}}_j^Q)\end{bmatrix} = \begin{bmatrix}\mathbf{0}_2\\ \mathbf{0}_2\end{bmatrix}$$



$$\mathbf{\Phi}^{RJ}(i,\bar{\mathbf{s}}_i^P,\bar{\mathbf{a}}_i,\bar{\mathbf{b}}_i,j,\bar{\mathbf{s}}_j^Q,\bar{\mathbf{c}}_j) = \begin{bmatrix}\mathbf{\Phi}^{SJ}(i,\bar{\mathbf{s}}_i^P,j,\bar{\mathbf{s}}_j^Q)\\ \mathbf{\Phi}^{\perp 1}(i,\bar{\mathbf{a}}_i,\bar{\mathbf{b}}_i,j,\bar{\mathbf{c}}_j)\end{bmatrix} = \begin{bmatrix}\mathbf{0}_3\\ \mathbf{0}_2\end{bmatrix}$$



$$\boldsymbol{\Phi}^{TJ} = \begin{bmatrix} \boldsymbol{\Phi}^{CJ}(i, \bar{\mathbf{s}}_i^P, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{s}}_j^Q, \bar{\mathbf{c}}_j) \\ \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, \text{const.}) \end{bmatrix}$$

$$\boldsymbol{\Phi}^{TJ}(i, \bar{\mathbf{s}}_i^P, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{s}}_j^Q, \bar{\mathbf{a}}_j, \bar{\mathbf{c}}_j) = \begin{bmatrix} \boldsymbol{\Phi}^{CJ}(i, \bar{\mathbf{s}}_i^P, \bar{\mathbf{a}}_i, \bar{\mathbf{b}}_i, j, \bar{\mathbf{s}}_j^Q, \bar{\mathbf{c}}_j) \\ \Phi^{DP1}(i, \bar{\mathbf{a}}_i, j, \bar{\mathbf{a}}_j, \text{const.}) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_4 \\ 0 \end{bmatrix}$$

- Take one time derivative of constraints $\boldsymbol{\Phi}$(q,t) to obtain the **velocity equation**:

$$\frac{d}{dt}\boldsymbol{\Phi}(\mathbf{q}, t) = \mathbf{0} \qquad \Rightarrow \qquad \frac{\partial \boldsymbol{\Phi}}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} + \frac{\partial \boldsymbol{\Phi}}{\partial t} = \mathbf{0} \qquad \Rightarrow \qquad \boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \underbrace{-\boldsymbol{\Phi}_t}_{\nu}$$

- The Jacobian has as many rows (m) as it has columns (nc) since for Kinematics Analysis, $NDOF = nc - m - nb = 0$

- Therefore, you have a linear system that you need to solve to recover $\dot{\mathbf{q}}$

$$\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \nu$$

- Summary: you have a linear system that you need to solve to recover

$$\boldsymbol{\Phi}_{\mathbf{q}}\dot{\mathbf{q}} = \nu$$

- Since NDOF=0, the coefficient matrix in Eq. (1) above is nonsingular

- At each time step, the value of ν changes, as well as the coefficient matrix $\boldsymbol{\Phi}_q$

- Recall the expression of the time derivative $\dot{\mathbf{\Phi}}$:

$$\dot{\mathbf{\Phi}} = \mathbf{\Phi_q}\dot{\mathbf{q}} + \mathbf{\Phi}_t$$

- Then,

$$\ddot{\mathbf{\Phi}} = \frac{d}{dt}\left[\frac{\partial \mathbf{\Phi}}{\partial \mathbf{q}}\dot{\mathbf{q}} + \mathbf{\Phi}_t\right]$$

- Using the notation $\mathbf{b}(\dot{\mathbf{q}}, \mathbf{q}, t) \equiv \mathbf{\Phi_q}\dot{\mathbf{q}} + \frac{\partial \mathbf{\Phi}}{\partial t}$ and the chain rule of differentiation,

$$\ddot{\mathbf{\Phi}} = \frac{\partial \mathbf{b}}{\partial \dot{\mathbf{q}}}\frac{d\dot{\mathbf{q}}}{dt} + \frac{\partial \mathbf{b}}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} + \frac{\partial \mathbf{b}}{\partial t} = \mathbf{\Phi_q}\ddot{\mathbf{q}} + \left[(\mathbf{\Phi_q}\dot{\mathbf{q}})_\mathbf{q} + \mathbf{\Phi}_{\mathbf{q}t}\right]\dot{\mathbf{q}} + [\mathbf{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} + \mathbf{\Phi}_{tt}] = \mathbf{0}$$

- Therefore,

$$\mathbf{\Phi_q}\ddot{\mathbf{q}} = \gamma \qquad \text{where} \qquad \gamma \equiv -\left(\mathbf{\Phi_q}\dot{\mathbf{q}}\right)_\mathbf{q}\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t}\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}$$

- Summary, the **acceleration equation**:

$$\ddot{\mathbf{\Phi}} = \frac{d^2}{dt^2}\mathbf{\Phi}(\mathbf{q}, t) = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{\Phi_q}\ddot{\mathbf{q}} = \underbrace{-(\mathbf{\Phi_q}\dot{\mathbf{q}})_\mathbf{q}\,\dot{\mathbf{q}} - 2\mathbf{\Phi}_{\mathbf{q}t}\,\dot{\mathbf{q}} - \mathbf{\Phi}_{tt}}_{\gamma}$$

- NOTE: Getting right-hand side of acceleration equation is tedious
  - Observation that simplifies the computation of $\boldsymbol{\gamma}$: note that $\boldsymbol{\gamma}$ is made up of everything in the expression of $\ddot{\mathbf{\Phi}}$ that does *not* depend on second time derivatives $\ddot{\mathbf{q}}$
    - Important observation, simplifies the task of deriving $\boldsymbol{\gamma}$

- Just like we pointed out for the velocity analysis, you also have to solve a linear system to retrieve the acceleration $\ddot{\mathbf{q}}$

$$\mathbf{\Phi_q}\ddot{\mathbf{q}} = \gamma$$