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ME 751

Problem 1. [SimEngine3D: MATLAB/Python/C] This problem builds on the problem in the previous assignment. The schematic of the mechanism is shown in Figure 1. The rigid body is subjected to a motion specified as $\theta(t) = \frac{\pi}{4}\cos(2t)$. Recycle as much as possible of the code you already generated. If need be, draw inspiration from ME751 colleagues who pushed their work in GitHub.

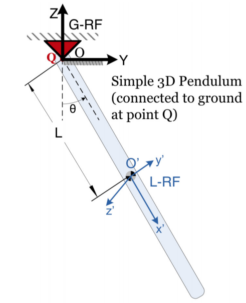


Figure 1: Pendulum with revolute joint.

Perform an Inverse Dynamics Analysis to compute the amount of torque that you would have to apply to the pendulum to make it move as indicated by the specified motion. Assume that $L = 2$, and the cross-section of the bar is a square of width 0.05. The density of the material is $\rho = 7,800$. The gravitational acceleration (not shown in the picture) is assumed to act in the opposite direction of the global z axis; its magnitude is $g = 9.81$. All units are SI. For this problem, please provide a plot that displays the value of the torque as a function of time for $t \in [0, 10]$.

$$\mathbf{M} = \begin{bmatrix} m_1\mathbf{I}_3 & \mathbf{0}_{3\times3} & \cdots & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & m_2\mathbf{I}_3 & \cdots & \mathbf{0}_{3\times3} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} & \cdots & m_{nb}\mathbf{I}_3 \end{bmatrix}_{3nb\times3nb} \qquad \mathbf{J^P} = \begin{bmatrix} 4\mathbf{G}_1^T\bar{\mathbf{J}}_1\mathbf{G}_1 & \mathbf{0}_{4\times4} & \cdots & \mathbf{0}_{4\times4} \\ \mathbf{0}_{4\times4} & 4\mathbf{G}_1^T\bar{\mathbf{J}}_1\mathbf{G}_1 & \cdots & \mathbf{0}_{4\times4} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{4\times4} & \mathbf{0}_{4\times4} & \cdots & 4\mathbf{G}_{nb}^T\bar{\mathbf{J}}_{nb}\mathbf{G}_{nb} \end{bmatrix}_{4nb\times4nb}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{p}_1^T & \mathbf{0}_{1\times4} & \cdots & \mathbf{0}_{1\times4} \\ \mathbf{0}_{1\times4} & \mathbf{p}_2^T & \cdots & \mathbf{0}_{1\times4} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{1\times4} & \mathbf{0}_{1\times4} & \mathbf{0}_{1\times4} & \mathbf{p}_{nb}^T \end{bmatrix}_{nb\times4nb} \qquad \mathbf{F} = \begin{bmatrix} \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_{nb} \end{bmatrix}_{3nb} \qquad \hat{\tau} = \begin{bmatrix} \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_{nb} \end{bmatrix}_{4nb}$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{nc} \end{bmatrix} \qquad \lambda^{\mathbf{P}} = \begin{bmatrix} \lambda_1^{\mathbf{P}} \\ \vdots \\ \lambda_{nb}^{\mathbf{P}} \end{bmatrix} \qquad \gamma^{\mathbf{P}} = \begin{bmatrix} -2\dot{\mathbf{p}}_1^T\dot{\mathbf{p}}_1 \\ \cdots \\ -2\dot{\mathbf{p}}_{nb}^T\dot{\mathbf{p}}_{nb} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M} & \mathbf{0}_{3nb\times4nb} & \mathbf{0}_{3nb\times nb} & \mathbf{\Phi}_\mathbf{r}^T \\ \mathbf{0}_{4nb\times3nb} & \mathbf{J^P} & \mathbf{P}^T & \mathbf{\Phi}_\mathbf{p}^T \\ \mathbf{0}_{nb\times3nb} & \mathbf{P} & \mathbf{0}_{nb\times nb} & \mathbf{0}_{nb\times nc} \\ \mathbf{\Phi_r} & \mathbf{\Phi_p} & \mathbf{0}_{nc\times nb} & \mathbf{0}_{nc\times nc} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{r}} \\ \ddot{\mathbf{p}} \\ \lambda^{\mathbf{P}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \hat{\tau} \\ \gamma^{\mathbf{P}} \\ \hat{\gamma} \end{bmatrix}$$

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{r}} + \mathbf{\Phi}_\mathbf{r}^T\lambda &= \mathbf{F} \\ \mathbf{J^P}\ddot{\mathbf{p}} + \mathbf{\Phi}_\mathbf{p}^T\lambda + \mathbf{P}^T\lambda^{\mathbf{P}} &= \hat{\tau} \end{aligned}$$

$$\mathbf{F}_i \equiv \mathbf{F}_i^m + \mathbf{F}_i^a \qquad \hat{\tau}_i \equiv 2\mathbf{G}_i^T(\bar{\mathbf{n}}_i^m + \bar{\mathbf{n}}_i^a) + 8\dot{\mathbf{G}}_i^T\bar{\mathbf{J}}_i\dot{\mathbf{G}}_i\mathbf{p}_i$$

EOM :

$$\begin{bmatrix} M & 0 & 0 & \Phi_r^T \\ 0 & J^P & P^T & \Phi_P^T \\ 0 & P & 0 & 0 \\ \Phi_r & \Phi_P & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{r} \\ \ddot{p} \\ \lambda^P \\ \lambda \end{bmatrix} = \begin{bmatrix} F \\ \hat{\tau} \\ \gamma^P \\ \hat{\gamma} \end{bmatrix}$$

$$M\ddot{r} + \phi_r^T \lambda = F \qquad\qquad F = F^a + F^m$$

$$J^p \ddot{p} + P^T \lambda^p + \phi_p^T \lambda = \hat{\tau} \qquad\qquad \tau = \tau^a + \tau^m$$

$$\begin{bmatrix} \phi_r^T & 0 \\ \phi_p^T & P^T \end{bmatrix} \begin{bmatrix} \lambda \\ \lambda^p \end{bmatrix} = \begin{bmatrix} F - M\ddot{r} \\ \hat{\tau} - J^p \ddot{p} \end{bmatrix}$$

$(3nb \times ngc)$ $(3nb \times nb)$

$(4nb \times nge)$ $(4nb \times nb)$

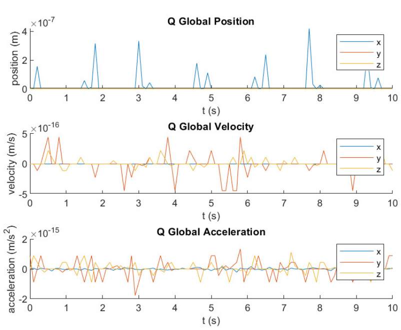
solve for λ, λ^p

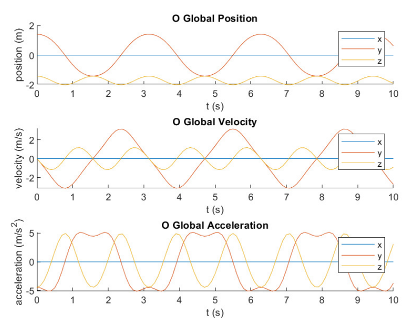
$$\vec{F} = -mg\hat{z} \quad \Rightarrow \quad \vec{F}_i = m_i \vec{g}$$

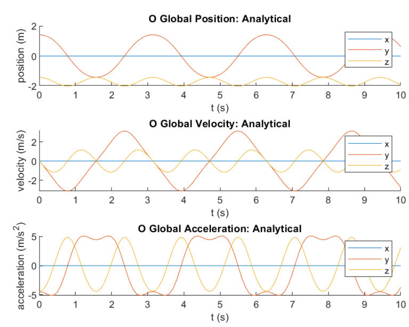
basic GCon Φ^α that has the Lagrange Multiplier λ_α are

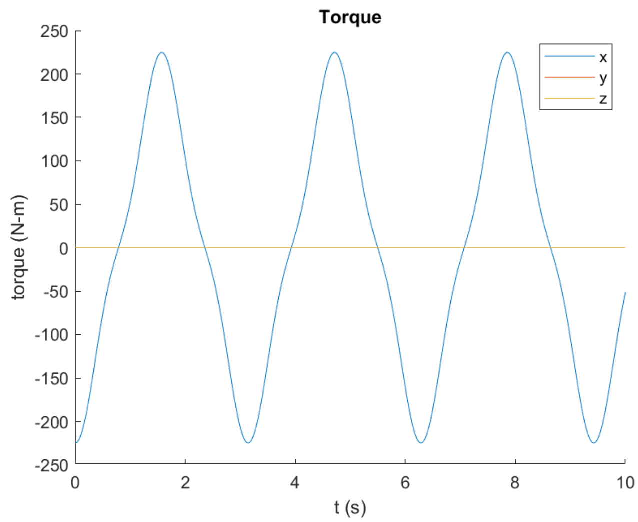
$$\mathbf{F}_i^{r,\alpha} = -[\Phi^\alpha_{\mathbf{r}_i}]^T \lambda_\alpha \qquad \bar{\mathbf{n}}_i^{r,\alpha} = -\bar{\mathbf{\Pi}}_i^T(\Phi^\alpha)\lambda_\alpha \qquad \bar{\Pi}^T = -\frac{1}{2} G \phi_p^T$$

$\uparrow$ local $\Rightarrow$ convert to global w/ $A(p)$









Problem 2. Consider the following IVP (discussed in class, see also handout example on the computation of the Jacobian):

$$\begin{cases} \dot{x} = 1 - x - \dfrac{4xy}{1+x^2} \\ \dot{y} = x(1 - \dfrac{y}{1+x^2}) \end{cases},$$

where

$$\begin{cases} x(0) = 0, \; y(0) = 2 \\ t \in [0, 20] \end{cases}.$$

Apply Backward Euler to find an approximation of the exact solution of this IVP. Generate plots of x and y, respectively, that you include as part of your HW.

following the handout

Backwards euler: $x_n = x_{n-1} + h\dot{x}_n$

$y_n = y_{n-1} + h\dot{y}_n$

$\dot{x} = \alpha - x - \dfrac{4xy}{1+x^2}$ $\Rightarrow$ $\dot{x}_n = \alpha - x_n - \dfrac{4x_n y_n}{1+x_n^2}$ $(\alpha = 1, \quad \beta = 1)$

$\dot{y} = \beta x\left(1 - \dfrac{y}{1+x^2}\right)$ $\quad \dot{y}_n = \beta x_n\left(1 - \dfrac{y_n}{1+x_n^2}\right)$

$x_n = x_{n-1} + h\dot{x}_n = x_{n-1} + h\left(\alpha - x_n - \dfrac{4x_n y_n}{1+x_n^2}\right)$

$y_n = y_{n-1} + h\dot{y}_n = y_{n-1} + h\left(\beta x_n\left(1 - \dfrac{y_n}{1+x_n^2}\right)\right)$

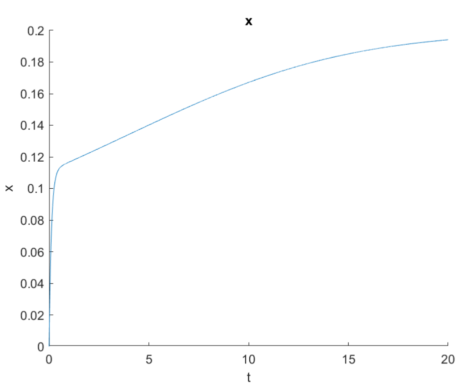
$$x_n - x_{n-1} - h\left(\alpha - x_n - \frac{4x_ny_n}{1+x_n^2}\right) = 0$$

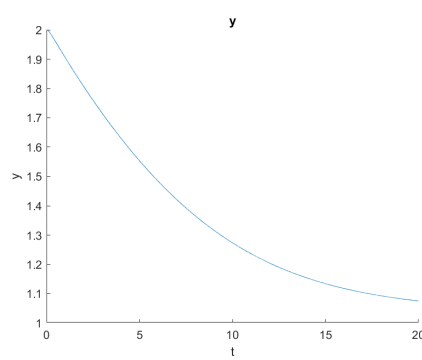
$$y_n - y_{n-1} - h\left(\beta x_n\left(1 - \frac{y_n}{1+x_n^2}\right)\right) = 0$$

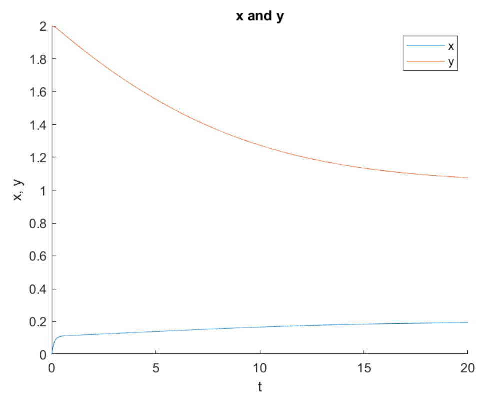
$$\Rightarrow \quad g = \begin{bmatrix} x_n(1+h) + \frac{4hx_ny_n}{1+x_n^2} - x_{n-1} - h\alpha \\ -h\beta x_n + y_n + \frac{h\beta x_ny_n}{1+x_n^2} - y_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

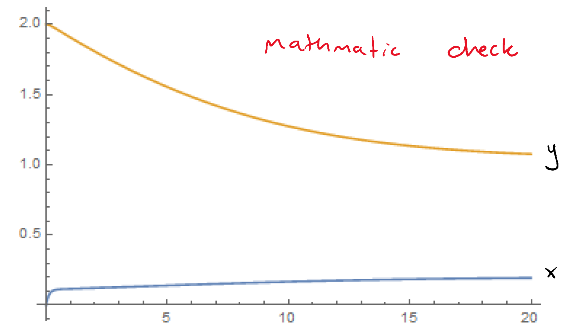
$$J = \begin{bmatrix} \frac{\partial g_1}{\partial x_n} & \frac{\partial g_1}{\partial y_n} \\ \frac{\partial g_2}{\partial x_n} & \frac{\partial g_2}{\partial y_n} \end{bmatrix} = \begin{bmatrix} 1 + h + 4hy_n\frac{1-x_n^2}{(1+x_n^2)^2} & \frac{4hx_n}{1+x_n^2} \\ -h\beta + h\beta\frac{1-x_n^2}{(1+x_n^2)^2} & 1 + \beta h\frac{x_n}{1+x_n^2} \end{bmatrix}$$

Newton raphson w/ $\Delta = J^{-1}g \quad \Rightarrow \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \Delta$









Problem 3. For this problem, you will have to perform a convergence analysis, a concept explained at the end of the problem. Consider the following IVP:

$$\dot{y} = -y^2 - \frac{1}{t^4},$$

where

$$\begin{cases} y(1) = 1 \\ t \in [1, 10] \end{cases}.$$

a) Prove that the exact solution of this IVP is $y(t) = \frac{1}{t} + \frac{1}{t^2}\tan(\frac{1}{t} + \pi - 1)$ by showing that it satisfies both the scalar ODE above and the IC specified.

b) Generate the Backward Euler convergence plot for the above IVP

c) Generate the BDF convergence plot for the above IVP. Note:

   i) Display the convergence plot in the same figure you used for the Backward Euler analysis

   ii) Use the 4th order BDF formula in this exercise

   iii) Use the exact solution above to generate the required starting points for the BDF formula

d) Measure the slope of the two plots and comment whether the two values come in line with your expectations

(a) $\dot{y} = -y^2 - \frac{1}{t^4}$

$y = \frac{1}{t} + \frac{1}{t^2} \tan\left(\frac{1}{t} + \pi - 1\right)$

$\dot{y} = -\frac{1}{t^2} - 2\frac{1}{t^3}\tan\left(\frac{1}{t} + \pi - 1\right) + \frac{1}{t^2}\sec^2\left(\frac{1}{t} + \pi - 1\right)\left(-\frac{1}{t^2}\right)$

$y^2 = \left(\frac{1}{t} + \frac{1}{t^2}\tan\left(\frac{1}{t} + \pi - 1\right)\right)^2 = \frac{1}{t^2} + \frac{2}{t^3}\tan\left(\frac{1}{t} + \pi - 1\right) + \frac{1}{t^4}\tan^2\left(\frac{1}{t} + \pi - 1\right)$

$$\cancel{-\frac{1}{t^2}} \cancel{-2\frac{1}{t^3}\tan\left(\frac{1}{t}+\pi-1\right)} - \cancel{\frac{1}{t^4}}\sec^2\left(\frac{1}{t}+\pi-1\right)$$

$$= \cancel{-\frac{1}{t^2}} \cancel{-\frac{2}{t^3}\tan\left(\frac{1}{t}+\pi-1\right)} - \cancel{\frac{1}{t^4}}\tan^2\left(\frac{1}{t}+\pi-1\right) - \cancel{\frac{1}{t^4}}$$

$$\frac{1}{t^4}\tan^2\left(\frac{1}{t}+\pi-1\right) - \frac{1}{t^4}\sec^2\left(\frac{1}{t}+\pi-1\right) = -\frac{1}{t^4}$$

$$\tan^2\left(\frac{1}{t}+\pi-1\right) - \sec^2\left(\frac{1}{t}+\pi-1\right) = -1 \quad \Rightarrow \quad -1 = -1 \quad \checkmark$$

$$\tan^2(x) - \sec^2(x) = -1 \qquad \left(\tan^2(x) - \sec^2(x) = -1 \quad \text{identity}\right)$$

$$\frac{\sin^2(x)}{\cos^2(x)} - \frac{1}{\cos^2(x)} = \frac{\sin^2(x)-1}{\cos^2(x)} = -1$$

$$\sin^2(x) - 1 = -\cos^2(x) \quad \Rightarrow \quad \sin^2(x) + \cos^2(x) = 1 \quad \checkmark$$

$$y(1) = \frac{1}{1} + \frac{1}{1^2}\tan\left(\frac{1}{1}+\pi-1\right) = 1 + \tan(\pi) = 1 \quad \checkmark$$

(b) $\dot{y} = -y^2 - \frac{1}{t^4}$

Backwards euler: $y_n = y_{n-1} + h\dot{y}_n \quad \Rightarrow \quad \dot{y}_n = \frac{y_n - y_{n-1}}{h}$

$$y_n = y_{n-1} + h\dot{y}_n = y_{n-1} + h\left(-y_n^2 - \frac{1}{t^4}\right)$$

$$g = y_n - y_{n-1} + hy_n^2 + \frac{h}{t^4} = 0$$

$$J = 1 + 2hy_n$$

p	k	β_0	α_0	α_1	α_2	α_3	α_4	α_5	α_6
1	1	1	1	-1					
2	2	2/3	1	-4/3	1/3				
3	3	6/11	1	-18/11	9/11	-2/11			
4	4	12/25	1	-48/25	36/25	-16/25	3/25		
5	5	60/137	1	-300/137	300/137	-200/137	75/137	-12/137	
6	6	60/147	1	-360/147	450/147	-400/147	225/147	-72/147	10/147

$$\sum_{i=0}^{k} \alpha_i y_{n-i} = h\beta_0 f(t_n, y_n)$$

BV4 :

		α			
4: β_0	0	1	2	3	4
$\frac{12}{25}$	1	$-\frac{48}{25}$	$\frac{36}{25}$	$-\frac{16}{25}$	$\frac{3}{25}$

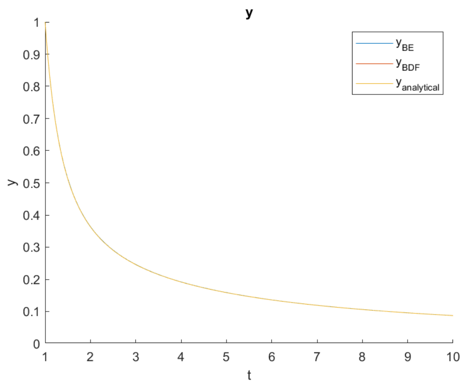
$$\sum_{i=0}^{k} \alpha_i \, y_{n-i} = h\,\beta_0 \; f(t_n, y_n)$$

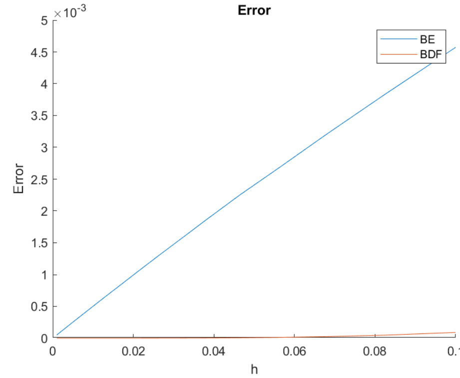
$$y_n - \frac{48}{25} y_{n-1} + \frac{36}{25} y_{n-2} - \frac{16}{25} y_{n-3} + \frac{3}{25} y_{n-4} = h \frac{12}{25} f$$

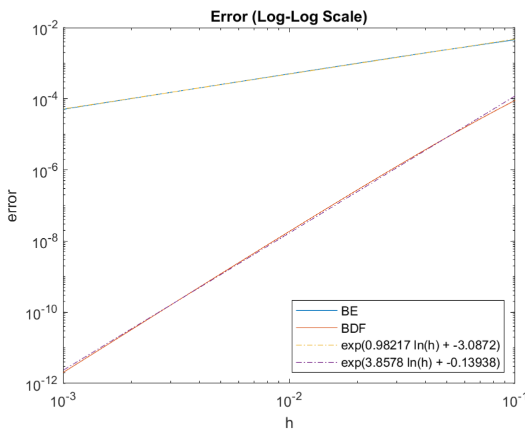
$$g = y_n - \frac{48}{25} y_{n-1} + \frac{36}{25} y_{n-2} - \frac{16}{25} y_{n-3} + \frac{3}{25} y_{n-4} - h \frac{12}{25} \left(- y_n^2 - \frac{1}{t^4} \right) = 0$$

$$= y_n - \frac{48}{25} y_{n-1} + \frac{36}{25} y_{n-2} - \frac{16}{25} y_{n-3} + \frac{3}{25} y_{n-4} + h \frac{12}{25} y_n^2 + h \frac{12}{25} \frac{1}{t^4} = 0$$

$$J = 1 + h \frac{24}{25} y_n$$







BE is first order

$\Rightarrow$ slope ≈ 1

BDF used is 4^{th} order

$\Rightarrow$ slope ≈ 4

$\Rightarrow$ this makes sense