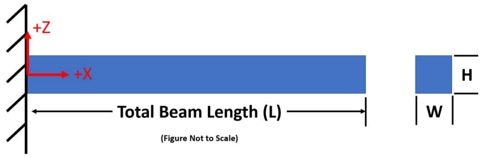


Figure 1: Cantilever Beam

For the following parts of this problem assume the following for the cantilever beam:

i) The gravitational acceleration (not shown in the picture) is assumed to act in the opposite direction of the global z axis; its magnitude is $g = 9.81\,m/s^2$.

ii) The Beam is made of steel with Young's Modulus $E = 2.0e11\,Pa$, density $\rho = 7700\,kg/m^3$, and Poisson's ratio $\nu = 0.3$.

iii) The total length of the beam is $0.5\,m$ and it has a square cross section $W = H = 0.003\,m$

iv) The shear correction factors, k_1 and $k_2 = 10\left(\frac{1+\nu}{12+11\nu}\right)$

v) The ANCF fully parameterized 2 node beam element that we used as our example during the lectures will be used to model this beam.

vi) The [3x24] shape function matrix for this beam is

$$\mathbf{S}^S(\mathbf{U}) = \begin{bmatrix} S_1^S(\mathbf{U}) & 0 & 0 \\ 0 & S_1^S(\mathbf{U}) & 0 \\ 0 & 0 & S_1^S(\mathbf{U}) \\ S_2^S(\mathbf{U}) & 0 & 0 \\ 0 & S_2^S(\mathbf{U}) & 0 \\ 0 & 0 & S_2^S(\mathbf{U}) \\ S_3^S(\mathbf{U}) & 0 & 0 \\ 0 & S_3^S(\mathbf{U}) & 0 \\ 0 & 0 & S_3^S(\mathbf{U}) \\ S_4^S(\mathbf{U}) & 0 & 0 \\ 0 & S_4^S(\mathbf{U}) & 0 \\ 0 & 0 & S_4^S(\mathbf{U}) \\ S_5^S(\mathbf{U}) & 0 & 0 \\ 0 & S_5^S(\mathbf{U}) & 0 \\ 0 & 0 & S_5^S(\mathbf{U}) \\ S_6^S(\mathbf{U}) & 0 & 0 \\ 0 & S_6^S(\mathbf{U}) & 0 \\ 0 & 0 & S_6^S(\mathbf{U}) \\ S_7^S(\mathbf{U}) & 0 & 0 \\ 0 & S_7^S(\mathbf{U}) & 0 \\ 0 & 0 & S_7^S(\mathbf{U}) \\ S_8^S(\mathbf{U}) & 0 & 0 \\ 0 & S_8^S(\mathbf{U}) & 0 \\ 0 & 0 & S_8^S(\mathbf{U}) \end{bmatrix}^T$$

where

$$S_1^S(\mathbf{U}) = \frac{2}{L^3}u^3 - \frac{3}{2L}u + \frac{1}{2}$$
$$S_2^S(\mathbf{U}) = \frac{1}{L^2}u^3 - \frac{1}{2L}u^2 - \frac{1}{4}u + \frac{L}{8}$$
$$S_3^S(\mathbf{U}) = v\left(-\frac{1}{L}u + \frac{1}{2}\right)$$
$$S_4^S(\mathbf{U}) = w\left(-\frac{1}{L}u + \frac{1}{2}\right)$$
$$S_5^S(\mathbf{U}) = -\frac{2}{L^3}u^3 + \frac{3}{2L}u + \frac{1}{2}$$
$$S_6^S(\mathbf{U}) = \frac{1}{L^2}u^3 + \frac{1}{2L}u^2 - \frac{1}{4}u - \frac{L}{8}$$
$$S_7^S(\mathbf{U}) = v\left(\frac{1}{L}u + \frac{1}{2}\right)$$
$$S_8^S(\mathbf{U}) = w\left(\frac{1}{L}u + \frac{1}{2}\right)$$

with $-\frac{L}{2} \le u \le \frac{L}{2}$, $-\frac{W}{2} \le v \le \frac{W}{2}$, and $-\frac{H}{2} \le w \le \frac{H}{2}$.

a) Take the provided shape function matrix, $\mathbf{S}^S(\mathbf{U})$, and write it in terms of normalized coordinates (ξ, η, ζ) (i.e. $\mathbf{S}^\xi$). Then take a partial derivate of the shape function matrix with respect to ξ (i.e $\mathbf{S}^\xi_{,\xi}$). Then take a partial derivate of the shape function matrix with respect to η (i.e $\mathbf{S}^\xi_{,\eta}$). Then take a partial derivate of the shape function matrix with respect to ζ (i.e $\mathbf{S}^\xi_{,\zeta}$). Write four separate functions to separately evaluate the normalized shape function matrix, $\mathbf{S}^\xi$, and the derivatives of the normalized shape function matrix ($\mathbf{S}^\xi_{,\xi}$, $\mathbf{S}^\xi_{,\eta}$, and $\mathbf{S}^\xi_{,\zeta}$) given inputs for the beam L, W, and H and the position within the beam in normalized coordinates (ξ, η, ζ). For your assignment submission for this part, report just the unique 8 terms for the analytical expression for $\mathbf{S}^\xi_{,\xi}$ that you calculated and then coded as part of your function to return $\mathbf{S}^\xi_{,\xi}$.

Do in Mathmatica for time saving and, more importantly, I can export the results straight to MATLAB format.

In[9]:= `Clear["Global`*"]`

In[10]:=
$$\text{SU} = \begin{pmatrix} \frac{2}{L^3}u^3 - \frac{3}{2L}u + \frac{1}{2} \\ \frac{1}{L^2}u^3 - \frac{1}{2L}u^2 - \frac{1}{4}u + \frac{L}{8} \\ v\left(-\frac{1}{L}u + \frac{1}{2}\right) \\ w\left(-\frac{1}{L}u + \frac{1}{2}\right) \\ -\frac{2}{L^3}u^3 + \frac{3}{2L}u + \frac{1}{2} \\ \frac{1}{L^2}u^3 + \frac{1}{2L}u^2 - \frac{1}{4}u - \frac{L}{8} \\ v\left(\frac{1}{L}u + \frac{1}{2}\right) \\ w\left(\frac{1}{L}u + \frac{1}{2}\right) \end{pmatrix};$$

In[11]:= $\text{S}\xi = \text{SU} \;/.\; \left\{u \to \frac{L}{2}\xi,\; v \to \frac{W}{2}\eta,\; w \to \frac{H}{2}\zeta\right\}$ // Simplify; $\text{S}\xi$ // MatrixForm

Out[11]//MatrixForm=
$$\begin{pmatrix} \frac{1}{4}\left(2 - 3\xi + \xi^3\right) \\ \frac{1}{8}L(-1+\xi)^2(1+\xi) \\ -\frac{1}{4}W\eta(-1+\xi) \\ -\frac{1}{4}H\zeta(-1+\xi) \\ \frac{1}{4}\left(2 + 3\xi - \xi^3\right) \\ \frac{1}{8}L(-1+\xi)(1+\xi)^2 \\ \frac{1}{4}W\eta(1+\xi) \\ \frac{1}{4}H\zeta(1+\xi) \end{pmatrix}$$

In[13]:= $\text{S}\xi\eta = \partial_\eta \text{S}\xi$ // Simplify; $\text{S}\xi\eta$ // MatrixForm

Out[13]//MatrixForm=
$$\begin{pmatrix} 0 \\ 0 \\ -\frac{1}{4}W(-1+\xi) \\ 0 \\ 0 \\ 0 \\ \frac{1}{4}W(1+\xi) \\ 0 \end{pmatrix}$$

In[14]:= $\text{S}\xi\zeta = \partial_\zeta \text{S}\xi$ // Simplify; $\text{S}\xi\zeta$ // MatrixForm

Out[14]//MatrixForm=
$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{4}H(-1+\xi) \\ 0 \\ 0 \\ 0 \\ \frac{1}{4}H(1+\xi) \end{pmatrix}$$

In[12]:= $\text{S}\xi\xi = \partial_\xi \text{S}\xi$ // Simplify; $\text{S}\xi\xi$ // MatrixForm

Out[12]//MatrixForm=
$$\begin{array}{cl} \left(\begin{array}{c} \frac{3}{4}\left(-1+\xi^2\right) \\ \frac{1}{8}L\left(-1-2\xi+3\xi^2\right) \\ -\frac{W\eta}{4} \\ -\frac{H\zeta}{4} \\ -\frac{3}{4}\left(-1+\xi^2\right) \\ \frac{1}{8}L\left(-1+2\xi+3\xi^2\right) \\ \frac{W\eta}{4} \\ \frac{H\zeta}{4} \end{array}\right) & \begin{array}{l} \leftarrow S^{\xi}_{,\xi 1} \\ \vdots \\ \\ \\ \\ \\ \\ \leftarrow S^{\xi}_{,\xi 8} \end{array} \end{array}$$

In[35]:= `ToMatlab[SU]`

Out[35]=
```
[(1/2)+(-3/2).*L.^(-1).*u+2.*L.^(-3).*u.^3;(1/8).*L+(-1/4).*u+( ...
  -1/2).*L.^(-1).*u.^2+L.^(-2).*u.^3;((1/2)+(-1).*L.^(-1).*u).*v;(( ...
  1/2)+(-1).*L.^(-1).*u).*w;(1/2)+(3/2).*L.^(-1).*u+(-2).*L.^(-3).* ...
  u.^3;(-1/8).*L+(-1/4).*u+(1/2).*L.^(-1).*u.^2+L.^(-2).*u.^3;((1/2) ...
  +L.^(-1).*u).*v;((1/2)+L.^(-1).*u).*w];
```

In[34]:= `ToMatlab[Sξ]`

Out[34]=
```
[(1/4).*(2+(-3).*xi+xi.^3);(1/8).*L.*((-1)+xi).^2.*(1+xi);(-1/4).* ...
  eta.*W.*((-1)+xi);(-1/4).*H.*((-1)+xi).*zeta;(1/4).*(2+3.*xi+(-1) ...
  .*xi.^3);(1/8).*L.*((-1)+xi).*(1+xi).^2;(1/4).*eta.*W.*(1+xi);( ...
  1/4).*H.*(1+xi).*zeta];
```

In[36]:= `ToMatlab[Sξξ]`

Out[36]=
```
[(3/4).*((-1)+xi.^2);(1/8).*L.*((-1)+(-2).*xi+3.*xi.^2);(-1/4).* ...
  eta.*W;(-1/4).*H.*zeta;(-3/4).*((-1)+xi.^2);(1/8).*L.*((-1)+2.*xi+ ...
  3.*xi.^2);(1/4).*eta.*W;(1/4).*H.*zeta];
```

In[37]:= `ToMatlab[Sξη]`

Out[37]=
```
[0;0;(-1/4).*W.*((-1)+xi);0;0;0;(1/4).*W.*(1+xi);0];
```

In[38]:= `ToMatlab[Sξζ]`

Out[38]=
```
[0;0;0;(-1/4).*H.*((-1)+xi);0;0;0;(1/4).*H.*(1+xi)];
```

b) Write a generic parameterized function to calculate the Mass Matrix, $\mathbf{M}$, for any instance of a fully parametrized 2 node beam element and use it to evaluate the Mass Matrix, $\mathbf{M}$, for an element the same dimensions as the beam shown in figure 1. Assume that the reference configuration for the beam is straight and undeformed with a node on each end and $\mathbf{r}_{,u}$, $\mathbf{r}_{,v}$, and $\mathbf{r}_{,w}$ are unit vectors aligned with the global X, Y, and Z axes respectively for each node. For your assignment submission for this part, report just the diagonal terms of the Mass Matrix in scientific notation. (Hint: $\mathbf{M}(1,1) = 1.29\text{e-}2$ and you will need to use 6 GQ points along the beam axis and 2 GQ point along each of the cross-section directions.)

M Diagonals =
1.29e-02
1.29e-02
1.29e-02
8.25e-05
8.25e-05
8.25e-05
8.66e-09
8.66e-09
8.66e-09
8.66e-09
8.66e-09
8.66e-09
1.29e-02
1.29e-02
1.29e-02
8.25e-05
8.25e-05
8.25e-05
8.66e-09
8.66e-09
8.66e-09
8.66e-09
8.66e-09
8.66e-09

$$\mathbf{M} = \int_{V_\xi} \rho\, \mathbf{S}^{\xi^{\mathsf{T}}} \mathbf{S}^{\xi} \left| \left[\mathbf{S}^{\xi}_{,\xi}\mathbf{e}_0 \quad \mathbf{S}^{\xi}_{,\eta}\mathbf{e}_0 \quad \mathbf{S}^{\xi}_{,\zeta}\mathbf{e}_0 \right] \right| dV_\xi$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \rho\, \mathbf{S}^{\xi^{\mathsf{T}}} \mathbf{S}^{\xi} \left| \left[\mathbf{S}^{\xi}_{,\xi}\mathbf{e}_0 \quad \mathbf{S}^{\xi}_{,\eta}\mathbf{e}_0 \quad \mathbf{S}^{\xi}_{,\zeta}\mathbf{e}_0 \right] \right| d\zeta\, d\eta\, d\xi$$

Where the Jacobian is

$$\mathbf{J}_{0\xi} = \frac{\partial \mathbf{S}^{\xi}}{\partial \boldsymbol{\xi}}\mathbf{e}_0 = \left[\mathbf{S}^{\xi}_{,\xi}\mathbf{e}_0 \quad \mathbf{S}^{\xi}_{,\eta}\mathbf{e}_0 \quad \mathbf{S}^{\xi}_{,\zeta}\mathbf{e}_0 \right]$$

Integration is done with Gauss Quadrature (https://pomax.github.io/bezierinfo/legendre-gauss.html)

$$\int_a^b f(x)dx = \sum_{i=1}^{\infty} w_i f(x_i) \simeq \sum_{i=1}^{n} w_i f(x_i)$$

i	weight - w_i	abscissa - x_i
1	1.0000000000000000	-0.5773502691896257
2	1.0000000000000000	0.5773502691896257

i	weight - w_i	abscissa - x_i
1	0.8888888888888888	0.0000000000000000
2	0.5555555555555556	-0.7745966692414834
3	0.5555555555555556	0.7745966692414834

i	weight - w_i	abscissa - x_i
1	0.6521451548625461	-0.3399810435848563
2	0.6521451548625461	0.3399810435848563
3	0.3478548451374538	-0.8611363115940526
4	0.3478548451374538	0.8611363115940526

i	weight - w_i	abscissa - x_i
1	0.5688888888888889	0.0000000000000000
2	0.4786286704993665	-0.5384693101056831
3	0.4786286704993665	0.5384693101056831
4	0.2369268850561891	-0.9061798459386640
5	0.2369268850561891	0.9061798459386640

i	weight - w_i	abscissa - x_i
1	0.3607615730481386	0.6612093864662645
2	0.3607615730481386	-0.6612093864662645
3	0.4679139345726910	-0.2386191860831969
4	0.4679139345726910	0.2386191860831969
5	0.1713244923791704	-0.9324695142031521
6	0.1713244923791704	0.9324695142031521

c) Write a generic parameterized function to evaluate the generalized force vector due to gravity for any instance of a fully parametrized 2 node beam element and use it to evaluate the generalized force vector due to gravity for this same element. Use the same number of GQ points as the hint for the mass matrix. For your assignment submission for this part, report the values of this entire vector.

Q_g =

0
0
-4.9050
0
0
-0.4088
0
0
0
0
0
0
0
0
-4.9050
0
0
0.4087
0
0
0
0
0
0

$$\mathbf{Q}^{Gravity} = \int_{V_\xi} \rho\, \mathbf{S}^{\xi^{\mathsf{T}}} \mathbf{F}_{Gravity} \left| \left[\mathbf{S}^{\xi}_{,\xi}\mathbf{e}_0 \quad \mathbf{S}^{\xi}_{,\eta}\mathbf{e}_0 \quad \mathbf{S}^{\xi}_{,\zeta}\mathbf{e}_0 \right] \right| dV_\xi$$

where $\mathbf{F}_{Gravity}$ is the force due to gravity per unit mass

```
F_g = [0; 0; -9.81];
M_tot = rho*L*W*H;
F_g_unitMass = F_g/M_tot;
```

With F_g = [0; 0; -9.81];

Q_g =

0
0
-0.1700
0
0
-0.0142
0
0
0
0
0
0
0
0
-0.1700
0
0
0.0142
0
0
0
0
0
0

d) Write a generic parameterized function to evaluate the generalized internal force vector for any instance of a fully parametrized 2 node beam element using a linear Hookean material with full integration of the entire stiffness matrix (5 GQ points along the beam axis and 3 along each of the cross-section directions without splitting the elasticity matrix). For your assignment submission for this part, report the generalized internal force vector for this same element when it its new nodal coordinates are $(0,0,0,\ 1,0,0,\ 0,1,0,\ 0,0,1,\ 0.5,0,0,\ 0,0,-1,\ 0,1,0,\ 1,0,0)$. (Hint the 1st term in the vector is 5.7951e5)

Q_inter =

1.0e+05 *

5.7951
0
3.3718
-0.1867
0
0.0412
0.0000
-0.1731
0.0000
-0.6392
0
0.0136
-5.7951
0
-3.3718
0.7624
0
0.0729
-0.0000
0.3461
-0.0000
-0.0554
0
0.2431

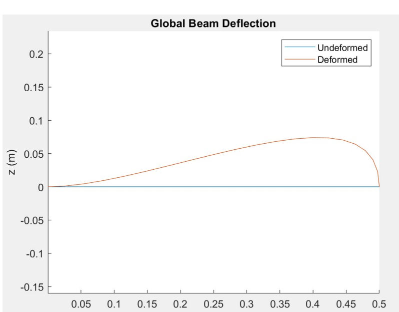
$$\begin{aligned}\mathbf{Q}^{Internal}_{Linear\ Hookean} = -\int_{V_\xi} \Bigg[&\left[\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\mathbf{S}_i^{\xi F^{\intercal}}\mathbf{S}_i^{\xi F}\mathbf{e}\right)\left(\frac{D_{ij}}{2}\left(\mathbf{e}^{\intercal}\mathbf{S}_j^{\xi F^{\intercal}}\mathbf{S}_j^{\xi F}\mathbf{e}-1\right)\right)\right] \\ &+\left(\left(\mathbf{S}_2^{\xi F^{\intercal}}\mathbf{S}_3^{\xi F}+\mathbf{S}_3^{\xi F^{\intercal}}\mathbf{S}_2^{\xi F}\right)\mathbf{e}\right)\left(D_{44}\left(\mathbf{e}^{\intercal}\mathbf{S}_2^{\xi F^{\intercal}}\mathbf{S}_3^{\xi F}\mathbf{e}\right)\right) \\ &+\left(\left(\mathbf{S}_1^{\xi F^{\intercal}}\mathbf{S}_3^{\xi F}+\mathbf{S}_3^{\xi F^{\intercal}}\mathbf{S}_1^{\xi F}\right)\mathbf{e}\right)\left(D_{55}\left(\mathbf{e}^{\intercal}\mathbf{S}_1^{\xi F^{\intercal}}\mathbf{S}_3^{\xi F}\mathbf{e}\right)\right) \\ &+\left(\left(\mathbf{S}_1^{\xi F^{\intercal}}\mathbf{S}_2^{\xi F}+\mathbf{S}_2^{\xi F^{\intercal}}\mathbf{S}_1^{\xi F}\right)\mathbf{e}\right)\left(D_{66}\left(\mathbf{e}^{\intercal}\mathbf{S}_1^{\xi F^{\intercal}}\mathbf{S}_2^{\xi F}\mathbf{e}\right)\right)\Bigg]\left(\left|\left[\mathbf{S}^\xi_{,\xi}\mathbf{e}_0\quad \mathbf{S}^\xi_{,\eta}\mathbf{e}_0\quad \mathbf{S}^\xi_{,\zeta}\mathbf{e}_0\right]\right|\right) dV_\xi\end{aligned}$$

$$\begin{aligned}\mathbf{F} &= \left[\left(J_{11}^{Inv}\mathbf{S}^\xi_{,\xi}+J_{21}^{Inv}\mathbf{S}^\xi_{,\eta}+J_{31}^{Inv}\mathbf{S}^\xi_{,\zeta}\right)\mathbf{e}\quad \left(J_{12}^{Inv}\mathbf{S}^\xi_{,\xi}+J_{22}^{Inv}\mathbf{S}^\xi_{,\eta}+J_{32}^{Inv}\mathbf{S}^\xi_{,\zeta}\right)\mathbf{e}\quad \left(J_{13}^{Inv}\mathbf{S}^\xi_{,\xi}+J_{23}^{Inv}\mathbf{S}^\xi_{,\eta}+J_{33}^{Inv}\mathbf{S}^\xi_{,\zeta}\right)\mathbf{e}\right] \\ &= \left[\mathbf{S}_1^{\xi F}\mathbf{e}\quad \mathbf{S}_2^{\xi F}\mathbf{e}\quad \mathbf{S}_3^{\xi F}\mathbf{e}\right]\end{aligned}$$

We'll use this notation for the combined shape function derivatives which account for a potentially deformed reference state

$$\mathbf{D} = \left(\frac{E\nu}{(1+\nu)(1-2\nu)}\right)\begin{bmatrix} \frac{1-\nu}{\nu} & 1 & 1 & 0 & 0 & 0 \\ 1 & \frac{1-\nu}{\nu} & 1 & 0 & 0 & 0 \\ 1 & 1 & \frac{1-\nu}{\nu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2\nu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2\nu}k_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2\nu}k_3 \end{bmatrix}$$

e) Write a generic parameterized function to calculate the global position of a point in the beam element given the beam's nodal coordinates and the normalized coordinates (ξ, η, ζ) of the particular point in the beam. Use this function to plot the entire beam axis (i.e ξ = -1 to 1) for the element in the same state as the general internal force vector was calculated for. Scale the plot so that the size of the x and z axis are the same size (i.e "axis equal" in MATLAB). Turn in this plot.



$r = S^\xi(\xi, \eta, \zeta)\, e$

f) Write a generic parameterized function to calculate the generalized force vector due to an external point force acting on the free tip of the beam on the beam axis for this same element. Evaluate this function to report the generalized external tip force vector using an applied tip force vector in global coordinates of $(10cos\,(45°)\,, 0, 10sin\,(45°))$N and turn in the values of this vector.

```
Q_point =

     0
     0
     0
     0
     0
     0
     0
     0
     0
     0
     0
     0
7.0711
     0
7.0711
     0
     0
     0
     0
     0
     0
     0
     0
     0
```

$$\mathbf{Q}^{External\ Point\ Force} = \sum_{n} \left(\left[\mathbf{S}^{\xi} \left(\boldsymbol{\xi}^{n}_{Application\ Point} \right) \right]^{\mathsf{T}} \mathbf{f}^{n}_{Applied} \right)$$

```
F_app =[10*cosd(45); 0; 10*sind(45)];
```

g) Turn in a copy of your code that calculate the items above along with a document that lists all of the items to be reported as stated in the items above.

h) Optional: Since this is the first time this assignment relatively new, please describe what aspects, if any, of this homework were helpful in clarifying the material from the lectures. Did you learn anything new from this assignment? Did you struggle with any aspects of this assignment or was it fairly straight forward? Any other feedback.

Some of the assignment was straightforward and helped but a few points and definitions were not as clear in the notes/lecture which gave some troubles. I.e., is the applied force mass normalized like the F_gravity term and are these supposed to be in the global frame or local, etc.