ME 614: Homework 2

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22/02/2017

1 Problem 1

In this problem, the irrational number π is multiplied with itself n number of times and the resultant product is divided n number of times by π. In case of infinite precision, we would expect the final product to be 1. However, due to finite arithmetic, we are left with a residual, which arises due to the truncation of digits in π. The following plot shows the variation of the absolute truncation error with n. For comparison, 3 float data types, with 16, 32 and 64 bit precision have been taken.

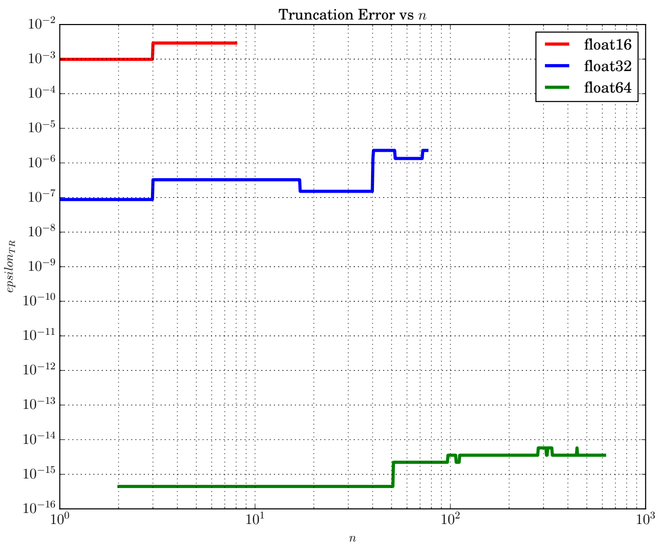

Figure 1: Absolute Truncation Error vs No. of multiplications

As seen from the plot, the truncation error naturally decreases with increasing order of precision. However, it can also be seen that the for a given precision, the truncation error increases with increasing n, until it goes out of bounds (as seen from the truncation of each line). This is a result of the truncation error propagating inward due to repeated truncations and round-off with each multiplication. It should be noted that for float64, the initial truncation errors are zero (within a float64 precision) and are not shown on the log-log plot.

2 Problem 2

2.1 Part a

Consider:
$u(x,t) = c_1 e^{-\omega_1^2 \alpha t} sin(\omega_1(x-ct)-\gamma_1) - c_2 e^{-\omega_2^2 \alpha t} cos(\omega_2(x-ct)-\gamma_2)$
Hence, the temporal and spatial derivatives (1^{st} and 2^{nd} order are given by-
$\frac{\partial u}{\partial t} = -\omega_1^2 \alpha c_1 e^{-\omega_1^2 \alpha t} sin(\omega_1(x-ct)-\gamma_1) + \omega_2^2 \alpha c_2 e^{-\omega_2^2 \alpha t} cos(\omega_2(x-ct)-\gamma_2) - c_1 c \omega_1 e^{-\omega_1^2 \alpha t} cos(\omega_1(x-ct)-\gamma_1) - c_2 c \omega_2 e^{-\omega_2^2 \alpha t} sin(\omega_2(x-ct)-\gamma_2)$
$\frac{\partial u}{\partial x} = c_1 \omega_1 e^{-\omega_1^2 \alpha t} cos(\omega_1(x-ct)-\gamma_1) + c_2 \omega_2 e^{-\omega_2^2 \alpha t} sin(\omega_2(x-ct)-\gamma_2)$
$\frac{\partial^2 u}{\partial x^2} = -c_1 \omega_1^2 e^{-\omega_1^2 \alpha t} sin(\omega_1(x-ct)-\gamma_1) + c_2 \omega_2^2 e^{-\omega_2^2 \alpha t} cos(\omega_2(x-ct)-\gamma_2)$
Putting this in the advection-diffusion equation, gives LHS=0

Hence, it has been verified that the given analytical solution indeed satisfies the partial differential equation.

2.2 Part b

The system was solved for using the initial conditions-
$u(x,0) = c_1 sin(\omega_1 x - \gamma_1) - c_2 cos(\omega_2 x - \gamma_2)$
using $c_1 = 2, c_2 = 2, m = 2, \gamma_1 = 2.0, \gamma_2 = 2.0, c = 2, \alpha = 0.5$.

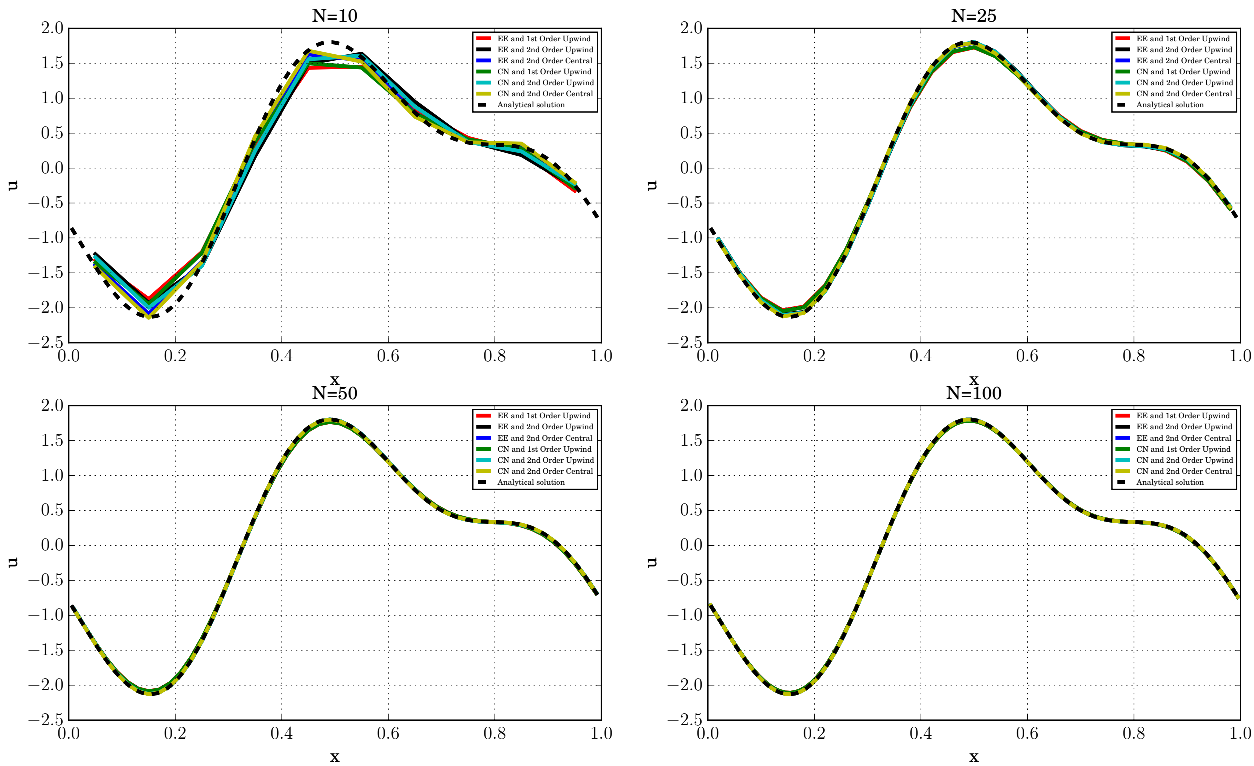

Figure 2: Solution at $t = T_f$

As seen from the above figure, the best scheme is the Crank-Nicolson time advancement with 2^{nd} order central for convection (Note that diffusion is always 2^{nd} order central) and the worst one is Explicit Euler with first order upwinding for convection (For a sufficiently fine grid, all schemes nearly collapse on to the analytical curve and this statement has been made based on how accurate each scheme is for the sparsest grid of 10 points)

2.3 Part c

The variation of the absolute truncation error for a fixed $dt = 10^{-5}$ and the scaled absolute truncation error (scaled with total number of iterations) for a fixed $dx = 10^{-2}$ was plotted:

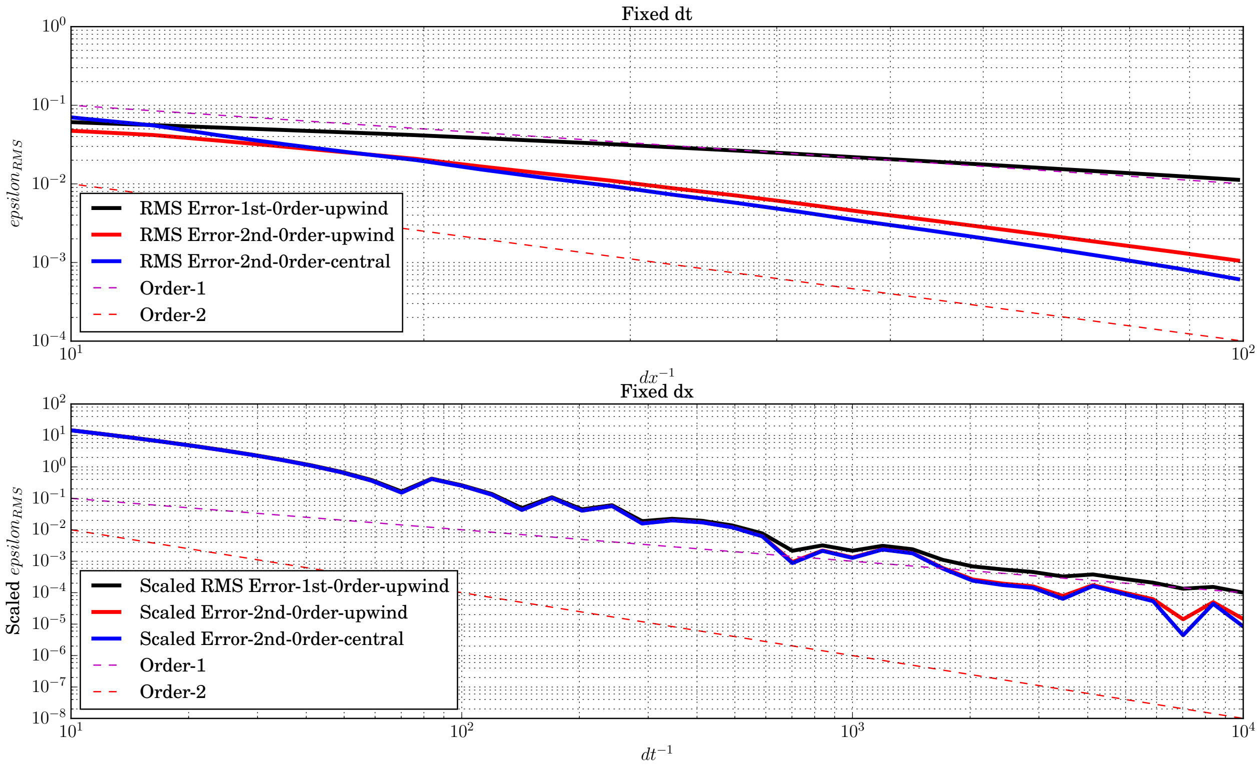

Figure 3: Variation of truncation error with dt and dx keeping the other parameter fixed

It can be seen that the order of accuracy of the schemes is more or less, as expected. A slight decrease in order of accuracy can be observed for the 2^{nd} order upwind method i.e. it does not strictly decay with second order accuracy.

2.4 Part d

The following figure shows the spy plots for the matrices A and B for N=10 points:

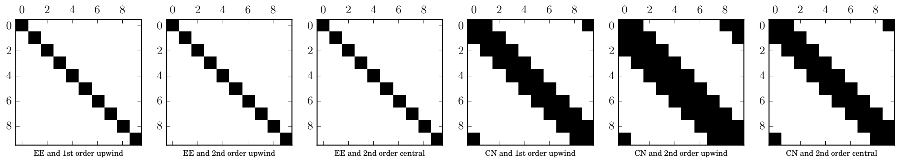

Figure 4: Spy plots of A for various schemes and N=10

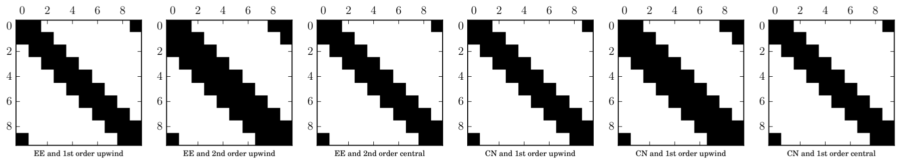

Figure 5: Spy plots of B for various schemes and N=10

2.5 Part e

The eigenvalues of the transition matrix $T = A^{-1}B$ strongly dictate the stability of the numerical scheme. Depending on the spectral radius (magnitude of the largest eigenvalue of T), one can see that the scheme is stable for C_c and C_α values within the filled part of the contours.

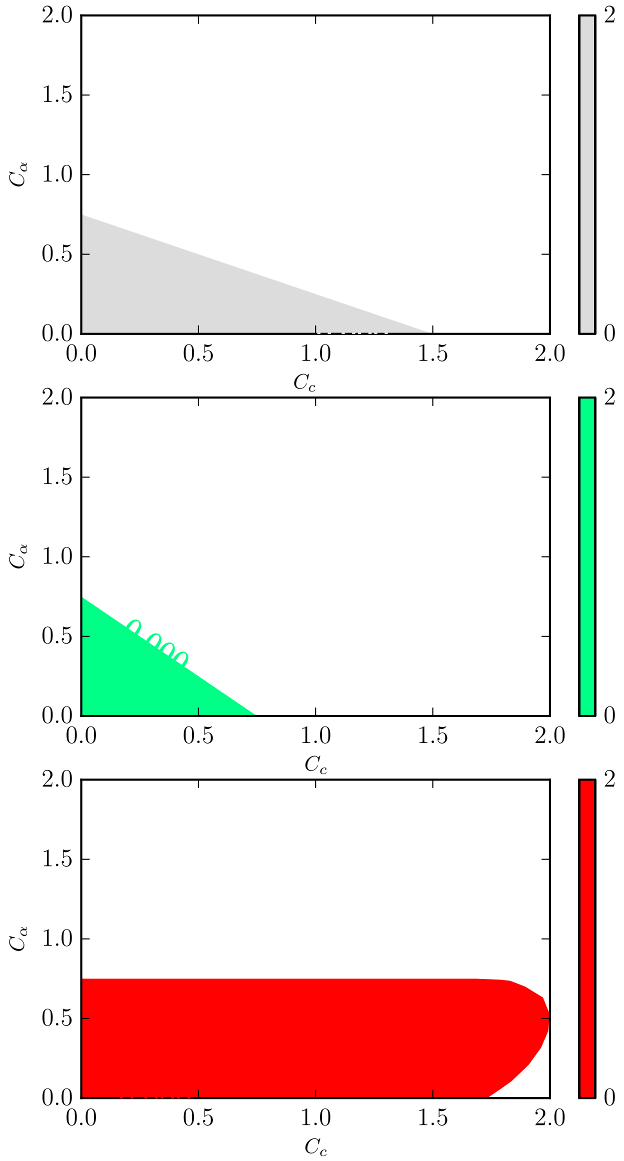

Figure 6: Spectral radius for (from top to bottom)- 1^{st} order upwind, 2^{nd} order upwind and 2^{nd} order central

3 Problem 3

3.1 Part a

In this problem, it can be seen (by running the commented code block in part a of problem 3 in the main code file) that the parameters a, c, ω, α, β have the following general effect on the transient solution:

c : Convects boundary effects into the domain.
ω : Affects frequency of oscillations of the solution.
a : Affects the magnitude of the oscillations.
α : Diffuses/smooths out the solution curves.
β : Dampens point-to-point oscillations in the solution.

The transient solution for the following parameters has been plotted in the figure below-

L = 10, Np = 100, c = 2.0, α = 1.0, β = 50, dt = 0.005, a = 1.0, ω = 20.0, T_f = 10.0

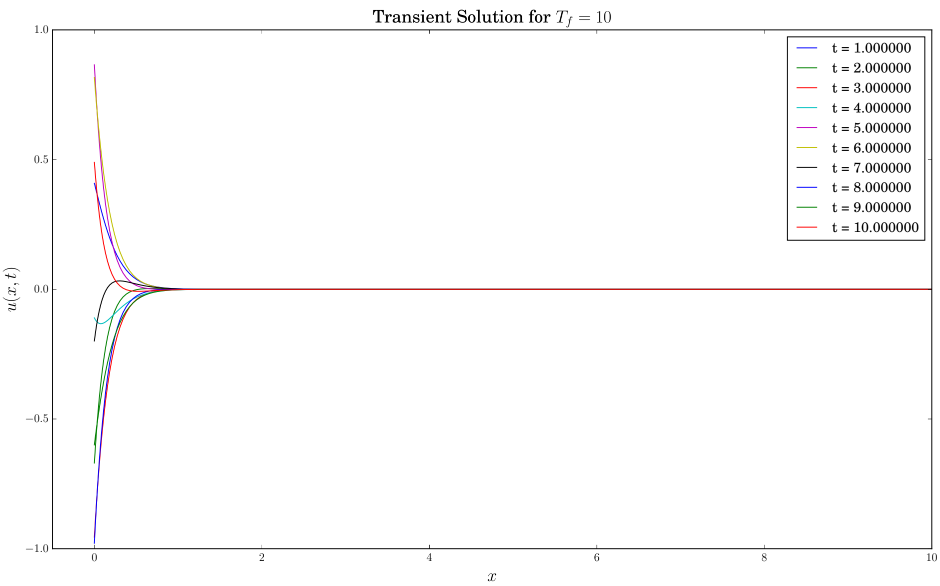

Figure 7: Transient Solutions for T_f = 10 plotted at an interval of 1 each

3.2 Part b

Here,
δ = f(a,α,β,c,ω)
Note that L is not considered as it doesn't matter as long as β is strong enough.
From the Buckingham-Pi theorem, the pi groups are:

$$\pi_1 = \frac{\delta\omega}{c}$$

,

$$\pi_2 = \frac{a\omega}{c}$$

$$\pi_3 = \frac{\alpha\omega}{c^2}$$

,

$$\pi_4 = \frac{\beta}{\omega}$$

As π_3 and π_4 are the significant groups, the effect of their variation (by varying α and β) is studied. The following plots show the variation of the π_1 with π_3 and π_4 keeping the other groups constant.

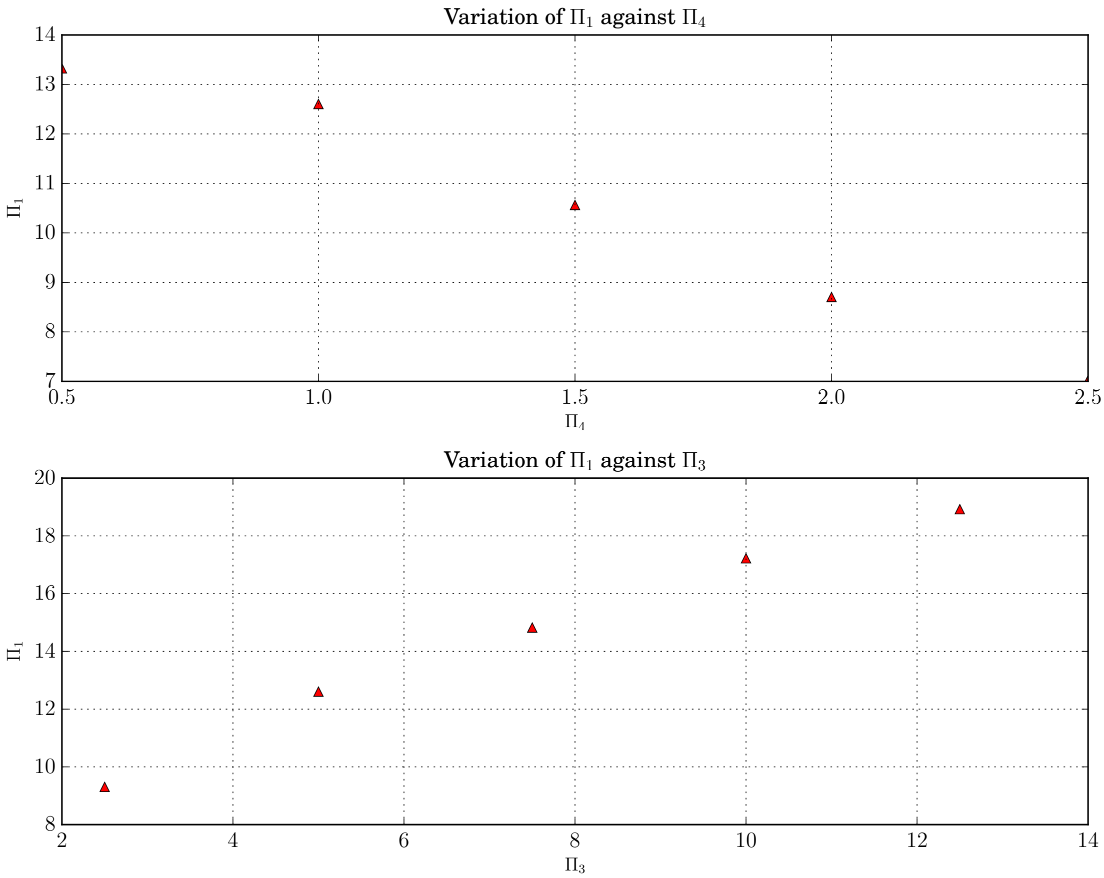

Figure 8: Variation π_1 with groups π_3 and π_4

As expected, π_1 increases with increase in π_3 and decreases with increase in π_4