Sesqui/Bi-Linear forms

1 Suppose X is right $\mathbb{K}$-module so that $X^* = \overline{X}^{tr}(\cong \overline{X^{tr}}$ is a right $\mathbb{K}$-module again. Let $X \xrightarrow{S_{right}} X^*$ be a right $\mathbb{K}$-linear transformation. Thus to each $x \in X$ we associate a left $\mathbb{K}$-linear transformation $\overline{X} \xrightarrow{s_{right}(x)} \mathbb{K}$. Recording $(S_{right}(x))(\overline{u}) \in \mathbb{K}$ as $< \overline{u} \mid S_{right}(x) >$ in the usual manner at each $\overline{u} \in \overline{X}$, we obtain $< \overline{z} \mid S_{right}(x + u\lambda) >=< \overline{z} \mid S_{right}(x) > + < \overline{z} \mid S_{right}(u) > \lambda$ ...(1) ($\because S_{right}$ *is right* $\mathbb{K}$*-linear)* at each $\overline{z} = \overline{X}$, $x, u \in X$, $\lambda \in \mathbb{K}$ and $< \overline{z + u\lambda} \mid S_{right}(x) >=< \overline{\lambda}\overline{u} + \overline{z} \mid S_{right}(x) >= \overline{\lambda} < \overline{u} \mid S_{right}(x) > + < \overline{z} \mid S_{right}(x) >$ ($\because S_{right}(x)$ *is left* $\mathbb{K}$*-linear)* ...(2) at each $\overline{u}$, $\overline{z} \in \overline{X}$, $x \in X$, $\lambda \in \mathbb{K}$. We now record $(S_{right}(x))(\overline{u}) \in \mathbb{K}$ as $S(x, u)$. This provides a function $X \times X \xrightarrow{s} \mathbb{K}$ which satisfies $S(x + u\lambda, z) = S(x + u\lambda, z) = S((x, z) + (su, z)\lambda$, and $S(x, z + u\lambda) = s(x, z) + \overline{\lambda}s(x, u)$ at $x, u, z \in X$, $\lambda \in \mathbb{K}$, when we read (1) and (2) in this notation. This will be written into a set of three equations: $S(x + u, z) = s(x, z) + s(x, z)$, $S(x, u + z) = s(x, u) + s(x, z)$ $s(x\lambda, u\mu) = \overline{\mu}s(x, u)\lambda$ .....(3)
at each x, u, $z \in X$, μ, $\lambda \in \mathbb{K}$ and made into a definition.

Definition 0.1. Let X be a right $\mathbb{K}$-module. A sesquilinear form on X is a function $X \times X \xrightarrow{s} \mathbb{K}$ which obeys the set of equations described in (3).

(”sesqui” means ”one and a half”; s is sesquilinear because it is ’linear in the first variable’ and conjugate-linear in the second variable. Since ’conjugate-linear’ is also called ’semi-linear’, this totals into ’one and a half linear’).

2. The preceding definition which says ’sesquilinear means linear in the first variable, and conjugate-linear in the second variable’ is the ’mathematician’s definition of sesquilinearity’. But we could have recorded $(s_{rihgt}(x))(\overline{u}) =< \overline{u} \mid S_right(x) >$ as $s(u,)$ rather than $s(x, u)$. Then (1) and (2) would read $s(z, x + u\lambda) = s(z, x) + s(z, u)\lambda$ and $s(z + u\lambda, x) = s(z, x) + \overline{\lambda}s(u, x)$ we would have then summarized them into $s(x + u, z) = s(x, z) + s(x, z)$, $s(x, u + z) = s(x, u) + s(x, z)$, $s(x\lambda, u\mu) = \overline{\lambda}s(x, u)\mu$ at $x, u, z \in X$, $\mu, \lambda \in \mathbb{K}$ ...(3)
Then ’sesquilinearity’ would still be ’one and a-half-linearity’ but it would be ’conjugate-linear in the

first variable, and linear in the second variable'. This is the 'physicists' definition of sesquilinearity'. Clearly, the two definitions are 'conjugate to each other': the physicists $s(u,x)$ is simply $\overline{s(u,x)}$ of 1. We note that $s(0,x) = s(0+0,x) = s(0,x) + s(0,x)$ so that $s(0,x) = 0$; similarly, $s(x,0) = 0$.

3. Now suppose $X \times X \xrightarrow{s} \mathbb{K}$ is a sesquilinear from. Define $\overline{X} \xrightarrow{right_\sigma(x)} X$ by writing $(right_\sigma(x))(\overline{U}) :=$ $s(x,u)$ at each $\overline{u} \in \overline{x}$ for each x. Then $(right_s(x))(\overline{\lambda}\overline{u}+) = s(x, z+u\lambda) = s(x,z) + \overline{\lambda}s(x,u) =$ $\overline{\lambda}(right_s(x))(\overline{u}) + (right_s(x))(\overline{z})$ which shows that $right_s(x)$ is a left $\mathbb{K}$-linear from on $\overline{x}$ i.e. $rihgt_s(x) \in \overline{X}^{tr} = X^*$ and provides $X \xrightarrow{right_s} X^*$ with $right_s(x) = s(x,u)$. Further, $(right_s(x+u\lambda))(\overline{z}) = s(x+u\lambda, z) = s(x,z) + s(u\lambda, z) = s(x,z) + s(u,z)\lambda(\because S$ is sesquilinear thus linear in the first variable)$= (right_s(x) + right_s(u)\lambda)(\overline{z})$ at each $\overline{z} \in \overline{X}$. This holds at each $\overline{z} \in \overline{X}$ so $X \xrightarrow{right_s} X^*$ is right $\mathbb{K}$-linear. This raises then a sesquilinear form s' defined by $s'(x,u) :=< \overline{u} \mid right_s(x) >= (rihgt_s(x))(\overline{u}) = s(x,u)$ at each $x, u \in X$. In other words $s' = s$. So when $X \times X \xrightarrow{s} \mathbb{K}$ is a given sesquilinear form, the $X \xrightarrow{right_s} X^*$ raised by s again raises s as its sesquilinear form. Conversely, if $X \xrightarrow{s_{right}} X^*$ is a given right $\mathbb{K}$-linear transformation, and thus raises the sesquilinear form $s(x,u) = (s_{right}(x))(\overline{u}) =< \overline{u} \mid s_{right}(x) >$, this s raises in its turn, $X \xrightarrow{right_s} X^*$ which computes as $(right_s(x))(\overline{u}) =< \overline{u} \mid right_s(x) >= s(x,u)$, and thus we conclude:

The correspondence $s \leftrightarrow s_{rihgt} = rihgt_s$ between the sesquilinear forms on X and the right $\mathbb{K}$-linear transformation $X \to X^*$ is unambiguously settled.

4. The calculation was done in 3 above for the 'mathematician's sesquilinear forms'. Since the correspondence between the 'physicists sesquilinear forms' and the 'mathematician's sesquilinear form' is also unambiguous, we conclude

A sesquilinear form $X \times X \xrightarrow{s} \mathbb{K}$ *(whether conjugate-linear in the first variable or in the second variable)* is equally well-recorded as a right $\mathbb{K}$-linear transformation $X \to X^*$.

5. If we take the trivial involution *(so that the ring $\mathbb{K}$ is in particular commutative)* the requirements (3) and ($\overline{3}$) collapse into $\beta(x+u,z) = \beta(x,z) + \beta(u,z)$, $\beta(x,u+z) = \beta(x,u) + \beta(x,z)$, $\beta(x\lambda, u\mu) = \lambda\beta(x,u)\mu = \mu\beta(x,u)\lambda = \lambda\mu\beta(x,u) = \beta(x,u)\mu\lambda$ ...(4) and we say that β is a bi-linear form on X.*(Of course, 'the physicists bilinear form' is the same as 'the mathematician's bilinear form).* Some authors choose to define bilinear forms as follows: Let Y be a left $\mathbb{K}$-module, X be a right $\mathbb{K}$-module and define a bilinear form $Y \times X \xrightarrow{\beta} \mathbb{K}$ by $\beta(y+v,x) = \beta(y,x) + \beta(v,x)$, $\beta(y, x+u) = \beta(y,x) + \beta(y,u)$ $\beta(\lambda y, x\mu) = \lambda\beta(y,z)\mu$ ...(4)

Then 'sesqui-linear forms' are derived (in the physicist's version) by taking Y to be $\overline{X}$ and writing $x \in X$ for $\overline{x} \in \overline{x}$.*(Taking* $Y = X^{tr}$ *the 'Dirac bracket'* $X^{tr} \times X \xrightarrow{<-1->} \mathbb{K}$ *which takes* (ϕ, x) *to* $< \phi \mid x >$ *is another example.)* We shall stick to our terminology taking the phisicist's version as standard.

(i) A sesquilinear form on a right $\mathbb{K}$-module X is a function $X \times X \xrightarrow{s} \mathbb{K}$ which is conjugate-linear in the first variable and linear in the second variable so that $s(x\lambda + u, z + w\mu) = \overline{\lambda}s(x, z) + s(u, z) + s(u, w)\mu + \overline{\lambda}s(x, w)\mu$, for each $x, u, z, w \in X,\ \lambda, \mu \in \mathbb{K}$ (SL)

(ii) When $\mathbb{K}$ is commutative (so that the adjective 'right' is irrelevant) the requirement (SL) can be written as $s(\lambda x + u, z + w\mu) = \overline{\lambda}s(x, z) + s(u, z) + s(u, w)\mu + \overline{\lambda}s(x, w)\mu$

(iii) When $\mathbb{K}$ is commutative and the involution is taken to be the trivial involution, we shall say 'bi-linear' in place of 'sesqui-linear' adn use the letter β; the requirement SL is then $s(\lambda x + u, z + w\mu) = \lambda s(x, z) + s(u, z) + s(u, w)\mu + \lambda s(x, w)\mu$

The all-important case $\mathbb{K} = \mathbb{C}$ will be covered by (ii) and (iii); complex vector space (=modules over $\mathbb{C}$) have both sesquilinear and bilinear forms and both will be needed. The case $\mathbb{K} = \mathbb{H}$*(the quaternion division ring)* will be covered by (i), though as the course develops, its importance will diminish.