School of Basic Sciences
I. I. T. MANDI

29^{th} April., 2014 8:00-8:50 a.m.

IC121 : Mechanics of Particles and Waves
Quiz II (20 marks)

Name:______ **Roll No:**______

Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9
2	2	3	2	1	2	1	2	1

Q10	Q11	Total
2	2	20

SUGGESTED SOLUTIONS

Instructions

- Fill the roll number and name before leaving the examination hall.
- Write your answers with proper and brief explanation.
- Write answers in the space provided after each question, if extra space is required you may use extra sheets, specify your roll number and name on each additional sheets also number each additional sheets.
- If additional sheets are used for answering mention the page number on which answer is written below the question.
- Make sure that 11 questions are printed.

Questions

1. Show that Taylor series expansion near the minimum of a potential, $V(x)$ at $x = 0$ can be approximated by potential of a harmonic oscillator, use $V(0) = 0$. Show that this relation is exact for a potential $U(u) = \frac{1}{2}a(u - u_0)^2$ where u_0 and a are constants. (2 marks)

Soln: Let $V(x)$ be any arbitrary potential

Expanding $V(x)$ in Taylor Series

$$V(x) = V(x_0) + \frac{dV}{dx}(x-x_0) + \frac{1}{2}\frac{d^2V}{dx^2}(x-x_0)^2 + \frac{1}{6}\frac{d^3V}{dx^3}(x-x_0)^3 + \cdots$$

for any point very near to minimum of the potential, we can neglect the higher order terms

So,

$$V(x) = V(x_0) + \frac{dV}{dx}(x-x_0) + \frac{1}{2}\frac{d^2V}{dx^2}(x-x_0)^2$$

Given $V(x_0) = 0$

and at minimum $\frac{dV}{dx} = 0$.,

Therefore,

$$V(x) = \frac{1}{2}\frac{d^2V}{dx^2}(x-x_0)^2$$

$$= \frac{1}{2}k(x-x_0)^2 = \frac{1}{2}kx^2\Big|_{x_0=0}$$

Now take the case of the potential

$$U(u) = \frac{1}{2}a(u-u_0)^2$$

$$U(u)\Big|_{u=u_0} = 0.$$

$$\frac{dU}{du}\Big|_{u=u_0} = a(u-u_0)\Big|_{u=u_0} = 0$$

$$\frac{d^2U}{du^2} = a\Big|_{u=u_0}.$$

$$\therefore \frac{d^nU}{du^n} = 0 \qquad n \geq 0.$$

$$\therefore U(u) = \frac{1}{2}a(u-u_0)^2$$

2. A certain oscillator satisfy the equation $\ddot{x} + 4x = 0$. Initially the particle is at the point $x = \sqrt{3}$ then it is projected toward the origin with speed 2. Show that the subsequent motion is given by the equation $x = \sqrt{3}\cos 2t - \sin 2t$. (2 marks)

Given, the eqn of the oscillator $\ddot{x} + 4x = 0$.

$\Rightarrow D^2 + 4 = 0 \Rightarrow D = \pm 2i$

The general solution

$x = Ae^{2it} + Be^{-2it}$

Initially : at $t = 0 \Rightarrow x = \sqrt{3}$ and $\frac{dx}{dt} = -2$

$\frac{dx}{dt} = 2iAe^{2it} - 2iBe^{-2it}$

$-2 = 2i(A - B)$

$\therefore A - B = i$

& $\sqrt{3} = A + B$.

$\Rightarrow A = \frac{\sqrt{3} + i}{2}$ $\quad B = \frac{\sqrt{3} - i}{2}$.

$\therefore$ the solution is

$x = Ae^{2it} + Be^{-2it} \Rightarrow A(\cos 2t + i\sin 2t) + B(\cos 2t - i\sin 2t)$

$= \cos 2t\,(A + B) + i\sin 2t\,(A - B)$

$x = \sqrt{3}\cos 2t - \sin 2t$

3. What is the molecular origin of frictional forces? When a particle of mass m moving subjected to frictional forces, write down the equation of motion using Newton's equation of motion. Show that kinetic energy of this particle decays exponentially. (3 marks)

Frictional forces are due to transfer of conserved quantities such as energy to external degrees of freedom interacting with the system.

Frictional forces are represented by a term that is proportional to velocity.

$$m\frac{d^2x}{dt^2} = F_{fric} = -b\frac{dx}{dt}$$

$$\therefore\ b = \frac{\text{force}}{\text{velocity}}$$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} = 0 \qquad \tau = \frac{m}{b}$$

$$\therefore\ \frac{d^2x}{dt^2} + \frac{1}{\tau}\frac{dx}{dt} = 0 \Rightarrow \frac{dv}{dt} + \frac{1}{\tau}v = 0$$

$$\int\frac{dv}{v} = -\int\frac{1}{\tau}dt$$

$$\log v = -t/\tau + c \quad : \text{at } t=0\ ;\ v = v_0$$

So $\log v_0 = c$

$$\log\frac{v}{v_0} = -t/\tau$$

$$v = v_0 e^{-t/\tau}$$

Kinetic energy $= \frac{1}{2}mv^2 = \frac{1}{2}mv_0 e^{-2t/\tau}$

4. What are the unit vectors of the generalized vector space in terms of which the Fourier expansion of a periodic function with periodicity 2π can be done. Prove the orthogonality relation between the unit vectors (show steps). (2 marks)

In complex representation

$$\hat{e}_n = \{e^{inx}\} \qquad n = -\alpha \text{ to } +\alpha.$$

$$\hat{e}_m\cdot\hat{e}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx}e^{imx}\,dx$$

at $m = n$.

$$\hat{e}_m \cdot \hat{e}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx = 1.$$

at $m \neq n$

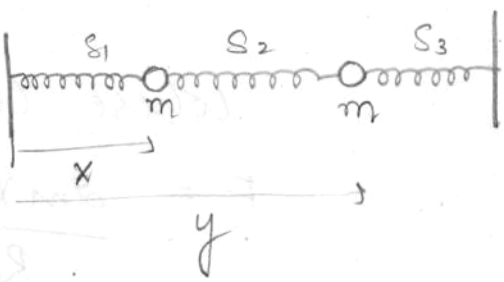
$$\hat{e}_m \cdot \hat{e}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)} dx.$$

$$= \frac{1}{2\pi} \left[\frac{e^{i(m-m)}}{n-m} \right]_{-\pi}^{+\pi} = 0.$$

$$\therefore \quad e^{in\pi} = -1 \qquad n = \pm \text{integer}$$

5. Consider two masses connected to each other and two walls by springs in one dimension, let the position of the springs be x and y from one of the walls. Also, let these springs have the same spring constant k. Find the coupled differential equation for motion of springs. From the geometry draw the schematic diagram of normal modes of this oscillator (with out solving the equation of motion). (1 mark)

Coordinates of two masses are x and y.



Displacement $S_1 = x$

$S_2 = y - x$

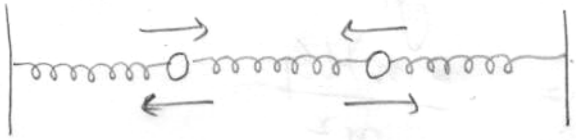
$S_3 = -y$

eqns of motion

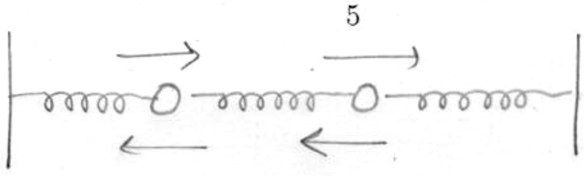
$$m\ddot{x} = -kx + k(y-x)$$

$$m\ddot{y} = -k(y-x) - ky$$

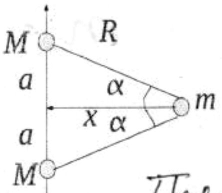
MODE I



MODE II



6. A test mass m experiences gravitational forces from two symmetrically placed masses M each, in the geometry given in the figure. The distance between the masses is $2a$. The test mass is placed at a distance x. Find the expression of magnitude and direction of force on the test particle. Show that for large x this expression reduced to the usual inverse square law form. (2 marks)



$$F = \frac{mMG}{R^2}$$

The components of force towards the center of mass exist.

$$F = \frac{mMG\cos\alpha}{R^2} + \frac{mMG\cos\alpha}{R^2}$$

where $\cos\alpha = \frac{x}{R}$

$$F = \frac{2mMGx}{R^3}$$

$$= \frac{2mMG\,x}{(a^2+x^2)^{3/2}} = \frac{2mMG}{x^2\left(1+\frac{a^2}{x^2}\right)^{3/2}}$$

for $a << x$.

$$\left(1+\frac{a^2}{x^2}\right) \simeq 1$$

So, at large distance

$$F = \frac{2mMG}{x^2}$$

7. Evaluate the integrals that involve delta functions (a) $\int_0^3 x^3\delta(x+1)dx$, (b) $\int_0^5 \cos x\delta(x-\pi)dx$ (1 mark)

(a) $\int_0^3 x^3\,\delta(x+1)\,dx = 0$ $\because$ $x=-1$ is not in the limit

(b) $= \int_0^5 \cos x\,\delta(x-\pi)\,dx = \cos(\pi) = -1$

8. Show for an inverse square law force $\vec{v} = \frac{\hat{r}}{r^2}$ the total divergence of the force produced at the origin, $r = 0$, is 4π (2 marks)

Given $\vec{v} = \frac{\hat{r}}{r^2}$

$\nabla v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \times \frac{1}{r^2}\right)\Big|_{r=0}$ is undefined

By Gauss' Divergence Theorem

$$\int_s \vec{v}\cdot d\sigma = \int_v \nabla\cdot\vec{v}\, d\tau$$

$$= \iint \frac{\hat{r}}{r^2}\, r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

$$= \int_0^\pi \sin\theta\, d\theta \cdot \int_0^{2\pi} d\phi = -\cos\theta\Big|_0^\pi\, \phi\Big|_0^{2\pi}$$

$$= 4\pi$$

9. What is a configuration space, for a system of 1000 particles in the three dimensional space, what is the dimension of a point in the configuration space of this system (1 mark)

Configuration space is a vector space of defined parameters that define a system (collection of particles).

The dimension of configuration space for 1000 particles is 3000 (3N) in 3-dimensions.

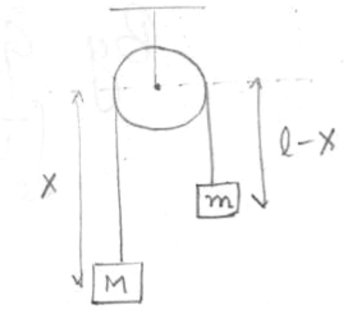
10. Write down the differences and similarities between Lagrangian and Newtonian approach to mechanics. (2 marks)

1. Lagrangian: Energy, work done, scalar, general coordinates, considers whole system

Newtonian: Vector method, force on each particle, Cartesian coordinate system

Both give same equations of motion

11. Give the expression of Lagrangian for an Atwood's machine, where two masses M and m are hanging at both ends of a massless string and pulley, find the equation of motion using the Lagrange's method. (2 marks)



$L = T - V$

Lagrange's equation of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$V = -Mgx - mg(l-x)$$

$$T = \frac{1}{2}(M+m)\dot{x}^2$$

$$L = T - V = \frac{1}{2}(M+m)\dot{x}^2 + Mgx + mg(l-x)$$

$$\frac{\partial L}{\partial x} = (M-m)g$$

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$(M+m)\ddot{x} - (M-m)g = 0.$$

$$\Rightarrow \quad \ddot{x} = \frac{(M-m)}{(M+m)}g$$