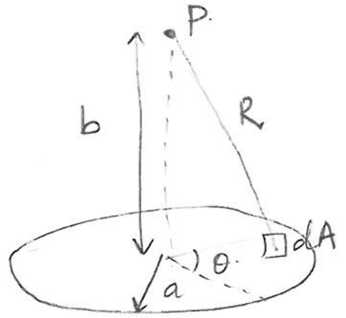
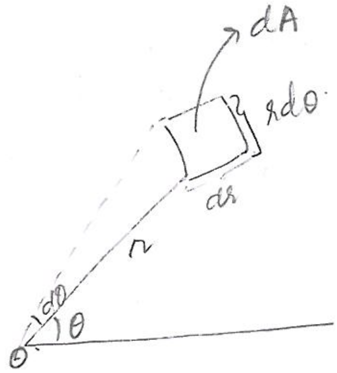
IC121: Mechanics of Particles and Waves
Tutorial - 4 Solutions

1. A particle of mass m is situated on the disc's axis which is uniform in mass and radius a. Find the gravitational force that the disc exerts on the particle.



Soln: Consider the element of area dA of the disc, which has mass $\frac{M}{\pi a^2} dA$ and attracts P with a force of magnitude

$$\vec{F} = \frac{m\left(\frac{M dA}{\pi a^2}\right)G}{R^2}$$



Resultant force exerted by the disc

$$\vec{F} = \frac{mMG}{\pi a^2}\int \frac{\cos\alpha}{R^2}\, dA$$

$$\vec{dA} = dr.r d\theta.$$

$$\frac{\cos\alpha}{R^2} = \frac{R\cos\alpha}{R^3} = \frac{b}{(r^2+b^2)^{3/2}}$$

Range of integration

$$r \rightarrow 0 \leq r \leq a$$

$$\theta \rightarrow 0 \leq \theta \leq 2\pi$$

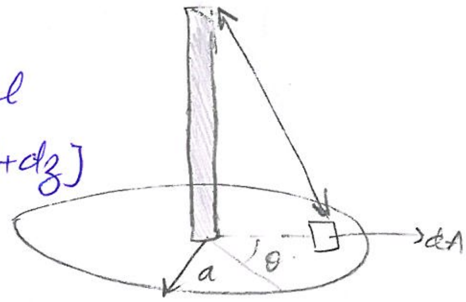
$$\vec{F} = \frac{mMG}{\pi a^2}\int_{r=0}^{r=a}\int_{\theta=0}^{2\pi} \frac{b \; r dr d\theta}{(r^2+b^2)^{3/2}}$$

$$= \frac{mMG}{\pi a^2}\int_{r=0}^{a} \frac{2\pi b r dr}{(r^2+b^2)^{3/2}} = \frac{2mMG}{a^2}\left(-b(r^2+b^2)^{-1/2}\right)_{r=0}^{r=a}$$

$$F = \frac{2mMG}{a^2}\left[1 - \frac{b}{(a^2+b^2)^{1/2}}\right]$$

2. A uniform disc has mass M and radius a and uniform rigid rod has mass M and length b. The rod is placed along the symmetry axis of the disc with one end in a contact with disc. Find the force required to pull the disc and rod apart. Use the result from previous problem.

Soln Let the rod occupy in the interval $(0 \leq z \leq b)$. Consider the element $[z, z+dz]$ of the rod which has mass $\frac{M'dz}{b}$



The force exerted by the disc on this element acts towards O and has magnitude. Using the result of previous soln.

$$F = \frac{2MM'G}{a^2 b}\left(1 - \frac{z}{(z^2+a^2)^{1/2}}\right)dz$$

∵ the mass is continuously distributed in rod, the resultant force becomes

$$F = \frac{2MM'G}{a^2 b}\int_a^b\left(1 - \frac{z}{(z^2+a^2)^{1/2}}\right)dz$$

$$= \frac{2mM'G}{a^2 b}\left[z - (z^2+a^2)^{1/2}\right]_b$$

$$= \frac{2MM'G}{a^2}\left[a+b-(a^2+b^2)^{1/2}\right]$$

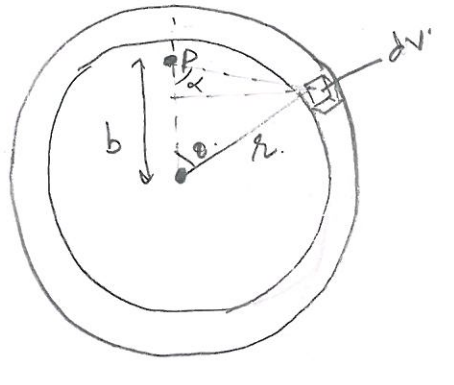
This ~~is~~ the force needed to pull the rod & disc apart.

3. Show that the gravitational force exerted on a particle inside a hollow symmetric sphere is zero.

Soln:

$$F = \frac{m(\rho\, dv)G}{R^2}$$

Resultant force



$$F = mG\int \frac{\rho\, dv \cos\alpha}{R^2}$$

$$\frac{\rho\cos\alpha}{R^2} = \frac{\rho R\cos\alpha}{R^3}$$

$$= \frac{\rho(r)(b - r\cos\alpha)}{(r^2+b^2-2rb\cos\theta)^{3/2}}$$

$$F = mG\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{2\pi} \frac{\rho(r)(b-r\cos\theta)}{(r^2+b^2-2rb\cos\theta)^{3/2}} \times r^2\sin\theta\, d\theta\, dr\, d\phi$$

Now $\int_0^{2\pi} d\phi = 2\pi$

$$F = 2\pi mG\int_{r=0}^{r=a} r^2\rho(r)\left\{\int_{\theta=0}^{\theta=\pi}\frac{(b-r\cos\theta)\sin\theta\, d\theta}{(r^2+b^2-2rb\cos\theta)^{3/2}}\right\}dr$$

Now, we solve θ integral keeping R constant we change variable from θ to R

$$R^2 = r^2+b^2-2br\cos\theta$$

limit range $\theta \rightarrow r-b < R < r+b$

$$2R\,dR = 2rb\sin\theta\,d\theta$$

$$\frac{2b}{2b} \times (b - r\cos\theta) = \frac{2b^2 - 2rb\cos\theta}{2b} = \frac{R^2 + (b^2 - r^2)}{2b}$$

$\therefore$ θ integral becomes

$$\int_{r-b}^{r+b} \frac{(R^2 + (b^2 - r^2))}{2bR^3} \cdot \frac{R\,dR}{rb}$$

$$= \frac{1}{2rb^2} \int_{r-b}^{r+b} \left(1 + \frac{b^2 - r^2}{R^2}\right) dR$$

$$= \frac{1}{2rb} \left[R + \frac{r^2 - b^2}{R}\right]_{R=r-b}^{R=r+b}$$

$$= \frac{1}{2rb^2} \left[r + b + \frac{(r-b)\cancel{(r+b)}}{\cancel{(r+b)}} - (r-b) - \frac{\cancel{(r-b)}(r+b)}{\cancel{(r-b)}}\right]$$

$$= \frac{1}{2rb^2} (2r - 2r) = 0.$$

Thus θ integral becomes zero.

So. $F = 0$.

$\Rightarrow$ Gravitational force exerted on a particle inside a hollow sphere is zero.

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4. Evaluate the following integrals:

(a) $$\int_6^2 (3x^2 - 2x - 1)\,\delta(x-3)\,dx$$

$$= 3(3)^2 - 2\times 3 - 1 = 20$$

(b) $$\int_0^5 \cos x.\ \delta(x - \pi)\,dx$$

$$= \cos\pi = -1$$

(c) $$\int_{-\infty}^{\infty} \ln(x+3)\, \delta(x+2)\, dx$$

$$= \ln(-2+3) = \ln 1 = 0$$

(d) $$\int_{-2}^{2} (2x+3)\, \delta(3x)\, dx$$

$$= \int_{-2}^{2} (2x+3)\, \frac{1}{3}\delta(x) = (2\times 0+3)\frac{1}{3} = 1$$

(e) $$\int_{0}^{2} (x^3+3x+2)\, \delta(1-x)\, dx$$

$$= \int_{0}^{2} (x^3+3x+2)\, \delta(x-1)\, dx = 1+3\times 1+2 = 6$$

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5) Show that $\delta(kx) = \frac{1}{|k|}\delta(x)$ where k is any non-zero constant.

Soln: For any arbitrary test function $f(x)$, consider the integral $\int_{-\infty}^{+\infty} f(x)\delta(kx)dx$

Let $y = kx$

$dy = k\,dx$

for $k>0$ integral is from $-\infty$ to $+\infty$

$k<0$ " " " $+\infty$ to $-\infty$

$$\int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx = \pm \int_{-\infty}^{\infty} f\left(\frac{y}{k}\right)\delta(y)\frac{dy}{k}$$

$$= \pm\frac{1}{k} f(0) = \frac{1}{|k|} f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(kx)\,dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right]dx$$

Under integral sign $\delta(kx)$ serves the same purpose as $\frac{1}{|k|}\delta(x)$

$\therefore \quad \delta(kx) = \frac{1}{|k|}\delta(x)$

6. Prove the following results

(a) $x\,\delta(x) = 0$

$= (x=0)\,\delta(x) = 0$

(b) $x\frac{d}{dx}\delta(x) = -\delta(x)$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\left(x\frac{d}{dx}\delta(x)\right) = x f(x)\,\delta(x)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{\infty}\frac{d}{dx}\left(x f(x)\,\delta(x)\right)dx$$

first term is zero as $\delta(x) = 0$ at $\pm\infty$

$$\frac{d}{dx}(x f(x)) = x\frac{df}{dx} + f$$

$$\int_{-\infty}^{\infty} f(x)\left(x\frac{d}{dx}\delta(x)\right)dx = -\int_{-\infty}^{\infty}\left(x\cancel{*}\frac{df}{dx} + f\right)\delta(x)\,dx$$

$$= 0 - f(0)$$

$$= -f(0)$$

$$= -\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx$$

$\therefore \quad x\frac{d}{dx}\delta(x) = -\delta(x)$

7. A sphere of radius R centered at the origin carries charge density $\rho(r,\theta) = \frac{kR}{r^2}(R-2r)\sin\theta$ where k is constant and r, θ are elliptical coordinates. Find the approximate potential for the points for the z-axis far from the sphere.

Soln: Given $\rho(r,\theta) = \frac{kR}{r^2}(R-2r)\sin\theta$

Monopole term

$$Q = \int \rho\, d\tau = kR \int \left(\frac{1}{r^2}(R-2r)\sin\theta\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= kR \int (R-2r)\sin\theta\, dr\, d\theta\, d\phi$$

r integral is

$$\int_0^R (R-2r)\,dr = (Rr - r^2)\Big|_0^R = R^2 - R^2 = 0$$

$$\Rightarrow Q = 0.$$

Dipole term

$$\int r\cos\theta\, \rho\, d\tau = kR \int (r\cos\theta)\left[\frac{1}{r^2}(R-2r)\sin\theta\right) r^2 \sin\theta\, dr\, d\theta\, d\phi$$

for θ integral.

$$\int_0^\pi \sin^2\theta \cos\theta\, d\theta = \left[\frac{\sin^3\theta}{3}\right]_0^\pi = 0$$

Quadrupole term

$$\int r^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)\rho\, d\tau = \frac{1}{2}kR \iiint r^2(3\cos^2\theta - 1) \times \left[\frac{1}{r^2}(R-2r)\sin\theta\right]$$

$$\times\, r^2 \sin\theta\, d\theta\, dr\, d\phi$$

r integral is

$$\int_0^R r^2 (R-2r)\,dr = \left[\frac{r^3 R}{3} - \frac{r^4}{2}\right]_0^R$$

$$= \frac{R^4}{3} - \frac{R^4}{2} = -\frac{R^4}{6}$$

θ integral

$$\int_0^\pi (3\cos^2\theta - 1)\sin^2\theta\, d\theta$$

$$= \int_0^\pi [3(1-\sin^2\theta) - 1]\sin^2\theta\, d\theta.$$

$$= \int_0^\pi (3 - 3\sin^2\theta - 1)\sin^2\theta\, d\theta.$$

$$= \int_0^\pi (2 - 3\sin^2\theta)\sin^2\theta\, d\theta$$

$$= 2\int_0^\pi \sin^2\theta\, d\theta - 3\int_0^\pi \sin^4\theta\, d\theta$$

$$= 2\times\frac{\pi}{2} - \frac{3\times 3\pi}{8} = \frac{-\pi}{8}$$

$$\phi \text{ integral} = \int_0^{2\pi} d\phi = 2\pi$$

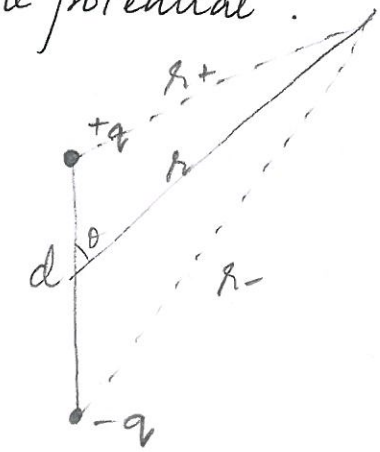
∴ Quadrupole term

$$\frac{1}{2}kR\left(-\frac{R^4}{6}\right)\left(-\frac{\pi}{8}\right)(2\pi) = \frac{k\pi^2 R^5}{48}$$

∴ The approximate potential is

$$V(z) = \frac{1}{4\pi\epsilon_0}\times\frac{k\pi^2 R^5}{48 z^3}$$

8. For an electrical dipole that consists of two equal and opposite charges $\pm q$ separated by a distance d. Determine the quadrupole and octupole terms in the potential.



Soln

Let r_- be the distance from q_- & r_+ be the distance from q_+

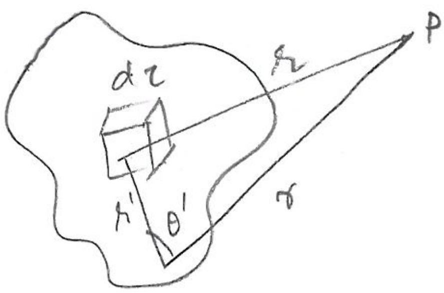
$$\therefore \quad V(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

$$r_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp r d \cos\theta$$

$$= r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

for any arbitrary localized charge distribution potential is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathcal{R}}\rho(r')\,d\tau'$$



$$r = r' + \mathcal{R}$$

$$\mathcal{R} = r - r'$$

$$\mathcal{R}^2 = r^2 + r'^2 - 2rr'\cos\theta'$$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]$$

put $\epsilon = \frac{r'}{r}\left(\frac{r'}{r} - 2\cos\theta'\right)$

where $\mathcal{R} = r\sqrt{1+\epsilon}$

$$\frac{1}{\mathcal{R}} = \frac{1}{r}(1+\epsilon)^{-1/2}$$

$$\frac{1}{r} = \frac{1}{\gamma}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \frac{5}{16}\epsilon^2 + \cdots\right)$$

$$\frac{1}{r} = \frac{1}{\gamma}\left[1 - \frac{1}{2}\left(\frac{\gamma'}{\gamma}\right)\left(\frac{\gamma'}{\gamma} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{\gamma'}{\gamma}\right)^2\left(\frac{\gamma'}{\gamma} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{\gamma'}{\gamma}\right)^3\left(\frac{\gamma'}{\gamma} - 2\cos\theta'\right)^3 + \cdots\right]$$

$$= \frac{1}{\gamma}\left[1 + \left(\frac{\gamma'}{\gamma}\right)\cos\theta + \left(\frac{\gamma'}{\gamma}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \left(\frac{\gamma'}{\gamma}\right)^3\left(\frac{5\cos^3\theta' - 3\cos\theta'}{2}\right) + \cdots\right.$$

$$\frac{1}{r} = \frac{1}{\gamma}\sum_{n=0}^{\infty}\left(\frac{\gamma'}{\gamma}\right)^n P_n(\cos\theta')$$

$$\gamma' = \frac{d}{2}.$$

$$\frac{1}{r_+} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2\gamma}\right)^n P_n(\cos\theta)$$

for r_- $\quad \theta \to \pi + \theta \qquad \cos\theta \to -\cos\theta$

$$\frac{1}{r_-} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2\gamma}\right)^n P_n(-\cos\theta)$$

$$P_n(-x) = (-1)^n P_n(x)$$

$$V = \frac{1}{4\pi\epsilon_0} q \left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{r}\sum_{n=0}^{\infty}\left(\frac{d}{2\gamma}\right)^n\left[P_n\cos\theta - P_n(-\cos\theta)\right]$$

for odd n

$$V = \frac{2q}{4\pi\epsilon_0 \gamma}\sum_{n=1,3,5}\left(\frac{d}{d\gamma}\right)^n P_n(\cos\theta)$$

for even n, $\quad V = 0$.

for $n=1$, dipole term

$$V_{dip} = \frac{2q}{4\pi\epsilon_0} \cdot \frac{1}{r} \cdot \frac{d}{2r} \left(P_1(\cos\theta) \right)$$

$$= \frac{q d \cos\theta}{4\pi\epsilon_0 r^2}$$

Quadrupole term $n=2$.

$$V_{qua} = 0$$

Octupole term $n=3$

$$V_{oct} = \frac{2q}{4\pi\epsilon_0 r} \left(\frac{d}{2r}\right)^3 P_3(\cos\theta)$$

$$= \frac{2qd^3}{4\pi\epsilon_0 \, 8 r^4} \times \frac{1}{2} \times \left(5\cos^3\theta - 3\cos\theta\right)$$

$$= \frac{q d^3}{4\pi\epsilon_0} \times \frac{1}{8 r^4} \left(5\cos^3\theta - 3\cos\theta\right)$$

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9. Find the equation of motion of the charge using Lagrange's method in an electrical circuit ($R=0$) with components of an inductor of inductance L is connected to the capacitor of capacitance C. Initially, the capacitor has charge Q. Compare the equation of motion with usual method in electrodynamics.

Soln.

$$K.E. = \frac{1}{2} L I^2.$$

$$P.E. = \frac{Q^2}{2C}$$

$$L = T - V = \frac{1}{2} L I^2 - \frac{Q^2}{2C}$$

where $I = \frac{dQ}{dt}$

Lagrange's eqn of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right) - \frac{\partial L}{\partial Q} = 0$$

$$L = \frac{1}{2} L \dot{Q}^2 - \frac{Q^2}{2C}$$

$$\frac{\partial L}{\partial \dot{Q}} = L\dot{Q}$$

$$\frac{\partial L}{\partial Q} = \frac{Q}{C}$$

$$\frac{d}{dt}(L\dot{Q}) + \frac{Q}{C} = 0$$

$$L\ddot{Q} + \frac{Q}{C} = 0$$

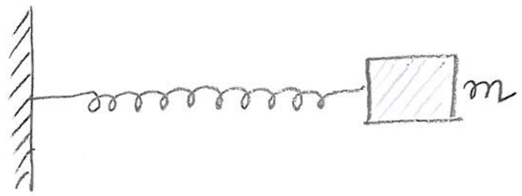
Equation of motion in electrodynamics

$$L\frac{dI}{dt} + \frac{1}{C}\int I\,dt = 0$$

$$L\ddot{Q} + \frac{Q}{C} = 0$$

10. Find the Lagrangian and Lagrange's eqn of motion for a spring mass system (simple harmonic oscillator).

Soln:



$$K.E. = \frac{1}{2} m\dot{x}^2.$$

$$P.E. = \frac{1}{2} k x^2$$

$$L = T - V$$
$$= \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

eqn. of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad ; \quad \frac{\partial L}{\partial x} = -kx$$

$$\frac{d}{dt}(m\dot{x}) + kx = 0$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m} x = 0$$

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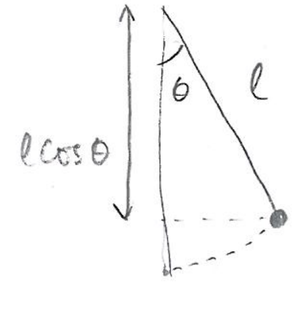
11. Find the Lagrangian and Lagrange's eqn of motion for a ~~spring mass system~~ simple pendulum.

Soln:

$$K.E. = \frac{1}{2} m (r\dot{\theta})^2$$

$$P.E. = mg(l - l\cos\theta)$$



$$L = T - V$$
$$= \frac{1}{2} m r^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) - mgl\sin\theta = 0$$

$$mr^2\ddot{\theta} - mgl\sin\theta = 0$$

$$\ddot{\theta} - \frac{gl}{r^2}\sin\theta = 0$$

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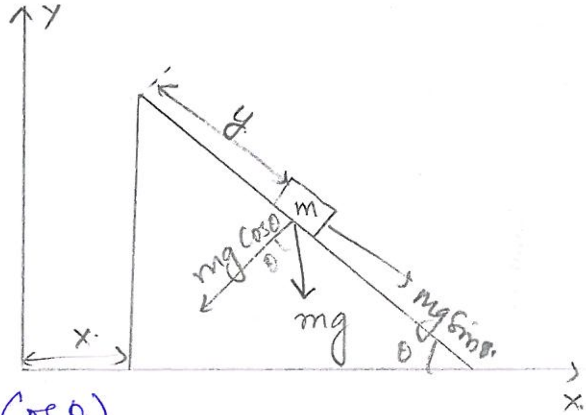
12. A block of mass m is held motionless on frictionless plane of mass M and the angle of inclination of the plane is θ. The block rests on a frictionless horizontal surface. The block is now released. What is the horizontal acceleration of the plane? Find the solution by both Newtonian and Lagrangian methods and compare.

Soln. velocity of M wrt $O = \dot{x}$

velocity of m wrt $O' = \dot{y}$

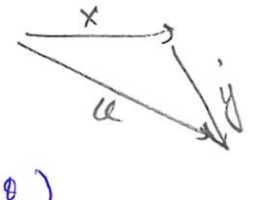
velocity of m wrt $O = \dot{x} + \dot{y}$



$$v^2 = \dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\theta$$

$$K.E. = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\theta\right)$$

$$V = -mgy\sin\theta$$



$$L = T - V$$

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + 2\dot{x}\dot{y}\cos\theta\right)$$

$$+ mg\sin\theta.$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} + m\cos\theta\,\dot{y}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} + m\cos\theta\,\dot{x}$$

$$\frac{\partial L}{\partial y} = mg\sin\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\left(M\dot{x} + m\dot{x} + m\dot{y}\cos\theta\right) = 0$$

$$M\ddot{x} + m\ddot{x} + m\ddot{y}\cos\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$\frac{d}{dt}\left(m\dot{y} + m\dot{x}\cos\theta\right) - mg\sin\theta = 0$$

$$\cancel{(M+m)\ddot{x}}\; m\cos\theta\,\ddot{x} + m\ddot{y} = mg\sin\theta.$$

$$\ddot{x} = -\ddot{y}\,\frac{m\cos\theta}{M+m}$$

$$m\cos\theta\left(-\ddot{y}\,\frac{m\cos\theta}{m+M}\right) + m\ddot{y} = mg\sin\theta$$

$$\ddot{y}\left(\frac{-m^2\cos^2\theta}{(M+m)} + m\right) = mg\sin\theta$$

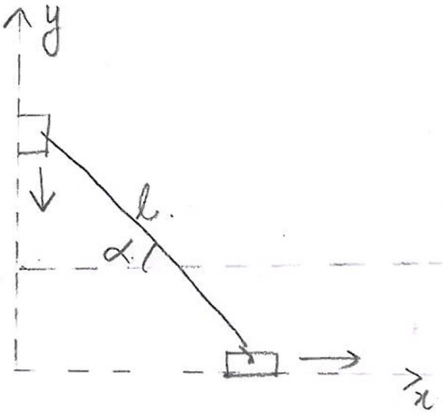
$$\ddot{y}\left(1 - \frac{m\cos^2\theta}{M+m}\right) = g\sin\theta$$

$$\dot{y} = \frac{(M+m)\,g\sin\theta}{M + m\sin^2\theta}.$$

$$\dot{x} = -\left(\frac{(M+m)\,g\sin\theta}{M+m\sin^2\theta}\right)\frac{m\cos\theta}{(M+m)}$$

$$\ddot{x} = -\frac{mg\sin\theta\cos\theta}{M+m\sin^2\theta}$$

13. Two blocks of equal mass are connected by a rigid bar of length l move without friction along the path shown. Find the eqn of motion



Soln:

$$x = l\cos\alpha$$

$$y = l\sin\alpha.$$

$$L = T - V$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$

$$\dot{x} = -l\sin\alpha\,\dot{\alpha}$$

$$\dot{y} = -l\cos\alpha\,\dot{\alpha}$$

$$T = \frac{1}{2}ml^2(\sin^2\alpha + \cos^2\alpha)\dot{\alpha}^2$$

$$= \frac{1}{2}ml^2\dot{\alpha}^2$$

$$V = mgy = mgl\sin\alpha.$$

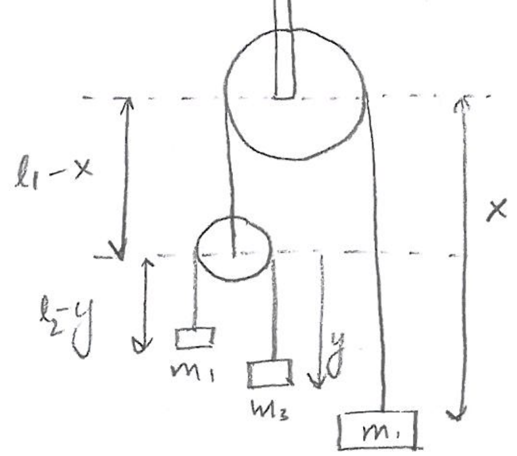
$$L = \frac{1}{2}ml^2\dot{\alpha}^2 - mgl\sin\alpha.$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\alpha}}\right) - \frac{\partial L}{\partial\alpha} = \frac{d}{dt}(ml^2\dot{\alpha}) + mgl\cos\alpha = 0$$

$$ml^2\ddot{\alpha} + mgl\cos\alpha = 0$$

$$\ddot{\alpha} + \frac{g}{l}\cos\alpha = 0$$

14. Consider the double pulley system shown. Find the equation of motion.



Soln

$$v_1 = \dot{x}$$

$$v_2 = \frac{d}{dt}(l_1 - x + y) = -\dot{x} + \dot{y}$$

$$v_3 = \frac{d}{dt}(l_1 - x + l_2 - y) = -\dot{x} - \dot{y}$$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{y} - \dot{x})^2 + \frac{1}{2} m_2 (-\dot{x} - \dot{y})^2$$

$$V = V_1 + V_2 + V_3$$

$$= -m_1 g x - m_2 g (l - (x+y)) - m_3 g (l_1 - x + l_2 - y)$$

$$L = T - V$$

$$= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 (\dot{y} - \dot{x})^2 + \frac{1}{2} m_3 (-\dot{x} - \dot{y})^2 + m_1 g x$$
$$+ m_2 g (l_1 - x + y) + m_3 g (l_1 - x + l_2 + y)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$m_1 \ddot{x} + m_2 \frac{d}{dt}(-(\dot{y} - \dot{x})) + m_3 \frac{d}{dt}(\dot{x} + \dot{y})$$
$$- m_1 g + m_2 g + m_3 g = 0$$

$$m_1 \ddot{x} + m_2 (\ddot{x} - \ddot{y}) + m_3 (\ddot{x} + \ddot{y}) = (m_1 - m_2 + m_3) g$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$

$$m_2 (\ddot{y} - \ddot{x}) + m_3 (\ddot{x} + \ddot{y})$$
$$= (m_2 - m_3) g.$$

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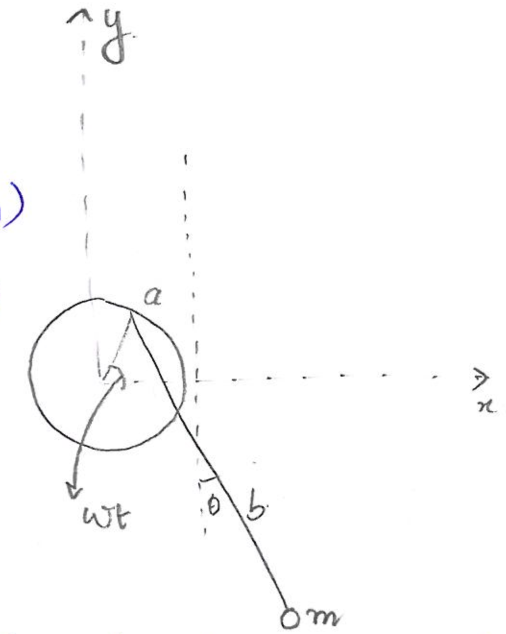
15. The point of support of a simple pendulum of length b moves on a massless rim of radius a rotating with constant velocity ω. Obtain the expression for velocity & acceleration of the mass m.

Soln. $x = a \cos \omega t + b \sin \theta$

$y = a \sin \omega t - b \cos \theta$

$\dot{x} = -a\omega \sin\omega t + b\dot{\theta}\cos\theta$

$\dot{y} = a\omega \cos\omega t + b\dot{\theta}\sin\theta$.



$\ddot{x} = -a\omega^2 \cos\omega t + b(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)$

$\ddot{y} = -a\omega^2 \sin\omega t + b(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta)$

$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$U = mgy$

$T = \frac{1}{2}m\left[(-a\omega\sin\omega t + b\dot{\theta}\cos\theta)^2 + (a\omega\cos\omega t + b\dot{\theta}\sin\theta)^2\right]$

$= \frac{1}{2}m\left[a^2\omega^2\sin^2\omega t + b^2\dot{\theta}^2\cos^2\theta - 2a\omega b\dot{\theta}\sin\omega t\cos\theta + a^2\omega^2\cos^2\omega t + b^2\dot{\theta}^2\sin^2\theta + 2a\omega b\dot{\theta}\cos\omega t\sin\theta\right]$

$= \frac{1}{2}m(a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\dot{\theta}\omega\sin(\theta - \omega t))$

$V = mg(a\sin\omega t \cancel{+} b\cos\theta)$

$L = T - V = \frac{m}{2}\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin(\theta - \omega t) - mg(a\sin\omega t - b\cos\theta)\right]$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mb^2\ddot{\theta} + mba\omega(\dot{\theta} - \omega)\cos(\theta - \omega t)$

$\frac{\partial L}{\partial \theta} = mba\dot{\theta}\omega\cos(\theta - \omega t) - mgb\sin\theta$.

$\ddot{\theta} = \frac{\omega^2 a}{b}\cos(\theta - \omega t) - \frac{g}{b}\sin\theta$
