ICI-121: Mechanics of Waves and Particles

Tutorial 3: Solutions

1. A certain oscillator satisfy the equation $\ddot{x}+4x=0$. Initially the particle is at the point $x=\sqrt{3}$ when it is projected towards the origin with speed 2. Show that the subsequent motion is given by the equation $x=\sqrt{3}\cos 2t-\sin 2t$. Deduce the amplitude of the oscillation. How long it will take by the particle to first reach the origin?

Soln: We can have standard solution in the form of

$$x = e^{\lambda t}$$

$$\therefore \frac{dx}{dt} = \lambda e^{\lambda t}$$

$$\frac{d^2x}{dt^2} = \lambda^2 e^{\lambda t}$$

Substituting in the eqn: $\ddot{x}+4x=0$ we get

$$\lambda^2+4=0$$

$$\therefore \lambda = \pm 2i$$

So $$x = Ae^{2it}+Be^{-2it}$$

Initially, at $t=0$; $x=\sqrt{3}$, $\frac{dx}{dt}=-2$.

$$\therefore A+B=\sqrt{3}$$

$$-2 = 2i(A-B)$$

$$\Rightarrow A-B = i$$

$$\therefore A = \frac{\sqrt{3}+i}{2}$$

$$B = \frac{\sqrt{3}-i}{2}$$

$$x = Ae^{2it}+Be^{-2it}$$

where $A = \frac{\sqrt{3}+i}{2}$ $\quad B = \frac{\sqrt{3}-i}{2}$

$$x = A(\cos 2t + i\sin 2t) + B(\cos 2t - i\sin 2t)$$

$$x = \cos 2t\,(A+B) + i\sin 2t\,(A-B)$$

$$x = \sqrt{3}\cos 2t - \sin 2t$$

The amplitude of oscillations

$$A = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$$

Time to reach origin (i.e. $x=0$)

$$\sqrt{3}\cos 2t - \sin 2t = 0$$

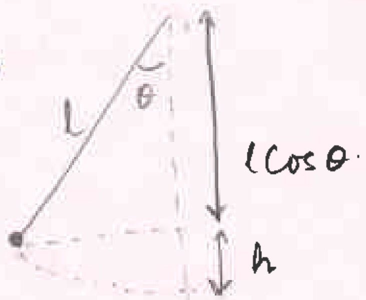
$$\tan 2t = \sqrt{3}$$

$$2t = \pi/3$$

$$t = \pi/6.$$

2 Arrive at the equation of motion for a simple pendulum from principle of conservation of energy with length of the pendulum l having mass m. The pendulum is released from an angle θ to the vertical.

Soln: Displacement of the mass from equilibrium



$$h = l - l\cos\theta$$

$$P.E. = mgh = mgl(1-\cos\theta)$$

$$K.E. = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2$$

$$E = K.E. + P.E. = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$\cos\theta = 1 - \frac{\theta^2}{2} + \frac{1}{24}\theta^4 + \cdots$$

for small θ, we can neglect terms of order greater than θ^2

So $\quad \cos\theta = 1 - \frac{\theta^2}{2}$

$\therefore \quad 1 - \cos\theta = \frac{1}{2}\theta^2$

So, $\quad E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}MgL\theta^2$

$$\frac{d\theta}{dt} = \dot{\theta} = \left(\frac{2E - mgl\theta^2}{ml^2}\right)^{1/2}$$

$$= \left(\frac{g}{l}\right)^{1/2}\left[\frac{2E}{mgl} - \theta^2\right]^2$$

Now, during the motion when the mass come to rest momentarily ($v=0 \Rightarrow K.E=0$) is at $\theta = \theta_0$

$$E = \frac{1}{2}mgl\theta_0^2$$

$$\therefore \theta_0^2 = \frac{2E}{mgl}$$

$$\frac{d\theta}{dt} = \left(\frac{g}{l}\right)^{1/2}\left[\theta_0^2 - \theta^2\right]^{1/2}$$

$$\frac{d\theta}{(\theta_0^2 - \theta^2)^{1/2}} = \left(\frac{g}{l}\right)^{1/2} dt$$

Initially, at $t=0$ let $\theta = \theta_1$.

$$\int_{\theta_1}^{\theta}\frac{d\theta}{(\theta_0^2-\theta^2)^{1/2}} = \left(\frac{g}{l}\right)^{1/2}\int_0^t dt$$

$$= \left[\sin^{-1}\frac{\theta}{\theta_0}\right]_{\theta_1}^{\theta} = \left(\frac{g}{l}\right)^{1/2} t$$

$$= \sin^{-1}\frac{\theta}{\theta_0} - \sin^{-1}\frac{\theta_1}{\theta_0} = \left(\frac{g}{l}\right)^{1/2} t$$

$$\sin^{-1}\frac{\theta}{\theta_0} = \left(\frac{g}{L}\right)^{1/2} t + \sin^{-1}\frac{\theta_1}{\theta_0}$$

$$\frac{\theta}{\theta_0} = \sin\left[\left(\frac{g}{l}\right)^{1/2} t + \sin^{-1}\frac{\theta_1}{\theta_0}\right]$$

$$\theta = \theta_0 \sin\left[\left(\frac{g}{l}\right)^{1/2} t + \sin^{-1}\frac{\theta_1}{\theta_0}\right]$$

Now put $\left(\frac{g}{l}\right)^{1/2} = \omega_0$

and $\sin^{-1}\frac{\theta_1}{\theta_0} = \phi$

So, $\theta = \underline{\theta_0 \sin^{-1}(\omega_0 t} + \phi)$

is the equation of motion of simple pendulum with small deflection.

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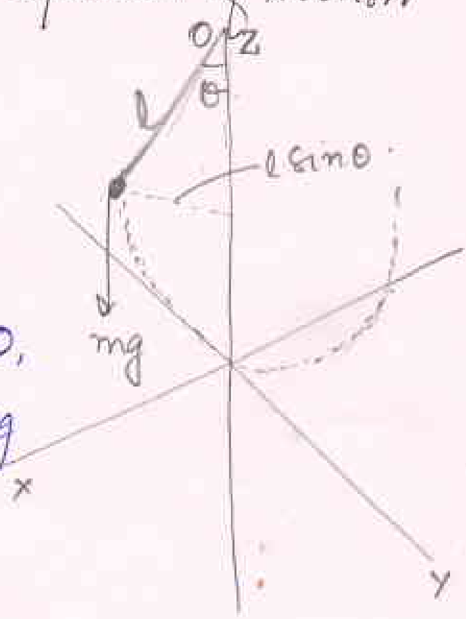
3. In the problem mentioned above find the equation of motion from balancing of balance of torques.



Soln: Consider x-axis normal to the plane of motion of the pendulum.

The torque N_x due to gravity about O, the point of suspension with force $F = mg$

$$N_x = (r \times F)_x$$
$$= l\, mg \sin\theta$$

Angular momentum about O

$$J_x = (r \times p)_x = (l \times ml\dot{\theta}) = -ml^2\dot{\theta}$$

Now, the rate of change of angular momentum = torque

$$\therefore\ ml^2\ddot{\theta} = -l mg \sin\theta$$

$$\Rightarrow\ \ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

for $\theta \ll 1$ $\quad \underline{\ddot{\theta} + \frac{g}{l}\theta = 0}$ $\Rightarrow$ which is the equation of motion

4 A mass m oscillate on a spring with spring constant k. The amplitude is d at $t=0$. At the moment when the mass is at the position $x = d/2$, while moving right it collides and sticks to another mass m. What is the amplitude of the new oscillation.

Soln: we find the velocity of mass just before the collision

The equation of motion will be of the form

$$x(t) = d\cos(\omega t + \phi)$$

where $\omega = \sqrt{\frac{k}{m}}$

Now, initially at $t=0$; $x(t) = d/2$

$$\therefore \frac{d}{2} = d\cos\phi$$

$$\Rightarrow \quad \phi = \pm\frac{\pi}{3}$$

$\therefore$ The velocity just before the collision

$$v(0) = \dot{x}(0) = -\omega d\sin\phi$$
$$= -\omega d\sin\left(\pm\frac{\pi}{3}\right)$$
$$= \mp\omega d\left(\frac{\sqrt{3}}{2}\right)$$

we take +ve sign as motion is towards right

Now after the mass sticks to another mass, the total mass becomes $2m$ which moves toward right with initial position $d/2$ attached to a spring of constant k and initial velocity as half of that of single mass (conservation of momentum in perfectly inelastic collision).

we use the standard solution of the form

$x(t) = C\cos\omega' t + D\sin\omega' t$

at $t=0$ $\quad x(t) = C.$

$\frac{dx(t)}{dt} = \omega' D$

where $\omega' = \sqrt{\frac{K}{2m}} = \frac{\omega}{\sqrt{2}}$

$x(0) = d/2 = C$

$v(0) = \frac{\sqrt{3}}{4}\omega d. = \omega' D \Rightarrow D = \frac{\sqrt{6}d}{4}$

$\therefore x(t) = \frac{d}{2}\cos\omega' t + \frac{\sqrt{6}d}{4}\sin\omega' t$

amplitude $= \sqrt{\frac{d^2}{4} + \frac{6d^2}{16}} = \underline{\underline{\sqrt{\frac{5}{8}}d}}$

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5. A particle of mass 5 kg moves along x-direction under the influence of two forces

(i) A force towards the origin with value 40 Nm^{-1}

(ii) A frictional force of 200N for $v = 10 m/s$.

Let $x(t=0) = 20m$ and $v = \dot{x}(t=0) = 10 m/s$. Find the differential equation of motion and the Solution.

Find also the amplitude of the vibration, period & frequency, ratio's of two successive amplitudes.

Soln (a) The equation of motion of particle under the influence of friction is :

$m\ddot{x} = -kx - \beta\dot{x}$

where $k = 40\ Nm^{-1}$

$F_{fric} = -\beta v = 200$

$v = \frac{200}{10} = 20\ Nsm^{-1}.$

$$\omega^2 = \frac{k}{m} = \frac{40}{5} = 8 s^{-2}$$

$$2\gamma = \frac{\beta}{m} = \frac{20}{5} = 4 s^{-1}$$

the equation of motion becomes

$$\ddot{x} + 2\gamma\dot{x} + \omega^2 x = 0$$

$$\ddot{x} + 4\dot{x} + 8x = 0$$

(b) We can see that $\omega^2 > \gamma^2$

It is the case of weak damping

The general solution of the differential eqn of a damped harmonic oscillator

$$x(t) = e^{-\gamma t}[A\cos(\Omega t) + B\sin(\Omega t)]$$

where $\Omega = \sqrt{\omega^2 - \gamma^2} = \sqrt{8-4} = 2 s^{-1}$

Initially, at $t=0$

$$x_0 = A = 20$$

$$\frac{dx}{dt} = -\gamma e^{-\gamma t}(A\cos(\Omega t) + B\sin(\Omega t)) + e^{-\gamma t} \times (-A\Omega\sin(\Omega t) + B\Omega\cos(\Omega t))$$

at $t=0$; $\dot{x} = 0$

$$0 = -\gamma x_0 + B\Omega \Rightarrow B = \frac{x_0\gamma}{\Omega} = \frac{20\times 2}{2} = 20 m$$

$$\therefore x(t) = 20(\cos\Omega t + \sin\Omega t)e^{-\gamma t} \quad m$$

Thus, by putting any value of t we can get corresponding positions of the particles.

(C) (i) Amplitude.

$$\sqrt{A^2+B^2} = \sqrt{20^2+20^2} = 20\sqrt{2}$$

(ii) frequency

$$\Omega = \sqrt{\omega_0^2 - \gamma^2} = 2\,s^{-1}.$$

(iii) Time period

$$T = \frac{2\pi}{\Omega} = \pi \text{ sec}.$$

(d) For two successive maximum elongations

$$x_n = 20\sqrt{2}\, e^{-\gamma}$$

$$x_{n+1} = 20\sqrt{2}\, e^{-\gamma(t + 2\pi/\Omega)}$$

$$\frac{x_n}{x_{n+1}} = e^{\gamma \times \frac{2\pi}{\Omega}}$$

$$\therefore \ln\left(\frac{x_n}{x_{n+1}}\right) = \frac{2\pi\gamma}{\Omega} = \gamma T$$

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6. An overdamped harmonic oscillator satisfies the equation $\ddot{x} + 10\dot{x} + 16x = 0$. At time $t=0$, the particle is projected from the point $x=1$ towards the origin with speed u. Find the solution of the problem. Show that the particle will reach the origin at some later time t if $\frac{u-2}{u-8} = e^{6t}$. How large must u such that the particle pass through the origin?

Soln: Let us suppose the standard solution

$$x = e^{\lambda t}$$

Substituting in the eqn of motion, we get

$$\lambda^2 + 10\lambda + 16 = 0$$

$$\Rightarrow (\lambda+8)(\lambda+2)=0$$

$$\Rightarrow \lambda = -2, -8$$

So, the general solution, we can write as

$$x = Ae^{-2t} + Be^{-8t}$$

Initially, at $t=0$; $x=1$; $\dot{x} = -u$

$$\therefore A+B=1$$

$$\frac{dx}{dt} = -2Ae^{-2t} - 8Be^{-8t}$$

$$-u = -2A - 8B$$

$$2A + 8B = u$$

Solving these to get

$$A = \frac{8-u}{6}$$

$$B = \frac{u-2}{6}$$

$$x = \frac{1}{6}(u-2)e^{-8t} - \frac{1}{6}(u-8)e^{-2t}$$

The particle is at origin (i.e. $x=0$) at time t

$$\frac{1}{6}(u-2)e^{-8t} - \frac{1}{6}(u-8)e^{-2t} = 0$$

$$\frac{u-2}{u-8} = e^{6t}$$

$$\log \frac{u-2}{u-8} = 6t$$

$$t = \frac{1}{6}\log\frac{u-2}{u-8}$$

for any value of t to exists

$$\frac{u-2}{u-8} > 0$$

$\therefore \ u > 8 \Rightarrow$ Thus, the particle will pass through origin if $u > 8$.

7 The exponential damping factor γ of a spring of a suspension system is one tenth of critical value. If the damping frequency is ω_0 (a) find the resonance frequency (b) quality factor (c) phase angle ϕ, when the system is driven at frequency $\omega = \omega_0/2$ (d) steady state amplitude at this frequency.

Soln

Given $\gamma = \gamma_{crit}/10 = \omega_0/10$

we know that, resonant frequency

$$\omega_r = \sqrt{\omega_0^2 - 2\gamma^2}$$

$$= \sqrt{\omega_0^2 - 2\left(\frac{\omega_0}{10}\right)^2} = \sqrt{\frac{49}{50}\omega_0^2} = 0.99\,\omega_0$$

(b) The quality factor

$$Q = \frac{\omega_d}{2\gamma}$$

For the case of weak damping

$$Q \simeq \frac{\omega_0}{2\gamma} = \frac{\omega_0}{2\left(\frac{\omega_0}{10}\right)} = 5$$

(c) when the system is driven at frequency $\omega = \frac{\omega_0}{2}$, then the phase angle

$$\tan\phi = \frac{2\gamma\omega_0}{\omega_0^2 - \omega^2} = \frac{2 \times \omega_0/10 \times \omega_0/2}{\omega_0^2 - \frac{\omega_0^2}{4}} = \frac{2}{15} = 0.133$$

$$\phi = \tan^{-1}(0.133) = 7.6^\circ$$

d The steady state amplitude at frequency $\omega = \frac{\omega_0}{2}$

$$A(\omega) = \frac{F_0/m}{[(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}$$

$$= \frac{F_0/m}{\left[\left(\omega_0^2 - \frac{\omega_0^2}{4}\right) + \frac{4\times\omega_0^2}{100}\times\frac{\omega_0^2}{4}\right]^{1/2}}$$

$$= \frac{F_0/m}{\sqrt{\left[\frac{9}{16}\omega_0^2 + \frac{\omega_0^2}{100}\right]}} = \frac{F_0/m}{0.7566\omega_0^2}$$

$$A(\omega = \omega_0/2) = \frac{1.322\,F_0}{m\omega_0^2}$$

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8. A critically damped oscillator with natural frequency ω and damping coefficient γ starts at position $x_0 > 0$. What is the maximum initial speed directed towards the origin and not to cross the origin?

Soln The eqn of motion in case of critical damping

$$x(t) = e^{-\gamma t}(A+Bt).$$

Given at $t=0$, $x(t) = x_0$; $v(t) = v_0$.

$$\frac{dx}{dt} = Be^{-\gamma t} - \gamma e^{-\gamma t}(A+Bt)$$

$$x_0 = A \; ; \quad v_0 = B - \gamma A = B - \gamma x_0$$

$$\therefore B = \gamma x_0 + v_0$$

$$x(t) = e^{-\gamma t}(x_0 + (v_0+\gamma x_0)t)$$

Now time at which $x(t)=0 \Rightarrow$ particle is at origin

$x(t) = 0 \Rightarrow e^{-\gamma t}(A+Bt) = 0 \Rightarrow t = -A/B = \frac{-x_0}{v_0+\gamma x_0}$

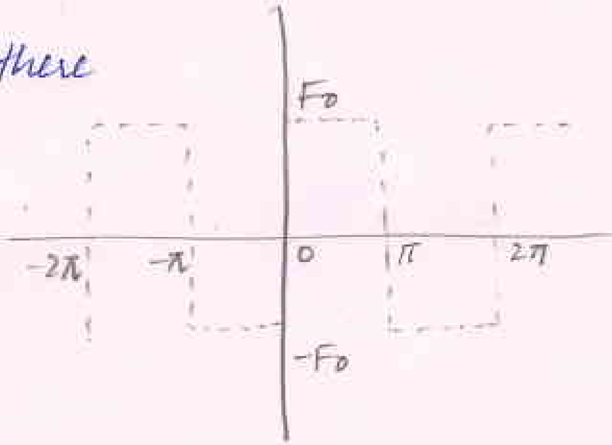
- Thus, mass will cross origin if $v_0 + \gamma x_0 < 0$

$\therefore$ Mass will NOT cross origin if $v_0 + \gamma x_0 \geq 0 \Rightarrow v_0 \geq -\gamma x_0$

$\therefore$ $V_{max} = v_0 = \gamma x_0 = \omega x_0$

9. Find the driven response of the damped linear oscillator for the case in which driving force $F(t)$ is periodic with period 2π and takes the values $F(t) = F_0$ $(0<t<\pi)$ and $F(t) = -F_0$ $(\pi<t<2\pi)$.

Soln: When damping is present and there is external force, the general equation of motion is



$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \Omega^2 x = F(t)$$

Given $F(t) = F_0$ $\quad (0<t<\pi)$

$= -F_0$ $\quad (\pi<t<2\pi)$

Now, it is a periodic function, to write it in the form of sine and cosine functions, we use Fourier's theorem

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

with fourier coefficients $\{a_n\}$ and $\{b_n\}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\cos nt\,dt = \frac{1}{\pi}\int_{-\pi}^{0}(-F_0)\cos nt\,dt + \frac{1}{\pi}\int_{0}^{\pi} F_0 \cos nt\,dt$$

$\because$ both integrals are zero for $n \geq 1$ and are equal and opposite for $n=0$

$\therefore$ $a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(t)\sin nt\,dt$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-F_0)\sin nt\,dt + \frac{1}{\pi}\int_{0}^{\pi}(F_0)\sin nt\,dt$$

$$= \frac{2F_0}{\pi}\int_{0}^{\pi}\sin nt\,dt$$

$$b_n = \frac{2F_0}{\pi}\left[-\frac{\cos nt}{n}\right]_0^{\pi}$$

$$= \frac{2F_0}{\pi}\left[\frac{1-\cos n\pi}{n}\right]$$

$$= \frac{2F_0}{\pi}\left(\frac{1-(-1)^n}{n}\right)$$

Thus, $\sum_{n=1}^{\infty} \frac{2F_0}{\pi}\left(\frac{1-(-1)^n}{n}\right)\sin nt$

To find the driven response of the oscillator to the force $m(b_n \sin nt)$ i.e. particular integral

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \Omega^2 x = b_n \sin nt$$

Now, we first replace the force term by the complex counterpart $b_n e^{ipt}$

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \Omega^2 x = b_n e^{ipt}$$

where C is complex constant

Let $x = ce^{ipt}$

putting it in above ~~comy~~ eqn.

$$c(-p^2 + 2kip + \Omega^2) = b_n$$

$$c = \frac{b_n}{\Omega^2 - p^2 + 2kip}$$

particular integral

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \Omega^2 x = 0$$

Let $x = Ce^{inx}$

$$\Rightarrow \quad \frac{dx}{dt} = Cin\,e^{inx}$$

$$\Rightarrow \quad \frac{d^2x}{dt^2} = -Cn^2 e^{inx}$$

$$\Rightarrow \quad \Omega^2 - n^2 + 2ik = 0.$$

$$\Rightarrow \quad \frac{b_n e^{int}}{\Omega^2 - n^2 + 2nik} \times \frac{\Omega^2 - n^2 - 2ikn}{\Omega^2 - n^2 - 2ikn}$$

$$= b_n e^{int}\left[\frac{(\Omega^2 - n^2) - 2ikn}{(\Omega^2 - n^2)^2 + 4k^2n^2}\right]$$

$$= b_n\left[\frac{(\Omega^2 - n^2)\sin nt + 2kn\cos nt}{(\Omega^2 - n^2)^2 + 4k^2n^2}\right]$$

Therefore, the driven response of oscillator to the force, so inserting b_n

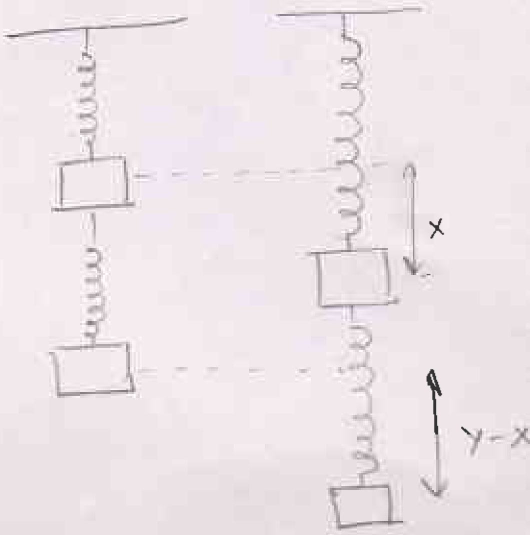
$$x = \frac{2F_0}{\pi}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n}\right)\left(\frac{(\Omega^2 - n^2)\sin nt + 2kn\cos nt}{(\Omega^2 - n^2)^2 + 4k^2n^2}\right)$$

10. A particle P of mass 3m is suspended from a fixed point O by a linear spring with strength α. A second particle Q of mass 2m is in turn suspended from P by a second spring of same strength. The system moves in the vertical straight line through O. Find the normal frequencies and the form of normal modes of the system.

Soln: Let x and y be the downward displacement of particles P and Q measured from the equilibrium positions.

The extensions in the springs are x and y-x

The equations of motions are



$$3m\ddot{x} = -\alpha x + \alpha(y-x)$$

$$2m\ddot{y} = -\alpha(y-x)$$

$$3\ddot{x} = -\frac{\alpha x}{m} + \frac{\alpha}{m}(y-x)$$

$$2\ddot{y} = -\frac{\alpha}{m}(y-x)$$

$$3\ddot{x} + 2n^2x - n^2y = 0$$

$$2\ddot{y} - n^2x + n^2y = 0 \qquad \text{where } n^2 = \frac{\alpha}{m}$$

To find the normal modes, let us suppose

$$x = A\cos(\omega t - \gamma)$$

$$y = B\cos(\omega t - \gamma)$$

Substituting in above eqns

$$\frac{dx}{dt} = -A\omega\sin(\omega t - \gamma)$$

$$\frac{d^2x}{dt^2} = -A\omega^2\cos(\omega t - \gamma)$$

$$\frac{dy}{dt} = -B\omega \sin(\omega t - \gamma)$$

$$\frac{d^2y}{dt^2} = -B\omega^2 \cos(\omega t - \gamma)$$

$$3(-A\omega^2) + 2n^2A - n^2B = 0$$

$$\Rightarrow A^2(-3\omega^2 + 2n^2) - Bn^2 = 0 \quad \text{--- (1)}$$

$$2(-B\omega^2) - n^2A + n^2B = 0 \quad \text{--- (2)}$$

$$-n^2A + B(n^2 - 2\omega^2) = 0$$

writing in matrix form

$$\begin{pmatrix} 2n^2 - 3\omega^2 & -n^2 \\ -n^2 & n^2 - 2\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

To have the non-trivial solution

$$\det \begin{pmatrix} 2n^2 - 3\omega^2 & -n^2 \\ -n^2 & n^2 - 2\omega^2 \end{pmatrix} = 0$$

$$(2n^2 - 3\omega^2)(n^2 - 2\omega^2) - n^4 = 0$$

$$= 2n^4 - 4n^2\omega^2 - 3n^2\omega^2 + 6\omega^4 - n^4 = 0$$

$$= 6\omega^4 - 7n^2\omega^2 + n^4 = 0$$

is quadratic in ω^2

$$\omega^2 = \frac{+7n^2 \pm \sqrt{49n^4 - 24n^4}}{12}$$

$$= \frac{7n^2 \pm \sqrt{25n^4}}{12}$$

$$= \frac{7n^2 \pm 5n^2}{12}$$

$$\omega^2 = n^2, \frac{1}{6}n^2$$

Thus, there are two normal modes

$$\omega_1 = n \qquad \omega_2 = \frac{n}{\sqrt{6}}$$

(i) for $\omega^2 = \frac{n^2}{6}$ is slow mode

So, amplitude A, B become

$$\frac{3n^2A}{2} - n^2B = 0$$

$$-n^2A + \frac{2}{3}n^2B = 0$$

which gives $3A = 2B$

Thus.

$$x = 2\delta \cos(\omega_1 t - \gamma)$$

$$y = 3\delta \cos(\omega_1 t - \gamma)$$

where δ is amplitude factor

γ is phase factor

In slow mode particles move in same direction.

(ii) for $\omega^2 = n^2$ fast mode.

$$-n^2A^2 - n^2B = 0$$

$$-n^2A^2 - n^2B = 0$$

$$A + B = 0$$

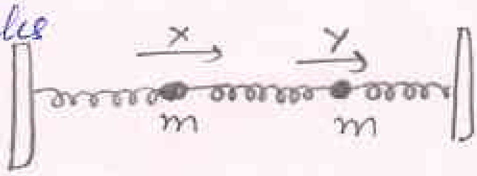
$$\therefore\ x = \delta \cos(\omega_2 t - \delta)$$

$$y = -\delta \cos(\omega_2 t - \delta)$$

In fast mode particles move in opposite direction.

11. Consider two masses m connected to each other and to two walls by springs. The three springs have the same spring constant k. Find the most general solution for the positions of masses as function to time. What are the normal coordinates? What are normal modes?

Soln Let the displacement of two particles be x and y



Then, the extensions in the 3 springs are

$x,\ y-x,\ -y$

They have spring constant k.

$\therefore$ eqns of motions

$$m\ddot{x} = -kx + k(y-x)$$
$$m\ddot{y} = -k(y-x) - ky$$
$$\ddot{x} + \frac{x(2k)}{m} - \frac{y(k)}{m} = 0$$
$$\ddot{y} - \frac{xk}{m} + \frac{2ky}{m} = 0$$

put $\frac{k}{m} = n^2$

$$\therefore\ \ddot{x} + 2n^2x - n^2y = 0$$
$$\ddot{y} - n^2x + 2n^2y = 0$$

The general solution is

$$x = A\cos(\omega t - \delta)$$
$$y = B\cos(\omega t - \delta)$$

$$\frac{dx}{dt} = -\omega A \sin(\omega t - \gamma)$$

$$\frac{dy}{dt} = -\omega B \sin(\omega t - \gamma)$$

$$\frac{d^2x}{dt^2} = -\omega^2 A \cos(\omega t - \gamma)$$

$$\frac{d^2y}{dt^2} = -\omega^2 B \cos(\omega t - \gamma)$$

putting in eqns of motion

$$-\omega^2 A + 2n^2 A - n^2 B = 0$$

$$-\omega^2 B - n^2 A + 2n^2 B = 0.$$

$$A(-\omega^2 + 2n^2) - n^2 B = 0$$

$$-n^2 A + B(2n^2 - \omega^2) = 0$$

Writing in matrix form, so that determinant is zero.

$$\begin{bmatrix} -\omega^2 + 2n^2 & -n^2 \\ -n^2 & 2n^2 - \omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

To have the non trivial solution

$$\det \begin{pmatrix} -\omega^2 + 2n^2 & -n^2 \\ -n^2 & 2n^2 - \omega^2 \end{pmatrix} = 0$$

$$(-\omega^2 + 2n^2) - (n^2)^2 = 0$$

$$(-\omega^2 + 2n^2 - n^2)(-\omega^2 + 2n^2 + n^2) = 0.$$

$$(-\omega^2 + n^2)(-\omega^2 + 3n^2) = 0$$

$$\therefore \omega^2 = n^2, 3n^2$$

The two normal modes with angular frequency $\omega = \sqrt{n}$ and $\sqrt{3}n$. These are normal frequencies

(i) Slow mode : $\omega^2 = n^2$

$$An^2 - n^2B = 0$$
$$A = B$$

Thus $A = \delta$, $B = \delta$.

$$\therefore \quad x = \delta \cos(\sqrt{n}t - \gamma)$$
$$y = \delta \cos(\sqrt{n}t - \gamma)$$

Two bodies move in same direction with equal amplitude.

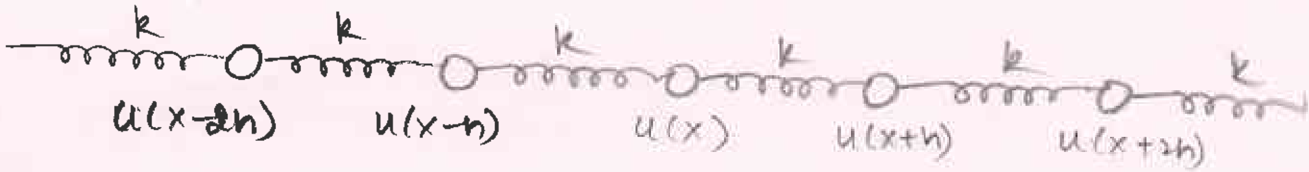
(ii) Fast mode : $\omega^2 = 3n^2$

$$-An^2 - n^2B$$
$$A = -B$$
$$x = \delta \cos(\sqrt{3}nt - \gamma)$$
$$y = -\delta \cos(\sqrt{3}nt - \gamma)$$

Two bodies move in opposite direction with equal amplitude.

12. Derive the equation of motion for system of n springs of spring constant K connected to each other by a mass having mass m. Let the length of each spring be h by balancing of the forces on each mass. Express the equation of motion in terms of the total length L, effective spring constant k, total mass of the n masses M. In the limit of the distance between masses approaches zero show that this equation of motion lead to wave equation $\dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{kL^2}{M}\dfrac{\partial^2 u(x,t)}{\partial x^2}$.

Soln :



If the force is applied at one end of this 1-d model, the mass at u(x) reacts with and is acted on by masses at u(x-h) and at u(x+h)

we use two fundamental Laws i) Newtons 2^{nd} Law ii) Hooke's Law.

Now $\tau(x,t)$ = force per unit length (for 1-d)

$$\tau(x,t) = \frac{m}{h}\left[\frac{v(x,t+\Delta t)-v(x,t)}{\Delta t}\right]$$

$v(x,t)$ we write in terms of position $u(x,t)$

$$v(x,t) = \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t}$$

$$v(x,t+\Delta t) = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}$$

$$\therefore \quad \tau(x,t) = \frac{m}{h\times \Delta t}\left[\frac{u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)}{\Delta t}\right] \quad (1)$$

Now using Hooke's Law: ~~force~~ strain is equal to bulk modulus times the increase in length divided by original lengt

Now force per unit length at any particle at pt x is determined by the action of the particles at pt x-h, x+h

Mathematically

$$\tau(x,t) = \tau(x+h,t) + \tau(x-h,t)$$

$$\tau(x,t) = \frac{k}{h}\left[u(x+h,t) - u(x,t)\right] + \left[-u(x,t) + u(x-h,t)\right]$$

$$= kh\left[\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}\right] \quad (2)$$

Now equating (1) & (2)

$$\frac{m}{h \times} \frac{\partial^2 u}{\partial t^2} = kh \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{kh^2}{m} \cdot \frac{\partial^2 u}{\partial x^2}$$

If we suppose that for N mass, each of density ρ the total length $L = Nh$, the total mass $M = Nm = \rho$ and the total stiffness of the array $K = k = N$, as h and $\Delta t \to 0$, so $\frac{kh^2}{m} = \frac{kL^2}{M} = \cancel{\frac{KL}{M}} = \frac{KL}{\rho}$

So. $$\frac{\partial^2 u}{\partial t^2} = \frac{KL}{\rho} \frac{\partial^2 u}{\partial x^2}$$

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