Lecture Notes, M261-004, Taylor's Formula for Two Variables

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We wrap up our chapter on differentiation with Taylor Series for functions of two variables. Taylor series are why we really care about derivatives and the basis for all their other applications, so this is an important subject even though we don't do much with them directly in this class. You may remember the Taylor Series for a function of one variable from Calculus II:

$$f(x+h) \approx f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$$

The first order Taylor series,

$$f(x+h) \approx f(x) + hf'(x)$$

is the linearization of f at x. We have already learned about the linearzation for a function of two variables:

$$f(x+h, y+k) \approx f(x,y) + h\frac{\partial f}{\partial x}(x,y) + k\frac{\partial f}{\partial y}(x,y) = f(x,y) + \nabla f(x,y) \cdot \langle h, k\rangle$$

For this lecture we will find the Quadratic Taylor approximation for a function of two variables, which uses the second-order partial derivatives.

In the case of a function of one variable, the second-order term comes from multiplying the second derivative by the displacement and dividing by two in a way that gives a quadratic in the displacement h.

$$\frac{h^2}{2}f''(x)$$

We need to do the same thing in the case of a function of two variables. Now, our second derivative is a matrix with all the second-order partial derivatives and the displacement is a vector instead of a scalar. The appropriate way to multiply them is by

$$\frac{1}{2}\begin{pmatrix} h & k \end{pmatrix}\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \frac{1}{2}\langle hf_{xx} + kf_{xy}, hf_{xy} + kf_{yy}\rangle \cdot \langle h, k\rangle = \frac{h^2}{2}f_{xx} + hkf_{xy} + \frac{k^2}{2}f_{yy}$$

We can also think of this as

$$\frac{1}{2}(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y})^2 f$$

I prefer the matrix multiplication approach, but as usual, you just need the formula for this class:

$$\frac{h^2}{2}f_{xx} + hkf_{xy} + \frac{k^2}{2}f_{yy}$$

This means that a second-order Taylor series for a function of two variables looks like

$$\begin{aligned} f(x+h, y+k) &\approx f(x,y) + \nabla f \cdot \langle h, k \rangle + \frac{1}{2} \begin{pmatrix} h & k \end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} \\ &= f(x,y) + hf_x + kf_y + \frac{h^2}{2} f_{xx} + hkf_{xy} + \frac{k^2}{2} f_{yy} \end{aligned}$$

Here are some examples:

Example 1. *Find the second-order Taylor series for*

$$f(x,y) = e^x \cos y$$

at the origin.

Example 2.

$$f(x,y) = \sin x \cos y$$

at $(\pi, \frac{3\pi}{2})$

Example 3.

$$f(x,y) = \ln(2x + y + 1)$$

at the origin.

Example 4.

$$f(x,y) = \cos(x^2 + y^2)$$

at the origin.