Indian Institute of Technology Mandi
भारतीय प्रौद्योगिकी संस्थान मण्डी

MA-201: B.Tech. II year

Odd Semester: 2011-12 **Exercise-5 Linear Algebra**

NOTE: The set of real numbers $\mathbb{R}$ is a field with addition and multiplication of real numbers as the binary compositions. If the underlying field is not mentioned then take $(\mathbb{R}, +, .)$ as the default field.

1. Consider the set $\mathbf{V} = \mathbb{C}^n$ of all ordered n-tuples of complex numbers. Prove that $\mathbf{V}$ is a vector space over the field $\mathbf{F} = \mathbb{C}$.

2. Let $\mathbf{V} = \mathbb{R}^+$ be the set of all positive real numbers. Define the operations of vector addition and scalar multication as follows:

$$\mathbf{u} + \mathbf{v} = \mathbf{u}.\mathbf{v} \; \forall \; \mathbf{u}, \mathbf{v} \in V$$
$$\alpha.\mathbf{u} = \mathbf{u}^\alpha \; \forall \; \mathbf{u}, \mathbf{v} \in \mathbf{V} \text{ and } \alpha \in \mathbb{R}$$

   Prove that $\mathbf{V}$ is a vector space over the field $\mathbf{F} = \mathbb{R}$.

3. Which of the following subsets of $\mathbf{V} = \mathbb{R}^4$ are vector spaces for co-ordinatewise addition and scalar multiplication ?

   (a) $S = \{x \in \mathbf{V} \mid x_4 = 0\}$
   (b) $S = \{x \in \mathbf{V} \mid x_1 = 1\}$
   (c) $S = \{x \in \mathbf{V} \mid x_2 > 0\}$
   (d) $S = \{x \in \mathbf{V} \mid x_3^2 \geq 0\}$
   (e) $S = \{x \in \mathbf{V} \mid x_1^2 < 0\}$
   (f) $S = \{x \in \mathbf{V} \mid 2x_1 + 3x_2 = 0\}$
   (g) $S = \{x \in \mathbf{V} \mid x_1 + \frac{3}{2}x_2 - 3x_3 + x_4 = 1\}$

4. In any vector space $\mathbf{V}$ prove that $\alpha.\mathbf{u} = \mathbf{0}_V$ iff either $\alpha = 0_F$ or $\mathbf{u} = \mathbf{0}_V$.

5. Let $\mathscr{P}$ be the set of all polynomials then find which of the following subsets of $\mathscr{P}$ are vector spaces over the field $\mathbf{F} = \mathbb{R}$.

   (a) $S = \{\mathbf{p} \in \mathscr{P} \mid \text{degree of } \mathbf{p} \leq n\}$
   (b) $S = \{\mathbf{p} \in \mathscr{P} \mid \text{degree of } \mathbf{p} = 3\}$
   (c) $S = \{\mathbf{p} \in \mathscr{P} \mid \text{degree of } \mathbf{p} \geq 4\}$
   (d) $S = \{\mathbf{p} \in \mathscr{P} \mid \mathbf{p}(1) = 0\}$
   (e) $S = \{\mathbf{p} \in \mathscr{P} \mid \mathbf{p}(2) = 1\}$
   (f) $S = \{\mathbf{p} \in \mathscr{P} \mid \mathbf{p}'(1) = 0\}$
   (g) $S = \{\mathbf{p} \in \mathscr{P} \mid \mathbf{p} \text{ has integer coefficients}\}$.

6. Let $\mathscr{C}[0,1]$ be the set of all continuous functions over the interval $[0,1]$. Which of the following subsets of $\mathscr{C}[0,1]$ are vector spaces over $\mathbf{F} = \mathbb{R}$?

   (a) $S = \{f \in \mathscr{C}[0,1] \mid f(0.5) = 0\}$
   (b) $S = \{f \in \mathscr{C}[0,1] \mid f(\frac{3}{4}) = 1\}$
   (c) $S = \{f \in \mathscr{C}[0,1] \mid f'(x) = xf(x)\}$
   (d) $S = \{f \in \mathscr{C}[0,1] \mid f(0) = f(1)\}$
   (e) $S = \{f \in \mathscr{C}[0,1] \mid f \text{ has maxima at } x = \frac{1}{2}\}$
   (f) $S = \{f \in \mathscr{C}[0,1] \mid f \text{has extrema at } x = \frac{1}{2}\}$
   (g) $S = \{f \in \mathscr{C}[0,1] \mid f(x) = 0 \text{ at finite number of points in } [0,1]\}$

7. Let $\mathbf{W} = \{(x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = 0\}$. Prove that $\mathbf{W}$ is a subspace of $\mathbb{R}^n$

8. Prove that $\mathbf{W} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{C}^n \mid \alpha_1 x_1 + \alpha_2 x_2 + \alpha x_3, \ldots, \alpha_n x_n = 0, \alpha_i' s \text{ are given constants}\}$ is a subspace of $\mathbb{C}^n$

9. Which of the following sets are subspaces of $\mathbb{R}^3$ over the field $\mathbf{F} = \mathbb{R}$?

(a) $\{(x_1, x_2, x_3) \mid x_1 x_2 = 0\}$

(b) $\{(x_1, x_2, x_3) \mid \sqrt{2}x_1 = \sqrt{3}x_2\}$

(c) $\{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$

(d) $\{(x_1, x_2, x_3) \mid x_1 = \sqrt{2}x_2, x_3 = 3x_2\}$

(e) $\{(x_1, x_2, x_3) \mid x_1 - 2x_2 = x_3 - \frac{3x_2}{2}\}$

10. Which of the following sets are subspace of $\mathscr{P}$ over the field $\mathbf{F} = \mathbb{R}$?

(a) $\{\mathbf{p} \in \mathscr{P} \mid$ degree of $\mathbf{p} = 4\}$

(b) $\{\mathbf{p} \in \mathscr{P} \mid$ degree of $\mathbf{p} \leq 3\}$

(c) $\{\mathbf{p} \in \mathscr{P} \mid$ degree of $\mathbf{p} \geq 5\}$

(d) $\{\mathbf{p} \in \mathscr{P} \mid$ degree of $\mathbf{p} \leq 4$ and $p'(0) = 0\}$

(e) $\{\mathbf{p} \in \mathscr{P} \mid \mathbf{p}(1) = 0\}$

11. which of the following sets are subspaces of $\mathscr{C}(a, b)$ over the field $\mathbf{F} = \mathbb{R}$?

(a) $\{f \in \mathscr{C}(a, b) \mid f(x_0) = 0, x_0 \in (a, b)\}$

(b) $\{f \in \mathscr{C}(a, b) \mid f'(x) = 0 \forall x \in (a, b)\}$

(c) $\{f \in \mathscr{C}(a, b) \mid f(\frac{a+b}{2}) = 1\}$

(d) $\{f \in \mathscr{C}(a, b) \mid f'(x) = x^2 f(x)\}$

(e) $\{f \in \mathscr{C}(a, b) \mid \int_a^b f(x)dx = 0\}$.

(f) $\{f \in \mathscr{C}(a, b) \mid 2f'''(x) + 3xf''(x) - f'(x) + x^2 f(x) = 0\}$.

12. If $\mathbf{U}$ and $\mathbf{W}$ are subspace of a vector space $\mathbf{V}$, prove that

(a) $\mathbf{U} \bigcap \mathbf{W}$ is a subspace of $\mathbf{W}$.

(b) $\mathbf{U} \bigcup \mathbf{W}$ is a subspace of $\mathbf{V}$ iff $\mathbf{U} \subset \mathbf{W}$ or $\mathbf{W} \subset \mathbf{U}$.