

Lecture 4 : Continuity and limits

Intuitively, we think of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ as continuous if it has a continuous curve. The term *continuous curve* means, intuitively, that the graph of f can be drawn without *jumps*, i.e., the graph can be drawn with a *continuous* motion of the pencil without leaving the paper.

Suppose a function $f : \mathbb{R} \rightarrow \mathbb{R}$ has a discontinuous graph as shown in the following figure.

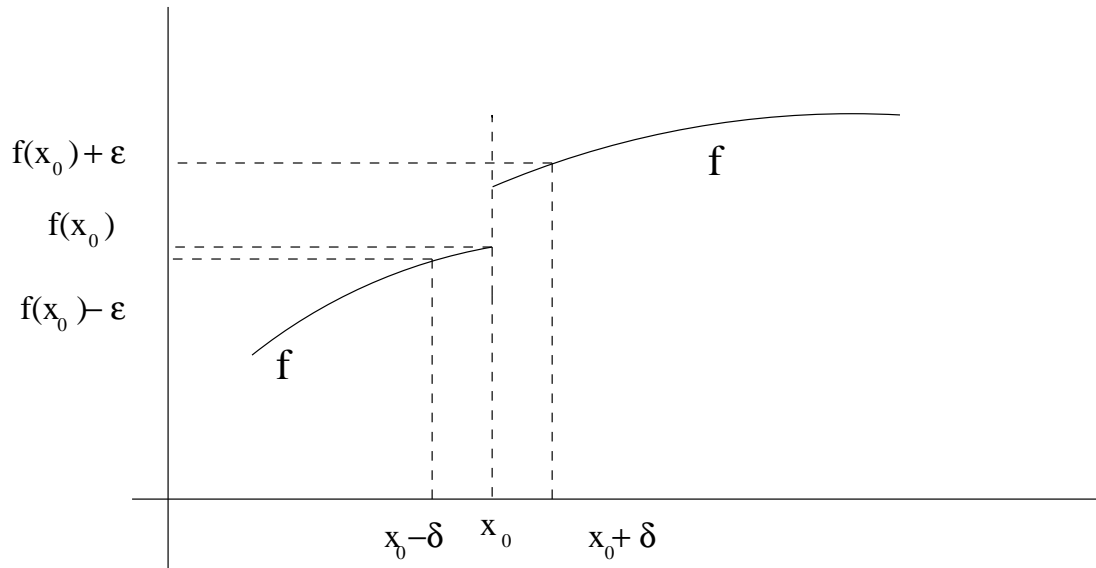


Figure 1: Discontinuous Graph

The graph is discontinuous at the point $(x_0, f(x_0))$, i.e., the function f is discontinuous at x_0 . Hence whenever x is close to x_0 from the right, $f(x)$ does not get close to $f(x_0)$. (The idea of getting close has already been discussed while dealing with convergent sequences). As shown in the figure, we can choose a neighbourhood $(f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$, $\epsilon_0 > 0$, at $f(x_0)$ such that if we take **any** neighbourhood $(x_0 - \delta, x_0 + \delta)$, $\delta > 0$, then the image of the interval $(x_0 - \delta, x_0 + \delta)$ does not lie inside $(f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$. In formal terms, **there exists** $\epsilon > 0$ such that **for all** $\delta > 0$, $|x - x_0| < \delta \not\Rightarrow |f(x) - f(x_0)| < \epsilon$. Hence if a function f is not continuous at x_0 , we have the above condition.

We will now give the formal definition of continuity of a function at a point (in the “ ϵ - δ language”).

Definition A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be continuous at a point $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

Roughly, f is continuous at x_0 if whenever x approaches x_0 , $f(x)$ approaches $f(x_0)$. In some cases when f is not continuous at x_0 , there may be a number A such that whenever x approaches x_0 , $f(x)$ approaches A . In this case we call such a number A the limit of f at x_0 . Formally, we have:

Definition : A number A is called the limit of a function f at a point x_0 if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - A| < \epsilon$ whenever $0 < |x - x_0| < \delta$. If such a number A exists then it is unique.

In this case we write $\lim_{x \rightarrow x_0} f(x) = A$. It is clear that $f(x_0)$ is the limit of f at x_0 if f is

continuous at x_0 .

The reader is advised to see the strong analogy between the definition of limit point and the definition of convergence of sequence.

Let us now characterize the continuity of a function at a point in terms of sequences.

Theorem 4.1 : *A real valued function f is continuous at $x_0 \in \mathbb{R}$ if and only if whenever a sequence of real numbers (x_n) converges to x_0 , then the sequence $(f(x_n))$ converges to $f(x_0)$.*

Proof: Suppose f is continuous at x_0 and $x_n \rightarrow x_0$. Let us show that $f(x_n) \rightarrow f(x_0)$. Let $\epsilon > 0$ be given. We must find N such that $|f(x_n) - f(x_0)| < \epsilon$ for all $n \geq N$. Since f is continuous at x_0 , there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Since $x_n \rightarrow x_0$, there exists N such that $|x_n - x_0| < \delta$ for all $n \geq N$. This N serves our purpose.

To prove the converse, let us assume the contrary that f is not continuous at x_0 . Then for some $\epsilon > 0$ and for each n , there is an element x_n such that $|x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - f(x_0)| \geq \epsilon$. This contradicts the fact that $x_n \rightarrow x_0$ implies $f(x_n) \rightarrow f(x_0)$. \square

Remark : To define the continuity of a function f at a point x_0 , the function f has to be defined at x_0 . But even if the function is not defined at x_0 , one can define the limit of a function at x_0 .

The proof of the following theorem is similar to the proof of the previous theorem.

Theorem 4.2: $\lim_{x \rightarrow x_0} f(x) = A$ if and only if whenever a sequence of real numbers (x_n) converges to x_0 , $x_n \neq x_0$ for all n , then the sequence $(f(x_n))$ converges to A .

Examples : 1. Define a function $f(x)$ such that $f(x) = 2x \sin(\frac{1}{x})$ when $x \neq 0$ and $f(0) = 0$. We will show that f is continuous at 0 using the $\epsilon - \delta$ definition and the sequential characterization.

Using the $\epsilon - \delta$ definition : Remember that for a given $\epsilon > 0$, we have to find a $\delta > 0$ (not the other way!). Note that here $x_0 = 0$ and

$$|f(x) - f(x_0)| = |2x \sin(\frac{1}{x}) - 0| \leq |2x| = 2|x - x_0|.$$

Suppose that ϵ is given. Choose any $\delta > 0$ such that $\delta \leq \frac{\epsilon}{2}$. Then we have

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta.$$

This shows that f is continuous at $x_0 = 0$.

Using the sequential characterization : Note that $|f(x)| \leq 2|x|$. Therefore, $f(x_n) \rightarrow f(0)$ whenever $x_n \rightarrow 0$. This proves that f is continuous at 0.

2. The function $f(x) = \sin(1/x)$ is defined for all $x \neq 0$. This function has no limit as $x \rightarrow 0$ because if we take $x_n = 2/\{\pi(2n+1)\}$ for $n = 1, 2, \dots$, then $x_n \rightarrow 0$ but $f(x_n) = (-1)^n$ which does not tend to any limit as $n \rightarrow \infty$.

3. Let $f(x) = 0$ when x is rational and $f(x) = x$ when x is irrational. We will see that this function is continuous only at $x = 0$. Let (x_n) be any sequence such that $x_n \rightarrow 0$. Because, $|f(x_n)| \leq |x_n|$, $f(x_n) \rightarrow f(0)$. Therefore f is continuous at 0.

Suppose $x_0 \neq 0$ and it is rational. We will show that f is not continuous at x_0 . Choose (x_n) such that $x_n \rightarrow x_0$ and all x'_n s are irrational numbers. Then $f(x_n) \nrightarrow f(x_0)$. This proves that f is not continuous at x_0 . When x_0 is irrational, the proof is similar.

Remark : In order to show that a function is not continuous at a point x_0 it is sufficient to produce one sequence (x_n) such that $x_n \rightarrow x_0$ but $f(x_n) \nrightarrow f(x_0)$. However, to show a function is continuous at x_0 , we have to show that $f(x_n) \rightarrow f(x_0)$ whenever $x_n \rightarrow x_0$ i.e, for every (x_n) such that $x_n \rightarrow x_0$.

Continuous function on a subset of \mathbb{R} : Let S be a subset of \mathbb{R} and $x_0 \in S$, we say that f is continuous at x_0 , if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in S$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$. Moreover, if f is continuous at each $x \in S$, then we say that f is continuous on S .

Limits at Infinity : Let $f : \mathbb{R} \rightarrow \mathbb{R}$. We say that $\lim_{x \rightarrow \infty} f(x) = A$ if for every $\epsilon > 0$, there exist $N > 0$ such that whenever $x \geq N$, we have $|f(x) - A| < \epsilon$.

Let $x_0 \in \mathbb{R}$. We say that $\lim_{x \rightarrow x_0} f(x) = \infty$ if for every M , there exists $\delta > 0$ such that wherever $|x - x_0| < \delta$ we have $f(x) > M$.

Problem 1: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be such that for every $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq |x - y|$. Show that f is continuous.

Solution : Let $x_0 \in \mathbb{R}$ and $x_n \rightarrow x_0$. Since $|f(x_n) - f(x_0)| \leq |x_n - x_0|$, $f(x_n) \rightarrow f(x_0)$. Therefore f is continuous at x_0 . Since x_0 is arbitrary, f is continuous everywhere.

Problem 2: Let $f : (-1, 1) \rightarrow \mathbb{R}$ be a continuous function such that in every neighborhood of 0, there exists a point where f takes the value 0. Show that $f(0) = 0$.

Solution : For every n , there exists $x_n \in (-\frac{1}{n}, \frac{1}{n})$ such that $f(x_n) = 0$. Since f is continuous at 0 and $x_n \rightarrow 0$, we have $f(x_n) \rightarrow f(0)$. Therefore, $f(0) = 0$.

Problem 3: Let $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every point $c \in \mathbb{R}$.

Solution : First note that $f(0) = 0$, $f(-x) = -f(x)$ and $f(x - y) = f(x) - f(y)$. Let $x_0 \in \mathbb{R}$ and $x_n \rightarrow x_0$. Then $f(x_n) - f(x_0) = f(x_n - x_0) \rightarrow f(0) = 0$ as f is continuous at 0 and $x_n - x_0 \rightarrow 0$.

Properties of Continuous Functions on a Closed Interval :

Definition : Let $S \subseteq \mathbb{R}$ and $f : S \rightarrow \mathbb{R}$. We say that f is bounded on S if the set $f(S) := \{f(x) : x \in S\}$ is a bounded subset of \mathbb{R} .

We will now see some properties of continuous functions on a closed interval.

Theorem 4.3 : If a function f is continuous on $[a, b]$ then it is bounded on $[a, b]$.

Proof: Suppose that f is not bounded on $[a, b]$. Then for each natural number n there is a point $x_n \in [a, b]$ such that $|f(x_n)| > n$, for all $n = 1, 2, \dots$. Since (x_n) is a bounded sequence, by Bolzano-Weierstrass theorem it has a convergent subsequence, say $x_{n_k} \rightarrow x_0 \in [a, b]$. By the continuity of f , we have $f(x_{n_k}) \rightarrow f(x_0)$. This contradicts the assumption that $|f(x_n)| > n$ for all n . Hence f is bounded on $[a, b]$. \square

We remark that if a function is continuous on an open interval (a, b) or on a semi-open interval of the type $(a, b]$ or $[a, b)$, then it is not necessary that the function has to be bounded. For example, consider the continuous function $\frac{1}{x}$ on $(0, 1]$.

In the next lecture we will see some other properties of continuous functions on a closed interval.