Classical and Quantum Computation of dual basis

It is hard to imagine that by passing to a different basis, a quantum printer suddenly becomes a quantum scanner.

1. Recall that if X is a vector space of dim $X = n < \infty$ and Y is a vector space dim $Y = m < \infty$ then for a linear transformation $X \xrightarrow{A} Y$, the matrix of A with respect to chosen coordinatizations is given by

$$(X,\ (e,\epsilon)) \xrightarrow[a=[a_i^j]_{m\times n}]{A} (Y,\ (d,\delta))$$

where ϵ is the dual basis to the basis e of X, δ is the dual basis to the basis d of Y and $a_i^j = \langle \delta^j \mid Ae_i\rangle$ is the entry at the intersection of the j-th row and the i-th column of the matrix. we have, for a given $x \in X$,

$$x = \sum_{i=0}^{n-1} e_k x^k,\ x^k = \langle \epsilon^k \mid x\rangle, \qquad ...(1)$$

$$y = Ax = \sum_{j=0}^{m-1} d_j y^j \ = \ \sum_{j=0}^{m-1} d_j \sum_{i=0}^{n-1} \langle \delta^j \mid Ae_i\rangle \ \langle \epsilon^i \mid x\rangle \qquad ...(2)$$

where $y^j \ = \ \langle \delta^j \mid y\rangle \ = \ \sum_{i=0}^{n-1} \langle \delta^j \mid Ae_i\rangle \ \langle \epsilon^i \mid x\rangle \qquad ...(3)$

We may use the symbol $[A]_e^d \ = \ a \ = \ [a_i^j]_{m\times n}$ for this matrix.

2. Now suppose X has two coordinate systems $(e,\ \epsilon)$ and $(e',\ \epsilon')$. Then we have $e_i' = \sum_{k=0}^{n-1} e_k\ b_i^k$ for uniquely given $b_i^k\ \in\ \mathbb{F}$ and also $e_i\ =\ \sum_{k=0}^{n-1} e_k'\ (b')_i^k$ for uniquely given $(b')_i^k\ \in\ \mathbb{F}$.

(i) Writing $Be_i\ =\ e_i'$ provides a linear transformation $X\ \xrightarrow{B}\ X$. Since a linear transformation is determined its values on the basis vectors e_i. Then we have $e_i'\ =\ Be_i\ =\ \sum_{k=0}^{n-1} e_k\ \langle\epsilon^k \mid Be_i\rangle$*(look at (1) above in paragraph 1)* so that $b_i^k\ =\ \langle\epsilon^k \mid Be_i\rangle$ *(because in $e_i' = \sum_{k=0}^{n-1} e_k\ b_i^k$, the scalars $b-i^k$ are uniquely given)*. Thus the matrix of

$$(X,\ (e,\epsilon)) \xrightarrow{B} (X,\ (e,\epsilon))$$

is $b\ =\ [b_i^k]_{n\times n}$. At the same time, $Be_i'\ =\ B\left(\sum_{k=0}^{n-1} e_k\ b_i^k\right)\ =\ \sum_{k=0}^{n-1}(B\ e_k)\ b_i^k\ = \sum_{k=0}^{n-1} e_k'\langle(\epsilon')^k \mid Be_i'\rangle$ so that $b_i^k\ =\ \langle(\epsilon')^k \mid Be_i'\rangle$ also $b\ =\ [b_i^k]_{n\times n}$ is also the matrix of

$$(X,\ (e',\epsilon')) \xrightarrow{\quad B \quad} (X,\ (e',\epsilon'))$$

And of course, since $Be_i = e'_i$, we have $b^k_i \ = \ \langle(\epsilon)^k \mid e'_i\rangle$ which says*(look at (1) above in paragraph 1)* that $b \ = \ [b^k_i]_{n\times n}$ is also the matrix of

$$(X,\ (e',\epsilon')) \xrightarrow{\quad Id \quad} (X,\ (e,\epsilon))$$

(ii) There are thus two ways of looking at the relation $e'_i = Be_i$

<u>**Passive**</u> There are two coordinate systems*(Call them two observers)*; the source basis $e = \{e_i\}$ is changed into the target basis $e' = \{e'_i\}$*(the observer $(e',\ \epsilon')$ replaces the observer $(e,\ \epsilon)$)*. The old coordinate system*(= the observer)* $(e,\ \epsilon)$ measures the coordinates of a vector x*(a point in the space)* as $\langle(\epsilon)^k \mid x\rangle = x^k$ and records it as $x = \sum_{k=0}^{n} |e_k\rangle\ \langle\epsilon^k \mid x\rangle$ while the new coordinate system $(e',\ \epsilon')$ records it as $x = \sum_{k=0}^{n} |e'_k\rangle\ \langle(\epsilon')^k \mid x\rangle$ measuring its coordinates as $\langle(\epsilon')^k \mid x\rangle$. This can be seen as $(X,\ (e',\epsilon')) \xrightarrow{Id} (X,\ (e,\epsilon))$ so that the vector x has not moved but the source basis e has been changed by the linear transformation B represented by the matrix $[b^k_i]_{n\times n}$ into the target basis e' and thus the matrix $[b^k_i]_{n\times n}$ can be called as the change of basis matrix; thus $b \ = \ [b^k_i]_{n\times n}$ works as follows:

Given a vector $x' = \sum_{i=0}^{n-1} e'_i(x')^i$ in terms of the target basis e', the application of b rewrites x' in terms of the source basis e

$$\begin{aligned} x &= \sum_{i=0}^{n-1} e'_i(x')^i \\ &= \sum_{i=0}^{n-1} |e'_i\rangle\ \langle(\epsilon')^i \mid x'\rangle \\ &= \sum_{i=0}^{n-1}\sum_{k=0}^{n-1} |e_k\rangle\ \langle\epsilon^k \mid Ide'_i\rangle\ \langle(\epsilon')^i \mid x'\rangle \end{aligned}$$

$$= \sum_{k=0}^{n-1} |e_k\rangle\ [\sum_{i=0}^{n-1} b^k_i(x')^i]$$

Let us note carefully that the arrow in $(X,\ (e',\epsilon')) \xrightarrow{Id} (X,\ (e,\epsilon))$ points from the 'target basis' e' to the 'source basis' 'e'. The 'source-target' vocabulary is with reference to the linear transformation B *(e_i gets transformed to $e'_i = Be_i$)*. Thus we have $B^e_e = B^{e'}_{e'} = P^e_{e'} = [Id]^e_{e'}$ in the notation of paragraph 1 page 1. The other interpretation is

<u>**Active**</u> There is only one coordinate system*(= observer)*, say $(e,\ \epsilon)$, for X. The $x = \sum_{i=0}^{n-1} e_i x^i$ moves to a different vector $u = Bx = \sum_{i=0}^{n-1} (Be_i)x^i = \sum_{i=0}^{n-1} e'_i x^i$ and the new vector u has components, with respect to $(e,\ \epsilon)$,

$$u^k = \langle \epsilon^k \mid u \rangle = \sum_{i=0}^{n-1} \langle \epsilon^k \mid Be_i \rangle x^i$$

So that $u = \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} e_k\ b^k_i x^i$.

(while of course, the k-th component of x is $\langle \epsilon^k \mid x \rangle = x^k$)

3. Taking the passive interpretation, let us write the change of basis matrix b from $\{e_i\}$ to $\{e'_i = Be_i\}$ as $P^e_{e'}$ so that $P^e_{e'}(e'_i) = \sum_{i=0}^{n-1} e_k\ b^k_i$ *(that is, e'_i has been written in terms of the $\{e_k\}$; see the summary at the end of page 2 above)* where $b^k_i = \langle \epsilon^k \mid e_i \rangle$.

Example 4.1 Consider $\mathbb{R}^3 = X$ the three-dimensional vector space over $\mathbb{R}$. Take $x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in X

then if $e = \left\{ e_0 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix},\ e_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}$ we have

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = e_0\,(7a - 3b + c) + e_1(-6a + 3b - c) + e_2(4a - 2b + c)$$

saying that each x is expressible uniquely as $\sum_{i=0}^{2} e'_i (x')^i$ as well and e' is also a basis. Then because $e'_0 = 2e_0 - e_1 + e - 2$, $e'_1 = -e_0 + e_1 + 0e_2$, $e'_2 = e_0 + 0e_1 + 2e_2$ the change of basis matrix which changes e into e' is $P^e_{e'} = b = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$ *(Obtained by writing components of $e'_o,\ e'_1,\ e'_2$ with respect to e as columns)*

Now $P^{e}_{e'}(\sum_{i=0}^{2} e'_i (x')^i) = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2a+2b-c \\ -8a+5b-2c \\ 3a-2b+c \end{bmatrix} = \begin{bmatrix} 7a+3b+c \\ -6a+3b-c \\ 4a-2b+c \end{bmatrix} = \sum_{i=0}^{2} e_i x^i$

which confirms

$$(X,\ (e', \epsilon')) \xrightarrow[P^{e}_{e'} = b = [b^j_i]]{Id} (X,\ (e, \epsilon))$$

in this case.

4. Consider now the matrix which changes e' into e. this will be the matrix b' given by

$$(X,\ (e, \epsilon)) \xrightarrow[P^{e'}_{e} = b' = [(b')^j_i]_{n\times n}]{Id} (X,\ (e', \epsilon'))$$

where $(b')^j_i = \langle (\epsilon')^j \mid e_i \rangle$ where $e_i = \sum_{k=0}^{n-1} e'_k (b')^k_i$ and we may write $B' e'_i = e_i$ which provides a linear transformation $X \xrightarrow{B'} X$ ($\because$ $e' = \{e'_i\}$ *is a basis and a linear transformation is determined by its values on the elements of a basis).* Then $BB'e'_i = Be_i = e'_i$ and $B'Be_i = Id = B'B$. We of course write B^{-1} for B'.

Then $(b')^j_i = \langle (\epsilon')^j \mid e_i \rangle = \langle (\epsilon')^j \mid B'e'_i \rangle$ presents b' as the matrix for B' as

$$(X,\ (e', \epsilon')) \xrightarrow[b' = [(b')^k_i]_{n\times n}]{B'} (X,\ (e', \epsilon'))$$

while $B'e_i = B'\left(\sum_{k=0}^{n-1} e'_k (b')^k_i\right) = \sum_{k=0}^{n-1} (B'e'_k)\,(b')^k_i = \sum_{k=0}^{n-1} e_k (b')^k_i = \sum_{k=0}^{n-1} e_k \langle \epsilon^k \mid B'e_i \rangle$

($\because$ $\sum_{k=0}^{n-1} e_k \langle \epsilon^k \mid B'e_i \rangle$ being the unique representation in terms of the basis e for $B'e_i$ like any vector $x = \sum_{k=0}^{n-1} e_k \langle \epsilon^k \mid x \rangle$)

we get $(b')^k_i = \langle \epsilon^k \mid B'e_i \rangle$ which means b' is a matrix of

$$(X,\ (e, \epsilon)) \xrightarrow[b' = [(b')^k_i]]{B'} (X,\ (e, \epsilon))$$

Example*(continued)* With the illustration above in 3, we have

$$P^{e'}_{e} = (P^{e}_{e'})^{-1} = b^{-1} = b' = [(b^{-1})^j_i] = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

which can be verified directly as above and we get

$$\begin{bmatrix} 2 & 2 & -1 \\ 2 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7a+3b+c \\ -6a+3b-c \\ 4a-2b+c \end{bmatrix} = \begin{bmatrix} -2a+2b-c \\ -8a+5b-2c \\ 3a-2b+c \end{bmatrix}$$ which confirms

$$(X,\ (e',\epsilon')) \xrightarrow[P_e^{e'} \ = b^{-1} = \ b' = \ [(b')_i^j]]{Id} (X,\ (e,\epsilon))$$

in this case.

5. The preceding discussion shows that a change of basis matrix is invertible. Now take an invertible matrix $a = [a_i^j]$, then $e_i = a^{-1}ae_i$ shows that each

$$\begin{aligned} x &= \sum_{k=0}^{n-1} e_i x^i \\ &= \sum_{k=0}^{n-1} a^{-1}(ae_i)x^i \\ &= \sum_{i=0}^{n-1}\left(\sum_{k=0}^{n-1} (ae_k)(a^{-1})_i^k\right)x^i \\ &= \sum_{i=0}^{n-1} (ae_k)\left(\sum_{k=0}^{n-1} (a^{-1})_i^k x^i\right) \\ &= \sum_{i=0}^{n-1} (ae_k)\lambda^k \quad (say) \end{aligned}$$

showing that each x is expressible as uniquely as a linear combination of the vectors $\{ae_k\}$ constitute a basis and a is the change of basis matrix from the source basis $\{e_i\}$ to the target basis $\{e_i' \ = \ ae_i\}$.*(To understand why we wrote $a^{-1}(ae_i)$ as $\sum_{i=0}^{n-1} (ae_k)(a^{-1})_i^k$, note that when B' is a linear transformation, the formula is $B'(e_i') \ = e_i \ = \sum_{i=0}^{n-1} (e_k')(b')_i^k$, here the application of a^{-1} produces a linear transformation B' and we have $ae_i \ = \ e_i'$ so that the matrix of B' here is $b' \ = a^{-1}$ itself and we have $e_i \ = \ a^{-1}(ae_i) \ = \sum_{i=0}^{n-1} (e_k')(b')_i^k \ = \sum_{i=0}^{n-1} (ae_k)(a^{-1})_i^k$).*

Thus a change of basis matrix is the same as an invertible matrix.

6. If $P_{e'}^e \ = \ b \ = \ [b_i^j]$ is the change of basis matrix changing e to e' the formula $e_i' \ = \ \sum_{j=0}^{n-1} (e_j)b_i^j$ says that we expand the vector e_i', that is, the $i-th$ vector of the target basis e', in terms of the source basis $e = \{e_j\}$; the components in this expansion*(namely, the scalars b_i^j)* then supply the i-th column of the matrix $P_{e'}^e \ = \ b$ which is $b_i \ = \begin{bmatrix} b_i^0 \\ \vdots \\ b_i^{n-1} \end{bmatrix}$

Example 0.1. Consider $\mathbb{K} = \mathbb{R}$, $\mathbb{X} = \mathbb{R}^2$; then $e = \left\{ e_0 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, e_1 = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right\}$ is a basis $\left(\because \begin{bmatrix} a \\ b \end{bmatrix} = e_0(-2a - \frac{3}{2}b) + e_1(a + \frac{1}{2}b) \textit{ uniquely expressed} \right)$

and $e' = \left\{ e_0 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, e_1 = \begin{bmatrix} 3 \\ 8 \end{bmatrix} \right\}$ is also a basis $\left(\because \begin{bmatrix} a \\ b \end{bmatrix} = e_0'(-8a + 3b) + e_1'(3a - b) \textit{ uniquely expressed} \right)$.

Then since $e_0' = -\frac{13}{2}e_0 + \frac{5}{2}e_1$ and $e_1' = -18e_0 + 7e_1$, we obtain the first column of $P^e_{e'}$ as $\begin{bmatrix} -13/2 \\ 5/2 \end{bmatrix}$ and the second column of $P^e_{e'}$ as $\begin{bmatrix} -18 \\ 7 \end{bmatrix}$ so that we have

$$P^e_{e'} = \begin{bmatrix} -13/2 & -18 \\ 5/2 & 7 \end{bmatrix}$$

Also, $P^{e'}_e$ *(the matrix which changes the basis e' to e, e' being the source basis and e being the target basis so that $P^{e'}_e$ is the matrix of $(X,\ (e,\epsilon)) \xrightarrow{Id} (X,\ (e',\epsilon'))$ and the linear transformation is $(X,\ (e',\epsilon')) \xrightarrow{B} (X,\ (e,\epsilon))$)* is given by

$$P^{e'}_e = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}.$$

(One can get it using the relationship $P^{e'}_e = (P^e_{e'})^{-1}$ since $\begin{bmatrix} -13/2 & -18 \\ 5/2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} -14 & -36 \\ 5 & 13 \end{bmatrix}$. But it is instructive to proceed ab initio and calculate $e_1 = -14e_1' + 5e_2'$, $e_2 = -36e_1' + 13e_2'$ directly).

7. Now suppose $(e", \epsilon")$ is a third coordinatization of X and $P^e_{e'} = b$, $P^{e'}_{e''} = c$ so that $b = [b^j_i]_{n\times n}$,

$c = [c^k_j]_{n\times n}$ are supplied by $e'_i = \sum_{j=0}^{n-1} (e_j)b^j_i$, $e_j" = \sum_{k=0}^{n-1} (e'_k)c^j_k$. We find

$$\begin{aligned} e" &= \sum_{k=0}^{n-1} \left(\sum_{l=0}^{n-1} e_l b^l_k\right) c^k_j \\ &= \sum_{l=0}^{n-1} e_l \left(\sum_{k=0}^{n-1} b^l_k c^k_j\right) \\ &= \sum_{l=0}^{n-1} e_l (bc)^l_j \end{aligned}$$

(recall that to get the entry $(bc)^l_j$ of the product matrix bc, you have to multiply the l-th row $[b^l_0 \dots b^l_{n-1}]$ of b to the j-th column $\begin{bmatrix} c^0 \\ \vdots \\ c^{n-1}_j \end{bmatrix}$ term by term and add up: that is, the formula for bc is given by $(bc)^l_j = \sum_{k=0}^{n-1} b^l_k c^k_j$)

Thus we find that $P^e_{e''} = bc$ which should be seen in the context of the composition

$(X,\ (e,\epsilon)) \xrightarrow[P^{e'}_{e''}=c]{Id} (X,\ (e'',\epsilon'')) \xrightarrow[P^e_{e'}=b]{Id} (X,\ (e,\epsilon)) =$
$(X,\ (e'',\epsilon'')) \xrightarrow[P^e_{e''}=bc]{Id} (X,\ (e,\epsilon))$

Notice that in the result $P^e_{e''} = P^e_{e'} = P^{e'}_{e''}$, e' gets erased when it occurs both as a subscript and a superscript.

8. Consider now a linear transformation $X \xrightarrow{T} X$ we recall that

$X \xrightarrow{Id} X = X \xrightarrow{\sum_{k=0}^{n-1} |e_k\rangle\langle\epsilon^k|} X$ *(the 'decomposition of identity')*

Then we have $X \xrightarrow{T} X = X \xrightarrow{\sum_{k=0}^{n-1} |e_k\rangle\langle\epsilon^k|} X \xrightarrow{T} X \xrightarrow{\sum_{l=0}^{n-1} |e_l\rangle\langle\epsilon^l|} X$ and hence

$$\begin{aligned} \langle(\epsilon')^j \mid Te'_i\rangle &= \left\langle (\epsilon')^j \mid \left[\sum_{l=0}^{n-1} |e_l\rangle\langle\epsilon^l|T\left(\sum_{k=0}^{n-1} |e_k\rangle\langle\epsilon^k|\right)(e'_i)\right]\right\rangle \\ &= \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}\langle(\epsilon')^j \mid e_l\rangle\langle\epsilon^l \mid Te_k\rangle\langle\epsilon^k \mid e'_i\rangle \\ &= \sum_{k=0}^{n-1}\sum_{l=0}^{n-1}(b^{-1})^j_l\langle\epsilon^l \mid Te_k\rangle b^k_i\rangle \qquad \dots(5) \end{aligned}$$

(we used $(b')^j_l = (b^{-1})^j_l = \langle(\epsilon')^j \mid e_l\rangle$ and $\langle\epsilon^k \mid e'_i\rangle = b^k_i$ where $b = P^e_{e'}$, $b^{-1} = P^{e'}_e$, b being the matrix of $(X,\ (e'',\epsilon'')) \xrightarrow[P^e_{e''}=bc]{Id} (X,\ (e,\epsilon))$ which changes the source basis e' to the target

basis e; these have been calculated earlier.)

What does this equation (5)say? It refers to two coordinatizations, (e,ϵ) and (e',ϵ') of X and two matrices representing the linear transformation T:

$t^l_k := \langle \epsilon^l \mid Te_k\rangle$ for $[T]^e_e$,

$s^j_i := \langle (\epsilon')^j \mid Te'_i\rangle$ for $[T]^{e'}_{e'}$.

(Compare with $A^d_e = a = [a^j_i];\ a^j_i = \langle \delta^j \mid Ae_i\rangle$ *in* $(X,\ (e,\epsilon)) \ \xrightarrow{A} (Y,\ (d,\delta)))$

There are two interpretations of (5)

(i) **<u>Passive</u>** There is one operator $X \xrightarrow{T} X$ and two coordinate systems*(= observers)*, (e,ϵ) and (e',ϵ'), for X. Given $x \in X$,

$$\begin{aligned}
x &= \sum_{i=0}^{n-1} |e'_i\rangle (x')^i, \\
\text{we have } Tx &= \sum_{i=0}^{n-1} |Te'_i\rangle (x')^i \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} |e'_j\rangle s^j_i (x')^i \\
&\quad \left(\because\ Te'_i = \sum_{j=0}^{n-1} e'_j \langle \epsilon'^j \mid Te'_i\rangle = \sum_{j=0}^{n-1} e'_j s^j_i \right) \\
&= \sum_{j=0}^{n-1} |e'_j\rangle \left(\sum_{i=0}^{n-1} s^j_i (x')^i \right),
\end{aligned}$$

so the components of Tx with reference to the coordinate system (e',ϵ') are $\sum_{i=0}^{n-1} s^j_i (x')^i$, $0 \le j \le n-1$;

$$\begin{aligned}
Tx &= \sum_{j=0}^{n-1} |e'_j\rangle (Tx)^j \\
&= \sum_{j=0}^{n-1} |e'_j\rangle \langle \epsilon'^j \mid Tx\rangle \\
&= \sum_{j=0}^{n-1} |e'_j\rangle \left(\sum_{i=0}^{n-1} s^j_i (x')^i \right) \quad \text{where } s^j_i = \langle \epsilon'^j \mid Te'_i\rangle.
\end{aligned}$$

For the same vector $x = \sum_{i=0}^{n-1} |e_i\rangle x^i = \sum_{i=0}^{n-1} |e_i\rangle\langle\epsilon^i \mid x\rangle$, we have

$$\begin{aligned}
Tx &= \sum_{i=0}^{n-1} |Te_i\rangle x^i \\
&= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} |e_j\rangle t^j_i x^i \\
&\quad \left(\because\ Te_i = \sum_{j=0}^{n-1} e_j\langle\epsilon^j \mid Te_i\rangle = \sum_{j=0}^{n-1} |e_j\rangle t^j_i\right) \\
&= \sum_{j=0}^{n-1} |e_j\rangle \left(\sum_{i=0}^{n-1} t^j_i x^i\right)
\end{aligned}$$

so the components of the same vector Tx with reference to (e, ϵ) are $\sum_{i=0}^{n-1} t^j_i x^i$, $0 \le j \le n-1$;

$$\begin{aligned}
Tx &= \sum_{j=0}^{n-1} |e_j\rangle (Tx)^j \\
&= \sum_{j=0}^{n-1} |e_j\rangle\langle\epsilon^j \mid Tx\rangle \\
&= \sum_{j=0}^{n-1} |e_j\rangle \left(\sum_{i=0}^{n-1} t^j_i x^i\right) \quad \text{where } t^j_i = \langle\epsilon^j \mid Te_i\rangle.
\end{aligned}$$

Thus no two vectors moves; the two coordinate systems*(= observers)* measure their coordinates.

(ii) **Active** There is only one coordinate system but there are two operators $X \xrightarrow{T,\ S} X$, $S := B^{-1}TB$, where B is an invertible operator on X which of course changes e_i to $Be_i = e'_i$ say. The matrix $[S]^e_e$ has entries

$$S^j_i = \langle\epsilon^j \mid Se_i\rangle = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} (b^{-1})^j_l t^l_k b^k_i$$

Each vector $x = \sum_{i=0}^{n-1} |e_i\rangle x^i$ moves under the action of S to

$$\begin{aligned}
Sx &= \sum_{i=0}^{n-1} |Se_i\rangle x^i \\
&= \sum_{j=0}^{n-1} |e_j\rangle (Sx)^j \text{ where } (Sx)^j = \sum_{i=0}^{n-1} s^j_i x^i \\
&\quad (s^j_i = \langle\epsilon^j \mid Se_i\rangle = \langle(\epsilon')^j \mid Te'_i\rangle = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} (b^{-1})^j_l t^l_k b^k_i).
\end{aligned}$$

that is, $S = B^{-1}TB$, and the $(j \times i)$-entry of the matrix of S is $s^j_i = \sum_{k=0}^{n-1}\sum_{l=0}^{n-1} (b^{-1})^j_l t^l_k b^k_i$, all matrices being calculated with respect to the same coordinate-system $(e,\ \epsilon)$.

(iii) The 'Active interpretation' prompts a definition:

Two $n \times n$ matrices S and T are called similar iff there exists an invertible $n \times n$ matrix B such that $S = B^{-1}TB$.

This is clearly an equivalence relation on $Mat_n(\mathbb{F})$

Reflexivity: $T = (Id_n)^{-1}T(Id_n)$

Symmetry: $S = B^{-1}TB$ iff $T = (B^{-1})^{-1}SB^{-1}$

Transitivity: $S = B^{-1}TB,\ R = A^{-1}SA$ ensure $R = (BA)^{-1}T(BA)$. Now taking T to be Id and noting $Id = \sum_{i=0}^{n-1} |e_i\rangle\langle|$, we find that the j-th component of a vector $x \in X$ with respect to $e' = (e',\ \epsilon)$ to be