## 0.1 Change of Basis

**Definition:** Let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis of a vector space $V$, and let $B' = \{v1, v2, \ldots, v_n\}$ be another basis. (For reference, we will call S the "old" basis and $B'$ the "new" basis) Since $B$ is a basis, each vector in the "new" basis $B'$ can be written uniquely as a linear combination of the vectors in $BS$; say,

$$\begin{array}{l} v_1 = a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n \\ v_2 = a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_n = a_{n1}u_1 + a_{2n}u_2 + \ldots + a_{2n}u_n \end{array}$$

Let $P$ be the transpose of the above matrix of coefficients; that is, let $P = [p_{ij}]$, where $p_{ij} = a_{ji}$. Then $P$ is called the change-of-basis matrix (or transition matrix) from the "old" basis $B$ to the "new" basis $B'$. The following remarks are in order.

**Remark 1:** The above change-of-basis matrix $P$ may also be viewed as the matrix whose columns are, respectively, the coordinate column vectors of the "new" basis vectors $v$ relative to the "old" basis $B$; namely, $P = [[v_1]_B, [v2]_B, \ldots, [v_n]_B]$.

**Remark 2:** Analogously, there is a change-of-basis matrix $Q$ from the "new" basis $B'$ to the "old" basis $B$. Similarly, $Q$ may be viewed as the matrix whose columns are, respectively, the coordinate column vectors of the "old" basis vectors u relative to the "new" basis $B'$; namely, $Q = [[u_l]_{B'}, [u_2]_{B'}, \ldots [u_n]B']$.

**Remark 3:** Since the vectors $v_1, v_2, \ldots v_n$, in the new basis $B'$ are linearly independent, the matrix $P$ is invertible . Similarly, $Q$ is invertible. In fact, we have the following proposition .

**Proposition:** Let $P$ and $Q$ be the above change-of-basis matrices. Then $Q = P^{-1}$.

**Example :** Consider the following two bases of $\mathbb{R}^2$

$B = \{u_1, u_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$, and $B' = \{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$,

1. Find the change-of-basis matrix $P$ from $B$ to the "new" basis $B'$.
2. Find the change-of-basis matrix $Q$ from the "new" basis $B'$ back to the "old" basis $B$.

**Solution:**

1. Write each of the new basis vectors of $B'$ as a linear combination of the original basis vectors $u_1$ and $u_2$ of $B$. We have

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow \begin{array}{rr} x+3y= & 1 \\ 2x+5y= & -1 \end{array} \Rightarrow \begin{array}{rr} x= & -8 \\ y= & 3 \end{array}$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow \begin{array}{rr} x+3y= & 1 \\ 2x+5y= & -2 \end{array} \Rightarrow \begin{array}{rr} x= & -11 \\ y= & 4 \end{array}$$

Thus $\begin{array}{rr} v_1 = & -8u_1 + 3u_2 \\ v_2 = & -11u_1 + 4u_2 \end{array}$ and hence $P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$.

Note that the coordinates of $v_1$ and $v_2$ are the columns, not rows, of the change-of-basis matrix $P$.

2. Here we write each of the "old" basis vectors $u_1$ and $u_2$ of $B'$ as a Linear combination of the "new" basis vectors $v_1$ and $v_2$ of $S'$. We have

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \begin{array}{rr} x+y= & 1 \\ -x-2y= & 2 \end{array} \Rightarrow \begin{array}{rr} x= & 4 \\ y= & -3 \end{array}$$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \begin{array}{rr} x+y= & 1 \\ -x-2y= & 2 \end{array} \Rightarrow \begin{array}{rr} x= & 11 \\ y= & -8 \end{array}$$

Thus $\begin{array}{rr} u_1 = & 4v_1 - 3v_2 \\ u_2 = & 11v_1 - 8v_2 \end{array}$ and hence $Q = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}$.

As expected , $Q = P^{-1}$. (In fact, we could have obtained $Q$ by simply finding $P^{-1}$.

**Example :** Consider the following two bases of $\mathbb{R}^3$

$E = \{e_1, e_2, e_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $B = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$,

1. Find the change-of-basis matrix $P$ from $E$ to the basis $B$.
2. Find the change-of-basis matrix $Q$ from the basis $B$ back to the "old" basis $E$.

**Solution:**

1. Since $E$ is the usual basis, we can immediately write each basis element of $E$ as a Linear combination of the basis elements of $B$. Specifically,

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1e_1 + 0e_2 + 1e_3$$

$$u_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 2e_1 + 1e_2 + 2e_3$$

$$u_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1e_1 + 2e_2 + 2e_3$$

$\Longrightarrow P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$. Again, the coordinates of $u_1, u_2, u_3$ appear as the columns in $P$. Observe that $P$ is simply the matrix whose columns are the basis vectors of $B$. This is true only because the original basis was the usual basis $E$.

2. The definition of the change-of-basis matrix $Q$ tells us to write each of the (usual) basis vectors in $E$ as a linear combination of the basis elements of $B$. This yields

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -2e_1 + 2e_2 - 1e_3$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -2e_1 + 1e_2 + 0e_3$$

$$e_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 3e_1 - 2e_2 + 1e_3$$

$$\Longrightarrow P = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}.$$

We emphasize that to find $Q$, we need to solve three $3 \times 3$ systems of linear equations each of $e_1, e_2, e_3$. Alternatively, we can find $Q = P^{-1}$ by forming the matrix $M = [P, I]$ and row reducing $M$ to row canonical form as

$$\Longrightarrow M = [P|I]$$

$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & 3 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] = [I|P^{-1}]$$

$$Q = P^{-1} = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix}$$

(Here we have used the fact that $Q$ is the inverse of $P$.)
The result in above is true in general. We state this result formally since it occurs often.

**Proposition :** The change-of-basis matrix from the usual basis $E$ of $\mathbb{R}^n$ to any basis $B$ of $\mathbb{R}^n$ is the matrix $P$ whose columns are, respectively, the basis vectors of $B$.
**Theorem :** Let $P$ be the change-of-basis matrix from a basis $B$ to a basis $B'$ in a vector space $V$. Then, for any linear operator $T$ on $V$,

$$[T]_{B'} = P^{-1}[T]_B P$$

That is, if $A$ and $B$ are the matrix representations of $T$ relative, respectively, to $B$ and $B'$, then $B = P^{-1}AP$.

**Example :** Consider the following two bases of $\mathbb{R}^3$

$$E = \{e_1, e_2, e_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}, \text{ and } B = \{u_1, u_2, u_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right\}.$$

1. Write $v = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ as a Linear combination of $u_1, u_2, u_3$ or, equivalently, find $[V]_B$. .

2. Let $P[v]_{B'} = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix}$, which may be viewed as a Linear operator on $\mathbb{R}^3$. Find the matrix $C$ that represents $A$ relative to the basis $B$.

**Solution:** The change-of-basis matrix $P$ from $E$ to $B$ and its inverse $P^{-1}$ were obtained in earlier example

1. One way to do this is to directly solve the vector equation $v = xu_1 + yu_2 + zu_3$,that is,

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Longrightarrow \begin{array}{ccccccc} x & + & 2y & + & z & = & 1 \\ & & y & + & 2z & = & 3 \\ x & + & 2y & + & 2z & = & 5 \end{array}.$$

The solution is $x = 7, y = -5, z = 4$, so $v = 7u_1 - 5u_2 + 4u_3$

$$\Longrightarrow [v]_B = \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}.$$

On the other hand, we know that $[v]_E = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, since $E$ is the usual basis, and we already know $P^{-1}$. Therefore, by

$$[v]_B = P^{-1}[v]_E = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

Thus, again, $[v]_B = 7u_1 - 5u_2 + 4u_3$.

2. The definition of the matrix representation of $A$ relative to the basis $B$ tells us to write each of $A(u_1), A(u_2), A(u_3)$ as a linear combination of the basis vectors $u_1, u_2, u_3$ of $B$. This yields

$$A(u_1) = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 11u_1 - 5u_2 + 6u_3$$

$$A(u_2) = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix} = 21 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 14 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 8 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 21u_1 - 14u_2 + 8u_3$$

$$A(u_3) = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = 17 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 8 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 17u_1 - 8u_2 + 2u_3$$

$$\Longrightarrow C = \begin{bmatrix} 11 & 21 & 17 \\ -5 & -14 & -8 \\ 6 & 8 & 2 \end{bmatrix}.$$

We emphasize that to find $C$, we need to solve three $3 \times 3$ systems of linear equations each of $A(u_1), A(u_2), A(u_3)$. On the other hand, since we know $P$ and $P^{-1}$, we can use the theorem That is,

$$C = P^{-1}AP = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 21 & 17 \\ -5 & -14 & -8 \\ 6 & 8 & 2 \end{bmatrix}.$$

This, as expected, gives the same result.