IC-121 : Mechanics of Particles and Waves
Tutorial-2.

1) Find the expression for velocity and acceleration in cylindrical coordinates.

Soln

In cylindrical coordinate system

$$x = \rho\cos\phi$$
$$y = \rho\sin\phi$$
$$z = z$$

The unit vectors are $\hat{\rho}, \hat{\phi}, \hat{z}$, which can be found out

$$\hat{\rho} = \frac{\frac{\partial \mathbf{r}}{\partial \rho}}{\left|\frac{\partial \mathbf{r}}{\partial \rho}\right|}$$

$$r = x\hat{x} + y\hat{y} + z\hat{z}$$
$$= \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y} + z\hat{z}$$

$$\frac{\partial r}{\partial \rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\left|\frac{\partial r}{\partial \rho}\right|^2 = \cos^2\phi + \sin^2\phi = 1.$$

$$\therefore \hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\frac{\partial r}{\partial \phi} = -\rho\sin\phi\,\hat{x} + \rho\cos\phi\,\hat{y}$$

$$\left|\frac{\partial r}{\partial \phi}\right|^2 = \rho^2$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\frac{\partial r}{\partial z} = \hat{z}$$

$$\left|\frac{\partial r}{\partial z}\right| = 1$$

$$\therefore \hat{z} = \hat{z}$$

Thus,

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$
$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$
$$\hat{z} = \hat{z}$$

$$\frac{\partial\hat{\rho}}{\partial\rho} = 0 \qquad \frac{\partial\hat{\phi}}{\partial\rho} = 0 \qquad \frac{\partial\hat{z}}{\partial\rho} = 0$$

$$\frac{\partial\hat{\rho}}{\partial\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \qquad \frac{\partial\hat{\phi}}{\partial\phi} = -\cos\phi\,\hat{x} - \sin\phi\,\hat{y} \qquad \frac{\partial\hat{z}}{\partial\phi} = 0$$

$$\frac{\partial\hat{\rho}}{\partial z} = 0 \qquad \frac{\partial\hat{\phi}}{\partial z} = 0 \qquad \frac{\partial\hat{z}}{\partial z} = 0$$

Path increment

$$d\vec{r} = d(\rho\hat{\rho} + z\hat{z})$$
$$= \rho\, d\hat{\rho} + \hat{\rho}\, d\rho + \hat{z}\, dz + z\, d\hat{z}$$
$$= \hat{\rho}\, d\rho + \rho\left[\frac{\partial\hat{\rho}}{\partial\rho}\cdot d\rho + \frac{\partial\hat{\rho}}{\partial\phi}\, d\phi + \frac{\partial\hat{\rho}}{\partial z}\, dz\right]$$
$$+ \hat{z}\, dz + \hat{z}\left[\frac{\partial\hat{z}}{\partial\rho}\, d\rho + \frac{\partial\hat{z}}{\partial\phi}\, d\phi + \frac{\partial\hat{z}}{\partial z}\, dz\right]$$

Substituting from above values, we get

$$dr = \hat{\rho}\, d\rho + \hat{\phi}\rho\, d\phi + \hat{z}\, dz$$

Time derivative of unit vector

$$\frac{\partial\hat{\rho}}{\partial t} = \cancel{\frac{\partial\hat{\rho}}{\partial\rho}}^{\,0}\cdot\frac{\partial\rho}{\partial t} + \frac{\partial\hat{\rho}}{\partial\phi}\cdot\frac{\partial\phi}{\partial t} + \cancel{\frac{\partial\hat{\rho}}{\partial z}}^{\,0}\cdot\frac{\partial z}{\partial t}$$

$$\frac{\partial\hat{\phi}}{\partial t} = \cancel{\frac{\partial\hat{\phi}}{\partial\rho}}^{\,0}\cdot\frac{\partial\rho}{\partial t} + \frac{\partial\hat{\phi}}{\partial\phi}\cdot\frac{\partial\phi}{\partial t} + \cancel{\frac{\partial\hat{\phi}}{\partial z}}^{\,0}\cdot\frac{\partial z}{\partial t}$$

$$\frac{\partial\hat{z}}{\partial t} = \cancel{\frac{\partial\hat{z}}{\partial\rho}}^{\,0}\cdot\frac{\partial\rho}{\partial t} + \cancel{\frac{\partial\hat{z}}{\partial\phi}}^{\,0}\cdot\frac{\partial\phi}{\partial t} + \cancel{\frac{\partial\hat{z}}{\partial z}}^{\,0}\cdot\frac{\partial z}{\partial t}$$

So,

$$\frac{\partial \hat{\rho}}{\partial t} = \left(-\sin\phi\, \hat{x} + \cos\phi\, \hat{y}\right)\frac{\partial \phi}{\partial t} = \hat{\phi}\,\dot{\phi}$$

$$\frac{\partial \hat{\phi}}{\partial t} = \left(-\cos\phi\, \hat{x} - \sin\phi\, \hat{y}\right)\frac{\partial \phi}{\partial t} = -\hat{\rho}\,\dot{\phi}$$

$$\frac{\partial \hat{z}}{\partial t} = 0$$

Velocity

$$\vec{v} = \dot{\vec{r}} = \overset{\hat{\phi}\dot{\phi}}{\dot{\hat{\rho}}}\rho + \hat{\rho}\dot{\rho} + \hat{z}\dot{z} + z\overset{0}{\dot{\hat{z}}}$$

$$= \hat{\rho}\dot{\rho} + \hat{\phi}\rho\dot{\phi} + \hat{z}\dot{z}$$

Acceleration

$$\vec{a} = \dot{\vec{v}} = \hat{\rho}\ddot{\rho} + \dot{\rho}\overset{\hat{\phi}\dot{\phi}}{\dot{\hat{\rho}}} + \hat{\phi}\rho\ddot{\phi} + \hat{\phi}\dot{\rho}\dot{\phi} + \rho\dot{\phi}\overset{-\hat{\rho}\dot{\phi}}{\dot{\hat{\phi}}} + \hat{z}\ddot{z} + \overset{0}{\dot{\hat{z}}}\dot{z}$$

$$= \hat{\rho}\ddot{\rho} + \dot{\rho}\hat{\phi}\dot{\phi} + \hat{\phi}\rho\ddot{\phi} + \hat{\phi}\dot{\rho}\dot{\phi} - \hat{\rho}\rho\dot{\phi}^2 + \hat{z}\ddot{z}$$

$$a = \hat{\rho}\left(\ddot{\rho} - \rho\dot{\phi}^2\right) + \hat{\phi}\left(\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi}\right) + \hat{z}\ddot{z}$$

2. Express the unit vectors in spherical polar coordinates $(\hat{r}, \hat{\theta}, \hat{\phi})$ in terms of Cartesian coordinates $(\hat{x}, \hat{y}, \hat{z})$. Work out the inverse formulas giving $(\hat{x}, \hat{y}, \hat{z})$ in terms of $(\hat{r}, \hat{\theta}, \hat{\phi})$.

Soln

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$= r\sin\theta\cos\phi\, \hat{x} + r\sin\theta\sin\phi\, \hat{y} + r\cos\theta\, \hat{z}$$

$$\hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} \qquad \hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|} \qquad \hat{\phi} = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$$

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\, \hat{x} + \sin\theta\sin\phi\, \hat{y} + \cos\theta\, \hat{z}$$

$$\left|\frac{\partial \vec{r}}{\partial r}\right| = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta = 1.$$

$$\frac{\partial r}{\partial \theta} = r\cos\theta\cos\phi\,\hat{x} + r\cos\theta\sin\phi\,\hat{y} + r\sin\theta\,\hat{z}$$

$$\left|\frac{\partial r}{\partial \theta}\right|^2 = r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta = r^2$$

$$\frac{\partial r}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{x} + r\sin\theta\cos\phi\,\hat{y}$$

$$\left|\frac{\partial r}{\partial \phi}\right|^2 = r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi = r^2\sin^2\theta$$

$$\hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{|\partial\vec{r}/\partial r|} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \quad \text{--- (1)}$$

$$\hat{\theta} = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \quad \text{--- (2)}$$

$$\hat{\phi} = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \quad \text{--- (3)}$$

Now, to calculate $\hat{x}, \hat{y}, \hat{z}$ in terms of $\hat{r}, \hat{\theta}, \hat{\phi}$

$$\sin\theta\,\hat{r} = \sin^2\theta\cos\phi\,\hat{x} + \sin^2\theta\sin\phi\,\hat{y} + \sin\theta\cos\theta\,\hat{z}$$

$$\cos\theta\,\hat{\theta} = \cos^2\theta\cos\phi\,\hat{x} + \cos^2\theta\sin\phi\,\hat{y} - \sin\theta\cos\theta\,\hat{z}$$

Add the two eqns.

$$\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \quad \text{--- (4)}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \quad \text{--- (5)}$$

(4) $\times \cos\phi$ − (5) $\times \sin\phi$

$$\underline{\hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta}} - \sin\phi\,\hat{\phi}$$

(4) $\times \sin\phi$ + (5) $\times \cos\phi$

$$\underline{\hat{y} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi}}$$

$\hat{z}$ = (1) $\times \cos\theta$ − (2) $\times \sin\theta$

$$\underline{\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}}$$

3. Express $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in spherical polar Coordinates.

Soln:

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial x}\right)\frac{\partial}{\partial \theta} + \left(\frac{\partial \phi}{\partial x}\right)\frac{\partial}{\partial \phi}$$

$$r = (x^2+y^2+z^2)^{1/2}$$

$$\theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2} \times \frac{2x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{(x^2+y^2+z^2)^{1/2}}$$

$$\frac{\partial \theta}{\partial x} = \frac{z^2}{x^2+y^2+z^2} \times \frac{1}{2z} \times \frac{2x}{(x^2+y^2)^{1/2}} = \frac{zx}{(x^2+y^2)^{1/2}(x^2+y^2+z^2)}$$

$$\frac{\partial \phi}{\partial x} = \frac{x^2}{x^2+y^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\therefore \frac{\partial}{\partial x} = \left(\frac{x}{(x^2+y^2+z^2)^{1/2}}\right)\frac{\partial}{\partial r} + \left(\frac{zx}{(x^2+y^2)^{1/2}(x^2+y^2+z^2)}\right)\frac{\partial}{\partial \theta} + \left(\frac{-y}{x^2+y^2}\right)\frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial y}\right)\frac{\partial}{\partial \theta} + \left(\frac{\partial \phi}{\partial y}\right)\frac{\partial}{\partial \phi}$$

$$\frac{\partial r}{\partial y} = \frac{y}{(x^2+y^2+z^2)^{1/2}}$$

$$\frac{\partial \theta}{\partial y} = \frac{yz}{(x^2+y^2)^{1/2}(x^2+y^2+z^2)}.$$

$$\frac{\partial \phi}{\partial z} = \frac{x^2}{(x^2+y^2)} \times \frac{1}{x} = \frac{x}{(x^2+y^2)}$$

$$\frac{\partial}{\partial y} = \left(\frac{y}{(x^2+y^2+z^2)^{1/2}}\right)\frac{\partial}{\partial r} + \frac{yz}{(x^2+y^2)^{1/2}(x^2+y^2+z^2)}\frac{\partial}{\partial \theta} + \left(\frac{x}{x^2+y^2}\right)\frac{\partial}{\partial \phi}$$

$\frac{\partial}{\partial z}$

$$\frac{\partial}{\partial z} = \frac{\partial r}{\partial z}\left(\frac{\partial}{\partial r}\right) + \frac{\partial \theta}{\partial z}\left(\frac{\partial}{\partial \theta}\right) + \left(\frac{\partial \phi}{\partial z}\right)\frac{\partial}{\partial \phi}$$

$$\frac{\partial r}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{1/2}}$$

$$\frac{\partial \theta}{\partial z} = \frac{-z^2}{(x^2+y^2+z^2)} \times \frac{\sqrt{x^2+y^2}}{z^2} = -\frac{(x^2+y^2)^{1/2}}{(x^2+y^2+z^2)}$$

$$\frac{\partial \phi}{\partial z} = 0$$

$$\frac{\partial}{\partial z} = \frac{z}{(x^2+y^2+z^2)^{1/2}}\left(\frac{\partial}{\partial r}\right) - \frac{(x^2+y^2)^{1/2}}{(x^2+y^2+z^2)}\left(\frac{\partial}{\partial \theta}\right)$$

4. Compute the gradient and Laplacian of the function $T = r(\cos\theta + \sin\theta\cos\phi)$. Check the Laplacian by converting T to Cartesian coordinates and using Laplacian in this coordinates system. Test the theorem of gradients $\int_a^b \vec{\nabla}T\cdot d\vec{l} = T(\vec{b}) - T(\vec{a})$ using the path shown $(0,0,0)$ to $(0,0,2)$

Soln

Gradient ∇T

$$\frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\phi}$$

$$= (\cos\theta + \sin\theta\cos\phi)\hat{r} + \frac{1}{r}\times r(-\sin\theta + \cos\theta\cos\phi)\hat{\theta} - \frac{1}{r\sin\theta} r\sin\theta\sin\phi\,\hat{\phi}$$

$$\nabla T \Rightarrow (\cos\theta + \sin\theta\cos\phi)\hat{r} + (-\sin\theta + \cos\theta\cos\phi)\hat{\theta} - \sin\phi\,\hat{\phi}$$

Laplacian $\nabla^2 T$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cos\theta + r^2\sin\theta\cos\phi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta(-r\sin\theta + r\cos\theta\cos\phi)\right) + \frac{1}{r^2\sin^2\theta}(-r\sin\theta\cos\phi)$$

$$= \frac{1}{r^2}\left[2r\cos\theta + 2r\sin\theta\cos\phi\right] + \frac{1}{r^2\sin\theta}\left[-2r\sin\theta\cos\theta + r\cos\phi\cos^2\theta - r\cos\phi\sin^2\theta\right] - \frac{1}{r}\frac{\cos\phi}{\sin\theta}$$

$$= \frac{2}{r}\cos\theta + \frac{2}{r}\sin\theta\cos\phi - \frac{2}{r}\cos\theta + \frac{1}{r}\cos\phi\frac{\cos^2\theta}{\sin\theta} - \frac{1}{r}\cos\phi\sin\theta - \frac{1}{r}\frac{\cos\phi}{\sin\theta}$$

$$= \frac{1}{r}\sin\theta\cos\phi + \frac{1}{r}\frac{\cos^2\theta}{\sin\theta}\cos\phi - \frac{1}{r}\frac{\cos\phi}{\sin\theta}$$

$$= \frac{1}{r}\frac{\cos\phi}{\sin\theta}\left[\sin^2\theta + \cos^2\theta - 1\right] = 0$$

Test of theorem of gradients

$$\int_a^b \vec{\nabla} T \cdot \vec{dl} = \vec{T}(b) - \vec{T}(a)$$

using path $(0,0,0)$ to $(0,0,2)$

In cartesian coordinates

$$T = y + z.$$

$$\int_a^b \nabla T \cdot dl = (\hat{y} + \hat{z}) \cdot (dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z})$$

$$= \int_a^b (dy + dz)$$

Along the given path $dx = dy = 0$

$$\int_0^2 dz = 2.$$

$T(b) = 2.$

$T(a) = 0$

$T(b) - T(a) = 2.$

5(a) Find the divergence of the function $\vec{v} = s(2+\sin^2\phi)\hat{\rho} + \rho\sin\phi\cos\phi\,\hat{\phi} + 3z\hat{z}$

(b) Test the divergence theorem for this function, using quarter cylinder (radius 2 height 5) (c) Find the curl of $\vec{v}$.

Soln

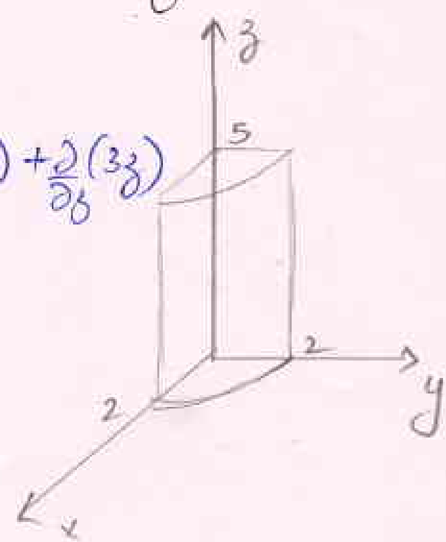
$$\nabla \cdot v = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, v_\rho) + \frac{1}{\rho}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho^2 (2+\sin^2\phi)\right) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\rho\sin\phi\cos\phi) + \frac{\partial}{\partial z}(3z)$$

$$= 2(2+\sin^2\phi) + (\cos^2\phi - \sin^2\phi) + 3$$

$$= 4 + 2\sin^2\phi + \cos^2\phi - \sin^2\phi + 3$$

$$= 8$$



(b) Divergence theorem

$$\int_V (\nabla\cdot v)\,d\tau = \oint_s v\cdot da$$

($z = 5$ ~~$\rho=2$~~)

LHS $= \int_V 8\rho\, d\rho\, d\phi\, dz$

$$= 8\int_0^2 \rho\, d\rho \int_0^{\pi/2} d\phi \int_0^5 dz$$

$$= 8\frac{\rho^2}{2}\times\frac{\pi}{2}\times 5$$

$$= 40\pi$$

RHS: $\oint_s v\cdot da$

(i) Top part at $z=5$: $dz=0$.

So $da = s\,d\rho\, d\phi\, \hat{z}$

$$v\cdot da = 3z\,\rho\, d\rho\, d\phi$$

$$= 15\int_0^2 \rho\, d\rho \int_0^{\pi/2} d\phi$$

$$= 15\times \frac{\rho^2}{2}\Big|_0^2\, \phi\Big|_0^{\pi/2} = 15\pi$$

(ii) bottom at $z=0$ $dz=0$

$$da = -s\,ds\,d\phi\,\hat{z}$$

$$v\cdot da = -3zs\,ds\,d\phi = 0.$$

(iii) back at $\phi=\pi/2$

$$da = d\rho\, dz\,\hat{\phi}$$

$$v\cdot da = \rho \sin\phi\cos\phi\, d\rho\, dz = 0$$

(iv) left at $\phi = 0$

$$da = -d\rho\, dz\,\hat{\phi}$$

$$V\cdot da = -8\sin\phi\cos\phi\, d\rho\, dz = 0$$

(v) front at $\rho = 2$

$$da = \rho\, d\phi\, dz\, \hat{\rho}$$

$$v \cdot da = \rho(2+\sin^2\phi)\rho\, d\phi\, dz$$

$$= 4(2+\sin^2\phi)\, d\phi\, dz.$$

$$\int v \cdot da = 4\int_0^{\pi/2}(2+\sin^2\phi)\, d\phi \int_0^5 dz$$

$$= 4\left(\pi + \frac{\pi}{4}\right) \times 5 = 25\pi$$

$$\oint v \cdot da = 15\pi + 25\pi = 40\pi$$

(c) $\nabla \times v = \left(\frac{1}{\rho}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho}\right)\hat{\phi} + \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial \phi}\right)\hat{z}$

$$= \left[\frac{1}{\rho}\frac{\partial}{\partial \phi}(3z) - \frac{\partial}{\partial z}(\rho \sin\phi \cos\phi)\right]\hat{\rho} + \left[\frac{\partial}{\partial z}\left(\rho(2+\sin^2\phi)\right) - \frac{\partial}{\partial \rho}(3z)\right]\hat{\phi}$$

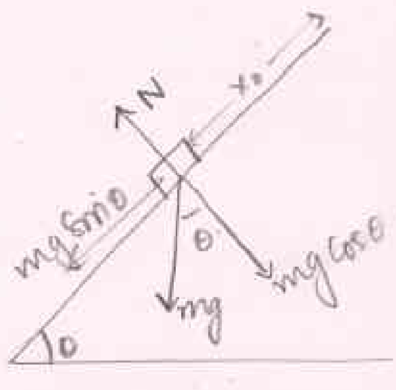
$$+ \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(\rho^2 \sin\phi \cos\phi) - \frac{\partial}{\partial \phi}\left(\rho(2+\sin^2\phi)\right)\right]\hat{z}$$

$$= \frac{1}{\rho}\left[2\rho \sin\phi \cos\phi - 2\rho \sin\phi \cos\phi\right] = 0$$

6. If a block slides without friction down a fixed inclined plane with $\theta = 30°$ what will be the block's acceleration. Find the expression for velocity of block after it moves from rest a distance x_0 down the plane.

Soln

Balancing the forces parallel and perpendicular to the plane



$$N = mg\cos\theta.$$

$$ma = mg\sin\theta.$$

$$a = g\sin\theta = g\sin 30° = g/2$$

(b) $v^2 = u^2 + 2as$.

$= 0 + 2 \times g \sin\theta \, x_0$

$v = \sqrt{2x_0 g \sin\theta}$

7 Find the velocity of a particle undergoing vertical motion under gravity in a medium having a retarding force proportional to velocity. Write the equation of motion of the system. Find its solution. Based on the solution comment on what happens to system at very long time and at $t \to \infty$.

Soln Let the particle falls down with initial velocity v_0 from height h.

Given that the retarding force is proportional to its velocity

$$F = m\frac{dv}{dt} = -mg - kmv$$

for downward motion $v < 0$, so $-kmv > 0$

$$\frac{dv}{g + kv} = -dt$$

Integrating (at $t = 0$, $v = v_0$)

$$\frac{1}{k} \ln(kv + g) = -t + c$$

$$kv + g = e^{-kt + kc}$$

$$v = \frac{dz}{dt} = -\frac{g}{k} + \frac{kv_0 + g}{k} e^{-kt}$$

Again integrating (at $t=0$, $z=h$)

$$\int_h^z dz = \int_0^t -\frac{g}{k}dt + \int_0^t \frac{kv_0+g}{k} e^{-kt} dt$$

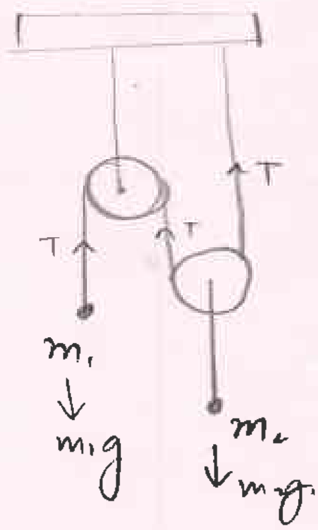
$$z-h = -\frac{gt}{k} + \left(\frac{kv_0+g}{k}\right)\left(\frac{-1}{k}\right)e^{-kt} + c$$

$$z = h - \frac{gt}{k} + \frac{kv_0+g}{k^2}(1-e^{-kt})$$

when the time becomes very long, the velocity approaches the limiting value $-g/k$ called terminal velocity.

8 Consider the pulley system with masses m_1 and m_2. What are the acceleration of the masses? what is the tension in the string? Consider the pulleys are massless.

Soln: Balancing the forces during the vertical motion of the two masses



$$T - m_1g = m_1a_1$$
$$2T - m_2g = m_2a_2$$

Now to get the relation in a_1 and a_2, we see that if m_1 moves $2d$ down m_2 move d distance up shared by two strings equally. So for motion of m_1 ie. y_1 & for m_2 it is y_2

$$y_1 = -2y_2$$
$$\ddot{y}_1 = -2\ddot{y}_2$$
$$\therefore a_1 = -2a_2$$

$$2m_1g - m_2g = -2m_1a_1 + m_2a_2$$

$$2m_1a_1 - m_2\left(\frac{-a_1}{2}\right) = m_2g - 2m_1g$$

$$a_1\left(2m_1 + \frac{m_2}{2}\right) = g\,(m_2 - 2m_1)$$

$$a_1 = g\,\frac{(2m_2 - 4m_1)}{4m_1 + m_2)}$$

$$m_2a_2 - 2m_1(-2a_2) = 2m_1g - m_2g$$

$$a_2(m_2 - 4m_1) = g\,(2m_1 - m_2)$$

$$a_2 = g\left(\frac{2m_1 - m_2}{m_2 + 4m_1}\right)$$

$$T = m_1g + m_1a\,.$$

$$= m_1\left(g + g\left(\frac{2m_2 - 4m_1}{4m_1 + m_2}\right)\right)$$

$$= m_1g\left[\frac{4m_1 + m_2 + 2m_2 - 4m_1}{4m_1 + m_2}\right]$$

$$T = \frac{3m_1m_2g}{4m_1 + m_2}$$

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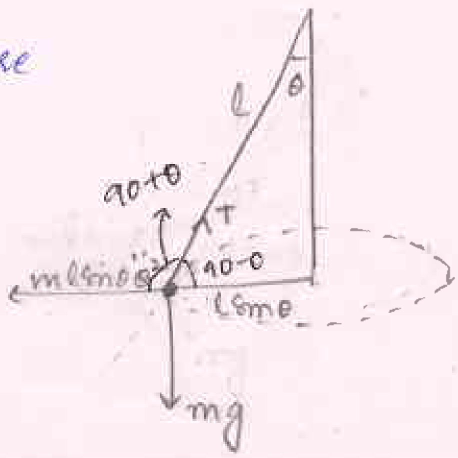
9. A mass m hangs from a massless string of length l. Conditions have set up such that the mass swing around a horizontal circle, with string making constant angle θ with the vertical, find the angular ~~velocity~~ frequency of the motion.

Soln: The forces that acts on the mass are shown



i) Tension T

(ii) horizontal outward centripetal force $ml\sin\theta\,\dot{\theta}^2$ $(\dot{\theta} = \omega)$

(iii) weight mg vertically downwards

Applying Lami's theorem for the equilibrium of 3 forces

$$\frac{T}{\sin 90} = \frac{mg}{\sin(90+\theta)} = \frac{ml\sin\theta\,\omega^2}{\sin(\pi-\theta)}$$

$$= \frac{\cancel{m}g}{\cos\theta} = \frac{\cancel{m}l\,\cancel{\sin}\theta\,\omega^2}{\sin\cancel{\theta}}$$

$$\omega = \sqrt{\frac{g}{l\cos\theta}}$$

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10 For a given initial speed, at what inclination angle should a ball be thrown so that it travels the maximum horizontal distance by the time returns to the ground? Assume that the ground is horizontal, and ball released from the ground level. What is the optimal angle if the ground is sloped upward at an angle θ?

Soln: when the ball is thrown with velocity V at an ~~g~~ angle θ
Along $x-y$ directions

$$\dot{x}(t) = V_x \quad \text{and} \quad \dot{y}(t) = V_y - gt$$

Integrating

$$x(t) = X + V_x t$$
$$y(t) = Y + V_y t - \frac{1}{2}gt^2$$

where $V_x = V\cos\theta$
$V_y = V\sin\theta$.

at $t = t/2$, the ball is at its height point $\therefore V_y = 0$

So $V_y = gt/2$

$$\therefore t = \frac{2V_y}{g}$$

horizontal distance $d = V_x t$

Now $t = \frac{2V_y}{g}$

So $d = \frac{2V_x V_y}{g} = \frac{V^2(2\sin\theta\cos\theta)}{g} = \frac{V^2\sin 2\theta}{g}$

for d to be maximum $\sin 2\theta = 1$.

$\therefore \theta = \pi/4$

$\therefore d_{max} = \frac{V^2}{g}$

(b) Now if ground is sloped at an angle β, then the eqn for the line of the ground is $y = (\tan\beta)x$

The path of the ball is given in terms of t by

$$x = (V\cos\theta)t$$

$$y = (V\sin\theta)t - \frac{1}{2}gt^2$$

$$y = (\tan\beta).V\cos\theta\, t$$

So $(\tan\beta)V\cos\theta\, t = (V\sin\theta)t - \frac{1}{2}gt^2$

or $\frac{1}{2}gt^2 + (V\tan\beta\cos\theta - V\sin\theta)t = 0$

$$t = 2V\frac{(\sin\theta - \tan\beta\cos\theta)}{g}$$

$$x = V\cos\theta . \frac{2V}{g}(\sin\theta - \tan\beta\cos\theta)$$

$$= \frac{2V^2}{g}(\sin\theta\cos\theta - \tan\beta\cos^2\theta)$$

To get the maximum displacement w.r.t θ

$$\frac{dx}{d\theta} = 0$$

$$\frac{dx}{d\theta} = \frac{2v^2}{g}\left(\cos 2\theta - \tan\beta\, 2\cos\theta(-\sin\theta)\right)$$

$$= \frac{2v^2}{g}\left(\cos 2\theta + \tan\beta \sin 2\theta\right) = 0$$

$$\tan\beta = -\frac{\cos 2\theta}{\sin 2\theta} = -\cot 2\theta$$

$$\therefore \quad = -\tan\left(\frac{\pi}{2} - 2\theta\right)$$

$$\therefore \quad \beta = -\left(\frac{\pi}{2} - 2\theta\right)$$

$$\theta = \frac{1}{2}\left(\beta + \frac{\pi}{2}\right)$$
