8-2-2005

Review: Convergence Tests for Infinite Series

When you are testing a series for convergence or divergence, it's helpful to run through your list of convergence tests if you don't see what to do immediately — just as you might run through your list of integration techniques when you're faced with computing an integral.

1. Known series: p-series and geometric series.
2. The Zero Limit Test.
3. The Integral Test.
4. Direct Comparsion.
5. Limit Comparison.
6. The Ratio Test.
7. The Root Test.

Here are some rough rules of thumb for deciding which convergence test to use. With the exception of the Zero Limit Test, these tests only apply to series with positive terms.

- Try the Zero Limit Test first, since it's usually easy to tell whether the terms go to 0.
- If the general term of a series is something you could integrate, think about using the Integral Test.
- Don't apply the Ratio Test to a series whose terms are rational functions. Use Limit Comparison instead.
- If you can see that a known series is "close to" your series, but you can't see how to relate them using inequalities, use Limit Comparison (in preference to Direct Comparison).
- If the terms of a series contain factorials, think about using the Ratio Test. (It's hard to handle factorials in other ways.)
- If the general term a_n contains products or quotients of powers of n (such as 7^n, but also things like e^{2n}), think about using the Root Test.

Example. Does the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^2}+1.01^n\right)$ converge or diverge?

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a p-series with $p=2>1$, so it converges. $\displaystyle\sum_{n=1}^{\infty}1.01^n$ is a geometric series with $r=1.01>1$, so it diverges.

Hence, the sum of the two series *diverges*. □

Example. Determine whether the series

$$\frac{7}{16}-\frac{7}{64}+\frac{7}{256}-\cdots+(-1)^n(7)\left(\frac{1}{4}\right)^n+\cdots$$

converges or diverges. If it converges, find its sum.

The series is geometric with ratio $-\frac{1}{4}$, so it converges. The sum is

$$\frac{7}{16} - \frac{7}{64} + \frac{7}{256} - \cdots + (-1)^n \left(\frac{1}{7}\right)\left(\frac{1}{4}\right)^n + \cdots = \frac{7}{16}\left(1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \cdots\right) =$$

$$\frac{7}{16} \cdot \frac{1}{1 - \left(-\frac{1}{4}\right)} = \frac{7}{16} \cdot \frac{4}{5} = \frac{7}{20}. \quad \square$$

Example. Does the series

$$\tan\frac{\pi}{3} + \tan\frac{\pi}{6} + \cdots + \tan\frac{\pi}{3n} + \cdots$$

converge or diverge?

The series is $\displaystyle\sum_{n=1}^{\infty} \tan\frac{\pi}{3n}$.

For $0 \le x < \frac{\pi}{2}$, $\tan x > x$. Thus,

$$\tan\frac{\pi}{3n} > \frac{\pi}{3n}.$$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{\pi}{3n}$ diverges, because it's ($\frac{\pi}{3}$ times) the harmonic series.

Therefore, $\displaystyle\sum_{n=1}^{\infty} \tan\frac{\pi}{3n}$ diverges by comparison. $\square$

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \left(\arcsin\frac{1}{n}\right)^n$ converge or diverge?

Apply the Root Test:

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\arcsin\frac{1}{n}\right) = \arcsin 0 = 0 < 1.$$

The series converges by the Root Test. $\square$

Example. Does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converge or diverge?

The terms are positive. The function $f(x) = \dfrac{1}{x(\ln x)^2}$ is continuous for $x \ge 2$. The derivative is

$$f'(x) = -\frac{1}{x^2(\ln x)^2} - \frac{2}{x^2(\ln x)^3},$$

which is clearly negative for $x \ge 2$. Thus, the terms of the series decrease. The hypotheses of the Integral Test are satisfied.

Compute the integral:

$$\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{1}{x \cdot u^2} \cdot x\,du = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u^2} =$$

$$\left[u = \ln x, \quad du = \frac{dx}{x}, \quad dx = x\,du; \quad x = 2, \quad u = \ln 2; \quad x = b, \quad u = \ln b\right]$$

$$\lim_{b\to\infty} \left[-\frac{1}{u}\right]_{\ln 2}^{\ln b} = \lim_{b\to\infty} \left(-\frac{1}{\ln b} + \frac{1}{\ln 2}\right) = \frac{1}{\ln 2}.$$

The integral converges, so the series converges, by the Integral Test. □

Example. Does the series

$$\frac{3}{2} + \frac{9}{8} + \cdot + \frac{3^n}{n \cdot 2^n} + \cdots$$

converge or diverge?

The series is $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n \cdot 2^n}$. Apply the Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\dfrac{3^{n+1}}{(n+1)\cdot 2^{n+1}}}{\dfrac{3^n}{n \cdot 2^n}} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)\cdot 2^{n+1}} \cdot \frac{n \cdot 2^n}{3^n} = \lim_{n\to\infty} \frac{3^{n+1}}{3^n} \cdot \frac{n}{n+1} \cdot \frac{2^n}{2^{n+1}} =$$

$$\lim_{n\to\infty} \frac{3}{2} \frac{n}{n+1} = \frac{3}{2} > 1.$$

The series diverges by the Ratio Test. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3+5}{7n^2-11n^3}$ converge or diverge?

$$\lim_{n\to\infty} \frac{4n^3+5}{7n^2-11n^3} = -\frac{4}{11} \neq 0.$$

The series diverges by the Zero Limit Test. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n \cdot n!}$ converge or diverge?

I'll use the Ratio Test. The limiting ratio is

$$\lim_{n\to\infty} \frac{\dfrac{(n+1)^{n+1}}{3^{n+1}\cdot (n+1)!}}{\dfrac{n^n}{3^n \cdot n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{3^{n+1}\cdot (n+1)!} \frac{3^n \cdot n!}{n^n} = \lim_{n\to\infty} \frac{3^n}{3^{n+1}} \cdot \frac{n!}{(n+1)!} \cdot \frac{(n+1)^{n+1}}{n^n} =$$

$$\lim_{n\to\infty} \frac{1}{3} \cdot \frac{1}{n+1} \cdot \left(\frac{n+1}{n}\right)^n \cdot (n+1) = \frac{1}{3} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n.$$

The limit is the well-known limit for e:

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

Therefore,

$$\frac{1}{3}\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = \frac{e}{3}.$$

The limiting ratio is less than 1, so the series converges, by the Ratio Test.

In case you've forgotten, here's the work for the limit. To compute $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n$, let $y = \left(1+\dfrac{1}{n}\right)^n$. Then

$$\ln y = \ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right).$$

So

$$\lim_{n\to\infty}\ln y = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\ln\left(1+\dfrac{1}{n}\right)}{\dfrac{1}{n}} = \lim_{n\to\infty}\frac{\left(\dfrac{1}{1+\dfrac{1}{n}}\right)\left(-\dfrac{1}{n^2}\right)}{-\dfrac{1}{n^2}} = \lim_{n\to\infty}\frac{1}{1+\dfrac{1}{n}} = 1.$$

Hence,

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} y = e^1 = e. \quad \square$$

Example. Does the series $\displaystyle\sum_{n=1}^{\infty}\frac{3^n+n^3}{7^n+10}$ converge or diverge?

I'll make the estimate $\dfrac{3^n+n^3}{7^n+10}\approx\dfrac{3^n}{7^n}$. Apply Limit Comparison:

$$\lim_{n\to\infty}\frac{\dfrac{3^n+n^3}{7^n+10}}{\dfrac{3^n}{7^n}} = \lim_{n\to\infty}\frac{3^n+n^3}{7^n+10}\frac{7^n}{3^n} = \lim_{n\to\infty}\frac{\dfrac{3^n+n^3}{3^n}}{\dfrac{7^n+10}{7^n}} = \lim_{n\to\infty}\frac{1+\dfrac{n^3}{3^n}}{1+\dfrac{10}{7^n}}.$$

Clearly, $\lim\limits_{n\to\infty}\dfrac{10}{7^n} = 0$. By L'Hôpital's rule,

$$\lim_{n\to\infty}\frac{n^3}{3^n} = \lim_{n\to\infty}\frac{3n^2}{3^n(\ln 3)} = \lim_{n\to\infty}\frac{6n}{3^n(\ln 3)^2} = \lim_{n\to\infty}\frac{6}{3^n(\ln 3)^3} = 0.$$

(I used the derivative formula $\dfrac{d}{dx}3^x = 3^x(\ln 3)$.) Hence,

$$\lim_{n\to\infty}\frac{1+\dfrac{n^3}{3^n}}{1+\dfrac{10}{7^n}} = \frac{1+0}{1+0} = 1.$$

The limit is a finite positive number. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n}{7^n}$ converges, because it's a geometric series with ratio $r = \dfrac{3}{7} < 1$. Therefore, the original series converges by Limit Comparison. $\square$

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{2^n+3^n}{4^n+5^n}$ converge or diverge?

$$\lim_{n\to\infty} \frac{\dfrac{2^n+3^n}{4^n+5^n}}{\dfrac{3^n}{5^n}} = \lim_{n\to\infty} \frac{2^n+3^n}{4^n+5^n} \cdot \frac{5^n}{3^n} = \lim_{n\to\infty} \frac{5^n}{4^n+5^n} \cdot \frac{2^n+3^n}{3^n} = \lim_{n\to\infty} \frac{1}{\dfrac{4^n+5^n}{5^n}} \cdot \frac{2^n+3^n}{3^n} =$$

$$\lim_{n\to\infty} \frac{1}{\dfrac{4^n}{5^n}+1} \cdot \left(\frac{2^n}{3^n}+1\right) = \lim_{n\to\infty} \frac{1}{\left(\dfrac{4}{5}\right)^n+1} \cdot \left(\left(\frac{2}{3}\right)^n+1\right) = \frac{1}{0+1} \cdot (0+1) = 1.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{5^n}$ converges, because it's geometric with ratio $r = \dfrac{3}{5} < 1$. Therefore, the original series converges by Limit Comparison. $\square$

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{e^n}}{\sqrt[3]{\pi^n}}$ converge or diverge?

$$\sum_{n=1}^{\infty} \frac{\sqrt{e^n}}{\sqrt[3]{\pi^n}} = \sum_{n=1}^{\infty} \left(\frac{\sqrt{e}}{\sqrt[3]{\pi}}\right)^n.$$

The series is geometric with ratio $\dfrac{\sqrt{e}}{\sqrt[3]{\pi}} \approx 1.12572 > 1$. Therefore, the series diverges. $\square$

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{26^n (n!)^3}{(3n)!}$ converge or diverge?

Apply the Ratio Test:

$$\lim_{n\to\infty} \frac{\dfrac{26^{n+1}((n+1)!)^3}{(3n+3)!}}{\dfrac{26^n(n!)^3}{(3n)!}} = \lim_{n\to\infty} \frac{26^{n+1}((n+1)!)^3}{(3n+3)!} \cdot \frac{(3n)!}{26^n(n!)^3} = \lim_{n\to\infty} \frac{26^{n+1}}{26^n} \cdot \frac{((n+1)!)^3}{(n!)^3} \cdot \frac{(3n)!}{(3n+3)!} =$$

$$\lim_{n\to\infty} 26 \cdot \left(\frac{(n+1)!}{n!}\right)^3 \cdot \frac{(3n)!}{(3n+3)!} =$$

$$\lim_{n\to\infty} 26 \cdot \left(\frac{(1)(2)\cdots(n)(n+1)}{(1)(2)\cdots(n)}\right)^3 \cdot \frac{(1)(2)\cdots(3n)}{(1)(2)\cdots(3n)(3n+1)(3n+2)(3n+3)} =$$

$$\lim_{n\to\infty} 26 \cdot (n+1)^3 \cdot \frac{1}{(3n+1)(3n+2)(3n+3)} = \lim_{n\to\infty} \frac{26(n+1)^3}{(3n+1)(3n+2)(3n+3)} = \frac{26}{27} < 1.$$

Therefore, the series converges by the Ratio Test. $\square$

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \arctan \frac{42^n}{41^n}$ converge or diverge?

$\dfrac{42^n}{41^n}$ is a geometric *sequence* with ratio $\dfrac{42}{41} > 1$, so $\dfrac{42^n}{41^n} \to \infty$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \arctan \frac{42^n}{41^n} = \frac{\pi}{2} \neq 0.$$

Hence, the series diverges, by the Zero Limit Test. □

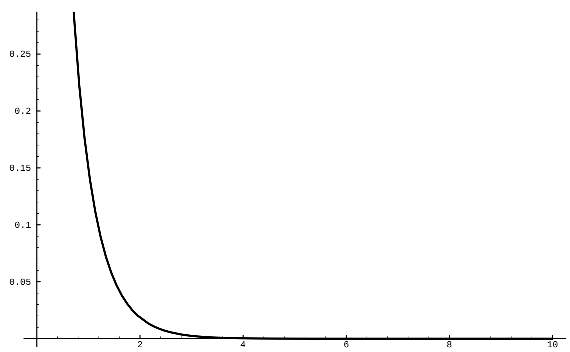
Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-n}\arcsin(e^{-n})}{\sqrt{1-e^{-2n}}}$ converge or diverge?

Note that as $n \to \infty$, $e^{-n} \to 0$, so

$$\lim_{n\to\infty} \frac{e^{-n}\arcsin(e^{-n})}{\sqrt{1-e^{-2n}}} = 0.$$

The Zero Limit Test fails.

The series has positive terms. If $f(x) = \dfrac{e^{-x}\arcsin(e^{-x})}{\sqrt{1-e^{-2x}}}$, then f is continuous for $x \geq 1$. Finally, the terms decrease, as is evident from the graph of f:



(It's possible, but messy, to show this by computing $f'(x)$.)
The hypotheses of the Integral Test are satisfied.

$$\int_1^{\infty} \frac{e^{-x}\arcsin(e^{-x})}{\sqrt{1-e^{-2x}}}\,dx = \lim_{b\to\infty} \int_1^b \frac{e^{-x}\arcsin(e^{-x})}{\sqrt{1-e^{-2x}}}\,dx = \lim_{b\to\infty} \int_?^? \frac{e^{-x}\arcsin u}{\sqrt{1-u^2}} \cdot \left(\frac{du}{-e^{-x}}\right) =$$

$$\left[u = e^{-x}, \quad du = -e^{-x}\,dx, \quad dx = \frac{du}{-e^{-x}}\right]$$

$$-\lim_{b\to\infty} \int_?^? \frac{\arcsin u}{\sqrt{1-u^2}}\,du = -\lim_{b\to\infty} \int_?^? \frac{w}{\sqrt{1-u^2}}(\sqrt{1-u^2}\,dw) = -\lim_{b\to\infty} \int_?^? w\,dw = -\lim_{b\to\infty} \left[\frac{1}{2}w^2\right]_?^? =$$

$$\left[w = \arcsin u, \quad dw = \frac{du}{\sqrt{1-u^2}}, \quad du = \sqrt{1-u^2}\,dw\right]$$

$$-\lim_{b\to\infty} \left[\frac{1}{2}(\arcsin u)^2\right]_?^? = -\lim_{b\to\infty} \left[\frac{1}{2}(\arcsin e^{-x})^2\right]_1^b = -\frac{1}{2}\lim_{b\to\infty} \left((\arcsin e^{-b})^2 - (\arcsin e^{-1})^2\right) =$$

$$-\frac{1}{2}\left(0 - (\arcsin e^{-1})^2\right) = \frac{1}{2}(\arcsin e^{-1})^2.$$

The integral converges, so the series converges, by the Integral Test. □

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{1+e^{-n^3}}{n}$ converge or diverge?

Notice that as $n \to \infty$, $1+e^{-n^3}$ *decreases* to 1. Thus,

$$1+e^{-n^3} \geq 1, \quad \text{so} \quad \frac{1+e^{-n^3}}{n} \geq \frac{1}{n}.$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is harmonic, so it diverges. Therefore, the original series diverges by direct comparison. $\square$

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}}$ converge or diverge?

For large n, $\dfrac{\sqrt{n(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}} \approx \dfrac{\sqrt{n^2}}{\sqrt{n^3}} = \dfrac{1}{n^{1/2}}$. Do a Limit Comparison:

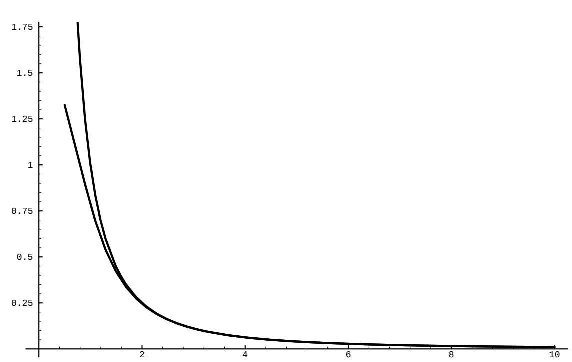
$$\lim_{n\to\infty} \frac{\dfrac{\sqrt{n(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}}}{\dfrac{1}{n^{1/2}}} = \lim_{n\to\infty} \frac{\sqrt{n(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}} \frac{n^{1/2}}{1} = \lim_{n\to\infty} \frac{\sqrt{n^2(n+1)}}{\sqrt{(n+2)(n+3)(n+4)}} =$$

$$\sqrt{\lim_{n\to\infty} \frac{n^2(n+1)}{(n+2)(n+3)(n+4)}} = \sqrt{1} = 1.$$

The limit is a finite positive number. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges, because it's a p-series with $p = \dfrac{1}{2} < 1$. Therefore, the original series diverges, by Limit Comparison. $\square$

Example. Does the series $\displaystyle\sum_{n=1}^{\infty} \arctan \frac{1}{n^2}$ converge or diverge?

Notice that the graphs of $\arctan \dfrac{1}{x^2}$ and $\dfrac{1}{x^2}$ appear to be very close.



I'll use Limit Comparison:

$$\lim_{n\to\infty} \frac{\arctan \dfrac{1}{n^2}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{\left(\dfrac{-2}{n^3}\right)\left(\dfrac{1}{1+\left(\dfrac{1}{n^2}\right)^2}\right)}{\dfrac{-2}{n^3}} = \lim_{n\to\infty} \frac{1}{1+\left(\dfrac{1}{n^2}\right)^2} = 1.$$

The limit is a finite positive number. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, because it's a p-series with $p = 2 > 1$. Therefore, the original series converges by Limit Comparison. $\square$
