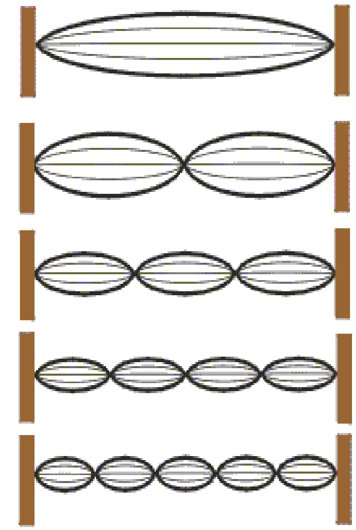
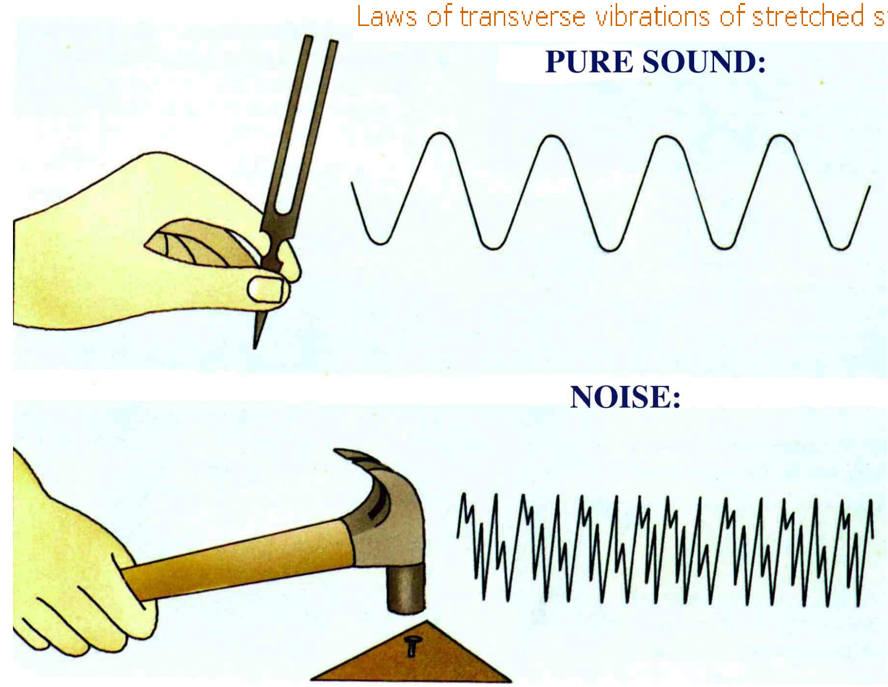
Oscillations and waves



Laws of transverse vibrations of stretched strings



List of reference

1) Classical dynamics of particles and systems by S T Thornton and J B Marion

2) Berkley Physics course I. Mechanics by Kittel, Knight,Ruderman,Helmholtz,Moyer

3) Classical mechanics by Kibble and Berkshire

4) Classical mechanics by Gregory

5) Introduction to classical mechanics with problems and solutions by D Morin

6) Mechanics by Hans and Puri

7) Vibrations and waves by A P French

8) The physics of vibrations and waves by H J Pain

9) Classical mechanics point particles and relativity by W Greiner

10) Advanced Engineering Mathematics by Kreyszing Chap 2

List of topics

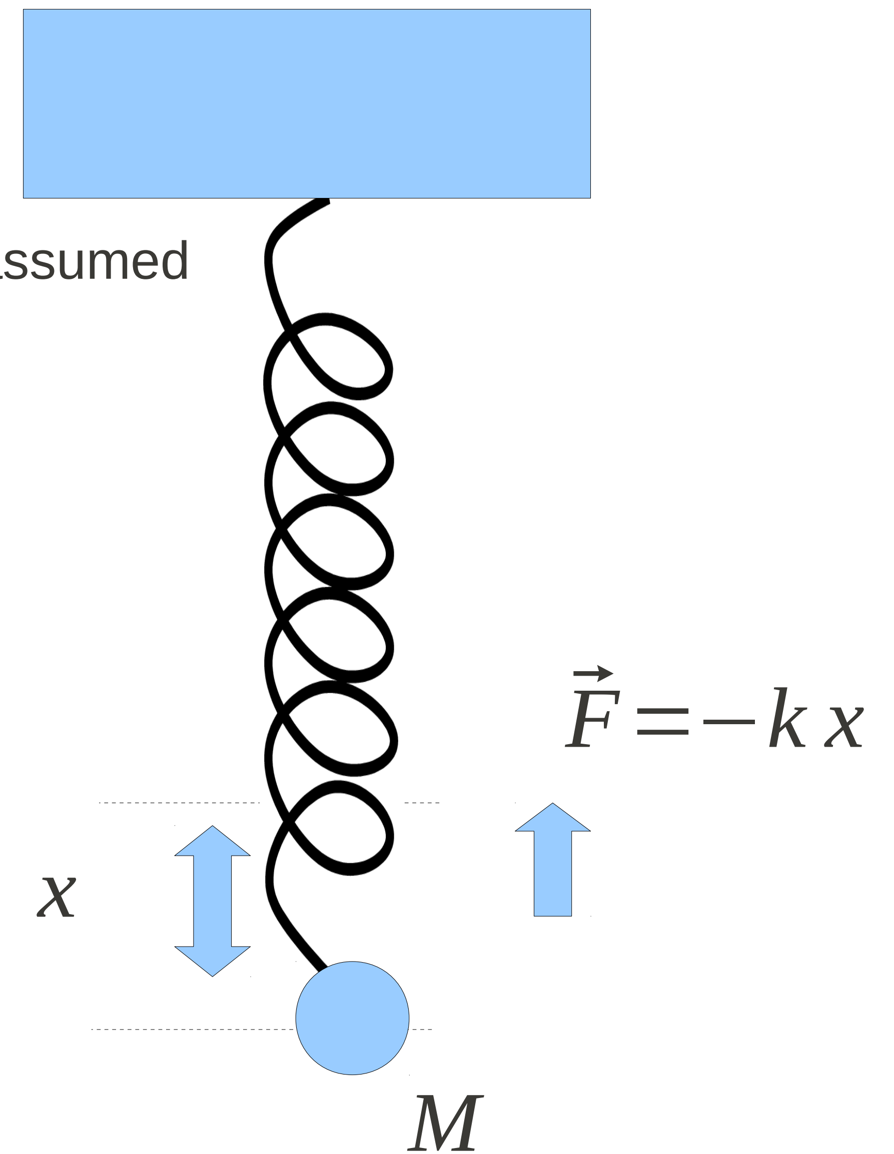
- Comparison of different oscillating systems
- Generalized representation of the potential energy surface
- Solution of the SHM
- Damping and damped harmonic motion
- Forced and driven harmonic motion
- Coupled oscillators
- Wave equation in one dimension and solution

Displaced spring with mass

$$\vec{F} = -k\,x\,\hat{x}$$

$$\frac{d^2 x}{d\,t} = -\frac{k}{M}\,x$$

No gravity is assumed



The differential equation with constant coefficients

The general solution is

$$x = A\,sin(\omega_0 t + \phi)$$

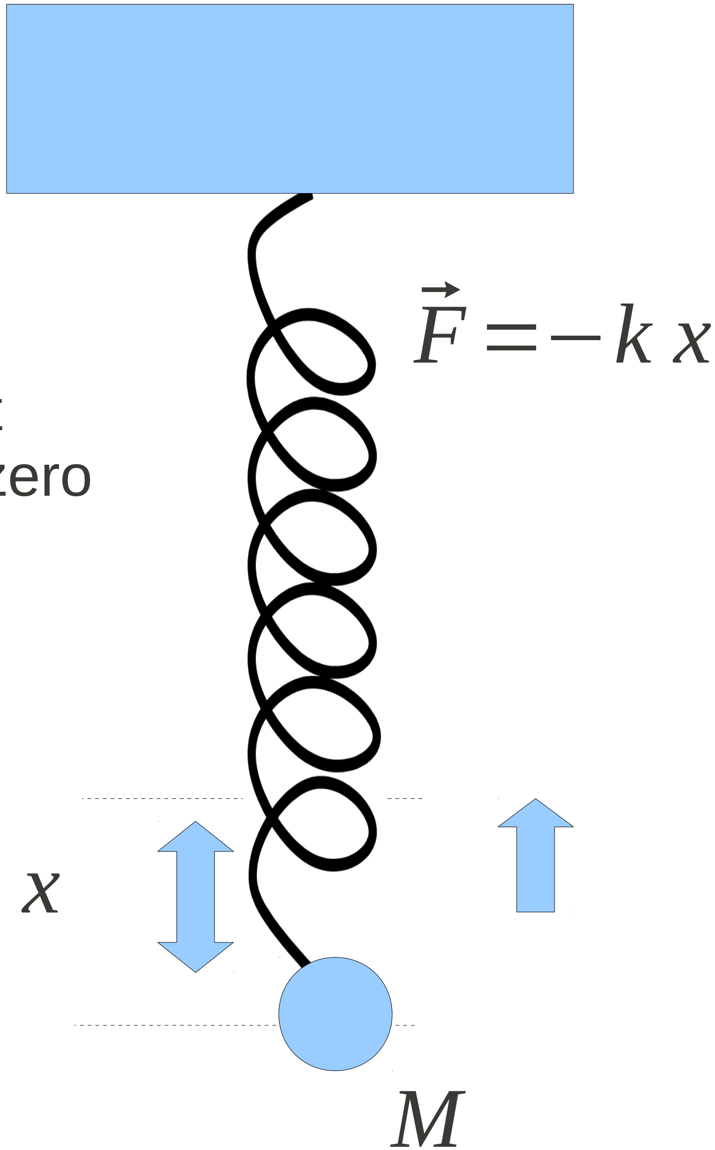
$$\omega_0 = \sqrt{\frac{k}{M}}$$

$$t=0\,;\, x = x_0 = A\,sin\,\phi\,;\, \frac{dx}{dt} = v_0 = \omega_0\,A\,cos\,\phi$$

Harmonic oscillator and energy conservation

Total energy of a harmonic oscillator is conserved

$$E=\overbrace{\frac{1}{2}k\,x^2}^{\text{P.E.}}+\overbrace{\frac{1}{2}m\,v^2}^{\text{K.E.}}$$



To arrive at equation total energy is to evaluated at maximum extension $x=x_0$ of the spring K.E. is zero

$$E=\frac{1}{2}k\,x_0^2$$

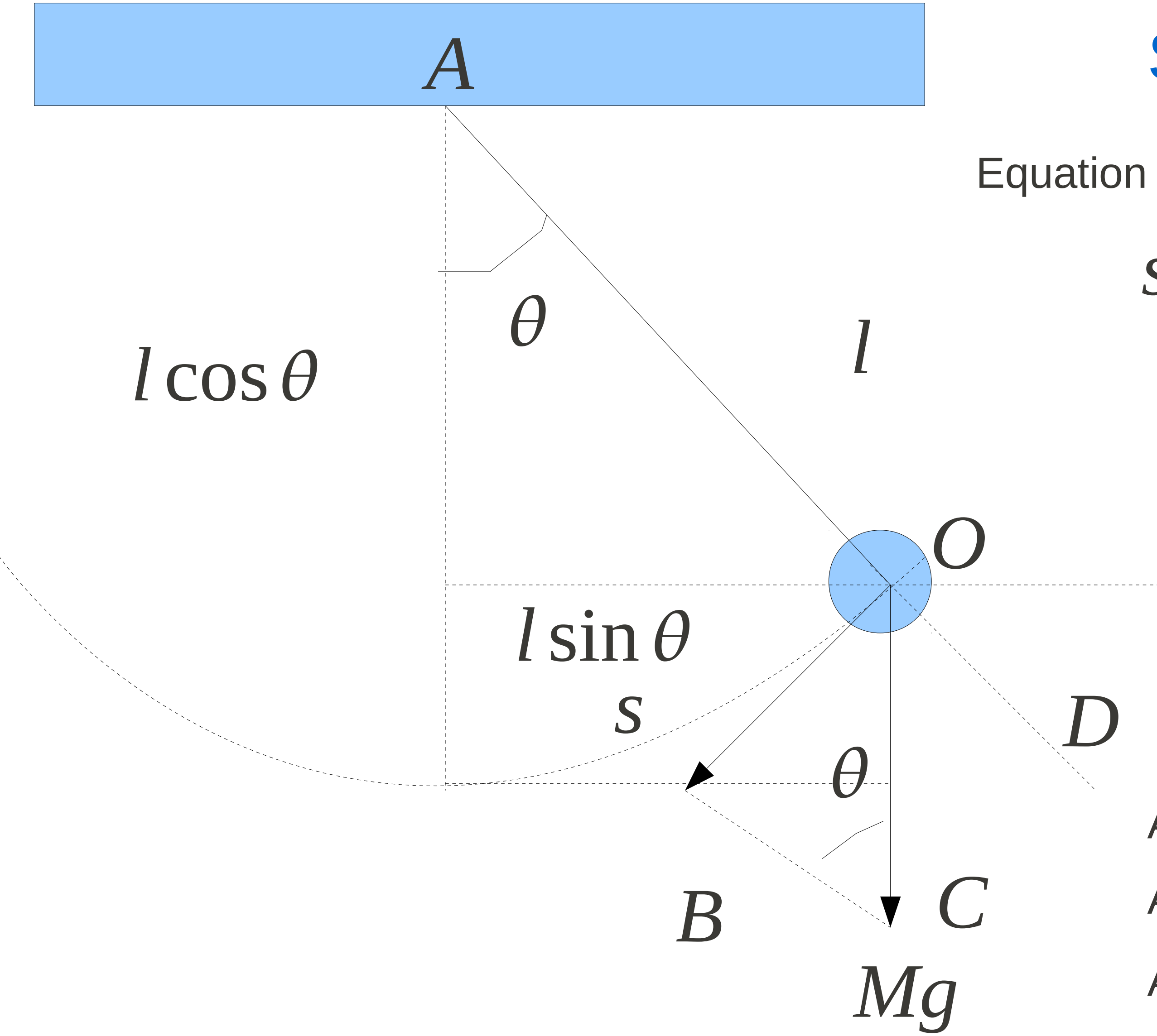
Now the total energy equation be

$$\frac{1}{2}k\,x_0^2=\frac{1}{2}k\,x^2+\frac{1}{2}m\left(\frac{d\,x}{d\,t}\right)^2$$

Rearranging the terms

$$\frac{d\,x}{d\,t}=\sqrt{\frac{k}{m}(x_0^2-x^2)}$$

Simple pendulum



Equation of motions by Newton's method

$$s = l\theta$$

$$v = \frac{ds}{dt} = l\frac{d\theta}{dt}$$

$$a = \frac{d^2 s}{dt^2} = l\frac{d^2\theta}{dt^2}$$

Angle AOB= angle DOB = 90

Angle COD is

Angle COB is $90^o - \theta$

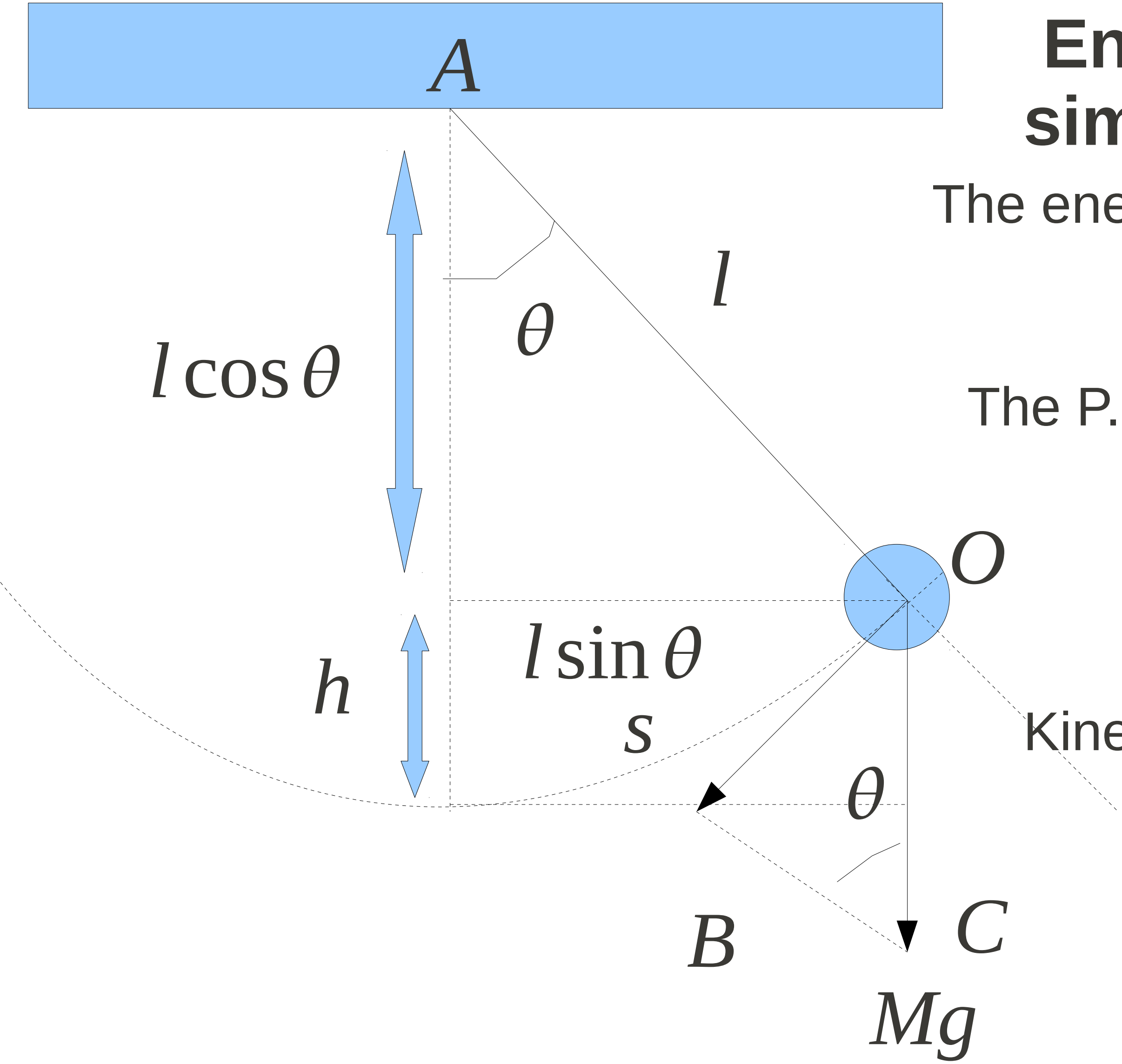
Angle OCB is θ

Direction of l is perpendicular to θ

$$Mg\sin\theta = -ML\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\ldots\ldots$$



Energy conservation in simple pendulum

The energy of simple pendulum

$$h = l - l\cos\theta$$

The P. E. of simple pendulum

$$U(h) = mgl(1-\cos\theta)$$

Kinetic energy of the simple pendulum

$$K = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

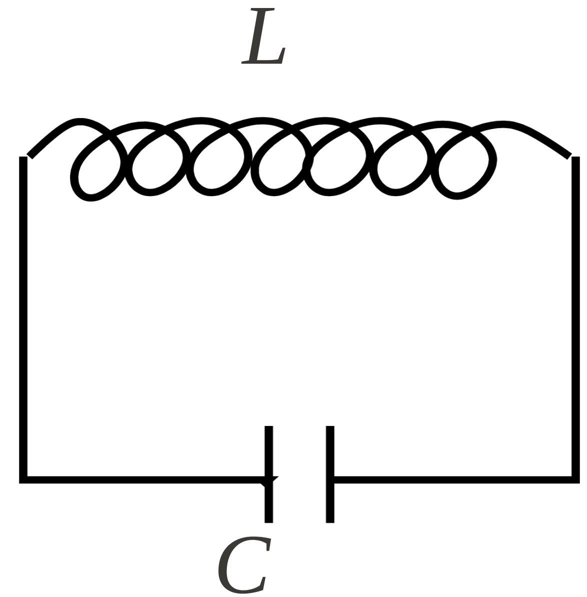
$$\cos\theta \simeq 1 - \frac{1}{2}\theta^2 \ldots\ldots\ldots$$

Total energy of the simple pendulum

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2$$

LC circuit



Voltage across the capacitance

$$V_c = \frac{Q}{C}$$ → charge

$$I = -\frac{dQ}{dt} \qquad Q = -\int I\, dt$$

Current flows in the opposite direction to decrease the charge in the capacitor

$$V_L = -L\frac{dI}{dt}$$

As sum of the voltage is equal to zero

$$-L\frac{dI}{dt} + \frac{Q}{C} = 0 \qquad L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0$$

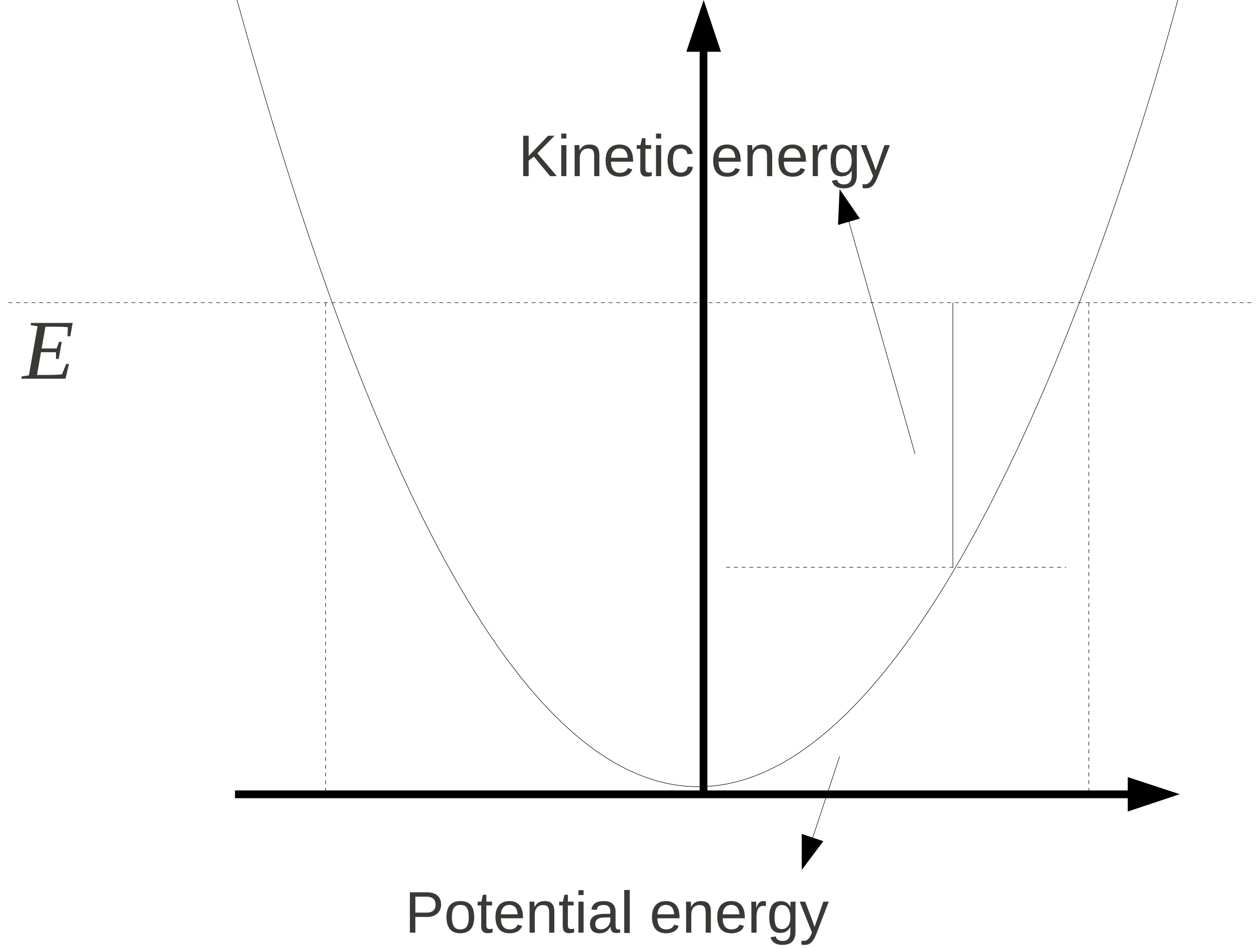
Alternatively the total energy stored is

$$E = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}L\left(\frac{dQ}{dt}\right)^2$$

$$E=\frac{1}{2}k\,x^2+\frac{1}{2}m\,\dot{x}^2$$

$$E=\frac{1}{2}mgl\,\theta^2+\frac{1}{2}m\,l^2\dot{\theta}^2$$

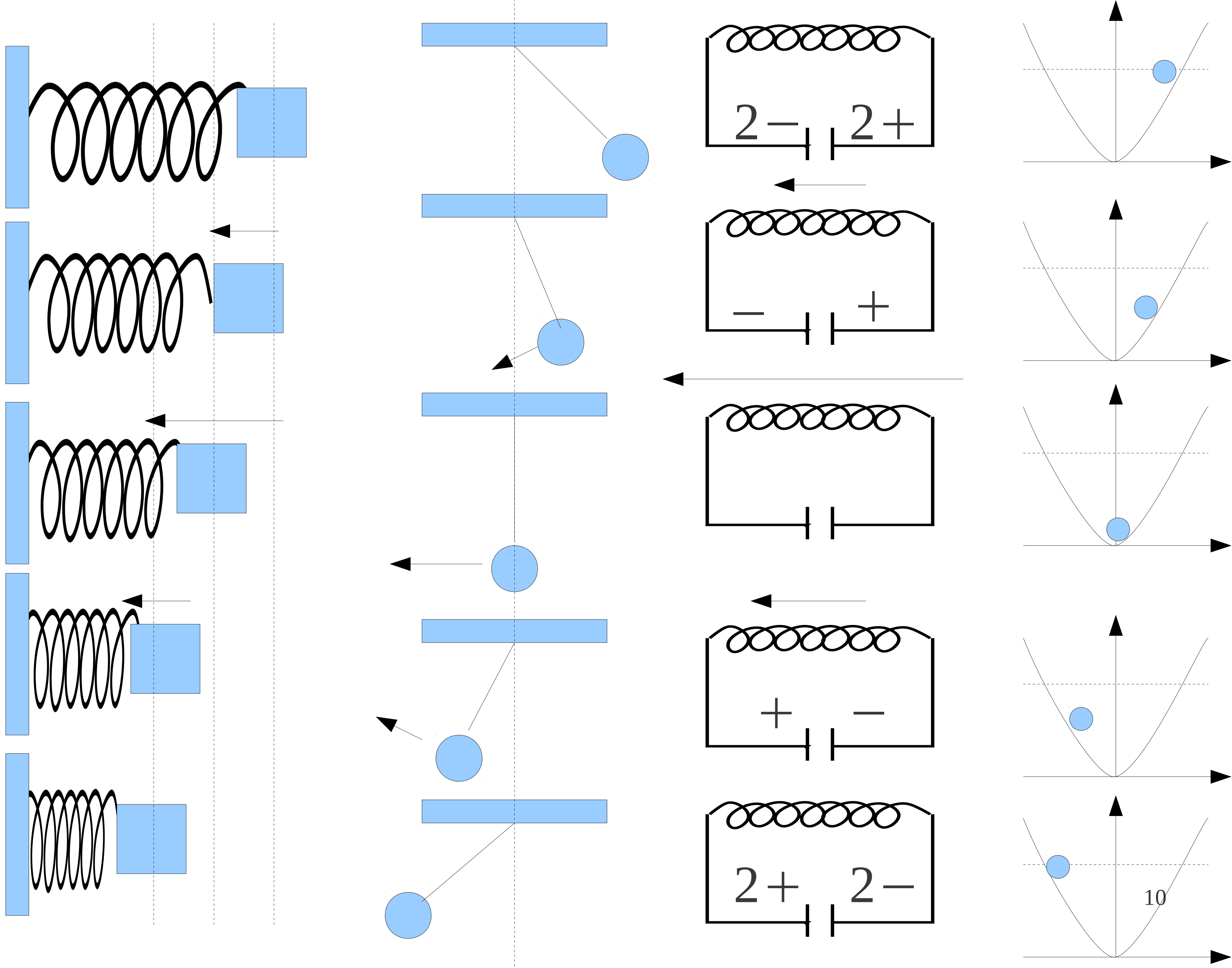
$$E=\frac{1}{2}\frac{Q^2}{C}+\frac{1}{2}L\left(\dot{Q}\right)^2$$



Three systems leads to same form of the differential equation with difference on the in the value of the constants

$$m\frac{d^2x}{dt^2}=-k\,x$$

The form of kinetic energy is general, however, is the potentials have any similarity ?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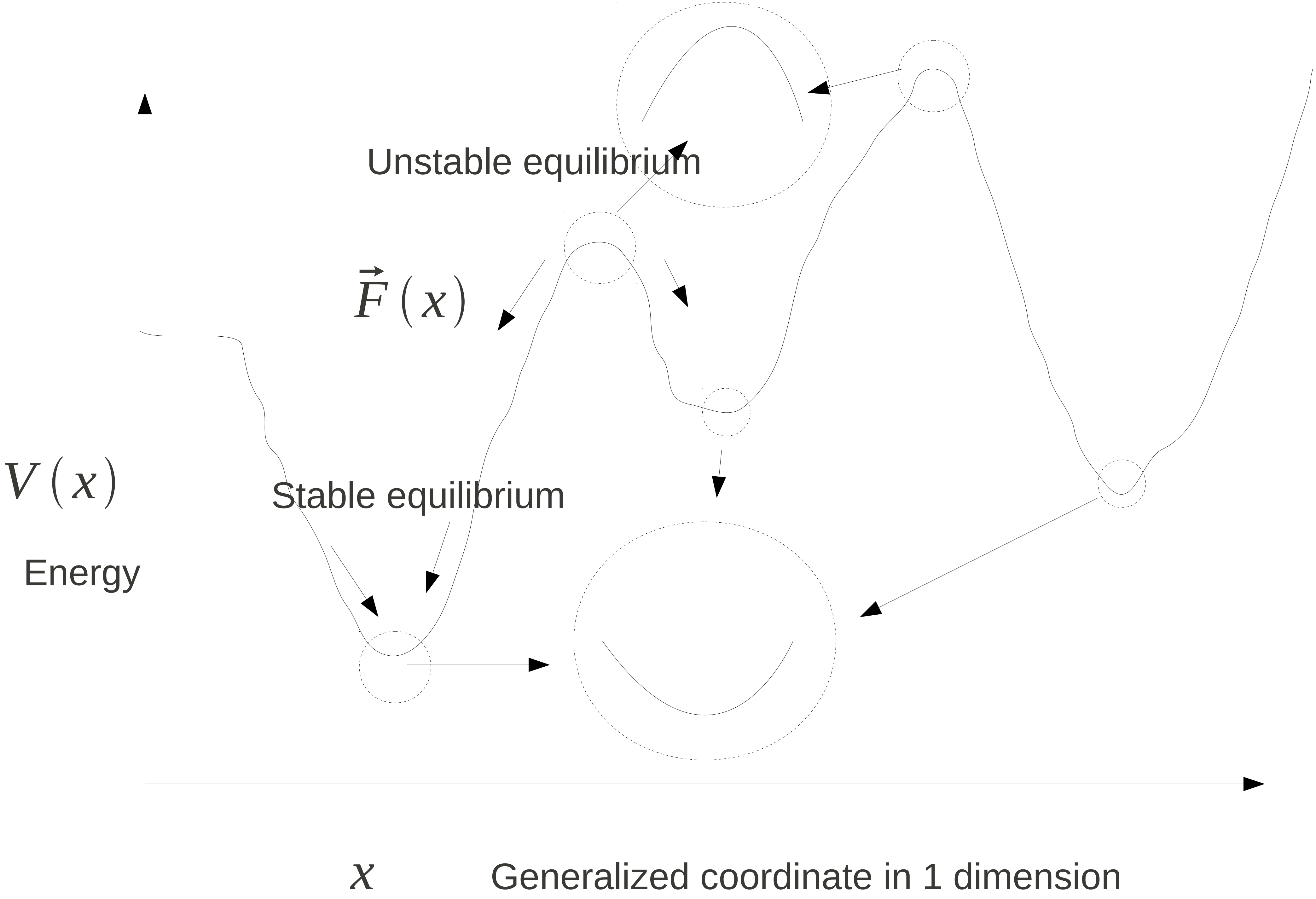
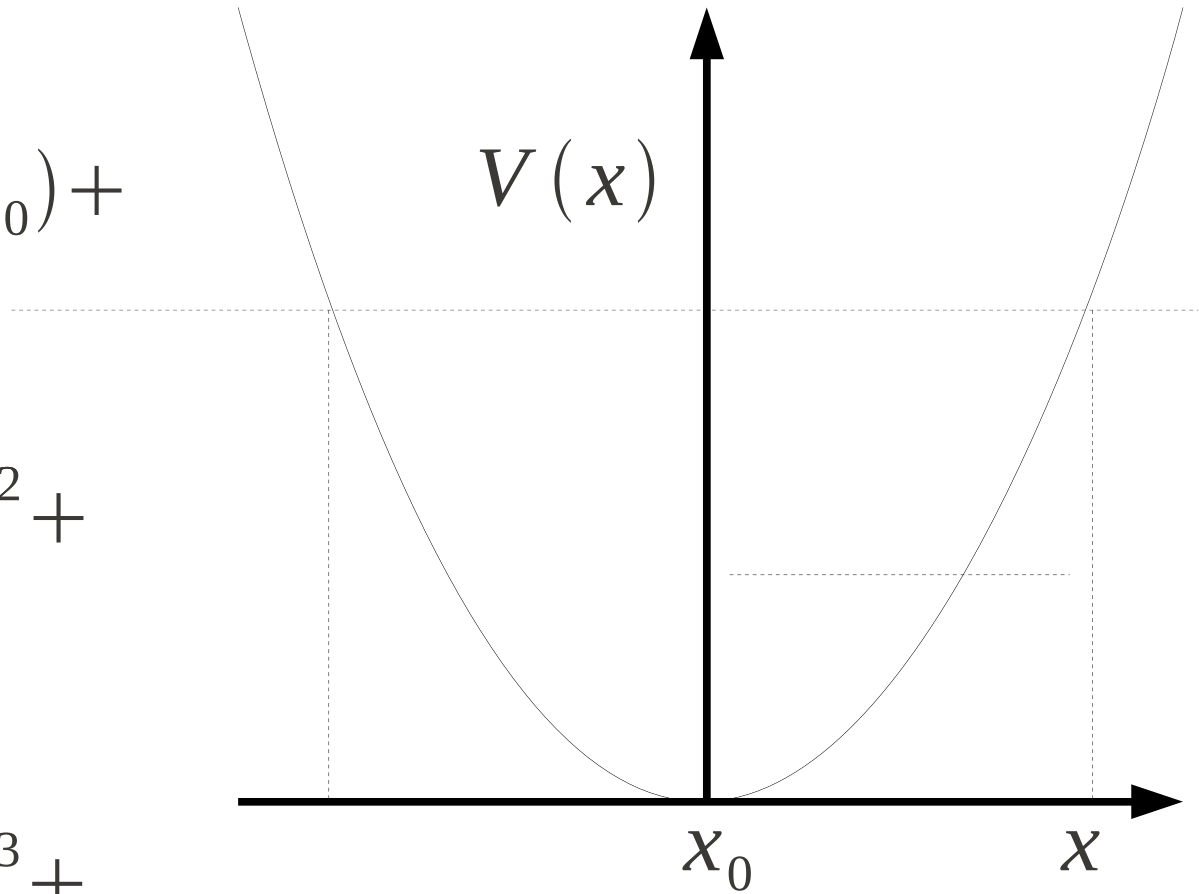

Unstable equilibrium
$\vec{F}(x)$
$V(x)$
Energy
Stable equilibrium
x
Generalized coordinate in 1 dimension

Taylor series expansion potential around potential energy minimum

$$V(x) = V(x_0) + \left.\frac{dV(x)}{dx}\right|_{x=0}(x-x_0) + \frac{1}{2!}\left.\frac{d^2V(x)}{dx^2}\right|_{x=0}(x-x_0)^2 + \frac{1}{3!}\left.\frac{d^3V(x)}{dx^3}\right|_{x=0}(x-x_0)^3 + \ldots\ldots\ldots\ldots$$



First differential is zero near potential energy minimum

$$\frac{dV(x)}{dx} = 0$$

Neglecting higher order terms

$$V(x) \simeq V(x_0) + \frac{1}{2!} \left. \frac{d^2 V(x)}{d x^2} \right|_{x=0} (x - x_0)^2$$

Potential of harmonic oscillator

$$V(x) = \frac{1}{2} k x^2$$

$$V(x) \simeq V(x_0) + \frac{1}{2!} \left. \frac{d^2 V(x)}{d x^2} \right|_{x=0} (x - x_0)^2$$

$$V(x) = V(\underset{0}{x_0}) + \frac{1}{2} k (x - \underset{0}{x_0})^2$$

No higher order terms

$$V(x) = \frac{1}{2} k (x)^2 \qquad \Rightarrow \left. \frac{d^2 V}{d x^2} \right|_{x=0} = k$$

Simple pendulum

$$U\left(h(\theta)\right)=mgl\left(1-\cos\theta\right) \qquad U(\theta)=mgl\left(1-\cos\theta\right)$$

$$\frac{dU}{d\theta}=mgl\left(\sin\theta\right) \qquad \frac{d^2U}{d\theta^2}=mgl\left(\cos\theta\right) \qquad \frac{d^3U}{d\theta^3}=mgl\left(-\sin\theta\right)$$

Using the equation for harmonic approximation of the potential

$$V(x)= \quad V(x_0)+\left.\frac{dV(x)}{dx}\right|_{x=0}(x-x_0)+$$

$$\frac{1}{2!}\left.\frac{d^2V(x)}{dx^2}\right|_{x=0}(x-x_0)^2+ \quad \frac{1}{3!}\left.\frac{d^3V(x)}{dx^3}\right|_{x=0}(x-x_0)^3+\ldots$$

When $\theta=0$ by substituting into the general expansion

$$U(\theta)=\frac{1}{2}mgl\,\cancelto{1}{\cos\theta}\left(\theta-\cancelto{0}{\theta_0}\right)^2$$

$$U(\theta)=\frac{1}{2}mgl\left(\theta\right)^2$$

Solution of the differential equation

$$m\frac{d^2x}{dt^2}=-kx \qquad \frac{d^2x}{dt^2}=-\frac{k}{m}x$$

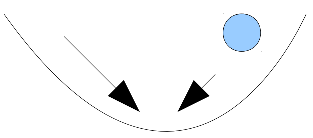
$$\frac{d^2x}{dt^2}=-\omega^2x$$

By method of constant coefficients the solution is

$$D^2=-\omega^2 \qquad \Rightarrow D=\mp i\omega$$

This is a second order differential equation, therefore two constants are to be specified

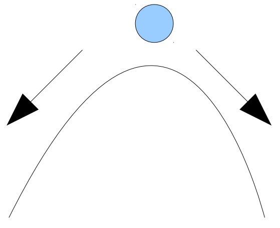
Convex surface



$$\Rightarrow x=Ae^{i\omega t}+Be^{-i\omega t}$$

When system is in unstable equilibrium we may write the equation as

$$\frac{d^2x}{dt^2}=\omega^2x$$



$$\Rightarrow x=Ae^{\omega t}+Be^{-\omega t}$$

$$\Rightarrow D=\mp\omega$$

Concave surface

Consider the general solution of the harmonic oscillator

$$x = A e^{i\omega t} + B e^{-i\omega t}$$

The coefficients A, B can be real or complex

The solution may be converted into sum of sine and cosine functions

We know from complex numbers that

$$e^{ia} = \cos a + i \sin a$$
$$e^{-ia} = \cos a - i \sin a$$

Using these relation the general solution becomes

$$x = A(\cos\omega t + i\sin\omega t) + B(\cos\omega t - i\sin\omega t)$$

by rearranging the coefficients

$$x = (A+B)\cos\omega t + i(A-B)\sin\omega t$$

$$x = c\cos\omega t + d\sin\omega t$$

$$c = A+B, d = i(A-B)$$

$$x = A e^{i\omega t} + B e^{-i\omega t}$$

$$x = c \cos \omega t + d \sin \omega t$$

another form of this equation is obtained by the substituting the relations

$$A = \frac{a}{2} e^{i\phi}, B = \frac{a}{2} e^{-i\phi}$$

in $x = (A+B) \cos \omega t + i(A-B) \sin \omega t$

$$x = \frac{a}{2}(e^{i\phi} + e^{-i\phi}) \cos \omega t - \frac{a}{2i}(e^{i\phi} - e^{-i\phi}) \sin \omega t$$

$$x = a(\cos \phi \cos \omega t - \sin \phi \sin \omega t)$$

This is the expansion of

$x = a \cos(\omega t + \phi)$ redefining the constant $\phi = -\theta$

$$x = a \cos(\omega t - \theta)$$

By defining constants in solution appropriately we can arrive at different forms of solutions

Let the solution differential equation

$$m\frac{d^2x}{dt^2}=-kx$$

$$m\frac{d^2z}{dt^2}+kz=0$$

be a complex number

$$z=x+iy$$

Both real and imaginary part of the solution must satisfy the differential equation for the oscillator independently

$$m\frac{d^2(x+iy)}{dt^2}+k(x+iy)=0$$

$$\left(m\frac{d^2x}{dt^2}+kx\right)+i\left(m\frac{d^2y}{dt^2}+ky\right)=0$$

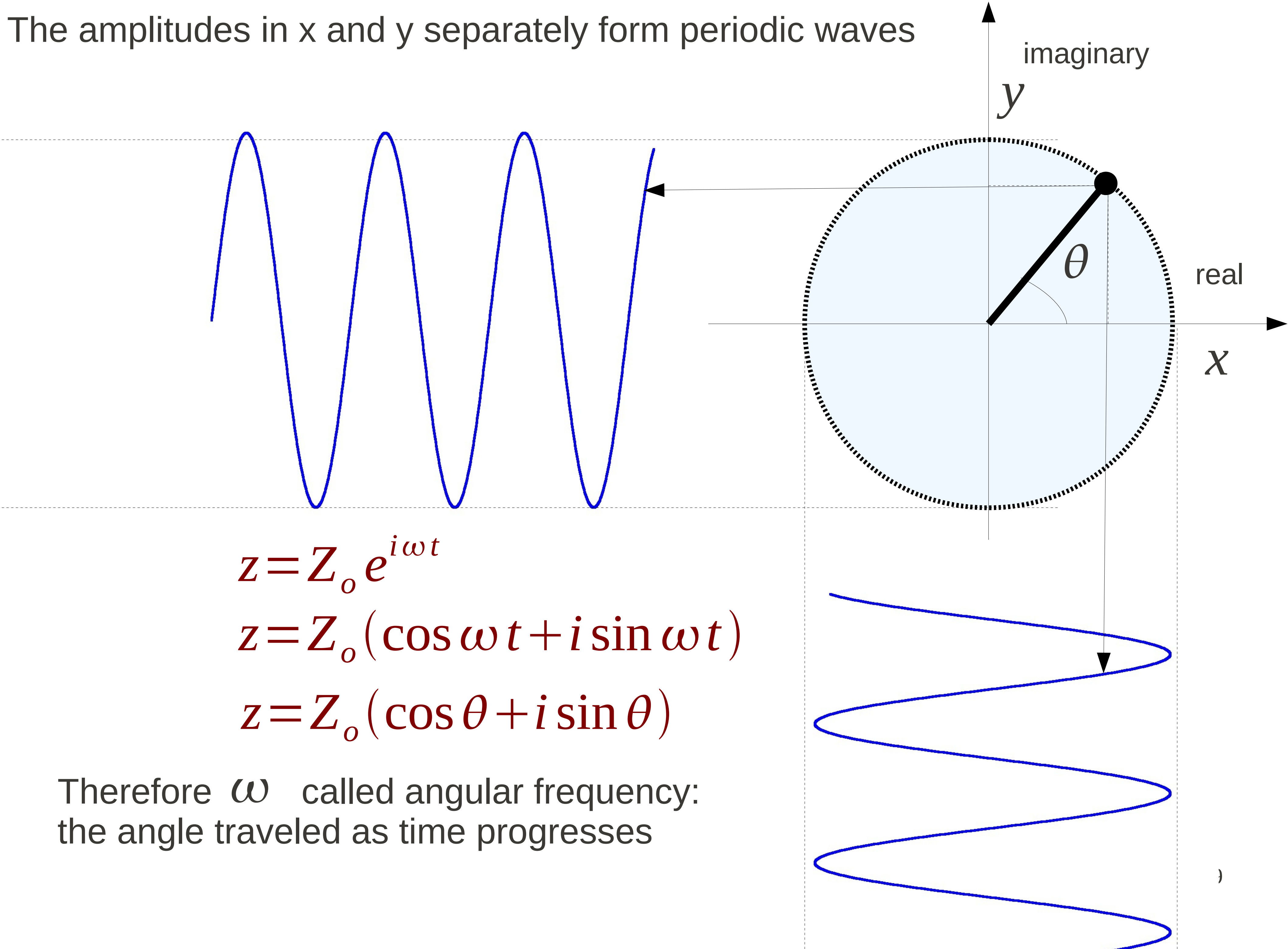
real

imaginary

$$z=Z_o e^{i\omega t}$$

any complex number can be expressed in this form

The amplitudes in x and y separately form periodic waves



$$z = Z_o e^{i\omega t}$$

$$z = Z_o(\cos\omega t + i\sin\omega t)$$

$$z = Z_o(\cos\theta + i\sin\theta)$$

Therefore ω called angular frequency:
the angle traveled as time progresses

$$x = a\cos(\omega t - \theta_1)$$

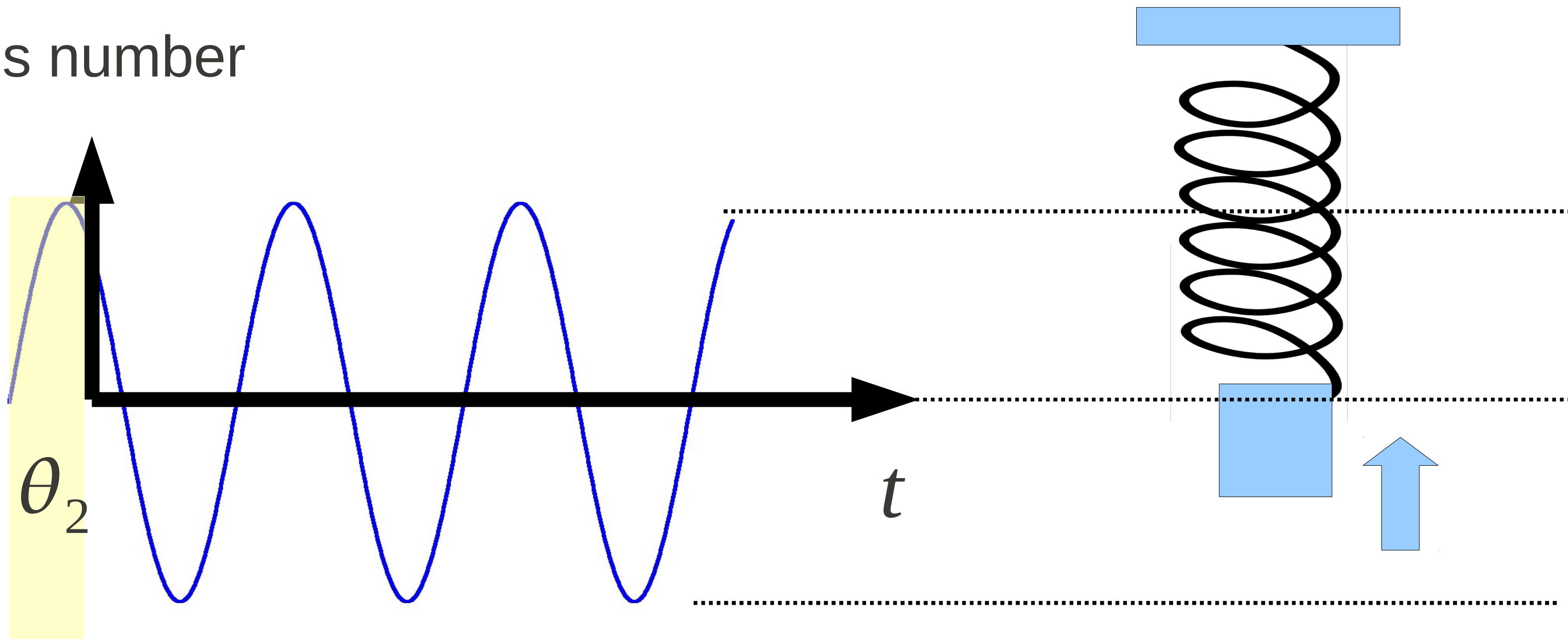
$$x = a\sin(\omega t - \theta_2)$$

phase of the oscillator

frequency of the oscillator

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$$

Dimensionless number



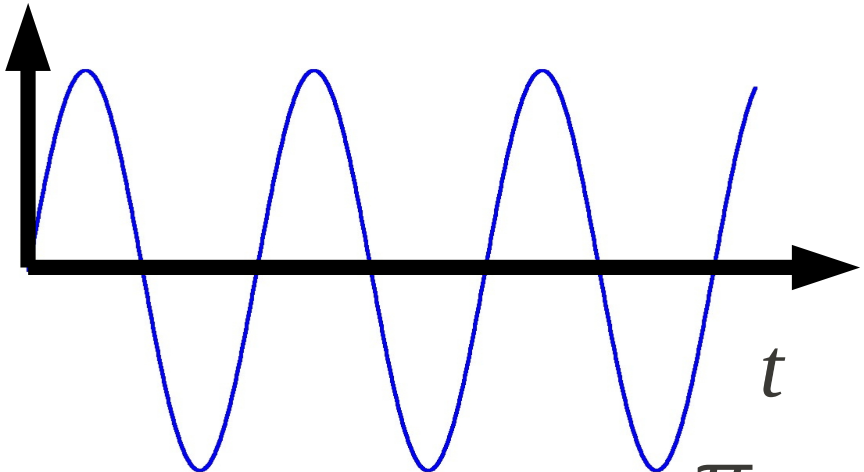
$$x = a\sin(\omega t - \theta_2)$$

f_0 has dimension of per unit time

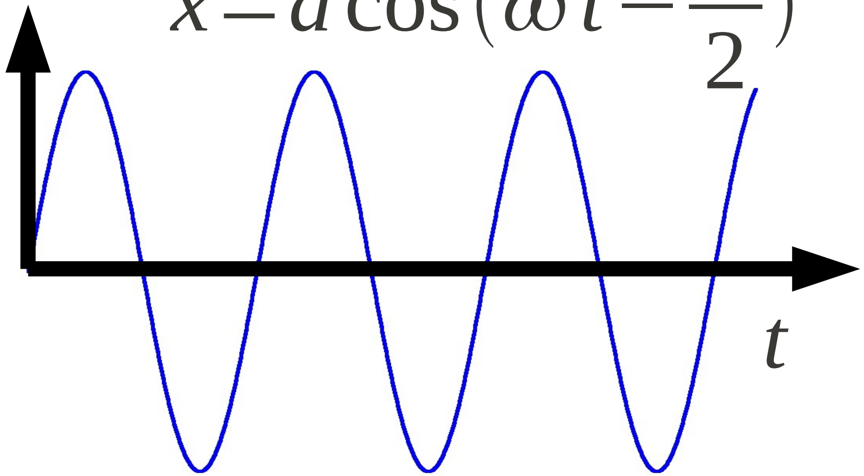
$$\frac{1}{f_0} = \frac{2\pi}{\omega}$$

is the period of one oscillation

$$x = a\sin(\omega t)$$



$$x = a\cos\left(\omega t - \frac{\pi}{2}\right)$$

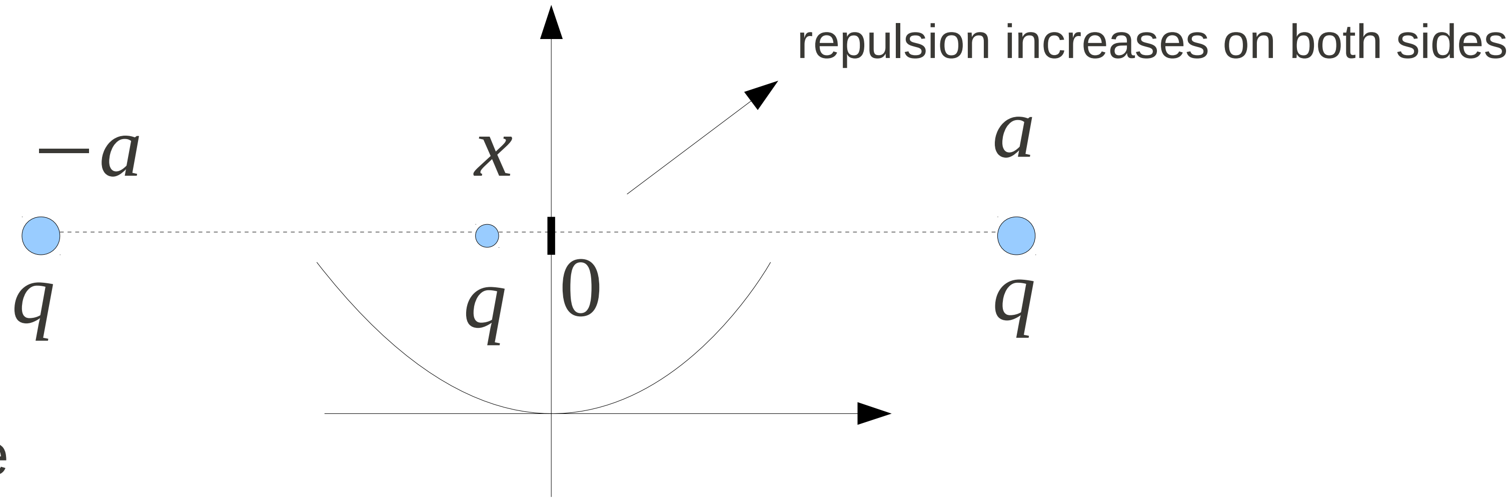


equivalence of sine and cosine representation

f_0 Has dimension of per unit time

$\frac{1}{f_0} = \frac{2\pi}{\omega}$ Is the period of one oscillation

Charged particle between two fixed charges - frequency of oscillation



Force on the particle

$$F=\frac{q^2}{4\pi\epsilon_0(a-x)^2}-\frac{q^2}{4\pi\epsilon_0(a+x)^2}$$

there are two opposing forces

The potential energy function

$$V=\frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{(a+x)}+\frac{1}{(a-x)}\right]=\frac{q^2}{4\pi\epsilon_0}\left[\frac{2a}{(a^2-x^2)}\right]$$

$$V=\frac{q^2}{4\pi\epsilon_0}\left[\frac{1}{(a+x)}+\frac{1}{(a-x)}\right]=\frac{q^2}{4\pi\epsilon_0}\left[\frac{2a}{(a^2-x^2)}\right]$$

The force constant is given by second derivative of the potential

$$\left.\frac{d^2V}{dx^2}\right|_{x=0}=k$$

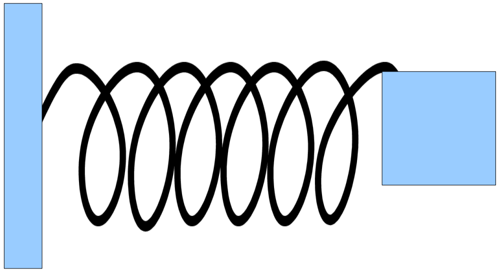
$$\left.\frac{d^2V}{dx^2}\right|_{x=0}=\left.\frac{q^2 a}{4\pi\epsilon_0}\left[\frac{16ax^2+4a}{(a^2-x^2)^2}\right]\right|_{x=0}=\frac{q^2}{\pi\epsilon_0 a^3}=k$$

The frequency of the oscillator

$$\omega^2=\frac{k}{m}=\frac{q^2}{\pi\epsilon_0 a^3 m}$$

Amplitude, frequency and period of a harmonic vibration

We have the general solution for harmonic oscillator



$$x(t) = A\cos\omega t + B\sin\omega t$$

Let $x(t=0) = x_0$ $\quad v(t=0) = v_0$ from this initial condition the constants of the solution can be determined

$$x(t=0) = x_0 = A$$

$$\dot{x}(t=0) = v_0 = -A\omega\sin\omega 0 + B\omega\cos\omega 0 = \omega B$$

$$B = \frac{v_0}{\omega}$$

Now by substituting for constants A and B

$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

A method of transforming the general solution into single constant and phase angle is

$$x(t) = A\cos\omega t + B\sin\omega t$$

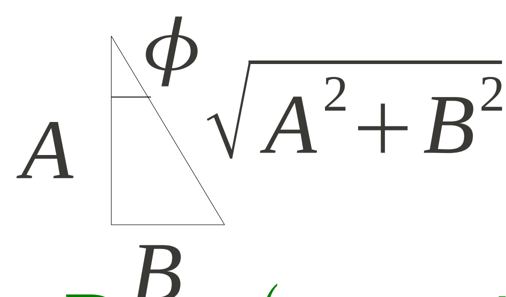
Dividing both terms with the quantity $\sqrt{A^2+B^2}$

$$x(t) = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos\omega t + \frac{B}{\sqrt{A^2+B^2}}\sin\omega t\right)$$

Consider a triangle with sides A and B then $\sqrt{A^2+B^2}$ gives the hypotenuse of a right angle triangle

$$\frac{A}{\sqrt{A^2+B^2}} = \cos\phi \qquad \Rightarrow \frac{B}{\sqrt{A^2+B^2}} = \sqrt{1-\cos^2\phi} = \sin\phi$$

$$D = \sqrt{A^2+B^2} \qquad \frac{B}{A} = \tan\phi$$



$$x(t) = D\left(\cos\phi\cos\omega t + \sin\phi\sin\omega t\right) = D\cos(\omega t - \phi)$$

A method of transforming the general solution into single constant and phase angle is

$$x(t) = D\left(\cos\phi\cos\omega t + \sin\phi\sin\omega t\right) = D\cos(\omega t - \phi)$$

Applying this transformation in this solution

$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t$$

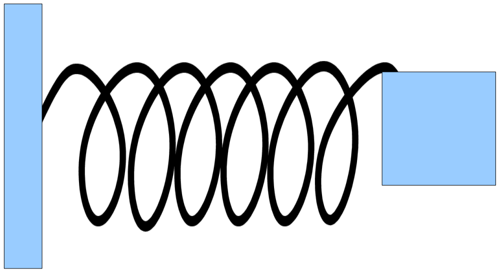
$$B = \frac{v_0}{\omega}$$

Applying this transformation in this solution

$$A = x_0$$

$$x(t) = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}\cos(\omega t - \phi)$$

Let the mass be $m = 2 \times 10^4 \, g$



Initial position $x_0(t=0) = 400 \, cm$

Initial velocity $v_0(t=0) = -150 \, cm/s$

Initial acceleration $a_0(t=0) = -1000 \, cm/s^2$

Force constant can be obtained from equation of motion for harmonic oscillator

$$F = ma = -kx$$

For harmonic oscillator $k = -\frac{F}{x}$,

$$\omega^2 = \frac{k}{m} = -\frac{F}{xm}, \qquad \Rightarrow \omega^2 = -\frac{a}{x} = 2.5 \, s^{-2}$$

$$\omega^2 = -\frac{a}{x} = 2.5\,s^{-2}$$

$$x_0(t=0) = 400\,cm$$

$$v_0(t=0) = -150\,cm/s$$

$$B = \frac{v_0}{\omega}$$

$$\frac{B}{A} = \tan\phi$$

$$A = x_0$$

$$x(t) = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}\cos(\omega t - \phi)$$

$$x(t) = 411.1\cos\left(2t/\sqrt{10} - 0.237\right)$$

The solution can give complete information of motion of the oscillator

Series connection of springs

When a force is applied the total variation of length is, respectively x_1, x_2 .

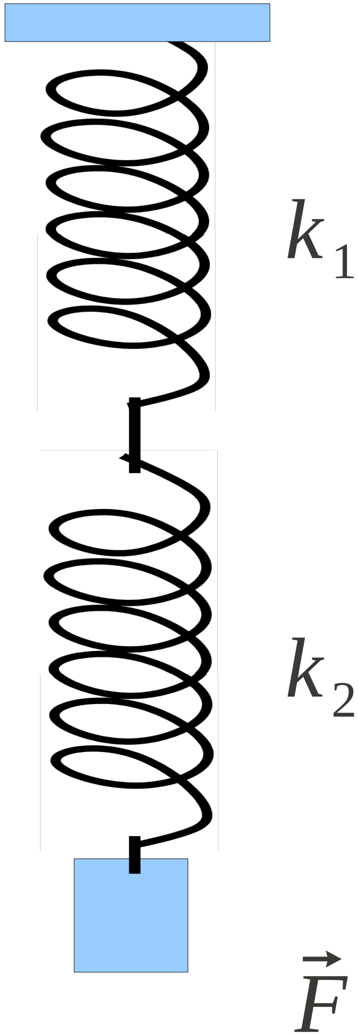
Let this system is be replaced with spring with effective spring constant. k

Now total displacement $F/k = x = x_1 + x_2$

$$\frac{F}{k} = \frac{F}{k_1} + \frac{F}{k_2}$$

$$\Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$k < k_1 , k < k_2$$



This can be generalized to n springs

$$\Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \ldots\ldots\ldots\ldots + \frac{1}{k_n} = \sum_{n=1}^{N} \frac{1}{k_i}$$

Parallel connection of springs

When force is applied the springs undergo same variation of length.

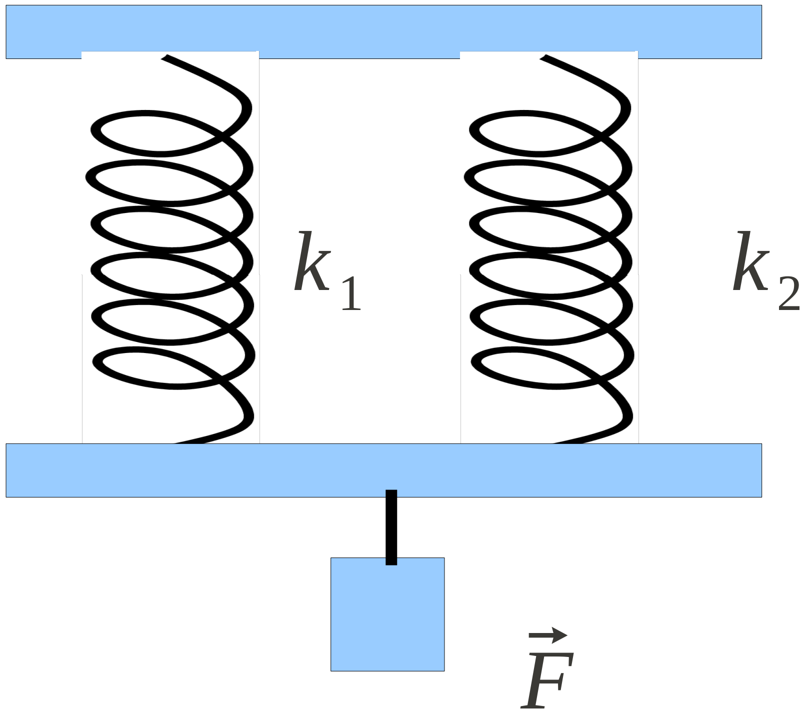
$$x = x_1 = x_2$$

$$F = k_1 x + k_2 x = k\,x$$

$$\Rightarrow k_1 + k_2 = k$$

In general for n springs

$$\Rightarrow k_1 + k_2 \ldots\ldots k_n = k$$



Average K.E and P. E.

Time average of the kinetic energy

Period of the oscillations

$$\langle K \rangle = \frac{1}{T}\int_0^T K(t)\,dt \qquad T = \frac{1}{f_0} = \frac{2\pi}{\omega}$$

The oscillation repeat over every period therefor average over a period is same as that over time till infinity

$$x = a\sin(\omega t - \theta_1 = 0)$$

$$\dot{x} = a\,\omega\cos(\omega t)$$

$$\langle K \rangle = \frac{\int_0^T \frac{1}{2} m \dot{x}^2\,dt}{T} = \frac{1}{2} m\omega^2 a^2 \frac{\int_0^{\frac{2\pi}{\omega}} \cos^2(\omega t)\,dt}{\frac{2\pi}{\omega}}$$

$$= \frac{1}{2} m\omega^2 a^2 I$$

integral

$$I=\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t)\,dt$$

We know following equivalence relation

$$I=\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\cos^2(\omega t)\,dt=\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}\sin^2(\omega t)\,dt$$

Summing sine and cosine integrals

$$2I=\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}}dt=1 \qquad \Rightarrow I=\frac{1}{2}$$

$$\Rightarrow\langle K\rangle=\frac{1}{2}m\omega^2a^2I=\frac{1}{4}m\omega^2a^2$$

the potential energy is given by

$$x = a\sin(\omega t - \theta_1 = 0)$$

$$P(t) = \frac{1}{2} k\, x(t)^2$$

where $x(t) = a\sin(\omega t)$

$$P(t) = \frac{1}{2} m\omega^2 a^2 \sin^2(\omega t)$$

$$k = m\omega^2$$

Time average of the potential energy over one cycle

$$\langle P \rangle = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{2} m\omega^2 a^2 \frac{\int_0^{\frac{2\pi}{\omega}} \sin^2(\omega t)\,dt}{\frac{2\pi}{\omega}}$$

$$\langle P \rangle = \frac{1}{4} m a^2 \omega^2$$

$$\langle K \rangle = \frac{1}{4} m\omega^2 a^2$$

Total energy of the system is

$$E = \langle E \rangle = \langle K \rangle + \langle P \rangle = \frac{1}{2} m a^2 \omega^2$$

Frictional forces in mechanical oscillators

Frictional forces are represented by a term that is proportional to velocity

$$F = ma$$

$$m\frac{d^2x}{dt^2} = F_{fric} = -b\frac{dx}{dt} = -b\dot{x}$$

A system only frictional force is acting

Is a positive constant – dimension = force/velocity

$$\tau = \frac{m}{b}$$

relaxation time of the system – dimension of time

$$m\left[\frac{d^2x}{dt^2} + \frac{1}{\tau}\frac{dx}{dt} = 0\right]$$

using $v = \dfrac{dx}{dt}$

In terms of velocities $\dfrac{dv}{dt} + \dfrac{1}{\tau}v = 0$

In terms of velocities $$\frac{dv}{dt}+\frac{1}{\tau}v=0$$

$$\frac{dv}{dt}=-\frac{1}{\tau}v$$

re-arranging $$\frac{dv}{v}=-\frac{1}{\tau}dt$$

integrating $$\log v=-\frac{1}{\tau}t+c$$ at $t=0$ $\quad c=\log v_0$

$$\log v=-\frac{1}{\tau}t+\log v_0$$

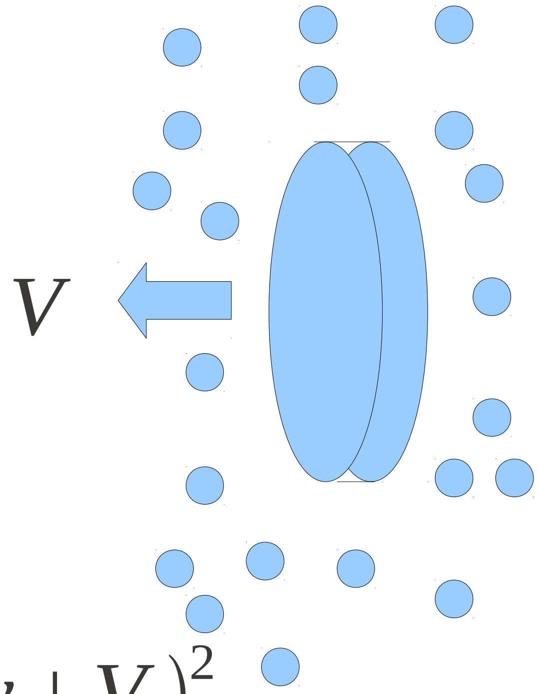
$$\log\left(\frac{v}{v_0}\right)=-t/\tau \qquad v(t)=v_0 e^{-t/\tau}$$

Exponential relaxation of velocity, leads to relaxation of the kinetic energy

$$K=\frac{1}{2}mv^2=\frac{1}{2}mv_0^2e^{-2t/\tau}$$

Molecular origin of friction

Consider a Plate moving 1d in a medium



$$v = \langle v_i \rangle$$

The disc moves slower than molecules

The pressure is proportional to the transfer of kinetic energy by collision of molecules

One side of the plate pressure $$P_1 \propto (v+V)^2$$

On the other side $$P_2 \propto (v-V)^2$$

Difference in pressure

$$P = P_1 - P_2 \propto 4vV \propto v$$

Drag is proportional to the plate velocity

Frictional forces in mechanical oscillators

Equation for particle that experience only frictional forces

$$m\frac{d^2x}{dt^2}=F_{fric}=-b\frac{dx}{dt}=-b\dot{x}$$

When such particle is subjected to harmonic restoration force

$$m\frac{d^2x}{dt^2}=F_{fric}+F_{harmonic}=-b\frac{dx}{dt}-kx$$

The desired differential equation is then

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

Solving it using method for linear differential equation with constant coefficients.

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

Using the substitution of the operator $D=\frac{d}{dx}$

$$mD^2x+bDx+kx=0$$

$$mD^2+bD+k=0$$

This is in the standard format of a quadratic equation

$$ax^2+bx+c=0 \qquad x=\frac{-b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

The algebraic solution is now given by

$$D=\frac{-b}{2m}\pm\frac{\sqrt{b^2-4mk}}{2m}$$

$$D=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$$ Where $\gamma=\frac{b}{2m}$ and $\omega_0^2=\frac{k}{m}$

This leads to two independent solution

ω_0 is the frequency of undamped oscillator

We know there can be three type of solutions

$$\Omega = \sqrt{\gamma^2 - \omega_0^2}$$

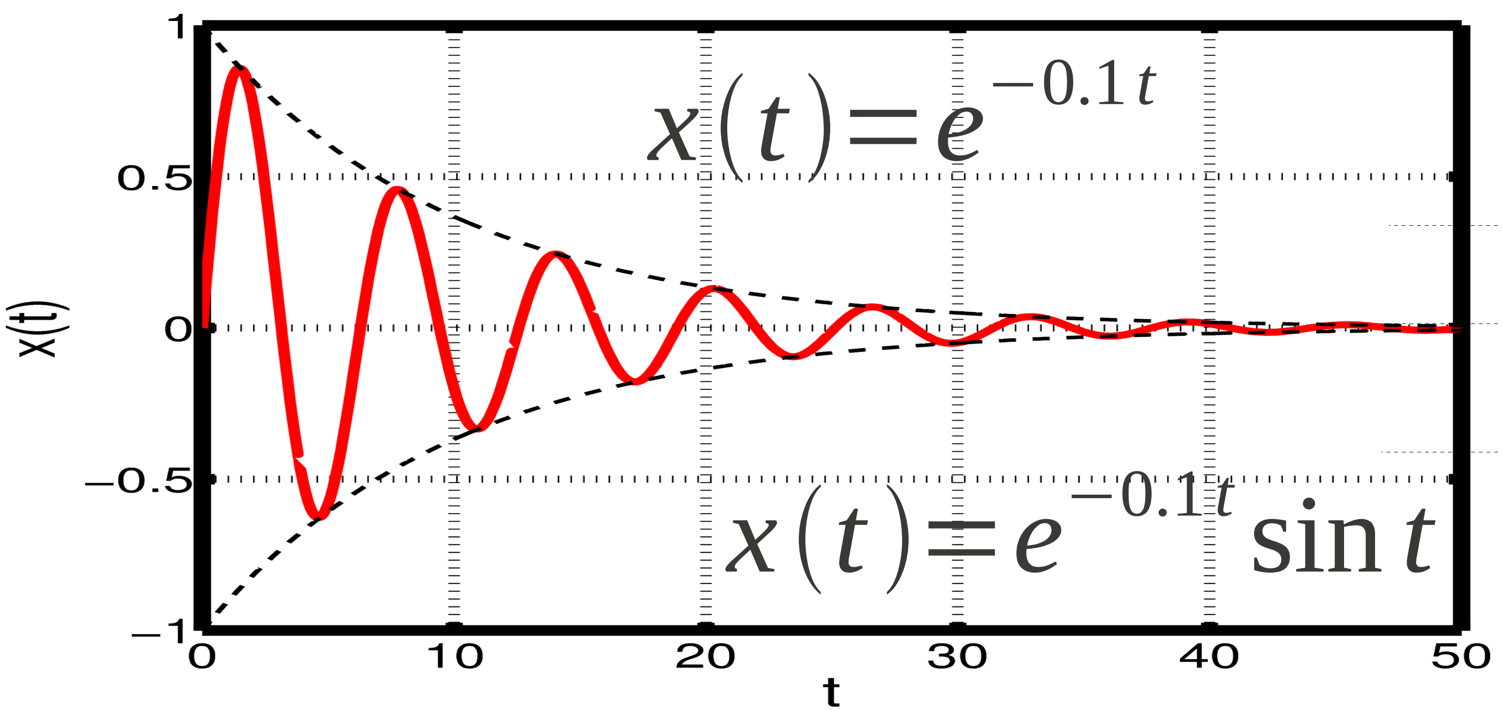
$$D = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} \text{ or } D = -\gamma \pm \Omega$$

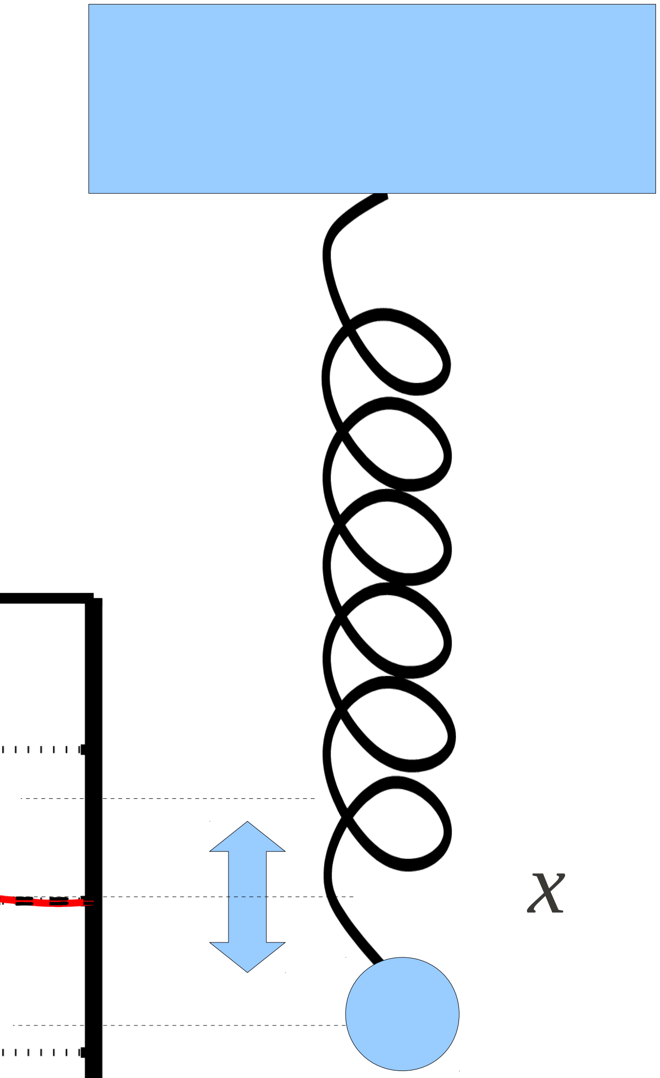
The general solution of the differential equation is

$$x(t) = e^{-\gamma t}\left(A e^{\Omega t} + B e^{-\Omega t}\right)$$

Case I : Underdamping $\gamma < \omega_0 , \quad \Omega^2 < 0$

The frequency of the oscillation is Ω





Analytical Mechanics by Fowls and Cassiday

In this case Ω is purely imaginary, let $\Omega = i\omega$

The solution is $x(t) = e^{-\gamma t}(Ae^{i\omega t} + Be^{-i\omega t})$

This may be converted into

$$x(t) = Ce^{-\gamma t}\cos(\omega t + \phi)$$

The real exponential function decays the amplitude to zero

$$\omega < \omega_0$$

The **relaxation time** of the oscillator is defined as time at which the amplitude reduces to $\frac{1}{e}$ of the original

$$\gamma t = 1$$

$$\gamma = \frac{b}{2m}$$

$$t = \tau = 1/\gamma = 2m/b$$

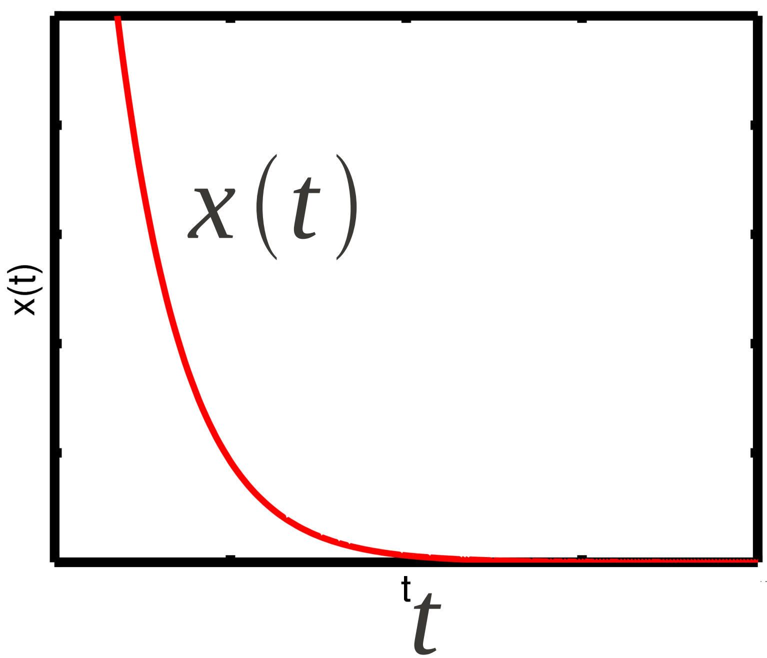
Case II :Overdamping $\Omega^2 > 0 \quad \gamma > \omega_0,$

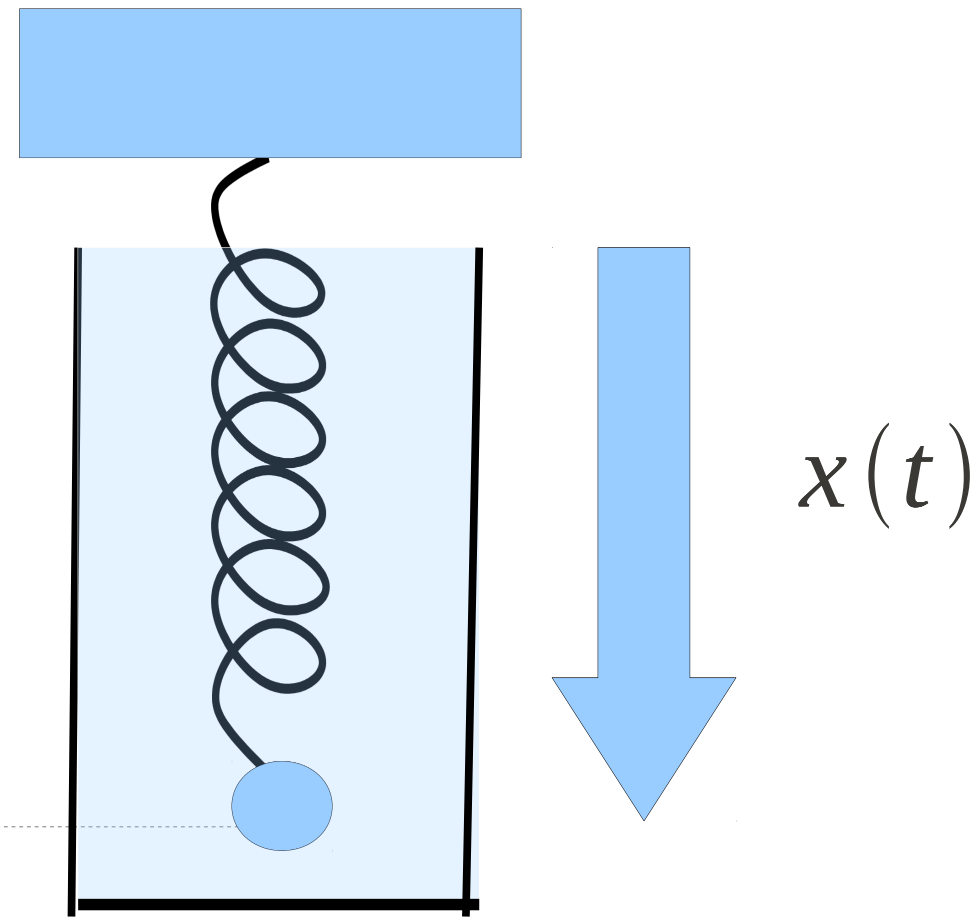
$$x(t) = e^{-\gamma t}\left(A e^{\Omega t} + B e^{-\Omega t}\right)$$

$$D = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$D = \gamma \pm \Omega$$

The motion goes to zero at large t





The compressed spring slowly relaxes to the equilibrium position in a viscous liquid

$$x(t)=Ae^{-(\gamma-\Omega)t}+Be^{-(\gamma+\Omega)t}$$

slower part

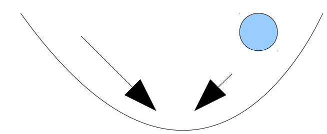
faster part

$$\Omega=\sqrt{\gamma^2-\omega_0^2}$$

γ slightly greater than $\omega_{0,}$ $\Rightarrow \Omega^2 \approx 0$

The two terms have roughly equal exponential decay

$$x(t)\approx Ce^{-(\gamma)t}$$



When $\gamma \gg \omega_0$

$$\Omega=\sqrt{\gamma^2-\omega_0^2}=\gamma\sqrt{1-\frac{\omega_0^2}{\gamma^2}}\approx\gamma\left(1-\frac{\omega_0^2}{2\gamma^2}\right)$$

$$x(t)=Ae^{-(\gamma-\Omega)t}+Be^{-(\gamma+\Omega)t}$$

negligible

$$x(t)=Ae^{-(\omega_0^2/2\gamma)t}+Be^{-(2\gamma+\omega_0^2/2\gamma)t}$$

$$x(t)\approx Ae^{-(\omega_0^2/2\gamma)t}$$

case of weak spring in highly viscous liquid

Case II :Critically damped $\Omega^2 = 0 \quad \gamma = \omega_0 ,$

The general equation for a damped harmonic oscillator is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$ since $\gamma = \omega_0 ,$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \gamma^2 x = 0$$

$$\gamma = \frac{b}{2m} \qquad \omega_0^2 = \frac{k}{m}$$

The general solution is not valid since the roots are the same

$$x(t) = e^{-\gamma t}\left(Ae^{\Omega t} + Be^{-\Omega t}\right) = C_1 e^{-\gamma t}$$

To get the second solution consider

$$x_1(t) = C_1 e^{-\gamma t}$$ be the first solution

Now set the ratio of the two solution be

$$x_2(t) = u\, x_1(t)$$

Now set

$$x_2(t)=u\,x_1(t)$$

$$\frac{d^2(u\,x_1(t))}{dt^2}+2\gamma\frac{d(u\,x(t))}{dt}+\gamma^2 u\,x(t)=0$$

$$\frac{d(u\,x_1(t))}{dt}=u\frac{d\,x_1(t)}{dt}+x_1(t)\frac{du}{dt}$$

$$\frac{d^2(u\,x_1(t))}{dt^2}=\frac{d^2u}{dt^2}x_1(t)+2\frac{du}{dt}\frac{d\,x(t)}{dt}+\frac{d^2x(t)}{dt^2}u$$

By substituting and rearranging

$$x_1(t)\frac{d^2u}{dt^2}+2\frac{du}{dt}\left(\frac{d\,x_1(t)}{dt}+\gamma\,x_1(t)\right)+$$

$$u\left(\frac{d^2x_1}{dt^2}+2\gamma\frac{d\,x_1}{dt}+\gamma^2x_1\right)=0$$

$$x_1(t)\frac{d^2u}{dt^2}+2\frac{du}{dt}\left(\cancelto{0}{\frac{dx_1(t)}{dt}+\gamma x_1(t)}\right)+u\left(\cancelto{0}{\frac{d^2x_1}{dt^2}+2\gamma\frac{dx_1}{dt}+\gamma^2x_1}\right)=0$$

Differential equation

$$\frac{dx_1(t)}{dt}=-\gamma x_1(t)$$

From first solution

$$\frac{dx_1(t)}{dt}=-\gamma C_1e^{-\gamma t} \qquad x_1(t)=C_1e^{-\gamma t}$$

Only term that survives in this equation, that is

$$\Rightarrow\frac{d^2u}{dt^2}=0 \qquad \Rightarrow\frac{du}{dt}=C_2 \qquad \Rightarrow u=C_2t+C_3$$

$$x_2(t)=C_1e^{-\gamma t}(C_2t+C_3)$$

Therefore the general solution is

$$x_2(t)=e^{-\gamma t}(A+Bt)$$

Lagrange's method of reduction of order

The solution of the differential equation

$$x(t)=e^{-\gamma t}(A+Bt)$$

Exponent factor wins over the linear term

$$\lim_{t\to\infty} x(t)=\lim_{t\to\infty} e^{-\gamma t}(A+Bt)=0$$

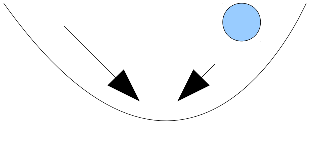
The motion converges to zero at quickest way

Compare it with the solution at over damped case

$$x(t)=Ae^{-(\gamma-\Omega)t}+Be^{-(\gamma+\Omega)t}$$

Here one term is slower than the critical damping case

Therefore it has application of design of shock absorbers, damping of gun recoiling, door closers etc..



Energy of damped harmonic oscillator

For a damped harmonic oscillator the energy is not conserved as friction dissipated energy

$$E(t)=E(0)+W(t)_{friction}$$

The total energy is

$$E(t)=K.E.+P.E. \qquad E(t)=\frac{1}{2}m\dot{x}^2+\frac{1}{2}k x^2$$

$$x(t)=C e^{-\gamma t}\cos(\omega t+\phi)$$

$$\dot{x}(t)=C e^{-\gamma t}\left[-\gamma\cos(\omega t+\phi)-\omega\sin(\omega t+\phi)\right]$$

$$\dot{x}(t)=C\omega e^{-\gamma t}\left[\frac{-\gamma}{\omega}\cos(\omega t+\phi)-\sin(\omega t+\phi)\right]$$

In case of small damping $\frac{\gamma}{\omega}\ll 1$

$$\dot{x}(t)=C\omega e^{-\gamma t}\sin(\omega t+\phi)$$

The total Kinetic Energy

$$K.E.=\frac{1}{2}m\dot{x}^2$$

$$=\frac{1}{2}mC^2\omega^2e^{-2\gamma t}\sin^2(\omega t+\phi)$$

The total Potential Energy

$$P.E.=\frac{1}{2}kx^2$$

$$x(t)=Ce^{-\gamma t}\cos(\omega t+\phi)$$

$$=\frac{1}{2}kC^2e^{-2\gamma t}\cos^2(\omega t+\phi)$$

$$\omega_0^2=\frac{k}{m}$$

$$E(t)=K.E.+P.E$$

$$E(t)=\frac{1}{2}mC^2e^{-2\gamma t}\left(\omega^2\sin^2(\omega t+\phi)+\frac{k}{m}\cos^2(\omega t+\phi)\right)$$

$$E(t)=\frac{1}{2}mC^2e^{-2\gamma t}\left(\omega^2\sin^2(\omega t+\phi)+\frac{k}{m}\cos^2(\omega t+\phi)\right)$$

When damping is small (under damping) $\gamma<\omega_0,$ $\Omega=i\omega$

$$\omega_0^2=\omega^2+\gamma^2$$

$$\Omega=\sqrt{\gamma^2-\omega_0^2}$$

$$=\omega^2\left(1+\left(\frac{\gamma}{\omega}\right)^2\right)\approx\omega^2 \qquad \gamma\ll\omega_0,$$

$$-\omega^2=\gamma^2-\omega_0^2$$

the frequency of oscillation depends on the friction of the system

$$E(t)=\frac{1}{2}mC^2\omega^2e^{-2\gamma t}\left(\sin^2(\omega t+\phi)+\cos^2(\omega t+\phi)\right)$$

$$E(t)=\frac{1}{2}mC^2\omega^2e^{-2\gamma t}=E_0e^{-2\gamma t}$$

In other two cases of solution system does not oscillate

Energy dissipation can also be calculated from expression for energy

$$\frac{dE(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = \dot{x}(m\ddot{x} + kx)$$

$$= \dot{x}(-b\dot{x})$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

We know that b has the dimension of force / velocity.

$$= \frac{Force \times distance}{time}$$

Rate of change of energy equal to rate at with energy is lost through the frictional forces.

$$E(t) = E_0 e^{-2\gamma t}$$

Relaxation time is defined as the time in energy drops to e^{-1} of the initial value

$$2\gamma t = 1 \qquad \Rightarrow t = \tau = \frac{1}{2\gamma}$$

as $\Rightarrow \gamma \to 0$ $\quad \tau \to \infty$

Quality factor Q of an oscillator

Defines the quality of the oscillations

$$Q = \frac{Energy\ stored\ \ in\ \ the\ system}{Energy\ dissipated\ per\ cycle}$$

Time required to oscillate through one cycle $\frac{T}{2\pi} = \frac{1}{\omega}$

$$E(t) = E_0 e^{-2\gamma t},$$

$$\frac{dE}{dt} = -2\gamma E_0 e^{-2\gamma t} = -2\gamma E$$

Energy lost per unit time is $\Delta E \approx \left| \frac{dE}{dt} \right| \Delta t,$

$$\Delta E \approx 2\gamma E \Delta t$$

Energy lost per unit time is $\Delta E \approx \left| \frac{dE}{dt} \right| \Delta t ,$

Since the time Δt for oscillation through one cycle is $\frac{1}{\omega}$ the energy dissipated per cycle is

$$= \frac{2\gamma E}{\omega}$$

$$Q = \frac{Energy\ stored\ in\ the\ system}{Energy\ dissipated\ per\ cycle}$$

$$Q = \frac{E}{\left(\frac{2\gamma E}{\omega} \right)} = \frac{\omega}{2\gamma} \approx \frac{\omega_0}{2\gamma}$$

Large Q value means good oscillator

Tuning fork has $Q \approx 10^3$ and superconducting microwave cavity has $Q \approx 10^7$

Damped and driven harmonic oscillator

Consider an oscillator in addition to have friction and harmonic forces, also subjected to external force

$$\gamma = \frac{b}{2m}$$

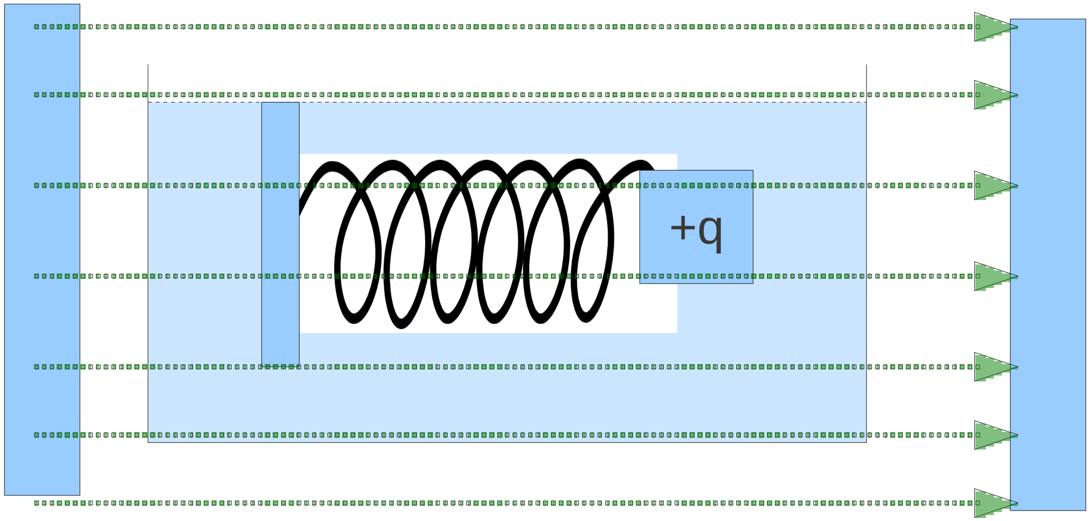
$$\omega_0^2 = \frac{k}{m}$$

$$m\frac{d^2x}{dt^2} = F_{fric} + F_{harmonic} + F_{driving}$$

$$= -b\frac{dx}{dt} - kx + F_{driving}$$

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t)$$

electric field $\vec{E}$



The differential equation is then

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F(t)}{m}$$

$$\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F(t)}{m}$$

operator

$$Lx = \frac{F(t)}{m}$$

We need only a particular solution of the problem, that is, the differential equation that satisfy the equating function.

$$Lx_0 = 0, \qquad \Rightarrow \frac{d^2 x_0}{dt^2} + 2\gamma \frac{dx_0}{dt} + \omega_0^2 x_0 = 0$$

Let the solution of the differential equation be $x_1(t)$ and let the solution of the homogeneous part be $x_0(t)$

$$Lx_1 = \frac{F(t)}{m}$$

Then $x(t) = x_1(t) + x_0(t)$ is also a solution to the equation

$$Lx = L(x_0 + x_1) = Lx_0 + Lx_1 = \frac{F(t)}{m}$$

Consider the case where driving forces are periodic

$$z = Z_o e^{i\omega t}$$

$$F(t) = F_1 \cos\omega_1 t \qquad z = Z_o(\cos\theta + i\sin\theta)$$

We know that $F(t) = F_1 \cos\omega_1 t = Real(F_1 e^{i\omega_1 t})$

There for more general representation of the periodic force is $F_1 e^{i\omega_1 t}$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = \frac{F_1}{m} e^{i\omega_1 t}$$

Now we consider a solution that is periodic in nature – same period as that of driving force- driving part is the final frequency of the oscillation

unknown amplitude

$$x_1(t) = a_1 e^{i(\omega_1 t - \phi_1)}$$

unknown phase

ω_1 driving frequency

The amplitude and phase of the oscillation is obtained by solving the differential equation

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=\frac{F_1}{m}e^{i\omega_1 t}$$

Now we consider a solution that is periodic in nature – same period as that of driving force- driving part is the final frequency of the oscillation

$$x_1(t)=a_1e^{i(\omega_1 t-\phi_1)}$$

$$\frac{dx_1}{dt}=a_1 i\omega_1 e^{i(\omega_1 t-\phi_1)}, \qquad \frac{d^2x_1}{dt^2}=-a_1\omega_1^2 e^{i(\omega_1 t-\phi_1)}$$

By substituting in the in the differential equation above

$$(-\omega_1^2+2i\gamma\omega_1+\omega_0^2)a_1e^{i(\omega_1 t-\phi_1)}=\frac{F_1}{m}e^{i\omega_1 t}$$

Rearranging the exponential part

$$(-\omega_1^2+2i\gamma\omega_1+\omega_0^2)a_1=\frac{F_1}{m}e^{i\phi_1}$$

Separating into real and imaginary parts

$$((\omega_0^2-\omega_1^2)+2i\gamma\omega_1)a_1=\frac{F_1}{m}(\cos\phi_1+i\sin\phi_1)$$

real part

$$(\omega_0^2-\omega_1^2)a_1=\frac{F_1}{m}\cos\phi_1,$$

imaginary part

$$2\gamma\omega_1 a_1=\frac{F_1}{m}\sin\phi_1$$

$$(\omega_0^2-\omega_1^2)a_1=\frac{F_1}{m}\cos\phi_1 \quad \text{and} \quad 2\gamma\omega_1 a_1=\frac{F_1}{m}\sin\phi_1$$

The magnitude of these oscillation is given by squaring and adding them

$$a_1=\frac{F_1/m}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}}$$

The phase between these oscillations taking ratio between imaginary and real parts of the solution

$$\tan\phi_1=\frac{2\gamma\omega_1}{\omega_0^2-\omega_1^2}$$

The real part of the particular solution of the equation is now

$$x_1(t)=a_1\cos(\omega_1 t-\phi_1)$$

Then the general solution is for a given system having small damping or under-damped systems

$$x(t)=x_1(t)+x_0(t)=a_1\cos(\omega_1 t-\phi_1)+Ce^{-\gamma t}\cos(\omega t+\phi)$$

$$x(t)=x_1(t)+x_0(t)=a_1\cos(\omega_1 t-\phi_1)+Ce^{-\gamma t}\cos(\omega t+\phi)$$

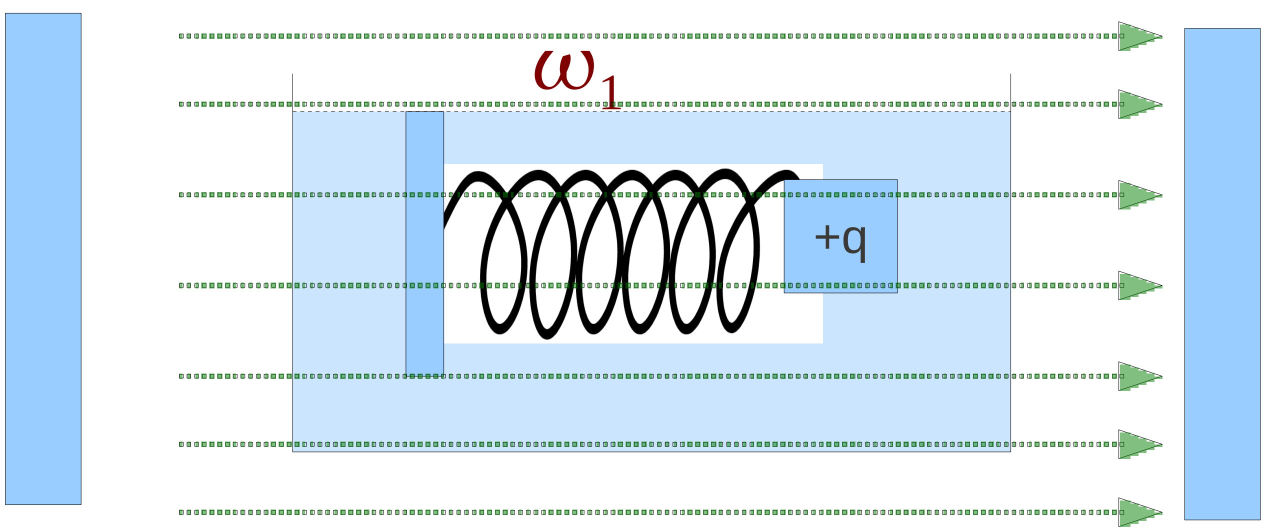
Transient term

(decays with time)

In long time the first term dominates and oscillator oscillates with frequency ω_1. Eventually driving frequency dominates.

In long time the solution becomes

$$x(t)=a_1\cos(\omega_1 t-\phi_1)$$



$$\vec{E}=\vec{E}_o\cos(\omega_1 t)$$

Oscillating electric field

$$x(t)=x_1(t)+x_0(t)=a_1\cos(\omega_1 t-\phi_1)+Ce^{-\gamma t}\cos(\omega t+\phi)$$

In long time the transient term vanishes, therefore the solution is

$$x(t)=a_1\cos(\omega_1 t-\phi_1)$$

$$x(t)=\frac{F_1/m}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}}\cos(\omega_1 t-\phi_1)$$

$$x(t)=\frac{F_1}{\sqrt{(m\omega_0^2-m\omega_1^2)^2+4\gamma^2 m^2\omega_1^2}}\cos(\omega_1 t-\phi_1)$$

By substitution of $R_m=2\gamma m,\quad m\omega_0^2=k$

$$x(t)=\frac{F_1}{\omega_1\sqrt{\left(\frac{k}{\omega_1}-m\omega_1\right)^2+R_m^2}}\cos(\omega_1 t-\phi_1)$$

$$x(t)=\frac{F_1}{\omega_1\sqrt{\left(\frac{k}{\omega_1}-m\omega_1\right)^2+R_m^2}}\cos(\omega_1 t-\phi_1)$$

We can define the mechanical impedance as

$$Z_m=\sqrt{\left(\frac{k}{\omega_1}-m\omega_1\right)^2+R_m^2}$$

Where mechanical reactance is given by $X_m=\left(\frac{k}{\omega_1}-m\omega_1\right)$

and mechanical resistance is given by $R_m=2\gamma m,$

$$x(t)=\frac{F_1}{\omega_1 Z_m}\cos(\omega_1 t-\phi_1)$$

Low driving frequency $\omega_1 \ll \omega_0$

The variation in the phase angle

$$\phi_1 = \tan^{-1}\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2} \qquad \phi_1 \to 0$$

Driving force and the resulting displacements are in phase

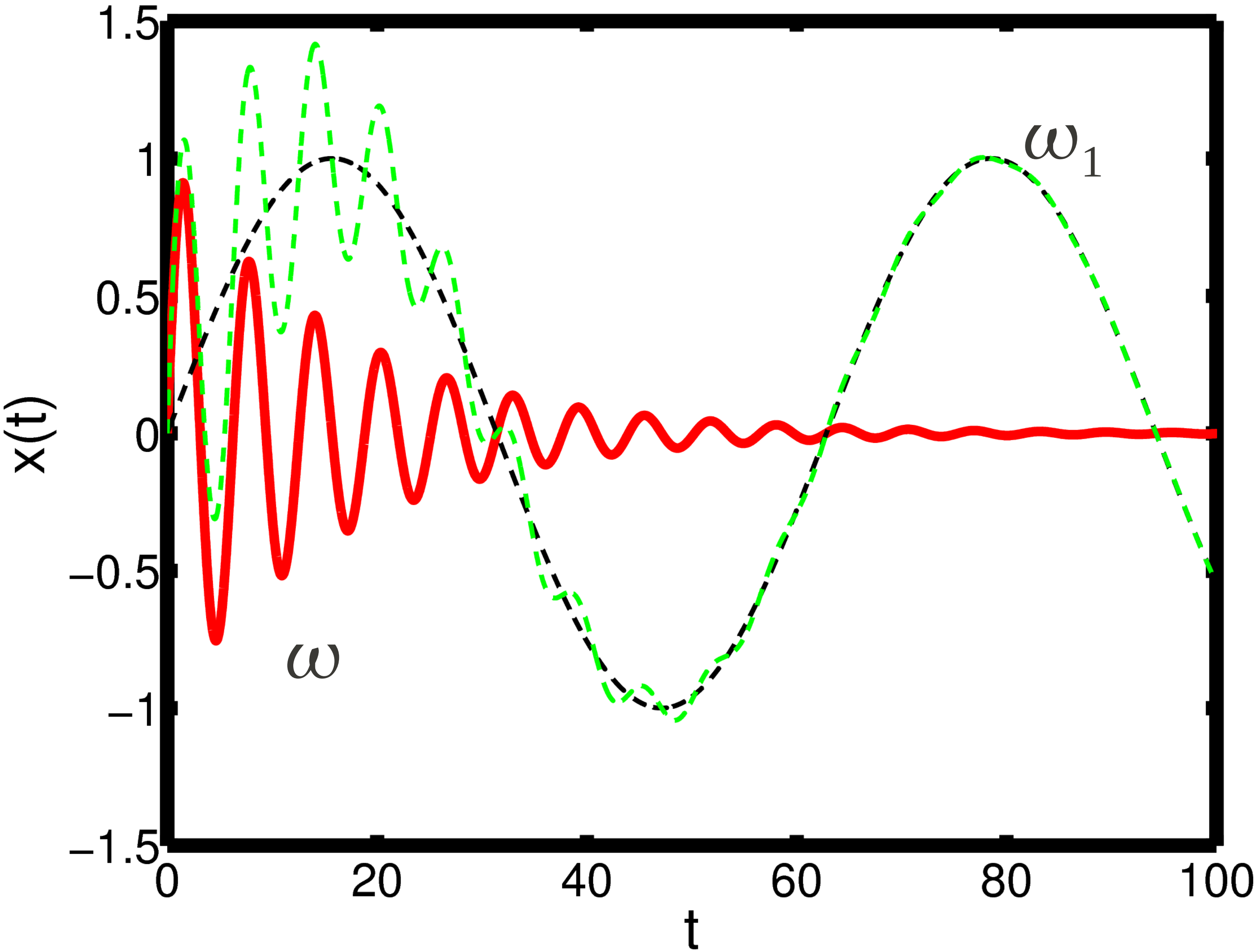
The amplitude changes to

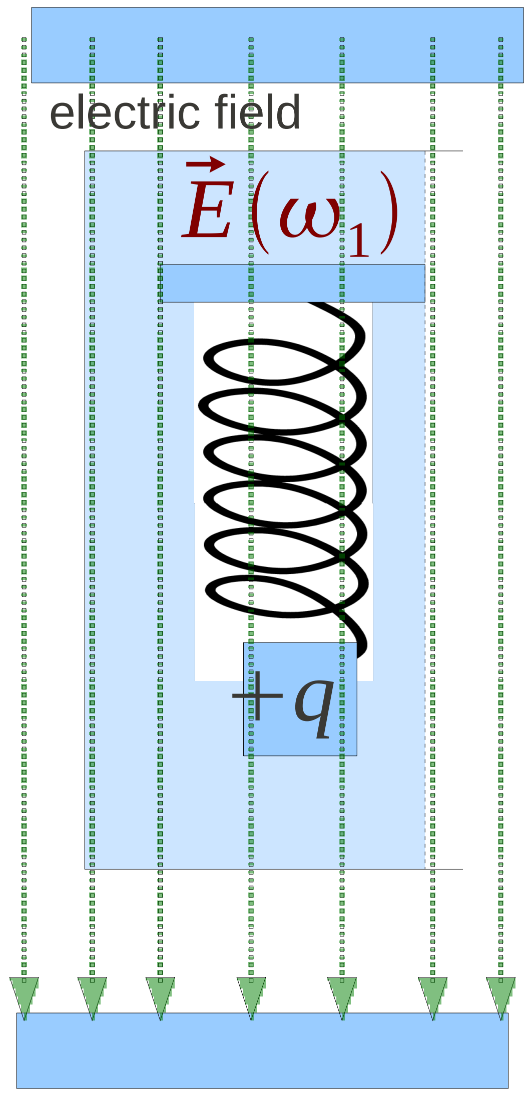
$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}} \approx \frac{F_1/m}{\omega_0^2} = \frac{F_1}{k}$$

$$x(t) = x_1(t) + x_0(t) = a_1 \cos(\omega_1 t - \phi_1) + C e^{-\gamma t} \cos(\omega t + \phi)$$

driving oscillation

Natural frequency of the damped oscillator





as driving frequency become very small $\omega_1 \ll \omega_0$ phase approaches $\phi_1 \rightarrow 0$

Resonance $\omega_1 = \omega_0$

The driving frequency ω_1 is same as that of the natural frequency, ω_0 the amplitude attains the maximum for a given driving force the condition is called resonance

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}} = \frac{F_1}{2m\gamma\omega_1}$$

$$\phi_1 = \tan^{-1}\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2} = \tan^{-1}\infty = \frac{\pi}{2}$$

In the absence of friction the amplitude become infinite.

The resonance occurs slightly less than that of $\omega_1 = \omega_0$

$$\frac{d a_1}{d\omega_1} = \frac{d}{d\omega_1}\left(\frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}\right) = 0$$

$$\frac{d\,a_1}{d\,\omega_1}=\frac{(F_1/m)\left(-4\,\omega_1(\omega_0^2-\omega_1^2)+8\,\gamma^2\,\omega_1\right)}{\left((\omega_0^2-\omega_1^2)^2+4\,\gamma^2\,\omega_1^2\right)^{3/2}}=0$$

That is $\left(-4\,\omega_1(\omega_0^2-\omega_1^2)+8\,\gamma^2\,\omega_1\right)=0$

$$\omega_0^2-\omega_1^2-2\,\gamma^2=0$$

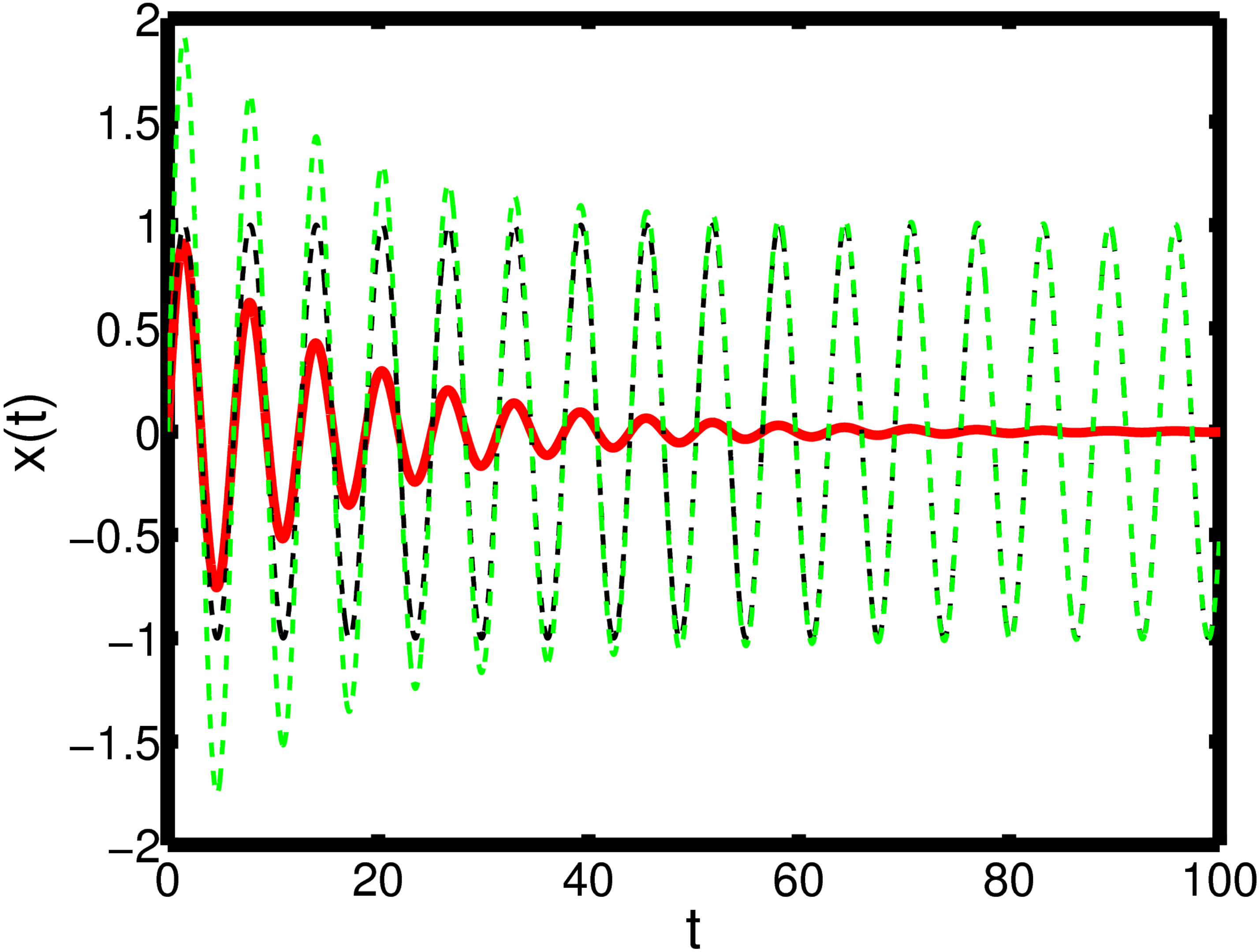
Solving it for ω_1 $\qquad \omega_1^2=\omega_0^2-2\,\gamma^2$

$$\omega_1=\omega_0\sqrt{1-\frac{2\gamma^2}{\omega_0^2}} \qquad\qquad R_m=2\,\gamma\,m\,,$$

$$\omega_1=\omega_0\sqrt{1-\frac{R_m^2}{2\,m^2\,\omega_0^2}}$$

Lesser damping more near to the natural frequency

$$x(t) = x_1(t) + x_0(t) = a_1 \cos(\omega_1 t - \phi_1) + C e^{-\gamma t} \cos(\omega t + \phi)$$



High driving frequency $\omega_1 \gg \omega_0$

The amplitude changes to

$$a_1 = \frac{F_1/m}{\sqrt{\omega_1^4 + 4\gamma^2\omega_1^2}} = \frac{F_1}{m\omega_1^2}$$

The variation in the phase angle

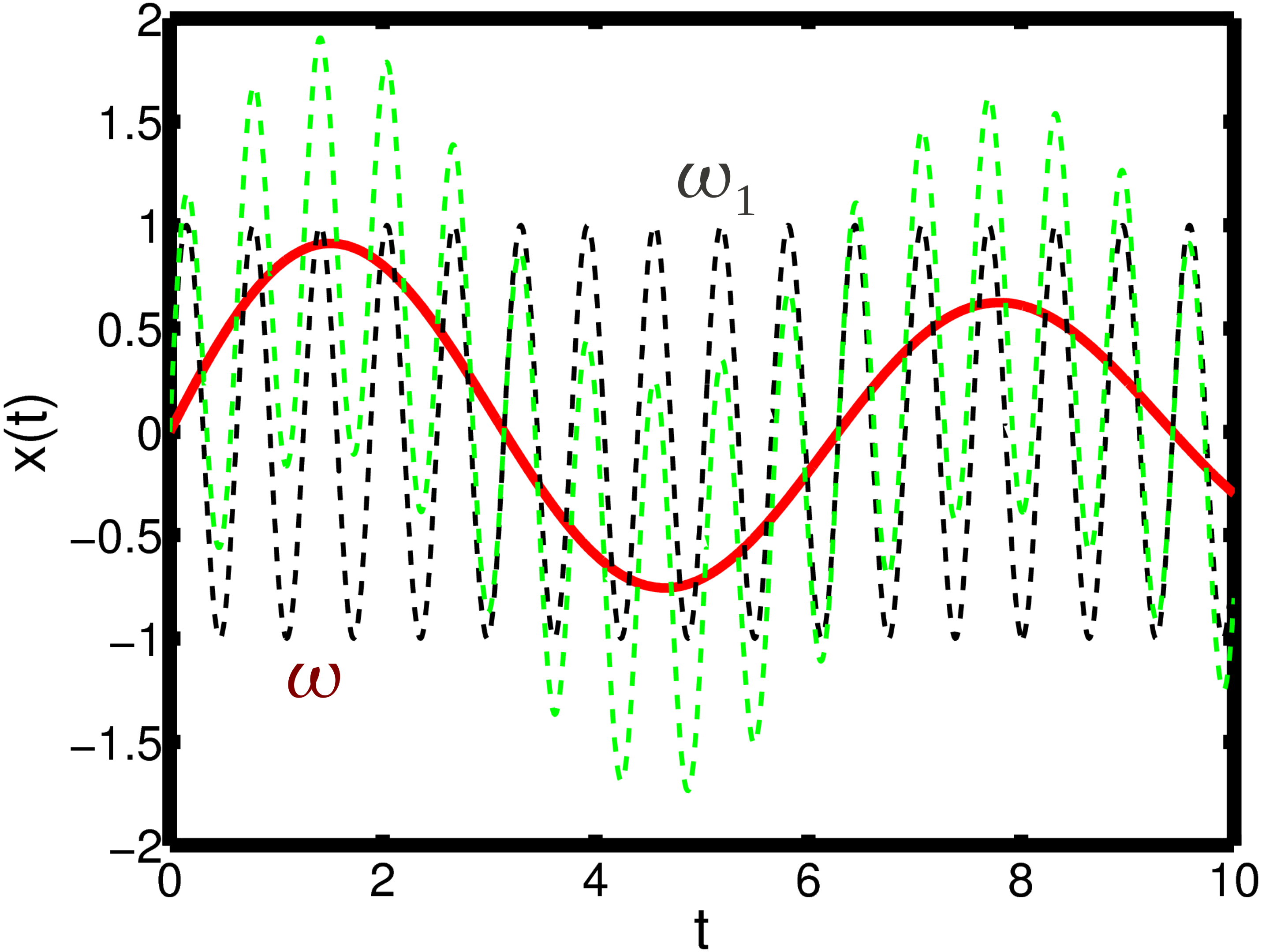
$$\phi_1 = \tan^{-1}\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2} = \tan^{-1} - 0 \qquad \phi_1 \rightarrow \pi$$

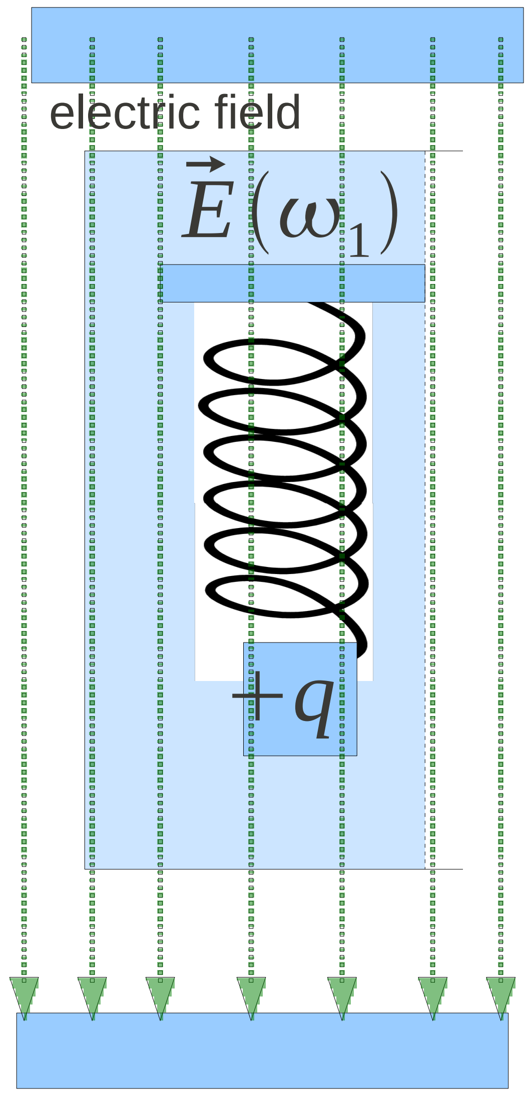
Driving force and the resulting displacements are out of phase

$$x(t) = x_1(t) + x_0(t) = a_1 \cos(\omega_1 t - \phi_1) + C e^{-\gamma t} \cos(\omega t + \phi)$$

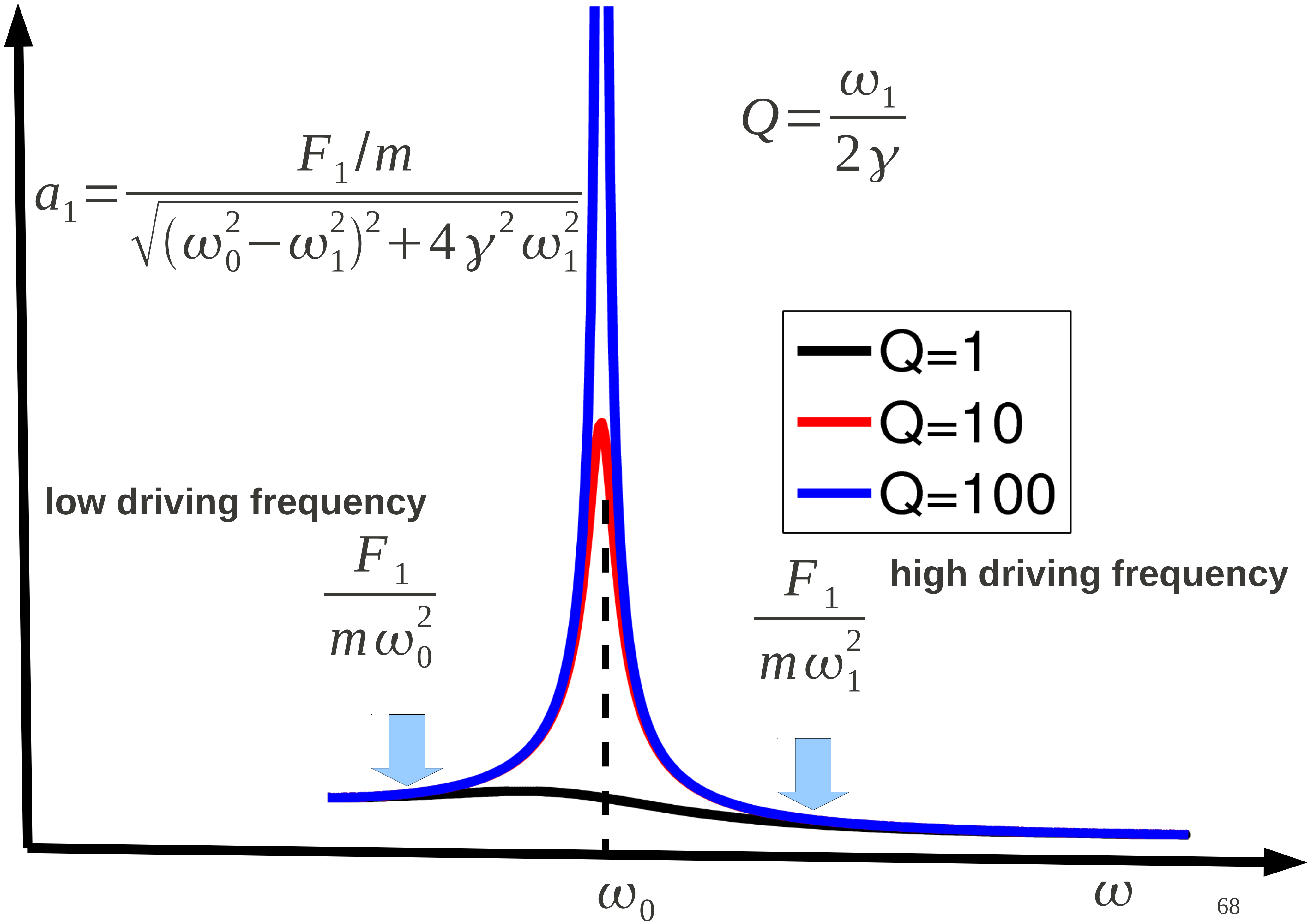
driving oscillation

Natural frequency of the damped oscillator

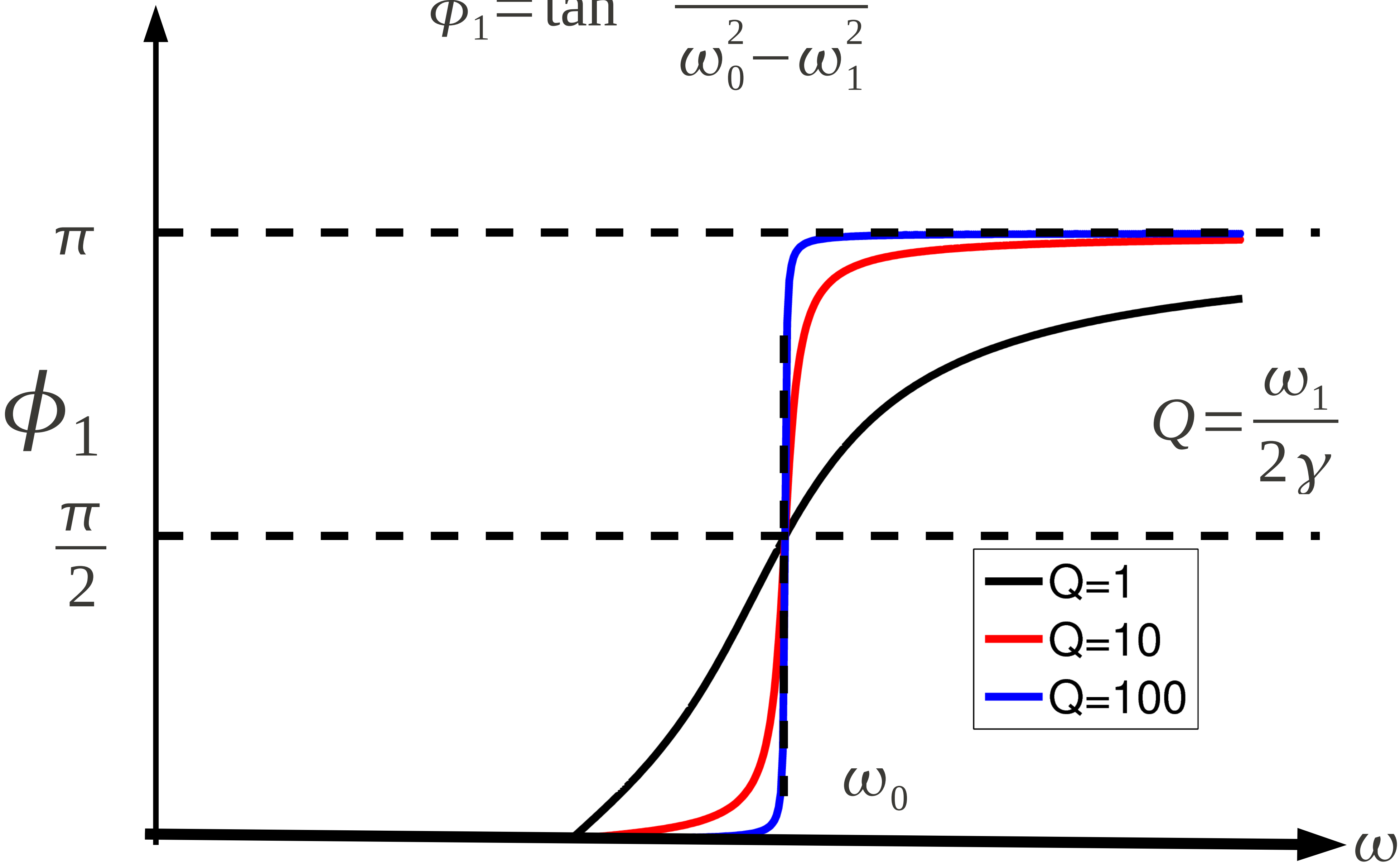




as driving frequency become very small $\omega_1 \gg \omega_0$ phase approaches $\phi_1 \to \pi$


$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2-\omega_1^2)^2+4\gamma^2\omega_1^2}}$
$Q = \frac{\omega_1}{2\gamma}$
Q=1
Q=10
Q=100
low driving frequency
$\frac{F_1}{m\omega_0^2}$
high driving frequency
$\frac{F_1}{m\omega_1^2}$
ω_0
ω

$$\phi_1 = \tan^{-1}\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}$$



Oscillator under different driving forces

$$\frac{d^2 x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x=\frac{F_1}{m}$$

A damped harmonic oscillator is driven by two driving forces $F_1(t), F_2(t)$

Corresponding displacements be $x_1(t), x_2(t)$

The differential equation for damped driven oscillator be modified as

$$\left[\frac{d^2}{dt^2}+2\gamma\frac{d}{dt}+\omega_0^2\right](x_1(t)+x_2(t))=\frac{F_1(t)}{m}+\frac{F_2(t)}{m}$$

Since the differential equations are **linear**, we may separate the equations

$$\left(\frac{d^2 x_1(t)}{dt^2}+2\gamma\frac{dx_1(t)}{dt}+\omega_0^2 x_1(t)\right)+\left(\frac{d^2 x_2(t)}{dt^2}+2\gamma\frac{dx_2(t)}{dt}+\omega_0^2 x_2(t)\right)=\frac{F_1(t)}{m}+\frac{F_2(t)}{m}$$

The solution has the form $x(t)=x_0(t)+x_1(t)+x_2(t)$

General periodic forces

Now we consider the case oscillator is driven by general periodic force as

$$F(t)=F_1 e^{i\omega_1 t}+F_2 e^{i\omega_2 t}+F_3 e^{i\omega_3 t}+\ldots..=\sum_{j=1}^{n} F_j e^{i\omega_j t}$$

The differential equation is now

$$\frac{d^2 x}{d t^2}+2\gamma\frac{d x}{d t}+\omega_0^2 x=\frac{1}{m}\sum_{j=1}^{n} F_j e^{i\omega_j t}$$

Due to the linear nature of the differential equation, the solution is

$$x(t)=\sum_{j=1}^{n} a_r e^{i(\omega_i t-\phi_i)}+C e^{-\gamma t}\cos(\omega t+\phi)$$

Now we look into usefulness of this general solution

Fourier's Theorem of periodic functions

Fourier's theorem of periodic functions states that any periodic function that has period 2π can be expanded as a series of the from

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$$

Where the constants are given by

$$a_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$n=0,1,2,3.......$$

$$b_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

x has the periodicity of 2π, i. e. (that is), $x=2\pi f t=\omega t$ for time dependent periodic functions

This may be also expressed in the complex series

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{inx}$$

The coefficients are now given by

$$c_n=\frac{1}{2}(a_n-i b_n) \qquad n>0$$

$$c_{-n}=\frac{1}{2}(a_n+i b_n)$$

$$c_0=\frac{1}{2}a_0$$

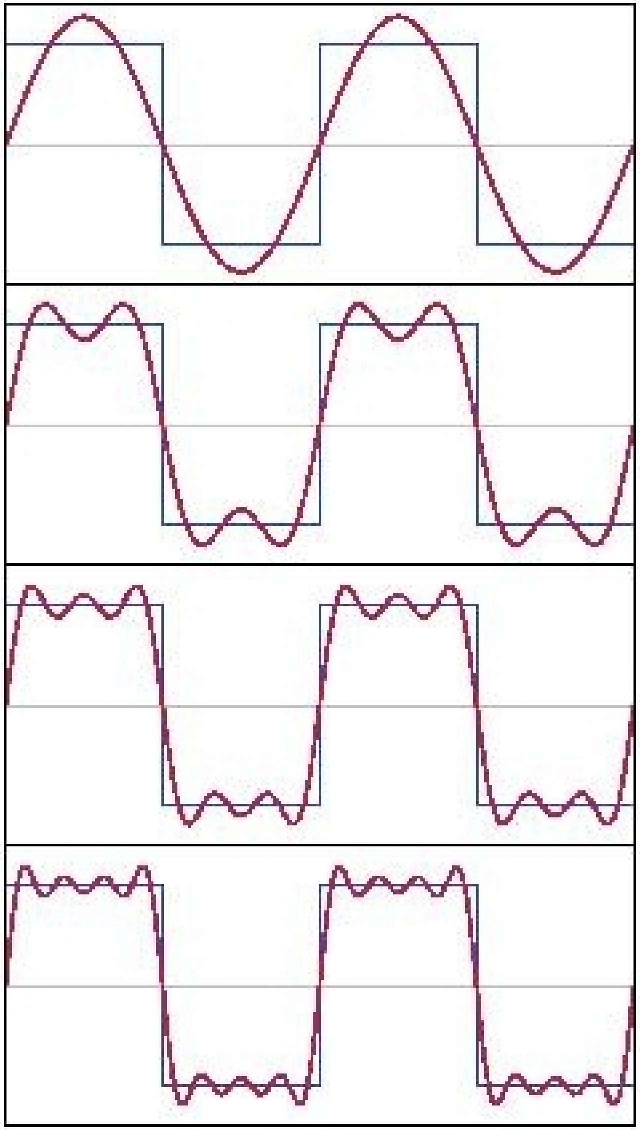
Alternatively we obtain the coefficient directly by integration

$$c_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

Fourier decomposition is an example of decomposing a function in the basis of infinite dimensional vector space

$$\{1, \sin(nx), \cos(nx)\}$$
$$n \in \{1, 2. \ldots \infty\}$$

Infinite dimensional vector space



Scalar product

$$b_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$

$$c_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

Vector that defines axis (like $(\hat{x}, \hat{y}, \hat{z})$ in Cartesian coordinates)

Orthogonality properties of unit vectors in a vector space

$$b_n(x)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

The space spanned by unit vectors of Fourier decomposition obeys the relation

$$\int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx=\begin{cases}\pi\,\delta_{mn}, & m\neq n\\ 0, & m=0\end{cases}$$

$$\int_{-\pi}^{\pi}\cos(mx)\cos(nx)\,dx=\begin{cases}\pi\,\delta_{mn}, & m\neq n\\ 2\pi, & m=n=0\end{cases}$$

$$\int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx=0$$ for all integral m and n

In complex representation of the Fourier series

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{imx})^{*}e^{inx}\,dx=\delta_{mn}$$

Fourier's Theorem of periodic functions for any arbitrary period

Fourier's theorem of periodic functions states that any periodic function that has period $2L$ can be expanded as a series of the from

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Where the constants are given by

$$a_n(x)=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx \qquad n=0,1,2,3.......$$

$$b_n(x)=\frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

during the change of period

$$x\rightarrow\frac{\pi x}{L}$$

x has the periodicity of $2L$ for periodic functions

In complex representation

$$f(x)=\sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \qquad c_n(x)=\frac{1}{2L}\int_{-L}^{L} f(x)\, e^{\frac{-in\pi x}{L}}\,dx$$

Any arbitrary periodic function can drive the oscillator

now the general equation of an oscillator under any periodic motion may be written as

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x=\frac{1}{m}\sum_{j=-\infty}^{\infty}F_j e^{in\omega t}$$

$$n=1,2,3.......$$

$$x=2\pi f t=\omega t$$

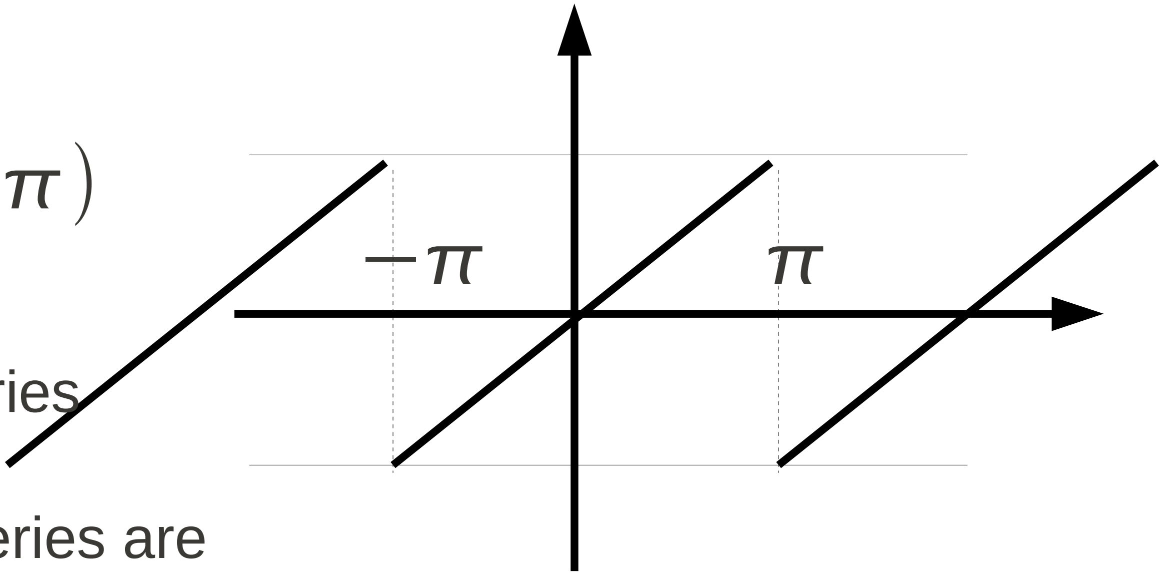
ω fundamental frequency $n\omega$ Higher harmonics

Two terms in this series are orthogonal with respect each other

$$\frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} e^{i(n-l)\omega} dt=\delta_{nl}$$

Damped oscillator driven by sawtooth wave oscillation

$$F(t) = F_0 t \qquad (-\pi < t < \pi)$$



This periodic function can be expanded in terms of Fourier series

The coefficients of the Fourier series are

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(t) \cos(nt)\, dt$$

Odd function $f(-t) = -f(t)$

$$\Rightarrow a_n = \frac{F_0}{\pi} \int_{-\pi}^{\pi} t \cos(nt)\, dt = 0$$

$$b_n = \frac{F_0}{\pi} \int_{-\pi}^{\pi} t \sin(nt)\, dt$$

even function $f(-t) = f(t)$

$$= \frac{2F_0}{\pi} \int_{0}^{\pi} t \sin(nt)\, dt$$

Limit changes

$$0$$

$$b_n = \frac{2F_0}{\pi}\left[t\left(\frac{-\cos(nt)}{n}\right)\right]_0^{\pi} - \frac{2F_0}{\pi}\int_0^{\pi}\left(\frac{-\cos(nt)}{n}\right)dt$$

$$b_n = -\frac{2F_0}{\pi}\left(\frac{-\pi(-1)^n - 0}{n}\right) = \frac{2F_0(-1)^{n+1}}{n}$$

Now the function can be expressed in Fourier series as

$$F(t) = \sum_{n=1}^{\infty}\frac{2F_0(-1)^{n+1}}{n}\sin(nt)$$

$$F(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$$

Differential equation for such a driven force is given by

Consider a damped harmonic oscillator is driven by sawtooth wave, then the equation for the driven oscillator is given by

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x=\sum_{n=1}^{\infty} b_n \sin(nt)$$

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x=\sum_{n=1}^{\infty} \frac{2F_0(-1)^{n+1}}{n}\sin(nt)$$

We know that for the equation for this form with two driving forces

$$\left[\frac{d^2}{dt^2}+2\gamma\frac{d}{dt}+\omega_0^2\right](x_1(t)+x_2(t))=\frac{F_1(t)}{m}+\frac{F_2(t)}{m}$$

$$x(t)=x_0(t)+x_1(t)+x_2(t)$$ General solution

Solution with out driving force

Solutions due to each driving force

Since infinite set of periodic forces are driving this oscillator, the solution of the harmonic oscillator under each driving force need to be determined

Solution with out driving force

$$\left[\frac{d^2 x}{d t^2} + 2\gamma \frac{d x}{d t} + \omega_0^2 x\right] = b_n \sin(n t)$$

Now set the driving frequency $\omega = 1$

For a general periodic force
$$\left[\frac{d^2 x}{d t^2} + 2\gamma \frac{d x}{d t} + \omega_0^2 x\right] = b_n e^{i(n t)}$$

Use trial solution of the type $x(t) = c_n e^{i n t}$

$$\frac{d x_1}{d t} = c_n i n e^{i n t}, \qquad \frac{d^2 x}{d t^2} = -c_n n_1^2 e^{i n t}$$

$$\left[-n^2 + 2 i \gamma n + \omega_0^2\right] c_n e^{i n t} = b_n e^{i n t}$$

$$\left[-n^2+2i\gamma n+\omega_0^2\right]c_n e^{int}=b_n e^{int}$$

$$c_n=\frac{b_n}{\left[-n^2+2i\gamma n+\omega_0^2\right]}$$

by rearranging

$$=\frac{b_n}{\left[\omega_0^2-n^2+2i\gamma n\right]}$$

taking complex conjugate of denominator and by multiplying with both numerator and denominator

$$=b_n\frac{(\omega_0^2-n^2-2i\gamma n)}{\left[(\omega_0^2-n^2)^2+4\gamma^2 n^2\right]}$$

The trial solution with value of the coefficient

$$x(t)=c_n e^{int} \qquad c_n=b_n\frac{(\omega_0^2-n^2-2i\gamma n)}{\left[(\omega_0^2-n^2)^2+4\gamma^2 n^2\right]}$$

$$x(t)=b_n\frac{(\omega_0^2-n^2)-2i\gamma n}{\left[(\omega_0^2-n^2)^2+4\gamma^2 n^2\right]}(\cos nt+i\sin nt)$$

The imaginary part of the trial solution

$$x(t)=\Im(c_n e^{int})=b_n\frac{((\omega_0^2-n^2)\sin nt-2\gamma n\cos nt)}{\left[(\omega_0^2-n^2)^2+4\gamma^2 n^2\right]}$$

This solution is for oscillator that is driven by nth periodic force

$$\left[\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x\right]=b_n\sin(nt)$$

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=\sum_{n=1}^{\infty}b_n\sin(nt)$$

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=\sum_{n=1}^{\infty}\frac{2F_0(-1)^{n+1}}{n}\sin(nt)$$

For each of this periodic force, we can get solution by using the general formula

$$x(t)=x_0(t)+x_1(t)+x_2(t)............+x_\infty(t)=\sum_{n=1}^{\infty}x_n(t)$$

Then the general solution for the sawtooth wave is

$$x(t)=\sum_{n=1}^{\infty}b_n\frac{((\omega_0^2-n^2)\sin nt-2\gamma n\cos nt)}{\left[(\omega_0^2-n^2)^2+4\gamma^2n^2\right]}$$

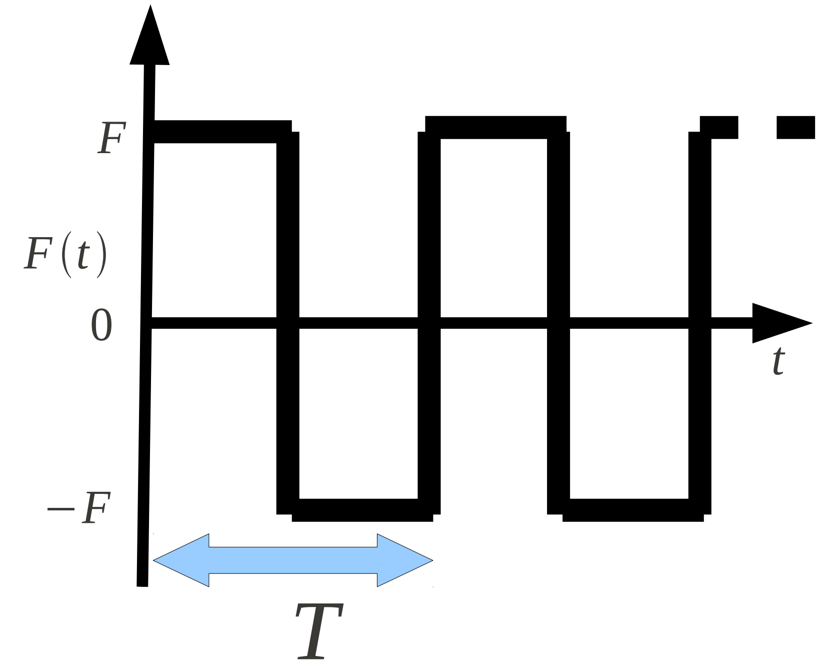
$$b_n=\frac{2F_0(-1)^{n+1}}{n}$$

$$\Rightarrow x(t)=\sum_{n=1}^{\infty}\frac{2F_0(-1)^{n+1}}{n}\frac{((\omega_0^2-n^2)\sin nt-2\gamma n\cos nt)}{\left[(\omega_0^2-n^2)^2+4\gamma^2n^2\right]}$$

Now take the case of an oscillator driven by square pulse

$$F(t)=\begin{cases} F, & nT<t\leq\left(n+\frac{1}{2}\right)T \\ -F, & \left(n-\frac{1}{2}\right)T<t\leq nT \end{cases}$$

$$n=0,\pm 1,\pm 2,........$$



Since $\omega=\frac{2\pi}{T}$

For finding coefficients of the Fourier expansion in the complex form

$$F_n=\frac{F}{T}\int_0^{T/2} e^{\frac{-2\pi int}{T}}\,dt-\frac{F}{T}\int_{T/2}^{T} e^{\frac{-2\pi int}{T}}\,dt$$

$$=I_1+I_2$$

$$I_1 = \frac{F}{T} \int_0^{T/2} e^{\frac{-2\pi i n t}{T}} \, dt$$

$$= \frac{F}{\cancel{T}} \frac{-\cancel{T}}{2\pi i n} \left[e^{\frac{-2\pi i n t}{T}} \right]_0^{T/2}$$

$$= \frac{-F}{2\pi i n} \left[e^{\frac{-2\pi i n (T/2)}{T}} - 1 \right]$$

$$= \frac{-F}{2\pi i n} \left[e^{-i\pi n} - 1 \right]$$

$$= \frac{-F}{2\pi i n} \left[(-1)^n - 1 \right]$$

$$= \frac{F}{2\pi i n} \left[1 - (-1)^n \right]$$

$$I_2 = \frac{-F}{T} \int_{T/2}^{T} e^{\frac{-2\pi i n t}{T}} dt \qquad = \frac{F}{2\pi i n} \left[1 - e^{\frac{-2\pi i n (T/2)}{T}} \right]$$

$$= \frac{F}{2\pi i n} \left[1 - (-1)^n \right]$$

$$F_n = \frac{F}{T} \int_{0}^{T/2} e^{\frac{-2\pi i n t}{T}} dt - \frac{F}{T} \int_{T/2}^{T} e^{\frac{-2\pi i n t}{T}} dt$$

$$= \frac{F}{2\pi i n} \left[1 - (-1)^n \right] + \frac{F}{2\pi i n} \left[1 - (-1)^n \right]$$

$$F_n = \frac{F}{i\pi n} \left[1 - (-1)^n \right]$$

$F_n = 0$ for $n = 0, \pm 2, \pm 4, \ldots\ldots\ldots$

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=\frac{1}{m}\sum_{n=-\infty}^{\infty}F_n e^{in\omega t}$$

Now we know the values of force coefficients for a square wave

$$F_n=F\frac{2}{i\pi n} \qquad n=\pm 1,\pm 3,\dots\dots\dots$$

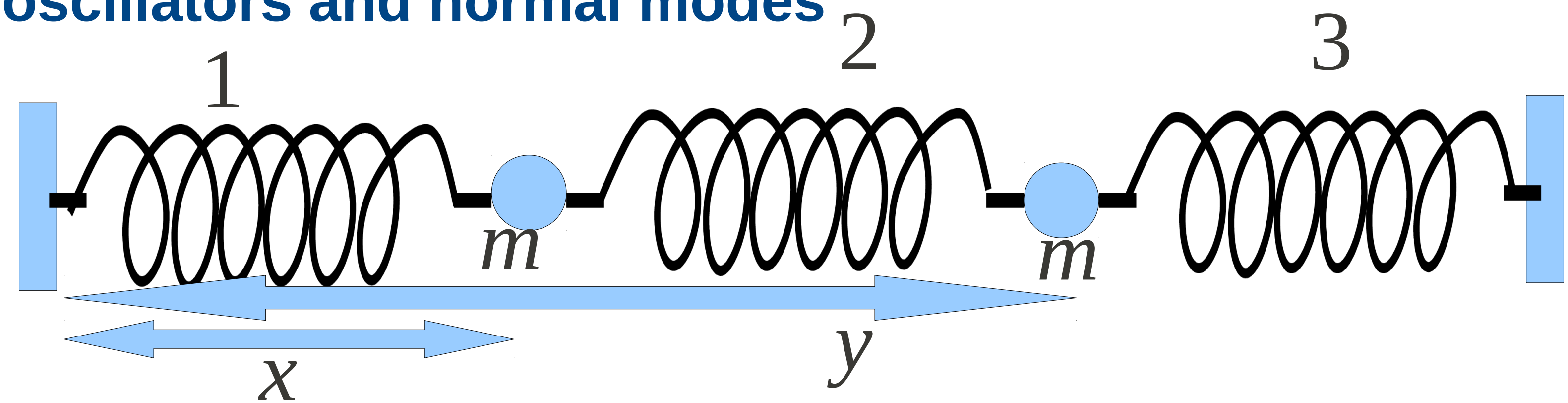
Change of n absorbs the conditions on the value of n

$$F_n=F\frac{2}{i\pi(2n+1)} \qquad n\rightarrow 2n+1$$

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=\frac{F}{mi\pi}\sum_{n=-\infty}^{\infty}\frac{1}{(2\mathrm{n}+1)}e^{in\omega t}$$

Solve it for one term in the coefficients and generalize for all terms to get the solution

Coupled oscillators and normal modes



When damping terms is absent in system that have may springs attached

Two coordinates are required to specify the system, therefore there are two degrees of freedom

The extension of the springs $(1,2,3)$ are respectively $(x, y-x, -y)$

Let the strength of the springs $(1,2,3)$ are respectively $(\alpha, 2\alpha, 4\alpha)$
(arbitrarily selected)

The restoring forces are $(\alpha x, 2\alpha(y-x), -4\alpha y)$

The equation motion then can be written as

By balancing the force on each particle

$$m\ddot{x}=-\alpha x+2\alpha(y-x),$$

$$m\ddot{y}=-2\alpha(y-x)-4\alpha y,$$

By re-arranging

$$\ddot{x}+3n^2x-2n^2y=0$$
$$\ddot{y}-2n^2x+6n^2y=0$$

$$n^2=\frac{\alpha}{m}$$

These are coupled second order homogeneous differential equations

Solution of the coupled differential equation

The solution can be obtained by assuming periodic trial solutions

$$x=A\cos(\omega t-\phi)$$
$$y=B\cos(\omega t-\phi)$$

These solutions are specific to this problem

$$\ddot{x}+3n^2x-2n^2y=0$$
$$\ddot{y}-2n^2x+6n^2y=0$$

$$x=A\cos(\omega t-\phi)$$

$$y=B\cos(\omega t-\phi)$$

$$\ddot{x}=-\omega^2A\cos(\omega t-\phi)$$

$$\ddot{y}=-\omega^2B\cos(\omega t-\phi)$$

Substituting the trial solution to the problem

$$-\omega^2A\cos(\omega t-\phi)+3n^2A\cos(\omega t-\phi)-$$
$$2n^2B\cos(\omega t-\phi)=0$$

$$-\omega^2B\cos(\omega t-\phi)t-2n^2A\cos(\omega t-\phi)+$$
$$6n^2B\cos(\omega t-\phi)=0$$

These equations are simplified into two algebraic equations

$$-\omega^2A+3n^2A-2n^2B=0$$
$$-\omega^2B-2n^2A+6n^2B=0$$

This simplified into two algebraic equations

$$-\omega^2 A+3n^2 A-2n^2 B=0$$

$$-\omega^2 B-2n^2 A+6n^2 B=0$$

There is two equations and two unknowns, the trivial solution is by equating the both constant to zero. These equations have non-trivial solution if the determinant of the coefficients is equal to zero

$$\begin{vmatrix} 3n^2-\omega^2 & -2n^2 \\ -2n^2 & 6n^2-\omega^2 \end{vmatrix}=0$$

$$(3n^2-\omega^2)(6n^2-\omega^2)-4n^4=0$$

$$\omega^4-9n^2\omega^2+14n^4=0$$

$$\omega^4 - 9n^2\omega^2 + 14n^4 = 0$$

Quadratic in ω^2 , this has two real roots

This is in the standard format of a quadratic equation

$$ax^2 + bx + c = 0 \qquad x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$a = 1, b = -9n^2, c = 14, x = \omega^2$$

$$\omega^2 = \frac{9n^2}{2} \pm \frac{5n^2}{2}$$

$$\omega^2 = 2n^2, \omega^2 = 7n^2$$

We see that there are two **normal modes** with frequencies given above and these frequencies are called **normal frequencies.**

Mode I

$$-\omega^2 A + 3n^2 A - 2n^2 B = 0$$

$$-\omega^2 B - 2n^2 A + 6n^2 B = 0$$

Substituting for the value of $\omega^2 = 2n^2$

$$n^2 A - 2n^2 B = 0 \qquad -2n^2 A + 4n^2 B = 0$$

This equation only define the ratio of the amplitudes $A = 2B$

Let $B = \delta$ then $A = 2\delta$

The solution becomes

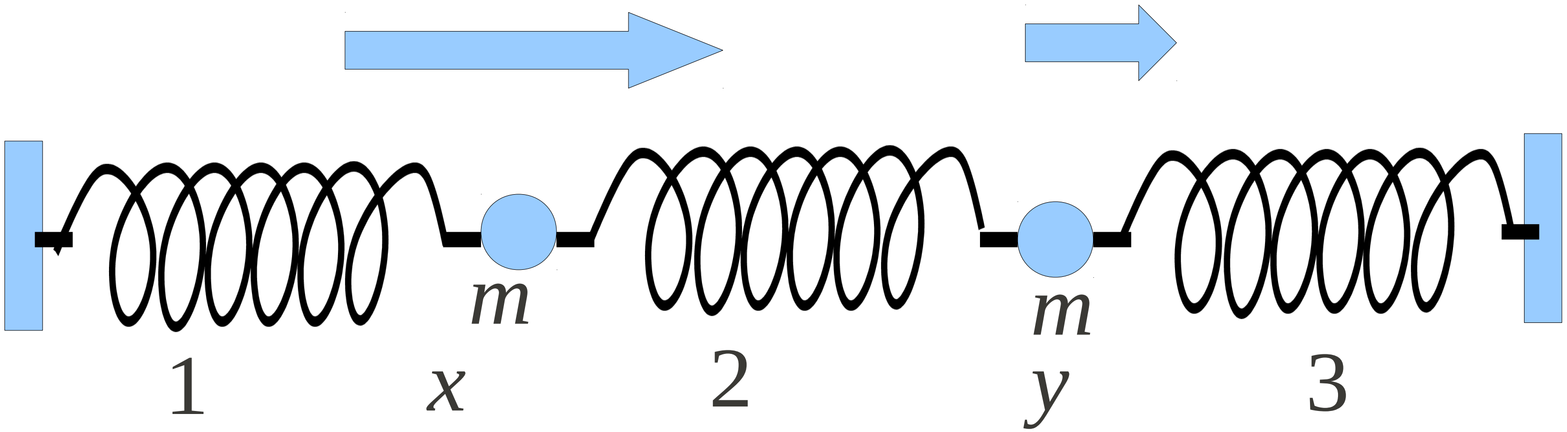
$$x = 2\delta\cos(\sqrt{2}nt - \phi)$$

$$y = \delta\cos(\sqrt{2}nt - \phi)$$

Direction of motion is same and amplitude of one is the double of the other

$$x = 2\delta\cos(\sqrt{2}\,n t - \phi)$$

$$y = \delta\cos(\sqrt{2}\,n t - \phi)$$



Direction of motion is same and amplitude of one is the double of the other, the corresponding collective motion of the particles are called a mode of the system

Mode II

$$-\omega^2 A + 3n^2 A - 2n^2 B = 0$$

$$-\omega^2 B - 2n^2 A + 6n^2 B = 0$$

Substituting for the value of $\omega^2 = 7n^2$

$$-n^2 B - 2n^2 A = 0 \qquad B = -2A$$

Let $A = \delta$ then $B = -2\delta$

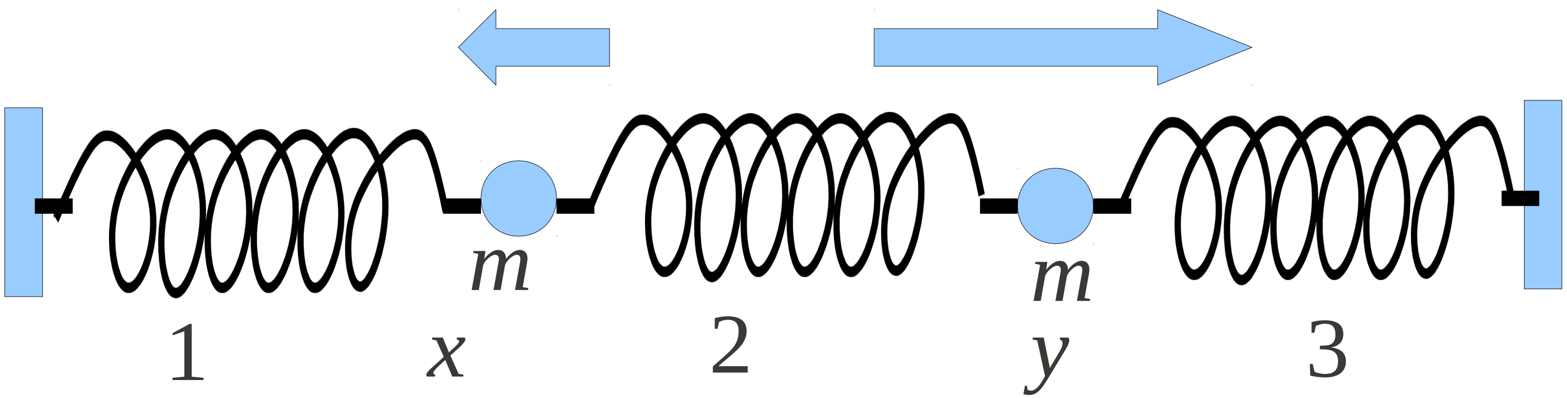
The solution in this case is

$$x = \delta \cos(\sqrt{7}\, n t - \phi)$$

$$y = -2\delta \cos(\sqrt{7}\, n t - \phi)$$

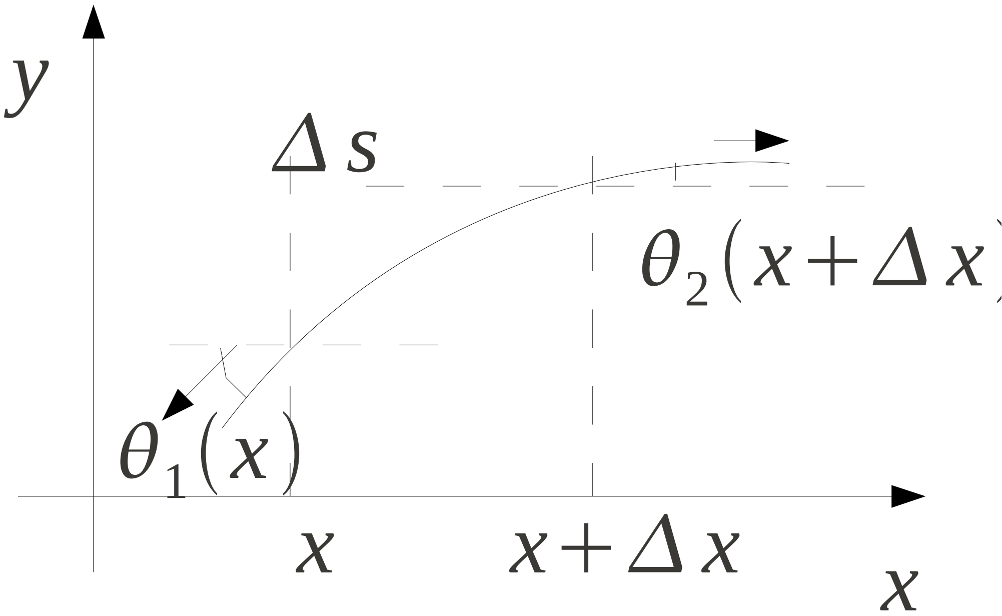
$$x = \delta \cos(\sqrt{7}\, n t - \phi)$$

$$y = -2\delta \cos(\sqrt{7}\, n t - \phi)$$



Direction of motion is opposite and amplitude of one is the double of the other, the corresponding collective motion of the particles are called a mode of the system

Equation of motion of string: wave-equation – By balancing of forces



Let the string have a total length of l

When the strings are displaced in the y direction tension force T try to restore it to original position

Consider a line element Δs

Component of the restoring force at x

$$F_y(x) = -T\sin\theta_1$$

$$F = ma \Rightarrow m\frac{d^2 r}{dt^2} = F$$

Component of the restoring force at $x+\Delta x$

$$F_y(x+\Delta x) = -T\sin\theta_2$$

Total tension along the line element

$$F_y = T\sin\theta_2 - T\sin\theta_1$$

Along the X direction the wire is pulled by the force

As an approximation we can take displacement along the x direction is 0

Now only acceleration component is $\frac{\partial^2 y}{\partial t^2}$

The mass of the element $\mu \Delta s$

The equation of motion of the system is obtained by combining all the these information using Newton's equation of motion

$$F = m a \Rightarrow m \frac{d^2 r}{d t^2} = F$$

$$F_y = \mu \Delta s \frac{\partial^2 y}{\partial t^2} = T \sin\theta_2 - T \sin\theta_1$$

Divide both sides by Δx

$$F_y = \mu \frac{\Delta s}{\Delta x} \frac{\partial^2 y}{\partial t^2} = \frac{T \sin\theta_2 - T \sin\theta_1}{\Delta x}$$

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\mu \frac{\Delta s}{\Delta x} \frac{\partial^2 y}{\partial t^2} = \mu \frac{\sqrt{\Delta x^2 + \Delta y^2}}{\Delta x} \frac{\partial^2 y}{\partial t^2} = \mu \sqrt{1 + \left(\frac{\Delta y}{\Delta x} \right)^2} \frac{\partial^2 y}{\partial t^2}$$

By rearranging and equating

$$\mu \sqrt{1 + \left(\frac{\Delta y}{\Delta x} \right)^2} \frac{\partial^2 y}{\partial t^2} = \frac{T \sin\theta_2 - T \sin\theta_1}{\Delta x}$$

Now the take the limit $\Delta x \rightarrow 0$

$$\mu \sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2} \frac{\partial^2 y}{\partial t^2} = T \frac{\partial \sin\theta}{\partial x}$$

In order to convert both sides to standard format of slope versus time variation of length – using the trigonometric relation

$$\sin\theta = \frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} \qquad = \frac{\partial y}{\partial x} / \sqrt{1 + \left(\frac{\partial y}{\partial x} \right)^2}$$

$$\mu\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\frac{\partial^2 y}{\partial t^2}=T\frac{\partial \sin\theta}{\partial x}\quad =T\frac{\partial}{\partial x}\left(\frac{\frac{\partial y}{\partial x}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\right)$$

Using relation $\sin\theta=\left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}\right)$

$$\mu\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\frac{\partial^2 y}{\partial t^2}\quad =T\left(\frac{\frac{\partial^2 y}{\partial x^2}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\right)\quad +T\left(\frac{2\left(\frac{\partial y}{\partial x}\right)^2}{\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right)^{3/2}}\right)$$

$$\mu\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\frac{\partial^2 y}{\partial t^2} = T\left(\frac{\frac{\partial^2 y}{\partial x^2}}{\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}}\right) + T\left(\frac{2\left(\frac{\partial y}{\partial x}\right)^2}{\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right)^{3/2}}\right)$$

multiplying both sides by the relation $\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}$

$$\mu\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right)\frac{\partial^2 y}{\partial t^2} = T\left(\frac{\partial^2 y}{\partial x^2}\right) + T\left(\frac{2\left(\frac{\partial y}{\partial x}\right)^2}{\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right)}\right)$$

there are terms that contain higher powers of slope

$$\mu\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right)\frac{\partial^2 y}{\partial t^2}=T\left(\frac{\partial^2 y}{\partial x^2}\right)+T\left(\frac{2\left(\frac{\partial y}{\partial x}\right)^2}{\left(1+\left(\frac{\partial y}{\partial x}\right)^2\right)}\right)$$

$0 \qquad 0$

Now neglecting all terms that involve higher powers of $\frac{\partial y}{\partial x}$

$$\mu\frac{\partial^2 y}{\partial t^2}=T\left(\frac{\partial^2 y}{\partial x^2}\right)$$

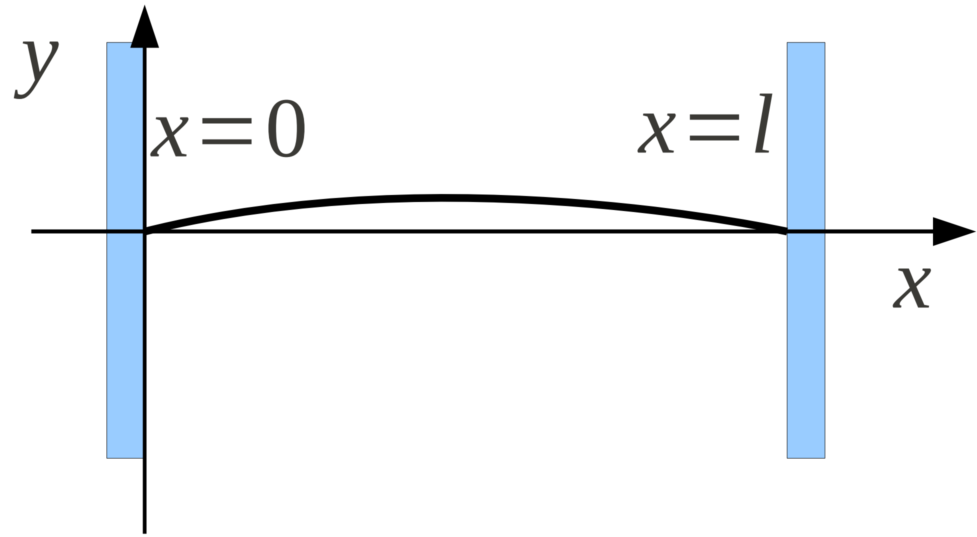
This is the desired wave equation

$$\frac{\partial^2 y}{\partial t^2}=c^2\left(\frac{\partial^2 y}{\partial x^2}\right) \qquad \text{where} \quad c^2=\frac{T}{\mu}$$

This is a second order partial differential equation called the **wave equation**

Solutions of the wave equation

For the solutions of the wave equation we need the initial values and boundary conditions



Consider a string clamped at the ends

$$x=0 \Rightarrow y(0,t)=0$$

$$x=l \Rightarrow y(l,t)=0$$

At $t=0$ the initial condition is specified by the condition of the form

$$y(x,0)=f(x) \qquad \left.\frac{\partial}{\partial t} y(x,t)\right|_{t=0}=0$$

Partial differential equation can be solved by the **method of separation variables**

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right) y(x,t)=0$$

In this method the important assumption is that the solution can be written in the product form

$$y(x,t) = X(x)\,T(t)$$

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \right) X(x)\,T(t) = 0$$

Since the product involve two function that have dependency only on one type of variable that is either position or time

$$\left(\frac{1}{c^2} X(x) \frac{\partial^2 T(t)}{\partial t^2} - T(t) \frac{\partial^2 X(x)}{\partial x^2} \right) = 0$$

Divide the equation by the product $X(x)\,T(t)$

$$\frac{1}{T(t)} \frac{\partial^2 T(t)}{\partial t^2} = \frac{c^2}{X(x)} \frac{\partial^2 X(x)}{\partial x^2}$$

Now we have two equations that are equated and both of them have dependency only on either time or space

Therefore we may equate both sides to a constant say $-\omega^2$

$$\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2}=\frac{c^2}{X(x)}\frac{\partial^2 X(x)}{\partial x^2}=-\omega^2$$

Now the two equations can be separated

$$\frac{\partial^2 T(t)}{\partial t^2}+\omega^2 T(t)=0 \qquad \frac{\partial^2 X(x)}{\partial x^2}+\frac{\omega^2}{c^2}X(x)=0$$

We are familiar with both of these equations – the general solution of these differential equations are given by

$$T(t)=A\sin\omega t+B\cos\omega t$$

$$X(x)=C\sin\frac{\omega}{c}x+D\cos\frac{\omega}{c}x$$

The values of the constants are determined by the boundary conditions

$$y(x,t)=\left(C\sin\frac{\omega}{c}x+D\cos\frac{\omega}{c}x\right)\left(A\sin\omega t+B\cos\omega t\right)$$

$$y(x,t)=\left(C\sin\frac{\omega}{c}x+D\cos\frac{\omega}{c}x\right)(A\sin\omega t+B\cos\omega t)$$

$$y(0,t)=D(A\sin\omega t+B\cos\omega t)$$

At one end of the string

$$x=0\Rightarrow y(0,t)=0 \qquad \Rightarrow D=0$$

Now the solution of the equation

$$y(x,t)=C\sin\left(\frac{\omega}{c}x\right)(A\sin\omega t+B\cos\omega t)$$

$$x=l\Rightarrow y(l,t)=0$$

$$\Rightarrow 0=C\sin\left(\frac{\omega}{c}l\right)(A\sin\omega t+B\cos\omega t)$$

$$\Rightarrow 0=C\sin\frac{\omega}{c}l$$

$$\Rightarrow 0 = C\sin\frac{\omega}{c}l \qquad \Rightarrow 0 = C$$

string has no displacement

This means the no displacement at the ends – that is possible when

$$n\pi = \frac{\omega l}{c} \qquad \Rightarrow \frac{n\pi}{l} = \frac{\omega}{c}$$

There are a series of angular frequencies that satisfies the boundary condition

$$\Rightarrow \frac{n\pi c}{l} = \omega_n$$

Since the string is a continuous system there are infinite frequencies that satisfy the boundary condition. Now the solution can be written as

$$y(x,t) = C\sin\frac{\omega}{c}x\left(A\sin\omega t + B\cos\omega t\right)$$

Wire is a system of n particles connected, so number of solutions are also infinite as $n \rightarrow \infty$

$$y(x,t) = C\sin\frac{n\pi}{l}x\left(A_n\sin\frac{n\pi c}{l}t + B_n\cos\frac{n\pi c}{l}t\right)$$

$$y(x,t)=C\sin\frac{\omega}{c}x\left(A\sin\omega t+B\cos\omega t\right)$$

The n th solution of the problem is given by where A_n, B_n gives the coefficients of the solution n

$$y_n(x,t)=C\sin\frac{n\pi}{l}x\left(A_n\sin\frac{n\pi c}{l}t+B_n\cos\frac{n\pi c}{l}t\right)$$

$$a_n=C\,A_n$$

$$b_n=C\,B_n$$

$$y_n(x,t)=\sin\frac{n\pi}{l}x\left(a_n\sin\frac{n\pi c}{l}t+b_n\cos\frac{n\pi c}{l}t\right)$$

The remaining two constants may found from time dependent boundary conditions

Now from the initial conditions $\left.\frac{\partial}{\partial t} y_n(x,t)\right|_{t=0} = 0$

$$y_n(x,t) = \sin\frac{n\pi}{l}x\left(a_n \sin\frac{n\pi c}{l}t + b_n \cos\frac{n\pi c}{l}t\right)$$

$$\left.\frac{\partial}{\partial t} y_n(x,t)\right|_{t=0} = 0 = \frac{n\pi c}{l}\sin\frac{n\pi}{l}x\left(a_n \cos\frac{n\pi c}{l}t - b_n \sin\frac{n\pi c}{l}t\right)$$

$$0 = \frac{n\pi c}{l}a_n \sin\frac{n\pi}{l}x$$

is satisfied for all x if $a_n = 0$

$$\Rightarrow y_n(x,t) = b_n \sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi c}{l}t\right)$$

Most general solution of the problem is linear superposition of all the individual solutions

$$y(x,t) = \sum y_n(x,t)$$

$$\Rightarrow y(x,t)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi c}{l}t\right)$$

$$=\sum_{n=1}^{\infty} b_n \sin(k_n x)\cos(\omega_n t)$$

The coefficients can be now calculated from given initial curve from Fourier coefficients.

$$y(x,0)=f(x)$$

The initial selection of coefficients is important to arrive at oscillating solutions, if we choose instead of $-\omega^2$ ω^2 as the constant the solution become exponentially decaying functions

$$y(x,t)=\left(Ae^{\omega t}+Be^{-\omega t}\right)\left(Ce^{(\omega/c)x}+De^{-(\omega/c)x}\right)$$

Then the solution of the boundary condition

$$x=0\Rightarrow y(0,t)=0 \qquad x=l\Rightarrow y(l,t)=0$$

$$C+D=0 \qquad \left(Ce^{(\omega/c)l}+De^{-(\omega/c)l}\right)=0$$

$$\Rightarrow C=D=0$$

This mean wire remains stationary that is a trivial solution