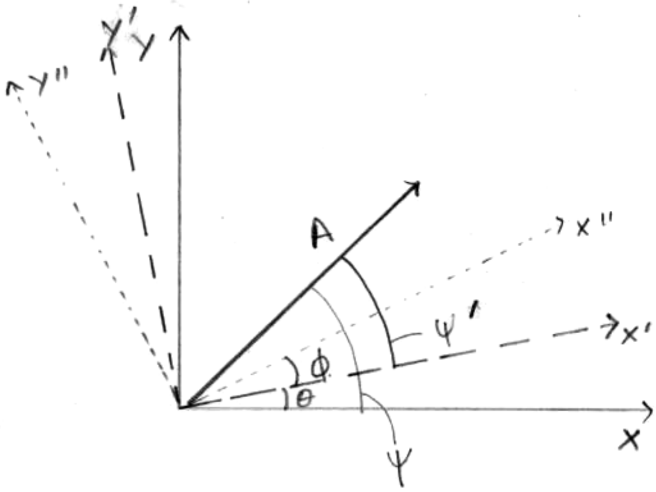
IC121: Mechanics of Particles and Waves

Tutorial - 1 Solutions

1. Using the transformation rules of vector rotation prove the familiar trigonometric identities.

$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$
$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$

<u>Soln</u>: Let $\vec{A}$ be any vector which makes an angle ψ with x-axis of x-y coordinate system



Now, $A_x = A\cos\psi$
$A_y = A\sin\psi$ —— ①

Now if the coordinate system is rotated by an angle θ

Let $\vec{A}$ makes an angle ψ' with x'-axis of rotated frame.

$\therefore$ $A_x' = A\cos\psi'$
$A_y' = A\sin\psi'$ —— ②

Now, $\psi = \psi - \theta$

$\therefore$ $A_x' = A\cos(\psi-\theta) = A\cos\psi\cos\theta + A\sin\psi\sin\theta$
$A_y' = A\sin(\psi-\theta) = A\sin\psi\cos\theta - A\sin\theta\cos\psi$

$A_x' = A_x\cos\theta + A_y\sin\theta$ —— ③
$A_y' = -A_x\sin\theta + A_y\cos\theta$

Now, if x'-y' frame is again rotated by an angle ϕ with respect to x'-y', to new position x''-y'' frame

So, equivalently we can write

$$A_x'' = A_x' \cos\phi + A_y' \sin\phi$$
$$A_y'' = -A_x \sin\phi + A_y' \cos\phi \quad \text{——— (4)}$$

$$A_x'' = \cos\phi\,(A_x \cos\theta + A_y \sin\theta) + \sin\phi\,(-A_x \sin\theta + A_y \cos\theta)$$
$$A_y'' = -\sin\phi\,(A_x \cos\theta + A_y \sin\theta) + \cos\phi\,(-A_x \sin\theta + A_y \cos\theta)$$

$$A_x'' = A_x(\cos\theta\cos\phi - \sin\theta\sin\phi) + A_y\,(\sin\theta\cos\phi + \cos\theta\sin\phi)$$
$$A_y = -A_x(\cos\theta\sin\phi + \sin\theta\cos\phi) + A_y\,(\cos\theta\cos\phi - \sin\theta\sin\phi) \quad \text{——— (5)}$$

Now, if we had rotated the original coordinate system by angles θ and ϕ consecutively, then we can write

$$A_x'' = A_x \cos(\theta+\phi) + A_y \sin(\theta+\phi) \quad \text{——— (6)}$$
$$A_y'' = -A_x \sin(\theta+\phi) + A_y \cos(\theta+\phi)$$

On comparing (8) and (6), we can write

$$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

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2. Show that scalar product of two dimensional vectors $\vec{a}\cdot\vec{b}$ is invariant under rotation.

Soln If $\vec{a}$ and $\vec{b}$ are two-dimensional vectors, then we can write

$$\vec{a} = a_x\,\hat{x} + a_y\,\hat{y} \quad \text{—— (1)}$$
$$\vec{b} = b_x\,\hat{x} + b_y\,\hat{y}$$

Now after rotation of the coordinate system by an angle ϕ the new components with respect to the rotated frame

$$\vec{a}' = a_x'\,\hat{x}' + a_y'\,\hat{y}'$$
$$\vec{b}' = b_x'\,\hat{x}' + b_y'\,\hat{y}' \quad \text{—— (2)}$$

The dot product

$$\vec{a}'\cdot\vec{b}' = a_x' b_x' + a_y' b_y'$$

$$a_x' = a_x \cos\phi + a_y \sin\phi$$
$$a_y' = -a_x \sin\phi + a_y \cos\phi$$

Similary for b_x' and b_y'

$$b_x' = b_x \cos\phi + b_y \sin\phi$$
$$b_y' = -b_x \sin\phi + b_y \cos\phi$$

Now

$$\vec{a}'\cdot\vec{b}' = (a_x \cos\phi + a_y \sin\phi)(b_x \cos\phi + b_y \sin\phi) + (-a_x \sin\phi + a_y \cos\phi)(-b_x \sin\phi + b_y \cos\phi)$$

$$= a_x b_x \cos^2\phi + a_y b_y \sin^2\phi + a_x b_y \cancel{\sin\phi\cos\phi} + a_y \cancel{b_x \sin\phi\cos\phi}$$
$$+ a_x b_x \sin^2\phi + a_y b_y \cos^2\phi - a_x b_y \cancel{\sin\phi\cos\phi} - \cancel{b_x a_y \cos\phi\sin\phi}$$

$$= a_y b_y + a_x b_x$$

Thus, we see that dot product remains invariant under rotation.

3. Find the gradient of the following functions

(a) $x^2 + y^3 + z^4$

(b) $x^2 y^3 z^4$

(c) $e^x \sin(y) \ln(z)$

Soln (a) $x^2 + y^3 + z^3$

$$\nabla f = \frac{\partial}{\partial x}(x^2+y^3+z^4)\hat{x} + \frac{\partial}{\partial y}(x^2+y^3+z^4)\hat{y} + \frac{\partial}{\partial z}(x^2+y^3+z^4)\hat{z}$$

$$= 2x\,\hat{x} + 3y^2\,\hat{y} + 4z^3\,\hat{z}$$

(c) $f = e^x \sin y \ln(z)$

$$\nabla f = \hat{x}\frac{\partial}{\partial x}(e^x \sin(y)\ln(z)) + \hat{y}\frac{\partial}{\partial y}(e^x \sin(y)\ln(z)) + \hat{z}\frac{\partial}{\partial z}(e^x \sin(y)\ln(z))$$

$$\Rightarrow \hat{x}\, e^x \sin(y)\ln(z) + \hat{y}\, e^x \cos y \ln(z) + \hat{z}\, e^x \frac{\sin(y)}{z}$$

(b) $f = x^2y^3z^4$

$$\nabla f = \hat{x}\frac{\partial}{\partial x}(x^2y^3z^4) + \hat{y}\frac{\partial}{\partial y}(x^2y^3z^4) + \hat{z}\frac{\partial}{\partial z}(x^2y^3z^4)$$

$$= 2xy^3z^4\,\hat{x} + 3x^2y^2z^4\,\hat{y} + 4x^2y^3z^3\,\hat{z}$$

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4 Find the directional derivative (gradient) of $f(x,y) = x^2 \sin 2y$ at point $(1, \pi/2)$ in the direction $v = 3\hat{x} - 4\hat{y}$.

Soln: The component of ∇f in the given direction is called directional derivative of the given function in a particular direction

Now $\nabla f(x,y) = \hat{x}\cdot\frac{\partial}{\partial x}(x^2\sin 2y) + \hat{y}\frac{\partial}{\partial y}(x^2 \sin 2y)$

$$= \hat{x}\, 2x \sin 2y + \hat{y}\, 2x^2 \cos 2y$$

Now $\nabla f(x,y)$ in the direction of $v = \left.\frac{(\nabla f)\cdot v}{|v|}\right|_{(1,\pi/2)}$

$$= \left.\frac{\hat{x}(2x\sin 2y) + \hat{y}(2x^2\cos 2y)\cdot(3\hat{x} - 4\hat{y})}{\sqrt{3^2+4^2}}\right|_{(1,\pi/2)}$$

$$= \left.\frac{6x\sin 2y - 8x^2\cos 2y}{5}\right|_{(1,\pi/2)}$$

$$\Rightarrow \frac{6\times 1 \sin 2\times\pi/2 - 8\times(1)^2 \cos 2\times(\pi/2)}{5} = \frac{6+8}{5}$$

$$= \frac{14}{5}$$

5. Find the derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $P_0(1,1,0)$ in the direction of $\vec{v} = 2\hat{x} - 3\hat{y} + 6\hat{z}$, in what direction does f change most rapidly at P_0?

Soln: To find the directional derivative of $f(x,y,z)$ at P_0 in direction of $\vec{v}$

$$\nabla f(x,y,z) = \hat{x}\frac{\partial}{\partial x}(x^3 - xy^2 - z) + \hat{y}\frac{\partial}{\partial y}(x^3 - xy^2 - z) + \hat{z}(x^3 - xy^2 - z)$$

$$= \hat{x}(3x^2 - y^2) + \hat{y}(-2xy) + \hat{z}(-1)$$

Now $$\left.\frac{\nabla f \cdot v}{|v|}\right|_{(1,1,0)} = \left.\frac{((3x^2-y^2)\hat{x} + (-2xy)\hat{y} - \hat{z})(2\hat{x} - 3\hat{y} + 6\hat{z})}{\sqrt{2^2+3^2+6^2}}\right|_{(1,1,0)}$$

$$\Rightarrow \left.\frac{2(3x^2-y^2) - 2xy(-3) - 6z}{7}\right|_{(1,1,0)}$$

$$= \frac{2(3\times1-1) - 2\times1\times1(-3) - 6\times0}{7}$$

$$= \frac{4+6-6}{7} = \frac{4}{7}$$

(b) We can write

$$df = (\nabla f)\cdot dl$$
$$= |\nabla f||dl|\cos\theta$$

Now for fixed $|dl|$, df is maximum when $\theta = 0$, i.e, if we move in same direction as ∇f, we get greatest df

Thus direction of ∇f at $P_0(1,1,0)$

$$\nabla f|_{P_0} = \left.(3x^2-y^2)\hat{x} - 2xy\,\hat{y} - \hat{z}\right|_{(1,1,0)}$$
$$= 2\hat{x} - 2\hat{y} - \hat{z}$$

6) Let $\vec{\nabla}\phi = (1+2xy)\hat{x} + (x^2+3y^2)\hat{y}$. Find the associated scalar field.

Soln: Given $\nabla\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$

On comparing, we get

$$\frac{\partial\phi}{\partial x} = 1+2xy \qquad \frac{\partial\phi}{\partial y} = x^2+3y^2 \qquad \frac{\partial\phi}{\partial z} = 0$$

$$d\phi = \nabla\phi \cdot dl$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

$$\int d\phi = \int(1+2xy)dx + \int(x^2+3y^2)dy$$

$$= \int dx + \int 2xy\,dx + \int x^2dy + 3y^2dy$$

$$= x + \frac{3y^3}{3} + \int(2xy\,dx + x^2dy)$$

$$\phi = x + \frac{3y^3}{3} + xy^2 + c$$

7. Prove that divergence of a curl is always zero. Also prove that curl of gradient is also zero always.

Soln: (a) Divergence of a curl $= \nabla\cdot(\vec{\nabla}\times\vec{v})$

Let $v = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

$$\nabla\times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial y \\ v_x & v_y & v_z \end{vmatrix} = \hat{x}\left[\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right] + \hat{y}\left[\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right] + \hat{z}\left[\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right]$$

$$\nabla\cdot(\nabla\times v) = \frac{\partial^2 v_z}{\partial x\partial y} - \frac{\partial^2 v_y}{\partial x\partial z} - \frac{\partial^2 v_z}{\partial y\partial x} + \frac{\partial^2 v_x}{\partial y\partial z} + \frac{\partial^2 v_y}{\partial y\partial x} + \frac{\partial^2 v_x}{\partial z\partial y} = 0.$$

Thus $\nabla\cdot(\nabla\times v) = 0$

(b) Curl of the gradient

$\nabla\times(\nabla\phi)$

$$\nabla\phi = \hat{x}\frac{\partial\phi}{\partial x} + \hat{y}\frac{\partial\phi}{\partial y} + \hat{z}\frac{\partial\phi}{\partial z}$$

$$\nabla\times(\nabla\phi) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$\hat{x}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right] + \hat{y}\left[\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right] + \hat{z}\left[\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right]$$

$= 0.$

Thus, $\nabla\times(\nabla\phi) = 0$

8. Let $T = xy^2$. Show that $\int_a^b \vec{\nabla}T\cdot d\vec{r} = T(\vec{b}) - T(\vec{a})$ (independent of path) between points $\vec{a} = (0,0,0)$ and $\vec{b} = (2,1,0)$ via two path

(i) path connecting two points along straight line

(ii) first move parallel to the x-axis till $x = 2$ and then move parallel to y axis till $y = 1$

Soln: Always $d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

(i) $a(0,0,0)$ $\vec{b} = (2,1,0)$

so $2y = x$

$\Rightarrow 2dy = dx$

$d\mathbf{r} = dx\,\hat{x} + \frac{dx}{2}\,\hat{y}$

$$\nabla T = \hat{x}\frac{\partial}{\partial x}(xy^2) + \hat{y}\frac{\partial}{\partial y}(xy^2)$$

$$= \hat{x}\,y^2 + \hat{y}\,2xy.$$

$$\nabla T \cdot d\mathbf{r} = \left(y^2\hat{x} + 2xy\,\hat{y}\right)\left(dx\,\hat{x} + \frac{dx}{2}\hat{y}\right)$$

$$= \left(\frac{x^2}{4}\hat{x} + \frac{2x \times x}{2}\hat{y}\right)\left(dx\,\hat{x} + \frac{dx}{2}\hat{y}\right)$$

$$= \frac{x^4}{4}dx + \frac{x^2}{2}dx$$

$$\int \nabla T \cdot dl = \int_0^2 \frac{x^2}{4}dx + \int_0^2 \frac{x^2}{2}dx \qquad \text{(as } x \text{ goes from 0 to 2)}$$

$$= \frac{x^3}{12}\Big|_0^2 + \frac{x^3}{3}\Big|_0^2 = 2$$

(ii) along x axis $(0 \to 2)$.

$y=0 \quad d\mathbf{r} = dx\,\hat{x}$

$T = xy^2 = 0$

$\int \nabla T \cdot d\mathbf{r} = 0$

along y axis $(0 \to 1)$ at $x=2$

$d\mathbf{r} = dy\,\hat{y}$

$\nabla T = y^2\hat{x} + 4y\hat{y}$

$\nabla T \cdot d\mathbf{r} = 4y\,dy$

$$\int \nabla T \cdot d\mathbf{r} = \int_0^1 4y\,dy = \frac{4y^2}{2}\Big|_0^1 = 2$$

Net $\int \nabla T \cdot d\mathbf{r} = 0 + 2 = 0$

Thus, we se that the integral of ∇T along any path is Same provided initial & final positions are Same.

9. Calculate the divergence of the following fields:

(i) $\vec{v} = e^x(\cos y\,\hat{x} + \sin y\,\hat{y})$

(ii) $\vec{v} = x^2\hat{x} + y^2\hat{y} + z^2\hat{z}$

(iii) $\vec{v} = v_1(y,z)\hat{x} + v_2(z,x)\hat{y} + v_3(x,y)\hat{z}$

Soln: (i) $\vec{v} = e^x(\cos y\,\hat{x} + \sin y\,\hat{y})$

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)\cdot\left(e^x\cos y\,\hat{x} + e^x\sin y\,\hat{y}\right)$$

$$= \frac{\partial}{\partial x}e^x\cos y + \frac{\partial}{\partial y}e^x\sin y$$

$$= e^x\cos y + e^x\cos y$$

$$= 2e^x\cos y$$

(ii) $\vec{v} = x^2\hat{x} + y^2\hat{y} + z^2\hat{z}$

$$\cancel{\vec{\nabla}\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)\left(x^2\cos y\,\hat{x} + e^x\sin y\,\hat{y}\right)}$$

$$\vec{\nabla}\cdot\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y}\right)\cdot\left(x^2\hat{x} + y^2\hat{y} + z^2\hat{z}\right)$$

$$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 2x + 2y + 2z.$$

(iii) $\vec{v} = v_1(y,z)\hat{x} + v_2(z,x)\hat{y} + v_3(x,y)\hat{z}$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}v_1(y,z) + \frac{\partial}{\partial y}v_2(z,x) + \frac{\partial}{\partial z}v_3(x,y)$$

$$= 0 + 0 + 0$$

$$= 0.$$

10 Find the Laplacian $\nabla^2 f$ for the following scalar fields

(i) $f = 4x^2 + 9y^2 + z^2$

(ii) $f = e^{2x} \sin 2y$

(iii) $f = xy/z$

Soln: (i) $f = 4x^2 + 9y^2 + z^2$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}(4x^2+9y^2+z^2) + \frac{\partial^2}{\partial y^2}(4x^2+9y^2+z^2) + \frac{\partial^2}{\partial z^2}(4x^2+9y^2+z^2)$$

$$= 8 + 18 + 2.$$

$$= 28$$

(ii) $f = e^{2x} \sin 2y.$

$$\nabla^2 f = \frac{\partial^2}{\partial x^2}(e^{2x}\sin 2y) + \frac{\partial^2}{\partial y^2}(e^{2x}\sin 2y) + \frac{\partial^2}{\partial z^2}(e^{2x}\sin 2y)$$

$$= 4e^{2x}\sin 2y - 4e^{2x}\sin 2y.$$

$$= 0.$$

(iii) $f = xy/z$

$$\nabla^2 f = \frac{\partial^2}{\partial x^2}\left(\frac{xy}{z}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{xy}{z}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{xy}{z}\right)$$

$$= 0 + 0 + \frac{2xy}{z^3}$$

$$= \frac{2xy}{z^3}$$

11 A vector field $\vec{A}$ and space curve $\vec{r} = \vec{r}(t)$ are given by
$\vec{A} = (3x^2 - 6xy)\hat{x} + (2y + 3xy)\hat{y} + (1 - 4xyz^2)\hat{z}$
and $\vec{r}(t) = t\hat{x} + t^2\hat{y} + t^3\hat{z}$. Evaluate line integral $\int \vec{A}\cdot d\vec{r}$ in the limits $t=0$ and $t=2$ (find the values of x, y, z in terms of t by comparing with $\vec{r}$).

Soln: we know that

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

Given $\vec{r} = \vec{r}(t) = t\hat{x} + t^2\hat{y} + t^3\hat{z}$

So, on comparing

$$x = t \; ; \; y = t^2 \; ; \; z = t^3$$

$$\therefore \frac{dx}{dt} = 1 \; ; \; \frac{dy}{dt} = 2t \; ; \; \frac{dz}{dt} = 3t^2$$

$$\frac{dr}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}$$

$$dr = \left(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right)dt$$

$$\int \vec{A}\cdot d\vec{r} = \int (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\left(\frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z}\right)dt$$

$$= \int\left[(3x^2 - 6xy)\frac{dx}{dt} + (2y + 3xy)\frac{dy}{dt} + (1 - 4xyz^2)\frac{dz}{dt}\right]dt$$

replace x, y, z in terms of t.

$$\int \vec{A}\cdot d\vec{r} = \int_0^2 \left[3t^2 - 6t^5 + (2t^2 + 3t^4)\cdot 2t + (1 - 4t^9)3t^2\right]dt$$

$$= \int_0^2 (6t^2 + 4t^3 - 12t^{11})\,dt$$

$$= (2t^3 - t^4 - t^{12})\Big|_0^2 = 2\cdot 2^3 - 2^4 - 2^{12}$$

$$= 2^5 - 2^{12} = -4064$$

12. Show that the vector field $\vec{A} = (2xy + z^3)\hat{x} + (x^2+2y)\hat{y} + (3xz^2-2)\hat{z}$ is independent of the path from $(1,-1,1)$ to $(2,1,2)$ for the integral $\int \vec{A}\cdot d\vec{r}$. Find the potential function $\phi(x,y,z)$.

<u>Soln</u>: we check $\nabla \times A$

$$\nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+z^3 & x^2+2y & 3xz^2-2 \end{vmatrix}$$

$$\Rightarrow \hat{x}\left(\frac{\partial}{\partial y}(3xz^2-2) - \frac{\partial}{\partial z}(x^2+2y)\right) - \hat{y}\left(\frac{\partial}{\partial x}(3xz^2-2) - \frac{\partial}{\partial z}(2xy+z^3)\right)$$
$$+ \hat{z}\left(\frac{\partial}{\partial x}(x^2+2y) - \frac{\partial}{\partial y}(2xy+z^3)\right)$$

$$\Rightarrow \hat{x}\left[0-0\right] - \hat{y}\left(3z^2-3z^2\right) + \hat{z}\left(2x-2x\right)\Big] = 0$$

Thus $\nabla \times A = 0$

Thus $\phi(r) = \int A\cdot dr$ is path-independent.

$$\therefore A = \nabla\phi$$
$$\int d\phi = \int \nabla\phi \cdot dr = \int A\cdot dr$$
$$= \int (2xy+z^3)dx + \int (x^2+2y)\,dy + \int (3xz^2-2)\,dz.$$
$$= \int 2xy\,dx + \int z^3 dx + \int x^2 dy + \int 2y\,dy$$
$$+ \int 3xz^2 dz - \int 2dz$$
$$\Rightarrow y^2 - 2z + \int (2xy\,dx + x^2 dy)$$
$$+ \int 3xz^2 dz + \int z^3 dx$$

$\Rightarrow y^2 - 2z + \int d(x^2y) + \int d(xz^3)$

$\Rightarrow y^2 - 2z + x^2y + xz^3$

$\because A = \nabla\phi$, the line integral is path independent.

The value can be determined as follows

$$\int_{(1,-1,1)}^{(2,1,2)} A \cdot dr = \int_{(1,-1,1)}^{(2,1,2)} \nabla\phi \cdot dr = \int_{(1,-1,1)}^{(2,1,2)} d\phi$$

$$= \phi(2,1,2) - \phi(1,-1,1)$$

$$= (4+1+16-4) - (-1+1+1-2) = 18$$

13. What is a conservative force field? Is the force field $\vec{F} = (3xy - y)\hat{x} - x\hat{y} + \frac{3}{2}x^2\hat{z}$ is conservative? If yes determine the potential V and the work done to move a particle from point (1,1,1) to (2,2,2) (A)

Soln: Conservative force field is a field whose integral along any path depends only on the initial and final positions of the path. Thus, for a closed path, the integral is zero. These are also called curl-less (or irrotational) fields.

By Stokes theorem

$$\int_s (\nabla \times v) \cdot dl = \oint v \cdot dl = 0$$

So $\nabla \times v = 0$

(b) $\vec{F} = (3xy - y)\hat{x} - x\hat{y} + \frac{3}{2}x^2\hat{z}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3xy - y & -x & \frac{3}{2}x^2 \end{vmatrix} = \hat{x}\left[\frac{\partial}{\partial y}\left(\frac{3x^2}{2}\right) - \frac{\partial}{\partial z}(-x)\right] + \hat{y}\left[\frac{\partial}{\partial z}(3xy - y) - \frac{\partial}{\partial x}\left(\frac{3}{2}x^2\right)\right] + \hat{z}\left[\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}(3xy - y)\right]$$

$$\Rightarrow \hat{x}\left[0 - 0\right] + \hat{y}\left[0 - 3x\right] + \hat{z}\left[-1 - 3x + 1\right]$$

$$= -3x\hat{y} - 3x\hat{z}$$

Hence, $\vec{F}$ is not a conservative field.

14 Check the divergence theorem using the function $\vec{v} = y^2\hat{x} + (2xy+z^2)\hat{y} + (2yz)\hat{z}$, the unit cube placed at origin $(0,0,0)$, $(1,1,1)$ are the diagonal points of the cube in positive quadrant.

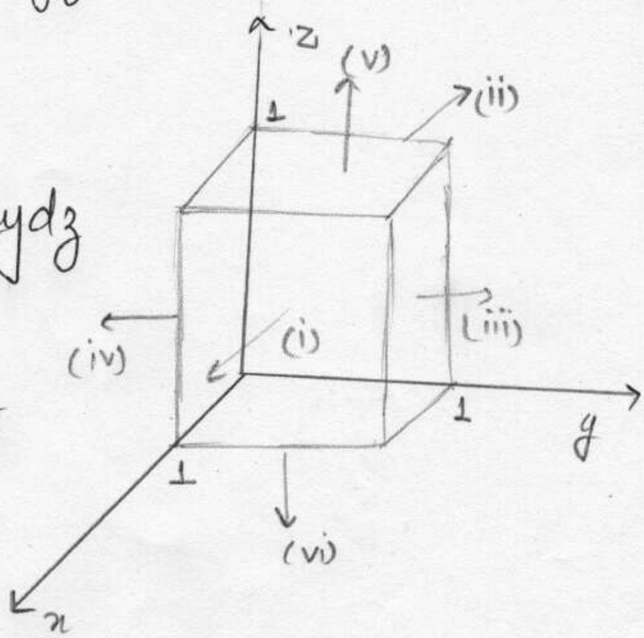
Soln: The divergence theorem states that

$$\int_V (\nabla\cdot v)\, d\tau = \oint_S v\cdot da$$

Now for LHS

$$\nabla\cdot v = \frac{\partial}{\partial x}y^2 + \frac{\partial}{\partial y}(2xy+z^2) + \frac{\partial}{\partial z}(2yz)$$

$$= 2x + 2y$$



$$\int_V (\nabla\cdot v)\, d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1 \left(\frac{x^2}{2}+xy\right)\Big|_0^1 dy\,dz$$

$$= 2\int_0^1\int_0^1 \left(\frac{1}{2}+y\right) dy\,dz$$

$$= 2\int_0^1 \left(\frac{1}{2}y+\frac{y^2}{2}\right)\Big|_0^1 dz$$

$$= 2\int_0^1 \left(\frac{1}{2}+\frac{1}{2}\right)dz = 2\int_0^1 dz = 2z\Big|_0^1 = 2$$

So, $\int_V \nabla\cdot v\, d\tau = 2$

RHS $\oint v\cdot da$

(i) $\int v\cdot da$ $\quad da = dy\,dz\,\hat{x}$

$$\int v\cdot da = \int_0^1\int_0^1 y^2\,dy\,dz = \frac{1}{3}$$

(ii) $da = -dy\,dz\,\hat{x}$

$$\int v \cdot da = -\int_0^1\int_0^1 y^2\,dy\,dz = \int_0^1 -\frac{y^3}{3}\Big|_0^1 dz \quad -\frac{1}{3}z\Big|_0^1 = -\frac{1}{3}$$

(iii) $$\int v \cdot da = \int_0^1\int_0^1 (2x+z^2)\,dx\,dz = \frac{4}{3}$$

$$\int_0^1 \left(\frac{2x^2}{2} + z^2 x\right)_0^1 dz = \int_0^1 (1+z^2)\,dz$$

$$= \left(z + \frac{z^3}{3}\right)_0^1 = \frac{4}{3}$$

(iv) $da = -dx\,dz\,\hat{y}$

$$\int v \cdot da = -\int_0^1\int_0^1 z^2\,dx\,dz$$

$$= -\int_0^1 (z^2 x)\Big|_0^1 dz = -\int_0^1 z^2\,dz = -\frac{z^3}{3}\Big|_0^1 = -\frac{1}{3}$$

(v) $da = dx\,dy\,\hat{z}$

$$\int v \cdot da = \int_0^1\int_0^1 2y\,dx\,dy\,\hat{z}$$

$$\int_0^1 (2yx)_0^1\,dy = \int_0^1 2y\,dy$$

$$= \frac{2y^2}{2}\Big|_0^1 = 1$$

(vi) $da = -dx\,dy\,\hat{z}$

$$\int v \cdot da = -\int_0^1\int_0^1 0\,dx\,dy = 0$$

So, total flux $= \oint_S v \cdot da = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2$

Hence, Divergence theorem is checked

15. Find the curl $\vec{\nabla}\times v$ of the following functions

(i) $\vec{v} = 2y\,\hat{x} + 5x\,\hat{y}$

(ii) $\vec{v} = xyz(x\hat{x} + y\hat{y} + z\hat{z})$

(iii) $\vec{v} = v_1(x)\,\hat{x} + v_2(y)\,\hat{y} + v_3(z)\,\hat{z}$

Soln: (i) $\vec{v} = 2y\,\hat{x} + 5x\,\hat{y}$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y & 5x & 0 \end{vmatrix}$$

$$\hat{x}\left(\frac{\partial 0}{\partial y} - \frac{\partial (5x)}{\partial z}\right) + \hat{y}\left(\frac{\partial}{\partial z}(2y) - \frac{\partial 0}{\partial x}\right) + \hat{z}\left(\frac{\partial}{\partial x}(5x) - \frac{\partial}{\partial y}(2y)\right)$$

$$= \hat{x}(0) + \hat{y}(0) + \hat{z}(5-2)$$

$$= 3\hat{z}$$

(ii) $\vec{v} = xyz(x\hat{x} + y\hat{y} + z\hat{z})$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & xy^2z & xyz^2 \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}(xyz^2) - \frac{\partial}{\partial z}(xy^2z)\right) + \hat{y}\left(\frac{\partial}{\partial z}(x^2yz) - \frac{\partial}{\partial x}(xyz^2)\right)$$

$$+ \hat{z}\left(\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial y}(x^2yz)\right)$$

$$= \hat{x}(xz^2 - xy^2) + \hat{y}(x^2y - yz^2) + \hat{z}(y^2z - x^2z)$$

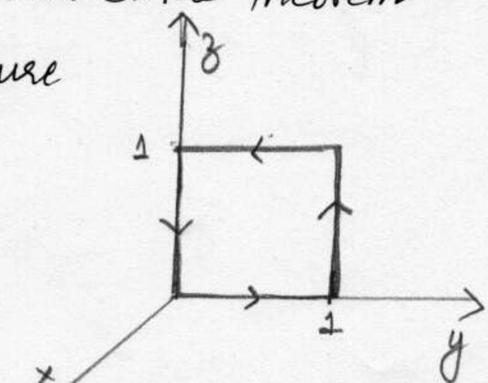
(iii) $\vec{v} = v_1(x)\,\hat{x} + v_2(y)\,\hat{y} + v_3(z)\,\hat{z}$

$$\nabla\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1(x) & v_2(y) & v_3(z) \end{vmatrix}$$

$$= \hat{x}\left(\frac{\partial}{\partial y}v_3(z) - \frac{\partial}{\partial z}v_2(y)\right) + \hat{y}\left(\frac{\partial}{\partial z}v_1(x) - \frac{\partial}{\partial x}v_3(z)\right) + \hat{z}\left(\frac{\partial}{\partial x}v_2(y) - \frac{\partial}{\partial y}v_1(x)\right)$$

$= 0.$

16. Suppose $\vec{v} = (2xz + 3y^2)\,\hat{y} + (4yz^2)\,\hat{z}$. Check Stoke's theorem for the square surface shown in the figure



Soln: The Stoke's Theorem states that

$$\int_s (\nabla\times v)\cdot da = \oint_p v\cdot dl$$

LHS: $\int_s (\nabla\times v)\, da$

$$\nabla\times v = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 2xz+3z^2 & 4yz^2 \end{vmatrix}$$

$$\hat{x}\left(\frac{\partial}{\partial y}(4yz^2) - \frac{\partial}{\partial z}(2xz+3y^2)\right) + \hat{y}\left(\frac{\partial}{\partial z}0 - \frac{\partial}{\partial x}(4yz^2)\right)$$

$$+ \hat{z}\left(\frac{\partial}{\partial x}(2xz+3y^2) - 0\right)$$

$$= \hat{x}(4z^2 - 2x) - 0\times\hat{y} + 2z\,\hat{z}$$

$$= (4z^2 - 2x)\,\hat{x} + 2z\,\hat{z}$$

$da = dy\,dz\,\hat{x}$ (direction given by right hand thumb rule)

Now for this surface $x=0$.

So $\int (\nabla \times v) \, da = \int (4z^2 \hat{x} + 2z \hat{z})(dy\,dz\,\hat{x})$

$= \int 4z^2 \, dy\,dz$

$= \int_0^1 4z^2 dz \int_0^1 dy = \frac{4z^3}{3}\Big|_0^1 \, y\Big|_0^1 = \frac{4}{3}$

Now for line integral, we break this into 4 segments going in counter clock-wise direction

(i) $x = 0$, $z = 0$

$v \cdot dl = 3y^2 dy$ $\int v \cdot dl = \int_0^1 3y^2 dy = 1$

(ii) $x = 0$, $y = 0$

$v \cdot dl = 4z^2 dz$ $\int v \cdot dl = \int_0^1 4z^2 dz = \frac{4}{3}$

(iii) $x = 0$, $z = 1$

$v \cdot dl = 3y^2 dy$ $\int v \cdot dl = \int_1^0 3y^2 dy = -1$

(iv) $x = 0$, $y = 0$

$v \cdot dl = 0$ $\int v \cdot dl = \int_1^0 0 \, dz = 0$

So, $\oint v \cdot dl = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}$

Hence, Stokes theorem is checked.

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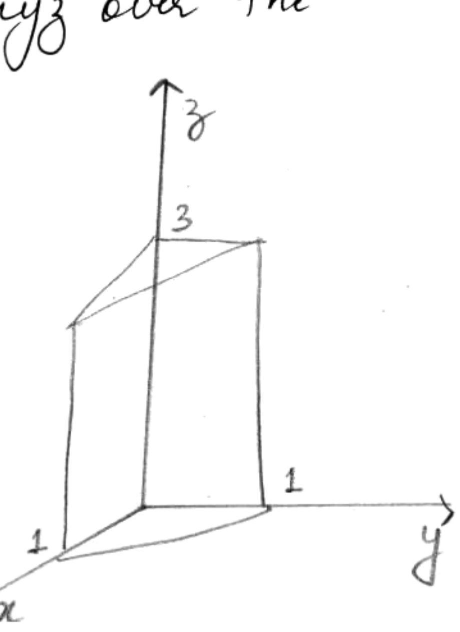
17. Calculate the volume integral of $T = xyz^2$ over the prism as shown in the figure



Soln We can do integral in any order

No, if y moves from 0 to 1 along the line $x + y = 1$, then x moves from 0 to $1-y$ & z independently moves from 0 to 3

$\therefore$ The volume integral of the function.

$$\int T\,d\tau = \int_0^3 z^2 \left\{ \int_0^1 y \left[\int_0^{1-y} x\,dx \right] dy \right\} dz$$

$$= \frac{1}{2}\int_0^3 z^2 dz \int_0^1 (1-y)^2 y\,dy$$

$$= \frac{1 \times z^3}{2 \times 3} \Big|_0^3 \int_0^1 (1-y^2-2y)\, y\,dy$$

$$= \frac{9}{2} \times \int_0^1 (y+y^3-2y^2)\,dy$$

$$= \frac{9}{2} \times \left[\frac{y^2}{2} + \frac{y^4}{4} - \frac{2y^3}{3} \right]_0^1$$

$$= \frac{9}{2} \times \left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right]$$

$$= \frac{9}{2} \times \left[\frac{6+3-8}{12} \right]$$

$$= \frac{9}{2} \times \frac{1}{12} = \frac{3}{8}$$

—— × —— × —— × ——