More on Sesqui/Bi-Linear forms

1. Consider now a sesquiliner form $X \times X \xrightarrow{s} \mathbb{K}$ and introduce $s^*(x,u) := \overline{s(u,x)}$. Then s^* is again a sesquilinear form $(s^*(x\lambda, u\mu) = \overline{s(u\mu, x\lambda)} = \overline{\overline{\mu}s(u,x)\lambda} = \overline{\lambda}\overline{s(u,x)}\mu = \overline{\lambda}s^*(x,u)\mu)$ Let $Y \in Cen\mathbb{K}$; this means $Y\lambda = \lambda Y$ for each $\lambda \in \mathbb{K}$ and hence $\overline{Y}\lambda = \overline{\overline{\lambda}\gamma} = \overline{\gamma\overline{\lambda}} = \lambda\overline{\gamma}$ for each $\lambda \in \mathbb{K}$. Thus $Y \in Cen(\mathbb{K})$ iff $\overline{\gamma} \in Cen\mathbb{K}$. Let $\Gamma(\mathbb{K}) := y \in Cen(\mathbb{K}) \mid y\overline{y} = 1$. Then $\Gamma(\mathbb{K})$ is not empty since $\pm 1 \in \Gamma(\mathbb{K})$. We make a definition. A sesquilinear form $X \times X \xrightarrow{s} \mathbb{K}$ is called $y - hermitian(y \in \Gamma(\mathbb{K})$ iff $s = \gamma s^*$, i.e, $s(x,u) = \gamma\overline{s(u,x)}$ at each $x, u \in X$.

(i) When $\gamma = 1$, we say that γ-hermitian form is hermitian, when $\gamma = -1$, we say it is skew-hermitian. Thus s is hermitian iff $s(x,u) = \overline{s(u,x)}$ and s is skew hermitian iff $s(x,u) = -\overline{s(u,x)}$

(ii) For the trivial involution *(thus* $\urcorner$ *is forced to be commutative)* we have bilinear forms and 'hermitian' is called 'symmetric'. Thus a bilinear form β is called symmetric iff $\beta(x,u) = \beta(u,x)$ and it is called skew symmetric iff $\beta(x,u) = -\beta(u,x)$.

2. When $X \xrightarrow{\tau} X$ is an isomorphism with $s(\tau x, \tau) = s(x,u)$, it is called an isometry on X; the term unitary transformation is also used. The unitary transformation is also used. The unitary transformations of X constitute a group under composition $(s(\tau^{-1}x, \tau u) = s(\tau(\tau^{-1}x), \tau(\tau^{-1}u)) = s(x,u))$ with id_X as the identity. *(A group is a set A with an associative binary operation(i.e. $a(bc) = (ab)c$ holds) in which each $a \in A$ has an inverse a^{-1} so that $aa^{-1} = 1 = a^{-1}a$; the existence of an identity is part of the definition. The nonzero quaternions from a group; in fact the nonzero elements of any division ring do. The eight quaternions $\pm e_i \mid i = 0,1,2,3$ also form a group; it is called the quaternion group. When a group is commutative, it is called 'abelian'; the additive fragment of any* *ring is an abelian group, the nonzero elements of a field form an abelian group).*

3. **(i)** Let $X\times X \xrightarrow{s} \mathbb{K}$ be a sesquilinear form. Then if $\mathbb{C}$ is a subring of $\mathbb{K}$ and the involution on $\mathbb{K}$ becomes involution of complex numbers i.e. conjugation on $\mathbb{C}$, we have *(since* $\bar{i} = i^{-1}$, $\overline{i^k} = i^{-k}$ *and* s *is conjugate-linear in the first variable).* $\sum_{k=0}^{3} s(x+ui^k, x+ui^k)i^k = \sum_{k=0}^{3}[s(x,x)+ s(x,u)i^k+s(u,x)i^{-k}+s(u,u)i^{-k}i^k]i^k = \sum_{k=0}^{3}[s(x,x)i^k+s(x,u)i^{2k}+s(u,x)i^{-k}i^k+s(u,u)i^k] = \sum_{k=0}^{3}[s(x,x)-s(x,u)+s(u,u)]i^k + \sum_{k=0}^{3} s(u,x) = [s(x,x)-s(x,u)+s(u,u)][1+i+i^2+i^3]+ 4s(u,x) = 4s(u,x)$ *(*$\because 1+i+i^2+i^3=0$*)* The relation $4s(u,x) = \sum_{k=0}^{3} s(x+ui^k, x+ui^k)i^k$ is known as the Polarization identity for the sesquilinear forms. **(ii)** The rings $\mathbb{K}=\mathbb{C}$, $\mathbb{K}=\mathbb{H}$(the quaternions) clearly satisfy the requirement for $\mathbb{K}$ in (i) above. There are in fact many other $\mathbb{K}$ *(e.g the ring of complex* $n\times m$ *matrices with conjugation on* $\mathbb{K}$ *defined as the conjugate-transposse)* for which the calculations in (i) are valid but we will not discuss the more general situation *(which needs* $\mathbb{K}$ *to be what is known as a* C^**-algebra).*

4. **(i)** If we write $s(x,x)$ as $Q(x)$, we associate a function $X \xrightarrow{X} \mathbb{K}$ called the quadratic form of s. Then the polarization identity says that $s(u,x) = \frac{1}{4}[\sum_{k=0}^{3} Q(x+ui^k, x+ui^k)i^k]$ so that the sesquilinear form is completely determined by its quadratic form.

(ii) Noting that $s^*(u,x) := \overline{s(x,u)}$ is also sesquilinear (6 on page 6 preceding) and the quadratic form for $s^*(u,x)$ is then $Q^*(x) := \overline{Q(x)}$, we conclude that s is γ-hermitian iff $Q^*(x) = \overline{Q(x)} = \gamma Q(x)$ In particular: A sesquilinear form s is hermitian/skewhermitian iff its associated quadratic form satisfies $Q(x) = \overline{Q(x)}/Q(x) = -\overline{Q(x)}$.

(iii) When $\mathbb{K}=\mathbb{C}$ or $\mathbb{H}$, we conclude by using (ii) ($\mathbb{H}$ =quaternions) A sesquilinear form $X\times X \xrightarrow{s} \mathbb{K}$ ($\mathbb{K}=\mathbb{C}$ or $\mathbb{H}$) is hermitian iff $Q(x)\in\mathbb{R}$.

(iv) Suppose $\mathbb{K}=\mathbb{C}$ or $\mathbb{H}$. Let the real part of $a\in\mathbb{K}$ be denoted by a_0(thus

$$a = \begin{cases} a_0 + a_1 i & \text{for } a\in\mathbb{K} \\ a_0 + i a_1 \end{cases}$$

and $a = a_0 + \underline{a}$ where $\underline{a} = e_1a_1 + e_2a_2 + e_3a_3$ for $a \in \mathbb{H}$) Then $a + \overline{a} = 2a_0$. If s is hermitian, we have $s(x\lambda + u\mu, x\lambda + u\mu) = s(x\lambda, x\lambda) + s(u\mu, x\lambda) + s(x\lambda, u\mu) + s(u\mu, umu)$

$= \overline{\lambda}s(x,x)\lambda + \overline{\mu}s(u,x)\lambda + \overline{\lambda}s(x,u)\mu + \overline{\mu}s(u,u)\mu$

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$(\because \overline{\overline{\lambda}s(x,u)\mu} = \overline{\mu}\,\overline{s(x,u)}\,\overline{\overline{\lambda}} = \overline{\mu}s(u,x)\lambda)$

$=|\lambda|^2 s(x,x) + 2(\overline{\lambda}s(x,u)\mu) + |\mu|^2 s(u,u)$ $(\because s(z,z) \in \mathbb{R}$ for each $z \in X$ and $\mathbb{R} = Cen(\mathbb{H})$; for $\mathbb{K} = \mathbb{C}$ the argument is obvious any way)

That is, $Q(x\lambda + u\mu) = |\lambda|^2 Q(x) + 2(\overline{\lambda}s(x,u)\mu)_0 + |\mu|^2 Q(u)$ *(every term in this is a real number)*

(v) For complex numbers, $a = a_0 + ia_1$ will be written $a = \underline{re}(a) + i\overline{im}(a)$. Then $ai = a_0i - a_1$ so $\underline{re}(ai) = -\underline{im}(a)$ and $\underline{im}(ai) = \underline{re}(a)$. For $X \times X \xrightarrow{s} \mathbb{C}$, we have $s(xi,u) = \overline{i}s(x,u) = -is(x,u)$. Thus $\underline{im}s(x,u) = -\underline{re}(is(x,u)) = \underline{re}s(xi,u)$. Further, s is hermitian iff $s(x,u) = \overline{s(u,x)}$ and hence we have, from $s(xi,u) = \overline{i}s(x,u)$, s is hermitian iff $s(xi,u) = \overline{i}\,\overline{s(u,x)} = \overline{s(u,x)i} = -is\overline{(u,x)} = -is(x,u)$ i.e. s is hermitian iff is is skew hermitian.