0.1 Change of Basis

Definition: Let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis of a vector space V, and let $B' = \{v1, v2, \ldots, v_n\}$ be another basis. (For reference, we will call S the "old" basis and B' the "new" basis) Since B is a basis, each vector in the "new" basis B' can be written uniquely as a linear combination of the vectors in BS; say,

$$\begin{array}{l} v_1 = a_{11}u_1 + a_{12}u_2 + \ldots + a_{1n}u_n \\ v_2 = a_{21}u_1 + a_{22}u_2 + \ldots + a_{2n}u_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_n = a_{n1}u_1 + a_{2n}u_2 + \ldots + a_{2n}u_n \end{array}$$

Let P be the transpose of the above matrix of coefficients; that is, let $P = [p_{ij}]$, where $p_{ij} = a_{ji}$. Then P is called the change-of-basis matrix (or transition matrix) from the "old" basis B to the "new" basis B'. The following remarks are in order.

Remark 1: The above change-of-basis matrix P may also be viewed as the matrix whose columns are, respectively, the coordinate column vectors of the "new" basis vectors v relative to the "old" basis B; namely, $P = [[v_1]_B, [v2]_B, \ldots, [v_n]_B]$.

Remark 2: Analogously, there is a change-of-basis matrix Q from the "new" basis B' to the "old" basis B. Similarly, Q may be viewed as the matrix whose columns are, respectively, the coordinate column vectors of the "old" basis vectors u relative to the "new" basis B'; namely, $Q = [[u_1]_{B'}, [u_2]_{B'}, \ldots [u_n]B']$.

Remark 3: Since the vectors $v_1, v_2, \ldots v_n$, in the new basis B' are linearly independent, the matrix P is invertible . Similarly, Q is invertible. In fact, we have the following proposition .

Proposition: Let P and Q be the above change-of-basis matrices. Then $Q = P^{-1}$.

Example : Consider the following two bases of $\mathbb{R}^2$

$B = \{u_1, u_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$, and $B' = \{v_1, v_2\} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$,

1. Find the change-of-basis matrix P from B to the "new" basis B'.
2. Find the change-of-basis matrix Q from the "new" basis B' back to the "old" basis B.

Solution:

1. Write each of the new basis vectors of B' as a linear combination of the original basis vectors u_1 and u_2 of B. We have

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow \begin{array}{rr} x + 3y = & 1 \\ 2x + 5y = & -1 \end{array} \Rightarrow \begin{array}{rr} x = & -8 \\ y = & 3 \end{array}$$

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 3 \\ 5 \end{bmatrix} \Rightarrow \begin{array}{rr} x + 3y = & 1 \\ 2x + 5y = & -2 \end{array} \Rightarrow \begin{array}{rr} x = & -11 \\ y = & 4 \end{array}$$

Thus $\begin{array}{rr} v_1 = & -8u_1 + 3u_2 \\ v_2 = & -11u_1 + 4u_2 \end{array}$ and hence $P = \begin{bmatrix} -8 & -11 \\ 3 & 4 \end{bmatrix}$.

Note that the coordinates of v_1 and v_2 are the columns, not rows, of the change-of-basis matrix P.

2. Here we write each of the "old" basis vectors u_1 and u_2 of B' as a Linear combination of the "new" basis vectors v_1 and v_2 of S'. We have

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \begin{array}{rr} x + y = & 1 \\ -x - 2y = & 2 \end{array} \Rightarrow \begin{array}{rr} x = & 4 \\ y = & -3 \end{array}$$

$$\begin{bmatrix} 3 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 1 \\ -2 \end{bmatrix} \Rightarrow \begin{array}{rr} x + y = & 1 \\ -x - 2y = & 2 \end{array} \Rightarrow \begin{array}{rr} x = & 11 \\ y = & -8 \end{array}$$

Thus $\begin{array}{rr} u_1 = & 4v_1 - 3v_2 \\ u_2 = & 11v_1 - 8v_2 \end{array}$ and hence $Q = \begin{bmatrix} 4 & 11 \\ -3 & -8 \end{bmatrix}$.

As expected , $Q = P^{-1}$. (In fact, we could have obtained Q by simply finding P^{-1}.

Example : Consider the following two bases of $\mathbb{R}^3$

$E = \{e_1, e_2, e_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $B = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$,

1. Find the change-of-basis matrix P from E to the basis B.
2. Find the change-of-basis matrix Q from the basis B back to the "old" basis E.

Solution:

1. Since E is the usual basis, we can immediately write each basis element of E as a Linear combination of the basis elements of B. Specifically,

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1e_1 + 0e_2 + 1e_3$$

$$u_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 2e_1 + 1e_2 + 2e_3$$

$$u_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1e_1 + 2e_2 + 2e_3$$

$\implies P = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix}$. Again, the coordinates of u_1, u_2, u_3 appear as the columns in P. Observe that P is simply the matrix whose columns are the basis vectors of B. This is true only because the original basis was the usual basis E.

2. The definition of the change-of-basis matrix Q tells us to write each of the (usual) basis vectors in E as a linear combination of the basis elements of B. This yields

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} - 1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -2u_1 + 2u_2 - 1u_3$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -2\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = -2u_1 + 1u_2 + 0u_3$$

$$e_3 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 3u_1 - 2u_2 + 1u_3$$

$$\implies Q = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}.$$

We emphasize that to find Q, we need to solve three 3×3 systems of linear equations each of e_1, e_2, e_3. Alternatively, we can find $Q = P^{-1}$ by forming the matrix $M = [P, I]$ and row reducing M to row canonical form as

$$\implies M = [P|I]$$
$$= \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & 3 \\ 0 & 1 & 0 & 2 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] = [I|P^{-1}]$$
$$Q = P^{-1} = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix}$$

(Here we have used the fact that Q is the inverse of P.)
The result in above is true in general. We state this result formally since it occurs often.

Proposition : The change-of-basis matrix from the usual basis E of $\mathbb{R}^n$ to any basis B of $\mathbb{R}^n$ is the matrix P whose columns are, respectively, the basis vectors of B.

Applications of Change-of-Basis Matrix

First we show how a change of basis affects the coordinates of a vector in a vector space V. **Theorem :** Let P be the change-of-basis matrix from a basis B to a basis B' in a vector space V. Then, for any vector $v \in V$, we have

$$P[v]_{B'} = [v]_B \text{ and hence } [v]_{B'} = P^{-1}[v_B]$$

Namely, if we multiply the coordinates of v in the original basis B by P, we get the coordinates of v in the new basis B'.

Remark 1: Although P is called the change-of-basis matrix from the old basis B to the new basis B', we emphasize that it is P that transforms the coordinates of v in the original basis B into the coordinates of v in the new basis B'.

Remark 2: Because of the above theorem, many texts call $Q = P^{-1}$, not P, the transition matrix from the old basis P to the new basis B'. Some texts also refer to Q as the change-of-coordinates matrix.

Theorem : Let P be the change-of-basis matrix from a basis B to a basis B' in a vector space V. Then, for any linear operator T on V,

$$[T]_{B'} = P^{-1}[T]_B P$$

That is, if A and B are the matrix representations of T relative, respectively, to B and B', then $B = P^{-1}AP$.

Example : Consider the following two bases of $\mathbb{R}^3$

$$E = \{e_1, e_2, e_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}, \text{ and } B = \{u_1, u_2, u_3\} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\right\}.$$

1. Write $v = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ as a Linear combination of u_1, u_2, u_3 or, equivalently, find $[v]_B$. .
2. Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix}$, which may be viewed as a Linear operator on $\mathbb{R}^3$. Find the matrix C that represents A relative to the basis B.

Solution: The change-of-basis matrix P from E to B and its inverse P^{-1} were obtained in earlier example

1. One way to do this is to directly solve the vector equation $v = xu_1 + yu_2 + zu_3$,that is,
$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + z \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Longrightarrow \begin{array}{ccccccc} x & + & 2y & + & z & = & 1 \\ & & y & + & 2z & = & 3 \\ x & + & 2y & + & 2z & = & 5 \end{array}.$$
The solution is $x = 7, y = -5, z = 4$, so $v = 7u_1 - 5u_2 + 4u_3$
$$\Longrightarrow [v]_B = \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix}.$$
On the other hand, we know that $[v]_E = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, since E is the usual basis, and we already know P^{-1}. Therefore, by
$$[v]_B = P^{-1}[v]_E = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$
Thus, again, $[v]_B = 7u_1 - 5u_2 + 4u_3$.

2. The definition of the matrix representation of A relative to the basis B tells us to write each of $A(u_1), A(u_2), A(u_3)$ as a linear combination of the basis vectors u_1, u_2, u_3 of B. This yields
$$A(u_1) = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} = 11 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 5 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 11u_1 - 5u_2 + 6u_3$$
$$A(u_2) = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 9 \end{bmatrix} = 21 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 14 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 8 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 21u_1 - 14u_2 + 8u_3$$
$$A(u_3) = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix} = 17 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - 8 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 17u_1 - 8u_2 + 2u_3$$
$$\Longrightarrow C = \begin{bmatrix} 11 & 21 & 17 \\ -5 & -14 & -8 \\ 6 & 8 & 2 \end{bmatrix}.$$
We emphasize that to find C, we need to solve three 3×3 systems of linear equations each of $A(u_1), A(u_2), A(u_3)$. On the other hand, since we know P and P^{-1}, we can use the theorem That is,
$$C = P^{-1}AP = \begin{bmatrix} -2 & -2 & 3 \\ 2 & 1 & -2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 \\ 2 & -4 & 1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 11 & 21 & 17 \\ -5 & -14 & -8 \\ 6 & 8 & 2 \end{bmatrix}.$$
This, as expected, gives the same result.

Example : Consider the following two basis of $\mathbb{R}^3$
$$E = \{e_1, e_2, e_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \text{ and } B = \{u_1, u_2, u_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\}.$$
Then find

1. The change of basis matrix P from E to B
2. The change of basis matrix Q from B back to E

Solution:

1. Since,
$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
Therefore, the change-of-basis matrix P from E to B is given by
$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

2. Expressing each vector of E as a linear combination of the basis vector of B by first finding the coordinate of an arbitrary vector $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \Longrightarrow \begin{array}{ccccccc} \alpha_1 & + & \alpha_2 & + & & = & u_1 \\ 2\alpha_1 & & 3\alpha_2 & + & \alpha_3 & = & u_2 \\ & + & 2\alpha_2 & + & 3\alpha_3 & = & u_3 \end{array}.$$

Solving for α_1, α_2 and α_3 we get

$$\begin{array}{ccccccc} \alpha_1 & = & 7u_1 & - & 3u_2 & + & u_3 \\ \alpha_2 & = & -6u_1 & + & 3u_2 & - & u_3 \\ \alpha_3 & = & 4u_1 & - & 2u_2 & + & u_3 \end{array}.$$

$$\Longrightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = (7u_1 - 3u_2 + u_3) \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + (-6u_1 + 3u_2 - u_3) \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + (4u_1 - 2u_2 + u_3) \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$\Longrightarrow [u]_B = \begin{bmatrix} 7u_1 & - & 3u_2 & + & u_3 \\ -6u_1 & + & 3u_2 & - & u_3 \\ 4u_1 & - & 2u_2 & + & u_3 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 6 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

$$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

Therefore, the change-of-basis matrix Q from B to E is given by

$$Q = \begin{bmatrix} -7 & -3 & 1 \\ 6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

Alternatively, we can find $Q = P^{-1}$ by forming the matrix $M = [P, I]$ and row reducing M to row canonical form as

$$\begin{aligned} \Longrightarrow M = [P|I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] & \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \\ & \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right] \\ & \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & 1 \\ 0 & 1 & 0 & -6 & 3 & -1 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right] \\ & = [I|P^{-1}] \end{aligned}$$

$$Q = P^{-1} = \begin{bmatrix} -7 & -3 & 1 \\ 6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$

Theorem : For any vector $v \in V$, $[T]_B^{B'}[v]_B = [T(v)]_{B'}$.

Theorem : Let P be the change-of-basis matrix from a basis B_1 to a basis B_1' in V, and let Q be the change-of-basis matrix from B_2 to a basis B_2' in V in U. Then, for any linear map $F : V \to U$

$$[T]_{B_1'}^{B_2'} = Q^{-1}[T]_{B_1}^{B_2} P$$