(7) $f: [0,1] \to \mathbb{R}$ cts funcn & $f(0) = f(1)$

To show: $\exists\ c \in [0, \frac{1}{2}]$ s.that $f(c) = f(c + \frac{1}{2})$

Soln let $g(x) = f(x) - f(x + \frac{1}{2})$

Now $g(0) = f(0) - f(\frac{1}{2})$, $g(1/2) = f(\frac{1}{2}) - f(1)$

$= f(\frac{1}{2}) - f(0)$ [$\because f(0) = f(1)$]

$= -g(0)$

$\Rightarrow g(0) g(\frac{1}{2}) < 0$

$\Rightarrow \exists$ atleast one $c \in [0, \frac{1}{2}]$ such that

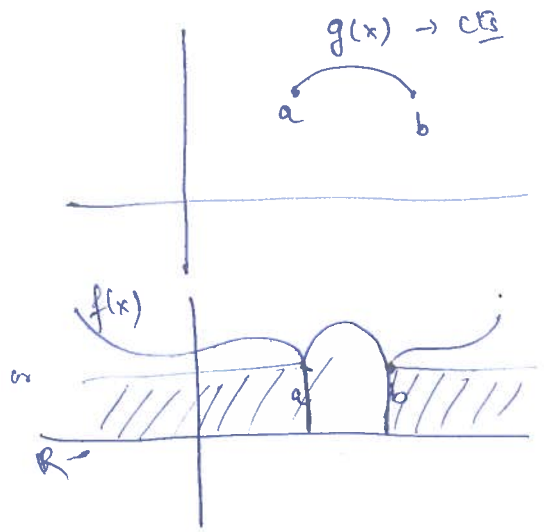
$g(c) = 0$

$\Rightarrow f(x) = f(x + \frac{1}{2})$

(8) Given: f is cts on $[a,b]$

To show: $\exists$ a funcn g, cts on $\mathbb{R}$ satisfying $g(x) = f(x)\ \forall x \in [a,b]$

Soln



let we consider a funcn $f(x)$ as

$$f(x) = \begin{cases} f(a) & (-\infty, a) \\ g(x) & [a,b] \\ f(b) & (b, \infty) \end{cases}$$

then clearly $g(x) = f(x)\ \forall x \in [a,b]$

& $g(x)$ is cts on $\mathbb{R}$.

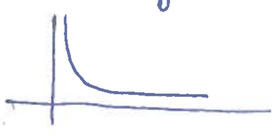
9 $f: [a,b] \to \mathbb{R}$ be a cts funcn.

(i) To prove Range(f) is closed & bdd interval.

Since f is a cts funcn, $\exists\ x_0, y_0 \in [a,b]$ such that $f(x_0) = m = \inf f$ & $f(y_0) = M = \sup f$. Suppose $x_0 < y_0$. By IVP, for every $\alpha \in [m, M]$ there exists $x \in [x_0, y_0]$ such that $f(x) = \alpha$. Hence $f([a,b]) = [m, M]$

(ii) If Domain of f is open instead of closed, → Range(f) may or may not be closed then

Consider $f: (0,1) \to \mathbb{R}$



Clearly f is cts bd
Range(f) isn't bdd, also

(5) If $f: R \to R$ be a funcn such that $|f(x) - f(y)| \leq (x-y)^2 \ \forall x, y \in R$ (3)

To show: f is constant

Pf If $x \neq y$, dividing by $|x-y|$ & taking lt $y \to x$

$$\Rightarrow \lim_{y \to x} \left| \frac{f(x) - f(y)}{x-y} \right| \leq \lim_{y \to x} |x-y| = 0$$

Thus $f'(x)$ exist & $f'(x) = 0$ for every $x \in R$

$\therefore f$ is constant. [$\because$ if $f'(x) = 0$ for all $x \in (a,b)$, then f is const] given f is differentiable in (a,b)]

(6) To show Every polynomial of odd degree has atleast one real root.

Pf Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_n \neq 0$ and $n \to$ odd

Then $p(x) = x^n \left(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n} \right)$

If $a_n > 0$, then $p(x) \to \infty$ as $x \to \infty$

& $p(x) \to -\infty$ as $x \to -\infty$

IVT [Let f be cts on $[a,b]$ & let $f(a) < S < f(b)$, then $\exists x$ such that $a < x < b$ & $f(x) = S$]

Thus by IVP, $\exists x_0$ such that $p(x_0) = 0$

Similar argument for $a_n < 0$ [$p(x) \to \infty$ as $x \to -\infty$ & $p(x) \to -\infty$ as $x \to \infty$] $\therefore$ By IVP $\exists x_0$ s.that $p(x_0) = 0$

(7) Let $f: R \to R$ satisfies $f(x+y) = f(x) + f(y)$ for each x, y in R.

(i) To show $f(nx) = nf(x)$ for all x in R & n in Z

Pf Now $f(x+y) = f(x) + f(y)$

$\Rightarrow f(0+0) = f(0) + f(0) \Rightarrow f(0) = f(0) + f(0)$

$\Rightarrow f(0) = 0$

Now, $f(0) = f(x-x) = f(x) + f(-x)$

$\Rightarrow 0 = f(x) + f(-x)$

$\Rightarrow f(-x) = -f(x)$

Now, $f(nx) = f(\underbrace{x + x + \cdots + x}_{n\text{-times}})$

$= \underbrace{f(x) + f(x) + \cdots + f(x)}_{n\text{-times}}$

$= nf(x)$

$\Rightarrow f(nx) = nf(x)$

(b) To show f is cts at 0 iff f is cts on R.

If f is cts on R $\Rightarrow f$ is cts at 0 (obvious)

Conversely, let f is cts at 0 $\to$ Now $f(x)$ is cts at $x = 0$

def let $y = x + a$

$\Rightarrow f(x) + f(a)$ ---- $x = 0$ ($\because$ some const is added & it doesn't affect cty)

$\Rightarrow f(x+a)$ is cts at $x = 0$

10 ~~To prove: if f is continuous at a, then $|f|$ is continuous at a.~~ (9

18 What are supremum & infimum of empty set?

Soln $\sup = -\infty$

$\inf = +\infty$

23 If A is non empty subset of R s.that $\sup A = \inf A$. Then what can you say about A.

Soln A must be singleton.

(4) Let $f: R \to R$ be a funcn & $x \in R$ is a fixed point of $f(x) = x$. Let f is diff & $f'(t) \neq 1 \ \forall \ t \in R$

To show f has atmost one fixed point

Pf On contrary let f has 2 fixed points t & s

ie $f(t) = t$ & $f(s) = s$.

Since f is diff on R $\therefore$ f is cts on R

$\therefore$ By Mean value thm, $\exists$ $c \in R$ s.that

$$f'(c) = \frac{f(t) - f(s)}{t - s}$$

$$= \frac{t - s}{t - s} = 1$$

$\Rightarrow f'(c) = 1$ which is a contradiction to given $f' \neq 1$

$\therefore$ f has almost one fixed point

(11) Construct a funcn which is discts everywhere except at 10 points

Soln

$$f(x) = \begin{cases} (x-1)(x-2) \cdots (x-10), & x \in Q \\ 0, & x \in R - Q \end{cases}$$

Clearly f is discts every where except at $x = 1, 2, \cdots 9, 10$.

(19) Let $f \to$ cts on $[a,b]$ & let $f(x)=0$, when x is rational.

<u>To prove</u>: $f(x)=0$ for every $x \in [a,b]$

<u>Pf</u> given $f(x)=0 \quad \forall \quad x \in \mathbb{Q} \cap [a,b]$

let $x \in \mathbb{R}/\mathbb{Q} \cap [a,b]$ (ie. irrationals)

then $\exists$ a sequence $\langle x_n \rangle$ in $\mathbb{Q} \cap [a,b]$ such that $\langle x_n \rangle \to x$ in (ie irrationals) $\mathbb{R}/\mathbb{Q} \cap [a,b]$

<u>Now</u> since $\langle x_n \rangle \longrightarrow x$

$\Rightarrow f(x_n) \to f(x)$ —— (i)

As $f(x)=0 \quad \forall \; x \in [a,b]$

$\Rightarrow f(x_n) = 0 \quad \{\because$ all x_n's $\in \mathbb{Q} \cap [a,b]\}$

$\Rightarrow$ ~~$0 \to x$~~

$\therefore f(x)=0$ (By (i))

(21) Given two non empty subsets A & B of $\mathbb{R}$ such that $\sup A = a$, $\sup B = b$.

We define $C = \{x+y \mid x \in A, y \in B\}$

<u>To show</u> $\sup C = a+b$.

<u>Soln</u> let $x \in A$ & $y \in B$

then $x \le a$ & $y \le b \Rightarrow x+y \le a+b \Rightarrow$ (i)

ie $z \le a+b$, $z \in C$ (By defn of C)

<u>Now</u> since a is lub of A $\Rightarrow a - \epsilon/2$ willn't be l.u.b of A

b B $\Rightarrow b - \epsilon/2$ B

$\Rightarrow a+b-\epsilon$ willn't be a l.u.b C.

or $z - \epsilon$ (ii)

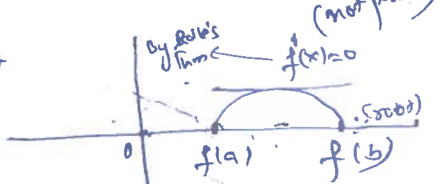
$\Rightarrow$ ~~$a+$~~ $a+b$ is l.ub of c

from (i) & (ii) ie. $\sup C = a+b$.

15 <u>To show</u> $x^{13}+7x^3-5=0$ has exactly one (real) root

Clearly root lies b/w $(0,1)$

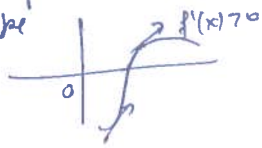
$\because f(0)f(1)<0$



<u>Now</u> $f'(x) = 13x^{12}+21x^2 > 0$

$\Rightarrow f(x)$ is rising

Clearly $f(x)$ is of type



$\Rightarrow$ slope will be +ve ie tangent to $f(x)$ will be in +ve x-axis dirn

17 Let $f:[a,b]\to R$ be differentiable

To show: f is const iff $f'(x)=0 \quad \forall\, x\in[a,b]$

Pf Since $f: \underset{(I)}{[a,b]}\to R$ is diff & let $f'(x)=0 \quad \forall\, x\in[a,b]$

To show f const

Since f is diff $\Rightarrow$ f is cts, we'll show $f(x)=f(a) \;\forall\, x\in I$

if $x\in I$, $x>a$ given ; By Mean Value thm to f on closed interval $[a,x]$,

we get a point c (depending on x, b/w a & x), such that

$$\frac{f(x)-f(a)}{x-a} = f'(c)$$

Since $f'(c)=0 \Rightarrow f(x)=f(a) \Rightarrow f$ is const

Conversely, $f(x)=f(a)$
$\Rightarrow f'(c)=0.$

19 If $0<x_1<1$, $x_{n+1}=1-\sqrt{1-x_n}$ for all $n\geq 1$

To show $\{x_n\}$ is dsing seq with limit 0.

Claim: $0<x_n<1$ for all $n\in N$

Induction As $n=1$, Nothing to prove

If $n=k$ holds ie. $0<x_k<1$.

then for $n=k+1$, we've $0<x_{k+1}=1-\sqrt{1-x_k}<1$ (By induction hypoth)
$\Rightarrow 0<x_{k+1}<1$ Claim proved

Using claim, we've

$$x_{n+1}-x_n = (1-x_n)-\sqrt{1-x_n} = \frac{x_n(x_n-1)}{(1-x_n)+\sqrt{1-x_n}} < 0 \quad \text{Since } 0<x_n<1$$

$\Rightarrow \{x_n\}$ is dsing seq.

Since $0<x_n<1$ for all $n\in N$ (ie. a monotonic bdd seq in R, is cgt)

$\therefore$ By Completeness of R

Hence $\{x_n\}$ is a cgt seq. let $\lim x_n = l$

$\Rightarrow l = 1-\sqrt{1-l} \Rightarrow (1-l)(l)=0 \Rightarrow l=0,1$

Since $\{x_n\}$ is dsing seq with $0<x_n<1 \quad \forall\, n\in N$

$\therefore \boxed{l=0}$

[Also $\lim\limits_{x\to 0} \frac{1-\sqrt{1-x}}{x}=\frac{1}{2}$ we've $\frac{x_{n+1}}{x_n}=\frac{1-\sqrt{1-x_n}}{x_n}\to\frac{1}{2}$]

16. Let f & g be funcns, cts on $[a,b]$
diff. on (a,b) & $f(a)=f(b)=0$

To show: $\exists$ a point $c \in (a,b)$ such that $g'(c)f(c)+f'(c)=0$

Soln define a funcn, $h(x)=f(x)e^{g(x)}$

then clearly $h(x)$ is also cts on $[a,b]$ & diff on (a,b)

Also $h(a)=f(a)e^{g(a)}=0$
$h(b)=f(b)e^{g(b)}=0$ $\left[\because \text{given } f(a)=f(b)=0\right]$

Then, by Rolle's thm $\exists$ $c \in (a,b)$ such that
$h'(c)=0$

ie. $f'(c)e^{g(c)}+f(c)e^{g(c)}g'(c)=0$

or $f'(c)+g'(c)f(c)=0$ Proved

20 $S = \{2^{-p} + 3^{-q} + 5^{-r}\}$ $\{p, q, r \in N\}$

$\sup S = \frac{1}{2} + \frac{1}{3} + \frac{1}{5}$ (when $p = q = r = 0$)

$\inf S = 0$ (when $p = q = r = \infty$)

22 To show $\lim_{n\to\infty} n^{1/n} = 1$.

Pf Since $n^{1/n} > 1$ for $n > 1$

$\therefore$ let $n^{1/n} = 1 + k$ for some $k > 0$, when $n > 1$

Hence $n = (1+k)^n$ for $n > 1$

$\therefore$ By Bernaulli thm, if $n > 1$, we've

$n = 1 + kn + \frac{1}{2}n(n-1)k^2 + \cdots \geq 1 + \frac{1}{2}n(n-1)k^2$

$\Rightarrow n \geq 1 + \frac{n(n-1)}{2}k^2$

$\Rightarrow (n-1) \geq \frac{n(n-1)}{2}k^2 \Rightarrow k^2 \leq \frac{2}{n}$

If $\epsilon > 0$, is given, from Archimedean property,

$\exists$ a natural no. 'N' s.that

$\frac{2}{n} < \epsilon^2$. It follows that if $n \geq \sup\{2, N\}$ then $\frac{2}{n} < \epsilon^2$ whence

$0 < n^{1/n} - 1 = k < \left(\frac{2}{n}\right)^{1/2} < \epsilon$

Since $\epsilon > 0$ is arbitrary $\Rightarrow$ $\boxed{\lim n^{1/n} = 1}$

25 To show (i) every cgt seq. is (ii) bdd, but converse not true.

first ~~let (i) ⇒ (ii)~~ To show (i) ⇒ (ii)

Let $\langle a_n \rangle$ be a sequence $\xrightarrow{cgs} l$

Let $\epsilon > 0$ be given.

Since $\{a_n\}$ cgs to l, $\therefore \exists$ a +ve integer m, such that

$|a_n - l| < \epsilon \quad \forall \; n \geq m$

$\Rightarrow l - \epsilon < a_n < l + \epsilon \quad \forall \; n \geq m$

let $t = \min\{a_1, a_2, \ldots, a_{m-1}, l - \epsilon\}$, $\Rightarrow t < a_n < T \;\forall n \in N$

$T = \max\{a_1, a_2, \ldots, a_{m-1}, l + \epsilon\}$ $\Rightarrow \{a_n\}$ is bdd

But Conversly let consider seq $\{a_n\} = (-1)^{n-1}$ bold, but not convergent.

$\{a_n\} = \{1, -1, 1, -1, \ldots\}$

26 To show : $\uparrow$ monotonic & bdd seq is cgt (7)

Pf Let $\{x_n\}$ be monot. $\uparrow$sing seq & let it is bdd

Let $\sup\{x_n\} = x$ Claim $\{x_n\} \to x$

$\because \sup\{x_n\} = x$

$\therefore$ for given $\epsilon > 0$ $\exists$ $m \in N$ s. that $x_m > x - \epsilon$

Since $\{x_n\}$ is mono $\uparrow$sing seq, sup → lub inf → glb.

(1) — $\therefore$ $x_n \geq x_m > x - \epsilon$ ($\because$ mono $\uparrow$sing) $\forall$ $n \geq m$

(2) — Also $x_n \leq x$ $\forall$ $n \in N$

(1) & (2) ⇒ $\therefore$ $x - \epsilon < x_n < x + \epsilon$

⇒ $|x_n - x| < \epsilon$ $\forall$ $n \geq m$

⇒ $\{x_n\} \to x$

Illy for mono ↓sing seq $\xrightarrow{cgs}$ infimum.

27 Let $f(0) = 0$, $f'(0) = 1$

To show $\lim_{x\to 0} \frac{1}{x}\{f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + \text{——} + f(\frac{x}{k})\} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}$.

Pf $\lim_{x\to 0} \frac{1}{x}\{f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + \text{——} + f(\frac{x}{k})\}$ $+ \frac{f(x/k)}{x}\}$

$= \lim_{x\to 0} \{\frac{f(x)}{x} + \frac{f(x/2)}{x} + \cdots$

$= \lim_{x\to 0} \{\frac{f(x) - f(0)}{x} + \frac{1}{2}\cdot\frac{f(x/2) - f(0)}{x/2} + \cdots + \frac{1}{k}\frac{f(x/k) - f(0)}{x/k}\}$

$= 1 + \frac{1}{2} + \frac{1}{3} + \text{—} + \frac{1}{k}$ $(\because f'(0) = 1)$

29 Investigate convergence of

$x_n = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \frac{1}{3^2+3} + \text{—} + \frac{1}{n^2+n}$ or $0 < \frac{1}{n^2+n} < \frac{1}{n^2} \to$ cgt ⇒ ans is also cgt

Soln $a_n = \frac{1}{n^2+n} = \frac{1}{n^2(1+\frac{1}{n})}$ $b_n = \frac{1}{n^2}$ ⇒ cgt ⇒ x_n cgt

let $b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ (finite)

28 Let $\{x_n\}$ be a sequence of strictly +ve real nos. such that (8
$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$. Then

Prove
(i) if $l<1$, then $\lim_{n\to\infty} x_n = 0$

Pf {Thm : If $X = \{x_n\}$ is a cgt sequence of real no's & if $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim \{x_n\} \geq 0$}

$\Rightarrow l \geq 0$
Let 'r' be a number such that $l < r < 1$

Let $\epsilon = r - l > 0$
$\exists$ a no. $k \in \mathbb{N}$ such that if $n \geq k$ then
$$\left| \frac{x_{n+1}}{x_n} - l \right| < \epsilon$$

if $n \geq k$, then $\frac{x_{n+1}}{x_n} < l + \epsilon = \cancel{l} + (r - \cancel{l}) = r$

$\therefore$ if $n \geq k$, we get
$$0 < x_{n+1} < x_n \cdot r < x_n r^2 < \cdots < x_k r^{n-k+1} \quad \left(x_k \cdot \frac{r^{n+1}}{r^k} \right)$$

let $c = \frac{x_k}{r^k}$ then

$$0 < x_{n+1} < c r^{n+1} \quad \text{for all } n \geq k$$

Since $0 < r < 1$. {$\because$ If $0<b<1$, then $\lim b^n = 0$}
$\Rightarrow \lim r^n = 0$ {$\because$ Let $x \in \mathbb{R}$. If $\{a_n\}$ is a seq of real no's with $\lim a_n = 0$ & if for some
ie. $\lim x_n = 0$ [If $\{x_n\}$ be a +ve real no's sequence of constant $c>0$ & some $m \in \mathbb{N}$, we've $|x_n - x| \leq c a_n$ $\forall n \geq m$
then $\lim x_n = x$.

(b) If $l > 1$, then $\lim_{n\to\infty} x_n = \infty$
Pf Since $l > 1$, we can find $r \in \mathbb{R}$ such that $1 < r < l$
As $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = l$ $\exists$ n_0 such that $\frac{x_{n+1}}{x_n} < l$ for all $n \geq n_0$
we get $n_0 \in \mathbb{N}$ such that $\cancel{x_{n+1}} > r x_n$ for all $n \geq n_0$

Illy $x_{n+n_0} > r^n x_{n_0}$.
Since $r > 1$, $\lim_{n\to\infty} r^n = \infty$ & $\therefore \lim_{n\to\infty} x_n = \infty$
(c) What will happen if $l = 1$.

(2) Let $x \in (a,b)$ & f is defined on (a,b). Consider

(a) $\lim_{h\to 0} |f(x+h) - f(x)| = 0$ (b) $\lim_{h\to 0} |f(x+h) - f(x-h)| = 0$

(i) To show: (a) $\Rightarrow$ (b)

Pf Since $\lim_{h\to 0} |f(x+h) - f(x)| = 0$

$\Leftrightarrow \lim_{h\to 0} |f(x-h) - f(x)| = 0$

Consider $|f(x+h) - f(x-h)| = |f(x+h) - f(x) + f(x) - f(x-h)|$

$\leq |f(x+h) - f(x)| + |f(x) - f(x-h)|$ (By a)

$\longrightarrow 0$ as $h \to 0$

So $\lim_{h\to 0} |f(x+h) - f(x-h)| = 0$

(ii) (b) $\not\Rightarrow$ (a) We contradict it by an example

Let $f(x) = \begin{cases} |x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$

then $\lim_{h\to 0} |f(0+h) - f(0-h)| = \lim_{h\to 0} |1-1| = 0$

But $\lim_{h\to 0} |f(0+h) - f(0)| = \lim_{h\to 0} ||h| - 1| = 1$ $\Rightarrow$ (b) $\not\Rightarrow$ (a)

(3) Let f is cts & defined on $\mathbb{R}$ & if $f(x) = f(x^2)$

To show: f is constant.

Pf Since $f(-x) = f((-x)^2) = f(x^2) = f(x)$ (given)

$\Rightarrow f$ is even function

To show f is const on $\mathbb{R}$, it is sufficient to show that f is const on $[0,\infty)$

Now Given any $x \in (0,\infty)$, since $f(x^2) = f(x)$ for all $x \in \mathbb{R}$

we've $f(x^{1/2^n}) = f(x)$ for all n

Hence $f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f(x^{1/2^n}) = f(\lim_{n\to\infty} x^{1/2^n})$ (By cty of f at 1)

($\because$ if $c > 0$ then $\lim c^{1/n} = 1$)

$= f(1)$ since $x \neq 0$

So $f(x) = f(1) = c$ for all $x \in (0,\infty)$

In addition, given a sequence $\{x_n\} \subseteq (0,\infty)$ such that $x_n \to 0$, we've

$c = \lim_{n\to\infty} c = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$ By cty of f at 0

$= f(0)$

$\Rightarrow f(0) = c$

$\Rightarrow f$ is constant

13 Given: f cts on $[a,b]$, differentiable on (a,b) & satisfies

$f^2(a) - f^2(b) = a^2 - b^2 \Rightarrow f^2(a) - a^2 = f^2(b) - b^2$ —(*)

To show: Equation $f'(x) f(x) = x$ has atleast one root in (a,b)

Solution: Let $h(x) = \frac{f^2(x)}{2} - \frac{x^2}{2}$

Since f is continuous on $[a,b]$

" " differentiable on (a,b)

$\Rightarrow$ (i) h is continuous on $[a,b]$ and (ii) differentiable on (a,b)

(iii) $h(a) = \frac{f^2(a) - a^2}{2}$ $\quad h(b) = \frac{f^2(b) - b^2}{2}$

$\therefore h(a) = h(b)$ (using *)

$\therefore$ All conditions of Rolle's theorem are satisfied

$\therefore \exists$ atleast one real $c \in (a,b)$ such that

$f'(x) = 0$ has atleast one root in (a,b)

i.e. $\frac{2 f(x) f'(x)}{2} - \frac{2x}{2} = 0$ has atleast one root in (a,b)

or $f(x) f'(x) = x$ has atleast one root in (a,b).