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Indian Institute of Technology Mandi
भारतीय प्रौद्योगिकी संस्थान मण्डी

Indian Institute of Technology Mandi

MA-201: B.Tech. II year

Odd Semester: 2011-12 **Exercise-6 Linear Algebra**

1. Which of the following subsets S of $\mathbb{R}^3$ are LI/LD over the field $\mathbb{R}$?
   (a) $S = \{(1,2,1),(-1,2,0),(5,-1,2)\}$
   (b) $S = \{(1,1,2),(-3,1,0),(1,-1,1),(1,2.-3)\}$
   (c) $S = \{(\frac{1}{2},\frac{1}{3},1),(0,0,0),(2,\frac{3}{4},-\frac{1}{3})\}$
   (d) $S = \{(1,0,0,0),(1,1,0,0),(1,1,1,1),(0,0,1,1)\}$
   (e) $S = \{(1,1,1,0),(3,2,2,1),(1,1,3,-2),(1,2,6,-5),(1,1,2,6,1)\}$
   (f) $S = \{(1,2,3,0),(-1,7,3,3),(1,-1,1,-1)\}$
   (g) $S = \{x^2-1, x+1, x-1\}$
   (h) $S = \{x, x^3-x, x^4+x^2, x+x^2+x^4+\frac{1}{2}\}$
   (i) $S = \{x, \sin x, \cos x\}$
   (j) $S = \{\sin x, \cos x, \sin x + 1\}$

2. Find a linearly independent subset A of S such that $[A] = [S]$, where S are given bellow:
   (a) $S = \{(1,0,0,0),(1,1,0,0),(1,1,1,1),(0,0,1,1)\}$
   (b) $S = \{(1,-1,2,0),(1,1,2,0),(3,0,0,1),(2,1,-1,0)\}$
   (c) $S = \{(1,1,1,0),(3,2,2,1),(1,1,3,-2),(1,2,6,-5),(1,1,2,6,1)\}$
   (d) $S = \{1, x+x^2, x-x^2, 3x\}$
   (e) $S = \{x, x^3-x, x^4+x^2, x+x^2+x^4+\frac{1}{2}\}$

3. Whenever a set S is LD, locate one of the vector that is in the span of the other. Where set S are
   (a) $S = \{(1,1,2),(-3,1,0),(1,-1,1),(1,2.-3)\}$
   (b) $S = \{(1,0,0,0),(1,1,0,0),(1,1,1,1),(0,0,1,1)\}$
   (c) $S = \{1, x+x^2, x-x^2, 3x\}$
   (d) $S = \{x, \sin x, \cos x\}$
   (e) $S = \{\ln x, \ln x^2, \ln x^3\}$

4. Let $S = \{(1,2,1),(1,1,-1),(4,5,-2)\}$. Determine which of the of the following vectors are in $[S]$:
   (a) (0,0,0) (b) (2,-1,-8) (c) (1,0,1)

5. Let $S = \{x^2, x^2+2x, x^2+2, 1-x\}$. Determine which of the of the following vectors are in $[S]$.

(a) $2x^3+3x^2+3x+7$
(b) x^4+7x+2
(c) $3x^2+x+5$
(d) $x^3-\frac{3}{2}x^2+\frac{x}{2}$
(e) $3x+2$
(f) x^3+x^2+2x+3

6. If S is a nonempty subset of a vector space V, prove that $[S]=S$ iff S is a subspace of V.

7. What is the span of

   (a) x-axis and y-axis in $\mathbb{R}^3$?
   (b) x-axis and xy-plane in $\mathbb{R}^3$?
   (c) xy-plane and yz-plane in $\mathbb{R}^3$?
   (d) x-axis and the plane $x+y=0$ in $\mathbb{R}^3$?

8. Find the intersection of the given sets **U** and **W** and determine whether it is a subspace.

   (a) $\mathbf{U}=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\geq 0\}$, $\mathbf{W}=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1\leq 0\}$
   (b) $\mathbf{U}=\{f\in\mathscr{C}(-2,2)\mid f(-1)=0\}$, $\mathbf{W}=\{f\in\mathscr{C}(-2,2)\mid f(1)=0\}$
   (c) $\mathbf{U}=\{f\in\mathscr{C}(-2,2)\mid \lim_{x\to 1}f(x)=0\}$ $\mathbf{W}=\{f\in\mathscr{C}(-2,2)\mid \lim_{x\to 2}f(x)=1\}$
   (d) $\mathbf{U}=\mathscr{P}$, $\mathbf{W}=\{f\in\mathscr{C}(-\infty,\infty)\mid f(-x)=f(x)\}$

9. Describe $\mathbf{A}+\mathbf{B}$ for the given subsets **A** and **B** of $\mathbb{R}^2$ and determine in each case whether it is a subspace or just a subset of $\mathbb{R}^2$.

   (a) $\mathbf{A}=\{(1,2),(0,1)\}, \mathbf{B}=\{(1,0),(3,-1)\}$
   (b) $\mathbf{A}=\{(\frac{1}{2},\frac{2}{3}\}$, $\mathbf{B}=$ segment joining $(-1,1)$ and $(2,3)$
   (c) $\mathbf{A}=\{(3,7), \mathbf{B}=\{t(-1,2)\mid 0\leq t\leq 1\}$
   (d) $\mathbf{A}=\{(2,4), \mathbf{B}=\{(x,y)\mid 2x+3y=1\}$
   (e) $\mathbf{A}=\{t(3,4)\mid 0\leq t\leq 1\}, \mathbf{B}=\{t(2,5)\mid 1\leq t\leq 2\}$
   (f) $\mathbf{A}=\{t(1,0)\mid \text{t is a scalar}\leq 1\}, \mathbf{B}=[(1,2)]$

10. Describe $\mathbf{A}+\mathbf{B}$ for the given subsets **A** and **B** of $\mathbb{R}^3$. Determine in each case whether $\mathbf{A}+\mathbf{B}$ is a subspace or just a subset of $\mathbb{R}^3$.

    (a) $\mathbf{A}=\{(1,2,1)\}, \mathbf{B}=\{t(1,2,0)\mid \text{t is a scalar}\}$
    (b) $\mathbf{A}=\{(1,-3,4)\}, \mathbf{B}=[(1,2,3),(0,0,1)]$
    (c) $\mathbf{A}=\{(\frac{1}{2},\frac{2}{3},1)\}, \mathbf{B}=\{(x,y,z)\mid 2x+3y+z=0\}$
    (d) $\mathbf{A}=[(1,0,-1)], \mathbf{B}=[(2,5,8),(2,3,4)]$

11. if **U** and **W** are two subspace of a vector space V, prove that $\mathbf{U}+\mathbf{W}=\mathbf{U}$ iff $\mathbf{W}\subset\mathbf{U}$.

12. Let **A** and **B** be two non-empty finite subsets of a vector space **V**. Then prove that

    (a) $[\mathbf{A}\bigcap\mathbf{B}]\subset[\mathbf{A}]\bigcap[\mathbf{B}]$
    (b) $[\mathbf{A}\bigcup\mathbf{B}]=[\mathbf{A}]+[\mathbf{B}]$.

Solution of Exercise-6- Linear Algebra

Question 1

(a) $S=\{(1,2,1),(-1,2,0),(5,-1,2)\}$

Here, we have $A=\begin{bmatrix}1&2&1\\-1&2&0\\5&-1&2\end{bmatrix}$; converting A into echelon form

we have $\sim\begin{bmatrix}1&2&1\\0&4&1\\0&-11&-3\end{bmatrix}\sim\begin{bmatrix}1&2&1\\0&4&1\\0&0&-3+\frac{11}{4}\end{bmatrix}\Rightarrow$ all the Rows of A are L.I. $\Rightarrow$ S is a L.I. set

(b) → set S is L.D. because in R^3 ~~(only three)~~ Cardinality of a Largest L.I. set is 3.

(c) → Set S is L.D. because Any set S containing zero vector is L.D.

(d) → Do as we have done in (a)

OR Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ s.t.

$\alpha(1,0,0,0)+\beta(1,1,0,0)+\gamma(1,1,1,1)+\delta(0,0,1,1)=(0,0,0,0)$.

$\Rightarrow\begin{array}{l}\alpha+\beta+\gamma=0\\0\cdot\alpha+\beta+\gamma+0\delta=0\\0\cdot\alpha+0\cdot\beta+\gamma+\delta=0\\0\cdot\alpha+0\cdot\beta+\gamma+\delta=0\end{array}\Rightarrow\begin{array}{l}\alpha+\beta+\gamma=0\\\beta+\gamma=0\\+\gamma+\delta=0\\\gamma+\delta=0\end{array}\Rightarrow$ $\alpha=0$ $\exists$ one free Variable $\Rightarrow$ one of the constants may take any value $\Rightarrow$ S is L.D.

OR Here $A=\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&1&1&1\\0&0&1&1\end{bmatrix}\sim\begin{bmatrix}1&1&1&1\\1&1&0&0\\0&0&1&1\\1&0&0&0\end{bmatrix}\sim\begin{bmatrix}1&1&1&1\\0&0&-1&-1\\-1&-1&0&0\\0&-1&-1&1\end{bmatrix}$

$\sim\begin{bmatrix}1&1&1&1\\0&0&-1&-1\\-1&-1&-1&-1\\0&-1&0&0\end{bmatrix}\sim\begin{bmatrix}1&1&1&1\\0&0&-1&-1\\0&0&0&0\\0&-1&0&0\end{bmatrix}\sim\begin{bmatrix}1&1&1&1\\0&-1&0&0\\0&0&-1&-1\\0&0&0&0\end{bmatrix}$

$\sim |A|=0\Rightarrow$ S is L.D.

(e) → S is L.D. because This is set of 5 elts in R^4.

(f) → check wheather they are L.I. or L.D. ~~as (a)~~ as in (a).

(g) Let $\alpha,\beta,\gamma\in\mathbb{R}$, s.t. $\alpha(x^2-1)+\beta(x+1)+\gamma(x-1)=0+0\cdot x+0x^2$.

$\Rightarrow\alpha x^2+(\beta+\gamma)x-\alpha+\beta-\gamma=0+0\cdot x+0x^2$

Comparing the Coefficients of like powers we get $\alpha=0$, $\beta+\gamma=0$ & $-\alpha+\beta-\gamma=0$

$\Rightarrow\alpha=\beta=\gamma=0\Rightarrow$ S is L.I.

(h) → Do as we have done in (g)

(i) $S = \{x, \sin x, \cos x\}$

take $\alpha, \beta, \gamma \in \mathbb{R}$ s.t. $\alpha x + \beta \sin x + \gamma \cos x = 0$ — (1)

at $\underline{x=0}$ $\Rightarrow \boxed{\gamma = 0}$

then (1) $\Rightarrow$ $\alpha x + \beta \sin x = 0$ — (2)

at $x = \pi/2$, $\alpha \cdot \frac{\pi}{2} + \beta = 0$ — (3)

differentiating (2) we get

$\alpha + \beta \cos x = 0$

$\underline{\text{at } x = \pi/2}$, $\alpha \cdot \frac{\pi}{2} = 0 \Rightarrow \boxed{\alpha = 0}$. & then (3) $\Rightarrow \beta = 0$

$\Rightarrow$ all $\alpha = \beta = \gamma = 0$

$\Rightarrow$ S is L.I

Similarly we can solve (j) as we have done in (i).

(1)

Question (2)

(a) Here $(1,1,1,1) = 0\cdot(1,0,0,0) + 1\cdot(1,1,0,0) + 1(0,0,1,1)$

$\Rightarrow$ $(1,1,1,1)$ is in linear Combination of $(1,0,0,0)$, $(1,1,0,0)$ & $(0,0,1,1)$.

$\Rightarrow A = \{(1,0,0,0), (1,1,0,0), (0,0,1,1)\}$ s.t. $[A] = [S]$.

Note similarly we can search for those vectors which are L.C of the vectors left & the deleting those vectors from the set we will get the required set A s.t

$$\boxed{[A] = [S]}.$$

Question (3)

(c) $S = \{\ln x, \ln x^2, \ln x^3\}$

$\ln x^2 = 2\ln x$ $\Rightarrow$ $\ln x^2$ & $\ln x^3$ both are L.C of

$\ln x^3 = 3\ln x$

the $\ln x$ $\Rightarrow$

4 (a) Let $\alpha, \beta, \gamma \in \mathbb{R}$ s.t

$$\alpha(1,2,1) + \beta(1,1,-1) + \gamma(4,5,-2) = (0,0,0).$$

$$\begin{aligned} \alpha+\beta+4\gamma &= 0 \\ 2\alpha+\beta+5\gamma &= 0 \\ \alpha-\beta-2\gamma &= 0 \end{aligned} \Rightarrow \begin{aligned} -\alpha-\gamma &= 0 \\ \& \; 3\alpha+2\gamma &= 0 \end{aligned} \Rightarrow \boxed{\alpha=0} \Rightarrow \beta=0 \\ \Rightarrow \boxed{\gamma=0}.$$

$\Rightarrow (0,0,0) \in [S]$.

Note Similarly we can check for other vectors.

Question 5 you can solve this question as you have solved the question 4.

Question 6 Let $S \neq \phi$ & S is a subspace of V.
We know that $S \subseteq V$ & $\forall$ $S \subseteq [S]$ as $[S]$ is collection of all possible Linear Combination of S i.e. & elements of S are nothing some special linear combination of elts of S. We can see

Now If S is a subspace of V. then
$\forall$ $x \in S$, $x \in L(S) = [x]$ as $L(S)$ is the collection of all the linear Combination of S. So.

$$\Rightarrow \quad S \subseteq [S] \text{ —(i)}$$

& S is a subspace, Hence Here if $S = \{x_1 \text{ — } x_n\}$

then $[S] = \left\{\sum_{i=1}^{n} \alpha_i x_i \in S \; ; \; \alpha_i \in F\right\}$

But $\sum_{i=1}^{n} \alpha_i x_i \in S$ as S is a subspace.

$\Rightarrow \quad [S] \subseteq S$ —(ii) from (i) & (ii) we have

$$\boxed{[S] = S}.$$

Conversely If $[S] = S$ then we know that
$[S] \subseteq S$ & $S \subseteq [S]$. & $[S]$ is always a subspace of V as $[S] \neq \phi$ as $0 \in [S]$.
& $\forall$ $x, y \in S$, $\alpha x + \beta y \in S$. as

$$S = \left\{\sum_{i=1}^{n} \alpha_i x_i \; ; \; x_i \in S\right\}.$$

Question (7) (a) x-axis is $\alpha(1,0)$; $\alpha \in \mathbb{R}$
y-axis is $\beta(0,1)$; $\beta \in \mathbb{R}$.

Then $\text{span}(x \,\&\, y \text{ axis}) = \{\alpha(1,0) + \beta(0,1) ; \alpha, \beta \in \mathbb{R}\}$

$= \{(\alpha, \beta) ; \alpha, \beta \in \mathbb{R}\}$

$\equiv \mathbb{R}^2$.

Ans is $\mathbb{R}^2 \rightarrow$ whole the ~~spa~~ plane.

Note – Similarly after seeing all possible Linear Combinations of the given set we see what is the actual Linear span (smallest subspace containing the set).

Question (8) (a) $U = \{(x_1, x_2) \in \mathbb{R}^2\}$, $W = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0\}$

Here clearly $U \cap W = \{(0,0)\}$. & It is clearly a subspace of $\mathbb{R}^2$ called zero subspace.

~~Qu Note~~ (b) $U \cap W = \{f \in [-2,2] \mid f(1) = f(-1) = 0\}$.

Note – similarly we can proceed for problems (c) & (d).

Question (9) (a) $A = \{(1,2), (0,1)\}$, $B = \{(1,0), (3,-1)\}$.

$A+B = \{(2,2), (4,1), (1,1), (3,0)\}$.

$A+B$ is Not a subspace of $\mathbb{R}^2$; because $(0,0) \notin A+B$.

10 (a) $A = \{(1,2,1)\}$, $B = \{t(1,2,0) \mid t \text{ is a scalar}\}$

Here $A+B = \{(1,2,1) + t(1,2,0) \mid t \text{ is a scalar}\}$

Here clearly $(0,0,0) \notin A+B \Rightarrow A+B$ is Not a subspace

Note- Similarly we can solve other parts also

11 Let $U+W = U$ then to prove $W \subset U$

Let $x+y \in U+W = U \Rightarrow x+y = z, z \in U$ & $x \in U, y \in V$.

$\Rightarrow x+y = z \Rightarrow y = z-x \in U \Rightarrow y \in U$

thus $x \in U, y \in U \Rightarrow x+y \in U+W$

$\Rightarrow$ ~~x+y~~ $y \in W \Rightarrow y \in U \Rightarrow \boxed{W \subseteq U}$

Now when $W \subseteq U \Rightarrow \forall x \in W \Rightarrow x \in U$

Now Let $y+x \in U+W \Rightarrow y+x \in U$ as $y \in U$ & $x \in U$

$\Rightarrow \boxed{U+W \subseteq U}$ & (i)

if $x \in U \Rightarrow x \in U+W \Rightarrow \boxed{U \subseteq U+W}$ (ii)

& (i) & (ii) together $\Rightarrow \boxed{U = U+W}$

Proved.

12 (i) Let $A \neq \phi$, $B \neq \phi$. $\subseteq V(F)$.

$A \cap B = \{x_i ; x_i \in A \text{ \& } x_i \in B\}$

$\Rightarrow [A \cap B] = \left\{ \sum \alpha_i x_i ; x_i \in A, \text{ \& } x_i \in B, \alpha_i \in F \right\}$

thus $\sum \alpha_i x_i \in [A]$ & $\sum \alpha_i x_i \in [B]$

$\Rightarrow \boxed{[A \cap B] \subseteq [A] \cap [B].}$ Proved

(ii) $A \cup B = \{x_i ; x_i \in A \text{ or } x_i \in B\}$

$\Rightarrow [A \cup B] = \{\sum \alpha_i x_i ; x_i \in A \text{ or } x_i \in B\}$

$\Rightarrow \sum \alpha_i x_i \in A$ or $\sum \alpha_i x_i \in B = \sum \alpha_i x_i \in [A] \cup [B]$

$\Rightarrow [A \cup B] \subseteq [A] + [B]$ because $[A \cup B] = A + B$.

& we know that

$\forall x \in \{[A] + [B]\} \Rightarrow x \in [A \cup B]$.

$\Longrightarrow \quad [A] + [B] \subseteq [A \cup B]$.

thus $\boxed{[A] + [B] = [A \cup B]}$ Ans.