Example: Let $T : \mathscr{C}(a,b) \to \mathbb{R}$ is defined as $T(f) = \int_a^b f(x)\,dx$. Here the range is the whole of $\mathbb{R}$, since every real number can be obtained as the algebraic area under some curve $y = f(x)$ from a to b. Therefore, it is an onto map. The kernel is the set of all those functions $'f'$ for which the area under the curve $y = f(x)$ from a to b is zero. It is difficult to say anything more than this about the kernel.

Example: Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix}$. $N(T)$ is the x_3-axis. So all points on the x_3-axis go into $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. So this map is not one-one.

Example: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix}$.

As $N(T) = \left[\left(\begin{array}{r} 1 \\ 1 \\ -1 \end{array}\right)\right]$. So, many points go into $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This again means T is not One-one.

Theorem *Let $(U, \oplus, \odot)$ and $(V, \boxplus, \boxdot)$ be two vector spaces over the* ***same*** *field $(\mathbb{F}, \oplus, \odot)$ and $T : U \to V$ be a linear transformation, Then*

1. *If T is one-one and $u_1, u_2, \ldots, u_n$ are linearly independent vectors of U, then $T(u_1), T(u_2), \ldots, T(u_n)$ are LI.*
2. *If $v_1, v_2, \ldots, v_n$, are linearly independent vectors of $R(T)$ and $u_1, u_2, \ldots u_n$, are vectors of U such that $T(u_1) = v_1, T(u_2) = v_2, \ldots T(u_n) = v_n$ then $u_1, u_2, \ldots u_n$ are LI.*

Proof:

1. Let T be one-one and $u_1, u_2, \ldots, u_n$ be linearly independent vectors in U. To prove that $T(u_1), T(u_2), \ldots, T(u_n)$ are LI, we assume that
$$\alpha_1 \boxdot T(u_1) \boxplus \alpha_2 \boxdot T(u_2) \boxplus \ldots \boxplus \alpha_n \boxdot T(u_n) = 0_V \tag{I}$$
or,
$$T(\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n) = 0_V$$
since T is linear.
So,
$$\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n = 0_U$$
since T is one-One.
But $u_1, u_2, \ldots, u_n$ are LI.
$$\Rightarrow \alpha_1 = \alpha_2 \ldots = \alpha_n = 0_{\mathbb{F}} \tag{II}$$
Now, from (I) and (II)
$$\alpha_1 \boxdot T(u_1) \boxplus \alpha_2 \boxdot T(u_2) \boxplus \ldots \boxplus \alpha_n \boxdot T(u_n) = 0_V \Rightarrow \alpha_i = 0_{\mathbb{F}}, i = 1, 2, \ldots n$$
$\Rightarrow T(u_1), T(u_2), ..., T(u_n)$ are LI.
2. Let $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ be as stated above in the statement of the theorem. To prove that $u_1, u_2, \ldots, u_n$ are LI, suppose
$$\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n = 0_U$$
Since T is linear, we have
$$T(\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n) = T(0_U) = 0_V \tag{III}$$
or,
$$\alpha_1 \boxdot T(u_1) \boxplus \alpha_2 \boxdot T(u_2) \boxplus \ldots \boxplus \alpha_n \boxdot T(u_n) = 0_V$$
or,
$$\alpha_1 \boxdot v_1 \boxplus \alpha_2 \boxdot v_2 \boxplus \oplus \ldots \boxplus \alpha_n \boxdot v_n = 0_V$$
But $v_1, v_2, \ldots, v_n$ are LI.
$$\Rightarrow \alpha_1 = \alpha_2 \ldots = \alpha_n = 0_{\mathbb{F}} \tag{IV}$$
Now, from (III) and (IV)
$$\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \oplus \ldots \oplus \alpha_n \odot u_n = 0_U \Rightarrow \alpha_i = 0_{\mathbb{F}}, i = 1, 2, \ldots n$$
$\Rightarrow u_1, u_2, \ldots, u_n$ are LI.

Theorem: Let $T: U \to V$ be a linear map. Then

(i) $R(T)$ is a subspace of V

(ii) $N(T)$ is a subspace of U

(iii) T is one-one iff $N(t)$ is a zero subspace of U

(iv) If $[u_1, u_2, \ldots, u_n] = U$, then $R(T) = [T(u_1), T(u_2), \ldots, T(u_n)]$.

(v) If U is finite-dimensional, then $dimR(T) \le dimU$.

Proof:

(i) Let $v_1, v_2 \in R(T)$. Then there exist vectors $u_1, u_2, \in U$ such that $T(u_1) = v_1$ and $T(u_2) = v_2$. So

$$v_1 \boxplus v_2 = T(u_1) \boxplus T(u_2) = T(u_1 \oplus u_2)$$

Since T is linear. But $u_1 \oplus u_2 \in U$, since U is a vector space. Hence, $v_1 \boxplus v_2$ is the image of an element of U. So $v_1 \boxplus v_2 \in R(T)$. In the same way, for all scalars $\alpha \in \mathbb{F}$,

$$\alpha \boxdot v_1 = \alpha \boxdot T(u_1) = T(\alpha \odot u_1)$$

Since T is linear. But $\alpha \odot u_1 \in U$, because U is a vector space. Hence, $\alpha \boxdot v_1 \in R(T)$. Thus, $R(T)$ is a subspace of V.

(ii) Let is $u_1, u_2 \in N(T)$. Then $T(u_1) = O_V$ and $T(u_2) = 0_V$, because this is precisely. the meaning of their being in $N(T)$. Now

$$\begin{aligned} T(u_1 \oplus u2) &= T(u_1) \boxplus T(u_2), \{\text{Since T is linear}\} \\ &= 0_V \boxplus 0_V \\ &= 0_V \end{aligned}$$

which shows that $u_1 \oplus u_2 \in N(T)$. Similarly, for all scalars $\alpha \in \mathbb{F}$, we have

$$\begin{aligned} T(\alpha \odot u_1) &= \alpha \odot T(u_1), \{\text{Since T is linear}\} \\ &= \alpha \odot 0_V \\ &= 0_V \end{aligned}$$

Which shows that $\alpha \odot u_1 \in N(T)$. Thus, $N(T)$ is a subspace of U.

(iii) Suppose T is one-one. Then $T(u) = T(v) \Rightarrow u = v$. If $U \in N(T)$, then $T(u) = 0_V = T(0_U)$. Therefore, $u = 0_U$. This means no nonzero vector u of U can belong to $N(T)$. Since 0_U in any case belongs to $N(T)$, it follows that $N(T)$ contains only 0_U and nothing else. Hence, $N(T)$ is the zero subspace of U.

Conversely, suppose $N(T) = 0_U$. Then, to prove that T is one-one, we have to prove that $T(u) = T(v) \Rightarrow u = v$. Suppose $T(u) = T(v)$. Then $T(u \ominus v) = T(u) \boxminus T(v) = 0_V$

So $u - v \in N(T) = O_U$. So $uv = 0_U$, i.e. $u = v$. This proves that T is one one.

(iv) Let $[u_1, u_2, \ldots, u_n] = U$. Then each vector u can be expressed as a linear combination of vectors $u_1, u_2, \ldots, u_n$.
The vectors $T(u_1), T(u_2) \ldots T(u_n) \in R(T)$. So, obviously, $[T(u_1), T(u_2), \ldots, T(u_n)] \subset R(T)$.
Let $v \in R(T)$. Then there exists a vector $u \in U$ such that $T(u) = v$. Since $u \in U = [u_1, u_2, \ldots, u_n]$, we have

$$u = \alpha_1 u_1 \oplus \alpha_2 u_2 \oplus \ldots \oplus \alpha_n u_n$$

Therefore, $v = T(u) = T(\alpha_1 u_1 \oplus \alpha_2 u_2 \oplus \ldots \oplus \alpha_n u_n)$
So, $v \in [T(u_1), T(u_2), \ldots, T(u_n)]$. This proves that

$$R(T) = [T(u_1), T(u_2), \ldots, T(u_n)]$$

(v) If U is finite dimensional vector space then from Rank-Nullity theorem we have

$$dimR(T) + dimN(T) = dimU$$

From above its clear that $dimR(T) \le dimU$ and equality holds when $N(T)$ is the zero subspace of U.

Example: Prove that the linear map $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(e_1) = e_1 - e_2, T(e_2) = 2e_2 + e_3, T(e_3) = e_1 + e_2 + e_3$ is neither one-one nor onto.

Solution: Given that

$$T(e_1) = e_1 - e_2 \Rightarrow T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$T(e_2) = 2e_2 + e_3 \Rightarrow T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

$$T(e_3) = e_1 + e_2 + e_3 \Rightarrow T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

To find a basis a basis for $R(T)$ we need to set which spans $R(T)$ and is LI also.
Since $[e_1, e_2, e_3] = \mathbb{R}^3$(= U the domain of T). Hence by previous theorem

$$R(T) = [T(e_1), T(e_2), T(e_3)] = \left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right].$$

It only left to check linear independence of the set $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Let,

$$\alpha_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & & & + & x_3 \\ -x_1 & + & 2x_2 & + & x_3 \\ & & x_2 & + & x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} \alpha_1 & & & + & \alpha_3 & = & 0 \\ -\alpha_1 & + & 2\alpha_2 & + & \alpha_3 & = & 0 \\ & & \alpha_2 & + & \alpha_3 & = & 0 \end{matrix}$$

Solving the above equation , we get $\alpha_1 = \alpha_2 = -\alpha_3$. Choosing $\alpha_3 = -1 \Rightarrow \alpha_2 = \alpha_1$

$$\Rightarrow 1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since a nonzero combination of the vectors $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is equal to the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Hence the set $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is LD.

Any one of the three vectors may be removed as all are having non zero coefficients (we will choose the third vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ arbitrarily).

This may be explained in the following way also.
Since,

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, $R(T) = \left[\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \right] \Rightarrow R(T) = 2$. Since $dim(\mathbb{R}^3) = 3$, $R(T)$ is a proper subset of $\mathbb{R}^3$ i.e. $R(T) \subset \mathbb{R}^3$. Hence, T is not onto.

To prove that T is not one-one, we check $N(T)$. By definition

$$N(T) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \; T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\Rightarrow T \left[x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1 & & +x_3 \\ -x_1 & +2x_2 & +x_3 \\ & x_2 & +x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} x_1 & & +x_3 & = 0 \\ -x_1 & +2x_2 & +x_3 & = 0 \\ & x_2 & +x_3 & = 0 \end{matrix}$$

Solving these, we get $x_1 = x_2 = -x_3$. Therefore,

$$N(T) = \left\{ \begin{pmatrix} x_1 \\ x_1 \\ -x_1 \end{pmatrix} \middle| \; x_1 \in \mathbb{R} \right\} = \left[\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right] \Rightarrow dim(N(T)) = 1$$

Hence, by last theorem T is not one-one.

Theorem *(Rank-Nullity Theorem) Let $T : U \to V$ be a linear map and U a finite-dimensional vector space. Then*

$$dim\ R(T) + dim\ N(T) = dim\ U$$

In other words,

$$r(T) + n(T) = dim\ U$$

or,

$$rank + nullity = dimension\ of\ the\ domain\ space.$$

Proof: $N(T)$ is a subspace of a finite-dimensional vector space U. Therefore, $N(T)$ is itself a finite-dimensional vector space. Let dim $N(T) = n(T) = n$ and dim $U = p(p \geq n)$.

Let $B = \{u_1, u_2, \ldots, u_n\}$ be a basis for $N(T)$.

Since $u_i \in N(T)$, therefore $T(u_i) = 0_V \forall i = 1, 2, \ldots, n$. B is LI in $N(T)$ and therefore in U. Extend this to a linearly independent set B' of U to a form basis for U.

Let $B' = \{u_1, u_2, \ldots, u_n, u_{n+1}, \ldots, u_p\}$ be a basis for U.

Consider the set $A = \{T(u_{n+1}), T(u_{n+2}), \ldots, T(u_p)\}$. Obviously $A \subseteq V$

We shall now prove that A is a basis for $R(T)$. Observe that, if this is proved, the proof of the theorem is over. Because, this means

dim $R(T) = p - n =$ dim U - dim $N(T)$,

It is therefore enough to prove

1. $[A] = R(T)$, and
2. A is LI.

1. Since $[B'] = U \Rightarrow R(T) = [T(u_1), T(u_2), \ldots, T(u_n), T(u_{n+1}), \ldots, T(u_p)]$.

   But $T(u_i) = 0_V$ for $i = l, 2, \ldots, n$. Hence

$$R(T) = [T(u_{n+1}), T(u_{n+2}), ..., T(u_p)] = [A]$$

2. Consider

$$\alpha_{n+1} \boxdot T(u_{n+1}) \boxplus \alpha_{n+2} \boxdot T(u_{n+2}) \boxplus \ldots \boxplus \alpha_p \boxdot T(u_p) = 0_V.$$

   Using the fact that T is linear, we get

$$T(\alpha_{n+1} \odot u_{n+1} \oplus \alpha_{n+2} \odot u_{n+2} \oplus \ldots \oplus \alpha_p \odot u_p) = 0_V.$$

   which means that

$$\alpha_{n+1} \odot u_{n+1} \oplus \alpha_{n+2} \odot u_{n+2} \oplus \ldots \oplus \alpha_p \odot u_p \in N(T).$$

   Therefore, $\alpha_{n+1} \odot u_{n+1} \oplus \alpha_{n+2} \odot u_{n+2} \oplus \ldots \oplus \alpha_p \odot u_p$ is a unique linear combination of the basis B for $N(T)$. Thus,

$$\alpha_{n+1} \odot u_{n+1} \oplus \alpha_{n+2} \odot u_{n+2} \oplus \ldots \oplus \alpha_p \odot u_p = \beta_1 \odot u_1 \oplus \beta_2 \odot u_2 \oplus \ldots \oplus \beta_n \odot u_n$$

   i.e.

$$\beta_1 \odot u_1 \oplus \beta_2 \odot u_2 \oplus \ldots \oplus \beta_n \odot u_n \ominus \alpha_{n+1} \odot u_{n+1} \ominus \alpha_{n+2} \odot u_{n+2} \ominus \ldots \ominus \alpha_p \odot u_p = 0_U$$

   B' being a basis for U is LI. Therefore,
   $\beta_1 = \beta_2 \odot \ldots \beta_n = \alpha_{n+1} = \alpha_{n+2} \ldots = \alpha_p = 0_{\mathbb{F}}$

   Hence,

$$\alpha_{n+1} \boxdot T(u_{n+1}) \boxplus \alpha_{n+2} \boxdot T(u_{n+2}) \boxplus \ldots \boxplus \alpha_p \boxdot T(u_p) = 0_V.$$
$$\Rightarrow \alpha_{n+1} = \alpha_{n+2} \ldots = \alpha_p = 0_{\mathbb{F}}.$$

   Hence, A is LI.

Example: Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear map defined by

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$
$$T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, T(e_4) = T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then verify that $r(T)+n(T) = \dim U(=\mathbb{R}^4) = 4$.

Solution:

We know that $R(T) = [T(e_1), T(e_2), T(e_3), T(e_4)] = \left[\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right]$

The set containing four vectors $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ is LD, because a set containing $n+1$ vectors in an n dimensional vector space is always LD. Here $n=3$ as $(\dim \mathbb{R}^3 = 3)$.

Let

$$\alpha_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\-1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\0\\0\end{pmatrix} + \alpha_4\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

$$\Rightarrow \begin{bmatrix} \alpha_1 & + & \alpha_2 & + & \alpha_3 & + & \alpha_4 \\ & & \alpha_2 & - & \alpha_3 & & \\ \alpha_1 & + & \alpha_2 & + & \alpha_3 & & \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

$$\Rightarrow \begin{array}{ccccccccc} \alpha_1 & + & \alpha_2 & + & \alpha_3 & + & \alpha_4 & = & 0 \\ \alpha_1 & - & \alpha_2 & & & & & = & 0 \\ \alpha_1 & + & \alpha_2 & + & & & \alpha_4 & = & 0 \end{array}.$$

Solving the above equation , we get $\alpha_3 = 0, \alpha_1 = \alpha_2, \alpha_4 = -2\alpha_2$. Choosing $\alpha_2 = 1 \Rightarrow \alpha_1 = 1, \alpha_4 = -2$

$$1\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\-1\\1\end{pmatrix} + 0\begin{pmatrix}1\\0\\0\end{pmatrix} - 2\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Since a nonzero combination of the vectors $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ is equal to the zero vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$. Hence the set $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$ is LD.

Any one of the **first, second or fourth** vectors may be removed as all of **first, second or fourth** are having non zero coefficients (we will choose the fourth vector $\begin{bmatrix}1\\0\\1\end{bmatrix}$ out of **first, second or fourth** arbitrarily).

This may be explained in the following way also.

Since,

$$1\begin{pmatrix}1\\1\\1\end{pmatrix} + 1\begin{pmatrix}1\\-1\\1\end{pmatrix} + 0\begin{pmatrix}1\\0\\0\end{pmatrix} = 2\begin{pmatrix}1\\0\\1\end{pmatrix}.$$

So that,

$$R(T) = \left[\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right].$$

So that

$$R(T) = \left[\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right].$$

To check whether the set $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ is LI, we suppose that

$$\alpha_1\begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}1\\-1\\1\end{pmatrix} + \alpha_3\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix} \alpha_1 & +\alpha_2 & +\alpha_3 \\ \alpha_1 & -\alpha_2 & \\ \alpha_1 & +\alpha_2 & \end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

Solving this, we get $\alpha_1 = 0 = \alpha_2 = \alpha_3$.

Hence the set $\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ are LI and $\dim R(T) = r(T) = 3$.

Now to find $N(T)$, we suppose that $T(u) = 0_{\mathbb{R}^3} = \begin{pmatrix}0\\0\\0\end{pmatrix}$.

If

$$u = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

then,

$$T(u) = T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = T \left(x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Now,

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 T \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 T \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 T \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 & +x_2 & +x_3 & +x_4 \\ x_1 & -x_2 & & \\ x_1 & +x_2 & + & x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving this, we get $x_1 = x_2 = -\frac{x_4}{2}, x_3 = 0$. So,

$$N(T) = \left\{ \begin{pmatrix} x_1 \\ x_1 \\ 0 \\ -2x_1 \end{pmatrix} \middle| \; x_1 \in \mathbb{R} \right\} = \left[\begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix} \right] \Rightarrow dim(N(T)) = 1$$

So $n(T) = \dim N(T) = 1$. Hence, $r(T) + n(T) = 3 + 1 = 4$, and the theorem is verified.