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Source: *Mathematics Magazine*, Vol. 57, No. 2 (Mar., 1984), pp. 93-94

Published by: [Mathematical Association of America](#)

Stable URL: <http://www.jstor.org/stable/2689590>

Accessed: 31/01/2011 09:50

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# The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof

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One of the most simple and successful techniques for solving systems of linear equations is to reduce the coefficient matrix of the system to **reduced row echelon form**. This is accomplished by applying a sequence of elementary row operations (see, e.g., [1, p. 5]) to the coefficient matrix until a matrix  $B$  is obtained which satisfies the following description:

If a row of  $B$  does not consist entirely of zeros then the first nonzero number in the row is a 1 (usually called a leading 1).

If there are any rows that consist entirely of zeros, they are grouped together at the bottom of  $B$ .

In any two successive non-zero rows of  $B$ , the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.

Each column of  $B$  that contains a leading 1 has zeros everywhere else.

The matrix  $B$  is said to be in reduced row echelon form.

It is well known that if  $A$  is an  $m \times n$  matrix and  $x$  is an  $n \times 1$  vector, then the systems  $Ax = 0$  and  $Bx = 0$  have the same solution set. However, the solution to the system  $Bx = 0$  may be read off immediately from the matrix  $B$ . For example, consider the system:

$$x_1 + 2x_2 + 4x_3 = 0$$

$$2x_1 + 3x_2 + 7x_3 = 0$$

$$3x_1 + 3x_2 + 9x_3 = 0$$

The matrix of coefficients is

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{pmatrix}$$

and through use of elementary row operations we obtain

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is seen to be in reduced row echelon form. Column 3 (unlike columns 1 and 2) does not contain a leading 1. We call such a column a **free** column, because in the solution to the system  $Bx = 0$  (and hence  $Ax = 0$ ), the variable corresponding to that column is a free parameter. Thus we can set  $x_3 = t$ , and read off the equations  $x_1 + 2t = 0$  (row 1) and  $x_2 + t = 0$  (row 2). The solution set is then  $x_1 = -2t$ ,  $x_2 = -t$ ,  $x_3 = t$ .

An important theoretical result is that the reduced row echelon form of a matrix is unique. Most texts either omit this result entirely or give a proof which is long and very technical (see [2, p. 56]). The following proof is somewhat clearer and less complicated than the standard proofs.

**THEOREM.** *The reduced row echelon form of a matrix is unique.*

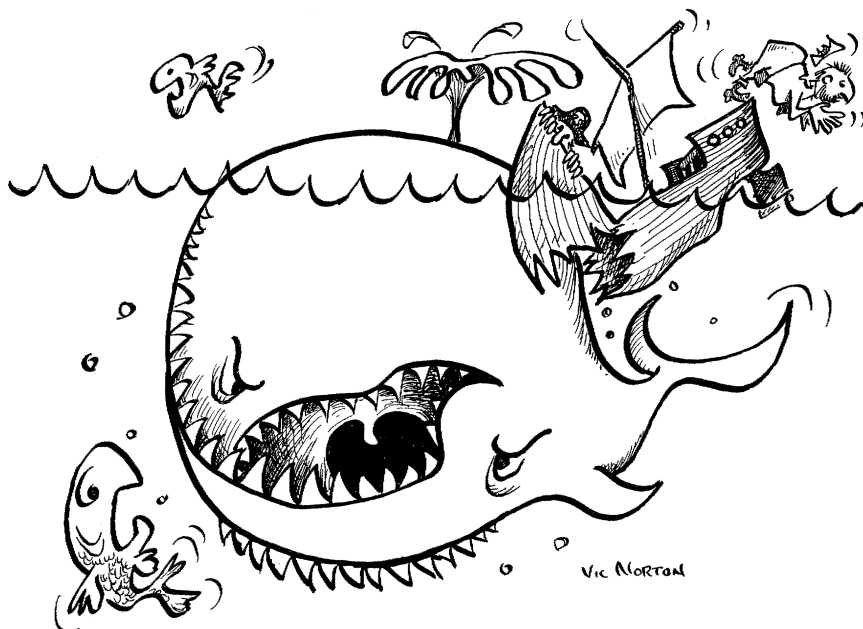
*Proof.* Let  $A$  be an  $m \times n$  matrix. We will proceed by induction on  $n$ . For  $n = 1$  the proof is obvious. Now suppose that  $n > 1$ . Let  $A'$  be the matrix obtained from  $A$  by deleting the  $n$ th column. We observe that any sequence of elementary row operations which places  $A$  in reduced

row echelon form also places  $A'$  in reduced row echelon form. Thus by induction, if  $B$  and  $C$  are reduced row echelon forms of  $A$ , they can differ in the  $n$ th column only. Assume  $B \neq C$ . Then there is an integer  $j$  such that the  $j$ th row of  $B$  is not equal to the  $j$ th row of  $C$ . Let  $u$  be any column vector such that  $Bu = 0$ . Then  $Cu = 0$  and hence  $(B - C)u = 0$ . We observe that the first  $n - 1$  columns of  $B - C$  are zero columns. Thus the  $j$ th coordinate of  $(B - C)u$  is  $(b_{jn} - c_{jn})u_n$ . Since  $b_{jn} \neq c_{jn}$  we must have  $u_n = 0$ . Thus any solution to  $Bx = 0$  or  $Cx = 0$  must have  $x_n = 0$ . It follows that both the  $n$ th columns of  $B$  and  $C$  must contain leading 1's, for otherwise those columns would be free columns and we could arbitrarily choose the value of  $x_n$ . But since the first  $n - 1$  columns of  $B$  and  $C$  are identical, the row in which this leading 1 must appear must be the same for both  $B$  and  $C$ , namely the row which is the first zero row of the reduced row echelon form of  $A'$ . Because the remaining entries in the  $n$ th columns of  $B$  and  $C$  must all be zero, we have  $B = C$ , which is a contradiction. This establishes the theorem.

We remark that this proof easily generalizes to the following proposition: *Let  $A$  be an  $m \times n$  matrix with row space  $W$ . Then there is a unique  $m \times n$  matrix  $B$  in reduced row echelon form such that the row space of  $B$  is  $W$ .* (For another proof, see [2, p. 56].)

## References

- [1] H. Anton, *Elementary Linear Algebra*, Wiley, New York, 1977.
- [2] K. Hoffman and R. Kunze, *Linear Algebra*, Prentice-Hall, Englewood Cliffs, New Jersey, 1961.



Riddle: What is non-orientable and lives in the sea?

—ROBERT MESSER

(See News and Letters if you give up.—ed.)