0.1 Linear Transformations

Definition : Let $(U, \oplus, \odot)$ and $(V, \boxplus, \boxdot)$ be two vector spaces over the **same** field $(\mathbb{F}, \bigoplus, \bigodot)$. Then the map $T : U \to V$ is said to be a linear transformation (map), if

$$T(u_1 \oplus u_2) = T(u_1) \boxplus T(u_2) \ \forall u_1, u_2 \in U \text{ and}$$
$$T(\alpha \odot u) = \alpha \boxdot T(u) \ \forall u \in U, \text{ and } \forall \alpha \in \mathbb{F}$$

Theorem : *Let $(U, \oplus, \odot)$ and $(V, \boxplus, \boxdot)$ be two vector spaces over the **same** field $(\mathbb{F}, \bigoplus, \bigodot)$. Then the map $T : U \to V$ be a linear transformation*

(i) $T(0_U) = 0_V$

(ii) $T(\ominus u) = \boxminus v$ *where* $v = T(u)$

(iii) $T(\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n) = \alpha_1 \boxdot T(u_1) \boxplus \alpha_2 \boxdot T(u_2) \boxplus \ldots \boxplus \alpha_n \boxdot T(u_n)$

In other words, a linear map T transforms the zero of U into the zero of V and the negative of every $u \in U$ into the negative of $T(u)$ in V.

Proof:

(i) We know that, $\forall u \in U \Rightarrow T(u) \in V$ and additive inverse of $T(u)$ in V, $\Rightarrow \boxminus T(u) \in V$

$$\begin{aligned} u &= u \oplus 0_U \\ \Rightarrow T(u) &= T(u \oplus 0_U) \\ T(u) &= T(u) \boxplus T(0_U) \end{aligned}$$

Adding $\boxminus T(u) \in V$ (additive inverse of $T(u)$) both the sides (from left), we get,

$$\begin{aligned} \boxminus(T(u)) \boxplus T(u) &= \boxminus(T(u)) \boxplus T(u) \boxplus T(0_U) \\ \Rightarrow \boxminus(T(u)) \boxplus T(u) &= (\boxminus(T(u) \boxplus T(u)) \boxplus T(0_U) \\ 0_V &= 0_V \boxplus T(0_U) \\ 0_V &= T(0_U) \end{aligned}$$

Hence linear map T maps the zero of U into the zero of V.

(ii) We know that $\ominus u \oplus u = 0_U$ also from (i) part, $T(0_U) = 0_V$. Therefore, we have,

$$\begin{aligned} T(0_U) &= T(\ominus u \oplus u) \\ \Rightarrow 0_V &= T(\ominus u) \boxplus T(u) \end{aligned}$$

Adding $\boxminus T(u) \in V$ (additive inverse of $T(u)$) both the sides (from right), we get,

$$\begin{aligned} \Rightarrow 0_V \boxminus T(u) &= T(\ominus u) \boxplus T(u) \boxminus T(u) \\ \Rightarrow \boxminus T(u) &= T(\ominus u) \boxplus (T(u) \boxminus T(u)) \\ \Rightarrow \boxminus T(u) &= T(\ominus u) \boxplus 0_V \\ \Rightarrow \boxminus T(u) &= T(\ominus u) \end{aligned}$$

Hence, a linear map T transforms the negative of every $u \in U$ into the negative of $T(u)$ in V.

(iii) From definition of T we know that,

$$T(\alpha \odot u) = \alpha \boxdot T(u)$$

and using the property

$$T(\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2) = \alpha_1 \boxdot T(u_1) \boxplus \alpha_2 \boxdot T(u_2)$$

Complete the proof by finite mathematical induction.

Remark In view of (iii), we get a standard technique of defining a linear transformation T on a finite-dimensional vector space. Suppose $B = \{u_1, u_2, \ldots u_n\}$ is a basis for U. Then any vector $u \in U$ can be expressed uniquely in the form

$$u = \alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n$$

So, if $T : U \to V$ is a linear map, then

$$\begin{aligned} T(u) &= T(\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n) \\ &= \alpha_1 \boxdot T(u_1) \boxplus \alpha_2 \boxdot T(u_2) \boxplus \ldots \boxplus \alpha_n \boxdot T(u_n). \end{aligned}$$

Thus $T(u)$ is known as soon as $T(u_1), T(u_2), \ldots, T(u_n)$ are known. This is formalized in the following theorem.

Theorem *A linear transformation T is completely determined by its values on the elements of a basis. Precisely, if $B = \{u_1, u_2, \ldots, u_n\}$ is an ordered basis for U and $v_1, v_2, \ldots, v_n$ be n vectors (not necessarily distinct) in V, then there exists a unique linear transformation $T : U \to V$ such that $T(u_i) = v_i$ for $i = 1, 2, \ldots, n$.*

Proof: Let $u \in$U. Then u can be expressed uniquely in the form

$$u = \alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n$$

We define

$$T(u) = \alpha_1 \boxdot v_1 \boxplus \alpha_2 \boxdot v_2 \boxplus \ldots \boxplus \alpha_n \boxdot v_n.$$

We now claim that this transformation T is the required transformation. To prove our claim, we have to show that

(i) T is linear

(ii) T satisfies $T(u_i) = v_i$ for $i = 1, 2, \ldots, n$, and

(iii) T is unique.

(ii) is obvious, since

$$u_i = 0 \odot u_1 \oplus 0 \odot u_2 \oplus \ldots \oplus 0 \odot u_{i-1} \oplus 1_{\mathbb{F}} \odot u_i \oplus 0 \odot u_{i+1} \oplus \ldots 0 \odot u_n,$$

and so

$$T(u_i) = 1_{\mathbb{F}} \boxdot v_i = v_i \; \forall i \, .$$

(iii) follows, because if there were another such linear map S with $S(u_i) = v_i$, then

$$\begin{aligned} S(u) &= S(\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n) \\ &= \alpha_1 \boxdot S(u_1) \boxplus \alpha_2 \boxdot S(u_2) \boxplus \ldots \boxplus \alpha_n \boxdot S(u_n). \\ &= \alpha_1 \boxdot v_1 \boxplus \alpha_2 \boxdot v_2 \boxplus \ldots \boxplus \alpha_n \boxdot v_n \\ &= T(u). \end{aligned}$$

This is true for every $u \in U$. $\Rightarrow S = T$

It only remains to prove (i), which is just a verification of the two relations

$$T(u \oplus v = T(u) \boxplus T(v) \text{ and}$$
$$T(\alpha \odot u) = \alpha \boxdot T(u)$$

for arbitrary $u, v \in U$ and all scalars α in $\mathbb{F}$.
Let $u, v \in U$. Then

$$\begin{aligned} u &= \alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n \\ v &= \beta_1 \odot u_1 \oplus \beta_2 \odot u_2 \oplus \ldots \oplus \beta_n \odot u_n, \end{aligned}$$

and we have

$$\begin{aligned} u \oplus v &= (\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n) \oplus (\beta_1 \odot u_1 \oplus \beta_2 \odot u_2 \oplus \ldots \oplus \beta_n \odot u_n) \\ &= (\alpha_1 \bigoplus \beta_1) \odot u_1 \oplus (\alpha_2 \bigoplus \beta_2) \odot u_2 \oplus \ldots \oplus (\alpha_n \bigoplus \beta_n) \odot u_n \end{aligned}$$

Hence, by the definition of T, we have

$$T(u \oplus v) = (\alpha_1 \bigoplus \beta_1) \odot v_1 \oplus (\alpha_2 \bigoplus \beta_2) \odot v_2 \oplus \ldots \oplus (\alpha_n \bigoplus \beta_n) \odot v_n \oplus$$

Also,

$$\begin{aligned} T(u) \boxplus T(v) &= (\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n) \boxplus (\beta_1 \odot v_1 \oplus \beta_2 \odot v_2 \oplus \ldots \oplus \bigoplus \beta_n \odot v_n) \\ &= (\alpha_1 \bigoplus \beta_1) \odot v_1 \oplus (\alpha_2 \bigoplus \beta_2) \odot v_2 \oplus \ldots \oplus (\alpha_n \bigoplus \beta_n) \odot v_n \end{aligned}$$

Therefore,

$$T(u \oplus v) = T(u) \boxplus T(v).$$

Again,

$$\begin{aligned} T(\alpha \odot u) &= T(\alpha \odot (\alpha_1 \odot u_1 \oplus \alpha_2 \odot u_2 \oplus \ldots \oplus \alpha_n \odot u_n)) \\ &= T((\alpha \circledcirc \alpha_1) \odot u_1 \oplus (\alpha \circledcirc \alpha_2) \odot u_2 \oplus \ldots \oplus (\alpha \circledcirc \alpha_n) \odot u_n) \\ &= (\alpha \circledcirc \alpha_1) \boxdot v_1 \boxplus (\alpha \circledcirc \alpha_2) \boxdot v_2 \boxplus \ldots \boxplus (\alpha \circledcirc \alpha_n) \boxdot v_n \\ &= \alpha \boxdot (\alpha_1 \boxdot v_1 \boxplus \alpha_2 \boxdot v_2 \boxplus \ldots \boxplus \alpha_n \boxdot v_n) \\ &= \alpha \boxdot T(u) \end{aligned}$$

Definition Let $(U, \oplus, \odot)$ and $(V, \boxplus, \boxdot)$ be two vector spaces over the **same** field $(\mathbb{F}, \bigoplus, \circledcirc)$ and $T : U \to V$ be a linear transformation.
The kernel (null space) of T is the set $N(T) = \{u \in U | T(u) = 0_V\}$. It is also denoted as kerT. In other words, $N(T)$ is the set of all those elements in U that are mapped by T into the zero of V. i.e the T-pre-image of 0_V.
The Range space of T is the set $R(T) = \{T(u) \in V | u \in U\}$.

Example: Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix}$. In this case $R(T)$ consists of vectors of the form $\begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix}$. We want to determine the vectors of $\mathbb{R}^2$ that are of this form. For this, take vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ and solve the equation
$\begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$

This means $x_1 - x_2 = a$ and $x_1 + x_3 = b$. Solving these, we get $x_2 = x_1 - a, x_3 = b - x_1$. Hence, $T \begin{bmatrix} x_1 \\ x_1 - a \\ b - x_1 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$. This shows that every vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ is in $R(T)$. In other words, $R(T) = \mathbb{R}^2$. So this is an onto map.

To determine the kernel, we solve the equation $T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This gives $x_1 = x_2 = -x_3$, i.e. all vectors of the form $\begin{bmatrix} x_1 \\ x_1 \\ -x_1 \end{bmatrix}$ will be mapped into zero. So $N(T) = \left\{ \begin{pmatrix} x_1 \\ x_1 \\ -x_1 \end{pmatrix} \middle| \, x_1 \in \mathbb{R} \right\} = \left[\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right] \Rightarrow dim(N(T)) = 1$. This is the subspace of $\mathbb{R}^3$ generated by $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.