Generalized Inverse

1. Theorem Given an $m \times n$ matrix A over $\mathbb{C}$, there exist unitary matrices U and V of order m and n respectively such that $A = UDV^*$ where

(i) $$D = \text{diag}[\lambda_1, \ldots, \lambda_r, 0, 0, \ldots, 0] \text{ if } m = n \qquad \ldots(1)$$

(ii) $$D = \left[\begin{array}{ccccc|ccc} \lambda_1 & & & & & 0 & \cdots & 0 \\ & \ddots & & & & & & \\ & & \lambda_r & & & \cdots & \cdots & \cdots \\ & & & \ddots & & & & \\ & & & & 0 & 0 & \cdots & 0 \end{array}\right]_{m\times n} \text{ if } m < n \qquad \ldots(2)$$

(iii) $$D = \left[\begin{array}{cccccc} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_r & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ \hline 0 & \cdots & & & \cdots & 0 \\ 0 & \cdots & & & \cdots & 0 \end{array}\right]_{m\times n} \text{ if } m > n \qquad \ldots(3)$$

where $\lambda_1, \ldots, \lambda_r$ are real and positive, $r \leq minm, n$. The matrix D in case (ii), (iii) is sometimes called a rectangular diagonal matrix.

Proof. The matrix A^*A is a positive semi definite Hermitian matrix(*prove it if you are not convinced*) and hence has non-negative eigenvalues, say they are $\lambda_1^2, \ldots, \lambda_r^2, 0, 0, \ldots, 0$(*no* λ_i *being zero*) with $r \leq \min(m, n)$. So there is a unitary matrix of order n such that

$$V * (A * A)V = diag[\lambda_1^2, \ldots, \lambda_r^2, 0, \ldots, 0]_{n\times n}$$

Put $AV = [\underline{x}_1 \ldots \underline{x}_n]$, $x_i \in \mathbb{C}^m$ are column vectors; we then find $x_i * x_j = \lambda_i^2 \neq 0$ for $i = j = 1, \ldots r$ and $= 0$ otherwise.

For $i = r+1, \ldots, n$ we have $x_i * x^i = 0$ which means $x_i = 0$ for $r+1 \le i \le n$.

Put $u_i = \lambda_i^{-1} x_i$ for $1 \le i \le r$. Then $\{u_1, \ldots, u_r\}$ is an orthonormal set in $\mathbb{C}^m$ and thus can be extended to an orthonormal basis $\{u_1, \ldots, u_r, u_{r+1}, \ldots, u_m\}$ in $\mathbb{C}^m$. Each of them is a column vector in $\mathbb{C}^m$ thus there is a matrix U formed with them which is unitary. We have

$$\begin{aligned} AV &= [x_1, \ldots, x_r, 0, \ldots, 0]_{m\times n} \\ &= [\lambda_1 u_1, \ldots, \lambda_r u_r, 0, \ldots, 0]_{m\times n} \\ &= UD \end{aligned}$$

which gives $A = UDV^*$ where D is in the form stated in the result.

Example 1. Take $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \\ 1 & 1-i \end{bmatrix}$ so that $A * A = 4 \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$ which has eigenvalues $\lambda_1^2 = 12, \lambda_2 = 0$ and the corresponding orthogonal set of eigenvectors $\begin{bmatrix} 1 \\ 1+i \end{bmatrix}, \begin{bmatrix} 1-i \\ -i \end{bmatrix}$. After normalization choose the unitary matrix V with these as columns:

$V := \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1-i \\ 1+i & -i \end{bmatrix}$ so that $V^*A^*AV = \begin{bmatrix} 12 & 0 \\ 0 & 0 \end{bmatrix}$ and since $AV = \sqrt{3} \begin{bmatrix} 1 & 0 \\ 1+i & 0 \\ 1 & 0 \end{bmatrix} =:$ $\begin{bmatrix} \lambda_1 u_1\ , & 0 \end{bmatrix}$ with $u_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix}$, we extend it to a basis of $\mathbb{C}^3$ $\{u_1, e_2, e_3\}$.

Applying Gram-schmidt to get the orthonormal basis in $\mathbb{C}^3$.

$x_1 = u_1$,

$x_2 = e_2 - \frac{(e_2|x_1)}{||x_1||^2} x_1 = \frac{1}{4} \begin{bmatrix} -1+i \\ 2 \\ -1+i \end{bmatrix}$,

$x_3 = e_3 - \frac{(e_3|x_1)}{||x_1||^2} x_1 - \frac{(e_3|x_2)}{||x_2||^2} x_2 = \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

After normalization we choose $\{u_1, u_2, u_3\}$ given by

$u_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix}$, $u_2 = \frac{\sqrt{2}}{4}\begin{bmatrix} -1+i \\ 2 \\ -1+i \end{bmatrix}$, $u_3 = \frac{\sqrt{2}}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ as an orthonormal basis to provide us with

$$U = \frac{\sqrt{2}}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1+i}{2} & -1 \\ \frac{1+i}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1+i}{2} & 1 \end{bmatrix}$$

Then $U^*AV = \left[\begin{array}{cc} 2\sqrt{3} & 0 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right] = D$ is a rectangular diagonal matrix.

2. Theorem Given an $m \times n$ matrix A over $\mathbb{C}$, there exists a unique $n \times m$ matrix $A^@$ satisfying $AA^@A = A$, $A^@AA^@ = A^@$ such that both $AA^@$ and $A^@A$ are Hermitian.

Proof. Suppose D is a diagonal or a rectangular diagonal matrix[*i.e., has one of the forms (1), (2), (3) given in 5.1*]

Define $D^@$ by

(i) $D^@ = \text{diag}[\lambda_1^{-1}, \ldots, \lambda_r^{-1}, 0, 0, \ldots, 0]$ if $m = n$ [Form(1)]

(ii) $$D^@ = \left[\begin{array}{cccccc} \lambda_1^{-1} & & & & & \\ & \ddots & & & & \\ & & \lambda_r^{-1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \\ \hline 0 & & \cdots & & \cdots & 0 \\ 0 & & \cdots & & \cdots & 0 \end{array}\right]_{m \times n}$$ if $m < n$ [Form(2)]

[*notice the switch from form (2) of D to form (3) of $D^@$*]

$$\text{(iii)} \quad D^{@} = \left[\begin{array}{ccccc|ccc} \lambda_1^{-1} & & & & & 0 & \cdots & 0 \\ & \ddots & & & & & & \\ & & \lambda_r^{-1} & & & \ldots & \ldots & \ldots \\ & & & \ddots & & & & \\ & & & & 0 & 0 & \cdots & 0 \end{array}\right]_{m\times n} \quad \text{if } m > n \quad [\text{ Form(3)}]$$

[*notice again the switch*]

Then by direct multiplication we get

$D^{@}D = \operatorname{diag}[1, \ldots, 1, 0, \ldots, 0]_{n\times n}$

$DD^{@} = \operatorname{diag}[1, \ldots, 1, 0, \ldots, 0]_{m\times m}$

which are obviously Hermitian. Moreover the requirement $DD^{@}D = D$ and $D^{@}DD^{@} = D^{@}$ can be directly verified. Thus in this case, the result is established.

When A is an arbitrary $m \times n$ matrix, by 5.1 above, there exist unitary matrices U and V of order m and n respectively such that $A = UDV^*$ and D is a diagonal or a rectangular diagonal matrix. Define the $n \times m$ matrix $A^{@}$ by $A^{@} := VD^{@}U^*$. Then $AA^{@} = UDV^*VD^{@}U^* = UDD^{@}U^*$ and $A^{@}A = VD^{@}U^*UDV^* = VD^{@}DV^*$ which shows that $AA^{@}$ and $A^{@}A$ are Hermitian because $DD^{@}$ and $D^{@}D$ are so. Moreover,

$$AA^{@}A = UDD^{@}U^*UDV^* = UDD^{@}DV^* = UDV^* = A, \quad \text{and}$$

$$A^{@}AA^{@} = VD^{@}DV^*VD^{@}U^* = VD^{@}DD^{@}U^* = VD^{@}U^* = A^{@}.$$

Thus all requirements are met and the existence of the desired $A^{@}$ has been proved.

It remains to show uniqueness. Suppose $A^{\sharp}$ is another candidate satisfying the three requirements. Then since the four matrices $AA^{@}, A^{@}A, AA^{\sharp}, A^{\sharp}A$ are all Hermitian, we have

$$AA^{@} = (A^{@})^*A^*, \quad A^{@}A = A^*(A^{@})^*$$

$$AA^{\sharp} = (A^{\sharp})^*A^*, \quad A^{\sharp}A = A^*(A^{\sharp})^*$$

and hence

$$\begin{aligned}
A^{@} &= A^{@}AA^{@} \\
&= A^*(A^{@})^*A^{@} \\
&= (A^*(A^\sharp)^*A^*)(A^{@})^*A^{@} \\
&= A^\sharp AA^{@}AA^{@} \\
&= A^\sharp AA^{@} \\
&= A^\sharp(A^{@})^*A^* \\
&= A^\sharp(A^{@})^*(A^*(A^\sharp)^*A^* \\
&= A^\sharp AA^\sharp AA^\sharp \\
&= A^\sharp AA^\sharp \\
&= A^\sharp
\end{aligned}$$

which establishes uniqueness