Chapter 1

Vector Spaces

1.1 Group

Binary Operator: A Binary operator on a non-empty set $\mathbf{S}$ is a map from its Cartesian product $\mathbf{S} \times \mathbf{S}$ to $\mathbf{S}$. Let $*$ be the binary operation on $\mathbf{S}$ then we have

$$* : \mathbf{S} \times \mathbf{S} \longrightarrow \mathbf{S}$$

Group: A non-empty set G, together with a binary operation $*$ is said to form a group, if it satisfies the following properties.

1. *Associativity:* $a * (b * c) = (a * b) * c \qquad \forall a, b, c \in G.$
2. *Existence of Identity:* $\exists$ an element $e \in G$ such that

   $$a * e = e * a = a \qquad \forall a \in G$$

   where, e is the identity element.
3. *Existence of inverse:* For every $a \in G$, $\exists\, a' \in G$ such that

   $$a * a' = a' * a = e$$

   Here, a' is called an inverse element of a.

Remarks

1. If $*$ is a binary operation on G then G is said to satisfy closure property.
2. Identity element for a group is unique.
3. Inverse of an element is also unique.
4. Existence of right identity and left inverse does not form a group.
5. Existence of left identity and right inverse also does not form a group.
6. In above definition, existence of right identity and right inverse is sufficient to form a group because right identity is also left identity and right inverse is also left inverse.
7. If a' be the inverse element of a then, $(a')' = a$.

1.1.1 Solved Examples

Example 1: *Consider V, the set of all $m \times n$ matrices with real entries.*

$$V = \{A | A = [a_{ij}]_{m\times n}, a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$$

Let $A, B \in V$
Operation $\oplus$ in V is defined as

$$A \oplus B = [a_{ij} + b_{ij}]_{m\times n}, a_{ij}, b_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \tag{1.1}$$

Show that the $(V, \oplus)$ is a group.

Solution: In order to prove V is a group, we check all its properties. Let $A, B, C \in V$, where $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$ $\forall\, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

1. **Binary composition in V :** Let $A = [a_{ij}]_{m\times n} \in V$ and $B = [b_{ij}]_{m\times n} \in V, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. By definition of $\oplus$ operator we have,

   $$A \oplus B = [a_{ij} + b_{ij}]_{m\times n}, a_{ij}, b_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.$$

   Since $a_{ij} + b_{ij} \in \mathbb{R}\ \forall i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Hence, $A \oplus B = [a_{ij} + b_{ij}]_{m\times n} \in V\ \ \forall\, A, B \in V$. Clearly, $\oplus$ is a binary operation on V i.e. V is closed under the operation $\oplus$.

   $$\oplus : V \times V \longrightarrow V$$

2. **Associativity of addition :** By definition of $\oplus$ operation in V, $\forall\, A, B, C \in V$, we have,
   $$\begin{aligned}(A \oplus B) \oplus C &= [a_{ij} + b_{ij}]_{m\times n} \oplus C\\ &= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n}\\ &= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n}\\ &= A \oplus [(b_{ij} + c_{ij})]_{m\times n}\\ &= A \oplus (B \oplus C)\end{aligned}$$
   Hence, $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

3. **Existence of additive identity :** Since, $0_V = [0]_{m\times n} \in V$, therefore by definition of $\oplus$ operation in V, $\forall\, A \in V$, we have,
   $$\begin{aligned}0_V \oplus A &= [0]_{m\times n} \oplus [a_{ij}]_{m\times n}\\ &= [(0 + a_{ij})]_{m\times n}\\ &= [a_{ij}]_{m\times n}\\ &= A\end{aligned}$$
   Also,
   $$\begin{aligned}A \oplus 0_V &= [a_{ij}]_{m\times n} \oplus [0]_{m\times n}\\ &= [(a_{ij} + 0)]_{m\times n}\\ &= [a_{ij}]_{m\times n}\\ &= A\end{aligned}$$
   Since $A \oplus 0_V = 0_V \oplus A = A$ holds $\forall A \in V$, therefore 0_V defined above is the additive identity.

4. **Existence of additive inverses :** $\forall\, A = [a_{ij}]_{m\times n} \in V$, $a_{ij} \in \mathbb{R}, \forall\, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n \implies -a_{ij} \in \mathbb{R}, \forall\, i = 1, 2, \ldots, m, j = 1, 2 \ldots, n$
   Let $A' = [-a_{ij}]_{m\times n}, a_{ij} \in \mathbb{R}, \forall\, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Then, by definition of $\oplus$ operation in V, we have,
   $$\begin{aligned}A \oplus A' &= [a_{ij}]_{m\times n} \oplus [-a_{ij}]_{m\times n}\\ &= [a_{ij} + (-a_{ij})]_{m\times n}\\ &= [a_{ij} - a_{ij}]_{m\times n}\\ &= [0]_{m\times n}\\ &= 0_V \in V\end{aligned}$$
   Also,

$$\begin{aligned} A' \oplus A &= [-a_{ij}]_{m\times n} \oplus [a_{ij}]_{m\times n} \\ &= [(-a_{ij}) + a_{ij}]_{m\times n} \\ &= [-a_{ij} + a_{ij}]_{m\times n} \\ &= [0]_{m\times n} \\ &= 0_V \in V \end{aligned}$$

Since $A \oplus A' = A' \oplus A = 0_V$, therefore A' defined above is the inverse of A.

Hence, $(V, \oplus)$ is a group.

Example 2: *Consider*

$$V = \mathbb{R}^+ = \{u \in \mathbb{R} | u > 0\}$$

Let $u, v \in V$.
Operation $\oplus$ *in* V *is defined as*

$$u \oplus v = u.v, \quad u, v \in V \tag{1.2}$$

where "." is usual multiplication.
Show that the $(V, \oplus)$ *is a group.*

Solution : In order to prove $V(=\mathbb{R}^+)$ is a group, we check all its properties. Let $u, v, w \in V(=\mathbb{R}^+)$.

1. **Binary Composition in V :** Let $u, v \in V$. By definition $\oplus$ operation.

$$u \oplus v = u.v, \ \forall u, v \in V$$

Since $u, v \in \mathbb{R} \implies u.v \in V$ (product of two positive real number is also a positive real number). clearly, $\oplus$ is a binary operation on V i.e. V is closed under the operation of vector addition.

$$\oplus : V \times V \longrightarrow V$$

2. **Associativity of addition :** $\forall u, v, w \in V$ we have,

$$\begin{aligned} (u \oplus v) \oplus w &= (u.v) \oplus w && \text{from eq(1.2)} \\ &= q \oplus w && \text{let } q = u.v \\ &= q.w && \text{from eq(1.2)} \\ &= (u.v).w && \text{Putting } q = u.v \\ &= u.(v.w) && \text{Multiplication is associative in } \mathbb{R} \\ &= u.(v \oplus w) && \text{from eq(1.2)} \\ &= u \oplus (v \oplus w) && \text{from eq(1.2)} \end{aligned}$$

Hence, $(u \oplus v) \oplus w = u \oplus (v \oplus w)$ holds $\forall u, v, w \in \mathbb{R}^+$.

3. **Existence of additive identity :** Since, $1 \in V(=\mathbb{R}^+)$ we have,
$v \oplus 1 = v.1 = v$ holds $\forall v \in V(=\mathbb{R}^+)$.
Also, $1 \oplus v = 1.v = v$ holds $\forall v \in V(=\mathbb{R}^+)$.
Clearly, 1 is the identity element of $(V, \oplus)$.

4. **Existence of additive inverses :** Let $v \in V(=\mathbb{R}^+) \implies v' = \dfrac{1}{v} \in V(=\mathbb{R}^+)$. Hence, $\begin{aligned} v \oplus v' &= v.\frac{1}{v} \\ &= 1 && \text{From equ} \end{aligned}$

Also,

$$\begin{aligned} v' \oplus v &= \frac{1}{v}.v \\ &= 1 && \text{From equation(1.2)} \end{aligned}$$

Hence, $\dfrac{1}{v}$ exist for all every $v \in V(=\mathbb{R}^+)$ and it is the inverse element.

Clearly, $(V, \oplus)$ is a group.

Example 3: *Consider P_n, the set of polynomials of degree less than or equal to n.*

$$P_n = \{p(x)|p(x) = (a_0 + a_1x + ... + a_nx^n) \text{ such that } a_i \in \mathbb{F}, \quad \forall i = 1, 2, 3.., n\}$$

Let $p(x) = a_0 + a_1x + ... + a_nx^n \in P_n$ and $q(x) = b_0 + b_1x + ... + b_nx^n \in P_n$
Operation $\oplus$ in P_n is defined as

$$\begin{aligned} p(x) \oplus q(x) &= (a_0 + a_1x + ... + a_nx^n) \oplus (b_0 + b_1x + ... + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + .. + (a_n + b_n)x^n \end{aligned} \tag{1.3}$$

Show that the $(P_n, \oplus, \odot)$ is a group.

Solution : In order to prove P_n is a vector space, we check all its properties. Let
$p(x) = (a_0 + a_1x + ... + a_nx^n)$
$q(x) = (b_0 + b_1x + ... + b_nx^n)$
$r(x) = (c_0 + c_1x + ... + c_nx^n)$
Let $p(x), q(x)$ and $r(x) \in P_n$.

Binary Composition in V : By definition of $\oplus$ operation,

$$\begin{aligned} p(x) \oplus q(x) &= (a_0 + a_1x + ... + a_nx^n) \oplus (b_0 + b_1x + ... + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + .. + (a_n + b_n)x^n \end{aligned}$$

and since $(a_i + b_i) \in \mathbb{R}$, $\forall i = 1, 2, \ldots n$. Clearly, $(a_0 + b_0) + (a_1 + b_1)x + .. + (a_n + b_n)x^n \in P_n$ $\forall a_i, b_i \in \mathbb{R}$ $\forall i = 1, 2 \ldots n$. Hence P_n is closed under the operation $(\oplus)$.

$$\oplus : P_n \times P_n \longrightarrow P_n$$

- **Associativity of addition:** By definition of $\oplus$ operation in V. $\forall p(x), q(x), r(x) \in P_n$, we have
  $(p(x) \oplus q(x)) \oplus r(x) =$
  $= ((a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + ... + (a_0 + b_0)) \oplus$
  $(c_0 + c_1x + ... + c_nx^n)$
  $= ((a_n + b_n) + c_n)x^n + ((a_{n-1} + b_{n-1}) + c_{n-1})x^{n-1} + ... + ((a_0 + b_0) + c_0)$
  $= (a_n + (b_n + c_n))x^n + (a_{n-1} + (b_{n-1} + c_{n-1}))x^{n-1} + ... + (a_0 + (b_0 + c_0))$
  $= p(x) \oplus ((b_n + c_n)x^n + (b_{n-1} + c_{n-1})x^{n-1} + ... + (b_0 + c_0))$
  $= p(x) \oplus (q(x) \oplus r(x))$

  Hence, $(p(x) \oplus q(x)) \oplus r(x) = p(x) \oplus (q(x) \oplus r(x)) \quad \forall p(x), q(x), r(x) \in P_n$.

- **Existence of additive identity :** Since $\mathbf{0(x)} = 0x^n + 0x^{n-1} + ... + 0 \in P_n$ i.e polynomial with all coefficients zero (*Zero Polynomial*), then $\forall p(x) \in P_n$ we have,
  $$\begin{aligned} p(x) \oplus \mathbf{0(x)} &= (a_0 + a_1x + ... + a_nx^n) \oplus (0x^n + 0x^{n-1} + ... + 0) \\ &= (a_n + 0)x^n + (a_{n-1} + 0)x^{n-1} + ... + (a_0 + 0) \\ &= a_nx^n + a_{n-1}x^{n-1} + ... + a_0 \\ &= p(x) \end{aligned}$$

  Similarly,
  $$\begin{aligned} \mathbf{0(x)} \oplus p(x) &= (0x^n + 0x^{n-1} + ... + 0) \oplus (a_0 + a_1x + ... + a_nx^n) \\ &= (0 + a_n)x^n + (0 + a_{n-1})x^{n-1} + ... + (0 + a_0) \\ &= a_nx^n + a_{n-1}x^{n-1} + ... + a_0 \\ &= p(x) \end{aligned}$$

  Hence, $p(x) \oplus \mathbf{0(x)} = \mathbf{0(x)} \oplus p(x) = p(x) \quad \forall p(x) \in P_n$.

- **Existence of additive inverses :** For each $p(x) \in P_n$ there exists a $\bar{p}(x)$ such that if $p(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_0$ then $\bar{p}(x) = -a_nx^n + (-a_{n-1}x^{n-1}) + ... + (-a_0)$. $\bar{p}(x) \in P_n$ because as $a_i \in \mathbb{R} \implies -a_i \in \mathbb{R}$ $\forall i = 1, 2, ...n$. we have,

  $p(x) \oplus \bar{p}(x) =$

$= (a_0 + a_1x + ... + a_nx^n) \oplus ((-a_nx^n) + (-a_{n-1}x^{n-1}) + ... + (-a_0))$
$= ((a_n - a_n)x^n + (a_{n-1} - a_{n-1})x^{n-1} + ... + (a_0 - a_0))$
$= 0x^n + 0x^{n-1} + ... + 0$
$= \mathbf{0(x)}$

Similarly,
$\bar{p}(x) \oplus p(x) =$
$= ((-a_nx^n) + (-a_{n-1}x^{n-1}) + ... + (-a_0)) \oplus (a_0 + a_1x + ... + a_nx^n)$
$= ((-a_n + a_n)x^n + (-a_{n-1} + a_{n-1})x^{n-1} + ... + (-a_0 + a_0))$
$= 0x^n + 0x^{n-1} + ... + 0$
$= \mathbf{0(x)}$

Hence, $p(x) \oplus \bar{p}(x) = \bar{p}(x) \oplus p(x) =$ $\mathbf{0(x)}$ exists for each $p(x) \in P_n$.

Hence $(P_n, \oplus)$ is a group.

1.2 Abelian Group

If in a group G commutative property also holds

$$a * b = b * a \qquad \forall a, b \in G$$

Then, such a group is called Abelian Group or Commutative group.

1.2.1 Solved examples

We just extend the previous examples (example 1,2 and 3), by checking its commutativity property and thereby checking whether they are abelian group or not. **Commutativity** property states that $\forall u, v \in V$ following holds.

$$u \oplus v = v \oplus u \tag{1.4}$$

Example 1: Commutativity of addition : By definition of $\oplus$ operation in V, $\forall\ A, B \in V$, we have,
$A \oplus B \ = [a_{ij} + b_{ij}]_{m \times n}$
$= [b_{ij} + a_{ij}]_{m \times n}$
$= B \oplus A$
Since $A \oplus B = B \oplus A$, therefore $\oplus$ is commutative.

Example 2: Commutativity of addition : By definition of $\oplus$ operation in V, $\forall\ u, v \in V$, we have,

$u \oplus v \ = u.v$
$= v.u$ { Multiplication of real numbers is commutative. }
$= v \oplus u$

Hence, $(\mathbb{R}^+, \oplus)$ is an abelian group.

Example 3: Commutativity of addition : By definition of $\oplus$ operation in V, $\forall\ p(x), q(x) \in P_n$, we have,

$p(x) \oplus q(x) \ = (a_0 + a_1x + ... + a_nx^n) + (b_0 + b_1x + ... + b_nx^n)$
$= ((a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + ... + (a_0 + b_0))$

Now by using commutativity property of real numbers we have,
$p(x) \oplus q(x) \ = ((b_n + a_n)x^n + (b_{n-1} + a_{n-1})x^{n-1} + ... + (b_0 + a_0))$
$= (b_0 + b_1x + ... + b_nx^n) \oplus (a_0 + a_1x + ... + a_nx^n)$
$= q(x) \oplus p(x)$

Hence, $(P_n, \oplus)$ is an abelian group.

1.3 Ring

A non empty set R, together with two binary operations $\oplus$ and $\odot$ is said to form a *Ring* if the following properties are satisfied:

- $(R, \oplus)$ should be an abelian group.
- Associativity property w.r.t to $\odot$ operation

$$a \odot (b \odot c) = (a \odot b) \odot c \qquad \forall a, b, c \in R \tag{1.5}$$

- Distributive Properties
  1. $a \odot (b \oplus c) = a \odot b \oplus a \odot c \qquad \forall a, b, c \in R$
  2. $(b \oplus c) \odot a = b \odot a \oplus c \odot a \qquad \forall a, b, c \in R$

1.3.1 Examples

1. Set of real numbers, rational numbers, integers form rings w.r.t usual addition and multiplication.
2. Set of all even numbers E forms a rings w.r.t usual addition and multiplication.

1.4 Other Definitions

*Note : **1** and **0** (ie. 1 and 0 in bold font) here after represents multiplicative and additive identity w.r.t $\odot$ and $\oplus$ operation respectively.*

Commutative Ring: A ring which follow commutative property w.r.t to $\odot$ operation is called commutative ring.

Ring with Unity: A ring with the unit element or multiplicative identity is called Ring with unity. i.e

$$a \odot \mathbf{1} = \mathbf{1} \odot a = a \quad \forall a \in R$$

Multiplicative identity or unit element is generally denoted by **1**.

Zero-divisor: Let R be a ring. An element $\mathbf{0} \neq a \in R$ is called zero divisor, if $\exists$ an element $\mathbf{0} \neq b \in R$ such that $a \odot b = \mathbf{0}$ or $b \odot a = \mathbf{0}$.

Integral domain: A commutative ring R is called Integral Domain if $a \odot b = 0$ in $R \implies$ either $a = \mathbf{0}$ or $b = \mathbf{0}$. In other words, a commutative ring R is called integral domain if R has no zero divisors

Unit: An element a in a ring R with unity, is called invertible (or a unit) w.r.t multiplication if $\exists$ some $b \in R$ such that

$$a \odot b = b \odot a = \mathbf{1}$$

Division Ring or Skew field: A ring R with unity is called division ring or skew field if non-zero elements of R form a group with respect to multiplication.

Field: A commutative division ring is called field.

1.4.1 Illustrations of above definitions

- Set of real numbers, rational numbers, integers form rings w.r.t usual addition and multiplication. These are all commutative rings with unity.

- Set E of all even numbers forms a commutative ring, **without unity** under usual addition and multiplication.

- Let M be the set of all 2×2 matrices of integers under matrix addition and matrix multiplications. It is easy to see M forms a ring where zero of ring or additive identity is $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. In-fact it forms a ring with unity with unit element $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. It is easy to verify that M does not follow commutative property since $A \odot B \neq B \odot A$ (REASON : Matrix multiplication is not commutative).

- Let F be the set of all continuous functions $f : \mathbb{R} \longrightarrow \mathbb{R}$. Then F forms a ring over addition and multiplication defined below

$$\forall f, g \in F$$
$$(f \oplus g)x = f(x) + g(x) \quad \forall x \in \mathbb{R}$$
$$(f \odot g)x = f(x)g(x) \quad \forall x \in \mathbb{R}$$

Zero of the ring is $\mathbf{0} : \mathbb{R} \longrightarrow \mathbb{R}$ such that $\mathbf{0}(x) = 0 \quad \forall x \in \mathbb{R}$
Also, additive inverse (inverse w.r.t $\oplus$ operation) for any $f \in F$ is a function $(-f) : \mathbb{R} \longrightarrow \mathbb{R}$ such that $(-f)(x) = -f(x)$.
In-fact, F would have unity also, namely the function $i : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $i(x) = 1$ for all $x \in \mathbb{R}$.

- *A division ring which is not a field.* Let M be the set of all 2×2 matrices of the type $\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ where a, b are complex numbers and $\bar{a}, \bar{b}$ are their conjugates i.e, if $a = x + iy$ then $\bar{a} = x - iy$. Then M is a ring with unity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ under matrix addition and matrix multiplication.
Any non zero element of M will be $\begin{bmatrix} x + iy & u + iv \\ -(u - iv) & x - iy \end{bmatrix}$ where x, y, u, v are not all zero.
One can check that the matrix $\begin{bmatrix} \frac{x - iy}{k} & -\frac{u + iv}{k} \\ \frac{u - iv}{k} & \frac{x + iy}{k} \end{bmatrix}$ where $k = x^2 + y^2 + z^2$, will be multiplicative inverse of the above non-zero matrix showing that M is a division ring. But M will not be field as it is not commutative as

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

But

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$$

- *Let $R = \{0, a, b, c\}$. Define $+$ and . on R by*

$+$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Then one can check that R forms a non-commutative ring without unity.

1.5 Vector Spaces

Let V be a non-empty set and $(\mathbb{F}, \oplus, \odot)$ be a field having following properties:

Vector addition Composition in V : There should be a binary operation $\oplus$ called **vector addition** on V which is defined as

$$\oplus : V \times V \longrightarrow V$$

Scalar Multiplication Composition in V over $\mathbb{F}$: There should be a **scalar multiplication** operation $\odot$ on V which is defined as

$$\odot : \mathbb{F} \times V \longrightarrow V$$

1. **Associativity of addition :** Associativity property states that $\forall u, v, w \in V$ following should holds.

$$(u \oplus v) \oplus w = u \oplus (v \oplus w)$$

2. **Existence of identity element:** $\exists\ e \in V$ such that $\forall v \in V$ following condition holds.

$$v \oplus e = e \oplus v = v$$

3. **Existence of additive inverses:** For every $v \in V$ there exists a $v' \in V$ such that

$$v \oplus v' = v' \oplus v = e$$

   where e is the identity element of $(V, \oplus)$ and v' is called additive inverse element of v.

4. **Commutativity of addition :** Commutativity property states that $\forall u, v \in V$ following should holds.

$$u \oplus v = v \oplus u$$

5. **Distributivity of addition over multiplication :** $\forall u, v \in V$ and $\forall \alpha \in \mathbb{F}$ following property should hold

$$\alpha \odot (u \oplus v) = \alpha \odot u \oplus \alpha \odot v$$

6. **Distributivity of multiplication over addition :** $\forall u, v \in V$ and $\forall \alpha, \beta \in \mathbb{F}$ following property should hold

$$(\alpha \oplus \beta) \odot v = \alpha \odot v \oplus \beta \odot v$$

7. **Associativity of multiplication :** $\forall \alpha, \beta \in \mathbb{F}$ and $\forall v \in V$ following should satisfy

$$(\alpha \odot \beta) \odot v = \alpha \odot (\beta \odot v)$$

8. **Multiplication of unit element of field :** If $\mathbf{1}$ is the unit element of $\mathbb{F}$. Then, $\forall u \in V$

$$\mathbf{1} \odot u = u$$

1.5.1 Solved Examples

Example 1: *Consider V, the set of all $m \times n$ matrices with real entries.*

$$V = \{A | A = [a_{ij}]_{m \times n}, a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$$

Let $A, B \in V$ and $\alpha \in \mathbb{R}$

Operation $\oplus$ in V is defined as

$$A \oplus B = [a_{ij} + b_{ij}]_{m \times n}, a_{ij}, b_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \tag{1.6}$$

Another operation $\odot$ in V is defined as

$$\alpha \odot A = [\alpha a_{ij}]_{m \times n}, \alpha \in \mathbb{F}, a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \tag{1.7}$$

Show that the $(V, \oplus, \odot)$ is a vector space over field $(\mathbb{F}, \oplus, \odot)$ where $\mathbb{F} = \mathbb{R}$. $\oplus$ and $\odot$ are the usual addition and multiplication over reals.

Solution : In order to prove V is a vector space, we check all its properties. Let $A, B, C \in V$, where $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ and $C = [c_{ij}]_{m\times n}$ $\forall\, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ and $\alpha, \beta \in \mathbb{F}$

Vector addition Composition in V : Let $A = [a_{ij}]_{m\times n} \in V$ and $B = [b_{ij}]_{m\times n} \in V, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. By definition of vector addition we have,

$$A \oplus B = [a_{ij} + b_{ij}]_{m\times n}, a_{ij}, b_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n.$$

Since $a_{ij} + b_{ij} \in \mathbb{R}\; \forall i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Hence, $A \oplus B = [a_{ij} + b_{ij}]_{m\times n} \in V \;\; \forall\, A, B \in V$. Clearly, $\oplus$ is a binary operation on V i.e. V is closed under the operation of vector addition.

$$\oplus : V \times V \longrightarrow V$$

Scalar Multiplication Composition in V over $\mathbb{F}$: By definition of scalar multiplication $\alpha a_{ij} \in \mathbb{F}\;\, \forall\, \alpha \in \mathbb{F}, a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Hence V is closed under the operation of scalar multiplication $\odot$ i.e.

$$\odot : \mathbb{F} \times V \longrightarrow V$$

1. **Associativity of addition :** Refer previous sections(section 1.1 and 1.2).

2. **Existence of additive identity :** Refer previous sections(section 1.1 and 1.2).

3. **Existence of additive inverses :** Refer previous sections(section 1.1 and 1.2).

4. **Commutativity of addition :** Refer previous sections(section 1.1 and 1.2).

5. **Distributivity of addition over multiplication :** By definition of scalar multiplication in V over $\mathbb{F}$, $\forall\, A \in V, \alpha, \beta \in \mathbb{F}$, we have,
$$\begin{aligned}(\alpha \oplus \beta) \odot A &= (\alpha \oplus \beta) \oplus [a_{ij}]_{m\times n}\\ &= (\alpha + \beta) \oplus [a_{ij}]_{m\times n}\\ &= [(\alpha + \beta) a_{ij}]_{m\times n}\\ &= [\alpha a_{ij} + \beta a_{ij}]_{m\times n}\\ &= [\alpha a_{ij}]_{m\times n} \oplus \beta [a_{ij}]_{m\times n}\\ &= \alpha \odot [a_{ij}]_{m\times n} \oplus \beta \odot [a_{ij}]_{m\times n}\\ &= \alpha \odot A \oplus \beta \odot A\end{aligned}$$
Hence $(\alpha \oplus \beta) \odot A = \alpha \odot A \oplus \beta \odot A$

6. **Distributivity of multiplication over addition :** By definition of scalar multiplication in V over $\mathbb{F}$, $\forall\, A, B \in V, \alpha \in \mathbb{F}$ we have,
$$\begin{aligned}\alpha \odot (A \oplus B) &= \alpha \odot ([a_{ij}]_{m\times n} \oplus [b_{ij}]_{m\times n})\\ &= \alpha \odot [a_{ij} + b_{ij}]_{m\times n}\\ &= [\alpha(a_{ij} + b_{ij})]_{m\times n}\\ &= [\alpha a_{ij} + \alpha b_{ij}]_{m\times n}\\ &= [\alpha a_{ij}]_{m\times n} \oplus [\alpha b_{ij}]_{m\times n}\\ &= \alpha \odot [a_{ij}]_{m\times n} \oplus \alpha \odot [b_{ij}]_{m\times n}\\ &= \alpha \odot A \oplus \alpha \odot B\end{aligned}$$
Hence $\alpha \odot (A \oplus B) = \alpha \odot A \oplus \alpha \odot B$

7. **Associativity of multiplication :** By definition of scalar multiplication in V over $\mathbb{F}$, $\forall\, A \in V, \alpha, \beta \in \mathbb{F}$ we have,
$$\begin{aligned}(\alpha \circledcirc \beta) \odot A &= (\alpha\beta) \odot [a_{ij}]_{m\times n}\\ &= [(\alpha\beta) a_{ij}]_{m\times n}\\ &= [\alpha(\beta a_{ij})]_{m\times n}\\ &= \alpha \odot [\beta a_{ij}]_{m\times n}\\ &= \alpha \odot (\beta \odot [a_{ij}]_{m\times n})\\ &= \alpha \odot (\beta \odot A)\end{aligned}$$
Hence $(\alpha \circledcirc \beta) \odot A = \alpha \odot (\beta \odot A)$

8. **Multiplication by unit element of the field:** Since $1 \in \mathbb{F}(=\mathbb{R})$, by definition of scalar multiplication in V over $\mathbb{F}$, $\forall\, A \in V$ we have,
$$\begin{aligned} 1 \odot A &= 1 \odot [a_{ij}]_{m\times n} \\ &= [1a_{ij}]_{m\times n} \\ &= [a_{ij}]_{m\times n} \\ &= A \end{aligned}$$
Hence $1 \odot A = A$

Since all the properties of a vector are satisfied, hence $(V, \oplus, \odot)$ is a vector space over the field $(\mathbb{F}, \oplus, \odot)$.

Example 2: *Consider*
$$V = \mathbb{R}^+ = \{u \in \mathbb{R} | u > 0\}$$
Let $u, v \in V$ and $\alpha \in \mathbb{R}$
Operation $\oplus$ in V is defined as
$$u \oplus v = u.v, \quad u, v \in V \tag{1.8}$$
where "." is usual multiplication. Another operation $\odot$ in V is defined as
$$\alpha \odot u = u^{\alpha}, \ \ \alpha \in \mathbb{F}, \quad u, v \in V \tag{1.9}$$
Show that the $(V, \oplus, \odot)$ is a vector space over field $(\mathbb{F}, \oplus, \odot)$ where $\mathbb{F} = \mathbb{R}$. $\oplus$ and $\odot$ are the usual addition and multiplication over $\mathbb{R}$.

Solution : In order to prove V is a vector space, we check all its properties. Let $u, v, w \in V$ and $\alpha, \beta \in \mathbb{F}$.

Vector addition Composition in V : Let $u, v \in V$. By definition of vector addition
$$u \oplus v = u.v, \ \ \forall u, v \in V$$
Since $u, v \in \mathbb{R} \implies u.v \in V$ (product of two positive real number is also a positive real number). clearly, $\oplus$ is a binary operation on V i.e. V is closed under the operation of vector addition.
$$\oplus : V \times V \longrightarrow V$$

Scalar Multiplication Composition in V over $\mathbb{F}$: By definition of scalar multiplication, $\alpha \in \mathbb{R}$ and $u \in V$,
$$\alpha \odot u = u^{\alpha}$$
Clearly, $\forall\, \alpha \in \mathbb{F}(=\mathbb{R})$ and $\forall\, u \in V(=\mathbb{R}^+), u^{\alpha} \in V(=\mathbb{R}^+)$. Hence, V is closed under is a scalar multiplication operation $\odot$ i.e
$$\oplus : \mathbb{F} \times V \longrightarrow V$$

1. **Associativity of addition :** Refer previous sections(section 1.1 and 1.2).
2. **Existence of additive identity :** Refer previous sections(section 1.1 and 1.2).
3. **Existence of additive inverses :** Refer previous sections(section 1.1 and 1.2).
4. **Commutativity of addition :** Refer previous sections(section 1.1 and 1.2).
5. **Distributivity of addition over multiplication :** By definition of scalar multiplication in V over $\mathbb{F}$, $\forall u, v \in V$ and $\forall \alpha \in \mathbb{F}$, we have
$$\begin{aligned} \alpha \odot (u \oplus v) &= \alpha \odot (u.v) \\ &= (u.v)^{\alpha} \\ &= u^{\alpha}.v^{\alpha} \\ &= (\alpha \odot u).(\alpha \odot v) \\ &= \alpha \odot u \oplus \alpha \odot v \end{aligned}$$
Hence, $\alpha \odot (u \oplus v) = \alpha \odot u \oplus \alpha \odot v$.

6. **Distributivity of multiplication over addition :** By definition of scalar multiplication in V over $\mathbb{F}$, $\forall u, v \in V(=\mathbb{R}^+)$ and $\forall \alpha, \beta \in \mathbb{R}(=\mathbb{F})$, we have

$$\begin{aligned}(\alpha \oplus \beta) \odot v &= (\alpha + \beta) \odot v \\ &= v^{\alpha+\beta} \\ &= v^{\alpha}.v^{\beta} \\ &= v^{\alpha} \oplus v^{\beta} \\ &= \alpha \odot v \oplus \beta \odot v\end{aligned}$$

Hence, $(\alpha \oplus \beta) \odot v = \alpha \odot v \oplus \beta \odot v$

7. **Associativity of multiplication :** By definition of scalar multiplication in V over $\mathbb{F}$, $\forall \alpha, \beta \in \mathbb{F}(=\mathbb{R})$ and $\forall v \in V(=\mathbb{R}^+)$ we have,

$$\begin{aligned}(\alpha \circledcirc \beta) \odot v &= (\alpha\beta) \odot v && \{(\circledcirc \text{ is usual multiplication over } \mathbb{R})\} \\ &= \gamma \odot v && \{\text{Let } \gamma = (\alpha\beta)\} \\ &= v^{\gamma} \\ &= v^{(\alpha\beta)} && \{\text{ Since, } \gamma = (\alpha\beta)\} \\ &= v^{(\beta\alpha)} && \{\text{ Using associative property of reals}\} \\ &= (v^{\beta})^{\alpha} \\ &= (\beta \odot v)^{\alpha} \\ &= \alpha \odot (\beta \odot v)\end{aligned}$$

Hence, $(\alpha \circledcirc \beta) \odot v = \alpha \odot (\beta \odot v)$.

8. Since $1 \in \mathbb{F}(=\mathbb{R})$, by definition of scalar multiplication in V over $\mathbb{F}, \forall\ v \in V$ we have,

$$\mathbf{1} \odot u = u^1 = u$$

Hence, $1 \odot u = u\ \ \forall u \in V$.

Since all the properties of a vector space is satisfied, hence $(V, \oplus, \odot)$ is a vector space over the field $(\mathbb{F}, \oplus, \circledcirc)$.

Example 3: *Consider P_n, the set of polynomials of degree less than or equal to n.*

$$P_n = \{p(x)|p(x) = (a_0 + a_1x + ... + a_nx^n) \text{ such that } a_i \in \mathbb{F}, \quad \forall i = 1, 2, 3.., n\}$$

Let $p(x) = a_0 + a_1x + ... + a_nx^n \in P_n$ and $q(x) = b_0 + b_1x + ... + b_nx^n \in P_n$ and $\alpha \in \mathbb{F}$
Operation $\oplus$ in P_n is defined as

$$\begin{aligned}p(x) \oplus q(x) &= (a_0 + a_1x + ... + a_nx^n) \oplus (b_0 + b_1x + ... + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + .. + (a_n + b_n)x^n\end{aligned} \tag{1.10}$$

Another operation $\odot$ in P_n is defined as

$$\alpha \odot (a_0 + a_1x + ... + a_nx^n) = (\alpha a_0 + \alpha a_1x + ... + \alpha a_nx^n) \tag{1.11}$$

Show that the $(P_n, \oplus, \odot)$ is a vector space over field $(\mathbb{F}, \oplus, \circledcirc)$ where $\mathbb{F} = \mathbb{R}$. $\oplus$ and $\circledcirc$ are the usual addition and multiplication over reals.

Solution : In order to prove P_n is a vector space, we check all its properties. Let

$$\begin{aligned}p(x) &= (a_0 + a_1x + ... + a_nx^n) \\ q(x) &= (b_0 + b_1x + ... + b_nx^n) \\ r(x) &= (c_0 + c_1x + ... + c_nx^n)\end{aligned}$$

Let $p(x), q(x)$ and $r(x) \in P_n$ and Let $\alpha, \beta \in \mathbb{F}$.

Vector addition Composition in V : By definition of vector addition,

$$\begin{aligned}p(x) \oplus q(x) &= (a_0 + a_1x + ... + a_nx^n) \oplus (b_0 + b_1x + ... + b_nx^n) \\ &= (a_0 + b_0) + (a_1 + b_1)x + .. + (a_n + b_n)x^n\end{aligned}$$

and since $(a_i + b_i) \in \mathbb{R},\ \ \forall i = 1, 2, \dots n$. Clearly, $(a_0 + b_0) + (a_1 + b_1)x + .. + (a_n + b_n)x^n \in P_n\ \ \forall a_i, b_i \in \mathbb{R}\ \ \forall i = 1, 2 \dots n$. Hence P_n is closed under the operation of vector addition $(\oplus)$.

$$\oplus : P_n \times P_n \longrightarrow P_n$$

Scalar Multiplication Composition in V over $\mathbb{F}$: By definition of scalar multiplication

$$\alpha \odot p(x) = \alpha \odot (a_0 + a_1x + ... + a_nx^n) = (\alpha a_0 + \alpha a_1x + ... + \alpha a_nx^n)$$

and since, $\alpha a_i \in \mathbb{R},\ \forall i = 1,2,\dots n,\ \implies\ (\alpha a_0 + \alpha a_1x + ... + \alpha a_nx^n) = \alpha \odot p(x) \in P_n\ \forall\ p \in P_n$. Hence, P_n is closed under the operation of scalar multiplication $\odot$ i.e

$$\odot : \mathbb{F} \times P_n \longrightarrow P_n$$

1. **Associativity of addition :** Refer previous sections(section 1.1 and 1.2).
2. **Existence of additive identity :** Refer previous sections(section 1.1 and 1.2).
3. **Existence of additive inverses :** Refer previous sections(section 1.1 and 1.2).
4. **Commutativity of addition :** Refer previous sections(section 1.1 and 1.2).
5. **Distributivity of addition over multiplication :** By definition of scalar multiplication in P_n over $\mathbb{F}$, $\forall p(x), q(x) \in P_n$ and $\forall \alpha \in \mathbb{F}(=\mathbb{R})$ we have
$\alpha \odot (p(x) \oplus q(x)) =$
$$\begin{aligned}
&= \alpha \odot ((a_0 + a_1x + ... + a_nx^n) \oplus (b_0 + b_1x + ... + b_nx^n)) \\
&= \alpha \odot ((a_0 + b_0) + (a_1 + b_1)x + ... + (a_n + b_n)x^n) && \{ \text{ from eq(1.10)}\} \\
&= (\alpha(a_0 + b_0) + \alpha(a_1 + b_1)x + ... + \alpha(a_n + b_n)x^n && \{ \text{ from eq(1.11)}\} \\
&= \alpha(a_0 + a_1x + ... + a_nx^n) + \alpha(b_0 + b_1x + ... + b_nx^n) \\
&= (\alpha \odot p(x)) + (\alpha \odot q(x)) && \{ \text{ from eq(1.11)}\} \\
&= \alpha \odot p(x) \oplus \alpha \odot q(x) && \{ \text{ from eq(1.10)}\}
\end{aligned}$$
Hence, $\alpha \odot (p(x) \oplus q(x)) = \alpha \odot p(x) \oplus \alpha \odot q(x)$.
6. **Distributivity of multiplication over addition :** By definition of scalar multiplication in P_n over $\mathbb{F}$, $\forall p(x), q(x) \in P_n$ and $\forall \alpha, \beta \in \mathbb{F}(=\mathbb{R})$ we have
$(\alpha \oplus \beta) \odot p(x)$
$$\begin{aligned}
&= (\alpha + \beta) \odot (a_0 + a_1x + ... + a_nx^n) \\
&= ((\alpha + \beta)a_0 + (\alpha + \beta)a_1x + ... + (\alpha + \beta)a_nx^n) && \{ \text{ from eq.(1.11)}\} \\
&= (\alpha(a_0 + a_1x + ... + a_nx^n)) + (\beta(a_0 + a_1x + ... + a_nx^n)) \\
&= \alpha \odot p(x) \oplus \beta \odot q(x) && \{ \text{ from eq.(1.11)}\}
\end{aligned}$$
Hence, $(\alpha \oplus \beta) \odot p(x) = \alpha \odot p(x) \oplus \beta \odot p(x)$.
7. **Associativity of multiplication :** By definition of scalar multiplication in P_n over $\mathbb{F}(=\mathbb{R})$, $\forall \alpha, \beta \in \mathbb{R}$ and $\forall p(x) \in P_n$ we have,
$$\begin{aligned}
(\alpha \circledcirc \beta) \odot p(x) &= (\alpha\beta) \odot (a_0 + a_1x + ... + a_nx^n) \\
&= \gamma \odot (a_0 + a_1x + ... + a_nx^n) && \{\text{Let } \gamma = (\alpha\beta)\} \\
&= \gamma a_0 + \gamma a_1x + ... + \gamma a_nx^n && \{\text{from equation (1.11)}\} \\
&= (\alpha\beta)a_0 + (\alpha\beta)a_1x + ... + (\alpha\beta)a_nx^n && \{\text{Since, } \gamma = (\alpha\beta)\}
\end{aligned}$$
Now, since $\alpha, \beta \in \mathbb{F}(=\mathbb{R})$ and $a_i \in \mathbb{R}\ \forall i = 1,2,3..n$, then by using associative property of reals we have,
$$\begin{aligned}
(\alpha \circledcirc \beta) \odot p(x) &= \alpha(\beta a_0) + \alpha(\beta a_1)x + ... + \alpha(\beta a_nx^n) && \text{Since, } \gamma = (\alpha\beta) \\
&= \alpha \odot ((\beta a_0) + (\beta a_1)x + ... + (\beta a_nx^n)) && \text{from equation (1.11)} \\
&= \alpha \odot (\beta \odot (a_0 + a_1x + ... + a_)) && \text{from equation (1.11)} \\
&= \alpha \odot (\beta \odot p(x))
\end{aligned}$$
Hence, $(\alpha \circledcirc \beta) \odot p(x) = \alpha \odot (\beta \odot p(x))$
8. **Multiplication by unit element of the field:** Since, $1 \in \mathbb{F}(=\mathbb{R})$, by definition of scalar multiplication in V over $\mathbb{F}, \forall v \in V$ we have, **1** or unit element of field $(\mathbb{R}, \oplus, \circledcirc)$ is 1. Hence,
$\mathbf{1} \odot p = 1 \odot p$
$= 1 \odot (a_0 + a_1x + ... + a_nx^n)$ where 1 is the unit element of $\mathbb{R}$.
$= (1.a_nx^n + 1.a_{n-1}x^{n-1} + ... + 1.a_0)$ from equation (1.11)
$= (a_0 + a_1x + ... + a_nx^n) = p$

Since all the properties of a vector space is satisfied, hence $(V, \oplus, \odot)$ is a vector space over the field $(\mathbb{F}, \oplus, \circledcirc)$.

1.5.2 Theorems

Theorem *Let* $(V, \oplus, \odot)$ *be a vector space over the field* $(\mathbb{F}, \boxed{\oplus}, \boxed{\circledcirc})$ *then*

1. $\alpha \odot 0_V = 0_V \forall\ \alpha \in \mathbb{F}$
2. $0_F \odot u = 0_V \forall\ u \in V$
3. $\boxminus 1 \odot u = \ominus u \forall\ u \in V$

Proof: Since $(V, \oplus, \odot)$ is a vector space over a field $(\mathbb{F}, \boxed{\oplus}, \boxed{\circledcirc})$ therefore both $(V, \oplus)$ and $(\mathbb{F}, \boxed{\oplus})$ are abelian groups.

1. Since $0_V = 0_V \oplus 0_V$ {Since 0_V is identity w.r.t. $\oplus$}
   $\implies \alpha \odot 0_V = \alpha \odot (0_V \oplus 0_V)\ \ \forall\ \alpha \in \mathbb{F}$
   $\implies \alpha \odot 0_V = \alpha \odot 0_V \oplus \alpha \odot 0_V$ {Since $\odot$ is distributed over $\oplus$}
   $\implies \alpha \odot 0_V \oplus 0_V = \alpha \odot 0_V \oplus \alpha \odot 0_V$ {Since 0_V is identity w.r.t. $\oplus$}
   $\implies 0_V = \alpha \odot 0_V$ {In a group, left cancellation law holds}

2. Since $0_F = 0_F \boxed{\oplus} 0_F$ {Since 0_F is identity w.r.t. $\boxed{\oplus}$}
   $\implies 0_F \odot u = (0_F \boxed{\oplus} 0_F) \odot u\ \ \forall\ u \in V$
   $\implies 0_F \odot u = 0_F \odot u \oplus 0_F \odot u$ {Since $\boxed{\oplus}$ is distributed over $\odot$}
   $\implies 0_F \odot u \oplus 0_V = 0_F \odot u \oplus 0_F \odot u$ {Since 0_V is identity w.r.t. $\oplus$}
   $\implies 0_V = 0_F \odot u$ {In a group, left cancellation law holds}

3. Since $\boxed{\ominus} 1 \boxed{\oplus} 1 = 0_F$ {Since $\boxed{\ominus}1$ is inverse of 1 w.r.t. $\boxed{\oplus}$}
   $\implies (\boxed{\ominus} 1 \boxed{\oplus} 1) \odot u = 0_F \odot u\ \ \forall\ u \in V$
   $\implies \boxed{\ominus} 1 \odot u \oplus 1 \odot u = 0_V$ {Since $0_F \odot u = 0_V \forall\ u \in V$}
   $\implies \boxed{\ominus} 1 \odot u \oplus u = 0_V$ {Since $1 \odot u = u \forall\ u \in V$}
   $\implies \boxed{\ominus} 1 \odot u = \ominus u$

1.6 Vector Sub-Spaces

Let $(V, \oplus, \odot)$ be a vector space over the field $(\mathbb{F}, \boxed{\oplus}, \boxed{\circledcirc})$ and S be a non-empty subset of V. If $(S, \oplus, \odot)$ is itself a vector space over the same field $(\mathbb{F}, \boxed{\oplus}, \boxed{\circledcirc})$ then $(S, \oplus, \odot)$ is called a subspace of $(V, \oplus, \odot)$.

1.6.1 Examples of Sub-Spaces

Example 1: *Consider a vector space V, the set of all $m \times n$ matrices with real entries. In section 1.5 we have proved that such a space is a vector space.*

$$V = \{A | A = [a_{ij}]_{mxn}, a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$$

Let $A, B \in V$ and $\alpha \in \mathbb{R}$
Operation $\oplus$ in V is defined as

$$A \oplus B = [a_{ij} + b_{ij}]_{m \times n}, a_{ij}, b_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \tag{1.12}$$

Another operation $\odot$ in V is defined as

$$\alpha \odot A = [\alpha a_{ij}]_{m \times n}, \alpha \in \mathbb{F}, a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \tag{1.13}$$

*Let us consider the set S of all **symmetric** $m \times n$ matrices with all real entries i.e.*

$$S = \{A | A = [a_{ij}]_{m \times n}, a_{ij} = a_{ji}, a_{ij} \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$$

Prove that $(S, \oplus, \odot)$ is a subspace of V.

Solution : From above definition of non-empty sets V and S, one can easily check that $S \subset V$. In order to prove S is a subspace of V, we have to verify that $(S, \oplus, \odot)$ itself is a vector space over same field $(\mathbb{F}, \oplus, \odot)$. Now, Let $A, B, C \in S$, where $A = [a_{ij}]_{m\times n} \& a_{ij} = aji, B = [b_{ij}]_{m\times n} \& b_{ij} = bji$ and $C = [c_{ij}]_{m\times n} \& c_{ij} = cji \; \forall \; i = 1, 2, \dots, m, j = 1, 2, \dots, n$.
We define the same operations $\oplus$ and $\odot$ in S.

Binary addition composition in S : Let $A = [a_{ij}]_{m\times n} \in S$ and $B = [b_{ij}]_{m\times n} \in S, i = 1, 2, \dots, m, j = 1, 2, \dots, n$. By definition of vector addition operation in S we have,

$$A \oplus B = [a_{ij} + b_{ij}]_{m\times n}, a_{ij}, b_{ij} \in \mathbb{R}, i = 1, 2, \dots, m, j = 1, 2, \dots, n.$$

Since $a_{ij} = a_{ji} \& b_{ij} = b_{ji} \implies a_{ij} + b_{ij} = a_{ji} + b_{ji} \in \mathbb{R} \; \forall i = 1, 2, \dots, m, j = 1, 2, \dots, n$. Hence, $A \oplus B = [a_{ij} + b_{ij}]_{m\times n} \in S \;\; \forall \; A, B \in S$. Clearly, $\oplus$ is a binary operation on S i.e. S is closed under the operation $\oplus$.

$$\oplus : S \times S \longrightarrow S$$

Scalar Multiplication Composition in S over $\mathbb{F}$: By definition of scalar multiplication $\alpha a_{ij} \in \mathbb{F} \;\forall\; \alpha \in \mathbb{F}, a_{ij} \in \mathbb{R}, i = 1, 2, \dots, m, j = 1, 2, \dots, n$. Hence S is closed under the operation of scalar multiplication $\odot$ i.e.

$$\odot : \mathbb{F} \times S \longrightarrow S$$

1. **Associativity of addition :** By definition of $\oplus$ operation in S, $\forall \; A, B, C \in S$, we have,
$$\begin{aligned}(A \oplus B) \oplus C &= [a_{ij} + b_{ij}]_{m\times n} \oplus C \\ &= [(a_{ij} + b_{ij}) + c_{ij}]_{m\times n} \\ &= [a_{ij} + (b_{ij} + c_{ij})]_{m\times n} \\ &= A \oplus [(b_{ij} + c_{ij})]_{m\times n} \\ &= A \oplus (B \oplus C)\end{aligned}$$
Hence, $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

2. **Existence of additive identity :** Since, $0_S = [0]_{m\times n} \in S$, therefore by definition of $\oplus$ operation in S, $\forall \; A \in S$, we have,
$$\begin{aligned}0_S \oplus A &= [0]_{m\times n} \oplus [a_{ij}]_{m\times n} \\ &= [(0 + a_{ij})]_{m\times n} \\ &= [a_{ij}]_{m\times n} \\ &= A\end{aligned}$$
Also,
$$\begin{aligned}A \oplus 0_S &= [a_{ij}]_{m\times n} \oplus [0]_{m\times n} \\ &= [(a_{ij} + 0)]_{m\times n} \\ &= [a_{ij}]_{m\times n} \\ &= A\end{aligned}$$
Since $A \oplus 0_S = 0_S \oplus A = A$ holds $\forall A \in S$, therefore 0_S defined above is the additive identity. Also, $0_S = 0_V$.

3. **Existence of additive inverses :** $\forall \; A = [a_{ij}]_{m\times n} \in S$, $a_{ij} \in \mathbb{R}, \forall \; i = 1, 2, \dots, m, j = 1, 2, \dots, n \implies -a_{ij} \in \mathbb{R}, \forall \; i = 1, 2, \dots, m, j = 1, 2 \dots, n$
Let $A' = [-a_{ij}]_{m\times n}, a_{ij} \in \mathbb{R}, \forall \; i = 1, 2, \dots, m, j = 1, 2, \dots, n$. Then, by definition of $\oplus$ operation in S, we have,
$$\begin{aligned}A \oplus A' &= [a_{ij}]_{m\times n} \oplus [-a_{ij}]_{m\times n} \\ &= [a_{ij} + (-a_{ij})]_{m\times n} \\ &= [a_{ij} - a_{ij}]_{m\times n} \\ &= [0]_{m\times n} \\ &= 0_S \in S\end{aligned}$$
Also,
$$\begin{aligned}A' \oplus A &= [-a_{ij}]_{m\times n} \oplus [a_{ij}]_{m\times n} \\ &= [(-a_{ij}) + a_{ij}]_{m\times n} \\ &= [-a_{ij} + a_{ij}]_{m\times n} \\ &= [0]_{m\times n} \\ &= 0_S \in S\end{aligned}$$
Since $A \oplus A' = A' \oplus A = 0_S$, therefore A' defined above is the inverse of A.

4. **Commutativity of addition :** By definition of $\oplus$ operation in S, $\forall\, A, B \in S$, we have,

$$\begin{aligned} A \oplus B &= [a_{ij} + b_{ij}]_{m\times n} \\ &= [b_{ij} + a_{ij}]_{m\times n} \\ &= B \oplus A \end{aligned}$$

Since $A \oplus B = B \oplus A$, therefore $\oplus$ is commutative.

5. **Distributivity of addition over multiplication :** By definition of scalar multiplication in S over $\mathbb{F}$, $\forall\, A \in S, \alpha, \beta \in \mathbb{F}$, we have,

$$\begin{aligned} (\alpha \oplus \beta) \odot A &= (\alpha \oplus \beta) \oplus [a_{ij}]_{m\times n} \\ &= (\alpha + \beta) \oplus [a_{ij}]_{m\times n} \\ &= [(\alpha + \beta) a_{ij}]_{m\times n} \\ &= [\alpha a_{ij} + \beta a_{ij}]_{m\times n} \\ &= [\alpha a_{ij}]_{m\times n} \oplus \beta [a_{ij}]_{m\times n} \\ &= \alpha \odot [a_{ij}]_{m\times n} \oplus \beta \odot [a_{ij}]_{m\times n} \\ &= \alpha \odot A \oplus \beta \odot A \end{aligned}$$

Hence, $(\alpha \oplus \beta) \odot A = \alpha \odot A \oplus \beta \odot B$

6. **Distributivity of multiplication over addition :** By definition of scalar multiplication in S over $\mathbb{F}$, $\forall\, A, B \in S, \alpha \in \mathbb{F}$ we have,

$$\begin{aligned} \alpha \odot (A \oplus B) &= \alpha \odot ([a_{ij}]_{m\times n} \oplus [b_{ij}]_{m\times n}) \\ &= \alpha \odot [a_{ij} + b_{ij}]_{m\times n} \\ &= [\alpha(a_{ij} + b_{ij})]_{m\times n} \\ &= [\alpha a_{ij} + \alpha b_{ij}]_{m\times n} \\ &= [\alpha a_{ij}]_{m\times n} \oplus [\alpha b_{ij}]_{m\times n} \\ &= \alpha \odot [a_{ij}]_{m\times n} \oplus \alpha \odot [b_{ij}]_{m\times n} \\ &= \alpha \odot A \oplus \alpha \odot B \end{aligned}$$

Hence $\alpha \odot (A \oplus B) = \alpha \odot A \oplus \alpha \odot B$

7. **Associativity of multiplication :** By definition of scalar multiplication in S over $\mathbb{F}$, $\forall\, A \in S, \alpha, \beta \in \mathbb{F}$ we have,

$$\begin{aligned} (\alpha \circledcirc \beta) \odot A &= (\alpha\beta) \odot [a_{ij}]_{m\times n} \\ &= [(\alpha\beta) a_{ij}]_{m\times n} \\ &= [\alpha(\beta a_{ij})]_{m\times n} \\ &= \alpha \odot [\beta a_{ij}]_{m\times n} \\ &= \alpha \odot (\beta \odot [a_{ij}]_{m\times n}) \\ &= \alpha \odot (\beta \odot A) \end{aligned}$$

Hence, $(\alpha \circledcirc \beta) \odot A = \alpha \odot (\beta \odot A)$

8. **Multiplication by unit element of the field:** Since $1 \in \mathbb{F}(= \mathbb{R})$, by definition of scalar multiplication in S over $\mathbb{F}$, $\forall\, A \in S$ we have,

$$\begin{aligned} 1 \odot A &= 1 \odot [a_{ij}]_{m\times n} \\ &= [1 a_{ij}]_{m\times n} \\ &= [a_{ij}]_{m\times n} \\ &= A \end{aligned}$$

Hence $1 \odot A = A$

Since all the properties of a vector space is satisfied, hence $(S, \oplus, \odot)$ is a vector space over the field $(\mathbb{F}, \oplus, \circledcirc)$. Since, $S \subset V$ hence S is the subspace of V.

Example 2: *Consider V, the set of all real valued functions defined on the interval [a,b]*

$$V = \{f | f : [a, b] \longrightarrow \mathbb{R}\}$$

Let $f, g \in V, x \in [a, b]$ and $\alpha \in \mathbb{R}$
Operation $\oplus$ in V is defined as

$$(f \oplus g)x = f(x) + g(x) \forall\, x \in [a, b] \tag{1.14}$$

Another operation $\odot$ in V is defined as

$$(\alpha \odot f)x = \alpha f(x) \forall\, x \in [a,b] \tag{1.15}$$

Let us consider the set S of all real valued continuous functions defined on the interval [a,b]

$$S = \{f | f : [a,b] \longrightarrow \mathbb{R}, f \text{is continuous on} [a,b]\}$$

Prove that $(S, \oplus, \odot)$ is a vector space over same field as that V i.e. $(\mathbb{F}(= \mathbb{R}), \oplus, \odot)$.

Solutions : From above definition of non-empty sets V and S, one can easily check that $S \subset V$. In order to prove S is a subspace of V, we have to verify that $(S, \oplus, \odot)$ itself is a vector space over same field $(\mathbb{F}, \oplus, \odot)$. Now, Let $f, g, h \in S$, hence $f(x), g(x)$ and $h(x)$ are real valued continuous functions.
We define the same operations $\oplus$ and $\odot$ in S also.

Binary addition composition in S : Let $f(x), g(x) \in S$. By definition of vector addition operation in S we have,

$$(f \oplus g)x = f(x) + g(x) \forall\, x \in [a,b] \tag{1.16}$$

Since, $f(x)$ and $g(x)$ are continuous functions then $f(x)+g(x)$ is also a continuous function. Hence, $\forall f(x), g(x) \in S$ then $f(x) + g(x) \in S$. Hence, $\oplus$ is a binary operation on S i.e. S is closed under the operation $\oplus$.

$$\oplus : S \times S \longrightarrow S$$

Scalar Multiplication in S over F : By definition of scalar multiplication, $\alpha \in \mathbb{F}$

$$(\alpha \odot f)x = \alpha f(x) \forall\, x \in [a,b] \tag{1.17}$$

$\forall\, \alpha \in \mathbb{F}, f(x) \in S, \quad \alpha f(x) \in S$. Hence S is closed under the operation of scalar multiplication $\odot$ i.e.

$$\odot : \mathbb{F} \times S \longrightarrow S$$

1. **Associativity of addition :** By definition of vector addition operation in S, $\forall\, f(x), g(x), h(x) \in S$, we have,
   $$\begin{aligned}(f \oplus g)x \oplus h(x) &= (f(x) + g(x)) \oplus h(x)\\ &= (f(x) + g(x)) + h(x)\\ &= f(x) + g(x) + h(x)\\ &= f(x) + (g(x) + h(x))\\ &= f(x) + (g \oplus h)x\\ &= f(x) \oplus (g \oplus h)x\end{aligned}$$
   Hence, $(f \oplus g)x \oplus h(x) = f(x) \oplus (g \oplus h)x$

2. **Existence of additive identity :** Since, $0_S = 0(x) = 0 \in S$, therefore by definition of $\oplus$ operation in S, $\forall\, f(x) \in S$, we have,
   $$\begin{aligned}0_S \oplus f(x) &= (0 \oplus f)x\\ &= 0(x) + f(x)\\ &= f(x)\end{aligned}$$
   Also,
   $$\begin{aligned}f(x) \oplus 0_S &= (f \oplus 0)x\\ &= f(x) + 0(x)\\ &= f(x)\end{aligned}$$
   Since $0_S \oplus f(x) = f(x) \oplus 0_S = f(x)$ holds $\forall f(x) \in S$, therefore 0_S defined above is the additive identity.

3. **Existence of additive inverses :** $\forall\, f(x) \in S$, Let $\bar{f}(x) = -f(x)$. Then, by definition of $\oplus$ operation in S, we have,
   $$\begin{aligned}f(x) \oplus \bar{f}(x) &= f(x) + \bar{f}(x)\\ &= f(x) - f(x)\\ &= 0_S \in S\end{aligned}$$
   Also,

$$\begin{aligned}\bar{f}(x) \oplus f(x) &= f(x) + \bar{f}(x) \\ &= -f(x) + f(x) \\ &= 0_S \in S\end{aligned}$$

Since $f(x) \oplus \bar{f}(x) = \bar{f}(x) \oplus f(x) = 0_S$, therefore $\bar{f}(x)$ defined above is the inverse of $f(x)$.

4. **Commutativity of addition :** By definition of vector addition in S, $\forall\, f, g \in S$, we have,
$$\begin{aligned}(f \oplus g)x &= f(x) + g(x) \\ &= g(x) + h(x) \\ &= (g \oplus h)x\end{aligned}$$
Since $(f \oplus g)x = (g \oplus h)x$, therefore $\oplus$ is commutative.

5. **Distributive of addition over multiplication :** By definition of scalar multiplication in S over $\mathbb{F}$, $\forall\, f(x) \in S, \alpha, \beta \in \mathbb{F}$, we have,
$$\begin{aligned}((\alpha \oplus \beta) \odot f)x &= ((\alpha \oplus \beta) \odot f)x \\ &= ((\alpha + \beta) \odot f)x \\ &= (\alpha + \beta) f(x) \\ &= (\alpha f(x) + \beta f(x)) \\ &= (\alpha \odot f)x + (\beta \odot f)x \\ &= (\alpha \odot f \oplus \beta \odot f)x\end{aligned}$$
Hence, $((\alpha \oplus \beta) \odot f)x = (\alpha \odot f \oplus \beta \odot f)x$

6. **Distributive of multiplication over addition :** By definition of scalar multiplication in S over $\mathbb{F}$, $\forall\, f(x), g(x) \in S, \alpha \in \mathbb{F}$ we have,
$$\begin{aligned}(\alpha \odot (f \oplus g))x &= \alpha((f + g)x) \\ &= \alpha(f(x) + g(x)) \\ &= (\alpha f(x) + \alpha g(x)) \\ &= (\alpha \odot f)x + (\beta \odot g)x \\ &= (\alpha \odot f \oplus \beta \odot f)x\end{aligned}$$
Hence, $\alpha \odot (f \oplus g)x = (\alpha \odot f \oplus \beta \odot f)x$

7. **Associativity of multiplication :** By definition of scalar multiplication in S over $\mathbb{F}$, $\forall\, A \in S, \alpha, \beta \in \mathbb{F}$ we have,
$$\begin{aligned}((\alpha \odot \beta) \odot f)x &= ((\alpha\beta) \odot f)(x) \\ &= (\alpha\beta)\ f(x) \\ &= \alpha(\beta f(x)) \\ &= \alpha \odot ((\beta \odot f)x)\end{aligned}$$
Hence, $((\alpha \odot \beta) \odot f)x = \alpha \odot ((\beta \odot f)x)$

8. **Multiplication by unit element of the field :** Since $1 \in \mathbb{F}(= \mathbb{R})$, by definition of scalar multiplication in S over $\mathbb{F}$, $\forall\, A \in S$ we have,
$$\begin{aligned}(1 \odot f)x &= 1\ f(x) \\ &= f(x)\end{aligned}$$
Hence, $(1 \odot f)x = f(x)$

Theorem *Let $(V, \oplus, \odot)$ be a vector space over the field $(\mathbb{F}, \oplus, \odot)$ and S be a non-empty subset of V. Then S will be a subspace of $(V, \oplus, \odot)$ iff the following two properties are satisfied:*

- $u \oplus v \in S \qquad \forall u, v \in S$
- $\alpha \odot v \in S \qquad \forall \alpha \in \mathbb{F}, v \in S$

Proof: Given that $(V, \oplus, \odot)$ be a vector space over a field $(\mathbb{F}, \oplus, \odot)$ and $S \subset V$.
First suppose that S is a subspace of $(V, \oplus, \odot)$ over the field $(\mathbb{F}, \oplus, \odot)$ then we have to show that above two conditions hold.
Since S is a subspace of $(V, \oplus, \odot)$ over the field $(\mathbb{F}, \oplus, \odot)$, therefore $(S, \oplus, \odot)$ itself is a vector space over the field $(\mathbb{F}, \oplus, \odot)$. Hence,

- $u \oplus v \in S \qquad \forall u, v \in S$

- $\alpha \odot v \in S \qquad \forall \alpha \in \mathbb{F}, v \in S$

Conversely suppose that the two conditions hold then we have to show that S will be a subspace of $(V, \oplus, \odot)$ over the field $(\mathbb{F}, \circledast, \circledcirc)$.
The first condition above implies that S satisfies closure property w.r.t. $\oplus$.
Since S is a sub-set of V hence S satisfies associativity and commutativity w.r.t. $\oplus$. i.e.

1. $(u \oplus v) \oplus w = u \oplus (v \oplus w) \quad \forall\ u, v, w \in S.$
2. $u \oplus v = v \oplus u \quad \forall\ u, v \in S.$

Also since, $\alpha \odot u \in S\ \ \forall \alpha \in F, u \in S.$
Choosing $\alpha = \mathbf{0}_F$ (the additive identity of the field $\mathbb{F}$), we get,
$\mathbf{0}_F \odot u \in S\ \ \forall\ \ u \in S \implies \mathbf{0}_V \in S$ $\{$Since $\mathbf{0}_F \odot u = \mathbf{0}_V\}$
Hence, S contains the additive identity.
Again, choosing $\alpha = \ominus\mathbf{1}$ $\{$the additive inverse of the unit element of the field $\mathbb{F}\}$, We get,
$\ominus\mathbf{1} \odot u \in S\ \ \forall\ \ u \in S \implies \ominus u \in S\ \ \forall\ \ u \in S,\quad$ $\{$Since $\ominus\mathbf{1} \odot u = \ominus u\}$
Hence, S contains additive inverse of all its elements.
Since all the properties of abelian group are satisfied, hence $(S, \oplus)$ is an abelian group.
The second condition above implies that S satisfies scalar multiplication property w.r.t. the external composition $\odot$.
Since S is a sub-set of V, the following properties would obliviously be satisfied

1. $\alpha \odot (u \oplus v) = \alpha \odot u \oplus \alpha \odot u \quad \{\forall\ \alpha \in F, u, v \in\}$
2. $(\alpha \circledast \beta) \odot u = \alpha \odot u \oplus \beta \odot u \quad \{\alpha, \beta \in F; u \in S\}$
3. $(\alpha \circledcirc \beta) \odot u = \alpha \odot (\beta \odot u) \quad \{\alpha, \beta \in F; u \in S\}$
4. $\mathbf{1} \odot u = u \quad \{\forall u \in S$ and $\mathbf{1}$ is the unit element of $\mathbb{F}$ $\}$.

Since, all the properties of a vector space is satisfied; hence S is a subspace.

REMARK : *As an application this theorem can be straight-away applied to show that last two examples are subspaces by showing only two properties namely closure w.r.t to addition and multiplication.*

Theorem *Let $(V, \oplus, \odot)$ be a vector space over a field $(\mathbb{F}, \circledast, \circledcirc)$ and $(U, \oplus, \odot)$ and $(W, \oplus, \odot)$be two subspaces of $(V, \oplus, \odot)$ over $(\mathbb{F}, \circledast, \circledcirc)$,then*

1. *$U \cap W$ is also subspace of V.*
2. *$U \cup W$ will be subspace of V iff either $U \subseteq W$ or $W \subseteq U$.*

Proof

1. Since $(U, \oplus, \odot)$ and $(W, \oplus, \odot)$ are subspaces of $(V, \oplus, \odot) \implies U \subseteq V$ and $W \subseteq V \implies U \cap W \subseteq V$. Also since $\mathbf{0}_U (= \mathbf{0}_V) \in U$ and $\mathbf{0}_W (= \mathbf{0}_V) \in W \implies U \cap W \neq \phi$. Hence $U \cap W$ is a non-empty subset of V. Now, to show that $U \cap W$ is a subspace it is sufficient to prove that:

   (a) $v_1 \oplus v_2 \in U \cap W\ \forall\ v_1, v_2 \in U \cap W$
   (b) $\alpha \odot v \in U \cap W\ \forall\ \alpha \in \mathbb{F}, v \in U \cap W$

   Let $v_1, v_2 \in U \cap W$
   $\implies v_1, v_2 \in U$ and $v_1, v_2 \in W$
   $\implies v_1 \oplus v_2 \in U$ and $v_1 \oplus v_2 \in W$ $\{$Since U and W are vector spaces$\}$
   $\implies v_1 \oplus v_2 \in U \cap W$
   Hence $U \cap W$ is closed w.r.t. $\oplus$
   Let $\alpha \in \mathbb{F}$, $u \in U \cap W \implies u \in U, u \in W$
   $\implies \alpha \odot u \in U$ and $\alpha \odot u \in W$ $\{$Since U and W are vector spaces$\}$
   $\implies \alpha \odot u \in U \cap W$
   Hence $U \cap W$ is closed w.r.t. $\odot$
   $\therefore (U \cap W, \oplus, \odot)$ is also subspace of $(V, \oplus, \odot)$ over the field $(\mathbb{F}, \circledast, \circledcirc)$.

2. First suppose that $U \subseteq W$ or $W \subseteq U$ then we have to prove that $U \cup W$ is a subspace of V
Now, $U \subseteq W$ or $W \subseteq U$
$\implies U \cup W = W$ or $U \cap W = U$
$\implies U \cup W$ is a subspace of V.

Conversely suppose that $(U \cup W, \oplus, \odot)$ is a subspace of $(V, \oplus, \odot)$ then we have to show that either $U \subseteq W$ or $W \subseteq U$.
Let us suppose that $U \nsubseteq W$ and $W \nsubseteq U$.
Since $U \nsubseteq W \implies \exists\ u \in U$ such that $u \notin W$.
Since $W \nsubseteq U \implies \exists\ w \in W$ such that $w \notin U$.
But since $u \in U$ and $w \in W \implies u, w \in U \cup W$.
Also by our assumption $(U \cup W, \oplus, \odot)$ is a subspace of $(V, \oplus, \odot)$ over $(\mathbb{F}, \oplus, \circledcirc)$
$\implies u \oplus w \in U \cup W$
Now two cases arise either $u \oplus w \in U$ or $u \oplus w \in W$
If $u \oplus w \in U$ then since $u \in U$ and U is a subspace $\implies \ominus u \in U$
$\implies (u \oplus w) \oplus (\ominus u) \in U$
$\implies w \in U$
Which is contradiction.
On the other hand if $u \oplus w \in W$ then since $w \in W$ and W is a subspace $\implies \ominus w \in W$
$\implies (u \oplus w) \oplus (\ominus w) \in W$
$\implies u \in W$
Which is again a contradiction. Since our initial we assume that $U \nsubseteq W$ and $W \nsubseteq U$ leads to a contradiction in either of the above two cases. Hence our initial assumption is wrong. So, either either $U \subseteq W$ or $W \subseteq U$