In this section, we introduce a test for the linear independence of a set of functions. For our purpose, let V be the vector space of all functions on $\mathbb{R}$ which are differentiable infinitely many times. Then one can easily see that V is not finite dimensional.

Let $f_1(x)$, $f_2(x)$, $\cdots$, $f_n(x)$ be n functions in V. The n functions are linearly independent in V if the linear equation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all $x \in \mathbb{R}$ implies that all $c_i = 0$. By taking the differentiation $n-1$ times, we obtain n equations:

$$c_1 f_1^{(i)}(x) + c_2 f_2^{(i)}(x) + \cdots + c_n f_n^{(i)}(x) = 0, \quad 0 \le i \le n-1,$$

for all $x \in \mathbb{R}$. Or, in a matrix form:

$$\begin{bmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & & & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The determinant of the coefficient matrix is called the **Wronskian** for $\{f_1(x), f_2(x), \cdots, f_n(x)\}$ and denoted by $W(x)$. Therefore, if there is a point $x_0 \in \mathbb{R}$ such that $W(x) \neq 0$, then the coefficient matrix is nonsingular at $x = x_0$, and so all $c_i = 0$. Therefore, if the Wronskian is nonzero at a point in $\mathbb{R}$, then $\{f_1(x), f_2(x), \cdots, f_n(x)\}$ are linearly independent.

Example For the sets of functions $F_1 = \{x, \cos x, \sin x\}$ and $F_2 = \{x, e^x, e^{-x}\}$, the Wronskians are

$$W_1(x) = \det \begin{bmatrix} x & \cos x & \sin x \\ 1 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} = x$$

and

$$W_2(x) = \det \begin{bmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{bmatrix} = 2x.$$

Since $W_i(x) \neq 0$ for $x \neq 0$, both F_i are linearly independent.