Find the dimension and a basis of the solution space $W$ of each homogeneous system:

$$\begin{array}{ccc}
\begin{aligned} x+2y+2z-s+3t&=0\\ x+2y+3z+s+t&=0,\\ 3x+6y+8z+s+5t&=0 \end{aligned} &
\begin{aligned} x+2y+z-2t&=0\\ 2x+4y+4z-3t&=0\\ 3x+6y+7z-4t&=0 \end{aligned} &
\begin{aligned} x+y+2z&=0\\ 2x+3y+3z&=0\\ x+3y+5z&=0 \end{aligned}\\
(a) & (b) & (c)
\end{array}$$

(*a*) Reduce the system to echelon form:

$$\begin{aligned} x+2y+2z-s+3t&=0\\ z+2s-2t&=0\\ 2z+4s-4t&=0 \end{aligned} \quad \text{or} \quad \begin{aligned} x+2y+2z-s+3t&=0\\ z+2s-2t&=0 \end{aligned}$$

The system in echelon form has two (nonzero) equations in five unknowns. Hence the system has $5-2=3$ free variables, which are $y, s, t$. Thus $\dim W = 3$. We obtain a basis for $W$:

(1) Set $y=1, s=0, t=0$ to obtain the solution $v_1 = (-2, 1, 0, 0, 0)$.

(2) Set $y=0, s=1, t=0$ to obtain the solution $v_2 = (5, 0, -2, 1, 0)$.

(3) Set $y=0, s=0, t=1$ to obtain the solution $v_3 = (-7, 0, 2, 0, 1)$.

The set $\{v_1, v_2, v_3\}$ is a basis of the solution space $W$.

(*b*) (Here we use the matrix format of our homogeneous system.) Reduce the coefficient matrix $A$ to echelon form:

$$A = \begin{bmatrix} 1 & 2 & 1 & -2\\ 2 & 4 & 4 & -3\\ 3 & 6 & 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2\\ 0 & 0 & 2 & 1\\ 0 & 0 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2\\ 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the system

$$\begin{aligned} x+2y+2z-2t&=0\\ 2z+t&=0 \end{aligned}$$

The free variables are $y$ and $t$, and $\dim W = 2$.

(i) Set $y=1, z=0$ to obtain the solution $u_1 = (-2, 1, 0, 0)$.

(ii) Set $y=0, z=2$ to obtain the solution $u_2 = (6, 0, -1, 2)$.

Then $\{u_1, u_2\}$ is a basis of $W$.

(*c*) Reduce the coefficient matrix $A$ to echelon form:

$$A = \begin{bmatrix} 1 & 1 & 2\\ 2 & 3 & 3\\ 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 0 & 5 \end{bmatrix}$$

This corresponds to a triangular system with no free variables. Thus 0 is the only solution, that is, $W = \{0\}$. Hence $\dim W = 0$.

Find a homogeneous system whose solution set $W$ is spanned by

$$\{u_1, u_2, u_3\} = \{(1, -2, 0, 3), \quad (1, -1, -1, 4), \quad (1, 0, -2, 5)\}$$

Let $v = (x, y, z, t)$. Then $v \in W$ if and only if $v$ is a linear combination of the vectors $u_1, u_2, u_3$ that span $W$. Thus form the matrix $M$ whose first columns are $u_1, u_2, u_3$ and whose last column is $v$, and then row reduce $M$ to echelon form. This yields

$$M = \begin{bmatrix} 1 & 1 & 1 & x \\ -2 & -1 & 0 & y \\ 0 & -1 & -2 & z \\ 3 & 4 & 5 & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & x \\ 0 & 1 & 2 & 2x+y \\ 0 & -1 & -2 & z \\ 0 & 1 & 2 & -3x+t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & x \\ 0 & 1 & 2 & 2x+y \\ 0 & 0 & 0 & 2x+y+z \\ 0 & 0 & 0 & -5x-y+t \end{bmatrix}$$

Then $v$ is a linear combination of $u_1, u_2, u_3$ if $\operatorname{rank}(M) = \operatorname{rank}(A)$, where $A$ is the submatrix without column $v$. Thus set the last two entries in the fourth column on the right equal to zero to obtain the required homogeneous system:

$$\begin{aligned} 2x + y + z \quad &= 0 \\ 5x + y \quad - t &= 0 \end{aligned}$$

Consider the following subspaces of $\mathbf{R}^5$:

$$U = \text{span}(u_1, u_2, u_3) = \text{span}\{(1, 3, -2, 2, 3), \quad (1, 4, -3, 4, 2), \quad (2, 3, -1, -2, 9)\}$$
$$W = \text{span}(w_1, w_2, w_3) = \text{span}\{(1, 3, 0, 2, 1), \quad (1, 5, -6, 6, 3), \quad (2, 5, 3, 2, 1)\}$$

Find a basis and the dimension of: (*a*) $U + W$, (*b*) $U \cap W$.

(*a*) $U + W$ is the space spanned by all six vectors. Hence form the matrix whose rows are the given six vectors, and then row reduce to echelon form:

$$\begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 1 & 4 & -3 & 4 & 2 \\ 2 & 3 & -1 & -2 & 9 \\ 1 & 3 & 0 & 2 & 1 \\ 1 & 5 & -6 & 6 & 3 \\ 2 & 5 & 3 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & -3 & 3 & -6 & 3 \\ 0 & 0 & 2 & 0 & -2 \\ 0 & 2 & -4 & 4 & 0 \\ 0 & -1 & 7 & -2 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -2 & 2 & 3 \\ 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following three nonzero rows of the echelon matrix form a basis of $U \cap W$:

$$(1, 3, -2, 2, 2, 3), \qquad (0, 1, -1, 2, -1), \qquad (0, 0, 1, 0, -1)$$

Thus $\dim(U + W) = 3$.

(*b*) Let $v = (x, y, z, s, t)$ denote an arbitrary element in $\mathbf{R}^5$. First find, say as in Problem 4.49, homogeneous systems whose solution sets are $U$ and $W$, respectively.

Let $M$ be the matrix whose columns are the $u_i$ and $v$, and reduce $M$ to echelon form:

$$M = \begin{bmatrix} 1 & 1 & 2 & x \\ 3 & 4 & 3 & y \\ -2 & -3 & -1 & z \\ 2 & 4 & -2 & s \\ 3 & 2 & 9 & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 1 & -3 & -3x + y \\ 0 & 0 & 0 & -x + y + z \\ 0 & 0 & 0 & 4x - 2y + s \\ 0 & 0 & 0 & -6x + y + t \end{bmatrix}$$

Set the last three entries in the last column equal to zero to obtain the following homogeneous system whose solution set is $U$:

$$-x + y + z = 0, \qquad 4x - 2y + s = 0, \qquad -6x + y + t = 0$$

Now let $M'$ be the matrix whose columns are the $w_i$ and $v$, and reduce $M'$ to echelon form:

$$M' = \begin{bmatrix} 1 & 1 & 2 & x \\ 3 & 5 & 5 & y \\ 0 & -6 & 3 & z \\ 2 & 6 & 2 & s \\ 1 & 3 & 1 & t \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & x \\ 0 & 2 & -1 & -3x + y \\ 0 & 0 & 0 & -9x + 3y + z \\ 0 & 0 & 0 & 4x - 2y + s \\ 0 & 0 & 0 & 2x - y + t \end{bmatrix}$$

Again set the last three entries in the last column equal to zero to obtain the following homogeneous system whose solution set is $W$:

$$-9 + 3 + z = 0, \qquad 4x - 2y + s = 0, \qquad 2x - y + t = 0$$

Combine both of the above systems to obtain a homogeneous system, whose solution space is $U \cap W$, and reduce the system to echelon form, yielding

$$\begin{aligned} -x + y + \ z &= 0 \\ 2y + 4z + \ s &= 0 \\ 8z + 5s + 2t &= 0 \\ s - 2t &= 0 \end{aligned}$$

There is one free variable, which is $t$; hence $\dim(U \cap W) = 1$. Setting $t = 2$, we obtain the solution $u = (1, 4, -3, 4, 2)$, which forms our required basis of $U \cap W$.