Echelon Form and Reduced Echelon Form

Our objective when solving a system of linear equations is to perform row operations to arrive at a matrix which is as much simplified as possible. Recall that through elementary operations, we were able to transform the system of equations

$$\begin{aligned} 2x_1 - 2x_2 + 8x_3 &= 10 \\ 4x_1 - 6x_2 + 2x_3 &= \ 4 \\ 2x_2 + 4x_3 &= \ 6 \end{aligned}$$

with augmented matrix $\begin{bmatrix} 2 & -2 & 8 & 10 \\ 4 & -6 & 2 & 4 \\ 0 & 2 & 4 & 6 \end{bmatrix}$ to the very (most) simplified matrix $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$.

This most simplified form is known as the *reduced echelon form.*

A rectangular matrix (does not need to be a square matrix) is in **echelon form** if it has the following properties:

1. each leading entry (the leftmost nonzero entry) of a row is in a column to the right of the leading entry of the row above it,
2. all nonzero rows are above any rows of all zeros (the zeros are at the bottom of the matrix),
3. all entries in a column below a leading entry are zero.

Example of a matrix in echelon form

Note that the leading entries are denoted by $LE := \Re\backslash\{0\}$ whereas the starred ($*$) entries may have any value including zero.

$$\begin{bmatrix} 0 & LE & * & * & * & * & * & * & * \\ 0 & 0 & 0 & LE & * & * & * & * & * \\ 0 & 0 & 0 & 0 & LE & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & LE & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & LE & * \end{bmatrix}$$

An echelon form is *not unique* as there can be infinite variations of the same augmented matrix.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form**:

4. the leading entry in any nonzero row is 1.
5. all entries above and below the leading 1, are zero.

Example of a matrix in reduced echelon form
Note that the starred ($*$) entries may have any value including zero.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

On the other hand, a reduced echelon matrix is in fact, *unique*.