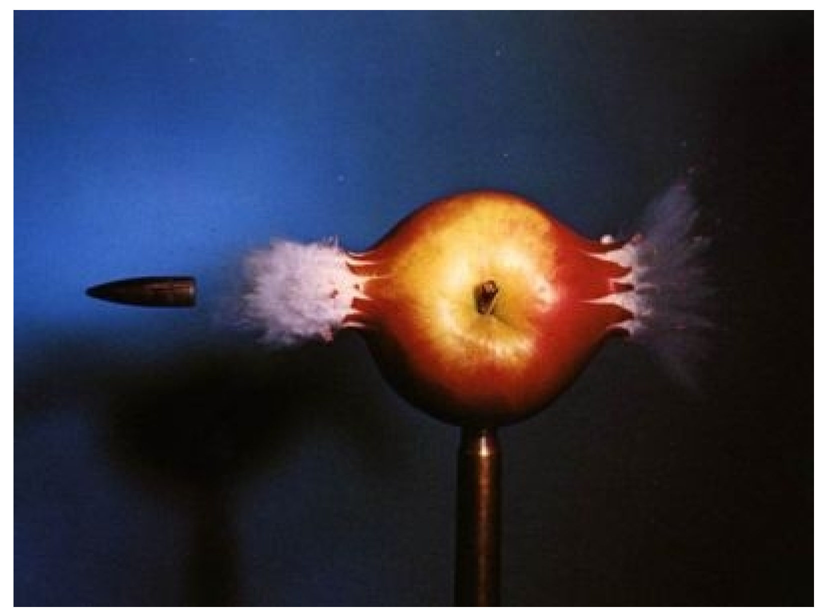
Laws of Newtonian mechanics



Size of objects

Decreasing size of particles

Classical mechanics

Mechanics of planets, and larger objects,
Mechanics of balls, apples simple pendulum etc..

Limiting case mechanics of Cells and macro molecules

$$10^{-9}\,meters\,(nano)$$

Quantum mechanics

Mechanics of atoms and molecule as limiting case

Mechanics of electron and nucleus, protons and neutrons etc..

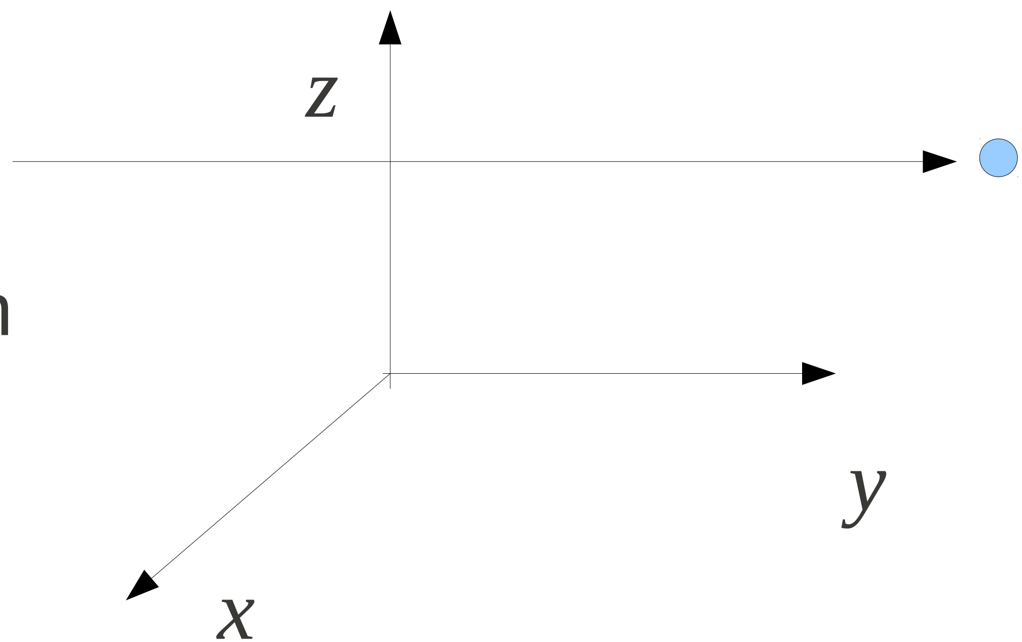
Any object that is smaller than atoms

Newton's first law

A body remains in a state of rest or of constant velocity when no external forces acting on it

$$\vec{a}=0 \quad \text{when} \quad \vec{F}=0$$

Time is uniform



Particle moves in straight line

Time can be measured not with respect uniform motion of isolated particle. It can be measured from rotation of an isolated particle.

It is also assumed that space is uniform everywhere

Newton's second law

The rate of change of momentum of a body is proportional to the force on the body.

$$\vec{F}=k\frac{d}{dt}(m\vec{v})=k\,m\frac{d\vec{v}}{dt}$$

Choose units such that $k=1$

Assuming $\frac{dm}{dt}=0$ $\Rightarrow \vec{F}=m\vec{a}=m\frac{d^2\vec{r}}{dt^2}$

When time t changes to $-t$ the dynamics is invariant under time reversal

In this equation the position of the body is represented by a point particle so that $\vec{r}$ give exact position of that particle.

Law of inertia: There exists in nature a unique class of mutually unaccelerated reference frames (the inertial frames) in which First Law is true.

$$\vec{F} = m\vec{a} = m\frac{d^2\vec{r}}{dt^2}$$

$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}$$

Represent force due to a cause, generally a number or a function

For Eg. force can be of origin gravitational, electrostatic... etc. This is called equation of motion for a point particle.

Atwood's machine one of example where equation of motion can be reduced to 1 dimension

Atwood's Machine (Application Newtonian dynamics) $\vec{F}=m\vec{a}$

What is the acceleration of the masses? What is the tension generated in the string?

$$m\frac{d\vec{v}}{dt}=\vec{T}-m\vec{g}$$

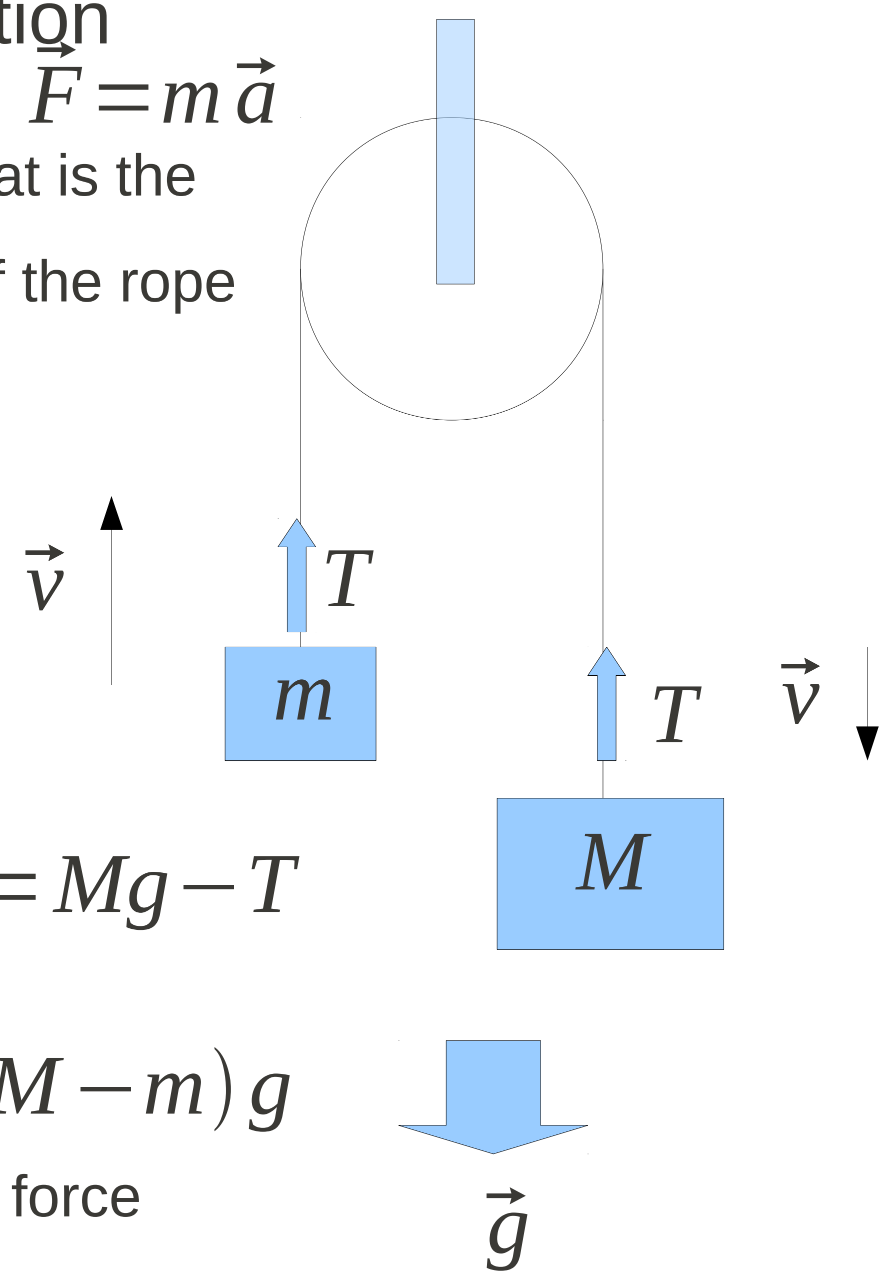
Tension of the rope

Converting it into 1d problem

Equation of motion for the mass m

$$m\frac{dv}{dt}=T-mg$$

Equation of motion for the mass M: $M\frac{dv}{dt}=Mg-T$



Solving for $\frac{dv}{dt}$ $\quad (M+m)\frac{dv}{dt}=(M-m)g$

$$\frac{dv}{dt}=\frac{(M-m)}{(M+m)}g$$

Net force

Net mass

Solving for T $\quad T=\left(\frac{2Mm}{M+m}\right)g$

Atwood's Machine with double-rope pulley

Equations of motion for mass M_1

(I) $$M_1 a_1 \hat{e} = -M_1 g \hat{e} + T \hat{e}$$

Acceleration for mass M_1

Unit vector along motion

(II)

Equations for mass M_2 (mass of second pulley)

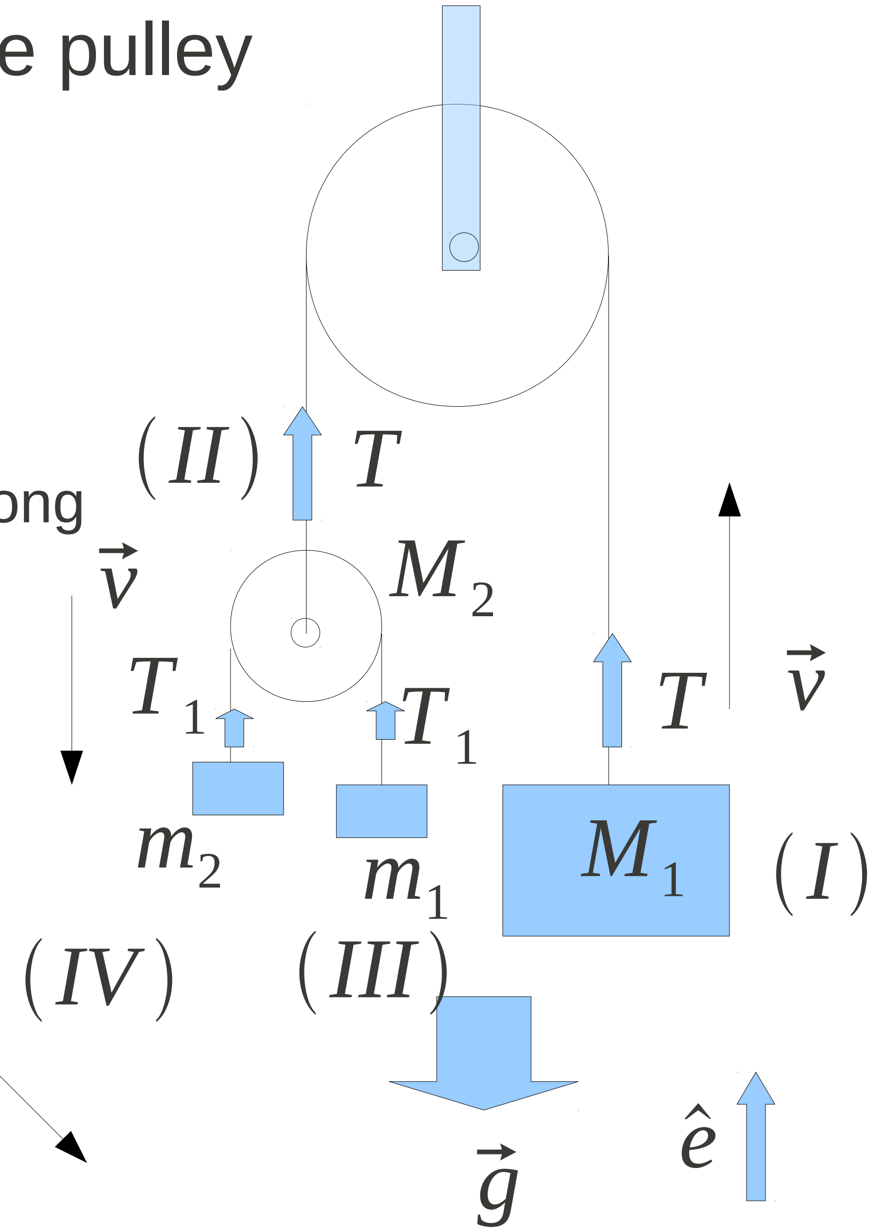
$$-M_2 a_1 \hat{e} = -M_2 g \hat{e} + T \hat{e} - 2 T_1 \hat{e}$$

(III) Equations for mass m_1

$$m_1 (a_2 - a_1) \hat{e} = -m_1 g \hat{e} + T_1 \hat{e}$$

(IV) Equations for mass m_2

$$m_2 (-a_2 - a_1) \hat{e} = -m_2 g \hat{e} + T_1 \hat{e}$$



Tension of second rope

$$M_1 a_1 \hat{e} = -M_1 g \hat{e} + T \hat{e} \qquad (1)$$

$$-M_2 a_1 \hat{e} = -M_2 g \hat{e} + T \hat{e} - 2T_1 \hat{e} \qquad (2)$$

$$m_1 (a_2 - a_1) \hat{e} = -m_1 g \hat{e} + T_1 \hat{e} \qquad (3)$$

$$m_2 (-a_2 - a_1) \hat{e} = -m_2 g \hat{e} + T_1 \hat{e} \qquad (4)$$

There are four equations and four unknowns $\{a_{1,} a_{2,} T, T_1\}$

$$(2) - (1)$$

$$(M_1 + M_2) a_1 = -(M_1 - M_2) g + 2T_1 \qquad (5)$$

$$(3) + (4)$$

$$-(m_1 + m_2) a_1 + (m_1 - m_2) a_2 = -(m_1 + m_2) g + 2T_1 \qquad (6)$$

To remove T_1

$$(6) - (5)$$

$$(M_1 + M_2 + m_1 + m_2) a_1 - (m_1 - m_2) a_2 =$$

$$(-M_1 + M_2 + m_1 + m_2) g \qquad (7)$$

$$(M_1+M_2+m_1+m_2)a_1-(m_1-m_2)a_2= (-M_1+M_2+m_1+m_2)g \qquad (7)$$

$$m_1(a_2-a_1)\hat{e}=-m_2 g\hat{e}+T_1\hat{e} \qquad (3)$$

$$m_2(-a_2-a_1)\hat{e}=-m_2 g\hat{e}+T_1\hat{e} \qquad (4)$$

$(4)-(3)$

$$-(m_1-m_2)a_1+(m_1+m_2)a_2=(m_1-m_2)g \qquad (8)$$

Now solve for acceleration from equation $(7),(8)$ - do this as an exercise

Now we get the equations for $a_{1,} a_2$ as

$$a_1=\frac{-M_1(m_1+m_2)+M_2(m_1+m_2)+4m_1m_2}{(m_1+m_2)(M_1+M_2)+4m_1m_2}g$$

$$a_2 = \frac{-2M_1(m_1 - m_2)}{(m_1 + m_2)(M_1 + M_2) + 4m_1 m_2} g$$

Solve also for the acceleration of masses m_1 and m_2. Also calculate tension in the both ropes

This a case where the acceleration is a constant with respect to time.

Now we look for problem with varying acceleration

Displaced spring with mass

$$\vec{F} = -k\,x\,\hat{x}$$

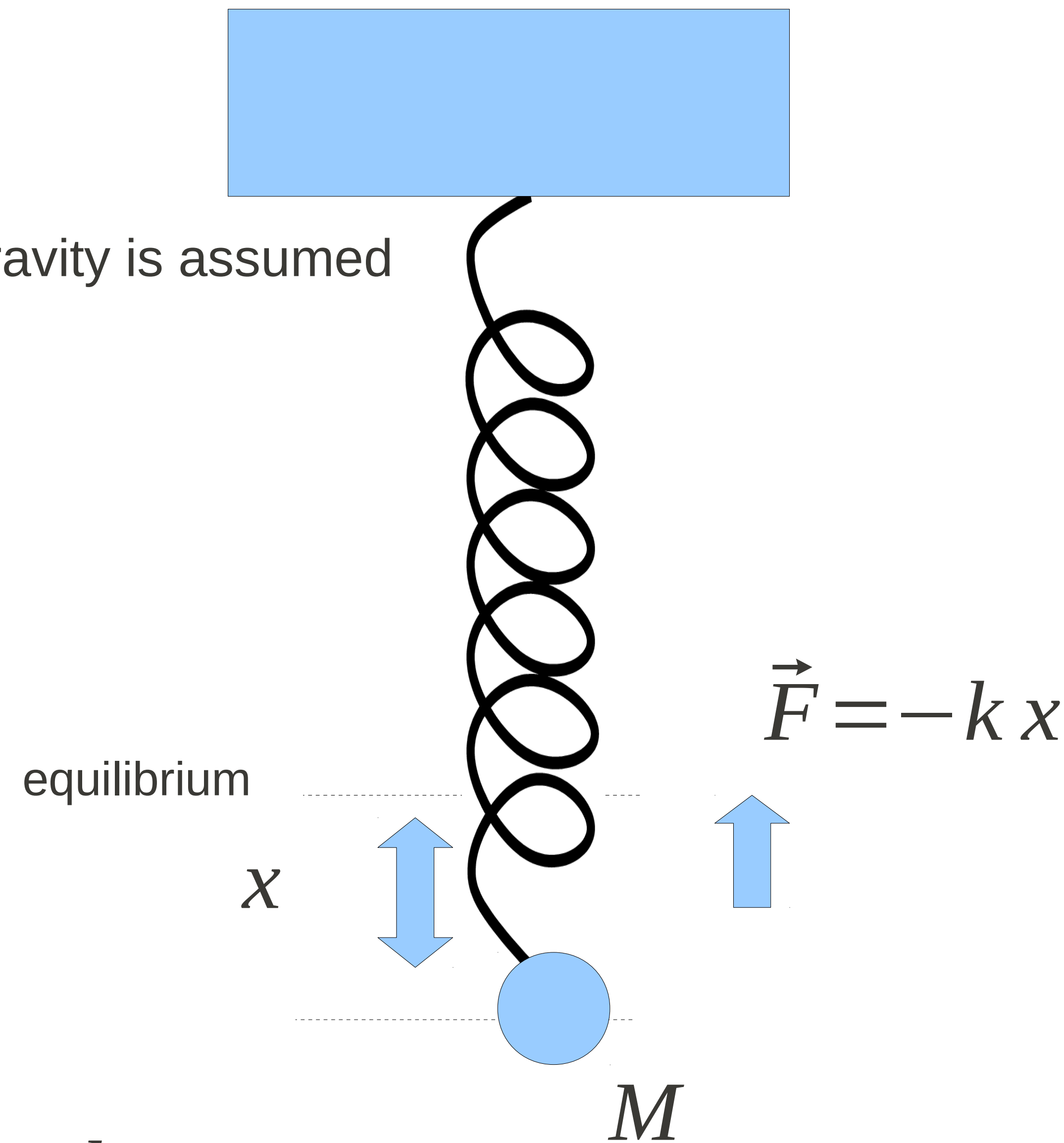
$$\frac{d^2 x}{d\,t} = -\frac{k}{M}\,x$$

No gravity is assumed

The differential equation with constant coefficients

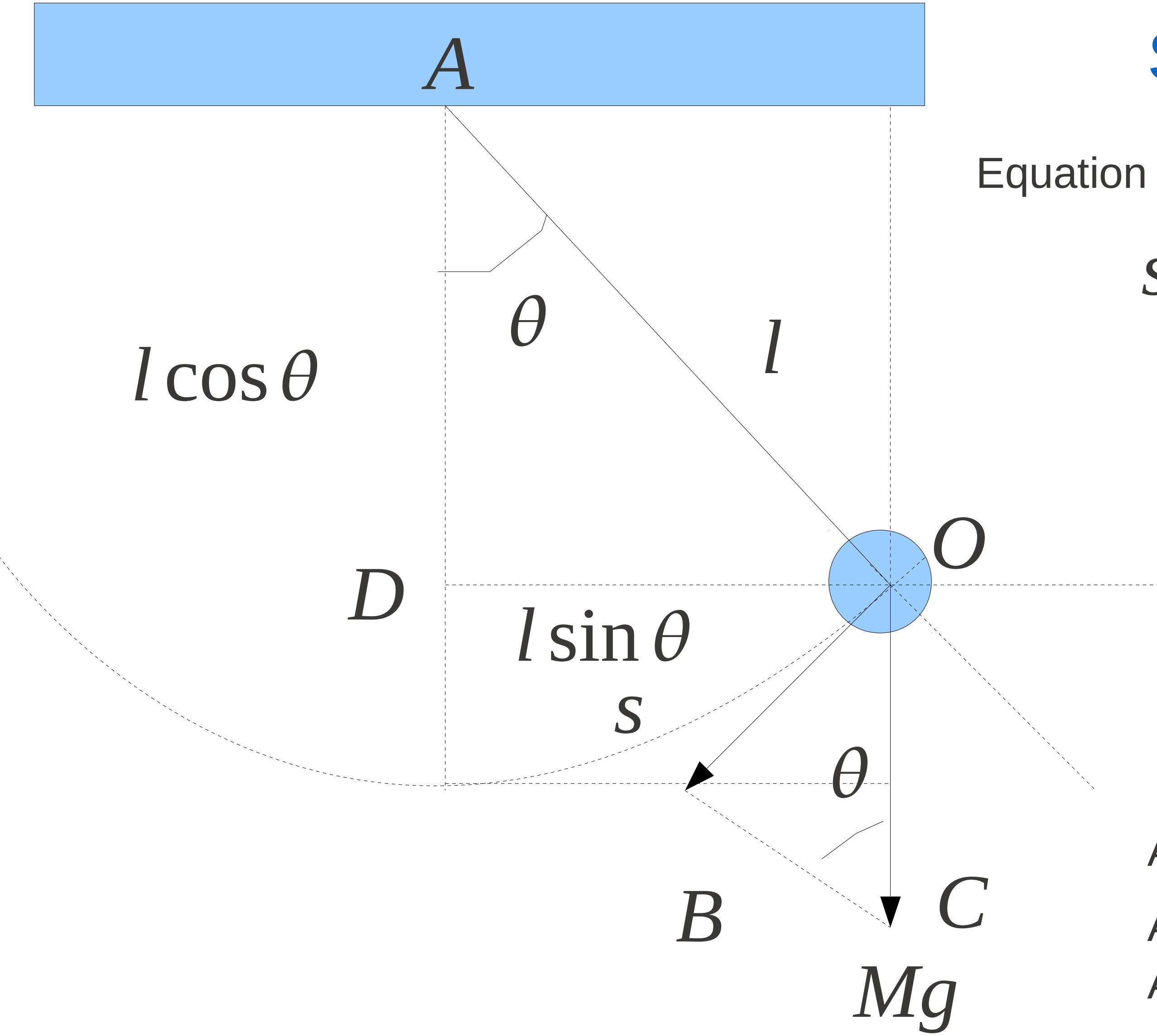
The general solution is

$$x = A\,sin(\omega_0 t + \phi)$$



$$\omega_0 = \sqrt{\frac{k}{M}}$$

$$t = 0\,;\, x = x_0 = A\,sin\,\phi\,;\, \frac{dx}{dt} = v_0 = \omega_0\,A\,cos\,\phi$$



Simple pendulum

Equation of motions by Newton's method

$$s = l\theta$$

$$v = \frac{ds}{dt} = l\frac{d\theta}{dt}$$

$$a = \frac{d^2 s}{dt^2} = l\frac{d^2\theta}{dt^2}$$

Angle AOB= angle DOB = 90

Angle AOD is $90^o - \theta$

Angle DOB is θ

Angle COB is $90^o - \theta$

Angle OCB is θ

Direction of l is perpendicular to the force component: radial and tangent directons

$$Mg\sin\theta = -Ml\frac{d^2\theta}{dt^2}$$

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots\ldots\ldots$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

Newton's third law

When ever two bodies interact the force $\vec{F}_{21}$ on that body 1 exerts on body 2 is equal and opposite to the force $\vec{F}_{12}$ that body 2 exerts on body 1

$$\vec{F}_{12} = -\vec{F}_{21}$$

$$m_1 \vec{a}_{12} = -m_2 \vec{a}_{21} \qquad \Rightarrow \frac{m_1}{m_2} = \frac{|\vec{a}_{21}|}{|\vec{a}_{12}|}$$

When three particles interact

$$\Rightarrow \frac{|\vec{a}_{21}|}{|\vec{a}_{12}|} \frac{|\vec{a}_{32}|}{|\vec{a}_{23}|} \frac{|\vec{a}_{13}|}{|\vec{a}_{31}|} = 1$$

For a particle that is under multiple interaction that create accelerations

$$\vec{F} = m\vec{a}_1 + m\vec{a}_2 + \ldots. + m\vec{a}_n = \vec{F}_1 + \vec{F}_2 + \ldots.. + \vec{F}_n$$

Dynamical quantities of Newtonian mechanics

Velocity is defined as $$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

Linear momentum $\vec{p}$ of the particle is defined as product of mass and velocity

$$\vec{p} = m\vec{v}$$

The particle may experience external force due interaction with other particles and fields such as gravitational or electromagnetic

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}} \qquad \Rightarrow \vec{F} = \frac{d}{dt}(m\vec{v})$$

$$\Rightarrow \vec{F} = \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt}$$ by product rule of differentiation

When the mass is a constant

$$\Rightarrow \vec{F} = m\frac{d\vec{v}}{dt} = m\vec{a}$$ where $$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}$$

Newton's equation of motion is valid for a inertial frame of reference

A frame reference fixed on earth is only an approximation to such a frame of reference

When external forces are absent

$$\vec{F}=\frac{d\,\vec{p}}{dt}=\dot{\vec{p}}=0$$

$$\vec{p}=constant$$

Law of conservation for the linear momentum: If total external force $\vec{F}$ is zero, then $\dot{\vec{p}}=0$ and the linear momentum $\vec{p}$ is conserved.

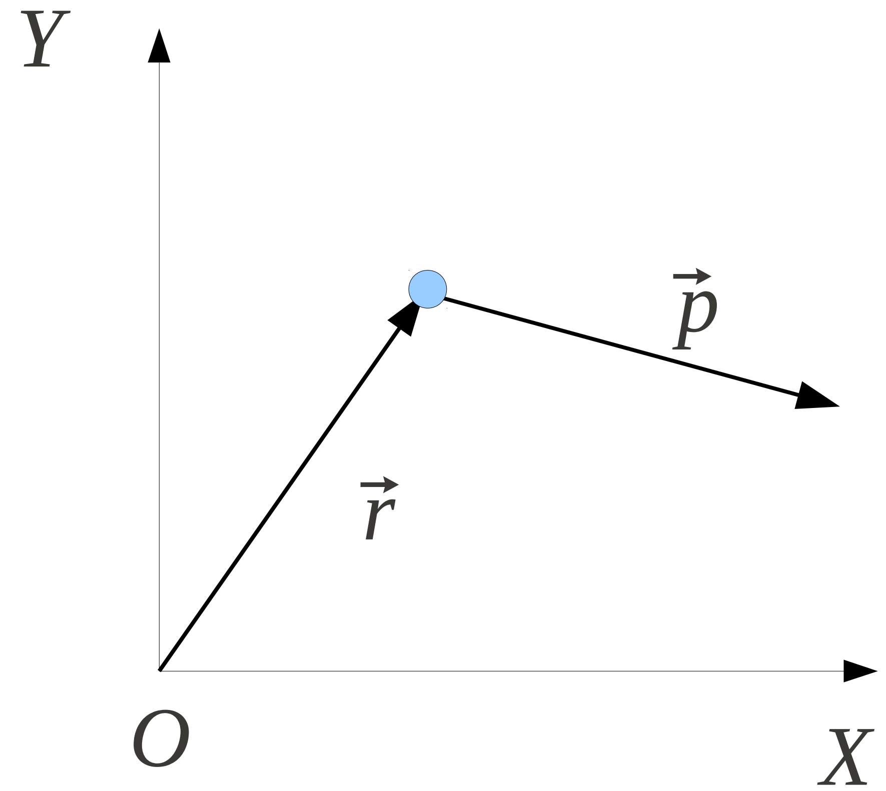
Also from the third law for a system of two particles $\vec{F}_{12}=-\vec{F}_{21}$

$$\frac{d\,\vec{p}_2}{d\,t}=\frac{-d\,\vec{p}_1}{d\,t} \qquad \frac{d\left(\vec{p}_2+\vec{p}_1\right)}{d\,t}=0 \qquad \left(\vec{p}_2+\vec{p}_1\right)=constant$$

Angular momentum

$$\vec{L}=\vec{r}\times\vec{p}$$

Defined with respect to origin O



Since angular momentum is pseudo vector the order of the definition is important

Let moment of force or torque be

$$\vec{N}=\vec{r}\times\vec{F}$$

$$\vec{r}\times\vec{F}=\vec{N}=r\times\frac{d}{dt}(m\vec{v})$$

Use the vector identity

$$\frac{d}{dt}(\vec{r}\times m\vec{v})=\overbrace{\vec{v}\times m\vec{v}}^{0}+\vec{r}\times\frac{d}{dt}(m\vec{v})$$

$$\vec{r}\times\vec{F}=\vec{N}=\vec{r}\times\frac{d}{dt}(m\vec{v})$$

$$\vec{N}=\frac{d}{dt}(\vec{r}\times m\vec{v})=\frac{d\vec{L}}{dt}$$

This yield conservation theorem of angular momentum

Law of conservation for the angular momentum: If total external torque $\vec{N}$ is zero, then $\dot{\vec{L}}=0$ and the angular momentum $\vec{L}$ is conserved.

Work done by a force in moving a particle from position 1 to 2

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{r}$$

Assuming constant mass $\quad d\vec{r} = \vec{v}\,dt \quad \vec{F} = \vec{a} = \frac{d\vec{v}}{dt}$

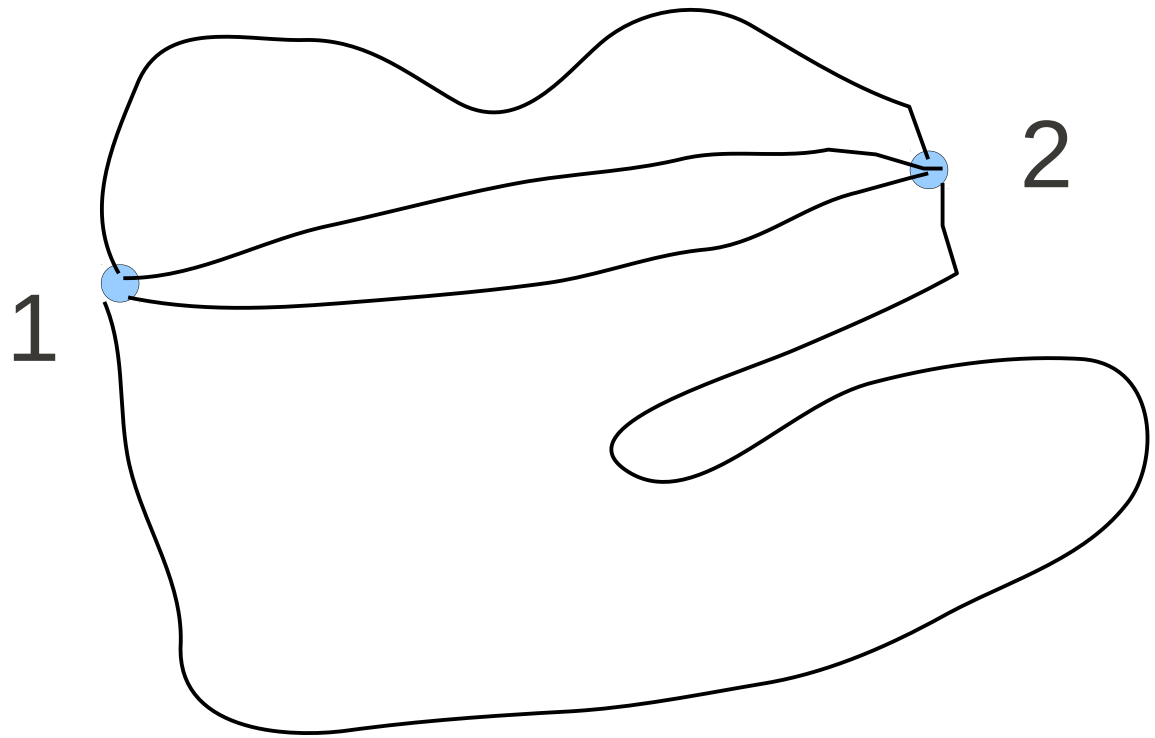
$$\int \vec{F} \cdot d\vec{r} = m \int \frac{d\vec{v}}{dt} \cdot \vec{v}\,dt = \frac{m}{2} \int \frac{d}{dt}\left(v^2\right) dt$$

Therefore $\quad W_{12} = \frac{m}{2}\left(v_2^2 - v_1^2\right)$

This is the expression for kinetic energy change of the particle. Therefore the change of kinetic energy is the work done on the particle

$$W_{12} = \left(T_2 - T_1\right)$$

Here the work done is independent of the path then force said to be conservative



Work done after returning to same position is zero irrespective of the path

$$\oint \vec{F} \cdot d\,\vec{r} = 0$$

If the frictional forces are present this cannot be true as energy dissipates under friction

By vector calculus the Force can now be expressed as gradient of a scalar function of position

$$\vec{F} = -\vec{\nabla} V(\vec{r})$$

Or by intuition we can consider change from position as

$$\vec{F} \cdot d\vec{r} = -dV(\vec{r}) \qquad \Rightarrow F = \frac{-dV(\vec{r})}{dr}$$

For conservative system the work done by the forces is

$$W_{12} = V_1 - V_2 \qquad W_{12} = (T_2 - T_1)$$

$$T_1 + V_1 = T_2 + V_2$$

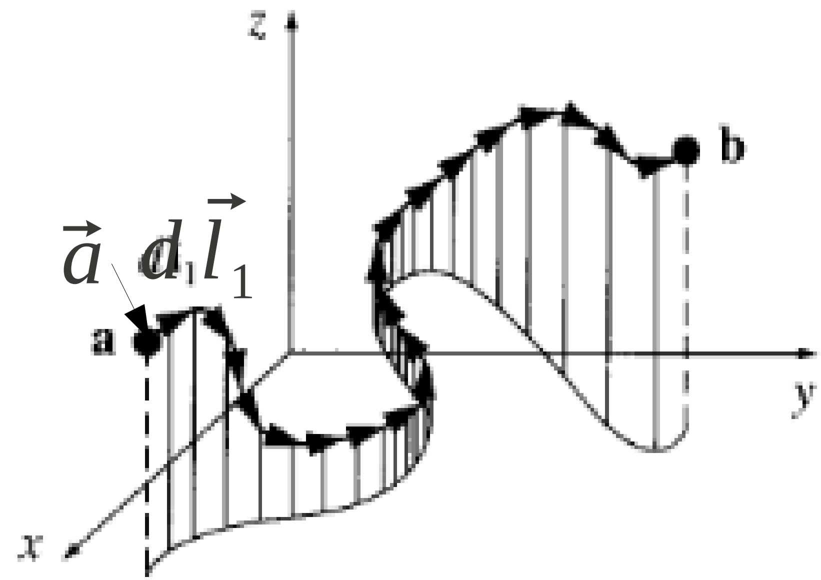
If the total forces acting on the particle are conservative, then total energy of the particle, $T + V$ is conserved.

Law of gradients

Suppose we have a scalar function $\phi = \phi(x, y, z)$

Starting at point $\vec{a}$ we move small distance $d\vec{l}_1$

$$d\phi = (\vec{\nabla}\phi)\cdot d\vec{l}_1$$



Total change T in going from $\vec{a}$ to $\vec{b}$

$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\phi\cdot d\vec{l} = \phi(\vec{a}) - \phi(\vec{b})$$

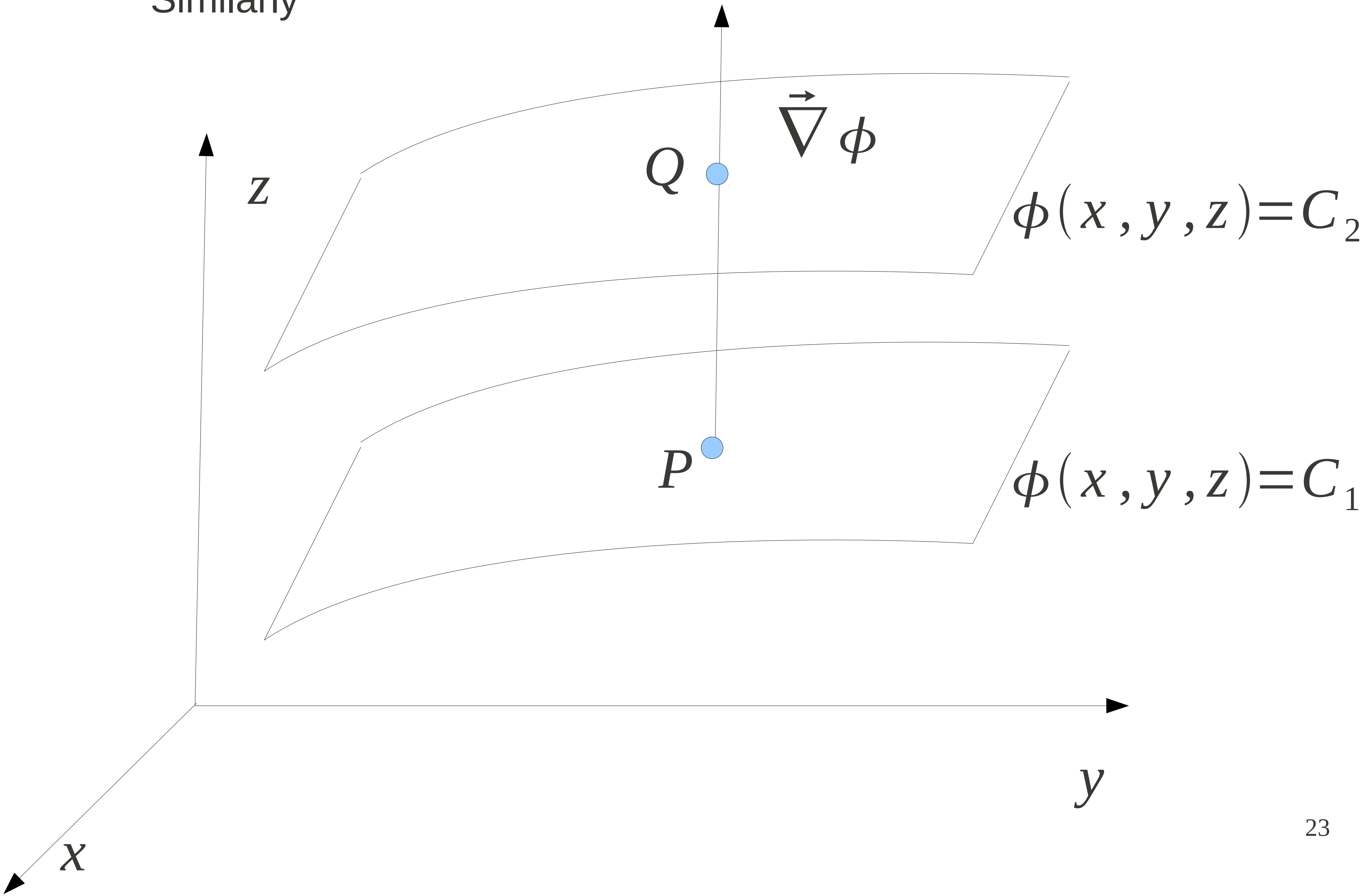
$$\int_{\vec{a}}^{\vec{b}} \vec{\nabla}\phi\cdot d\vec{l}$$

Is independent of the path taken from $\vec{a}$ to $\vec{b}$

$$\oint \vec{\nabla}\phi\cdot d\vec{l} = 0$$

$$d\,\phi = C_1 - C_2 = \Delta C = \vec{\nabla}\,\phi \cdot d\,\vec{r}$$

Similarly

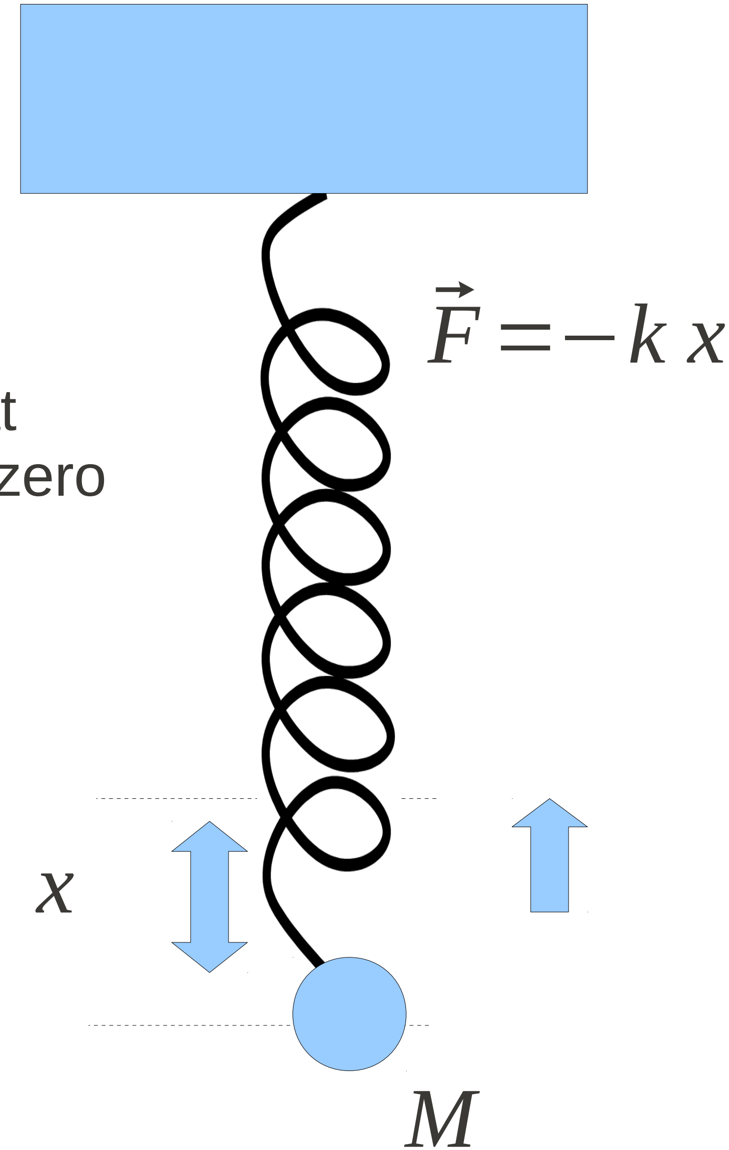


Harmonic oscillator and energy conservation

Total energy of a harmonic oscillator is conserved

P.E. K.E.

$$E = \frac{1}{2} k x^2 + \frac{1}{2} m v^2$$



To arrive at equation total energy is to evaluated at maximum extension $x = x_0$ of the spring K.E. is zero

$$E = \frac{1}{2} k x_0^2$$

Now the total energy equation be

$$\frac{1}{2} k x_0^2 = \frac{1}{2} k x^2 + \frac{1}{2} m \left(\frac{d x}{d t} \right)^2$$

Rearranging the terms

$$\frac{d x}{d t} = \sqrt{\frac{k}{m} (x_0^2 - x^2)}$$

$$\frac{dx}{dt} = \sqrt{\frac{k}{m}} \sqrt{(x_0^2 - x^2)}$$

Integrating $\displaystyle \int \frac{dx}{\sqrt{(x_0^2 - x^2)}} = \int \sqrt{\frac{k}{m}}\, dt$

The solution is $\displaystyle \sin^{-1}\frac{x}{x_0} = \sqrt{\frac{k}{m}}\, t + C$

Rearrange to get $\displaystyle \frac{x}{x_0} = \sin\left(\sqrt{\frac{k}{m}}\, t + \phi\right)$

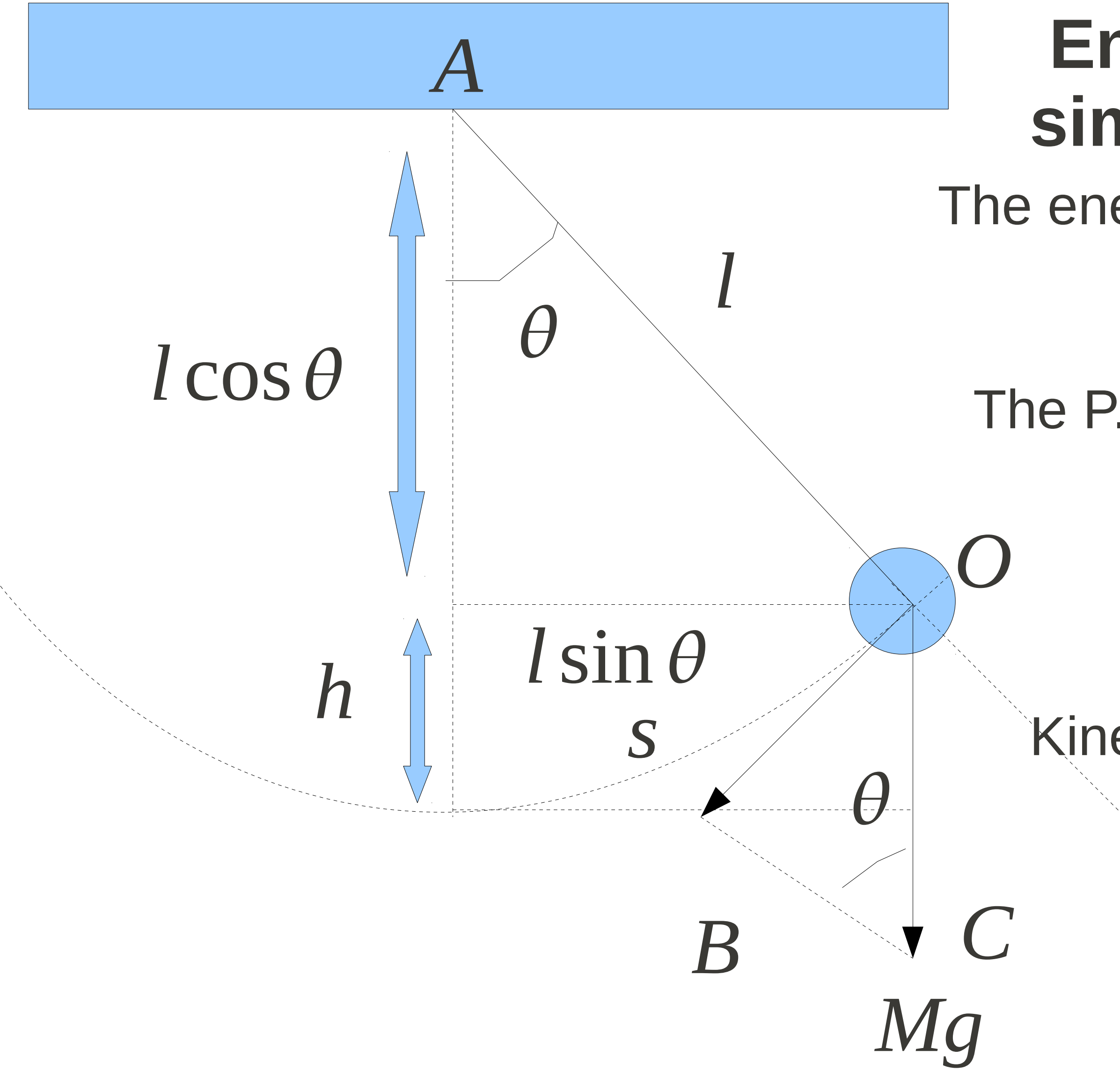
Or differentiate total energy function $\displaystyle E = \frac{1}{2} k x^2 + \frac{1}{2} m \left(\frac{dx}{dt}\right)^2$

$$0 = kx\frac{dx}{dt} + m\frac{dx}{dt}\frac{d^2x}{dt^2}$$

$$\Rightarrow m\frac{d^2x}{dt^2} = -kx$$

Now it can be solve in the usual way

Energy conservation in simple pendulum



The energy of simple pendulum

$$h = l - l\cos\theta$$

The P. E. of simple pendulum

$$U(h) = mgl(1-\cos\theta)$$

Kinetic energy of the simple pendulum

$$K = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

$$\cos\theta \simeq 1 - \frac{1}{2}\theta^2 \ldots\ldots\ldots$$

Total energy of the simple pendulum

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

$$E = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2$$

General rectilinear motion – equation of motion

Consider the motion of a point mass under potential

$$V = V(x) = -\int_0^x F(x')\,dx'$$

In one dimensional case potential always exist since

$$\vec{\nabla} \times \vec{F}(x) = 0$$

Also implies no rotational motion in one dimension

$$E = T + V = \frac{1}{2} m v^2 + V(x) = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + V(x)$$

$$\Rightarrow \left(\frac{dx}{dt}\right) = \pm\sqrt{\frac{2}{m}(E - V(x))}$$

$$\Rightarrow t = C \pm \int_{x_1}^{x} \frac{dx}{\sqrt{\frac{2}{m}(E - V(x))}}$$

Mechanics of system of particles

Let there be n particles in the system

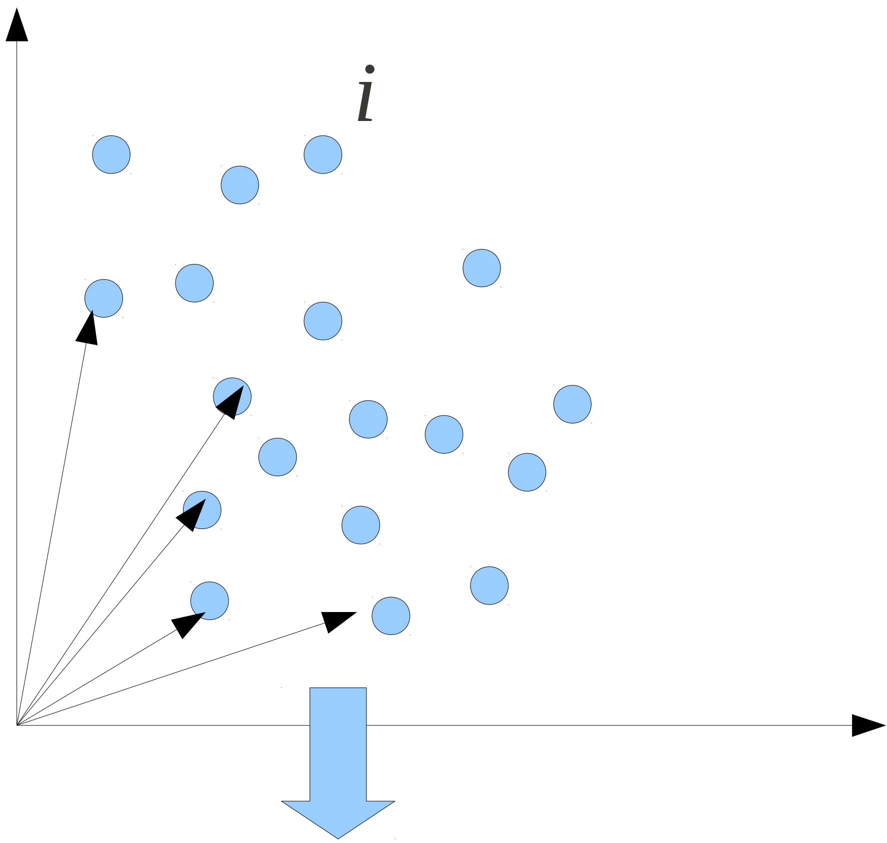
$$\Rightarrow \sum_{i} = \sum_{i=1}^{n} = \sum_{j=1}^{n}$$

short notation for the summation

$$\Rightarrow \sum_{i} \vec{r}_i = \vec{r}_1 + \vec{r}_2 + \ldots + \vec{r}_n = \sum_{i=1}^{n} \vec{r}_i$$

short notation for the double summation

$$\sum_{i,j} = \sum_{i=1\; j=1}^{i=n,\, j=n} = \sum_{i=1}^{i=n} \sum_{j=1}^{j=n}$$

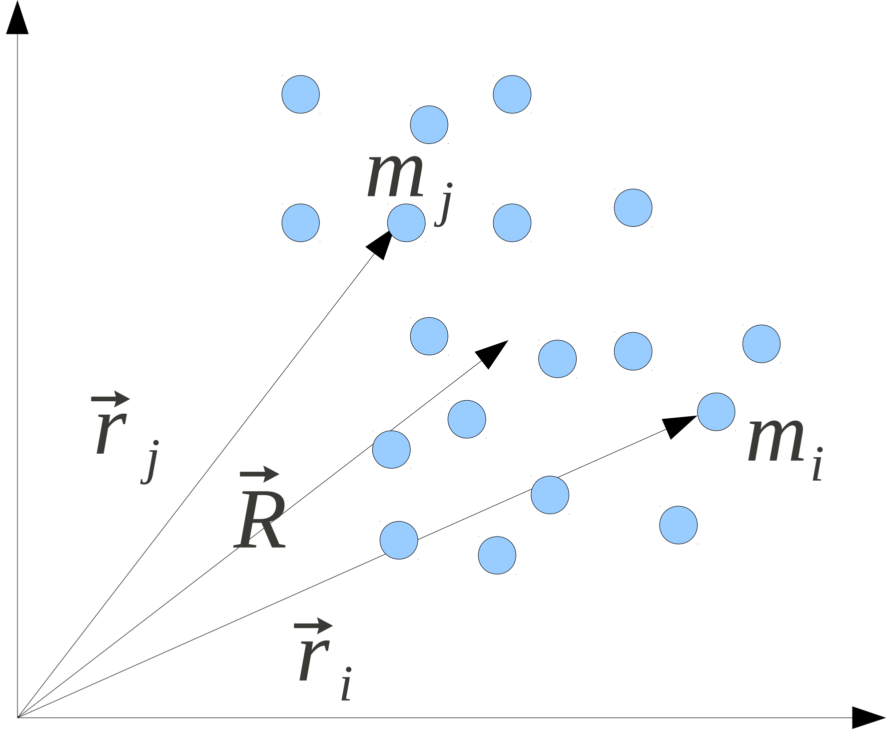


Representation of few position vectors

Let us define a vector $\vec{R}$ called center of mass of the system given by the relation

$$\vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$$

$$M\,\vec{R} = \sum_i m_i \vec{r}_i$$



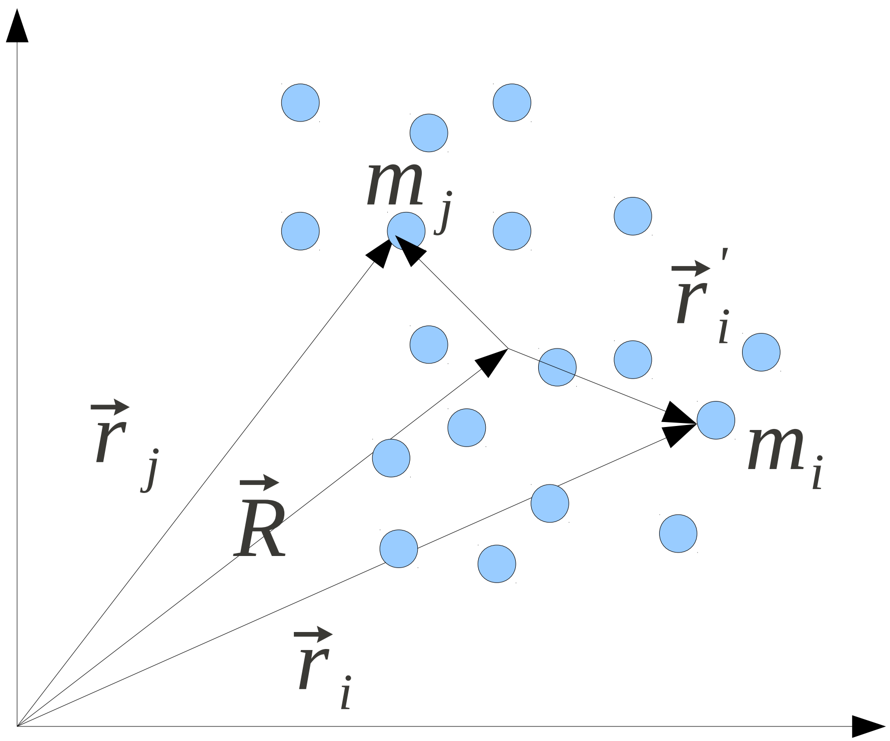
If the origin is shifted to center of mass

$$\vec{r}_i^{\,\prime} = \vec{R} - \vec{r}_i$$

$$\sum_i m_i \vec{r}_i^{\,\prime} = \sum_i m_i (\vec{R} - \vec{r}_i)$$

$$= \vec{R} \sum_i m_i - \sum_i m_i \vec{r}_i$$

$$= M\,\vec{R} - M\,\vec{R} = 0$$

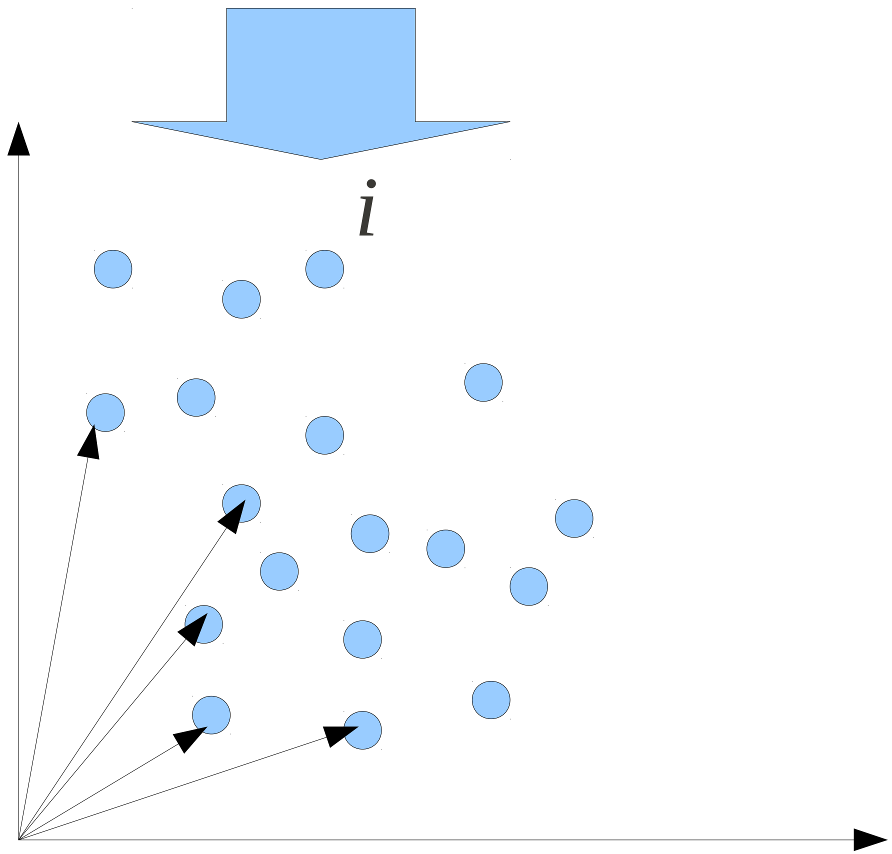


$$\sum_i m_i \vec{r}_i^{\,\prime} = 0$$

The origin is shifted to center of mass the sum of moments vanish

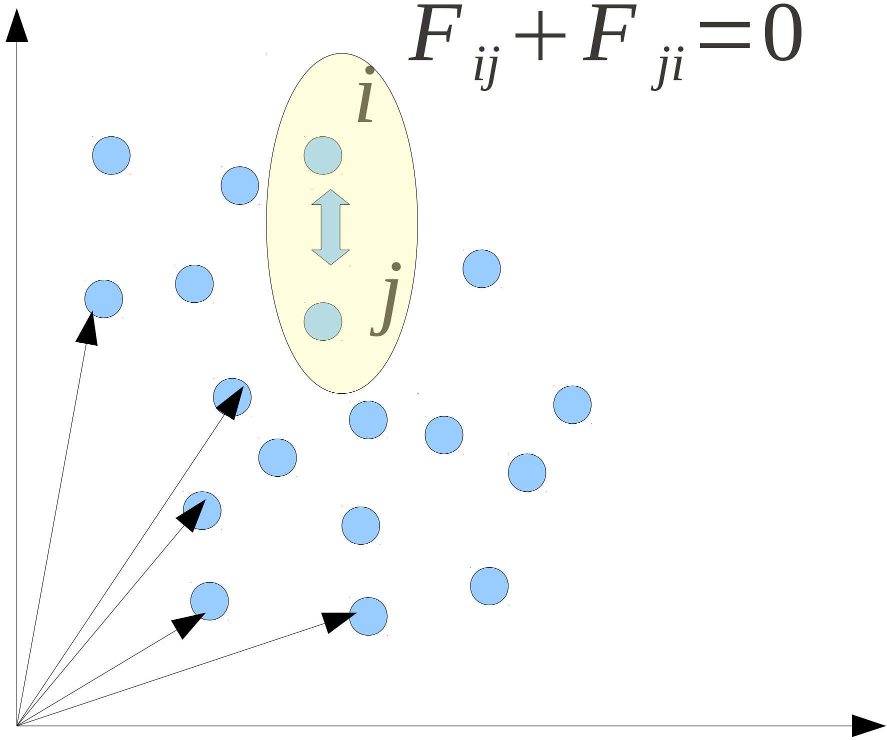
$\vec{R}$ is thus called center of mass of the system that is roughly the center of gravity of the system.

Forces on a system of particles



External forces, Eg. Gravity, Electrostatic, Magnetic... etc

$$\vec{F}_{ij} + \vec{F}_{ji} = 0$$



Internal forces, Eg. Gravity, Electrostatic, Magnetic, friction ... etc

Consider a system particles, there are forces due to internal interaction as well as external forces acting on the system

For ith particle the force experienced is

$$\sum_j \vec{F}_{ij} + \vec{F}_i^{\,e} = \dot{\vec{p}}_i$$

Now for a system of particles the total force acting on it is obtained by summing over all particles

$$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{\,e} + \sum_{i,j,i\neq j} \vec{F}_{ij}$$

Newton's equation of motion for a system of particles

The sum of the internal forces vanish due to third law of Newton- that is, law of action and reaction

$$\vec{F}_{ij} + \vec{F}_{ji} = 0$$

$$\sum_i m_i \vec{r}\,'_i = 0$$

The origin is shifted to center of mass the sum of moments vanish

$\vec{R}$ is thus called center of mass of the system that is roughly the center of gravity of the system.

In the Newton's equation of motion last term vanishes

$$\frac{d^2}{dt^2}\sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^e + \overbrace{\sum_{i,j,i\neq j} \vec{F}_{ij}}^{0}$$

$$\vec{F}_{ij} + \vec{F}_{ji} = 0$$

$$M\frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^e = \vec{F}^e$$

The system behaves as if all forces are acting on the center of mass

Law of conservation for the linear momentum: If total external force is zero, then total linear momentum is conserved.

Examples are exploding shell and rocket propulsion

Total linear momentum of the system is

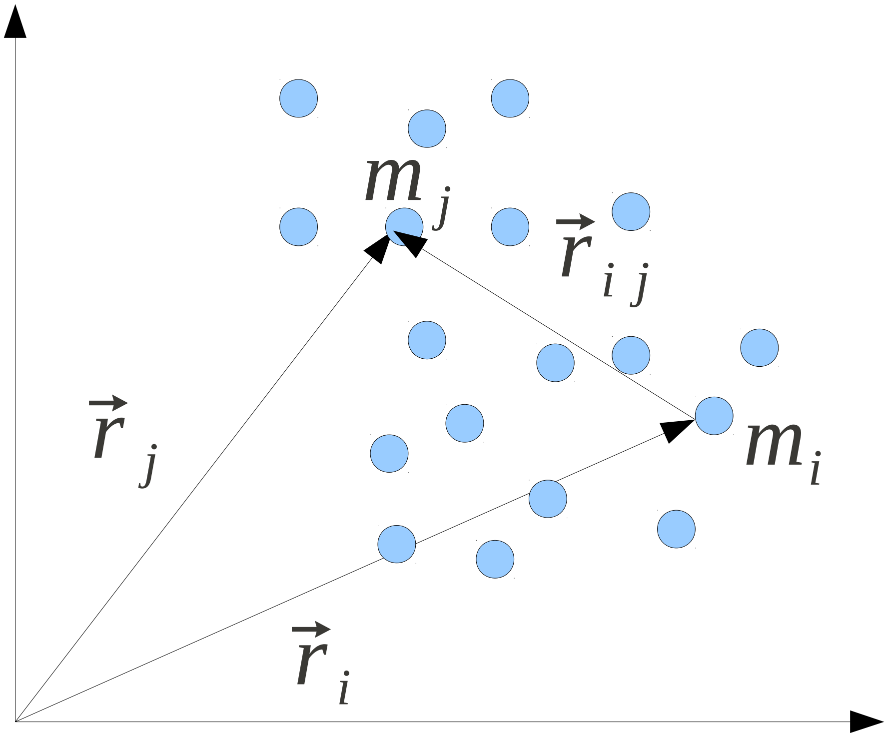
$$\vec{P}=\sum_i m_i \frac{d\,\vec{r}_i}{d\,t}=M\,\frac{d\,\vec{R}}{d\,t}$$

which is expressed sum of momentum of each individual particles angular momentum a particle

$$\vec{L}_i = \vec{r}_i \times \vec{p}_i$$

Total angular momentum of system of particles by vector addition

$$\vec{L} = \sum_i \vec{L}_i = \sum_i \vec{r}_i \times \vec{p}_i$$



$$\frac{d}{dt}(\vec{r} \times m\vec{v}) = \vec{v} \times m\vec{v} + \vec{r} \times \frac{d}{dt}(m\vec{v})$$

Total rate of change of angular momentum of system of particles

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \dot{\vec{p}}_i = \sum_i \frac{d}{dt}(\vec{r}_i \times \vec{p}_i)$$

Total forces acting on i th particle $\quad \dot{\vec{p}} = \vec{F}_i = \sum_j \vec{F}_{ij} + F_i^e$

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^e + \sum_{i,j,i \neq j} \vec{r}_i \times \vec{F}_{ji}$$

$$\vec{r}_i \times \vec{F}_{ji} + \vec{r}_j \times \vec{F}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \qquad \vec{F}_{ij} + \vec{F}_{ji} = 0$$

The vector $(\vec{r}_i - \vec{r}_j)$ is represented with $\vec{r}_{ij}$

$$(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} = \vec{r}_{ij} \times \vec{F}_{ji}$$

If all the internal forces between two particles, in addition to being equal and opposite also lie along the line joining particles (**known as strong law of action and reaction**) then all these cross products vanish

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^e + \underbrace{\sum_{i,j,i \neq j} \vec{r}_i \times \vec{F}_{ji}}_{0}$$

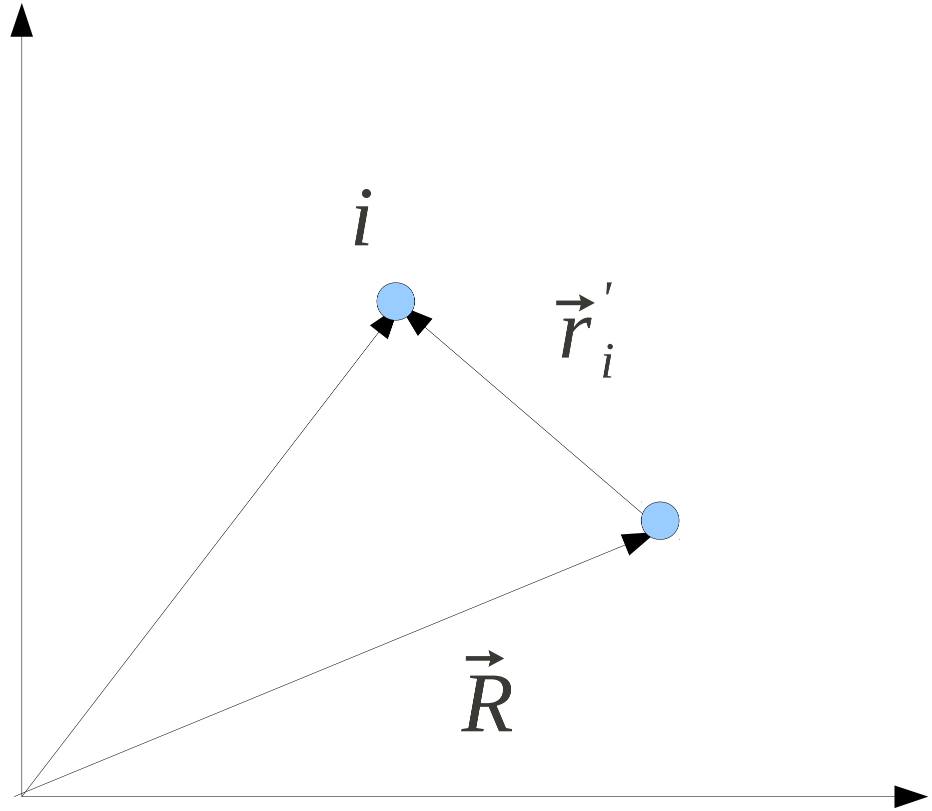
$\vec{r}_{ij} || \vec{F}_{ji}$

vectors are parallel

$$\dot{\vec{L}} = \sum_i \vec{r}_i \times \vec{F}_i^e = \vec{N}^e$$

$\vec{L}$ Is a constant in the case when of the external torque is zero

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$$



$$\vec{r}_i = \vec{r}_i' + \vec{R}$$

by taking time derivative of this equation

$$\vec{v}_i = \vec{v}_i' + \vec{v}$$

where $\vec{v} = \dfrac{d\vec{R}}{dt}$

is the velocity of the center of mass relative to the origin and $\vec{v}' = \dfrac{d\vec{r}_i'}{dt}$

is velocity of i'th particle relative to the center of mass

$$\vec{L}=\sum_i \vec{r}_i\times\vec{p}_i=\sum_i (\vec{r}_i^{\,\prime}+\vec{R})\times m_i(\vec{v}_i^{\,\prime}+\vec{v})$$

$$\vec{r}_i=\vec{r}_i^{\,\prime}+\vec{R}$$

$$\vec{v}_i=\vec{v}_i^{\,\prime}+\vec{v}$$

Now the total angular momentum of the particles

$$\vec{L}=\sum_i \vec{R}\times m_i\vec{v}+\sum_i \vec{r}_i^{\,\prime}\times m_i\vec{v}_i^{\,\prime}+\left(\sum_i m_i\vec{r}_i^{\,\prime}\right)\times\vec{v}+\vec{R}\times\frac{d}{dt}\sum_i m_i\vec{r}_i^{\,\prime}$$

0

Last two terms vanish since that involve term $\sum_i m_i\vec{r}_i^{\,\prime}$

$$\sum_i m_i\vec{r}_i^{\,\prime}=0$$

Radius vector of the center of mass in a coordinate system whose origin is center of mass itself

Then the remaining terms are

$$\vec{L}=\vec{R}\times M\vec{v}+\sum_i \vec{r}_i^{\,\prime}\times m_i\vec{v}_i^{\,\prime}$$

$$\vec{L} = \vec{R} \times M\,\vec{v} + \sum_i \vec{r}_i^{\,'} \times m_i \vec{v}_i^{\,'}$$

Total angular momentum about center of mass $\vec{R} \times M\,\vec{v}$

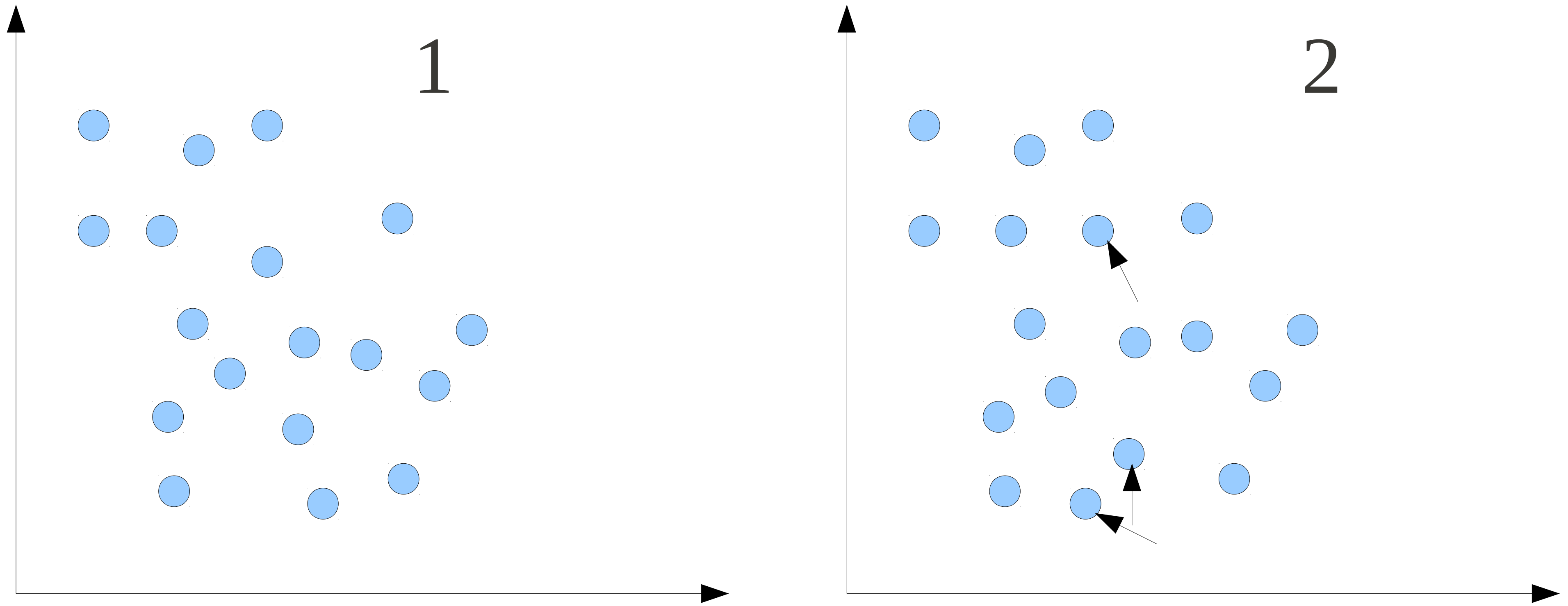
Angular momentum about center of mass $\sum_i \vec{r}_i^{\,'} \times \vec{p}^{\,'}$

The general $\vec{L}$ depends on center of mass vector $\vec{R}$

Now consider the energy equation for a system of particles

Work done by all forces in moving the system from configuration 1 to 2

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i$$



$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\,\vec{s}_i$$

Total forces acting on i th particle $\vec{F}_i = \sum_j \vec{F}_{ij} + F_i^e$

$$W_{12} = \sum_i \int_1^2 \vec{F}_i^e \cdot d\,\vec{s}_i + \sum_{i,\,j,\,i \neq j} \int_1^2 \vec{F}_{ji} \cdot d\,\vec{s}_i$$

Again equation of motion can be used to reduce the integrals to

$$\sum_i \int_1^2 \vec{F}_i \cdot d\,\vec{s}_i = \sum_i \int_1^2 m_i \dot{\vec{v}}_i \cdot \vec{v}_i\, dt = \sum_i \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right)$$

Remember the equation for work done for single particle

$$\int \vec{F} \cdot d\,\vec{r} = m \int \frac{d\,\vec{v}}{d\,t} \cdot \vec{v}\, dt = \frac{m}{2} \int \frac{d}{dt}(v^2)\, dt$$

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\,\vec{s}_i = \sum_i T_i^2 - T_i^1 = T_2 - T_1$$

Making use of transformation to center of mass coordinates

$$\vec{r}_i = \vec{r}_i' + \vec{R} \quad \text{and} \quad \vec{v}_i = \vec{v}_i' + \vec{v}$$

Kinetic energy can be expanded as

$$T = \frac{1}{2}\sum_i m_i (\vec{v} + \vec{v}_i') \cdot (\vec{v} + \vec{v}_i')$$

$$T = \frac{1}{2}\sum_i m_i v^2 + \frac{1}{2}\sum_i m_i v'^2 + \vec{v} \cdot \frac{d}{dt}\left(\sum_i m_i \vec{r}_i'\right)$$

The last term vanishes due to equation $\sum_i m_i \vec{r}_i' = 0$

$$T = \frac{1}{2}\sum_i m_i v^2 + \frac{1}{2}\sum_i m_i v_i'^2 \quad = \frac{1}{2} M v^2 + \frac{1}{2}\sum_i m_i v_i'^2$$

K. E of center of mass

K. E about center of mass

When external forces can be derived from gradient of a potential

$$\sum_i \int_1^2 \vec{F}_i^e \cdot d\vec{s}_i = \int_1^2 -\vec{\nabla}_i V_i \cdot d\vec{s}_i = -\sum_i V_i \Big|_1^2$$

The subscript $\vec{\nabla}_i$ means the derivative is with respect to $\vec{r}_i$

If the internal forces are conservative, then mutual forces between the particles i and j , $\vec{F}_{ij}, \vec{F}_{ji}$ can be obtained from the inter particle potential function V_{ij}

$$V_{ij} = V\left(\left|\vec{r}_i - \vec{r}_j\right|\right)$$

The forces are equal and opposite

$$\vec{F}_{ji} = -\vec{\nabla}_i V_{ij} = \vec{\nabla}_j V_{ij} = \vec{F}_{ij}$$

$$\vec{\nabla} V_{ij}(|\vec{r}_i - \vec{r}_j|) = (\vec{r}_i - \vec{r}_j)\, f$$

When all the forces are conservative the second term in the following equation

$$W_{12} = \sum_i \int_1^2 \vec{F}_i^e \cdot d\,\vec{s}_i + \sum_{i,j,i\neq j} \int_1^2 \vec{F}_{ji} \cdot d\,\vec{s}_i$$

becomes

$$-\int_1^2 (\vec{\nabla}_i V_{ij} \cdot d\,\vec{s}_i + \vec{\nabla}_j V_{ij} \cdot d\,\vec{s}_j)$$

If the difference vector $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ and the gradient $\vec{\nabla}_{ij}$ denote the gradient with respect to $\vec{r}_{ij}$ then

$$\vec{\nabla}_i V_{ij} = \vec{\nabla}_{ij} V_{ij} = -\vec{\nabla}_j V_{ij}$$

$$d\,\vec{s}_i - d\,\vec{s}_j = d\,\vec{r}_i - d\,\vec{r}_j = d\,\vec{r}_{ij}$$

Transformation to a new operator with shift of position

$$\vec{\nabla}_i \longrightarrow \vec{\nabla}_{ij}$$

$$\frac{\partial}{\partial x_i} = \frac{\partial}{\partial (x_i - x_j)} \frac{\partial (x_i - x_j)}{\partial x_i} = \frac{\partial}{\partial (x_i - x_j)}$$

$$-\vec{\nabla}_j \longrightarrow \vec{\nabla}_{ij}$$

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial (x_i - x_j)} \frac{\partial (x_i - x_j)}{\partial x_j} = -\frac{\partial}{\partial (x_i - x_j)}$$

$$-\int_1^2 (\vec{\nabla}_i V_{ij} \cdot d\vec{s}_i + \vec{\nabla}_j V_{ij} \cdot d\vec{s}_j)$$

This integral now becomes

$$= -\int_1^2 (\vec{\nabla}_{ij} V_{ij} \cdot d\vec{s}_i - \vec{\nabla}_{ij} V_{ij} \cdot d\vec{s}_j$$

$$= -\int_1^2 (\vec{\nabla}_{ij} V_{ij} \cdot (d\vec{s}_i - d\vec{s}_j))$$

$$d\vec{s}_i - d\vec{s}_j = d\vec{r}_i - d\vec{r}_j = d\vec{r}_{ij}$$

$$= -\int_1^2 (\vec{\nabla}_{ij} V_{ij} \cdot (d\vec{r}_i - d\vec{r}_j))$$

$$= -\int_1^2 \vec{\nabla}_{ij} V_{ij} \cdot d\vec{r}_{ij}$$

The total work arising from the internal forces

$$=-\frac{1}{2}\sum_{i,j;i\neq j}\int_1^2 \vec{\nabla}_{ij}V_{ij}\cdot d\vec{r}_{ij}=-\frac{1}{2}\sum_{i,j;i\neq j}V_{ij}\Big|_1^2$$

1/2 is introduced to avoid over counting of the pair of interaction, that is V_{12} and V_{21} should be counted only once

If external as well as internal forces are derived from potentials then total potential energy can be written as

$$V=\sum_i V_i+\frac{1}{2}\sum_{i,j;i\neq j}V_{i,j}$$

Now we have total work done in terms of K. E as

$$W_{12}=\sum_i\int_1^2\vec{F}_i\cdot d\vec{s}_i=\sum_i T_i^2-T_i^1=T_2-T_1$$

The total change in the potential energy, while the work is done is

$$W_{12} = -\sum_i V_i \Big|_1^2 - \frac{1}{2} \sum_{i,j;i\neq j} V_{i,j} \Big|_1^2 = V_1 - V_2$$

$$W_{12} = \sum_i \int_1^2 \vec{F}_i \cdot d\vec{s}_i = \sum_i T_i^2 - T_i^1 = T_2 - T_1$$

Combining both

$$T_2 - T_1 = V_1 - V_2$$

$$T_1 + V_1 = T_2 + V_2 \qquad E_1 = E_2$$

Combining both we arrive at conclusion that total energy is conserved

Reference for module “Newtonian Mechanics”

1) Classical dynamics of particles and systems by S T Thornton and J B Marion

Chapter 2 Basic introduction

2) Classical Mechanics by H Goldstein

Pages 1 – 6 : Conservation Laws for many particle systems

3) Classical mechanics point particles and relativity Walter Greiner

Part II