IC121: Mechanics of Particles and Waves

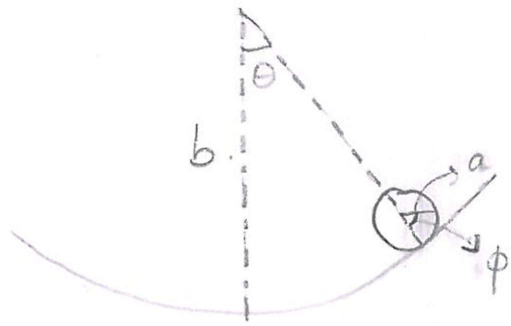
Tutorial - 5: Solutions

1. A solid cylinder with center G and radius 'a' rolling on rough inside surface of a fixed cylinder with center O' and radius b (b > a). Find the Lagrange's equations of motion and deduce the period of small oscillations

Soln: For a cylinder that rolls without slipping

$$(b-a)\theta = a\phi$$
$$(b-a)\dot{\theta} = a\dot{\phi}$$

K.E. = K.E. of translation of CM + K.E. of rotation about CM



$$x = (b-a)\cos\theta$$
$$y = (b-a)\sin\theta.$$
$$\dot{x} = -\dot{\theta}(b-a)\sin\theta.$$
$$\dot{y} = \dot{\theta}(b-a)\cos\theta$$

$$K.E. = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\phi}^2$$
$$= \frac{1}{2}m[(b-a)\dot{\theta}]^2 + \frac{1}{2}\left(\frac{m}{2}a^2\dot{\phi}^2\right)$$
$$= \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{4}a^2\left[\frac{(b-a)}{a}\dot{\theta}\right]^2$$
$$= \frac{3}{4}m(b-a)^2\dot{\theta}^2$$

P.E. $(R-(R-r)\cos\theta)mg$

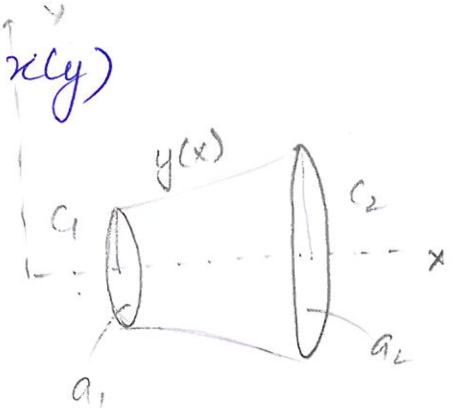
$L = T - V$
$$= \frac{3}{4}m(b-a)^2\dot{\theta}^2 + (b-a)mg\cos\theta - Rmg$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \Rightarrow \frac{d}{dt}\left(\frac{3}{2}m(b-a)^2\dot{\theta}\right) + mg(b-a)\sin\theta = 0$$
$$\Rightarrow \frac{3}{2}m(b-a)^2\ddot{\theta} + mg(b-a)\sin\theta = 0$$

for small θ, $\sin\theta = \theta$ $\therefore$ $\boxed{\ddot{\theta} - \frac{2}{3}\frac{g\theta}{(b-a)} = 0}$

2. A surface of revolution has two parallel rings at its boundary. What should be the shape of the surface so that it has minimum possible area. Let the rotation takes place around x-axis and the curve that connects the rings be represented by $y = y(x)$. Let the boundary conditions be $c_1 = y(a_1)$, $c_2 = y(a_2)$.

Consider the curve rotating about x axis as $x(y)$



The surface area of small segment of curve

$$= 2\pi y \, ds$$

Now the total surface are

$$A = \int_{a_1}^{a_2} 2\pi y \sqrt{1 + x'} \, dy \quad \text{where } x' = \frac{dx}{dy}$$

Now Lagrangian

$$\mathcal{L} \propto y\sqrt{1+x'^2}$$

Using Euler-Lagrangian equations

$$\frac{d}{dy}\left(\frac{\partial L}{\partial x'}\right) = \frac{\partial L}{\partial x}$$

$$\frac{\partial L}{\partial x'} = \frac{y x'}{\sqrt{1+x'^2}}$$

$$\frac{\partial L}{\partial x} = 0$$

$$\therefore \frac{d}{dy}\left(\frac{y x'}{\sqrt{1+x'^2}}\right) = 0$$

$\therefore \quad \frac{y x'}{\sqrt{1+x'^2}} = \text{constant} \left(\text{say } \frac{1}{b}\right)$

$\therefore \quad \frac{y^2 x'^2}{(1+x'^2)} = \frac{1}{b^2}$

$x'^2 (y^2 b^2 - 1) = 1$

$x' = \frac{1}{\sqrt{(by)^2 - 1}} \Rightarrow dx = \frac{1}{\sqrt{b^2 y^2 - 1}} dy$

Using the integral formula $\int \frac{1}{\sqrt{z^2-1}} dz = \cosh^{-1} z$

$$x = \frac{1}{b} \cosh^{-1}(yb) - d$$

$$bx + bd = \cosh^{-1}(yb)$$

$$y = \frac{1}{b} \cosh(bx + bd)$$

Using boundary conditions

$y(a_1) = c_1$

$y(a_2) = c_2$

$$c_1 = \frac{1}{b} \cosh b(a_1 + d)$$

$$c_2 = \frac{1}{b} \cosh b(a_2 + d)$$

Using these, we can get the value of constants

So, eqn

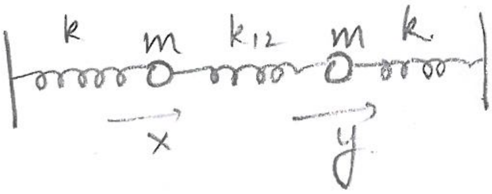
$$y = \frac{1}{b} \cos(bx + bd)$$

is the equation of Catenary.

3. Two masses each having mass m are connected by a spring to each other and by springs to fixed position spring 1, 2, 3 having spring constants k_1, k_{12}, k respectively. Find the Eigenfrequencies of the system using the Lagrangian method of small oscillations.

Soln

$$K.E. = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}m\dot{x}_2^2$$



$$P.E. = \frac{1}{2}kx_1^2 + \frac{1}{2}kx_2^2 + \frac{1}{2}k_{12}(x_2-x_1)^2$$

$$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}k(x_1^2 + x_2^2) - \frac{1}{2}k_{12}(x_2-x_1)^2$$

Using Lagrange's eqn of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_1}\right) - \frac{\partial \mathcal{L}}{\partial x_1} = 0$$

$$\frac{d}{dt}(m\dot{x}_1) + kx_1 - k_{12}(x_2 - x_1) = 0$$

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_2}\right) - \frac{\partial \mathcal{L}}{\partial x_2} = 0$$

$$\frac{d}{dt}(m\dot{x}_2) + kx_2 + k_{12}(x_2 - x_1) = 0$$

So eqn of motion becomes

$$m\ddot{x}_1 + kx_1 - k_{12}x_2 + k_{12}x_1 = 0$$

$$\& \quad m\ddot{x}_2 + kx_2 + k_{12}x_2 - k_{12}x_1 = 0$$

Let the solution be

$$x_1 = a_1\cos\omega t$$
$$x_2 = a_2\cos\omega t$$
$$\ddot{x}_1 = -\omega^2 a_1\cos\omega t$$
$$\ddot{x}_2 = -\omega^2 a_2\cos\omega t$$

Substituting these in Lagranges eqn of motion

$-m\omega_1^2 a_1 \cos\omega t + k a_1 \cos\omega t - k_{12} a_2 \cos\omega t + k_{12} a_1 \cos\omega t = 0$

$\Rightarrow a_1 \cos\omega t\,(-m\omega^2 + k + k_{12}) + a_2 \cos\omega t\,(-k_{12}) = 0$

& $-m\omega_2^2 a_2 \cos\omega t + k a_2 \cos\omega t + k_{12} a_2 \cos\omega t - k_{12} a_1 \cos\omega t = 0$

$\Rightarrow a_1 \cos\omega t\,(-k_{12}) + a_2 \cos\omega t\,(-m\omega^2 + k + k_{12}) = 0$

writing the coefficients of $a_1 \cos\omega t$ & $a_2 \cos\omega t$ in the form of determinant

$$\begin{vmatrix} -m\omega^2 + k + k_{12} & -k_{12} \\ -k_{12} & -m\omega^2 + k + k_{12} \end{vmatrix} = 0.$$

$\Rightarrow (-m\omega^2 + k + k_{12})^2 = k_{12}^2$

$\Rightarrow -m\omega^2 + k + k_{12} = \pm k_{12}$

$\Rightarrow m\omega^2 = k + k_{12} \pm k_{12}$

$\omega^2 = \frac{1}{m}(k + k_{12} \pm k_{12})$

$\omega_0 = \sqrt{\frac{k + k_{12} \pm k_{12}}{M}}$

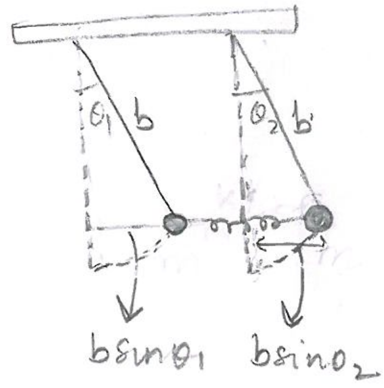
$\omega_1 = \sqrt{k/M}$ $\qquad \omega_2 = \sqrt{\frac{k + 2k_{12}}{M}}$

4. Determine the eigenfrequencies and describe the normal mode motion for two pendula of equal length b and equal masses m connected by a spring of force constant k. Choose the generalised coordinates and solve using Lagrange's method for small oscillations.

Soln

$K.E. = \frac{1}{2}m(b\dot{\theta}_1)^2 + \frac{1}{2}m(b\dot{\theta}_2)^2$

$P.E. = mgb(1-\cos\theta_1) + mgb(1-\cos\theta_2)$
$\quad + \frac{1}{2}k(b\sin\theta_1 - b\sin\theta_2)^2$



Now for small θ

$\sin\theta = \theta$ and $\cos\theta = 1 - \frac{1}{2}\theta^2$

Substituting above.

$V = \frac{mgb}{2}\theta_1^2 + \frac{mgb}{2}\theta_2^2 + \frac{1}{2}k(b\theta_1 - b\theta_2)^2$

$= \frac{mgb}{2}(\theta_1^2 + \theta_2^2) + \frac{1}{2}kb^2(\theta_1 - \theta_2)^2$

$L = T - V$

$= \frac{1}{2}mb^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - \frac{mgb}{2}(\theta_1^2 + \theta_2^2) - \frac{1}{2}kb^2(\theta_1 - \theta_2)^2$

Let $\theta_1 = a_1\cos\omega t$

$\theta_2 = a_2\cos\omega t$

$\ddot{\theta}_1 = -\omega^2 a_1\cos\omega t$

$\ddot{\theta}_2 = -\omega^2 a_2\cos\omega t$

Using Lagranges equations of motion

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$

$\Rightarrow \frac{d}{dt}(mb^2\dot{\theta}_1) + mgb\theta_1 + kb^2(\theta_1 - \theta_2) = 0$

$= mb^2\ddot{\theta}_1 + mgb\theta_1 + kb^2(\theta_1 - \theta_2) = 0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$

$\frac{d}{dt}(mb^2\dot{\theta}_2) + mgb\theta_2 - kb^2(\theta_1 - \theta_2) = 0$

$= mb^2\ddot{\theta}_2 + mgb\theta_2 - kb^2(\theta_1 - \theta_2) = 0$

Substituting the value of $\theta_1, \theta_2, \ddot{\theta}_1, \ddot{\theta}_2$ in Lagrange's equation of motion

$$(-mb^2\omega^2 + mgb + kb^2)\,a_1 \cos\omega t - kb^2 a_2 \cos\omega t = 0$$

& $$-kb^2 a_1 \cos\omega t + (-mb^2\omega^2 + mgb + kb^2)\,a_2 \cos\omega t = 0$$

writing the coefficients of $a_1 \cos\omega t$ & $a_2 \cos\omega t$ in the form of determinant.

$$\begin{vmatrix} -mb^2\omega^2 + mgb + kb^2 & -kb^2 \\ -kb^2 & -mb^2\omega^2 + mgb + kb^2 \end{vmatrix} = 0$$

$$\Rightarrow (-mb^2\omega^2 + mgb + kb^2)^2 = (kb^2)^2$$

$$\Rightarrow -mb^2\omega^2 + mgb + kb^2 = \pm kb^2$$

$$mb^2\omega^2 = mgb + kb^2 \pm kb^2$$

$$\omega^2 = \frac{mgb + kb^2 \pm kb^2}{mb^2}$$

$$\omega_1^2 = \frac{mgb}{mb^2} = \frac{g}{b}$$

$$\omega_2^2 = \frac{mgb + 2b^2k}{mb^2} = \frac{g}{b} + \frac{2k}{m}$$

5. Determine the eigen-frequencies and describe the normal mode of a symmetrical linear triatomic molecule similar to CO_2. The central atom has mass M and the symmetrical atoms have masses m. Both longitudinal & transverse vibrations are possible.

Soln :

According to center of mass position conservation

$$m(x_1+x_3)+Mx_2=0$$

$$x_2 = -\frac{m}{M}(x_1+x_3)$$

$$K.E. = \frac{1}{2}m\dot{x}_1^2+\frac{1}{2}M\dot{x}_2^2+\frac{1}{2}m\dot{x}_3^2$$

$$\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_3^2+\frac{1}{2}M.\frac{m^2}{M^2}(\dot{x}_1^2+\dot{x}_3^2+2\dot{x}_1\dot{x}_3)$$

$$P.E. = \frac{1}{2}K(x_2-x_1)^2+\frac{1}{2}K(x_3-x_2)^2$$

Let $q_1 = x_1+x_3$

$q_2 = x_3-x_1$ $\qquad x_2 = -\frac{m}{M}q_1$

$\Rightarrow$ $x_1 = \frac{1}{2}(q_1-q_2)$

$x_3 = \frac{1}{2}(q_1+q_2)$

$$K.E. = \frac{1}{2}m\times\frac{1}{4}(\dot{q}_1-\dot{q}_2)^2+\frac{1}{2}\frac{m}{4}(\dot{q}_1+\dot{q}_2)^2+\frac{1}{2}M\left(\frac{-m}{M}\dot{q}_1\right)^2$$

$$\Rightarrow \frac{1}{8}m(\dot{q}_1^2+\dot{q}_2^2-2\dot{q}_1\dot{q}_2)+\frac{1}{8}m(\dot{q}_1^2+\dot{q}_2^2+2\dot{q}_1\dot{q}_2)$$

$$+\frac{1}{2}\frac{m^2}{M}\dot{q}_1^2$$

$$= \frac{1}{4}m\dot{q}_1^2+\frac{1}{2}\frac{m^2}{M}\dot{q}_1^2+\frac{1}{4}m\dot{q}_2^2$$

$$\Rightarrow \dot{q}_1^2\left[\frac{m}{4}+\frac{m^2}{2M}\right]+\frac{\dot{q}_2^2 m}{4}$$

$$P.E. = \frac{1}{2}k\left[(x_2-x_1)^2+(x_3-x_2)^2\right]$$

$$= \frac{1}{2}k\left[x_2^2+x_1^2-2x_1x_2+x_3^2+x_2^2-2x_2x_3\right]$$

$$= \frac{1}{2}k\left[x_1^2+2x_2^2+x_3^2-2x_2(x_1+x_3)\right]$$

$$= \frac{1}{2}k\left[\frac{1}{4}(q_1-q_2)^2+2\left(\frac{-m}{M}q_1\right)^2+\frac{1}{4}(q_1+q_2)^2-2\left(\frac{-m}{M}\right)q_1^2\right]$$

$$= \frac{1}{2}k\left[\frac{1}{4}(q_1^2+q_2^2-2q_1q_2)+\frac{2m^2}{M^2}q_1^2+\frac{1}{4}(q_1^2+q_2^2+2q_1q_2) + \frac{2m}{M}q_1^2\right]$$

$$= \frac{1}{2}k\left[q_1^2\left(\frac{1}{4}+\frac{2m^2}{M^2}+\frac{1}{4}+\frac{2m}{M}\right)+q_2^2\left(\frac{1}{4}+\frac{1}{4}\right)\right]$$

$$= \frac{1}{2}k\left[q_1^2\left(\frac{1}{2}+\frac{2m}{M}+\frac{2m^2}{M^2}\right)+\frac{1}{2}kq_2^2\right]$$

$$\Rightarrow q_1^2k\left[\frac{1}{4}+\frac{m}{M}+\frac{m^2}{M^2}\right]+\frac{1}{4}kq_2^2$$

$$\Rightarrow q_1^2k\left(\frac{1}{2}+\frac{m}{M}\right)^2+\frac{1}{4}kq_2^2$$

$$L = T-V = \dot{q}_1^2\left(\frac{mM+2m^2}{4M}\right)+\frac{m}{4}\dot{q}_2^2-\left(\frac{m}{M}+\frac{1}{2}\right)^2kq_1^2 + \frac{1}{4}kq_2^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)-\frac{\partial L}{\partial q_1}=0$$

$$\left(\frac{mM+2m^2}{2M}\right)\ddot{q}_1+\left(\frac{m}{M}+\frac{1}{2}\right)^2kq_1=0 \quad \text{—(1)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)-\frac{\partial L}{\partial q_2}=0$$

$$\cancel{\left(\frac{mM+2m^2}{2M}\right)\ddot{q}_1} \Rightarrow \frac{m}{2}\ddot{q}_2+\frac{1}{2}kq_2=0 \quad (2)$$

Let $q_1 = a_1 \cos\omega t$

$q_2 = a_2 \cos\omega t$

$\ddot{q}_1 = -\omega^2 a_1 \cos\omega t$

$\ddot{q}_2 = -\omega^2 a_2 \cos\omega t$

Substituting in Lagrange's Eqns of motion.

$$-\left(\frac{mM + 2m^2}{2M}\right)\omega^2 a_1 \cos\omega t + k\left(\frac{m}{M} + \frac{1}{2}\right)^2 a_1 \cos\omega t = 0$$

& $$-\frac{m}{2}\omega^2 a_2 \cos\omega t + \frac{1}{2} k a_2 \cos\omega t = 0$$

writing the coefficients of $a_1 \cos\omega t$ and $a_2 \cos\omega t$ in determinant form

$$\begin{vmatrix} -\omega^2\left(\frac{mM+2m^2}{2M}\right) + \left(\frac{m}{M} + \frac{1}{2}\right)^2 k & 0 \\ 0 & -\frac{m\omega^2}{2} + \frac{1}{2}k \end{vmatrix} = 0$$

$$\Rightarrow \left[-\omega^2\left(\frac{mM+2m^2}{2M}\right) + \left(\frac{m}{M} + \frac{1}{2}\right)^2\right] \times \left[-\frac{m\omega^2}{2} + \frac{k}{2}\right] = 0$$

$$\omega^2 = \frac{k}{m}$$

& $$\omega^2 = \left(\frac{m}{M} + \frac{1}{2}\right)^2 k \Big/ \left(\frac{mM + 2m^2}{2M}\right)$$

$$= \left(\frac{m}{M} + \frac{1}{2}\right)^{\cancel{2}} k \Big/ m\left(\frac{1}{2} + \frac{m}{M}\right)$$

$$= \left(\frac{1}{M} + \frac{1}{2m}\right)k \Rightarrow \left(\frac{2m+M}{2mM}\right)k$$

$$\omega_1^2 = \frac{k}{m} \quad \& \quad \omega_2^2 = \left(\frac{2m+M}{2mM}\right)k$$

6. Find the Hamilton's equation of motion of an LC circuit that have no resistance contained in it. Initially the capacitor is charged to q. What is the generalised momentum of the system

Soln

Lagrangian $L = T - V = \frac{1}{2} L\dot{q}^2 - \frac{q^2}{2C}$

Hamiltonian $\mathcal{H} = p_i \dot{q}_i - L$

Generalised momentum $P = \frac{\partial L}{\partial \dot{q}} = L\dot{q}$

$$\Rightarrow \dot{q} = P/L$$

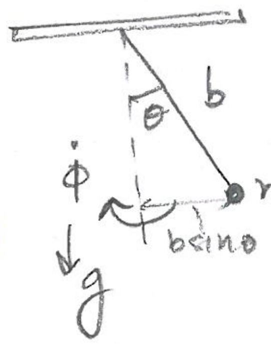
$$\mathcal{H} = P.\frac{P}{L} - \left(\frac{1}{2}L\left(\frac{P}{L}\right)^2 - \frac{q^2}{2C}\right)$$

$$= \frac{1}{2}\frac{P^2}{L} + \frac{1}{2C}q^2$$

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{P}{L}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -q/C$$

7. Use Hamilton's eqn of motion for a spherical pendulum of mass m and length b.



Soln

$T = \frac{1}{2} mb^2\dot{\theta}^2 + \frac{1}{2} mb^2 \sin^2\theta\, \dot{\phi}^2$

$V = -mgb\cos\theta$

$L = T - V = \frac{1}{2} mb^2\dot{\theta}^2 + \frac{1}{2} mb^2 \sin\theta\, \dot{\phi}^2 + mgb\cos\theta$

$P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mb^2\dot{\theta}$

$P_\phi = \frac{\partial L}{\partial \dot{\phi}} = mb^2 \sin^2\theta\, \dot{\phi}$

$$\mathcal{H} = T + V = \frac{1}{2} mb^2 \frac{P_\theta^2}{(mb^2)^2} + \frac{1}{2}\frac{mb^2\sin^2\theta}{(mb^2\sin^2\theta)^2} P_\phi^2 - mgb\cos\theta$$

$$= \frac{p_\theta^2}{2mb^2} + \frac{p_\phi^2}{2mb^2\sin^2\theta} - mgb\cos\theta$$

$$\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mb^2}$$

$$\dot{\phi} = \frac{\partial H}{\partial P_\phi} = \frac{P_\phi}{mb^2 \sin^2\theta}.$$

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = \frac{P_\phi^2 \cos\theta}{mb^2 \sin^3\theta} - mgb \sin\theta$$

$$\dot{P}_\phi = -\frac{\partial H}{\partial \phi} = 0$$

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