Introduction to analytical mechanics

Vectorial or Newtonian mechanics

- Isolates particle by particle
- The next is to start from the force on each particle (vector function)

Analytical mechanics

- Analytical mechanics considers system as a whole
- It starts from the work function (K.E. and P.E.) (scalar function)

List of reference

1) Classical dynamics of particles and systems by S T Thornton and J B Marion (TM)

2) Classical mechanics by Kibble and Berkshire (KB)

3) Classical mechanics by Gregory(GY)

4) Introduction to classical mechanics with problems and solutions by D Morin (DM)

5) Classical mechanics Systems of particles and Hamiltonian dynamics by W Greiner(GREH)

6) Classical dynamics of particles and systems by Thorton and Marion (TM)

7) Differential equations with applications and Historical notes by George F Simmons (GFS)

8) Classical mechanics by H Goldstein (HG)

9) Variational Principles of Mechanics by C Lanczos (CL)

10) Thermodynamics and an introduction to thermostatics by H B Callen (HBC)

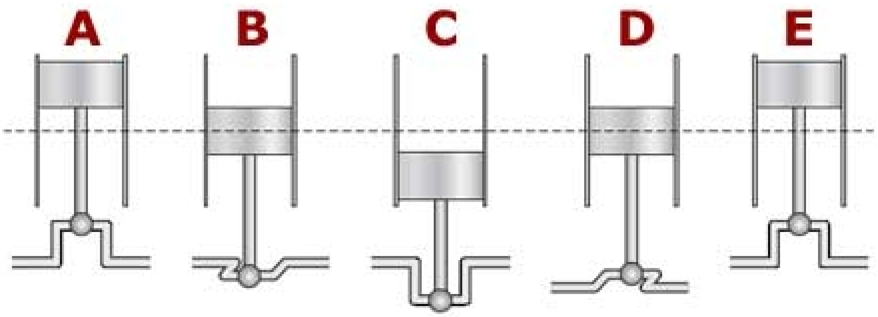
Degrees of freedom

Number of coordinate that can vary independently is called the **degrees of freedom**

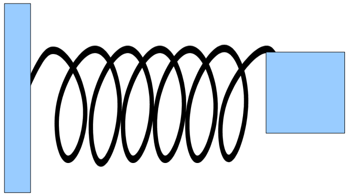
Examples

1 degree of freedom

piston $\vec{r}=\vec{r}\,(x)$



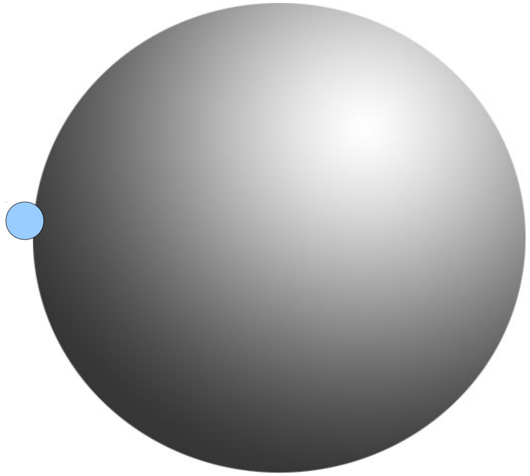
spring



2 degree of freedom

Particle constrained to move on the surface of a sphere

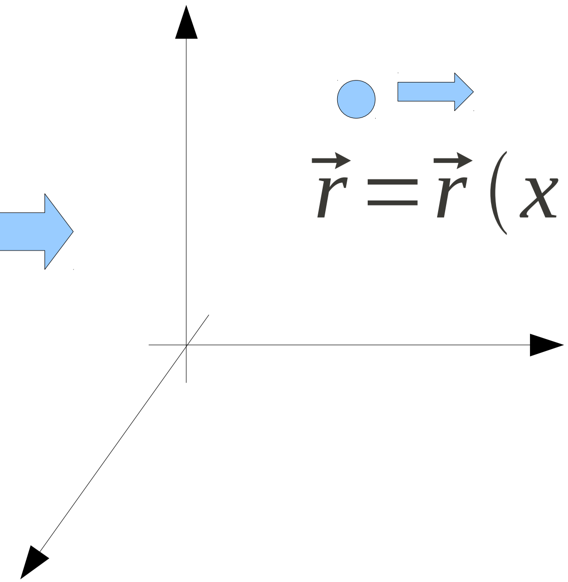
$$\vec{r}=\vec{r}\,(\theta\,,\phi)$$



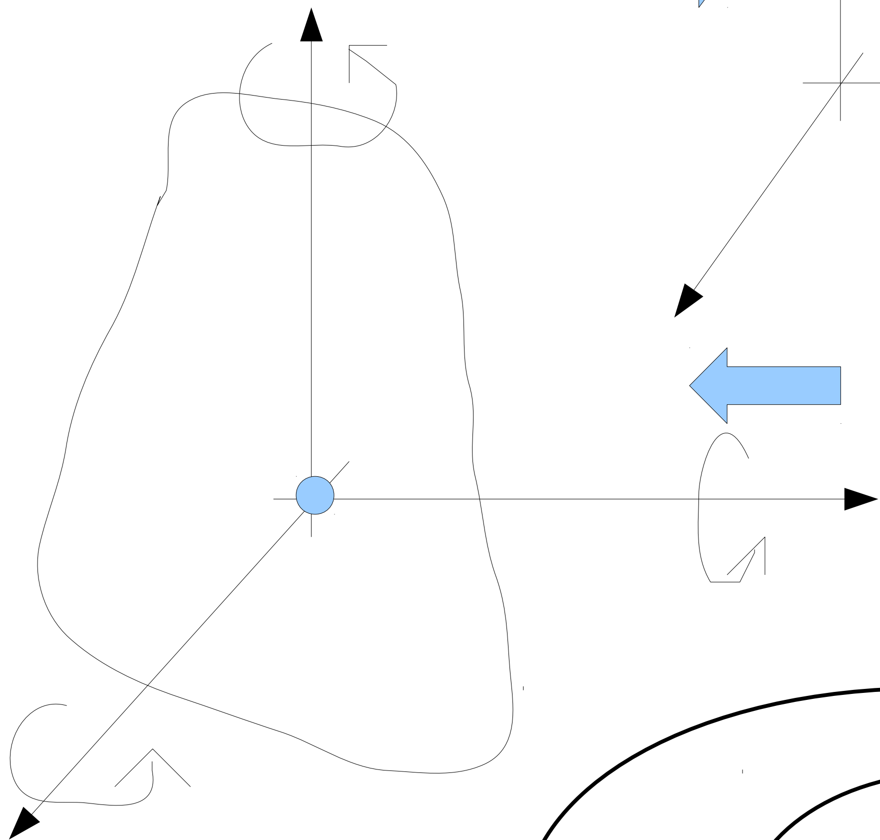
Degrees of freedom may not be Cartesian (by Rene Descartes)

3 degree of freedom

Particle motion in free space

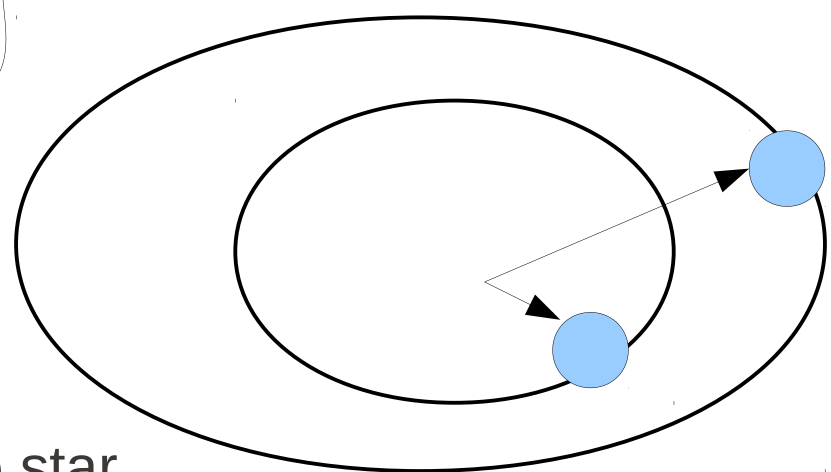


$$\vec{r} = \vec{r}\,(x, y, z)$$



Rotation of rigid body with one point fixed in space

4 degree of freedom

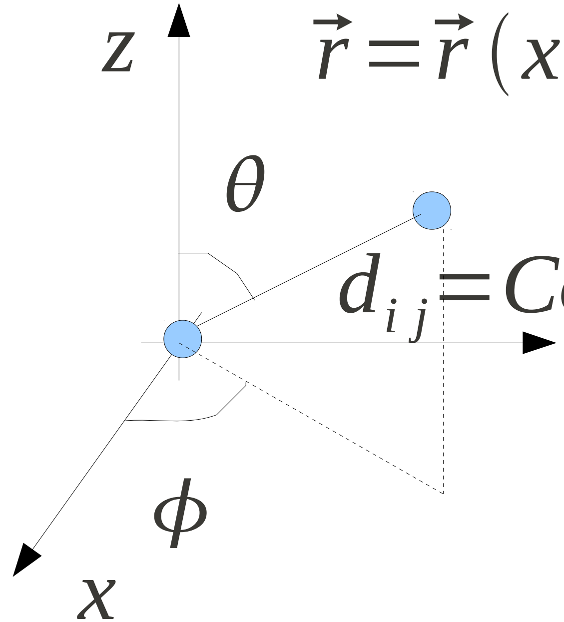


Orbital motion of a double star in a plane

$$\vec{r} = \vec{r}\,(r_{1,} \theta_{1,} r_{2,} \theta_{2})$$

5 degree of freedom

$$\vec{r} = \vec{r}(x, y, z, \theta, \phi)$$



Two particle distance between which are fixed

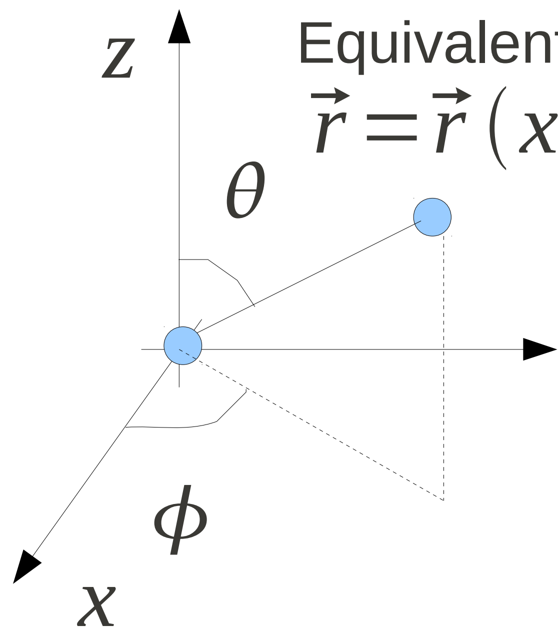
$$d_{ij} = Constant$$

6 degree of freedom

$$\vec{r} = \vec{r}(x, y, z, r, \theta, \phi)$$

Equivalent representations

$$\vec{r} = \vec{r}(x_{1,} y_{1,} z_{1,} x_{2,} y_{2,} z_2)$$



Two particle distance between which are not fixed or constrained

Generalized coordinates

In real life problems motion of particles are restricted or subjected to certain condition.

$$x = x(q_1, q_2, q_3) \qquad q_1 = q_1(x, y, z)$$

$$y = y(q_1, q_2, q_3) \qquad q_2 = q_2(x, y, z)$$

$$z = z(q_1, q_{2,} q_3) \qquad q_3 = q_3(x, y, z)$$

These coordinates can be expressed in terms of different representations, for example transformation between Cartesian and spherical polar coordinates for e.g. in a problem of orbital motion

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Consider a system of N particles that is not subjected to any constraints

$$x_i, y_i, z_i, (i=1,2,3,.........N)$$

Each particle has three degrees of freedom, therefore N has $3N$ degrees freedom

$$x_1 = f_1(q_1, q_2, ...q_{3N})$$

$$.......................................$$

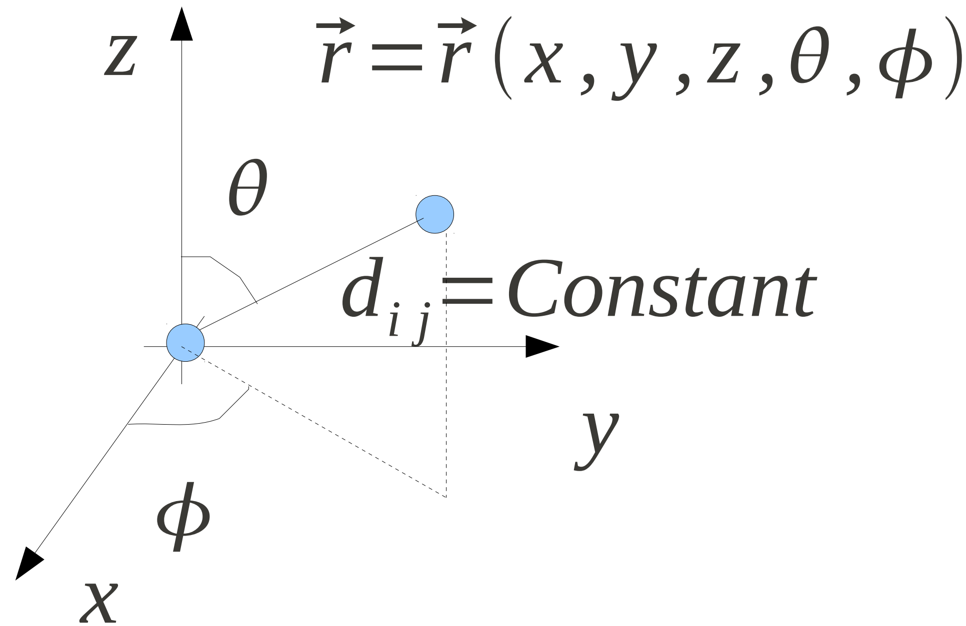
$$.......................................$$

$$z_n = f_{3N}(q_1, q_2, ...q_{3N})$$

These parameters that represent a degrees freedom may not have dimension of length

The number of degrees of freedom reduces as constraints are imposed on the system

Constraints



Consider a diatomic molecule or a dumbbell, let the distance between two objects

$$d_{ij}=Constant$$

Six degrees of freedom of two particles reduces to five due to a constraint

$$r=d_{ij}\Rightarrow r-d_{ij}=0$$

In general if there are m equations of constraint of type of system having n degrees of freedom

$$g_1(q_{1,}q_2.....q_n)=0$$

$$....................$$

$$g_m(q_{1,}q_2.....q_n)=0$$

$$r-d_{ij}=0$$

$$\sqrt{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2}=0$$

example

$g_1 \longrightarrow$ function of coordinates

The number of degrees of freedom reduces to n'

$$n'=n-m$$

Classification of constraints

Constraints that can be expressed as equations of the form shown below is called **holonomic constraints**

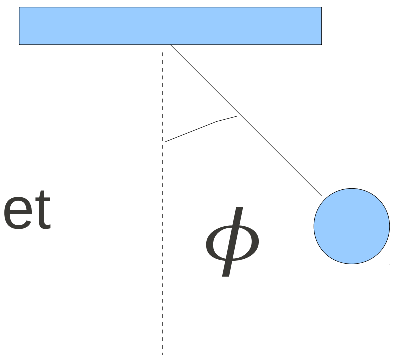
$$g_1(q_{1,} q_2 ..... q_n, t) = 0$$

A simple example of which is two particle attached by a rigid rod

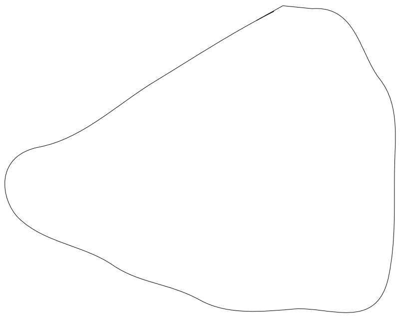
$$r - d_{i\,j} = 0$$

An explicit example is a pendulum attached to origin of the coordinate system

$$x^2 + y^2 - l^2 = 0$$



Another example is the degrees of freedom of a rigid body- let there be n particles in the system



Total number degrees of freedom $3n$

Total number of constraints are $n(n-1)/2$

The distance between first particle and others give $n-1$ equations

The distance between second particle and others give $n-2$ equations

$$n-1+n-2+n-3............=\frac{n(n-1)}{2}$$

The constraint equations are far more than the degrees of freedom however these equations are not independent.

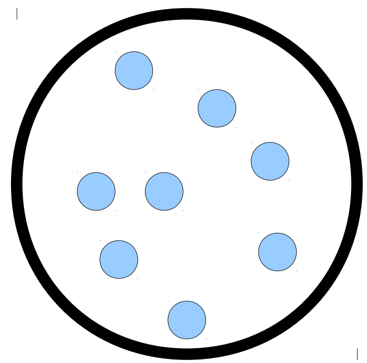
However we can calculate rigid body by empirical relation

There are three translational degrees of freedom+ three rotational degrees of freedom that means 6 degrees of freedom

Constraints that cannot be expressed as equations of the form shown below is called **non-holonomic constraints.**

They are expressed as inequalities. Gas molecules contained in container

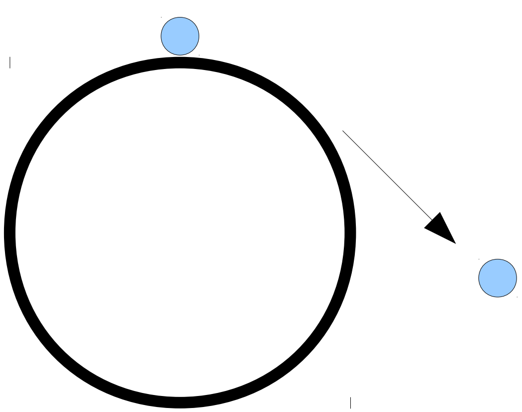
$$r^2 - a^2 \geq 0$$



Radius of the sphere

Position of the particle

Particle rolling from top of sphere



Configuration space

To find the evolution of a particle in 3d space we can represent them as function to time

$$x=f(t), y=g(t), z=h(t)$$

This is for three degrees of freedom, to represent many particle degrees freedom as a point in the corresponding number of dimension of the problem we can increase the dimension of the space. E.g For two particles 6 dimensional space.

For n degrees of freedom

$$q_1=q_1(t), q_2=q_2(t), \ldots\ldots\ldots\ldots q_n=q_n(t)$$

Such a space is called **configurational space** , in such a space a point represent the configuration of a system of particles

Mechanical quantities in generalized coordinates

We know that a mass point can be expressed in terms of n generalized coordinates

$$\vec{r}_i=\vec{r}_i(q_1,q_2,q_{3,.}....,q_n,t)$$

Time derivative is given by

$$\dot{\vec{r}}_i=\frac{\partial\vec{r}_i}{\partial q_1}\frac{\partial q_1}{\partial t}+........+\frac{\partial\vec{r}_i}{\partial q_n}\frac{\partial q_n}{\partial t}+\frac{\partial\vec{r}_i}{\partial t}$$

Due to the explicit dependence on time

For example consider the function

$$\vec{r}_i=\sin\theta(t)\quad\hat{\theta}(t)+k\quad t\quad\hat{r}(t)$$

The differential of this term consists of a term shown below

$$\dot{\vec{r}}_i=...+k\quad\hat{r}(t)+....$$

If there is no explicit dependence on time

$$\dot{\vec{r}}_i = \frac{\partial \vec{r}_i}{\partial q_1}\frac{\partial q_1}{\partial t} + \ldots\ldots\ldots + \frac{\partial \vec{r}_i}{\partial q_n}\frac{\partial q_n}{\partial t} + \underbrace{\cancel{\frac{\partial \vec{r}_i}{\partial t}}}_{0}$$

In compact form

$$\dot{\vec{r}}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

The generalized velocities does not necessary to have the dimension of distance by time

$$\vec{r}_i = \vec{r}_i(x_j)$$

$$x_j \in \{x_j, y_j, z_j\}$$

generalized velocities

Focusing on one component of the velocity with out explicit dependence on time

$$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\dot{q}_j$$

Acceleration is obtained by differentiating with respect to time once more

$$\ddot{x}_i = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right)\dot{q}_j + \frac{\partial x_i}{\partial q_j}\ddot{q}_j$$

$$\ddot{x}_i = \sum_{j=1}^{n} \frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right)\dot{q}_j + \frac{\partial x_i}{\partial q_j}\ddot{q}_j$$

$$\frac{d}{dt}\left(\frac{\partial x_i}{\partial q_j}\right) = \sum_{k=1}^{n} \frac{\partial^2 x_i}{\partial q_j \partial q_k}\dot{q}_k$$

$$\ddot{x}_i = \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 x_i}{\partial q_j \partial q_k}\dot{q}_k \dot{q}_j + \sum_{j=1}^{n} \frac{\partial x_i}{\partial q_j}\ddot{q}_j$$

The expression for acceleration in terms of the generalized coordinates

Now we may compute the differential change of length as

$$d\vec{r}_i = \sum_{i=1}^{n} \frac{\partial \vec{r}_i}{\partial q_i} dq_i$$

The work done can also be written in terms of the generalized coordinates

$$dW = \sum_{i=1}^{n} \vec{F}_i \cdot d\vec{r}_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) dq_j$$

$$= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) dq_j = \sum_{j=1}^{n} Q_j dq_j$$

Last step is by exchange of the sums

Q_j is called generalized force: Generalized force may not have dimensions of force

$$Q_j = \left(\sum_{i=1}^{n} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right)$$

However, $\sum_{j=1}^{n} Q_j dq_j$ always have dimensions of work

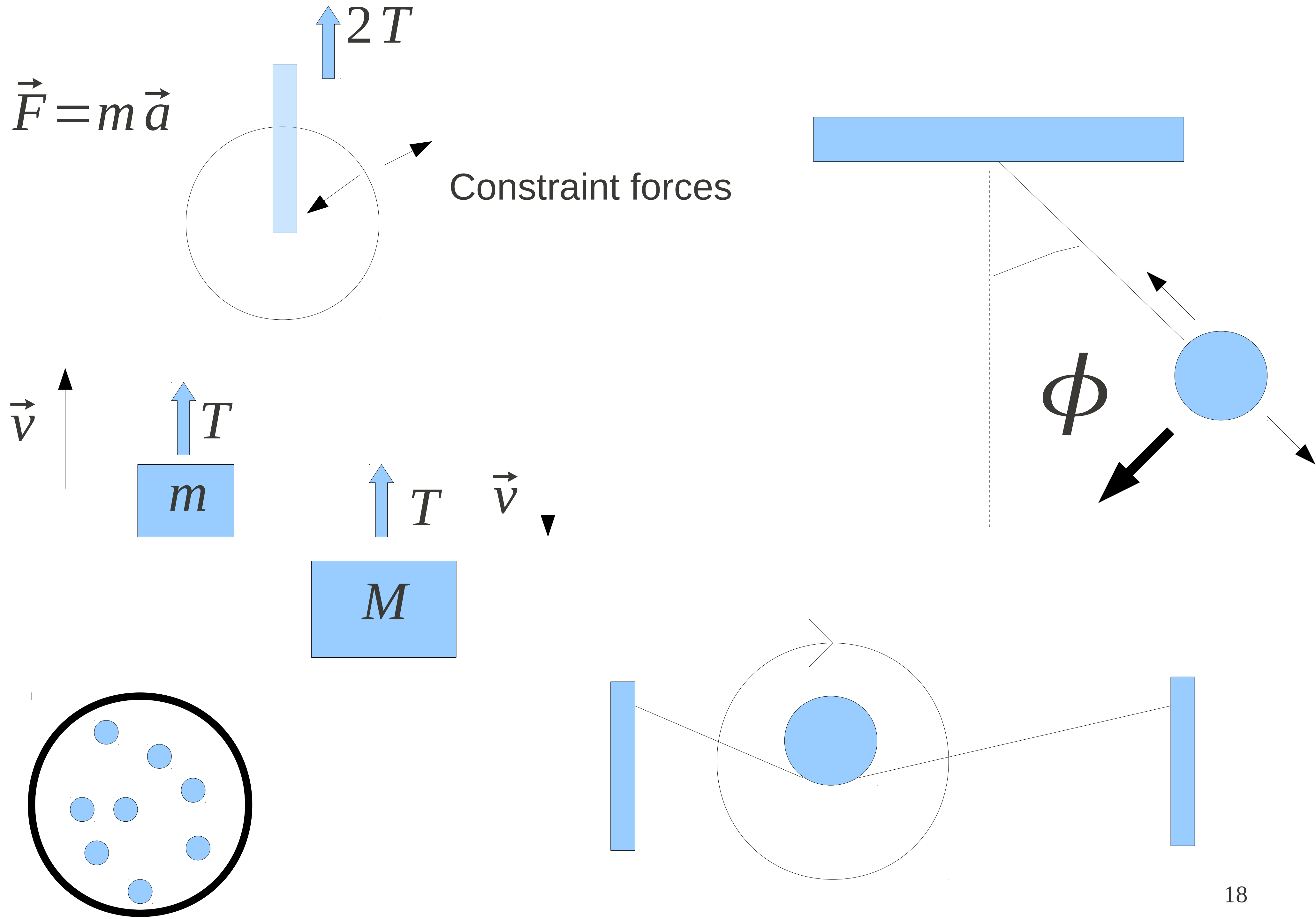
For conservative systems there is no explicit time dependence

$$dW = \sum_{j=1}^{n} \frac{\partial W}{\partial q_j} dq_j \;\; = \sum_{j=1}^{n} Q_j dq_j$$

$$\sum_{j=1}^{n} \left(Q_j - \frac{\partial W}{\partial q_j} \right) dq_j = 0$$

That means each term in the sum is equal to zero $\quad Q_j = \frac{\partial W}{\partial q_j}$

Components of generalized forces is obtained as derivative of work done with respect to generalized coordinates


$\vec{F} = m\vec{a}$
$2T$
Constraint forces
$\vec{v}$
T
m
T
$\vec{v}$
M
ϕ

Principle of virtual displacements

GREH: Chap14, KB:Chap 10, GY Chap 12

virtual displacements are infinitesimal displacements $\delta \vec{r}$ of the system that is compatible with constraints imposed on the system

Consider a system that is in equilibrium $\vec{F}_i = 0$ on each mass points

Work done on each mass points is zero $\sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0$

Separate the total force into constraint forces and reaction forces

$$\sum_i \left(\vec{F}_i^a + \vec{F}_i^z \right) \cdot \delta \vec{r}_i = 0$$

The constraint reaction is in many cases perpendicular to the direction of motion

$$\Rightarrow \vec{F}_i^z \cdot \delta \vec{r}_i = 0$$

When they are not perpendicular sum of them should vanish

$$\sum_i \vec{F}_i^z \cdot \delta \vec{r}_i = 0$$

Then $\sum_i \vec{F}_i^a \cdot \delta \vec{r}_i = 0$

For holonomic constraints, the effect of the constraint reaction is obtained from

$$g_i(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n, t) = 0$$

$$\vec{r}_i = \vec{r}_i(x_j)$$

$$x_j \in \{x_j, y_j, z_j\}$$

The constraint forces are derived from gradient of constraint equation

$$\vec{F}^z_{ji} = \lambda_i \frac{\partial g_i(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n, t)}{\partial \vec{r}_j} = \lambda_i \left[\frac{\partial g_i}{\partial x_j} \frac{\partial x_j}{\partial \vec{r}_j} + \ldots \right]$$

The constraint force on j the particle by the i'th constraint

$$= \lambda_i \left[\frac{\partial g_i}{\partial x_j} \frac{1}{\frac{\partial \vec{r}_j}{\partial x_j}} + \ldots \right] = \lambda_i \left[\frac{\partial g_i}{\partial x_j} \frac{1}{\hat{x}} + \ldots \right] = \lambda_i \left[\frac{\partial g_i}{\partial x_j} \hat{x} + \ldots \right]$$

$$= \lambda_i \vec{\nabla}_j g_i$$

Unknown factor

Total constraint reaction should be such that sum of which should vanish

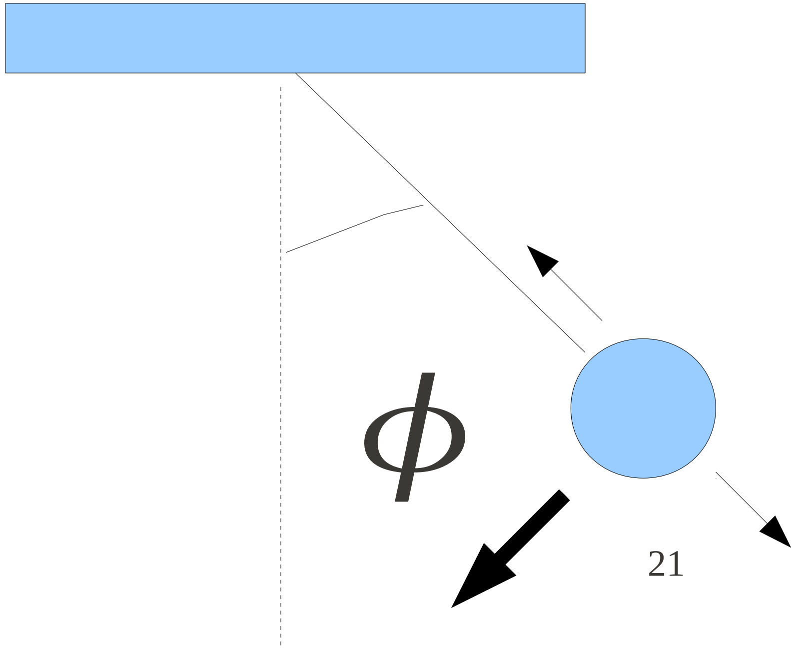
Total constrained forces are

$$\vec{F}_j^z = \sum_{i=1}^{m} \vec{F}_{ij}^z = \sum_{i=1}^{m} \lambda_i \frac{\partial g_i(\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots, \vec{r}_n, t)}{\partial \vec{r}_j}$$

Virtual work performed by the constraints are now given by substituting $\vec{F}_j^z$

$$\delta W = \sum_{j=1}^{n} \vec{F}_j^z \cdot \delta \vec{r}_j = \sum_{j=1}^{n} \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial \vec{r}_j} \cdot \delta \vec{r}_j = \sum_{j=1}^{n} \lambda_i \delta g_i$$

$$\delta g_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial \vec{r}_j} \delta \vec{r}_j$$



Since virtual displacements are compatible with the constraints $\delta g_i = 0$

$$\delta W = \sum_{j=1}^{n} \vec{F}_j^z \cdot \delta \vec{r}_j \quad = \sum_{j=1}^{n} \lambda_i \delta g_i$$

Where $$\delta g_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial \vec{r}_j} \delta \vec{r}_j$$

Since virtual displacements are compatible with the constraints

$$\delta g_i = 0$$

That implies $$\vec{F}_j^z \cdot \delta \vec{r}_j = 0$$

Total work done by the constraint forces are zero

$$\Rightarrow \delta W = \sum_{j=1}^{n} \vec{F}_j^z \cdot \delta \vec{r}_j = 0$$

The individual sum of the terms may not be zero however the total sum must be zero. For holonomic constraints each individual term in the sum vanishes

Principle of virtual work allow us to work on statics

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i = 0$$

When a work is done at non-equilibrium case

By Newton's laws $\vec{F}_i = \dot{\vec{p}}_i$

By condition $\delta W = \sum_{j=1}^{n} \vec{F}_j^z \cdot \delta \vec{r}_j = 0$

constraint forces do not perform any work, therefore

in the sum $\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$ each individual term vanishes

$$\Rightarrow \sum_i (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$$

This is the condition for equilibrium for a system, here work is done by action forces only

This is called **d' Alembert's Principle**

Conditions of balance

$$\delta W = \sum_{j=1}^{n} \vec{F}_j^z \cdot \delta \vec{r}_j = 0$$

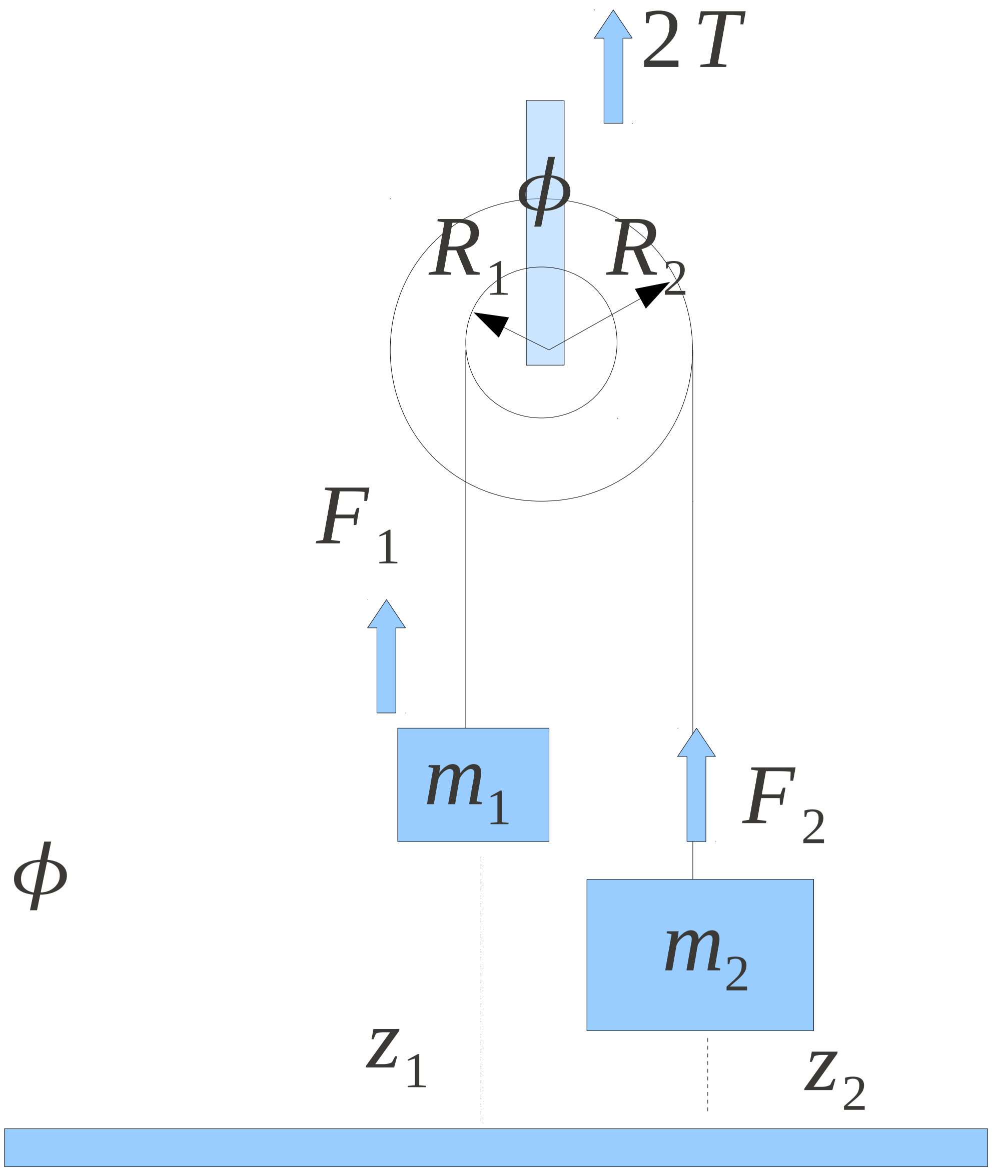
Constraint forces are string tensions

Equilibrium the torques are balanced

$$D_1 = R_1 F_1^z = D_2 = R_2 F_2^z$$

Change in displacements

$$\delta z_1 = R_1 \delta \phi \qquad \delta z_2 = -R_2 \delta \phi$$



Total work done

$$F_1^z \delta z_1 + F_2^z \delta z_2 = (F_1^z R_1 - F_2^z R_2) \delta \phi = (D_1 - D_2) \delta \phi = 0$$

When radius are equal the string tensions should be equal to achieve equilibrium

Therefore $$\sum_j \vec{F}_j^a \cdot \delta \vec{r}_j = 0$$

It follows that $$m_1 g \delta z_1 + m_2 g \delta z_2 = 0$$

Displacements are correlated by the condition

$$\delta z_1 = R_1 \delta \phi \qquad \delta z_2 = -R_2 \delta \phi$$

$$\Rightarrow (m_1 R_1 - m_2 R_2) \delta \phi = 0$$

or

$$m_1 R_1 = m_2 R_2$$

This is the conditions for equilibrium

Lagrange's equation of motion

Expressing the d' Alembert's principle of virtual work in generalized coordinates

$$\sum_i (\vec{F}_i^a - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$$

The removal of constraint forces is by generalized coordinates

$$\sum_i \vec{F}_i \cdot \delta \vec{r}_i \quad = \sum_i \vec{F}_i \cdot \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = 0$$

We know that $$Q_j = \left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right)$$

By the exchange of indexes

$$\sum_j \left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) dq_j \quad = \sum_j Q_j dq_j$$

By the exchange of indexes

$$Q_j = \left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right)$$

$$\sum_j \left(\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \right) dq_j \quad = \sum_j Q_j dq_j$$

Now we convert the second term in the d'Alembert's equation also in generalized coordinates

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$$

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i \quad = \sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i$$

$$= \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

As mass of the point particles are not varying with time

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

By addition and subtraction of the same term we can rearrange following term as

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \boxed{\sum_i \frac{d}{dt}(m_i \dot{\vec{r}}_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j} + \sum_i m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)}$$

$$- \sum_i m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

In the compact form

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \sum_i m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \sum_i m_i \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

Now to arrive at form of the kinetic energy

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right) = \frac{\partial}{\partial q_j}\vec{v}_i$$

$$\sum_i (\vec{F}_i^a - \dot{\vec{p}}_i)\cdot\delta\vec{r}_j = 0$$

By substituting in the equation of reaction term in the d Alembert's equation becomes

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \sum_i m_i \vec{v}_i \cdot \left(\frac{\partial \vec{v}_i}{\partial q_j}\right)$$

Example in polar coordinate system

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right)$$

$$d\vec{r} = d(r\hat{r}) = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$$

First perform partial differentiation with generalized coordinates

$$\frac{\partial \vec{r}}{\partial r} = \hat{r}, \qquad \frac{\partial \vec{r}}{\partial \theta} = r\,\hat{\theta}$$

then differentiate with time

$$\frac{d}{dt}\frac{\partial \vec{r}}{\partial r} = \dot{\hat{r}} = \dot{\theta}\,\hat{\theta}, \qquad \frac{d}{dt}\frac{\partial \vec{r}}{\partial \theta} = \dot{r}\,\hat{\theta} + \dot{\hat{\theta}}\,r = \dot{r}\,\hat{\theta} - r\,\dot{\theta}\,\hat{r},$$

To prove now differentiate first with time

$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$$

Now perform the partial differentiation with respect to generalized coordinates

$$\frac{\partial \dot{\vec{r}}}{\partial r} = \dot{\theta}\,\hat{\theta} = \frac{d}{dt}\frac{\partial \vec{r}}{\partial r},$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$$

$$\frac{\partial \dot{\vec{r}}}{\partial \theta} = \dot{r}\,\frac{\partial \hat{r}}{\partial \theta} + r\,\dot{\theta}\,\frac{\partial \hat{\theta}}{\partial \theta} = \dot{r}\,\hat{\theta} - r\,\dot{\theta}\,\hat{r} = \frac{d}{dt}\frac{\partial \vec{r}}{\partial \theta}$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \boxed{\sum_i \frac{d}{dt}\left(m_i \dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right)} - \sum_i m_i \vec{v}_i \cdot \left(\frac{\partial \vec{v}_i}{\partial q_j}\right)$$

We know $\dot{\vec{r}}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \frac{\partial \vec{r}_i}{\partial t},$ $\quad d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$

$$\Rightarrow \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

since $\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial \vec{r}_i}{\partial t}\right) = 0$

that is, the partial differentiation does not have have any generalized velocity dependence

Now insert the relation

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \boxed{\sum_i \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right)} - \sum_i m_i \vec{v}_i \cdot \left(\frac{\partial \vec{v}_i}{\partial q_j}\right)$$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \boxed{\sum_i \frac{d}{dt}\left(m_i \vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j}\right) - \sum_i m_i \vec{v}_i \cdot \left(\frac{\partial \vec{v}_i}{\partial q_j}\right)}$$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_j} \sum_i \left(\frac{1}{2} m_i \vec{v}_i^{\,2}\right)\right) - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2}\right)$$

Where $\sum_i \frac{1}{2} m_i \vec{v}_i^{\,2} = T$ is the kinetic energy of the system

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right)$$

Now this equation is expressed solely in terms system kinetic energy, in addition, the vectorial quantities are replaced by generalized coordinates.

Inserting this back to the equation $\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right)$$

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_j \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j$$

We have expressed the work done by the forces is now expressed in terms of **Kinetic energy of the system and generalized coordinates**

Now we can re express the d Alembert's principle as follows

We know $\sum_i \vec{F}_i \cdot d\vec{r}_i = \sum_j Q_j dq_j$

and $\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_i = 0$

$$\sum_i (\vec{F}_i - \dot{\vec{p}}_i) \cdot \delta \vec{r}_j = 0$$

By substitution $$\sum_j \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0$$

Since the q_j are independent generalized coordinates this equation is satisfied when each term in the coefficients vanish

$$\Rightarrow \left(\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right) = 0 \qquad j = 1,2,\ldots\ldots m$$

Now assume that all forces can be derived from potential V

(conservative force field) $\vec{F}_i = -\vec{\nabla}_i V$

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$Q_j = -\sum_i \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

This may be rearranged using

$$\sum_i \left(\frac{\partial V}{\partial x_i}\hat{x} + \frac{\partial V}{\partial y_i}\hat{y} + \frac{\partial V}{\partial z_i}\hat{z} \right) \cdot \left(\frac{\partial x_i}{\partial q_j}\hat{x} + \frac{\partial y_i}{\partial q_j}\hat{y} + \frac{\partial z_i}{\partial q_j}\hat{z} \right)$$

$$= \sum_i \left(\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_j} \right) = \frac{\partial V}{\partial q_j}$$

$$Q_j = -\frac{\partial V}{\partial q_j}$$

Using this relation in equation $\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right) = 0$

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} \right) = 0$$

$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial (T-V)}{\partial q_j} \right) = 0$$

Potential is independent of generalized velocity $\Rightarrow \frac{\partial V}{\partial \dot{q}_j} = 0$

$$\Rightarrow \left(\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j} \right) - \frac{\partial (T-V)}{\partial q_j} \right) = 0$$

The function $L = T - V$ called **Lagrangian** of the system

$$\Rightarrow \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} \right) = 0 \qquad j = 1,2,\ldots\ldots m$$

From **d Alembert's principle** we arrive at an equation of motion that is derivable from kinetic energy and potential energy of the system

$$\Rightarrow\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right)=0$$

This formalism can be interpreted in terms of generalized momentum and generalized force

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$ Generalized momentum

For a free particle with no potential energy

$$L=T-V=\frac{1}{2}m v^2-0$$

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right)=\frac{\partial L}{\partial v}=\frac{\partial\left(\left(1/2\right)m v^2\right)}{\partial v}=m\vec{v}$$ (momentum)

The equation of motion of the free particle is then

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right)=0\Rightarrow m\dot{\vec{v}}=0$$

$$\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}\right)=0$$

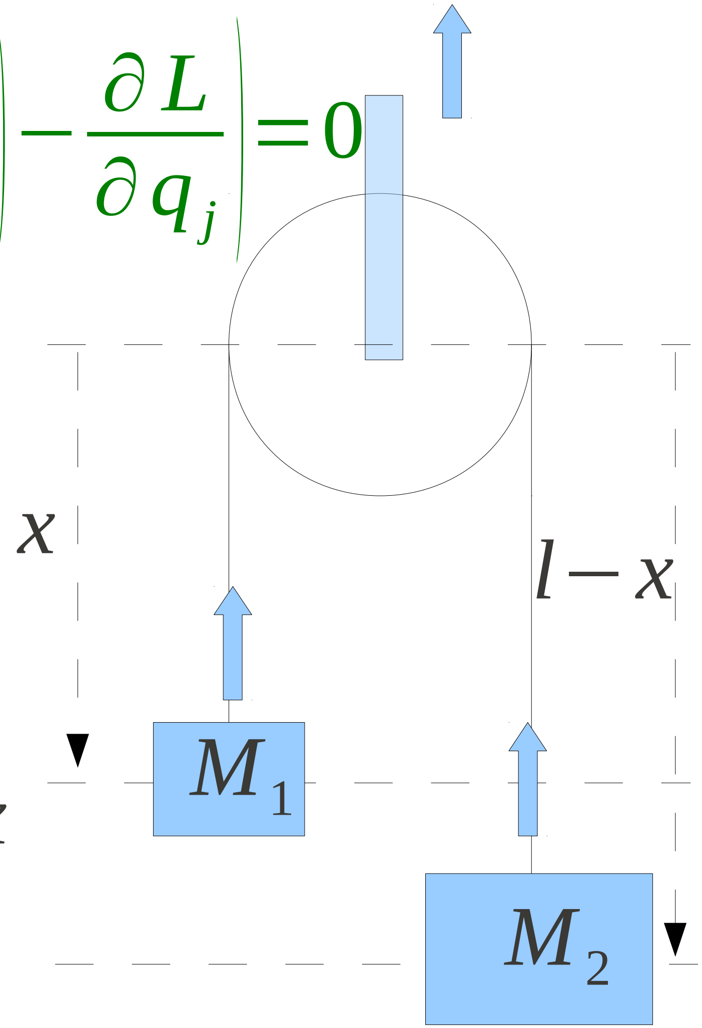
Generalized force

The equation of motion is then given by $\vec{F}_i = \dot{\vec{p}}_i$

We have found a simpler method of arriving at equation of motion from energy

Atwood's machine – equation of motion using Lagrangian $L=T-V$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0$$



$$V=-M_1 g x-M_2 g(l-x)$$

$$T=\frac{1}{2}(M_1+M_2)\dot{x}^2$$

$$L=T-V$$

$$=\frac{1}{2}(M_1+M_2)\dot{x}^2+M_1 g x+M_2 g(l-x)$$

$$\frac{\partial L}{\partial x}=(M_1-M_2)g,\ \frac{\partial L}{\partial \dot{x}}=(M_1+M_2)\dot{x}$$

Lagrange's equation of motion

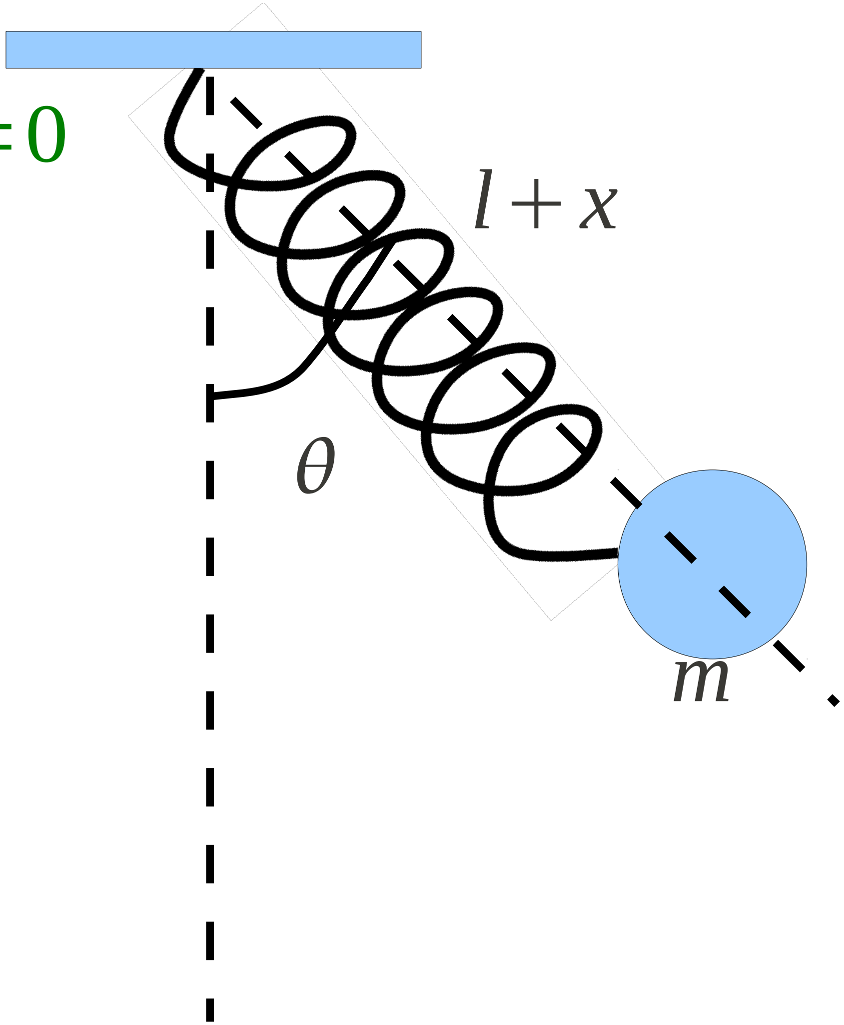
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x}$$

$$(M_1+M_2)\ddot{x}=(M_1-M_2)g$$

$$\ddot{x}=\frac{(M_1-M_2)}{(M_1+M_2)}g$$

Spring pendulum

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0$$



Mass of bead m

Length of the spring $l+x(t)$

Kinetic energy of the system

$$T=\frac{1}{2}m(\dot{x}^2+(l+x)^2\dot{\theta}^2)$$

Potential energy of the system has effect of gravity and potential due to spring

$$V(x,\theta)=-mg(l+x)\cos\theta+\frac{1}{2}k\,x^2$$

Lagrangian of the system is then $L=T-V$

$$L=\frac{1}{2}m(\dot{x}^2+(l+x)^2\dot{\theta}^2)+mg(l+x)\cos\theta-\frac{1}{2}k\,x^2$$

The Lagrange's equation of motion has the beauty that it can be applied separately to each degree of freedom

$$L=\frac{1}{2}m(\dot{x}^2+(l+x)^2\dot{\theta}^2)+mg(l+x)\cos\theta-\frac{1}{2}kx^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=\frac{\partial L}{\partial x}$$

Applying Lagrange's equation for x

$$\Rightarrow m\ddot{x}=m(l+x)\dot{\theta}^2+mg\cos\theta-kx$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=\frac{\partial L}{\partial \theta}$$

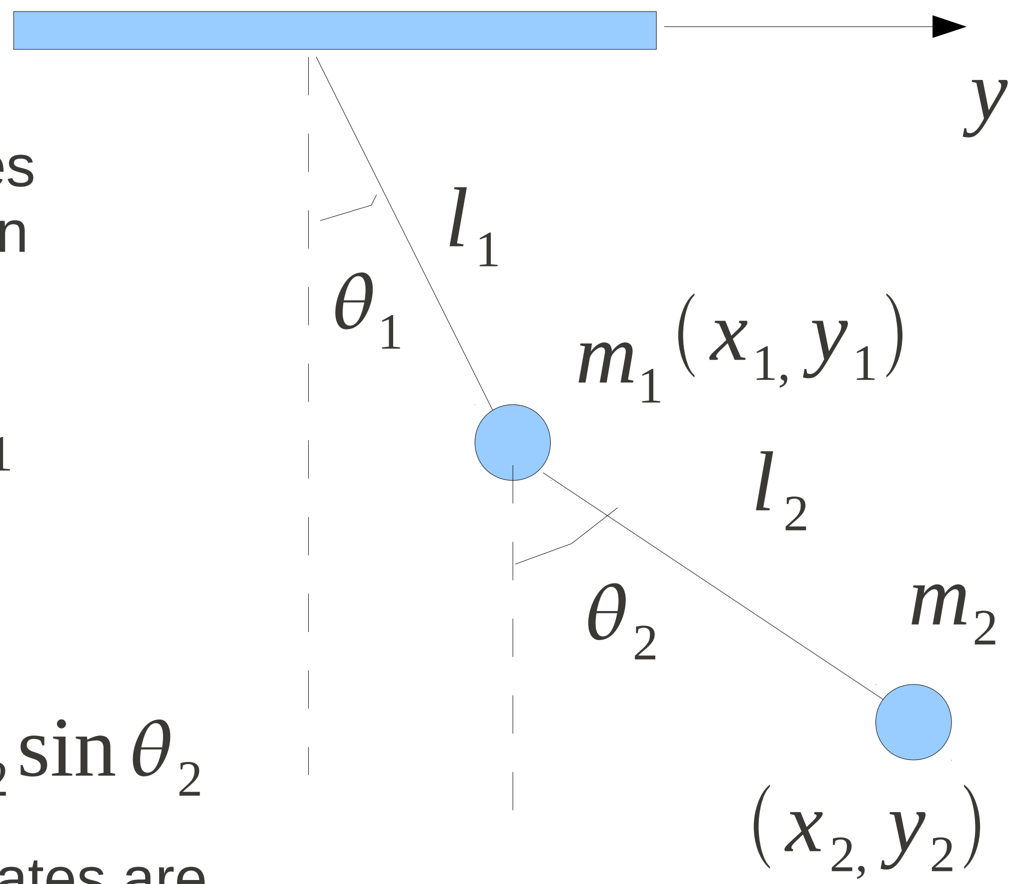
Applying Lagrange's equation for θ

$$\Rightarrow \frac{d}{dt}\left(m(l+x)^2\dot{\theta}\right)=-mg(l+x)\sin\theta$$

$$\Rightarrow m(l+x)^2\ddot{\theta}+2m(l+x)\dot{x}\dot{\theta}=-mg(l+x)\sin\theta$$

$$\Rightarrow m(l+x)\ddot{\theta}+2m\dot{x}\dot{\theta}=-mg\sin\theta$$

Double pendulum



The appropriate generalized coordinates are the angles. In terms of the Cartesian coordinates

$$x_1 = l_1 \cos\theta_1 \qquad y_1 = l_1 \sin\theta_1$$

$$x_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

$$y_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

The velocities of the Cartesian coordinates are

$$\dot{x}_1 = -l_1 \dot{\theta}_1 \sin\theta_1 \qquad \dot{y}_1 = l_1 \dot{\theta}_1 \cos\theta_1$$

$$\dot{x}_2 = -l_1 \dot{\theta}_1 \sin\theta_1 - l_2 \dot{\theta}_2 \sin\theta_2 , \quad \dot{y}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_2 \cos\theta_2$$

The kinetic energy of the system is now given by

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$T=\frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2)+\frac{1}{2}m_1(\dot{x}_2^2+\dot{y}_2^2)$$

$$x_1=l_1\cos\theta_1 \qquad y_1=l_1\sin\theta_1$$

$$x_2=l_1\cos\theta_1+l_2\cos\theta_2 \qquad y_2=l_1\sin\theta_1+l_2\sin\theta_2$$

$$\dot{x}_1=-l_1\dot{\theta}_1\sin\theta_1 \qquad \dot{y}_1=l_1\dot{\theta}_1\cos\theta_1$$

$$\dot{x}_2=-l_1\dot{\theta}_1\sin\theta_1-l_2\dot{\theta}_2\sin\theta_2, \quad \dot{y}_2=l_1\dot{\theta}_1\cos\theta_1+l_2\dot{\theta}_2\cos\theta_2$$

$$T=\frac{1}{2}m_1 l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2(l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$

The potential energy of the system is

$$V=m_1 g\overbrace{\left[l_1+l_2-l_1\cos\theta_1\right]}^{\text{height}}+m_2 g\overbrace{\left[l_1+l_2-(l_1\cos\theta_1+l_2\cos\theta_2)\right]}^{\text{height}}$$

$$T=\frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2)+\frac{1}{2}m_1(\dot{x}_2^2+\dot{y}_2^2)$$

Kinetic energy of the system is

$$T=\frac{1}{2}m_1 l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2(l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$

potential energy of the system is

$$V=m_1 g\left[l_1+l_2-l_1\cos\theta_1\right]+m_2 g[l_1+l_2-(l_1\cos\theta_1+l_2\cos\theta_2)]$$

The Lagrangian of the double pendulum is given by

$$L=T-V$$

$$L=\frac{1}{2}m_1 l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2(l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2))$$

$$-m_1 g\left[l_1+l_2-l_1\cos\theta_1\right]+m_2 g[l_1+l_2-(l_1\cos\theta_1+l_2\cos\theta_2)]$$

$$L=\frac{1}{2}m_1 l_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\left(l_1^2\dot{\theta}_1^2+l_2^2\dot{\theta}_2^2+2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right)$$
$$-m_1 g\left[l_1+l_2-l_1\cos\theta_1\right]+m_2 g\left[l_1+l_2-(l_1\cos\theta_1+l_2\cos\theta_2)\right]$$

The two Lagrangian equations for the two generalized coordinates are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_1}\right)-\frac{\partial L}{\partial\theta_1}=0 \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\frac{\partial L}{\partial\theta_2}=0$$

$$\frac{\partial L}{\partial\theta_1}=-m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)-m_1 g l_1\sin\theta_1-m_2 g l_1\sin\theta_1$$

$$\frac{\partial L}{\partial\dot{\theta}_1}=m_1 l_1^2\dot{\theta}_1+m_2 l_1^2\dot{\theta}_1+m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2)$$

$$\frac{\partial L}{\partial\theta_2}=m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)-m_2 g l_2\sin\theta_2$$

$$\frac{\partial L}{\partial\dot{\theta}_2}=m_2 l_2^2\dot{\theta}_2+m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2)$$

The Lagrange's equations are

$$m_1 l_1^2 \ddot{\theta}_1 + m_2 l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$

$$= -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$

$$= m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2$$

or

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)$$

$$= -(m_1 + m_2) g l_1 \sin\theta_1$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)$$

$$= -m_2 g l_2 \sin\theta_2$$

We may solve this equation for a special case where lengths of the rods and masses are equal for small oscillations.

$$m_1 = m_2 = m \qquad l_1 = l_2 = l$$

Then equations reduces to

$$2 l_1 \ddot{\theta}_1 + l \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - l \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) = -2 g \sin\theta_1$$

$$l \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + l \ddot{\theta}_2 - l \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) = -g \sin\theta_2$$

For small oscillations

$$\sin\theta = \theta \qquad \cos\theta = 1 \qquad \dot{\theta}^2 = 0$$

$$2 l \ddot{\theta}_1 + l \ddot{\theta}_2 = -2 g \theta_1$$

$$l \ddot{\theta}_1 + l \ddot{\theta}_2 = -g \theta_2$$

We have arrived at a set of couple differential equations, using the trial solution

$$\theta_1 = A_1 e^{i\omega t} \qquad \theta_2 = A_2 e^{i\omega t}$$

$$2l\ddot{\theta}_1 + l\ddot{\theta}_2 = -2g\theta_1 \qquad l\ddot{\theta}_1 + l\ddot{\theta}_2 = -g\theta_2$$

By substitution we obtain

$$2(g - l\omega^2)A_1 - l\omega^2 A_2 = 0$$

$$-l\omega^2 A_1 + (g - l\omega^2)A_2 = 0$$

To have non trivial solution the determinant of the coefficients must vanish

$$\begin{vmatrix} 2(g - l\omega^2) & -l\omega^2 \\ -l\omega^2 & g - l\omega^2 \end{vmatrix} = 0$$

The equation becomes $\quad l^2\omega^4 - 4lg\omega^2 + 2g^2 = 0$

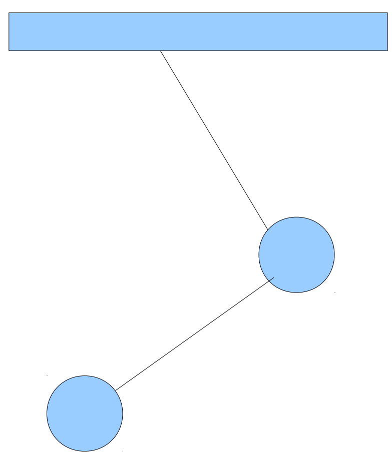
$$\Rightarrow \omega^2 = \frac{4lg \pm \sqrt{16l^2g^2 - 8l^2g^2}}{2l^2} = (2 \pm \sqrt{2})\frac{g}{l}$$

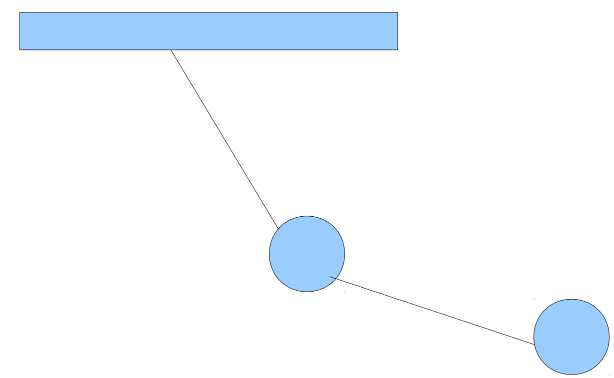
The equation becomes

$$\Rightarrow \omega_1^2 = (2+\sqrt{2})\frac{g}{l}, \omega_2^2 = (2-\sqrt{2})\frac{g}{l}$$

The two modes of the equation are by solving for amplitudes

$$\Rightarrow \omega_1^2 : A_2 = -\sqrt{2} A_{1,} \omega_2 : A_2 = \sqrt{2} A_1$$





Lagrange's equation of motion velocity dependent potentials

So far we are considering Lagrangian for the systems where forces can be derived from a potential $\vec{F}(\vec{r}) = -\vec{\nabla} V(\vec{r})$

Now take the case of Lorentz force in electrodynamics

$$\vec{F}^e = e\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Electric field

Magnetic field

In such case we call the force conservative if we can express the force in terms of generalized potentials

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j} \quad \text{where} \quad V = V(q_j, \dot{q}_j, t)$$

$V = V(q_j, \dot{q}_j, t)$ is then called generalized potential

GREH: Chap17,TM: Chap 7, HG:Chap2

The Lagrange's equation of motion in this case is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=Q_j$$

Substituting the value of generalized potential

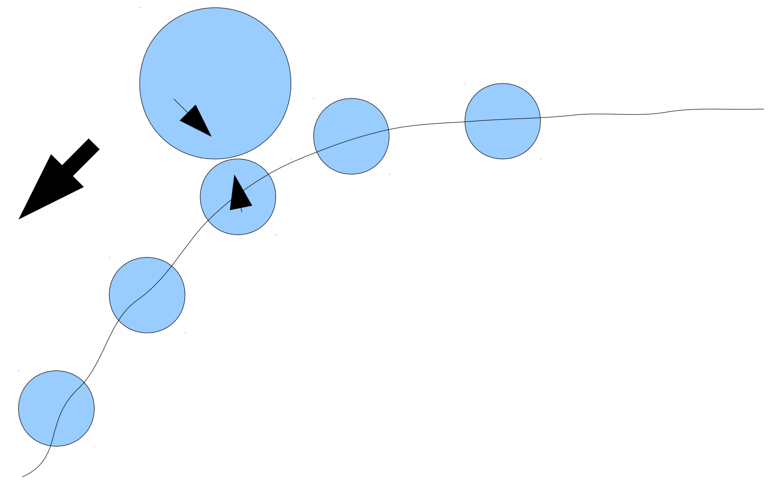
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=-\frac{\partial V}{\partial q_j}+\frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j}$$

Then in this case also the Lagrange's equation of motion in the regular form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0$$

Lagrange's equation of motion with frictional forces

A part of the energy is transferred to the sliding surface. That will appear as frictional forces



Then generalized forces can be split into two parts

$$Q_j = Q_j^{(c)} + Q_j^{(f)}$$

May be derived from a potential of the form

$$V = V(q_j, \dot{q}_j, t)$$

Then generalized forces can be obtained as

$$Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j^f$$

GREH: Chap17, KB:Chap 10, GY Chap 12

Then generalized forces can be expressed in terms of friction coefficients

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j^f \qquad Q_j^f=-\sum_{k=1}^{n} f_{jk}\dot{q}_k$$

The energy transferred to can be expressed as kinetic energy

$$D=\frac{1}{2}\sum_{k=1,l=1}^{n} f_{jk}\dot{q}_k\dot{q}_l$$

Then $\qquad Q_j^f=-\dfrac{\partial D}{\partial \dot{q}_j}$ $\qquad$ dissipation function

Lagrange's equation of motion is then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)+\frac{\partial D}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0$$

Work done by the dissipative forces per unit time

$$\frac{dW}{dt} = \sum_j Q_j^{(f)} \dot{q}_j = -\sum_{j,k} f_{jk} \dot{q}_j \dot{q}_k = -2\mathrm{D}$$

Energy consumed by the frictional forces.

$$\frac{dE}{dt} = \frac{d(T+V)}{dt} = -2D$$

Consider a particle undergoes friction due to air during a projectile motion

$$V = mgz$$

Dissipation is proportional to velocity

$$D = \frac{1}{2}\alpha(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

The Lagrangian is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Using the Lagrangian with dissipation function

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)+\frac{\partial D}{\partial \dot{q}_j}-\frac{\partial L}{\partial q_j}=0$$

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-mgz \qquad D=\frac{1}{2}\alpha(\dot{x}^2+\dot{y}^2+\dot{z}^2)$$

The equation of motion becomes.

$$m\ddot{x}+\alpha\dot{x}=0,$$

$$m\ddot{y}+\alpha\dot{y}=0$$

$$m\ddot{z}+\alpha\dot{z}+mg=0$$

Conservation theorems

Consider a single mass point moving under influence of potential that is a function position $L = L(q_j, \dot{q}_j, t)$

Consider the Lagrangian of free particles $L = \frac{1}{2}\sum_i m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial T}{\partial \dot{x}_i} - \frac{\partial V}{\partial \dot{x}_i}$$

let $i = 1$

$$= \frac{\partial T}{\partial \dot{x}_i} = \frac{\partial}{\partial \dot{x}_1}\sum_i \frac{1}{2} m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) = m_1 \dot{x}_1$$

Therefore we get the momentum associated with a particular degree of freedom

Therefore we can define the generalized momentum associated with the coordinate as

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

This momentum is called **canonical momentum** or **conjugate momentum**

HG:Chap2, TM: Chap 7

If the Lagrangian does not contain a given coordinate q_j the coordinate is said to be cyclic or ignorable

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0$$

We have $0=\frac{\partial L}{\partial q_j}$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)=0$$

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right)=constant$$

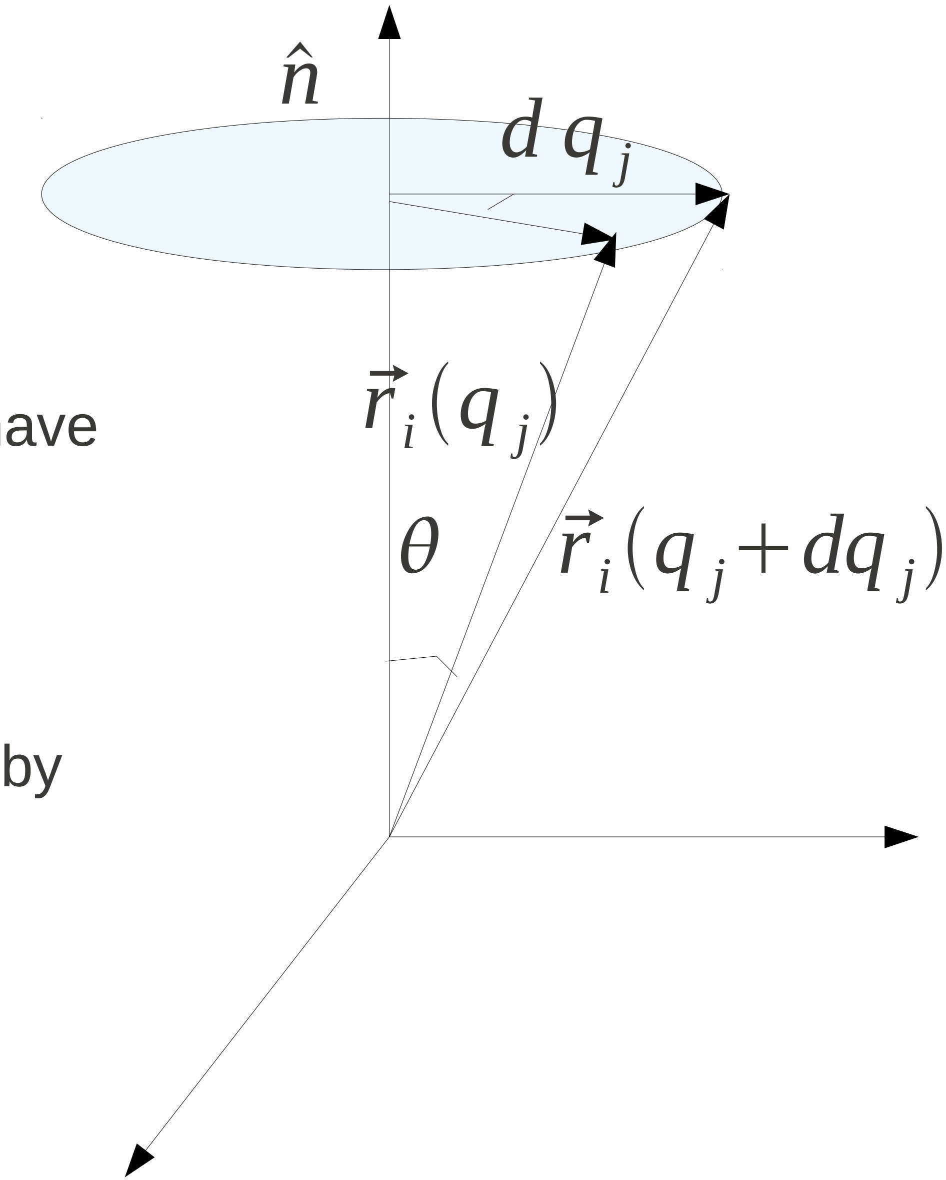
We have $p_j=\frac{\partial L}{\partial \dot{q}_j}$

$$\Rightarrow p_j=constant$$

Generalized momentum conjugate to a cyclic coordinate is conserved

Generalized Forces experienced by a particle that is subjected to torque can be given as

$$Q_j = \sum_i \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_j}$$



Generalized coordinate q_j can have general meaning of angular displacement or angle

Infinitesimal rotation is now given by

$$|d\,\vec{r}_i| = r \sin\theta \, dq_j$$

$$\left|\frac{d\,\vec{r}_i}{dq_j}\right| = r \sin\theta$$

$$\Rightarrow \frac{d\,\vec{r}_i}{dq_j} = \hat{n} \times \vec{r}_i$$

With this the generalized force becomes

$$Q_j = \sum_i \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_j} \qquad \frac{d\vec{r}_i}{dq_j} = \hat{n} \times \vec{r}_i$$

$$Q_j = \sum_i \vec{F}_i \cdot \hat{n} \times \vec{r}_i \quad = \sum_i \hat{n} \cdot \vec{r}_i \times \vec{F}_i$$

$$= \hat{n} \cdot \sum_i \vec{N}_i \quad = \hat{n} \cdot \vec{N}$$

The generalized force are equal to total torque experienced

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

No velocity dependence

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = Q_j$$

Now looking at the dependence of kinetic energy on the velocity

$$\vec{p}_j = \frac{\partial T}{\partial \dot{q}_j} = \sum_i m_i \vec{v}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i m_i \vec{v}_i \cdot (\hat{n} \times \vec{r}_i)$$

$$= \sum_i \hat{n} \cdot \vec{r}_i \times m_i \vec{v}_i = \hat{n} \cdot \sum_i \vec{L}_i$$

$$= \hat{n} \cdot \vec{L}$$

When generalized force become torque, naturally the generalized momentum become angular momentum. Therefore, the angular momentum is conserved same way as linear momentum, i. e., conservation of cyclic coordinates.

Conservation of energy

Consider a Lagrangian of the form $L = L(q_j, \dot{q}_j, t)$

Total time derivative is

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial q_j}\frac{dq_j}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}$$

From Lagrange's equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$

$$\Rightarrow \frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt} + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j}\right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\dot{q}_j \frac{\partial L}{\partial \dot{q}_j} \right) + \frac{\partial L}{\partial t}$$

$$\Rightarrow \frac{d}{dt} \underbrace{\left(\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L \right)}_{\text{Energy function}} + \frac{\partial L}{\partial t} = 0$$

$$h(q_1 \ldots q_n, \dot{q}_1 \ldots \dot{q}_n, t) = \left(\sum_j \overbrace{\dot{q}_j}^{\text{velocity}} \overbrace{\frac{\partial L}{\partial \dot{q}_j}}^{\text{momentum}} - L \right)$$

$$\Rightarrow \frac{dh}{dt} = -\frac{\partial L}{\partial t}$$

If Lagrangian has no explicit dependence on time then the energy function is conserved, for conservative system the energy function is the total energy of system.

We may test this by using Lagrangian of a harmonic oscillator

$$h(q_{1.}..q_n,\dot{q}_1...\dot{q}_n,t)=\left(\sum_j \dot{q}_j\frac{\partial L}{\partial \dot{q}_j}-L\right)$$

$$L=T-V=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^2$$

Energy function

$$h(x,\dot{x})=\left(\dot{x}\frac{\partial L}{\partial \dot{x}_j}-L\right)$$

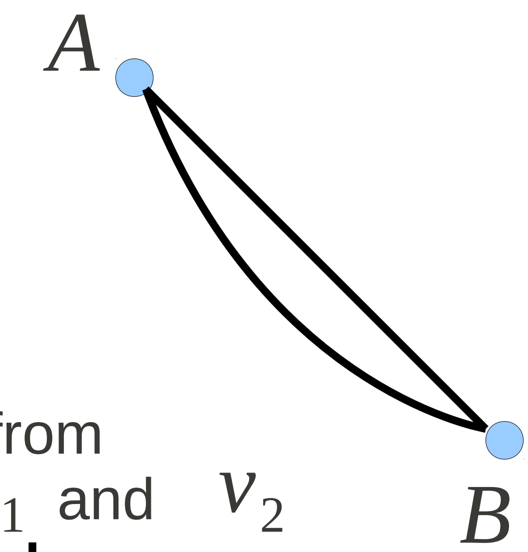
$$=\left(m\dot{x}^2-\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2\right)$$

$$=\left(\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2\right)=T+V=E$$

TM: Chap 7, GY: Chap13, GFS:Chap 1, GFS:Chap9, HG:Chap 2

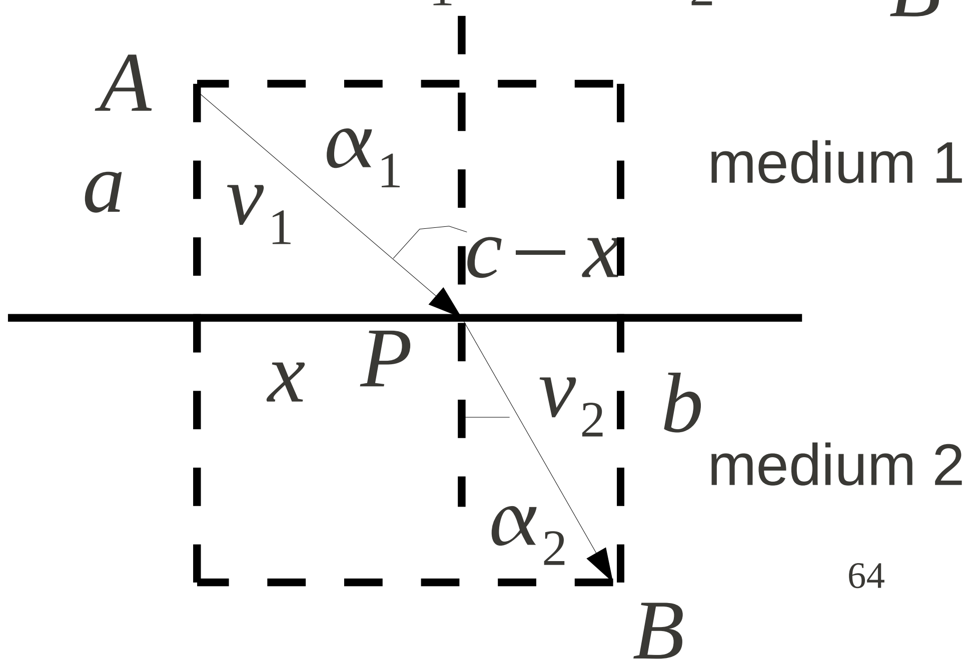
Brachistochrone problem - shortest time problem

Find the curve that gives the shortest time path between points A and B



1696 – Johann Bernoulli generalized the problem by introducing all arbitrary shaped curves between the points.

Consider another problem in optics : A light ray enters from medium 1 to medium 2, let respective velocities are v_1 and v_2



Total time required for the journey is

$$T=\frac{\sqrt{a^2+x^2}}{v_1}+\frac{\sqrt{b^2+(c-x)^2}}{v_2}$$

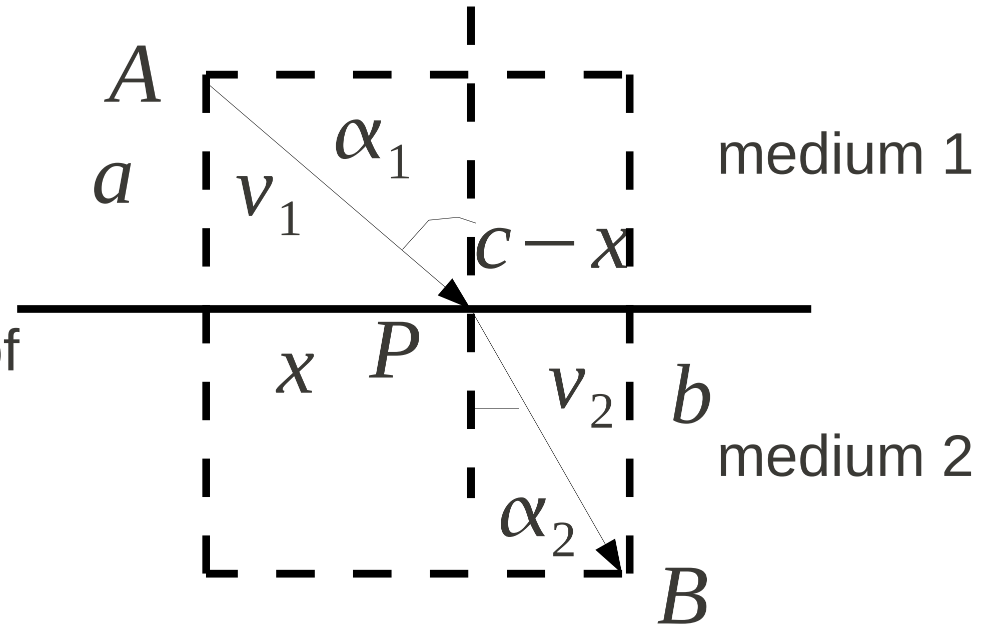
Shortest time is now given by

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{a^2+x^2}} + (-1)\frac{c-x}{v_2\sqrt{b^2+(c-x)^2}} = 0$$

therefore

$$\frac{x}{v_1\sqrt{a^2+x^2}} = \frac{c-x}{v_2\sqrt{b^2+(c-x)^2}}$$

$$\frac{\sin\alpha_1}{v_1} = \frac{\sin\alpha_2}{v_2}$$



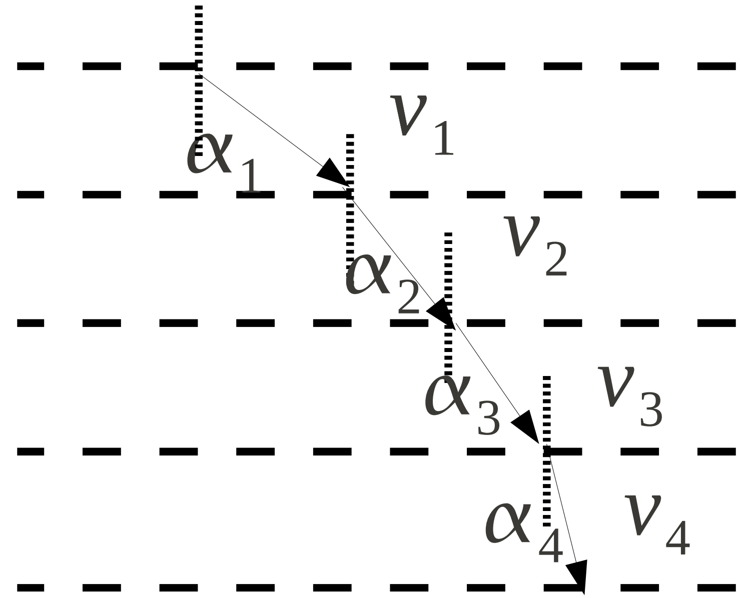
We have arrived at famous Snell's law of refraction

$$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{v_1}{v_2} = \frac{n_2}{n_1}$$

The assumption that light travels from one point to another along the path requiring shortest time is known as Fermat's principle of least time

Now we apply this theorem to continuously varying media

$$\frac{\sin\alpha_1}{v_1}=\frac{\sin\alpha_2}{v_2}=\frac{\sin\alpha_3}{v_3}=\frac{\sin\alpha_4}{v_4}$$



In general

$$\frac{\sin\alpha}{v}=constant$$

Now in the problem of least time of travel for the bead: let the bead knows this path having least time

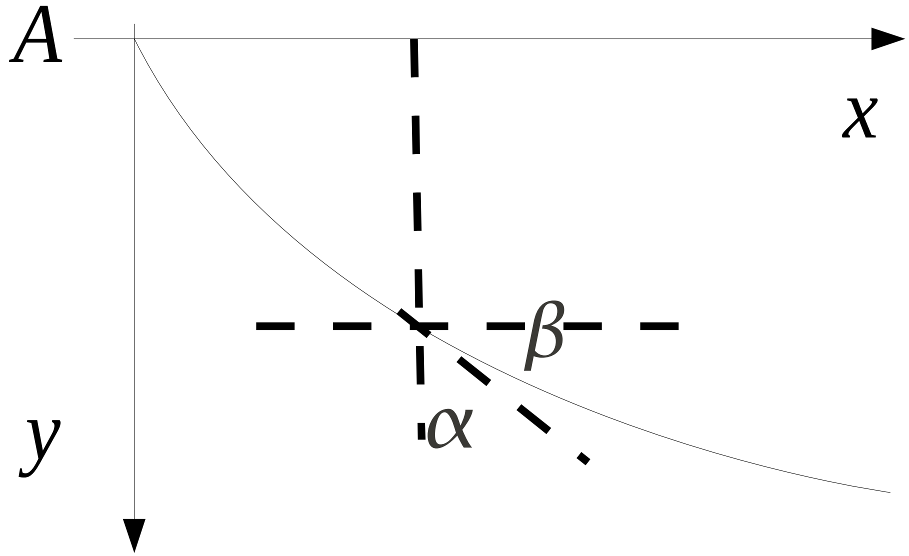
The velocity of the bead solely determined by the potential energy at any particular point not by the path.

$$\frac{1}{2}mv^2=mgy$$ ¿

$$v=\sqrt{2gy}$$

$$\sin\alpha=\cos\beta=\frac{1}{\sec\beta}=\frac{1}{\sqrt{1+\tan^2\beta}}$$

$$=\frac{1}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}$$



From the principle of optics

$$\frac{\sin\alpha}{v}=c_1$$

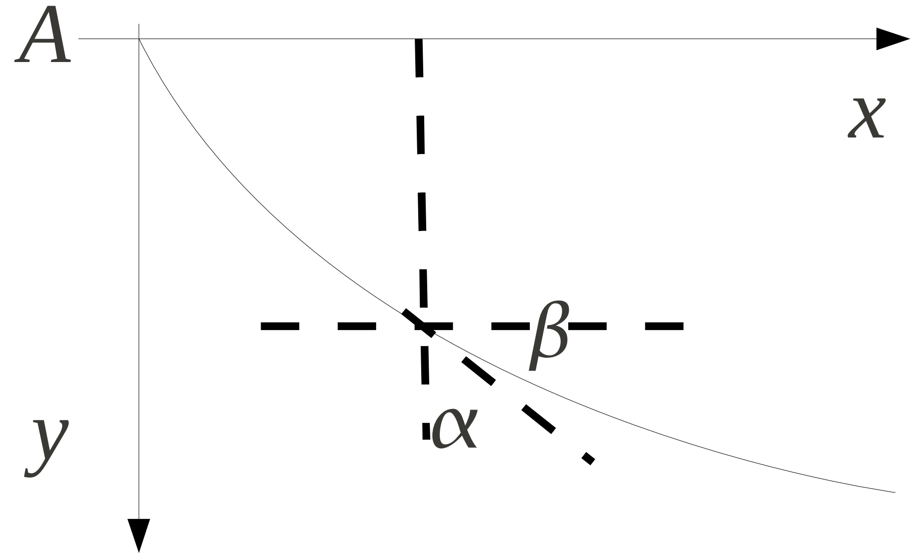
The velocity of the bead

$$v=\sqrt{2gy}$$

Substituting in $\frac{\sin\alpha}{v}=c_1$

$$\Rightarrow\frac{1}{\sqrt{2gy}\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=c_1$$

We have



$$\Rightarrow \frac{1}{\sqrt{2gy}\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = c_1$$

Squaring and rearranging and keeping new constant as

$$\Rightarrow y\left(1+\left(\frac{dy}{dx}\right)^2\right) = c$$

$$\Rightarrow \frac{dy}{dx} = \left(\frac{c-y}{y}\right)^{1/2}$$

$$\Rightarrow dx = \left(\frac{y}{c-y}\right)^{1/2} dy$$

Now use change of variable to ϕ

$$dx = \left(\frac{y}{c-y}\right)^{1/2} dy$$

$$\tan\phi = \left(\frac{y}{c-y}\right)^{1/2}$$

$$(c-y)\tan^2\phi = y$$

$$y = \left(c\frac{\tan^2\phi}{1+\tan^2\phi}\right)$$

$$y = c\sin^2\phi$$

Change of variable gives $$dy = 2c\sin\phi\cos\phi\, d\phi$$

$$dx = \tan\phi\, dy$$

By substituting

$$= 2\mathrm{c}\sin^2\phi\, d\phi$$

$$= c(1-\cos 2\phi)\, d\phi$$

$$dx = c(1 - \cos 2\phi)\, d\phi$$

By integration $$x = \frac{c}{2}(2\phi - \sin 2\phi) + c_2$$

the coordinate system shows that the curve pass through the origin

When $\phi = 0$ $x = y = 0$

$$\Rightarrow c_2 = 0$$

$$\Rightarrow x = \frac{c}{2}(2\phi - \sin 2\phi)$$

$$y = c \sin^2 \phi$$

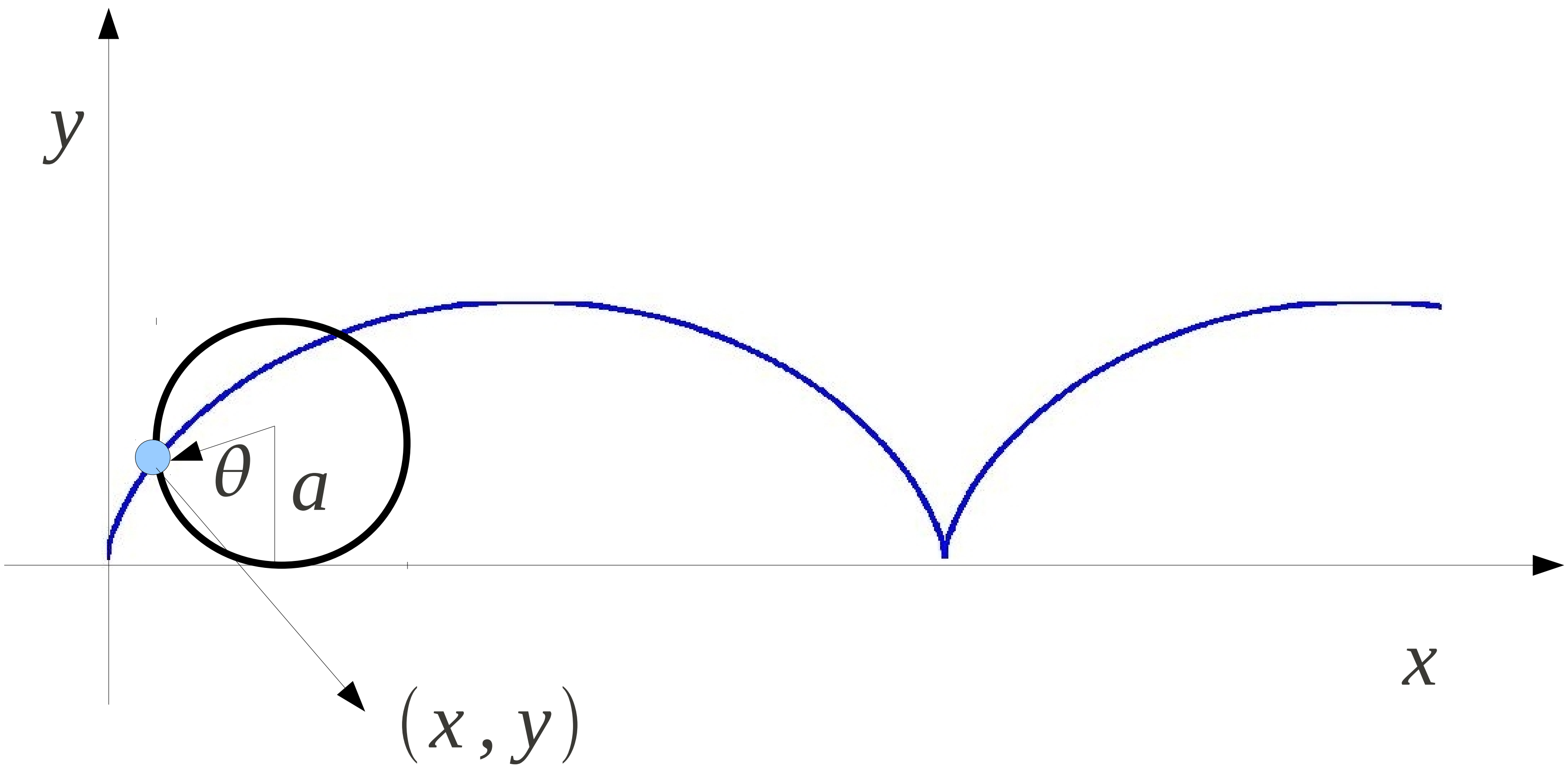
$$y = c \sin^2 \phi = \frac{c}{2}(1 - \cos 2\phi)$$

$$\Rightarrow x=\frac{c}{2}(2\phi-\sin 2\phi) \qquad y=\frac{c}{2}(1-\cos 2\phi)$$

Now let $a=c/2 \qquad \theta=2\phi$

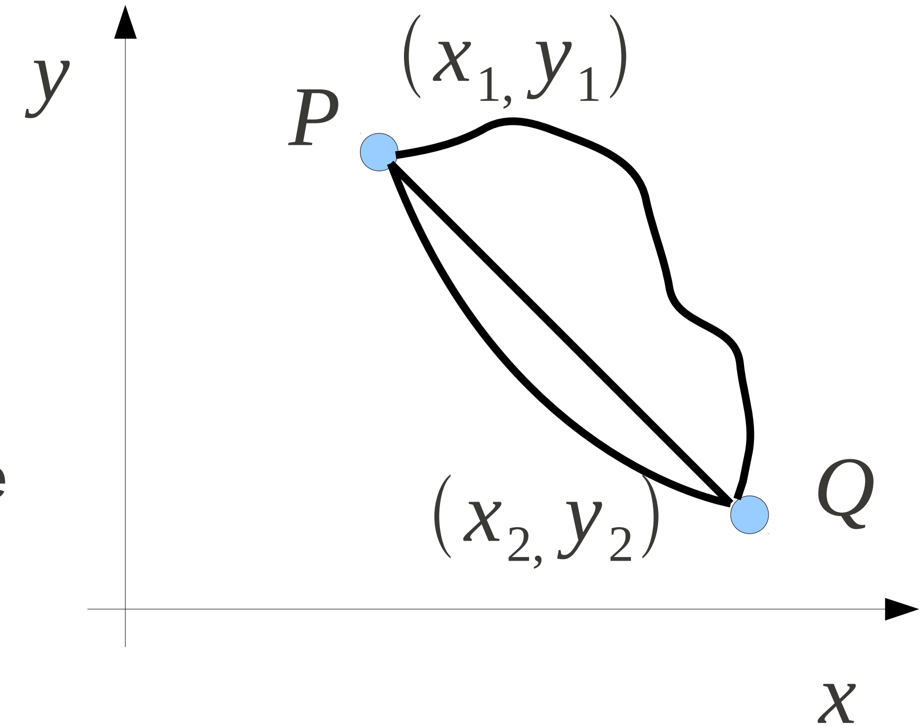
The standard equation of the cycloid

$$\Rightarrow x=a(\theta-\sin\theta), \qquad y=a(1-\cos\theta)$$



This Bernoulli's solution to problem. This is the source development of problem of **calculus of variations**

calculus of variations



Typical questions that are addressed are

(1) What is the shortest path between the points?

(2) Which curve will generate smallest area when rotated about x axis?

(3) what is the path of descent with shortest time? (brachistochrone problem)

Consider a family of functions $y = y(x)$ that satisfies boundary condition

$$y_1 = y(x_1) \qquad\qquad y_2 = y(x_2)$$

Next is to find the function that minimizes an integral of the form

$$I(y) = \int_{x_1}^{x_2} f(x, y, y')\, dx$$

$$I(y)=\int_{x_1}^{x_2} f(x,y,y')\,dx$$

In the length of the curve problem the integral is

$$I(y)=\int_{x_1}^{x_2} ds=\int_{x_1}^{x_2}\sqrt{dx^2+dy^2}=\int_{x_1}^{x_2}\sqrt{1+y'^2}\,dx$$

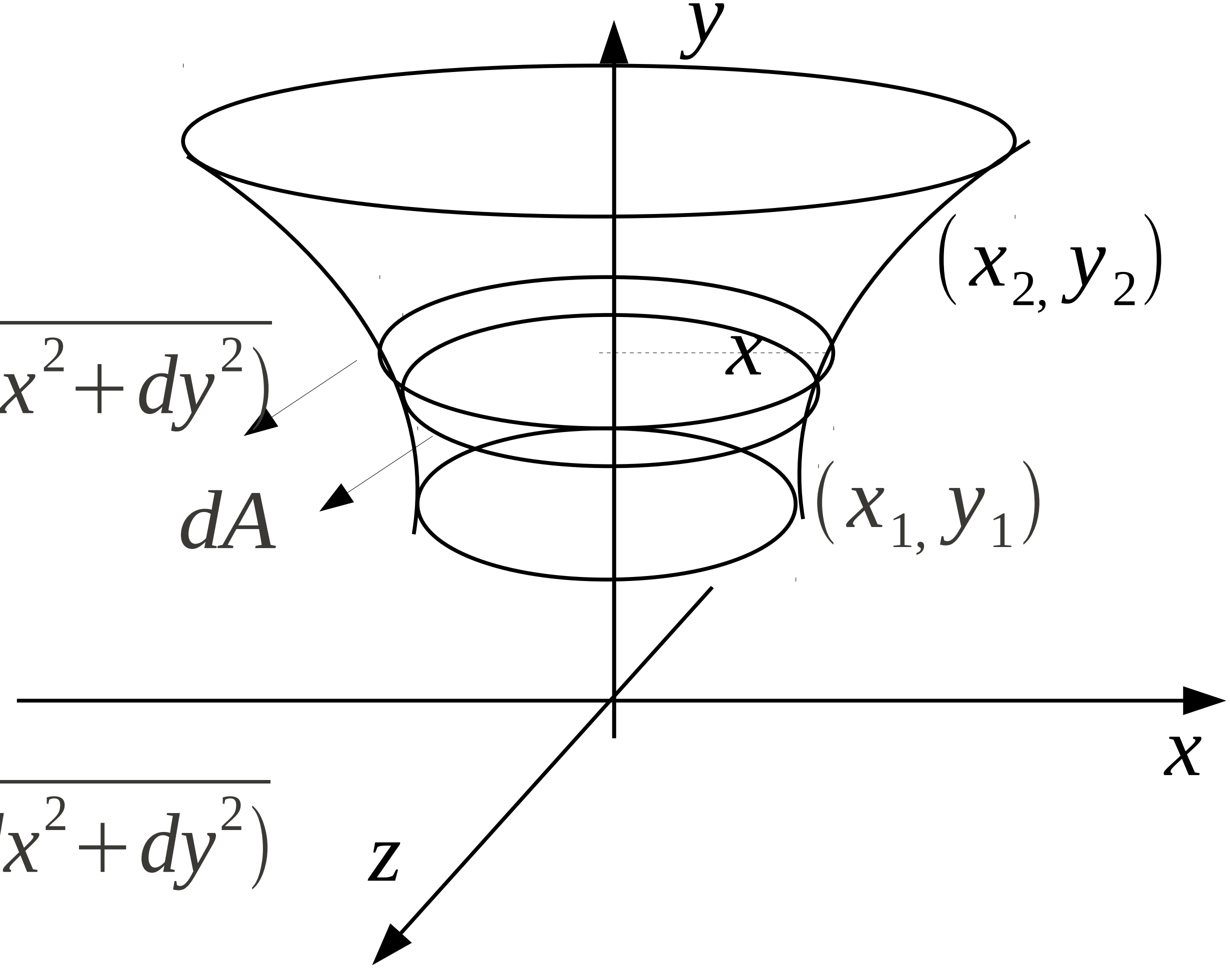
$$y'=\frac{dy}{dx}$$

This general problem can answer other problems also

Area of surface revolution – in this case the function that is to be minimized is

$$I(y)=\int_{x_1}^{x_2} 2\pi x\sqrt{1+y'^2}\,dx$$

$$ds = \sqrt{(dx^2 + dy^2)}$$

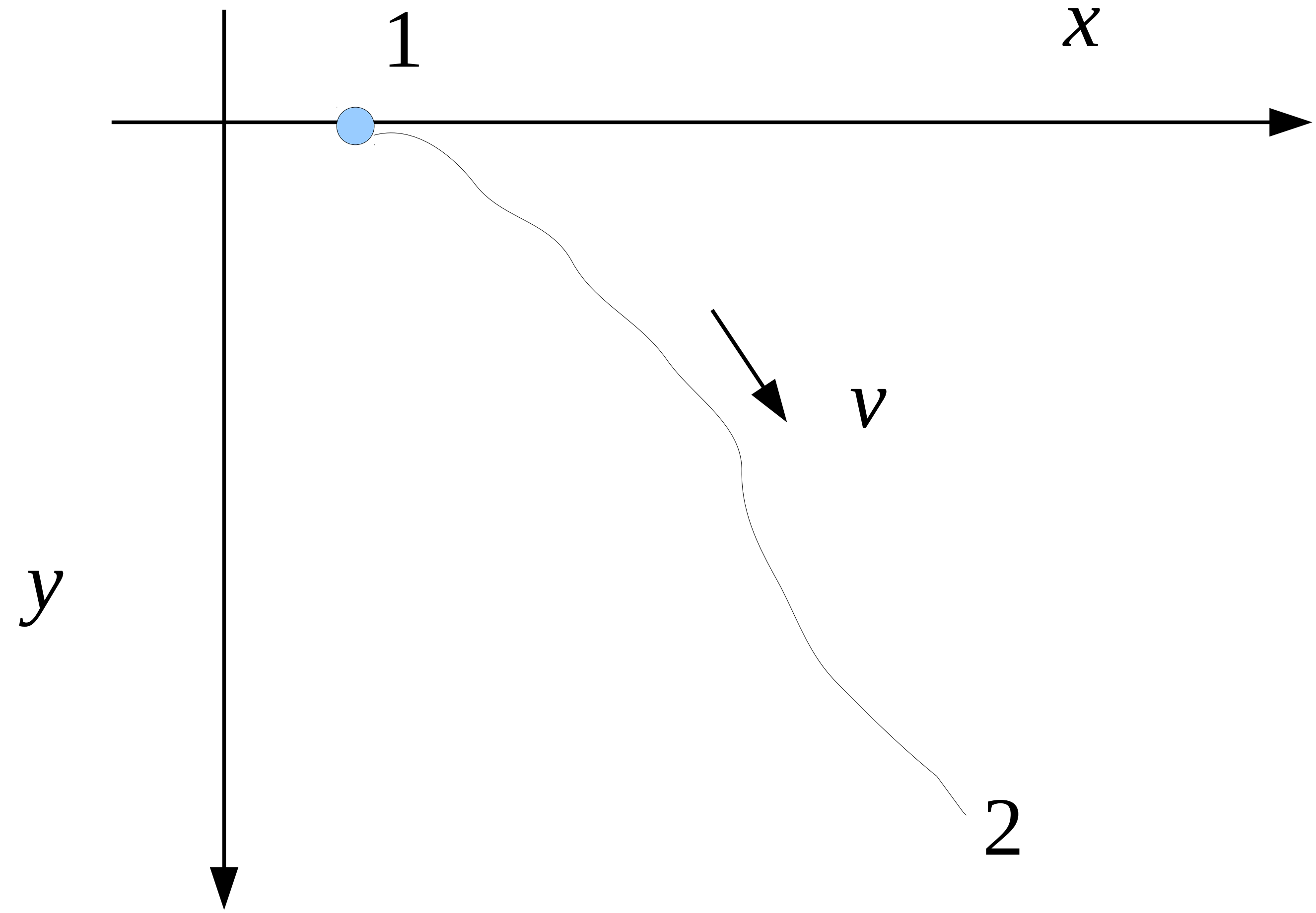


$$dA = 2\pi x\, ds = 2\pi x \sqrt{(dx^2 + dy^2)}$$

$$= 2\pi x \sqrt{(1 + (y')^2)}\, dx$$

$$A = 2\pi \int_{x_1}^{x_2} x \sqrt{(1 + (y')^2)}\, dx$$

Brachistochrone problem



For quickest descent, keeping the point P as origin

$$v = \sqrt{2 g y} \qquad v = \frac{ds}{dt}$$

Total time of descent is integral of $dt = \dfrac{ds}{v}$

$$dt = \frac{ds}{v} \qquad v = \sqrt{2gy}$$

For quickest descent, the integral becomes

$$I(y) = \int_1^2 \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} dx$$

In general we can find an admissible function that minimizes certain property

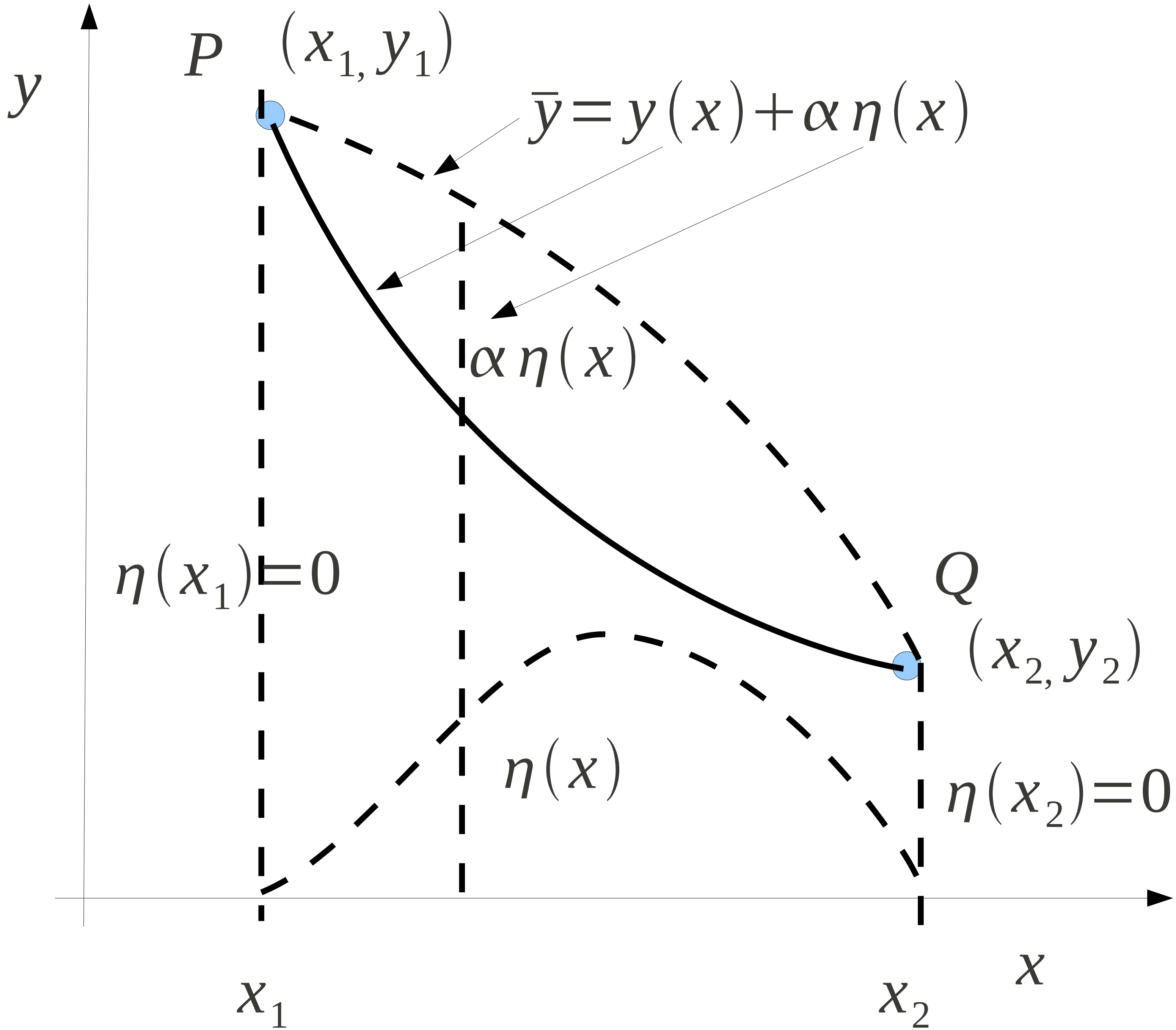
$$I(y) = \int_{x_1}^{x_2} f(x, y, y') dx \qquad y' = \frac{dy}{dx}$$

Since $y = y(x)$ is the minimum function and deviation will increase the value of the integral. Let the deviation be

$$\bar{y} = y(x) + \alpha \eta(x)$$

Family of admissible functions

Small parameter


y
P
(x_1, y_1)
$\bar{y}=y(x)+\alpha\eta(x)$
$\alpha\eta(x)$
$\eta(x_1)=0$
Q
(x_2, y_2)
$\eta(x)$
$\eta(x_2)=0$
x
x_1
x_2

The new value of the integral be $\quad \bar{y} = y(x) + \alpha\,\eta(x)$

$$I(\alpha) = \int_{x_1}^{x_2} f(x, \bar{y}, \bar{y}')\,dx$$

$$= \int_{x_1}^{x_2} f(x, y + \alpha\,\eta(x), y' + \alpha\,\eta'(x))\,dx$$

When $\alpha = 0$ this integral reduces to $\quad I(y) = \int_{x_1}^{x_2} f(x, y, y')\,dx$

This function which then minimizes the function can be obtained by minimizing the differential of integral when $\alpha = 0$. Therefore

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}')\,dx$$

$$0$$

$$\frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}') = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \alpha} + \frac{\partial f}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial f}{\partial \bar{y}'}\frac{\partial \bar{y}'}{\partial \alpha}$$

$$\frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}') = \frac{\partial f}{\partial \bar{y}} \frac{\partial \bar{y}}{\partial \alpha} + \frac{\partial f}{\partial \bar{y}'} \frac{\partial \bar{y}'}{\partial \alpha}$$

using $\bar{y} = y(x) + \alpha \eta(x)$

$$\frac{\partial \bar{y}}{\partial \alpha} = \eta(x)$$

$$\Rightarrow \bar{y}' = y'(x) + \alpha \eta'(x)$$

$$\frac{\partial \bar{y}'}{\partial \alpha} = \eta'(x)$$

By substituting the results

$$\frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}') = \frac{\partial f}{\partial \bar{y}} \eta(x) + \frac{\partial f}{\partial \bar{y}'} \eta'(x)$$

therefore $I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}') dx$

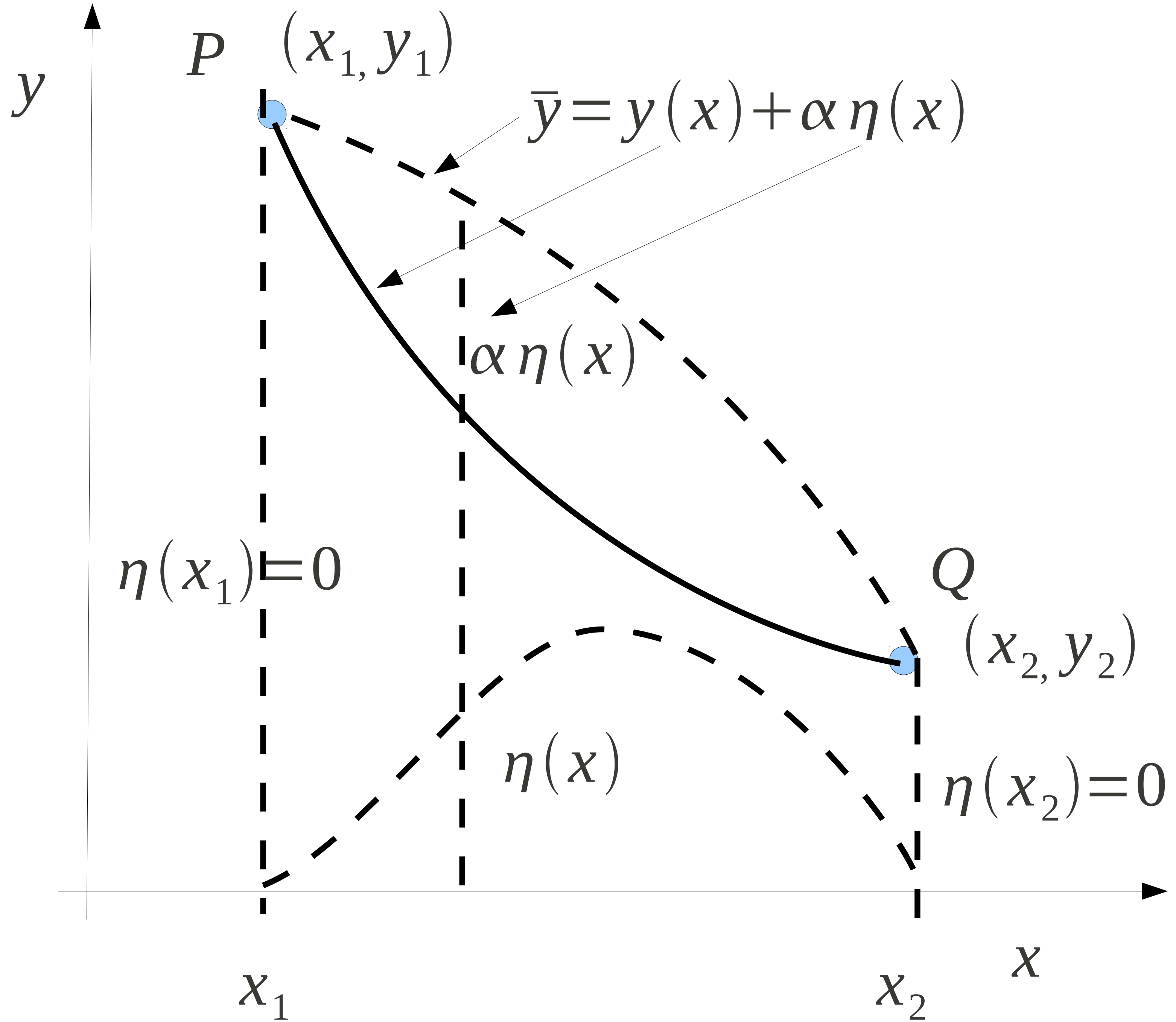
therefore

$$I'(\alpha) = \int_{x_1}^{x_2} \frac{\partial}{\partial \alpha} f(x, \bar{y}, \bar{y}') dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial \bar{y}} \eta(x) + \frac{\partial f}{\partial \bar{y}'} \eta'(x) \right] dx$$

$I'(\alpha = 0) = 0$ Using condition of minimization

$$\Rightarrow \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx = 0$$


y
P
(x_1, y_1)
$\bar{y} = y(x) + \alpha\eta(x)$
$\alpha\eta(x)$
$\eta(x_1) = 0$
Q
(x_2, y_2)
$\eta(x)$
$\eta(x_2) = 0$
x
x_1
x_2

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y} \eta(x) + \frac{\partial f}{\partial y'} \eta'(x) \right] dx = 0$$

The derivative of $\eta'(x)$ can be eliminated by integration by parts

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \eta'(x)\, dx = \overbrace{\left[\eta(x) \frac{\partial f}{\partial y'} \right]_{x_1}^{x_2}}^{0} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \frac{\partial f}{\partial y'}$$

since $\eta(x_1) = 0 \qquad \eta(x_2) = 0$

$$\int_{x_1}^{x_2} \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right] dx = 0$$

The integral mush vanish for any arbitrary $\eta(x)$ function, therefore

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

(1) The shortest distance between two points

Consider the integral

$$I(y)=\int_{x_1}^{x_2} ds=\int_{x_1}^{x_2} \sqrt{dx^2+dy^2}=\int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx$$

Now the function that is to be minimized is $f(y)=\sqrt{1+y'^2}$

We have Euler's equation $\dfrac{\partial f}{\partial y}-\dfrac{d}{dx}\dfrac{\partial f}{\partial y'}=0$

$$\frac{\partial f}{\partial y}=0 \qquad \frac{\partial f}{\partial y'}=\frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'}=\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)=0$$

$$\frac{d}{dx}\left(\frac{y'}{\sqrt{1+y'^2}}\right)=0$$

$$\Rightarrow\left(\frac{y'}{\sqrt{1+y'^2}}\right)=c$$

by rearranging

$$\Rightarrow y'=a$$ also $$a=\left(\frac{c}{\sqrt{1-c^2}}\right)$$

$$\Rightarrow\frac{dy}{dx}=a$$

The solution of the differential equation now becomes

$$y=ax+b$$

This is the equation of a straight line

(3) The brachistochrone problem – shortest time of descent

http://mathworld.wolfram.com/BrachistochroneProblem.html

$$I(y) = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

$$f = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}$$

Since this function does not contain explicit dependence on time we may rearrange the Euler equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Multiplying with y'

$$\frac{\partial f}{\partial y} y' - \frac{d}{dx}\frac{\partial f}{\partial y'} y' = 0$$

http://mathworld.wolfram.com/BeltramiIdentity.html

The derivative of the function with respect to x is given by

$$\frac{df}{dx}=\frac{\partial f}{\partial y}y'+\frac{\partial f}{\partial y'}y''+\frac{\partial f}{\partial x}$$

rearranging

$$\frac{\partial f}{\partial y}y'=-\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y'}y''+\frac{df}{dx}$$

Substituting this result in Euler's equation for the first term

$$\frac{\partial f}{\partial y}y'-\frac{d}{dx}\frac{\partial f}{\partial y'}y'=0$$

$$-\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y'}y''+\frac{df}{dx}-\frac{d}{dx}\frac{\partial f}{\partial y'}y'=0$$

$$-\frac{\partial f}{\partial x}+\frac{df}{dx}-\frac{\partial f}{\partial y'}y''-\frac{d}{dx}\frac{\partial f}{\partial y'}y'=0$$

$$-\frac{\partial f}{\partial x}+\frac{d f}{d x}-\frac{\partial f}{\partial y'} y''-\frac{d}{dx}\frac{\partial f}{\partial y'} y'=0$$

$$-\frac{\partial f}{\partial x}+\frac{d f}{d x}+\frac{d}{dx}\left(-y'\frac{\partial f}{\partial y'}\right)=0$$

rearranging

$$-\frac{\partial f}{\partial x}+\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=0$$

When function f has no explicit dependence on x , $\frac{\partial f}{\partial x}=0$

$$\frac{d}{dx}\left(f-y'\frac{\partial f}{\partial y'}\right)=0$$

$$\Rightarrow\left(f-y'\frac{\partial f}{\partial y'}\right)=C$$ (constant)

$$\left(f - y' \frac{\partial f}{\partial y'}\right) = C$$

$$f = (1 + y'^2)^{(1/2)} (2 g y)^{-1/2}$$

$$\frac{\partial f}{\partial y'} = y' (1 + y'^2)^{(-1/2)} (2 g y)^{-1/2}$$

$$y' \frac{\partial f}{\partial y'} = y'^2 (1 + y'^2)^{(-1/2)} (2 g y)^{-1/2}$$

$$\left(f - y' \frac{\partial f}{\partial y'}\right) = C$$

$$= (1 + y'^2)^{(1/2)} (2 g y)^{-1/2} - y'^2 (1 + y'^2)^{(-1/2)} (2 g y)^{-1/2}$$

$$= (1 + y'^2)^{(-1/2)} (2 g y)^{-1/2} (1 + y'^2 - y'^2)$$

$$= (1 + y'^2)^{(-1/2)} (2 g y)^{-1/2} = C$$

$$\left(1+y'^2\right)^{(-1/2)}(2gy)^{-1/2}=C$$

Comparing this result with the differential equation we obtained using Snell's law – we finds that we have arrived at same differential equation

$$\Rightarrow \frac{1}{\sqrt{2gy}\sqrt{1+\left(\frac{dy}{dx}\right)^2}}=c_1$$

Hamilton's principle

From the calculus of variation we know that if we are minimizing integral of a function, we can arrive at a differential equation, which is very similar to Lagrangian equation of motion. In order to present Lagrange's equation of motion in mechanics in the same language as Euler's equation we need to find out the quantity that is to be minimized.

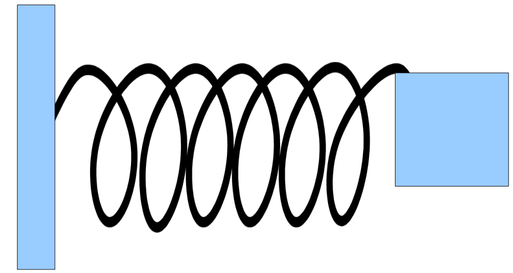
Hamilton's principle:

The motion of the system is from time t_1 to t_2 is such that the line integral (called action or the action integral),

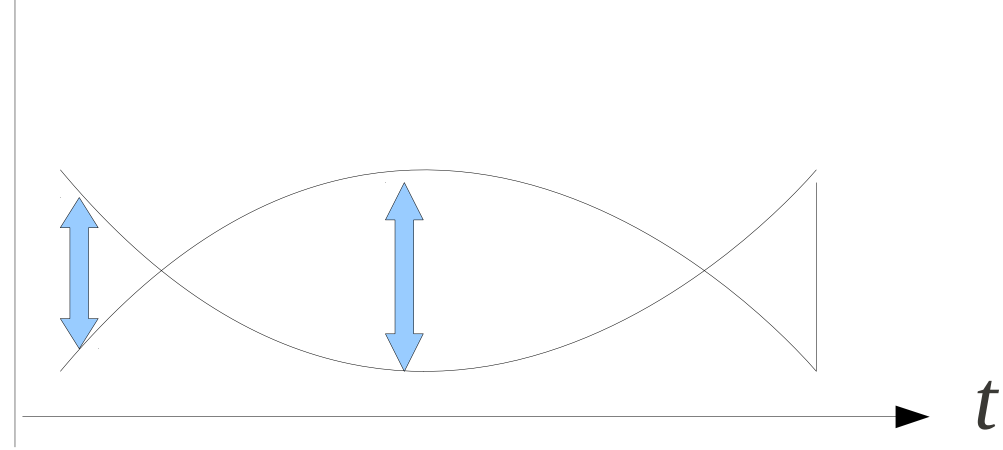
$$I = \int_{t_1}^{t_2} L\, dt$$

where $L = T - V$ has stationary value for the actual path of motion

$$\Rightarrow \delta I = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\, dt = 0$$



$$L = T - V$$



The total energy is conserved – therefore the change in the kinetic energy must be equal to potential energy

$$E = T + V \qquad \Rightarrow dT = -dV$$

$$K = \frac{1}{2} m \omega^2 a^2 \cos^2(\omega t)$$

$$dT = -dV$$

$$V = \frac{1}{2} k a^2 \sin^2(\omega t)$$

Transfer energy between forms of energy is governed by the shape of the potential energy function, if it contains only positions the action integral can be considered constant

From Euler equation for any function whose integral is to be minimized

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0$$

For the case of the dynamics of particles

$$\delta I=\delta\int_{t_1}^{t_2} L(q_i,\dot{q}_i,t)\,dt=0$$

By the following selection of the variables

$$x\rightarrow t \qquad y_i\rightarrow q_i \qquad f(y_i,\dot{y}_i,x)\rightarrow L(q_i,\dot{q}_i,t)$$

This can be proved for n variable system by this transformation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)-\frac{\partial L}{\partial q_i}=0$$

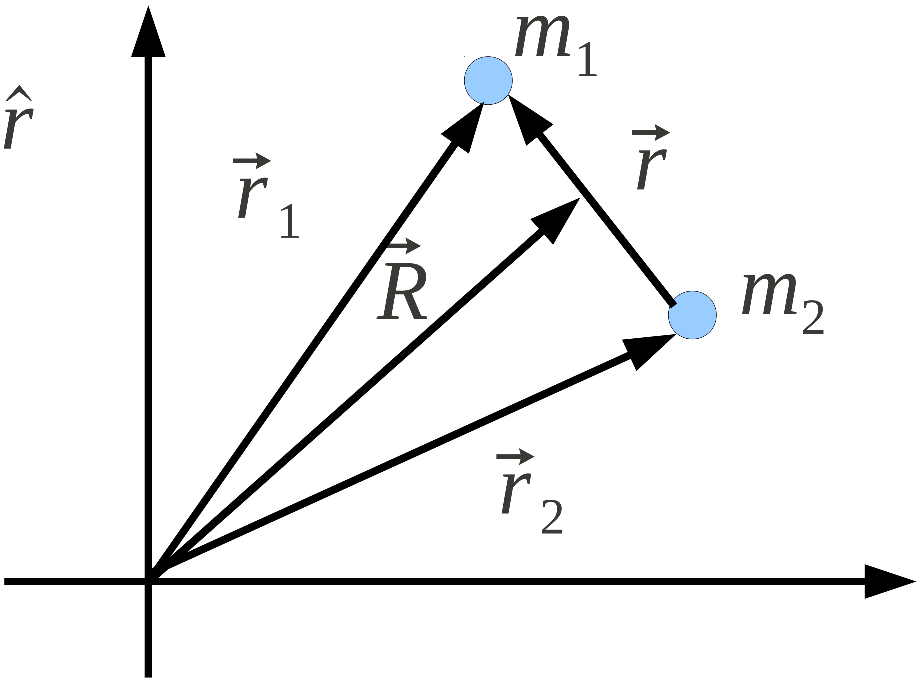
Central force problem – with two body

The **central force problem** means direction of the force is radially and whose magnitude depends only on the distance between the particles. That is there is no angular dependence for the force

$$\vec{F}(\vec{r}) = -\vec{\nabla} V(r)$$

For two body problem force is along the radial vector that connects two particles

$$\vec{F}(\vec{r}) = -\frac{dV(r)}{dr}\hat{r}$$



Examples of central forces
Gravitational and electrostatic forces

$$V(r) \propto \frac{1}{r}$$

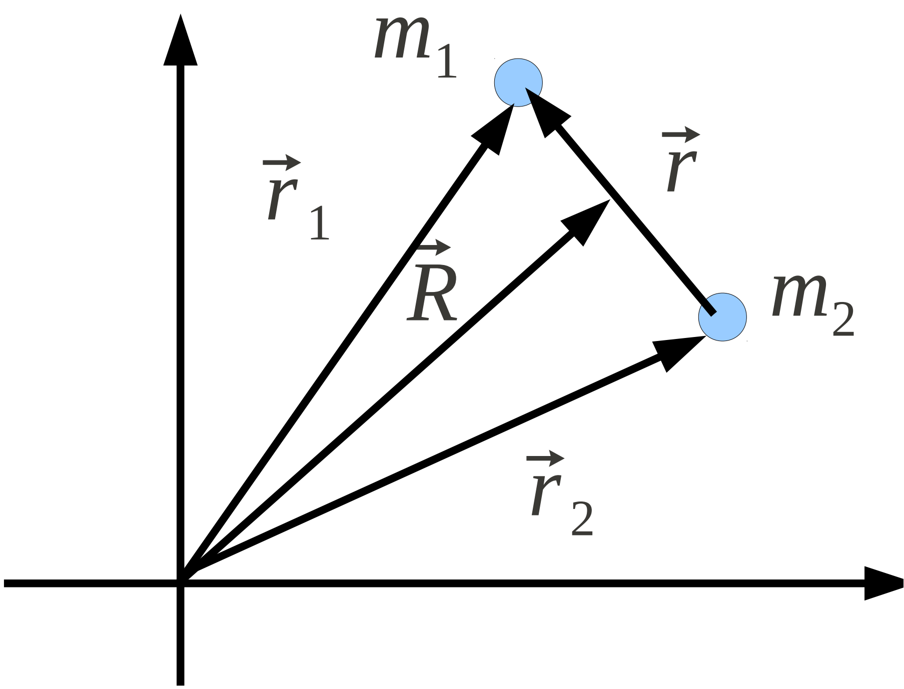
Spring forces

$$V(r) \propto (r-l)^2$$

Equilibrium length

Coordinate system and Newton's equation of motion

Let the mass of the particles be m_1 and m_2 and the position vectors are $\vec{r}_1$ and $\vec{r}_2$



If the particles influence each other by gravitational field by Newton's equation of motion

$$m_1 \ddot{\vec{r}}_1 = m_1 \vec{g} + \vec{F}$$

$$m_2 \ddot{\vec{r}}_2 = m_2 \vec{g} - \vec{F}$$

Gravitational field or acceleration due gravity that due external interactions

Change variable to a new set of variables, center of mass $\vec{R}$ and relative position $\vec{r}$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad \vec{r} = \vec{r}_1 - \vec{r}_2$$

The Newton's equation we can express in terms of center of mass by adding equations of motion for particle 1 and 2

$$m_1 \ddot{\vec{r}}_1 = m_1 g + \vec{F}$$

$$m_2 \ddot{\vec{r}}_2 = m_2 g - \vec{F}$$

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

The sum of two above equations gives

$$m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2 = m_1 g + m_2 g$$

where

$$\ddot{\vec{R}} = \frac{m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2}{m_1 + m_2}$$

$$(m_1 + m_2) \frac{m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2}{m_1 + m_2} = (m_1 + m_2) g$$

and

$$M = m_1 + m_2$$

$$M \ddot{\vec{R}} = M g$$

This is the equation of motion of the center of mass

If there is no external force acting on the system

$$M \ddot{\vec{R}} = M g = 0 \qquad M \dot{\vec{R}} = constant$$

The linear momentum of the system is conserved

$$m_1 \ddot{\vec{r}}_1 = m_1 g + \vec{F}$$

$$m_2 \ddot{\vec{r}}_2 = m_2 g - \vec{F}$$

Equation of motion in terms of the relative distance of the particles is given by eliminating the term that contains g by multiplying with the masses and subtracting

$$m_1 m_2 \ddot{\vec{r}}_1 = m_1 m_2 g + m_2 \vec{F}$$

$$m_1 m_2 \ddot{\vec{r}}_2 = m_1 m_2 g - m_1 \vec{F}$$

Subtracting from each other

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$m_1 m_2 (\ddot{\vec{r}}_1 - \ddot{\vec{r}}_2) = (m_1 + m_2) \vec{F}$$

$$\mu \ddot{\vec{r}} = \vec{F}$$

Where μ is the reduced mass of the system

We have separated the problem in to two equations of motion

Coordinate system transformation

From the general equation of the definition of the center of mass, the reverse transformation can be obtained from

$$\vec{R}=\frac{m_1\vec{r}_1+m_2\vec{r}_2}{m_1+m_2} \quad =\frac{m_1\vec{r}_1+m_2(\vec{r}_1-\vec{r})}{m_1+m_2}$$

$$\vec{r}=\vec{r}_1-\vec{r}_2$$

$$=\frac{(m_1+m_2)\vec{r}_1-m_2\vec{r}}{m_1+m_2}$$

$$\vec{r}_1-\vec{r}=\vec{r}_2$$

$$M=m_1+m_2$$

$$=\vec{r}_1-\frac{m_2\vec{r}}{M}$$

$$\Rightarrow\vec{R}+\frac{m_2\vec{r}}{M}=\vec{r}_1$$

Similarly(deduce yourself)

$$\vec{R}-\frac{m_1\vec{r}}{M}=\vec{r}_2$$

Kinetic energy in the new coordinate system

These transformations can be used to separate the Kinetic energy as two parts: one that of center of mass and second is with respect to the relative coordinates

$$T = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2}$$

$$\vec{r}_1 = \vec{R} + \frac{m_2 \vec{r}}{M}$$

$$\vec{r}_2 = \vec{R} - \frac{m_1 \vec{r}}{M}$$

$$T = \frac{1}{2} m_1 \left(\dot{\vec{R}} + \frac{m_2 \dot{\vec{r}}}{M} \right)^2 + \frac{1}{2} m_2 \left(\dot{\vec{R}} - \frac{m_1 \dot{\vec{r}}}{M} \right)^2$$

$$T = \left(\frac{1}{2} m_1 \dot{\vec{R}}^2 + \cancel{\frac{m_1 m_2 \dot{\vec{r}}}{M}} \dot{\vec{R}} + \frac{1}{2} m_1 \frac{m_2^2 \dot{\vec{r}}^{\,2}}{M^2} \right) + \left(\frac{1}{2} m_2 \dot{\vec{R}}^2 - \cancel{\frac{m_1 m_2 \dot{\vec{r}}}{M}} \dot{\vec{R}} + \frac{1}{2} m_2 \frac{m_1^2 \dot{\vec{r}}^{\,2}}{M^2} \right)$$

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} m_1 \frac{m_2^2 \dot{\vec{r}}^{\,2}}{M^2} + \frac{1}{2} m_2 \frac{m_1^2 \dot{\vec{r}}^{\,2}}{M^2}$$

$$\vec{r}_1 = \vec{R} + \frac{m_2 \vec{r}}{M} \qquad \vec{r}_2 = \vec{R} - \frac{m_1 \vec{r}}{M}$$

We re-arrange the expression for kinetic energy to obtain a convenient form

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} m_1 \frac{m_2^2 \dot{\vec{r}}^2}{M^2} + \frac{1}{2} m_2 \frac{m_1^2 \dot{\vec{r}}^2}{M^2}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$= T^{CM} + \frac{1}{2} \frac{m_1 m_2}{M^2} \dot{\vec{r}}^2 (m_1 + m_2)$$

$$\mu = \frac{m_1 m_2}{M}$$

$$= T^{CM} + \frac{1}{2} \frac{m_1 m_2}{M} \dot{\vec{r}}^2$$

$$= T^{CM} + \frac{1}{2} \mu \dot{\vec{r}}^2$$

Since the problem is not best solved in terms of the Cartesian coordinates, it is easier to approach the problem through Lagrangian methods

Two-body problem: by Lagrange's equation of motion

The Lagrangian of the problem can be written as

$$L = T - V$$

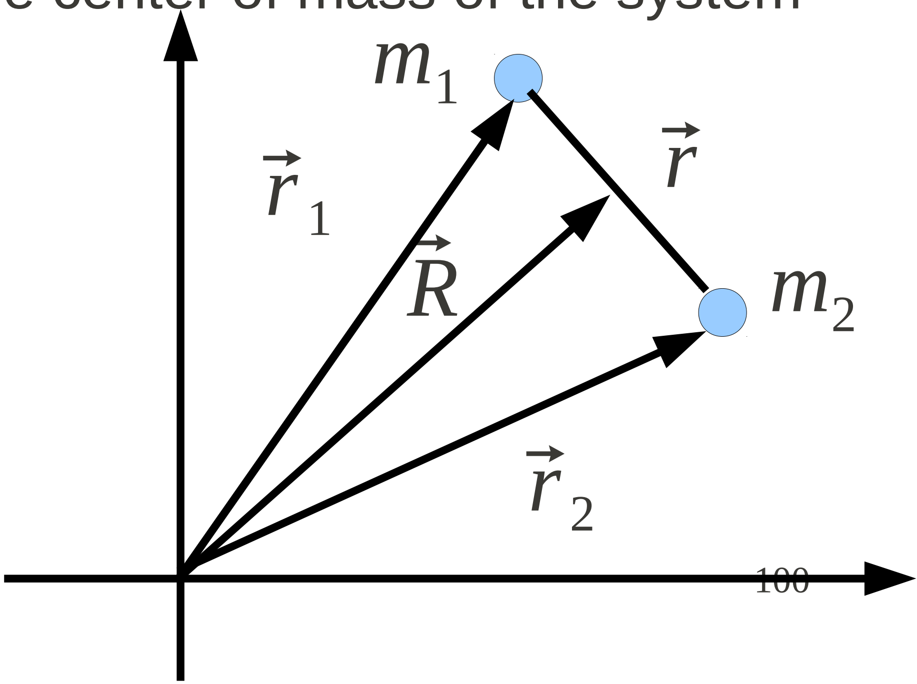
$$L = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)$$

The coordinate $\vec{R}$ is cyclic as the Lagrangian does not depend on this coordinate

We already know from the Newtonian mechanics that in such a system when external forces are absent the total momentum of the system is conserved, a finite value of $\dot{\vec{R}}$ means the center of mass of the system is moving with constant velocity.

Dropping cyclic coordinate from the Lagrangian

$$L = \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)$$



The interaction between the particles only depend in the distance between the particles, the problem have circular symmetry.

We have shown that angular momentum of a system of of particles with no external torque is conserved.

$$\vec{L} = \vec{r} \times \vec{p} = constant$$

Since Lagrangian obtained is that of single particle having reduced mass we can formulate the problem as that of a single particle with one coordinate.

In polar coordinate Lagrangian

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

The angle variable is a cyclic coordinate the momentum corresponding to this coordinate is conserved

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

$$\frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 = \frac{d}{dt}(\mu r^2 \dot{\theta})$$

$$\Rightarrow \mu r^2 \dot{\theta} = l$$

This gives the magnitude of the angular momentum.

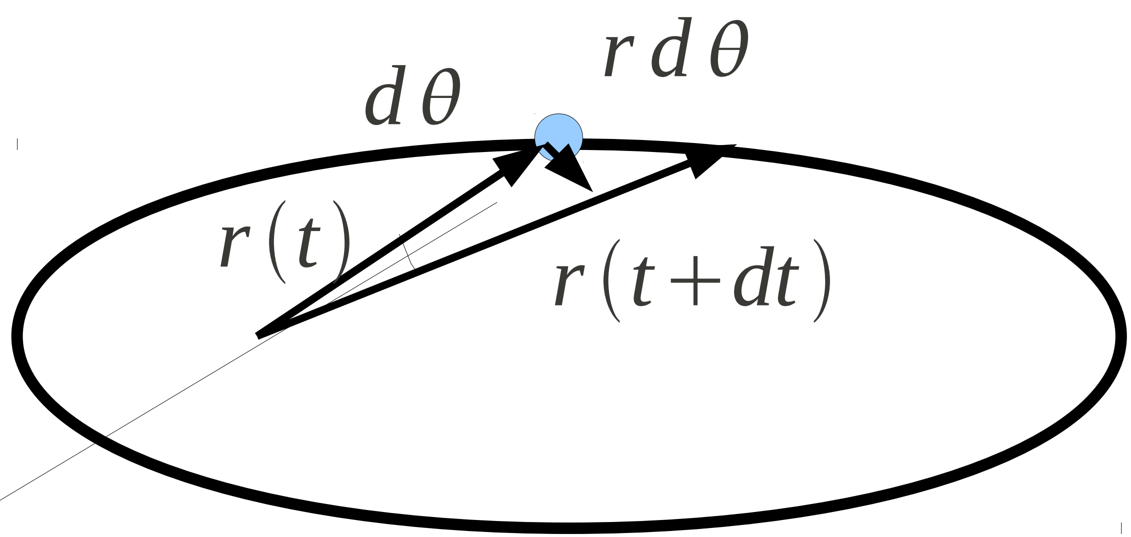
This may be restated as

$$\frac{d}{dt}\left(\frac{1}{2} r^2 \dot{\theta}\right) = 0$$

Area of the orbit is given by

$$A = \frac{1}{2} r (r\, d\theta)$$

$$\Rightarrow \frac{dA}{dt} = 0$$



Conservation of the angular momentum is equivalent to state that areal velocity is a constant

Conservation of the angular momentum is equivalent to state that areal velocity is a constant; this is the statement of **Kepler's second law of planetary motion**. This law states that the radius vector sweeps out equal areas in equal times

For the coordinate r

$$L=\frac{1}{2}\mu\left(\dot{r}^2+r^2\dot{\theta}^2\right)-U(r)$$

The Lagrange's equation of motion for generalized coordinate r

$$\frac{d}{dt}(\mu\dot{r})-\mu r\dot{\theta}^2+\frac{\partial U}{\partial r}=0$$

Angular momentum is given by $l=\mu r^2\dot{\theta}$

By squaring and rearranging $\mu r\dot{\theta}^2=\frac{l^2}{\mu r^3}$ $\quad f(r)=-\frac{\partial U}{\partial r}$

By substituting the equation of motion for r is

$$\mu\ddot{r}-\frac{l^2}{\mu r^3}=f(r)$$

Conservation of energy in two body problem

$$\mu \ddot{r} - \frac{l^2}{\mu r^3} = f(r) \qquad f(r) = -\frac{\partial U}{\partial r}$$

The equation of motion in r may be rewritten as

$$\mu \ddot{r} = -\frac{d}{dr}\left[\frac{l^2}{2\mu r^2} + U(r)\right]$$

Multiply both sides by $\dot{r}$

$$\mu \ddot{r}\dot{r} = -\dot{r}\frac{d}{dr}\left[\frac{l^2}{2\mu r^2} + U(r)\right]$$

Using the relation $\quad \mu \ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right)$ and $\quad \frac{d\,g(r)}{dt} = \frac{d\,g(r)}{dr}\frac{dr}{dt}$

$$\frac{d}{dt}\left(\frac{1}{2}\mu \dot{r}^2\right) = -\frac{d}{dt}\left[\frac{l^2}{2\mu r^2} + U(r)\right]$$

The equation of motion is now given by

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right)=-\frac{d}{dt}\left[\frac{l^2}{2\mu r^2}+U(r)\right]$$

rearranging

$$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2+\frac{l^2}{2\mu r^2}+U(r)\right)=0$$

$$\left(\frac{1}{2}\mu\dot{r}^2+\frac{l^2}{2\mu r^2}+U(r)\right)=Constant$$

Statement of conservation of energy of the system

$$\left(\frac{1}{2}\mu\dot{r}^2+\frac{l^2}{2\mu r^2}+U(r)\right)=Constant$$

This is further confirmed by writing the expression of energy in the system

$$E=\frac{1}{2}\mu\dot{\vec{r}}^2+U(r)=\frac{1}{2}\mu\left(\dot{r}^2+r^2\dot{\theta}^2\right)+U(r)$$

$$=\frac{1}{2}\mu\dot{r}^2+\frac{l^2}{2\mu r^2}+U(r)$$

$$\mu r^2\dot{\theta}=l$$

$$\mu r^2\dot{\theta}^2=\frac{l^2}{\mu r^2}$$

Therefore the equation of motion becomes

$$\left(\frac{1}{2}\mu\dot{r}^2+\frac{l^2}{2\mu r^2}+U(r)\right)=E$$

By inverting this relation for rate of change of radial distance we have

$$\dot{r}=\sqrt{\frac{2}{\mu}\left(E-U(r)-\frac{l^2}{2\mu r^2}\right)}$$

Two body problem equation of motion radial part

$$\frac{dr}{dt}=\sqrt{\frac{2}{\mu}\left(E-U(r)-\frac{l^2}{2\mu r^2}\right)}$$

By rearranging

$$dt=\frac{dr}{\sqrt{\frac{2}{\mu}\left(E-U(r)-\frac{l^2}{2\mu r^2}\right)}}$$

At $t=0$ let r have initial value r_0 , now the state change be described by the integral

$$t=\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{2}{\mu}\left(E-U(r)-\frac{l^2}{2\mu r^2}\right)}}$$

Two body problem equation of motion angular part

This equation can be solved for any arbitrary potential by first evaluating this integral and inverting the relation for time dependent variation of distance.

In the similar way the solution of angular part of the integral also can be evaluated from equation

$$\mu r^2 \dot{\theta} = l$$

$$d\theta = \frac{l\, dt}{\mu r^2(t)}$$

Assuming the initial value of the angle be θ_0

$$\theta = l \int_0^t \frac{dt}{\mu r^2(t)} + \theta_0$$

Two body problem equation – power law potentials

Solution with power law potentials $U(r) \propto r^{-1}$

Converting time integrals to theta integrals by change of variable

$$\mu r^2 \dot{\theta} = l$$

$$\mu r^2 d\theta = l\, dt$$

$$\frac{d}{dt} = \frac{l}{\mu r^2} \frac{d}{d\theta}$$

Second derivative with respect to time

$$\frac{d^2}{dt^2} = \frac{l}{\mu r^2} \frac{d}{d\theta} \left(\frac{l}{\mu r^2} \frac{d}{d\theta} \right)$$

Second derivative with respect to time

$$\frac{d^2}{dt^2} = \frac{l}{\mu r^2} \frac{d}{d\theta} \left(\frac{l}{\mu r^2} \frac{d}{d\theta} \right)$$

Lagrange's equation $\quad \mu \ddot{r} - \dfrac{l^2}{\mu r^3} = f(r) \quad$ now becomes

$$\frac{l}{r^2} \frac{d}{d\theta} \left(\frac{l}{\mu r^2} \frac{dr}{d\theta} \right) - \frac{l^2}{\mu r^3} = f(r)$$

For further simplification we notice that

$$\frac{1}{r^2} \frac{dr}{d\theta} = -\frac{d(1/r)}{d\theta}$$

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{\mu r^2}\frac{dr}{d\theta}\right)-\frac{l^2}{\mu r^3}=f(r)$$

by using the relation $$\frac{1}{r^2}\frac{dr}{d\theta}=-\frac{d(1/r)}{d\theta}$$

$$-\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{\mu}\frac{d(1/r)}{d\theta}\right)-\frac{l^2}{\mu r^3}=f(r)$$

By change of variable to $$r=\frac{1}{u}$$

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{\mu}\frac{du}{d\theta}\right)+\frac{l^2}{\mu r^3}=-f\left(\frac{1}{u}\right)$$

$$\frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{\mu}\frac{du}{d\theta}\right)+\frac{l^2}{\mu r^3}=-f\left(\frac{1}{u}\right)$$

By change of variable fully to u using relation $r=\frac{1}{u}$

$$\frac{l^2 u^2}{\mu}\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)+\frac{l^2 u^3}{\mu}=-f\left(\frac{1}{u}\right)$$

$$\frac{l^2 u^2}{\mu}\left(\frac{d^2 u}{d\theta^2}+u\right)=-f\left(\frac{1}{u}\right)$$

In order to get solution the exact form of this force function is required

In case of inverse square law potential

$$U(r)=-\frac{k}{r}$$

Solutions to inverse square law potential

$$U(r) = -\frac{k}{r}$$

$$f(r) = -\frac{k}{r^2} \qquad f(u) = -k u^2$$

$$\left(\frac{d^2 u}{d\theta^2} + u\right) = \frac{-\mu}{l^2 u^2} f\left(\frac{1}{u}\right)$$

$$= \frac{-\mu}{l^2 u^2}(-k u^2) = \frac{k\mu}{l^2}$$

$$\left(\frac{d^2 u}{d\theta^2} + u\right) = \frac{k\mu}{l^2}$$

$$\left(\frac{d^2 u}{d\theta^2}+u\right)=\frac{k\mu}{l^2}$$

Now change the variable to $y=u-\frac{k\mu}{l^2}$ $\qquad u=\frac{1}{r}$

$$\Rightarrow\left(\frac{d^2 y}{d\theta^2}+y\right)=0$$

We know now this the linear differential equation for a harmonic oscillator

The solution has the form $y=B\cos(\theta-\theta_0)$ $\qquad y=\frac{1}{r}-\frac{k\mu}{l^2}$

$$\frac{1}{r}-\frac{\mu k}{l^2}=B\cos(\theta-\theta_0)$$

$$\Rightarrow\frac{1}{r}=\frac{\mu k}{l^2}\left(1+e\cos(\theta-\theta_0)\right) \qquad e=B\frac{l^2}{\mu k}$$

The solution has the form where constant of the integration has to evaluated

The constants may be obtained by direct evaluation of the integral

$$dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}} \qquad l\,dt = \mu r^2 d\theta$$

Eliminating the time variable from the integral using relation involving angular momentum

$$d\theta = \frac{l\,dr}{\mu r^2 \sqrt{\frac{2}{\mu}\left(E - U(r) - \frac{l^2}{2\mu r^2}\right)}}$$

$$d\theta = \frac{dr}{r^2 \sqrt{\left(\frac{2\mu E}{l^2} - \frac{2\mu U(r)}{l^2} - \frac{1}{r^2}\right)}}$$

$$\theta=\theta_0+\int_{r_0}^{r}\frac{dr}{r^2\sqrt{\left(\frac{2\mu E}{l^2}-\frac{2\mu U(r)}{l^2}-\frac{1}{r^2}\right)}}$$

By change of integration variable to $u=\frac{1}{r}$ $\quad du=-\frac{1}{r^2}dr$

$$\theta=\theta_0-\int_{u_0}^{u}\frac{du}{\sqrt{\left(\frac{2\mu E}{l^2}-\frac{2\mu U(r)}{l^2}-u^2\right)}}$$

Solutions to inverse square law potential

$$U(r)=-\frac{k}{r} \qquad f(r)=-\frac{k}{r^2} \qquad f(u)=-ku^2$$

$$\theta=\theta_0-\int_{u_0}^{u}\frac{du}{\sqrt{\left(\frac{2\mu E}{l^2}+\frac{2\mu k u}{l^2}-u^2\right)}}$$

$$\theta=\theta_0-\int_{u_0}^{u}\frac{du}{\sqrt{\left(\frac{2\mu E}{l^2}+\frac{2\mu k u}{l^2}-u^2\right)}}$$

Using the standard integral

$$\int\frac{dx}{\sqrt{a+bx+cx^2}}=\frac{1}{\sqrt{-c}}\cos^{-1}\left(-\frac{b+2cx}{\sqrt{q}}\right)$$

$$q=b^2-4ac$$

$$a=\frac{2\mu E}{l^2}\qquad b=\frac{2\mu k}{l^2}\qquad c=-1$$

$$q=\left(\frac{2\mu k}{l^2}\right)^2+4\left(\frac{2\mu E}{l^2}\right)\qquad=\left(\frac{2\mu k}{l^2}\right)^2\left(1+\frac{2El^2}{\mu k^2}\right)$$

$$\int \frac{dx}{\sqrt{a+bx+cx^2}} = \frac{1}{\sqrt{-c}} \cos^{-1}\left(-\frac{b+2cx}{\sqrt{q}}\right)$$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{\sqrt{\left(\frac{2\mu E}{l^2} + \frac{2\mu k u}{l^2} - u^2\right)}}$$

$$a = \frac{2\mu E}{l^2} \qquad b = \frac{2\mu k}{l^2} \qquad c = -1$$

$$\theta = \theta_0 - \cos^{-1}\left(-\frac{\frac{2\mu k}{l^2} - 2u}{\left(\frac{2\mu k}{l^2}\right)\sqrt{\left(1 + \frac{2El^2}{\mu k^2}\right)}}\right)$$

$$\theta=\theta_0-\cos^{-1}\left(\frac{\frac{ul^2}{\mu k}-1}{\sqrt{\left(1+\frac{2El^2}{\mu k^2}\right)}}\right)$$

$$\left(\frac{ul^2}{\mu k}-1\right)=\sqrt{\left(1+\frac{2El^2}{\mu k^2}\right)}\cos\left(\theta-\theta_0\right)$$

$$u=\frac{\mu k}{l^2}\left(1+\sqrt{\left(1+\frac{2El^2}{\mu k^2}\right)}\cos\left(\theta-\theta_0\right)\right)$$

$$\frac{1}{r}=\frac{\mu k}{l^2}\left(1+\sqrt{\left(1+\frac{2El^2}{\mu k^2}\right)}\cos\left(\theta-\theta_0\right)\right)$$

$$\frac{1}{r}=\frac{\mu k}{l^2}\left(1+\sqrt{\left(1+\frac{2El^2}{\mu k^2}\right)}\cos\left(\theta-\theta_0\right)\right)$$

The solution can be expressed in terms of the convenient form

$$\frac{1}{r}=C\left(1+e\cos\left(\theta-\theta_0\right)\right)$$

The parameter e is called **eccentricity** of the particle motion, this defines the shape of the orbits

$$e=\sqrt{\left(1+\frac{2El^2}{\mu k^2}\right)}$$

$$\frac{1}{r} = C\left(1 + e\cos\left(\theta - \theta_0\right)\right)$$

$$e = \sqrt{\left(1 + \frac{2El^2}{\mu k^2}\right)}$$

e is called **eccentricity** of the orbit

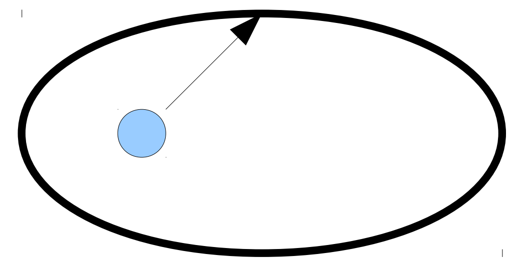
Limit of the orbits

$$\frac{1}{r} = \frac{\mu k}{l^2}\left(1 + e\cos\left(\theta - \theta_0\right)\right)$$

The value of the cosine reaches 1 the distance has minimum value

The value of the cosine reaches 1 the distance has minimum value

$$r = \frac{l^2}{\mu k \left(1 + e \cos\left(\theta - \theta_0\right)\right)}$$



$$r_{min} = \frac{l^2}{\mu k \left(1 + e\right)}$$

The value of the cosine reaches -1 the distance has maximum value

$$r_{max} = \frac{l^2}{\mu k \left(1 - e\right)}$$

Case I

$$e = 0 \qquad E = -\frac{\mu k^2}{2 l^2} \qquad e = \sqrt{\left(1 + \frac{2 E l^2}{\mu k^2}\right)}$$

Here the $r_{max} = r_{min}$ therefore the nature of the orbit is **circular**

Case II $0<e<1$ $-\frac{\mu k^2}{2l^2}<E<0$

The resulting motion is an **ellipse** the particle oscillates between maximum and minimum value of the orbits

Case III $e=1$ $$e=\sqrt{\left(1+\frac{2El^2}{\mu k^2}\right)}=1$$

$E=0$

The energy of the system is zero the resulting orbit is **parabola**

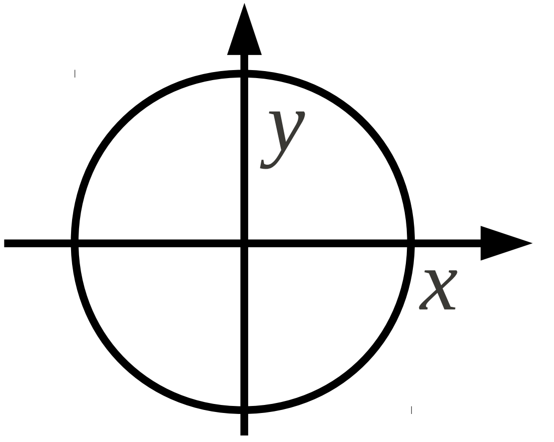
$$r_{max}=\frac{l^2}{\mu k\left(1-e\right)}=\infty$$

Case IV $e>1$ $$e=\sqrt{\left(1+\frac{2El^2}{\mu k^2}\right)}=1>1$$

$E>0$

The maximum radius become negative that mean motion become unbounded

The nature of the orbits can be proved as follows



$$\frac{1}{r}=\frac{\mu k}{l^2}\left(1+e\cos\left(\theta-\theta_0\right)\right)$$

$$g=\frac{l^2}{\mu k} \qquad \frac{x}{r}=\cos\left(\theta-\theta_0\right)$$

$$\frac{1}{r}=\frac{1}{g}\left(1+\frac{e\,x}{r}\right) \qquad g=r\left(1+\frac{e\,x}{r}\right)$$

$$g=r+e\,x \qquad \left(g-e\,x\right)^2=r^2$$

$$g^2-2g\,e\,x+e^2=x^2+y^2$$

Now by changing the value of e we can arrive at equation for different type of orbits.

Classification of the orbits

The solution of the two body problem requires evaluation of two integrals, which are not easy for any arbitrary function of position

$$dt=\frac{dr}{\sqrt{\frac{2}{\mu}\left(E-U(r)-\frac{l^2}{2\mu r^2}\right)}} \qquad \theta=l\int_0^t \frac{dt}{\mu r(t)^2}+\theta_0$$

It is possible to obtain the explicit nature of the solutions by analyzing the energy and angular momentum of the system

$$E=\frac{1}{2}mv^2+V(r)$$

By inverting the relation for energy we get magnitude of velocity of radial vector

$$v=\sqrt{\frac{2}{m}(E-V(r))}$$

This together with the angular part that comes from analysis of angular momentum give complete information time dependency of the radial vector

$$\mu\ddot{r}-\frac{l^2}{\mu r^3}=f(r)$$

The differential equation for the radial vector give two types of forces

$$f'=f(r)+\frac{l^2}{\mu r^3}$$

from the potential

from the angular momentum

Therefore differential equation for the radial part of the distance vector give two types of forces

Equation of motion is now $\mu\ddot{r}=f'(r)$

This is Newton's equation of motion for one degree of freedom. The effective potential is now can be expressed as

$$V'=U(r)+\frac{l^2}{2\mu r^2}$$

Such that that $$-\frac{\partial V'}{\partial r}=f(r)+\frac{l^2}{\mu r^3}$$

Such that that $$\left(\frac{1}{2}\mu\dot{r}^2+\frac{l^2}{2\mu r^2}+U(r)\right)=Constant$$

$$V'=U(r)+\frac{l^2}{2\mu r^2}$$

$$\Rightarrow\left(\frac{1}{2}\mu\dot{r}^2+V'\right)=Constant$$

For analyzing the energy of the system we take a specific case of a potential

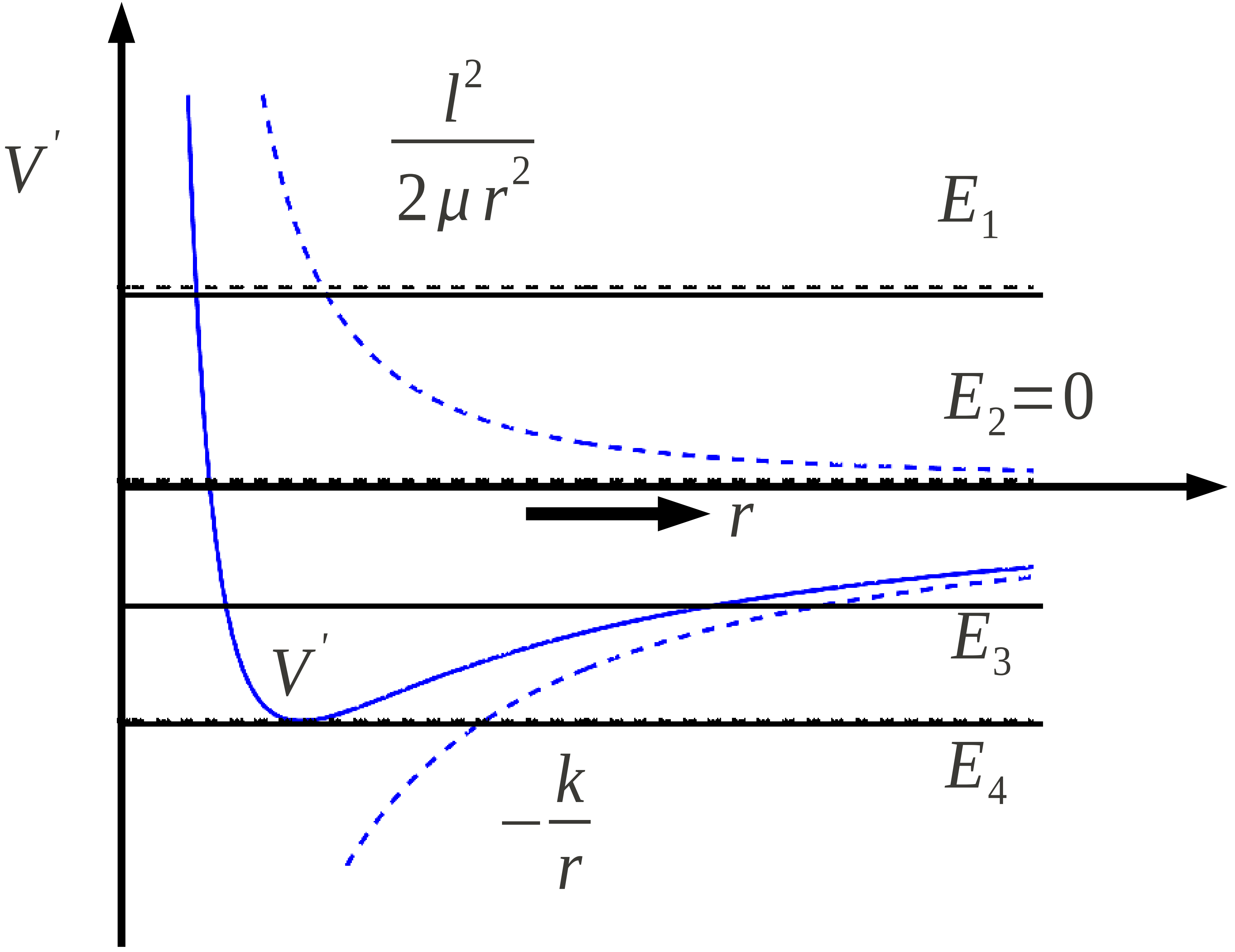
$$f=-\frac{k}{r^2}$$

Corresponding potential energy is given by $V=-\frac{k}{r}$

Now the effective potential becomes

$$V'=-\frac{k}{r}+\frac{l^2}{2\mu r^2}$$

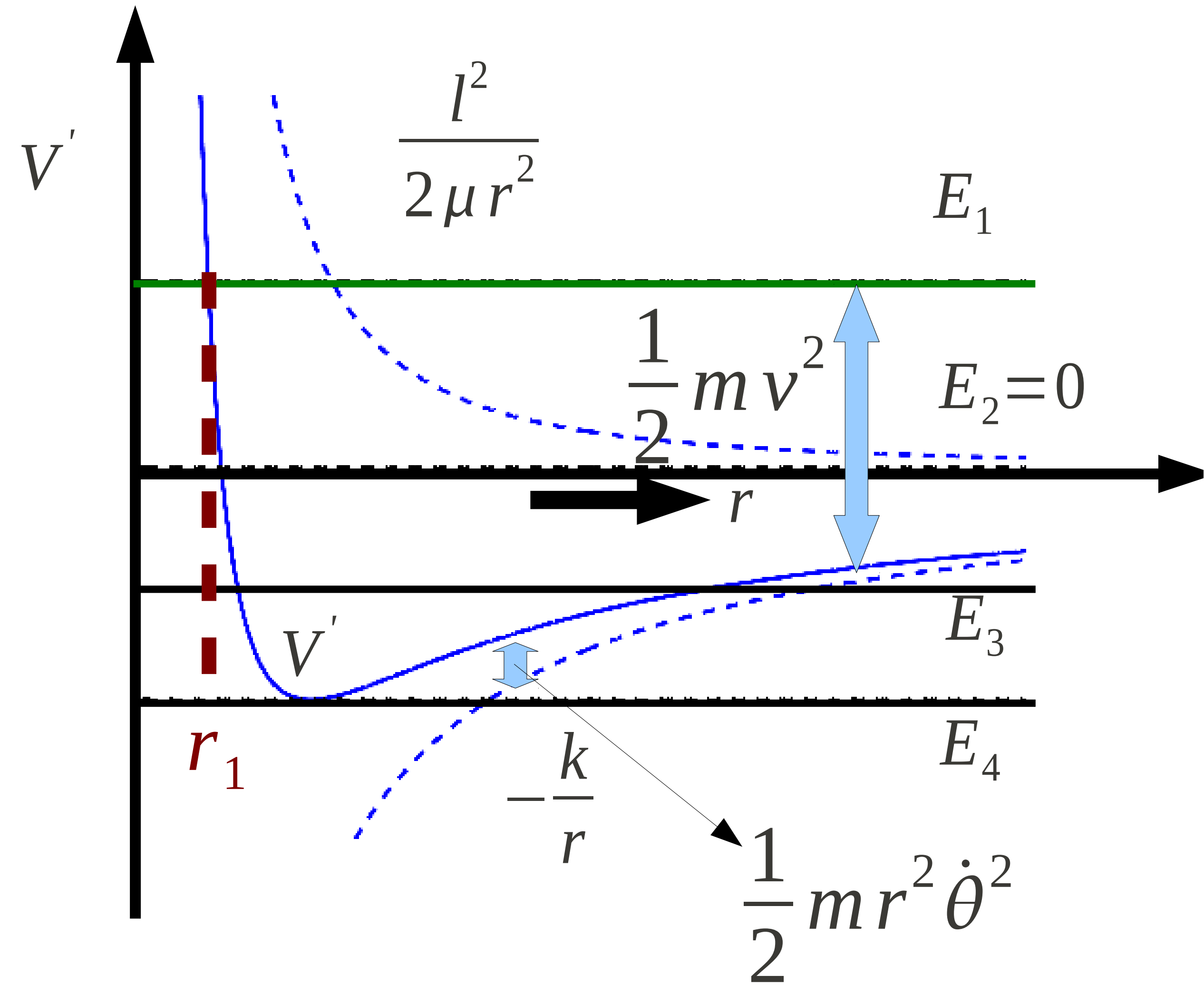
The potential can be analyzed graphically

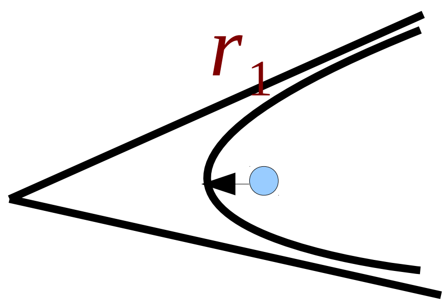


V'
$\frac{l^2}{2\mu r^2}$
E_1
$E_2 = 0$
r
V'
E_3
E_4
$-\frac{k}{r}$

Now consider a particle that have energy E_1

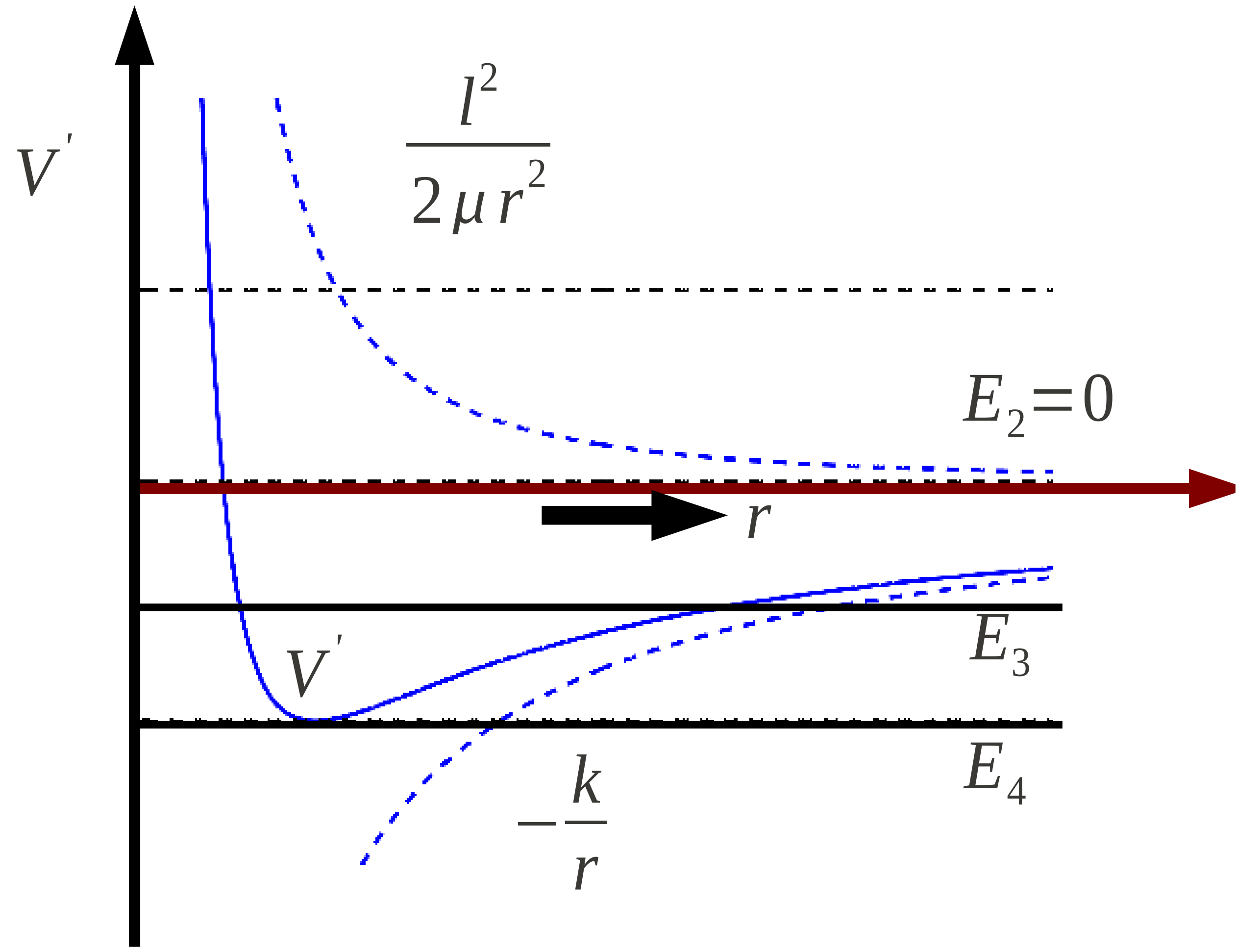
This particle can never come closer than the radius r_1 . The kinetic energy is always positive and there for all the energy will be converted into kinetic part at this point. Motion is not bound as there is no upper limit to r.



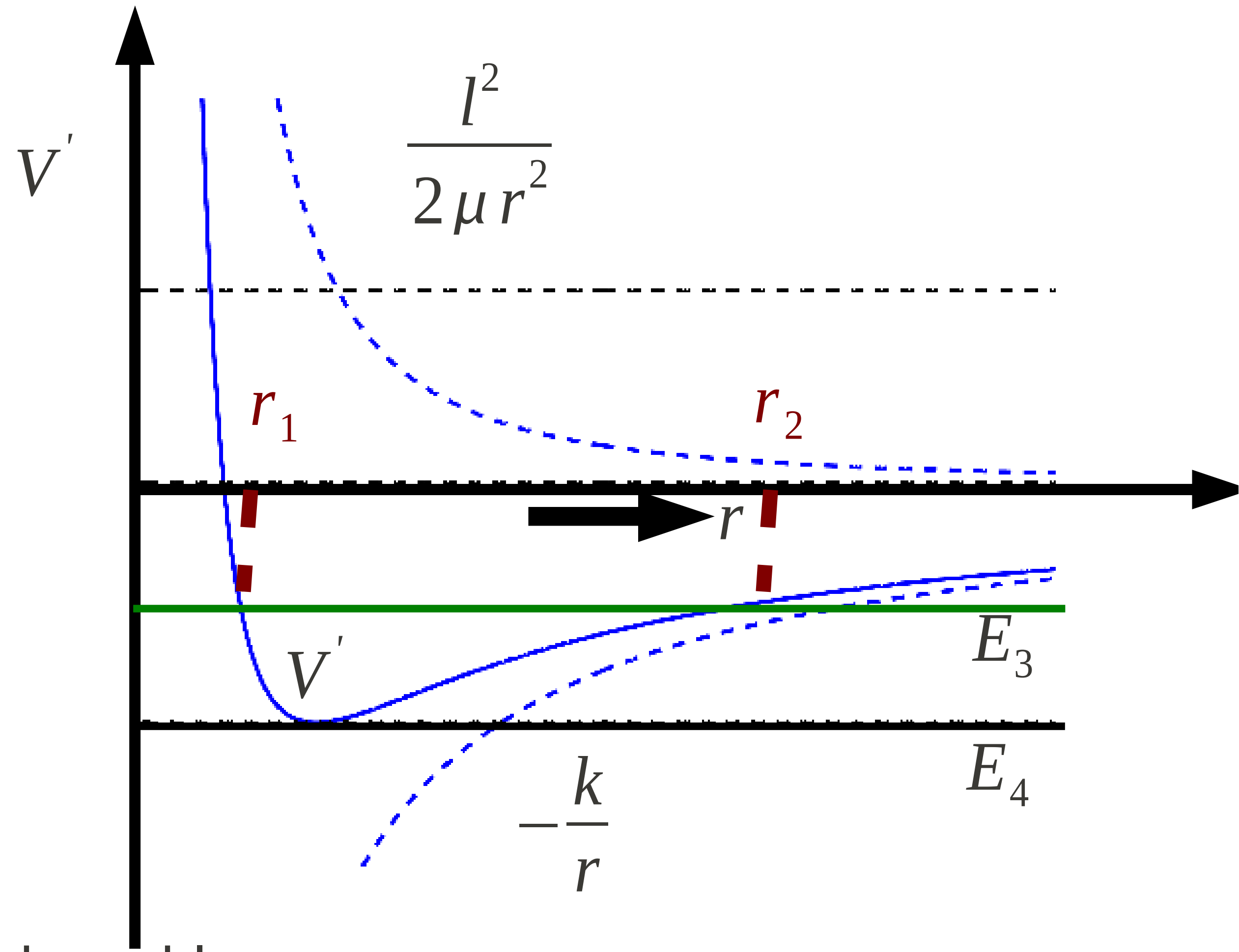


The kinetic energy part is remaining in the total energy at all distances that is greater than r_1 .

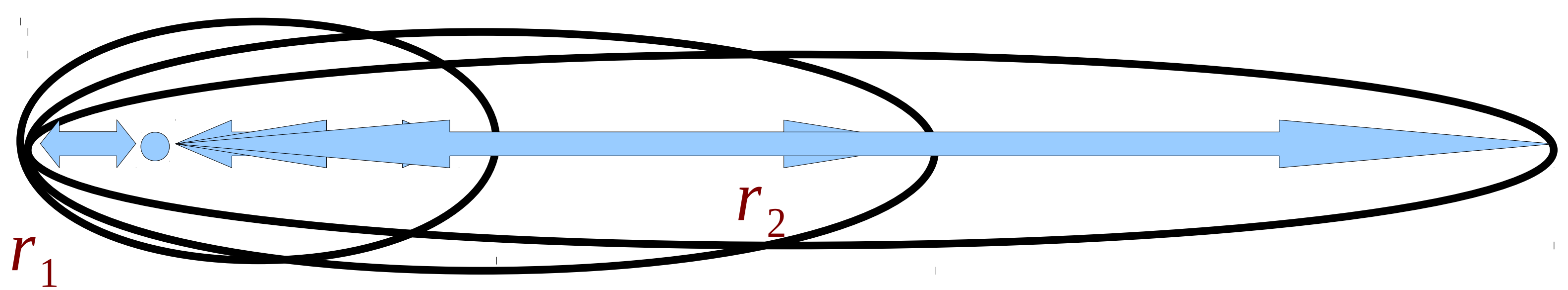
Now take the case where energy is given by $E_2 = 0$ this is the boundary case of unbound motion.



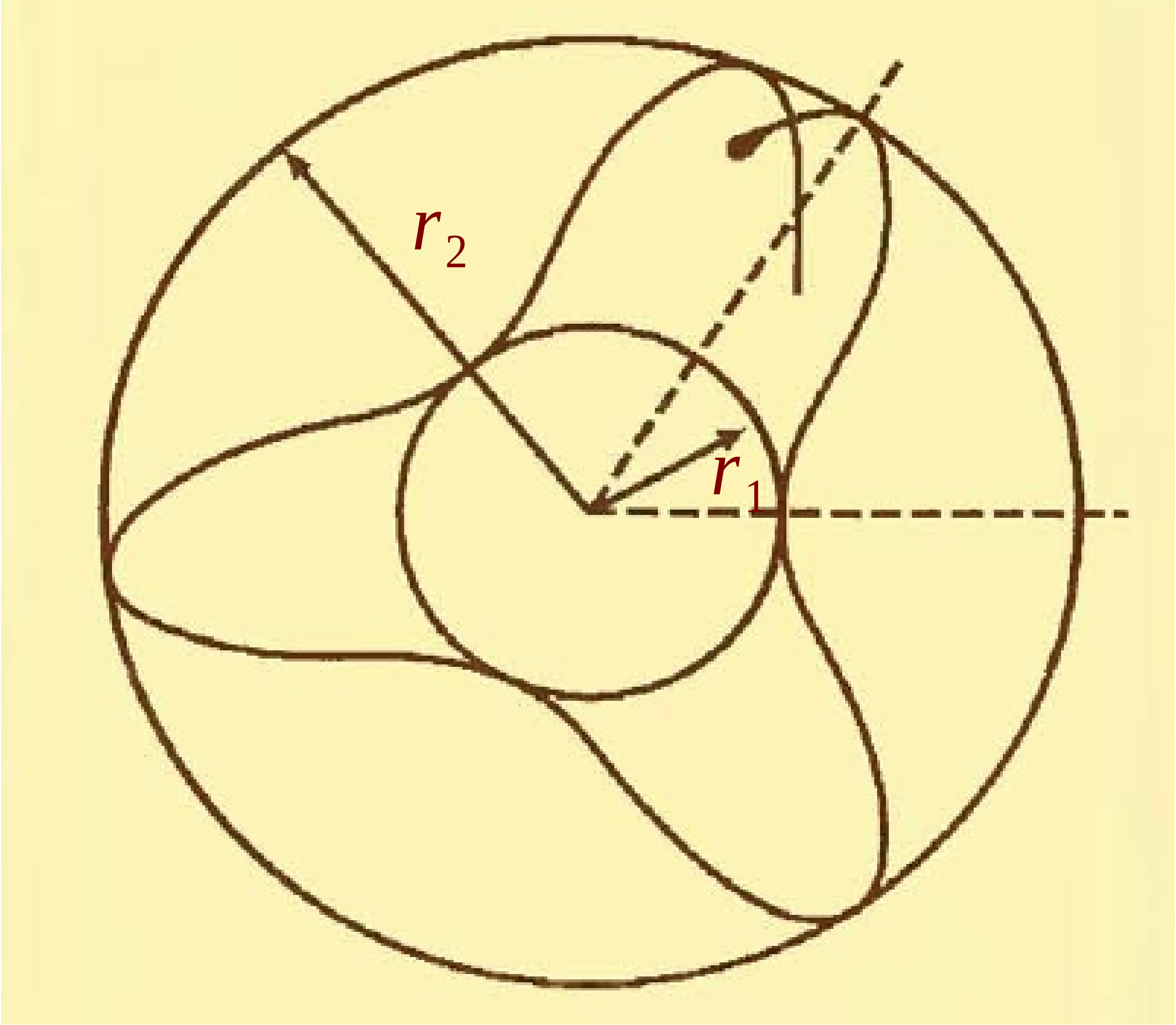
Now take the case where energy is given by E_3 . In this case there are two turning points for the motion r_1 and r_2. These distances are known as **apsidal distances.**



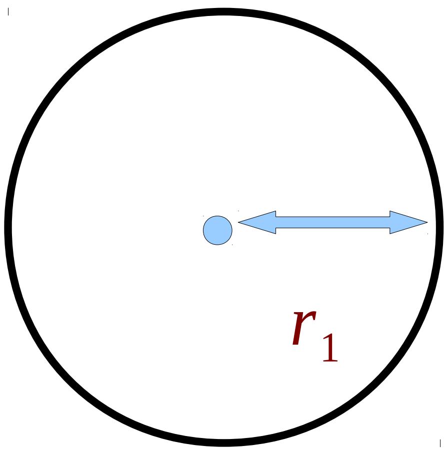
Closed orbits of the two body problem

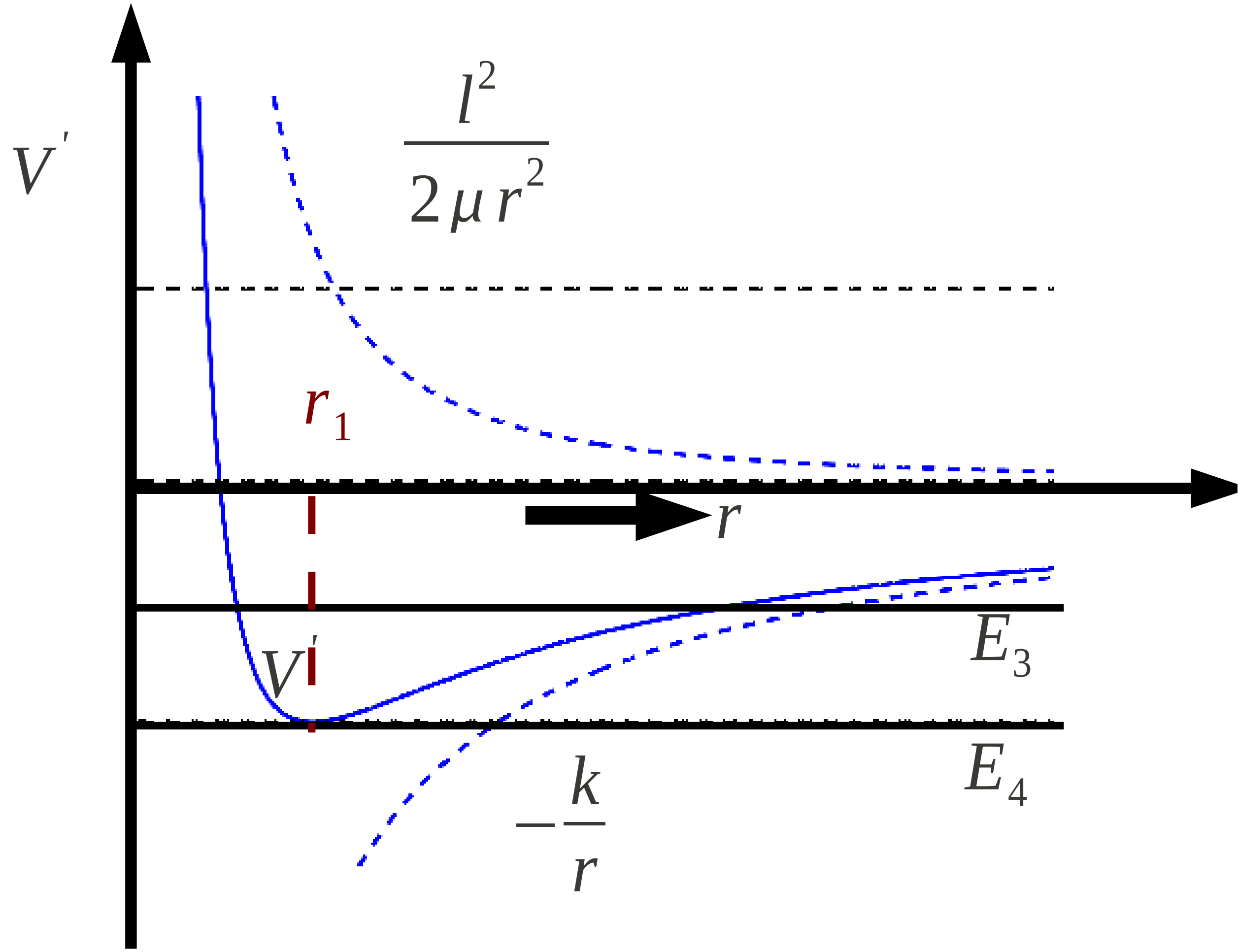


It is not necessary that bound motion means closed orbits. Example of bound motion without closed orbits.



Now take the case where energy is given by E_4 . In this case there are one turning points for the motion r_1 this corresponds to the circular orbit





This is the case when the centrifugal terms exactly balances the centripetal forces

$$0=f'=f(r)+\frac{l^2}{\mu r^3}$$

$$l=\mu r^2\dot{\theta}$$

$$f(r) = -\frac{l^2}{\mu r^3} \qquad l = \mu r^2 \dot{\theta}$$

$$\Rightarrow f(r) = -\mu r \dot{\theta}^2$$

In general the division of orbits into category of, open, bounded and circular only possible for potentials

(1) that falls off slower than $\frac{1}{r^2}$ as $r \to \infty$

(2) that is infinite slower than $\frac{1}{r^2}$ as $r \to 0$

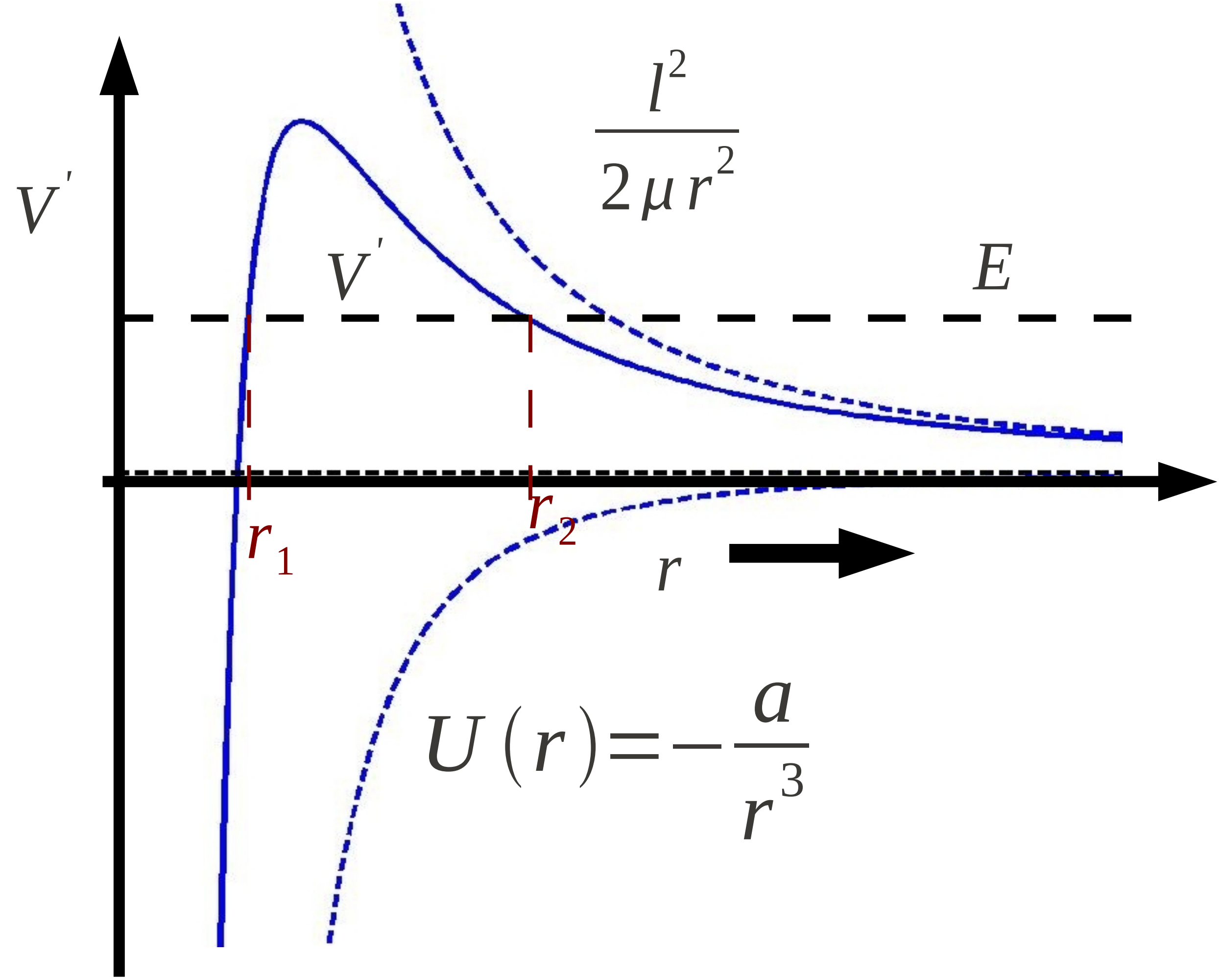
If the potential does not satisfy this criteria then the classification of the orbits cannot be done in the categories described above

Consider a potential of type

$$U(r) = -\frac{a}{r^3}$$

$$f = -\frac{3a}{r^4}$$

This potential has bound states when energy of the system is E, bound state depends on the radius of the system



(1) when radius $r < r_1$: bound state trajectory passing through center of force

(2) when radius $r_1 < r < r_2$: bound state trajectory but particle is forbidden fall in the potential hole

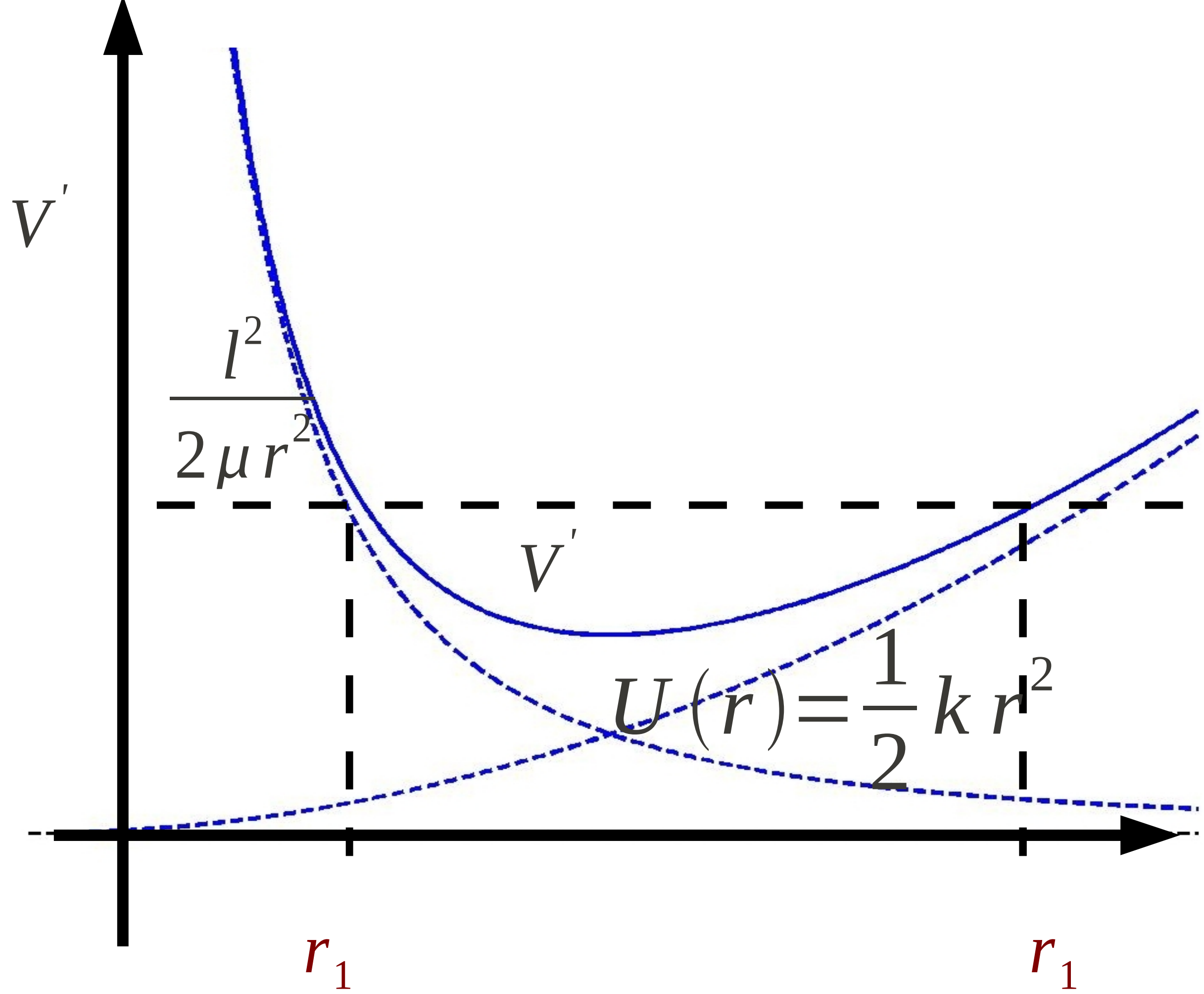
(3) when radius $r > r_2$: unbound state

Consider a harmonic potential

$$U(r) = \frac{1}{2} k r^2$$

$$f = -k r$$

When angular momentum is zero then it a straight line motion



The motion is always bounded for all physically possible energies with elliptic or circular orbits

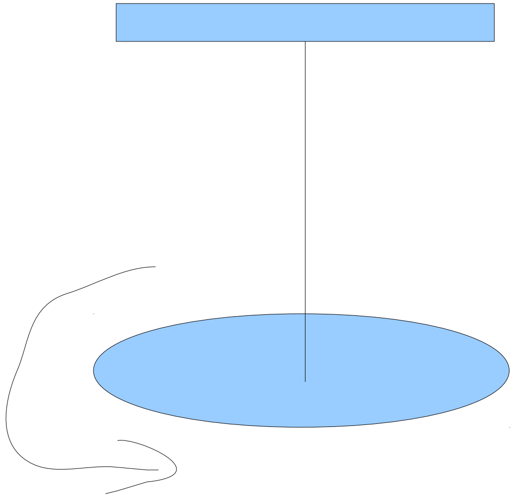
Small Oscillations

When the system is displaced infinitesimally around it's stable equilibrium it executes small oscillations

Applications molecular vibrations, electrical circuits, mechanical assemblies, bridges etc..

It is possible to develop a theory of small oscillations in terms of generalized coordinates

Example - torsional oscillations



Example – vibrations of polyatomic molecules

Consider a mechanical system with n degrees of freedom

$$\boldsymbol{q} = (q_{1,} q_{2,} ...., q_n)$$

Each degrees of freedom obey Lagrange's equation of motion

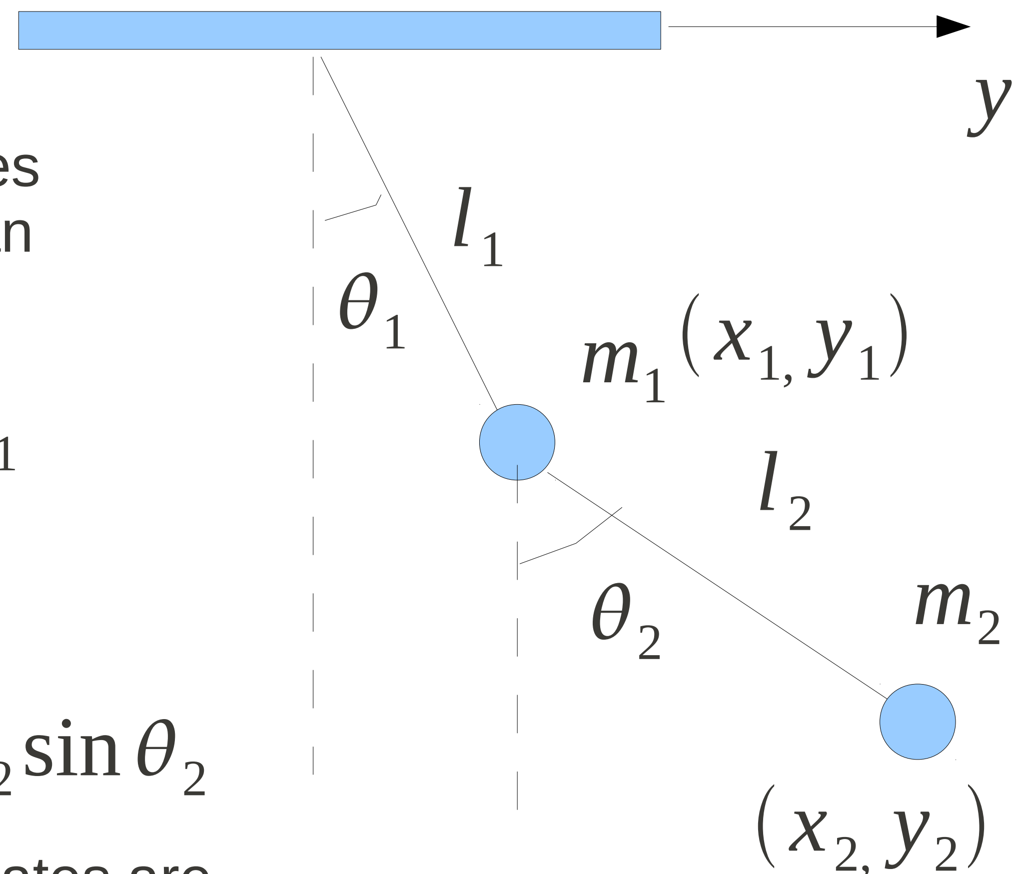
$$\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} \right) = 0$$

Now kinetic energy of the system has the standard form in the generalized coordinates

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij}(\boldsymbol{q}) \dot{q}_i \dot{q}_j$$

This statement include terms involve velocity cross terms also. Such a generalized square of velocity term may be found in case of double pendulum.

Double pendulum



The appropriate generalized coordinates are the angles. In terms of the Cartesian coordinates

$$x_1 = l_1 \cos\theta_1 \qquad y_1 = l_1 \sin\theta_1$$

$$x_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2$$

$$y_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

The velocities of the Cartesian coordinates are

$$\dot{x}_1 = -l_1 \dot{\theta}_1 \sin\theta_1 \qquad \dot{y}_1 = l_1 \dot{\theta}_1 \cos\theta_1$$

$$\dot{x}_2 = -l_1 \dot{\theta} \sin\theta_1 - l_2 \dot{\theta} \sin\theta_2 , \qquad \dot{y}_2 = l_1 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_1 \cos\theta_2$$

The kinetic energy of the system is now given by

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_1 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$T = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 (l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2))$$

This is an example of where kinetic energy can be expressed in terms of mixed/cross terms

$$T = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij}(\boldsymbol{q}) \dot{q}_i \dot{q}_j$$

Now the Lagrange's equation of motion is given by

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\sum_i \left(t_{ij}(\boldsymbol{q}) \dot{q}_i\right)\right) = \left(\sum_i \left(\frac{d t_{ij}(\boldsymbol{q})}{dt} \dot{q}_i + t_{ij}(\boldsymbol{q}) \ddot{q}_i\right)\right)$$

$$\frac{\partial T}{\partial q_j} = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial t_{ij}(\boldsymbol{q})}{\partial q_j} \dot{q}_i \dot{q}_j$$

Lagrange's equation of motion with new form of the Kinetic energy is

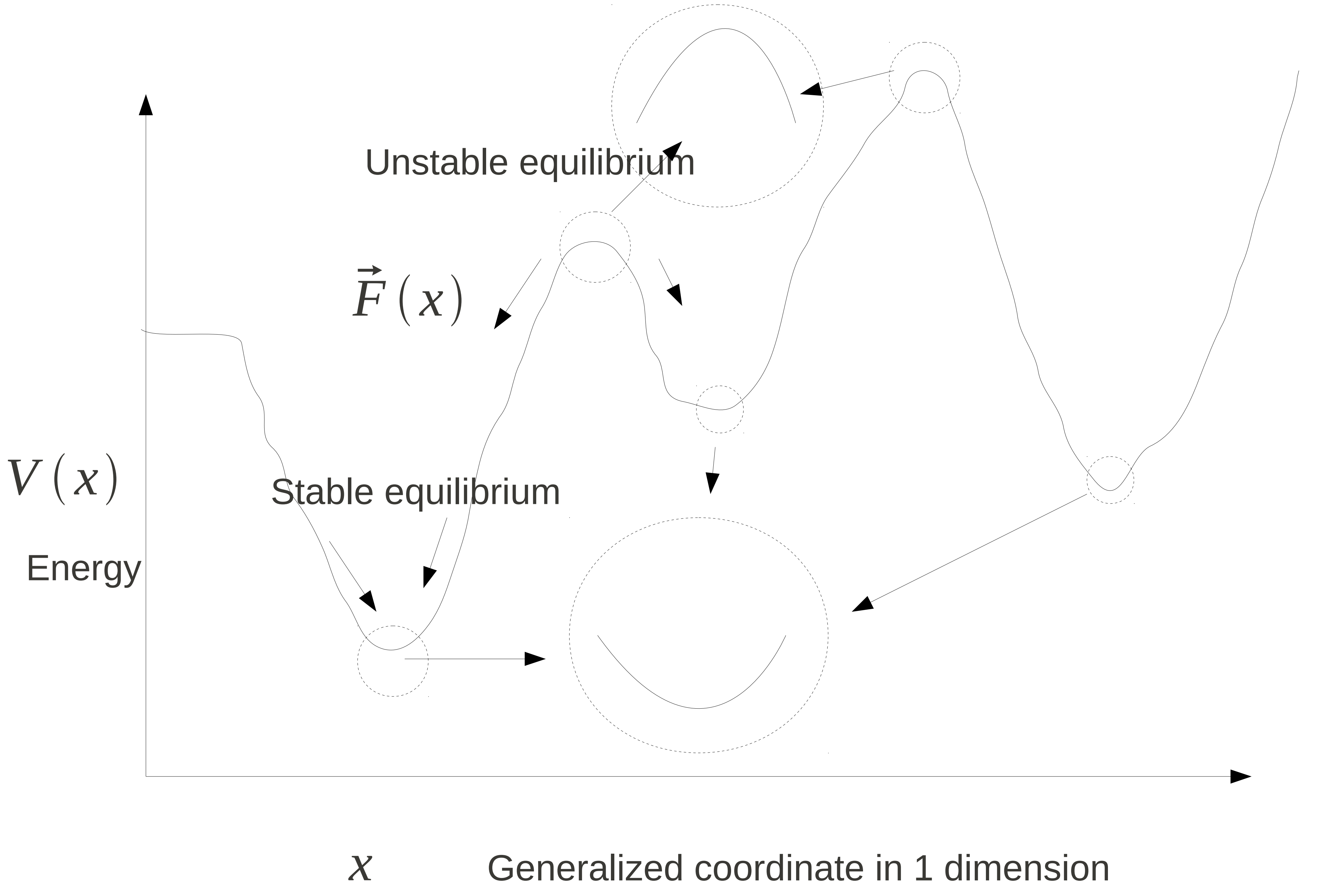
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)-\frac{\partial T}{\partial q_j}=-\frac{\partial V}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)=2\left(\sum_i\left(\frac{d t_{ij}(\boldsymbol{q})}{dt}\dot{q}_i+t_{ij}(\boldsymbol{q})\ddot{q}_i\right)\right)$$

$$\frac{\partial T}{\partial q_j}=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial t_{ij}(\boldsymbol{q})}{\partial q_j}\dot{q}_i\dot{q}_j$$

$$\left(\sum_i\left(\frac{d t_{ij}(\boldsymbol{q})}{dt}\dot{q}_i+t_{ij}(\boldsymbol{q})\ddot{q}_i\right)\right)-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial t_{ij}(\boldsymbol{q})}{\partial q_j}\dot{q}_i\dot{q}_j=-\frac{\partial V}{\partial q_j}$$

In general a potential can be approximated using by expansion in the generalized coordinates


Unstable equilibrium
$\vec{F}(x)$
$V(x)$
Stable equilibrium
Energy
x
Generalized coordinate in 1 dimension

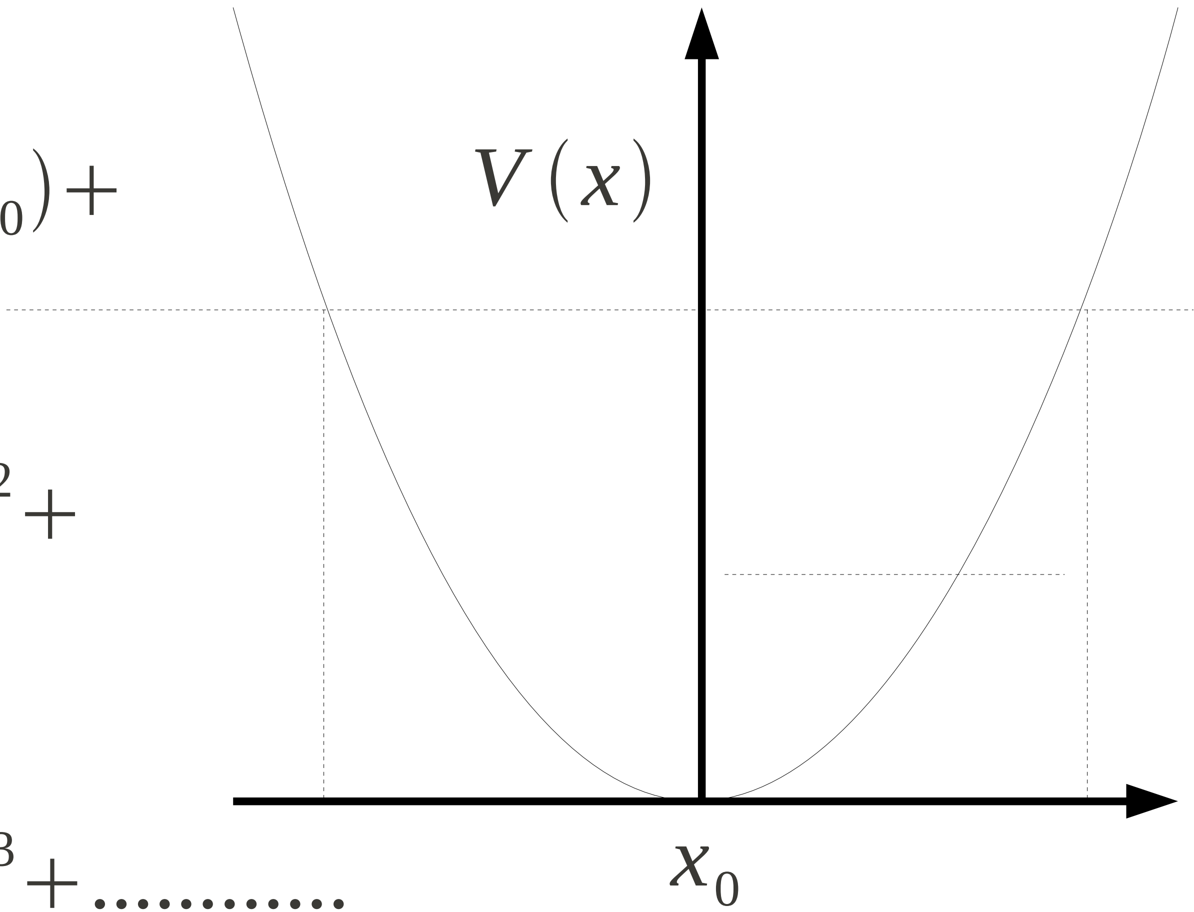
Taylor series expansion potential around potential energy minimum

$$V(x)=$$

$$V(x_0)+\left.\frac{dV(x)}{dx}\right|_{x=0}(x-x_0)+$$

$$\frac{1}{2!}\left.\frac{d^2V(x)}{dx^2}\right|_{x=0}(x-x_0)^2+$$

$$\frac{1}{3!}\left.\frac{d^3V(x)}{dx^3}\right|_{x=0}(x-x_0)^3+\ldots\ldots\ldots\ldots$$



First differential is zero near potential energy minimum

$$\frac{dV(x)}{dx}=0$$

Neglecting higher order terms

$$V(x) \simeq V(x_0) + \frac{1}{2!} \left. \frac{d^2 V(x)}{d x^2} \right|_{x=0} (x - x_0)^2$$

Potential of harmonic oscillator

$$V(x) = \frac{1}{2} k x^2$$

$$V(x) \simeq V(x_0) + \frac{1}{2!} \left. \frac{d^2 V(x)}{d x^2} \right|_{x=0} (x - x_0)^2$$

$$V(x) = V(\overset{0}{x_0}) + \frac{1}{2} k (x - \overset{0}{x_0})^2$$

No higher order terms

$$V(x) = \frac{1}{2} k (x)^2 \qquad \Rightarrow \frac{d^2 V}{d x^2} = k$$

Simple pendulum

$$U(h(\theta)) = mgl(1-\cos\theta) \qquad U(\theta) = mgl(1-\cos\theta)$$

$$\frac{dU}{d\theta} = mgl(\sin\theta) \qquad \frac{d^2U}{d\theta^2} = mgl(\cos\theta) \qquad \frac{d^3U}{d\theta^3} = mgl(-\sin\theta)$$

Using the equation for harmonic approximation of the potential

$$V(x) = \quad V(x_0) + \left.\frac{dV(x)}{dx}\right|_{x=0} (x-x_0) +$$

$$V(x_0) + \left.\frac{dV(x)}{dx}\right|_{x=0} (x-x_0) + \quad \frac{1}{3!} \left.\frac{d^3V(x)}{dx^3}\right|_{x=0} (x-x_0)^3 + \ldots$$

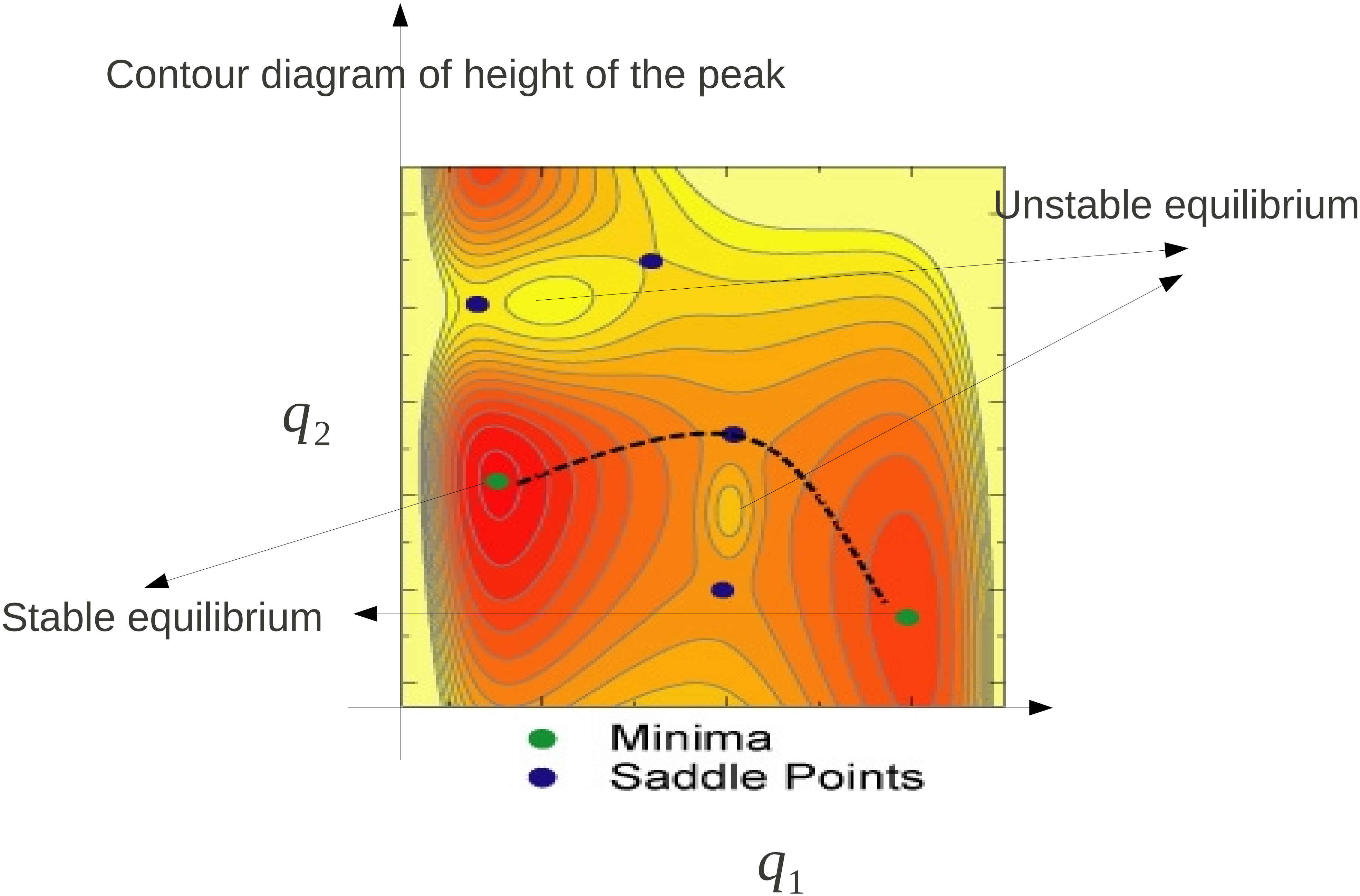
When $\theta = 0$ by substituting into the general expansion

$$1 \qquad\qquad 0$$

$$U(\theta) = \frac{1}{2} mgl \cos\theta (\theta-\theta_0)^2$$

$$U(\theta) = \frac{1}{2} mgl (\theta)^2$$

Contour diagram of height of the peak



Let q_1 and q_2 be the generalized coordinates of the system

General approximation for the potential that depend on two coordinates

$$V(q_{1,}q_2)=V(0,0)+\left(\frac{\partial V}{\partial q_1}q_1+\frac{\partial V}{\partial q_2}q_2\right)$$

$$+\frac{1}{2}\left(\frac{\partial^2 V}{\partial q_1^2}q_1^2+2\frac{\partial^2 V}{\partial q_1\partial q_2}q_1 q_2+\frac{\partial^2 V}{\partial q_2^2}q_2^2\right)+.....$$

Very near the equilibrium the first order derivatives become zero

$$V(q_{1,}q_2)=\frac{1}{2}\left(\frac{\partial^2 V}{\partial q_1^2}q_1^2+2\frac{\partial^2 V}{\partial q_1\partial q_2}q_1 q_2+\frac{\partial^2 V}{\partial q_2^2}q_2^2\right)$$

$$V(q_{1,}q_2)=\frac{1}{2}\left(v_{11}q_1^2+2v_{12}q_1 q_2+v_{22}q_2\right)$$

Now for system having n degrees of freedom

$$V=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}v_{ij}q_i q_j$$ where $$v_{ij}=\left.\frac{\partial^2 V}{\partial q_i\partial q_j}\right|_{\boldsymbol{q}=0}$$

General approximate form of K.E.

$$T=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}(\boldsymbol{q})\dot{q}_i\dot{q}_j$$

If we expand the Kinetic energy around $\boldsymbol{q}=0$

$$T=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}(\boldsymbol{q}=0)\dot{q}_i\dot{q}_j+\ldots\ldots \quad =\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}\dot{q}_i\dot{q}_j+\ldots\ldots$$

Now we have approximate form of the kinetic energy and potential energy around equilibrium position

$$T=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}\dot{q}_i\dot{q}_j \qquad V=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}q_i q_j$$

A close inspection of the form of the kinetic energy and potential energy shows that can be expressed in the matrix form

$$T=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}\dot{q}_i\dot{q}_j=\dot{\boldsymbol{q}}'\boldsymbol{T}\dot{\boldsymbol{q}}$$

$$T=\frac{1}{2}\begin{bmatrix}\dot{q}_1 & \dot{q}_2 & \dot{q}_3 & \cdots & \dot{q}_n\end{bmatrix}\begin{bmatrix} t_{11} & t_{12} & t_{13} & \cdots & t_{1n}\\ t_{21} & t_{22} & t_{23} & \cdots & t_{2n}\\ t_{31} & t_{32} & t_{33} & \cdots & t_{3n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ t_{n1} & t_{n2} & t_{n3} & \cdots & t_{nn}\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\\ \dot{q}_3\\ \cdots\\ \dot{q}_n\end{bmatrix}$$

$$V=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}q_i q_j=\boldsymbol{q}'\boldsymbol{V}\boldsymbol{q}$$

$$V=\frac{1}{2}\begin{bmatrix}q_1 & q_2 & q_3 & \cdots & q_n\end{bmatrix}\begin{bmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1n}\\ v_{21} & v_{22} & v_{23} & \cdots & v_{2n}\\ v_{31} & v_{32} & v_{33} & \cdots & v_{3n}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ v_{n1} & v_{n2} & v_{n3} & \cdots & v_{nn}\end{bmatrix}\begin{bmatrix}q_1\\ q_2\\ q_3\\ \cdots\\ q_n\end{bmatrix}$$

With approximate values of the potential and kinetic energy we may rewrite the Lagrange's equation of motion

$$\left(\sum_{i=1}^{n}\left(\frac{d\,t_{ij}(\boldsymbol{q})}{dt}\dot{q}_i+t_{ij}(\boldsymbol{q})\ddot{q}_i\right)\right)-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial t_{ij}(\boldsymbol{q})}{\partial q_j}\dot{q}_i\dot{q}_j=-\frac{\partial V}{\partial q_j}$$

$$\frac{\partial V}{\partial q_j}=\sum_{i=1}^{n}v_{ij}q_i$$

$$\left(\sum_{i=1}^{n}\left(\cancelto{0}{\frac{d\,t_{ij}}{dt}}\dot{q}_i+t_{ij}\ddot{q}_i\right)\right)-\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\cancelto{0}{\frac{\partial t_{ij}}{\partial q_j}}\dot{q}_i\dot{q}_j=-\sum_{i=1}^{n}v_{ij}q_i$$

$$\sum_{i=1}^{n}t_{ij}\ddot{q}_i+v_{ij}q_i=0$$

There are n such equations are there, these may be expressed in matrix form

$$\boldsymbol{T}\ddot{\boldsymbol{q}}+\boldsymbol{V}\boldsymbol{q}=0$$

Linearized coupled oscillations

$$\boldsymbol{T}\cdot\ddot{\boldsymbol{q}}+\boldsymbol{V}\cdot\boldsymbol{q}=0$$

These are n coupled second order differential equations satisfied by equations for the generalized coordinates

Assuming that coordinates undergo harmonic oscillations- we use a trial solution for the normal modes.

$$q_j=a_j\cos(\omega t-\gamma)$$

This oscillation gives normal mode of the system. In the normal mode vibration all the coordinates vary with same frequency ω and phase γ . But different coordinate have different amplitudes during the oscillation.

In the matrix form

$$\boldsymbol{q}=\boldsymbol{a}\cos(\omega t-\gamma)$$

$$\boldsymbol{T}\cdot\ddot{\mathbf{q}}+\boldsymbol{V}\cdot\boldsymbol{q}=0 \qquad\qquad \boldsymbol{q}=\boldsymbol{a}\cos(\omega t-\gamma)$$

Substituting in the matrix equation

$$\sum_{i=1}^{n}\left(v_{ji}-\omega^2 t_{ji}\right)a_i=0$$

In the matrix form $\left(\boldsymbol{V}-\omega^2\boldsymbol{T}\right)\cdot\boldsymbol{a}=0$

There are n algebraic equation coordinate amplitudes

In order to obtain the non-trivial solution of the problem the determinant of the matrix should vanish that is

$$\left|\boldsymbol{V}-\omega^2\boldsymbol{T}\right|=0$$

For n generalized coordinates we get n roots for this equation, the solutions are eigenfrequencies of the system