## 0.1 Matrices of Linear Transformations

**Definition** Let $T : V \to W$ be a linear transformation from an $n$-dimensional vector space $V$ to an $m$-dimensional vector space $W$. Take ordered bases $B_1 = \{v_1, v_2, \dots, v_n\}$ for $V$ and $B_2 = \{w_1, w_2 ..., w_m\}$ for $W$. Then each vector $T(v_i)$ in $W$ is expressed uniquely as a linear combination of the vectors $w_1, w_2 ..., w_m$ in the basis $B_2$ for $W$, say

$$\begin{aligned} T(v_1) &= a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m \\ T(v_2) &= a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ T(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m \end{aligned}$$

or, in short form

$$T(v_j) = \sum_{i=1}^{m} a_{ij}w_i, 1 \le j \le n$$

for some scalars $a_{ij}, (i = 1, ..., m; j = 1, ..., n)$.

**Remark:** Notice the indexing order of $a_{ij}$ in this expression.

**Theorem** Let $T : V \to W$ be a linear transformation from an $n$-dimensional vector space $V$ to an $m$-dimensional vector space $W$. For fixed ordered bases $B_1$ for $V$ and $B_2$ for $W$, the coordinate vector $[T(v)]_{B_2}$ of $T(v)$ with respect to $B_2$ is given as a matrix product of the associated matrix $[T]_{B_1}^{B_2}$ of $T$ and $[v]_{B_1}$ ie.,

$$[T(v)]_{B_2} = [T]_{B_1}^{B_2}\ [v]_{B_1}$$

The associated matrix $[T]_{B_1}^{B_2}$ is given as

$$[T]_{B_1}^{B_2} = [[T(v_1)]_{B_2}\ [T(v_2)]_{B_2}\ \dots\ [T(v_n)]_{B_2}]$$

The coordinate vector $[T(v)]_{B_2}$ of $T(v)$ with respect to the basis $B_2$ can be written as a column vector

$$[T(v_j)]_{B_2} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

Now for any vector $v \in V$, we have scalars $x_1, x_2 ..., x_n$ such that

$$\begin{aligned} v &= \sum_{j=1}^{n} \alpha_j v_j \\ \Rightarrow T(v) &= T(\sum_{j=1}^{n} \alpha_j v_j) \\ &= \sum_{j=1}^{n} \alpha_j T(v_j) \\ &= \sum_{j=1}^{n} \alpha_j \sum_{i=1}^{m} a_{ij} w_i \\ &= \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \alpha_j a_{ij} \right) w_i \\ &= \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} \alpha_j \right) w_i \end{aligned}$$

Therefore, the coordinate vector of $T(v)$ with respect to the basis $B_2$ is

$$[T(v)]_{B_2} = \begin{bmatrix} \sum_{j=1}^{n} a_{1j}\alpha_j \\ \sum_{j=1}^{n} a_{2j}\alpha_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj}\alpha_j \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = [T]_{B_1}^{B_2}\ [v]_{B_1}$$

where $[v]_{B_1} = [x_1, x_2, \dots, x_n]^T$ is the coordinate vector of $v$ with respect to the basis $B_1$ in $V$. (In this sense, we say that matrix multiplication by $A$ represents the transformation $T$.)

**Remark:** Note that $A = [a_{ij}]_{m \times n} = [T]_{B_1}^{B_2}$ is the matrix whose column vectors are just the coordinate vectors $[T(v)]_{B_2}$ of $T(v)$ with respect to the basis $B_2$.

**Remark:** Moreover, for the fixed bases $B_1$ for $V$ and $B_2$ for $W$, the matrix $A$ associated with the linear transformation $T$ with respect to these bases is unique, because the coordinate expression of a vector with respect to a basis is unique. Thus, the assignment of the matrix $A$ to a linear transformation $T$ is well-defined.

**Definition** The matrix $A$ is called the associated matrix for $T$ (or matrix representation of $T$) with respect to the bases $B_1$ and $B_2$, and denoted by $A = [T]_{B_1}^{B_2}$.

**Example** Let $T : \mathbb{R}^2 \to \mathbb{R}$ be the linear transformation defined by $T(x, y) = (x + 2y, 0, 2x + 3y)$ with respect to the standard bases $B_1$ and $B_2$ for $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Then

$$\begin{aligned} T(e_1) &= T(1, 0) = (1, 0, 2) = 1e_1 + 0e_2 + 2e_3, \\ T(e_2) &= T(0, 1) = (2, 0, 3) = 2e_1 + 0e_2 + 3e_3. \end{aligned}$$

Hence, $[T]_{B_1}^{B_2} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 2 & 3 \end{bmatrix}$.

If $B_{2'} = \{e_3, e_2, e_1\}$, then $[T]_{B_1}^{B_2'} = \begin{bmatrix} 2 & 3 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}$.

**Example** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator defined by $T(x, y) = (2x + 3y, 4x - 5y)$. Then

1. Find the matrix representation of $T$ relative to the basis $B = \{u_1, u_2\} = \{(1, 2), (2, 5)\}$.
2. Find the matrix representation of $T$ relative to the (usual) basis $E = \{e_1, e_2\} = \{(1, 0), (0, 1)\}$.

**Solution**

1. 
   - First we find $T(u_1)$, and then write it as a Linear combination of the basis vectors $u_1$ and $u_2$. (For notational convenience, we use column vectors.)
     We have $T(u_1) = T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -6 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
     $\Longrightarrow \begin{array}{rr} x + 2y = & 8 \\ 2x + 5y = & -6 \end{array}$
     Solving the system we get, $x = 52, y = -22$. Hence $T(u_1) = 52u_1 - 22u_2$.
   - Next we find $T(u_2)$, and then write it as a Linear combination of the basis vectors $u_1$ and $u_2$.
     We have $T(u_2) = T\left(\begin{bmatrix} 2 \\ 5 \end{bmatrix}\right) = \begin{bmatrix} 19 \\ -17 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
     $\Longrightarrow \begin{array}{rr} x + 2y = & 19 \\ 2x + 5y = & -17 \end{array}$
     Solving the system we get, $x = 129, y = -55$. Hence $T(u_2) = 129u_1 - 55u_2$.
     Now we write coordinate of $T(u_1)$ and $T(u_2)$ as column to get the matrix representation of $T$corresponding to the basis $B = \{u_1, u_2\}$
     $\Longrightarrow [T]_B = \begin{bmatrix} 52 & 129 \\ -22 & -55 \end{bmatrix}$.

2. Now we find $T(e_1)$, and then write it as a Linear combination of the usual basis vectors $e_1$ and $e_2$ and then find $T(e_2)$,and then write it as a Linear combination of the usual basis vectors $e_1$ and $e_2$
   $T(e_1) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2T(e_1) + 4T(e_2)$
   $T(e_2) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -5 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ - $5\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 3T(e_1) - 5T(e_2)$
   Now we write coordinate of $T(e_1)$ and $T(e_2)$ as column to get the matrix representation of $T$ with respect to the standard basis $E = \{e_1, e_2\}$
   $\Longrightarrow [T]_E = \begin{bmatrix} 2 & 3 \\ 4 & -5 \end{bmatrix}$.

**Example:** Let $V$ be the vector space of functions with basis $B = \{sint, cost, e^{3t}\}$, and let $T : V \to V$ be the differential operator defined by $T(f(t)) = \dfrac{d(f(t))}{dt}$. Fins the matrix representation of $T$.

**Solution:** We compute the matrix representing $T$ in the basis $B$.

$$\begin{array}{rrr} D(sin(t)) = & cos(t) = & 0(sin(t)) + 1(cos(t)) + 0(e^{3t}) \\ D(cos(t)) = & -sin(t) = & -1(sin(t)) + 1(cos(t)) + 0(e^{3t}) \\ D(e^{3t}) = & 3e^{3t} = & 0(sin(t)) + 0(cos(t)) + 3(e^{3t}) \end{array}$$

$$\Longrightarrow \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

**Remark:** Note that the coordinates of $T(sin(t)), T(cos(t)), T(e^{3t})$ form the columns, not the rows, of $[T]_B$.

**Matrix Mappings and Their Matrix Representation**

**Example:** Consider the following matrix $A$, which may be viewed as a linear operator on $\mathbb{R}^2$, and basis $B$ of $\mathbb{R}^2$ defined as $A = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}$, $B = \{u_1, u_2\} = \left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}\right\}$

Find the matrix representation of $A$ relative to the basis $B$.

**Solution:** First we write $A(u_1)$ as a linear combination of $u_1$ and $u_2$. We have, $A(u_1) = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -6 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\Longrightarrow \begin{array}{rr} x + 2y = & -1 \\ 2x + 5y = & -6 \end{array}$

Solving the above system we get, $x = 7, y = -4$. Hence $A(u_1) = 7u_1 - 4u_2 = 7\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ - $4\begin{bmatrix} 2 \\ 5 \end{bmatrix}$. Hence coordinate vectors of $Au_1$ corresponding to basis $B$ are given by $\begin{bmatrix} 7 \\ -4 \end{bmatrix}$.

Next we write $A(u_2)$ as a linear combination of $u_1$ and $u_2$. We have, $A(u_2) = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \end{bmatrix} = x\begin{bmatrix} 1 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}$

$\Longrightarrow \begin{array}{rr} x + 2y = & -4 \\ 2x + 5y = & -7 \end{array}$

Solving the above system we get, $x=-6, y=1$. Hence $A(u_2)=-6u_1+u_2=-6\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2\\5\end{bmatrix}$ Hence coordinate vectors of $Au_2$ corresponding to basis $B$ are given by $\begin{bmatrix}-6\\1\end{bmatrix}$.

Writing the coordinate vectors of $A(u_1)$ and $A(u_2)$ as column vectors, we get the matrix representation of $A$. $\Longrightarrow [A]_B = \begin{bmatrix}7 & -6\\-4 & 1\end{bmatrix}$.

**Remark:** Suppose we want to find the matrix representation of $A$ relative to the usual basis $E=\{e_1,e_2\}=\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$ of $\mathbb{R}^2$. We have

$$A(e_1)=\begin{bmatrix}3 & -2\\4 & -5\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}3\\4\end{bmatrix}=3\begin{bmatrix}1\\0\end{bmatrix}+4\begin{bmatrix}0\\1\end{bmatrix}=3e_1+4e_2$$
$$A(e_2)=\begin{bmatrix}3 & -2\\4 & -5\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-2\\-5\end{bmatrix}=-2\begin{bmatrix}1\\0\end{bmatrix}\text{ - }5\begin{bmatrix}0\\1\end{bmatrix}=-2e_1-5e_2$$

Writing the coordinate vectors of $A(e_1)$ and $A(e_2)$ as column vectors, we get the matrix representation of $A$. $\Longrightarrow [A]_E = \begin{bmatrix}3 & -2\\4 & -5\end{bmatrix}$.

Note that $[A]_E$ is the original matrix $A$. This result is true in general: The matrix representation of any $n\times n$ square matrix $A$ over a field of real numbers relative to the usual basis $E$ of $\mathbb{R}^n$ is the matrix $A$ itself; that is, $[A]_E = A$

**Steps for finding matrix representations**

The first Step 1 is optional. It may be useful to use it in Step $2(b)$, which is repeated for each basis vector. The input is a linear operator $T$ on a vector space $V$ and a basis $B=\{u_1,u_2,\dots,u_n\}$ of $V$. The output is the matrix representation $[T]_B$ of $T$.

1. Find a formula for the coordinates of an arbitrary vector $V$ relative to the basis $B$.
2. Repeat for each basis vector $u_i$ $i=1,2,\dots n$ in $B$:
   (a) Find $T(u_i)$.
   (b) Write $T(u_i)$ as a linear combination of the basis vectors $u_1,u_2,\dots,u_n$.
3. Form the matrix $[T]_B$ whose columns are the coordinate vectors in Step 2(b).

**Example:** Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be defined by $T(x,y)=(2x+3y,4x-5y)$, relative to the basis $B=\{u_1,u_2\}=\left\{\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}2\\-5\end{bmatrix}\right\}$

**Solution:**

**(Step 1)** First find the coordinates of any arbitrary vector $(a,b)\in\mathbb{R}^2$ relative to the basis $B$. We have

$$\begin{bmatrix}a\\b\end{bmatrix}=x\begin{bmatrix}1\\-2\end{bmatrix}+y\begin{bmatrix}2\\-5\end{bmatrix}\Rightarrow \begin{matrix}x+2y= & a\\-2x-5y= & b\end{matrix}\Rightarrow\begin{matrix}x+2y= & a\\-y= & 2a+b\end{matrix}$$

Solving for $x$ and $y$ in terms of $a$ and $b$ yields $\begin{matrix}x= & 5a+2b\\y= & -2a-b\end{matrix}$. Thus

$$\begin{bmatrix}a\\b\end{bmatrix}=(5a+2b)\begin{bmatrix}1\\2\end{bmatrix}+(-2a-b)\begin{bmatrix}2\\5\end{bmatrix}$$

**(Step 2)** Now we find $T(u_1)$ and write it as a linear combination of $u_1$ and $u_2$ using the above formula for $\begin{bmatrix}2\\5\end{bmatrix}$, and then we repeat the process for $T(u_2)$. We have $T(u_1)=T(\left(\begin{bmatrix}1\\-2\end{bmatrix}\right)=\begin{bmatrix}-4\\-14\end{bmatrix}=8\begin{bmatrix}1\\2\end{bmatrix}\text{ - }6\begin{bmatrix}2\\5\end{bmatrix}=8u_1-6u_2$

$$T(u_2)=T(\left(\begin{bmatrix}2\\-5\end{bmatrix}\right)=\begin{bmatrix}-11\\33\end{bmatrix}=11\begin{bmatrix}1\\2\end{bmatrix}\text{ - }11\begin{bmatrix}2\\5\end{bmatrix}=11u_1-11u_2$$

**(Step 3)** Finally, we write the coordinates of $T(u)_1$ and $T(u)_2$ as columns to obtain the required matrix $[T]_B$ as

$\Longrightarrow [A]_B=\begin{bmatrix}8 & 11\\-6 & -11\end{bmatrix}$.

**Example :** Consider the linear operator $T$ on $\mathbb{R}^2$ and the basis $T$ of last example that is, $T(x,y)=(2x+3y,4x-5y)$ and $B=\{u_1,u_2\}=\left\{\begin{bmatrix}1\\-2\end{bmatrix},\begin{bmatrix}2\\-5\end{bmatrix}\right\}$, Let $v=\begin{bmatrix}5\\-7\end{bmatrix}$, and so $T(v)=\begin{bmatrix}11\\-55\end{bmatrix}$ Using the formula from last example, we get $[v]_B=\begin{bmatrix}11\\-3\end{bmatrix}$ and $[T(v)]_B=\begin{bmatrix}55\\-33\end{bmatrix}$ verify $[T]_B[v]_B=[T(v)]_B$ for this vector $v$ (where $[T]$ is obtained from last example).

$$[T]_B[v]_B=\begin{bmatrix}8 & 11\\-6 & -11\end{bmatrix}\begin{bmatrix}11\\-3\end{bmatrix}=\begin{bmatrix}55\\-33\end{bmatrix}=[T(v)]_B.$$