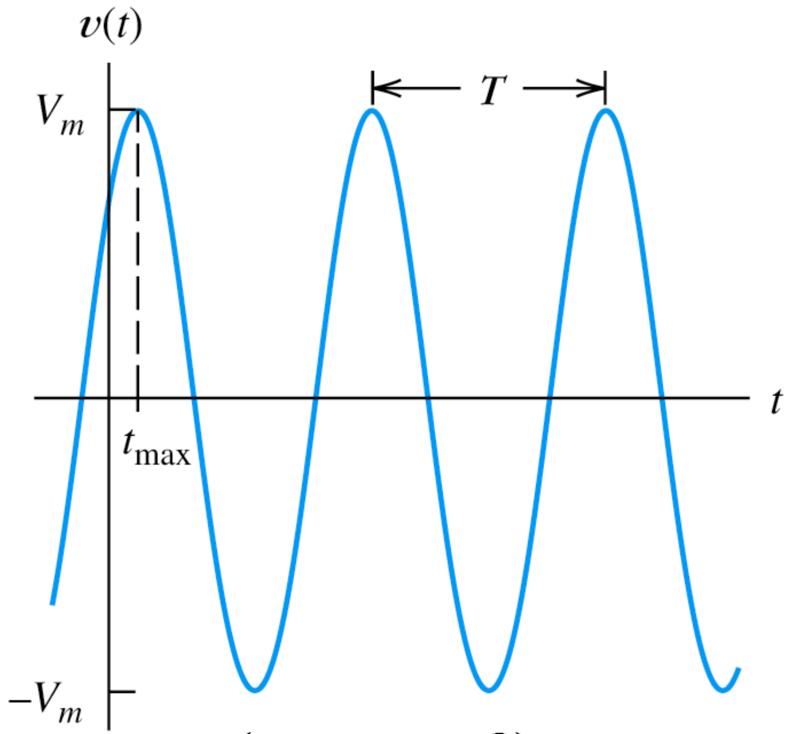
STEADY STATE SINUSOIDAL ANALYSIS

Overview

1. Identify the frequency, angular frequency, peak value, rms value, and phase of a sinusoidal signal.
2. Solve steady-state ac circuits using phasors and complex impedances.
3. Compute power for steady-state ac circuits.
4. Find Thévenin and Norton equivalent circuits.
5. Determine load impedances for maximum power transfer.
6. Solve balanced three-phase circuits.

Sinusoidal Waveform



$$v(t) = V_m \cos(\omega t + \theta)$$

SINUSOIDAL CURRENTS AND VOLTAGES

- V_m is the **peak value**
- ω is the **angular frequency** in radians per second
- θ is the **phase angle**
- T is the **period**

Frequency

$$f = \frac{1}{T}$$

Angular frequency

$$\omega = \frac{2\pi}{T}$$

$$\omega = 2\pi f$$

$$\sin(z) = \cos(z - 90^{\circ})$$

Root-Mean-Square Values

$$V_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$$

$$I_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)dt}$$

$$P_{\mathrm{avg}} = \frac{V_{\mathrm{rms}}^2}{R}$$

$$P_{\mathrm{avg}} = I_{\mathrm{rms}}^2 R$$

RMS Value of a Sinusoid

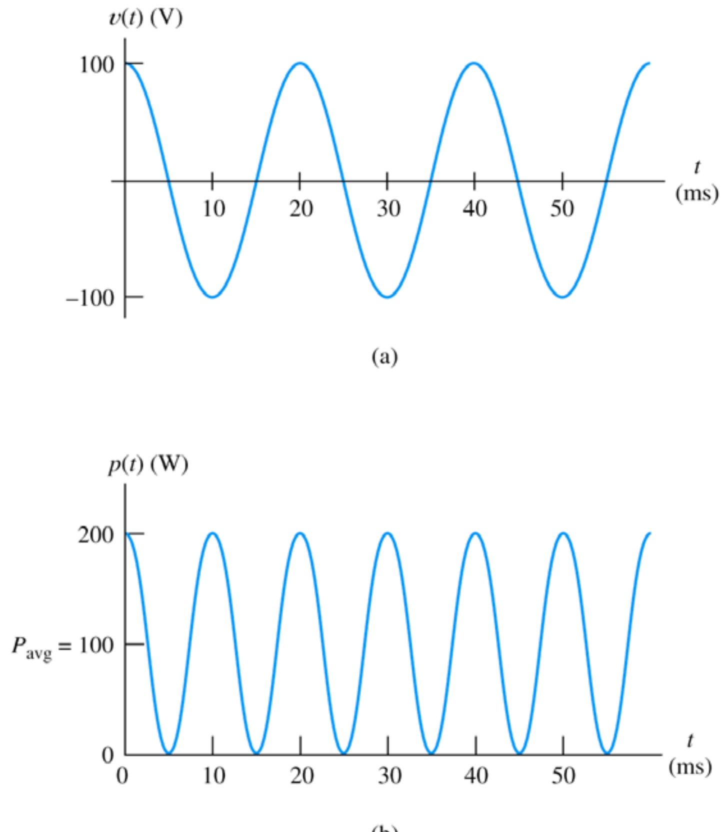
$$V_{\text{rms}} = \frac{V_m}{\sqrt{2}}$$

The rms value for a sinusoid is the peak value divided by the square root of two. This is not true for other periodic waveforms such as square waves or triangular waves.

Note: The typical voltage value quoted is the rms, not the peak value.

Example Exercise

- $V(t)=100\cos(100\pi t)$ to 50 ohm resistance



(b)

Phasor Definition

$$\text{Time function}: \quad v_1(t) = V_1 \cos(\omega t + \theta_1)$$

$$\text{Phasor}: \quad \mathbf{V}_1 = V_1 \angle \theta_1$$

Adding Sinusoids Using Phasors

Step 1: Determine the phasor for each term.

Step 2: Add the phasors using complex arithmetic.

Step 3: Convert the sum to polar form.

Step 4: Write the result as a time function.

Using Phasors to Add Sinusoids

$$v_1(t) = 20\cos(\omega t - 45^\circ)$$

$$v_2(t) = 10\sin(\omega t + 60^\circ)$$

$$\mathbf{V}_1 = 20\angle -45^\circ$$

$$\mathbf{V}_2 = 10\angle -30^\circ$$

$$\begin{aligned}\mathbf{V}_\mathrm{s} &= \mathbf{V}_1 + \mathbf{V}_2 \\ &= 20\angle -45^\circ + 10\angle -30^\circ \\ &= 14.14 - j14.14 + 8.660 - j5 \\ &= 23.06 - j19.14 \\ &= 29.97\angle -39.7^\circ\end{aligned}$$

$$v_s(t) = 29.97\cos\left(\omega t - 39.7^\circ\right)$$

Phasors as Rotating Vectors

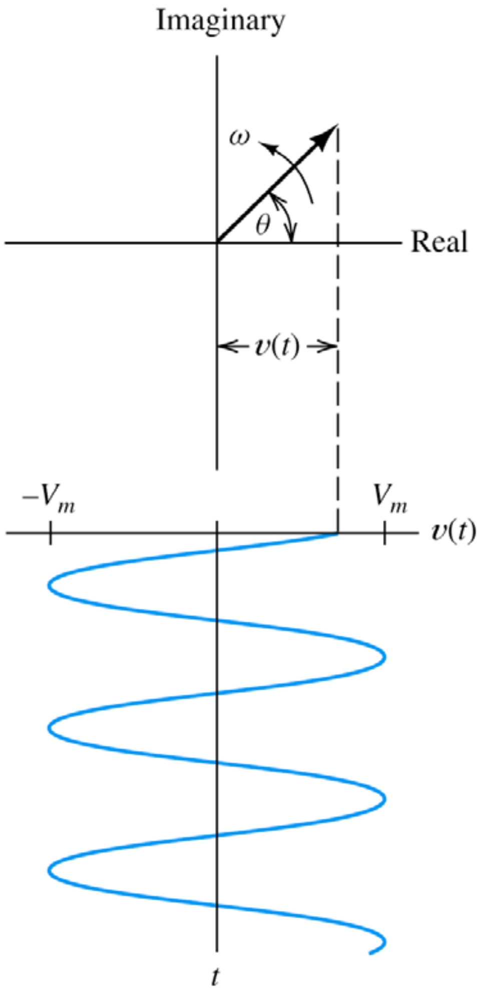


Figure 5.4 A sinusoid can be represented as the real part of a vector rotating counterclockwise in the complex plane.

Sinusoids can be visualized as the real-axis projection of vectors rotating in the complex plane. The phasor for a sinusoid is a snapshot of the corresponding rotating vector at $t = 0$.

Phase Relationships

To determine phase relationships from a phasor diagram, consider the phasors to rotate counterclockwise. Then when standing at a
fixed point, if $\mathbf{V}_1$ arrives first followed by $\mathbf{V}_2$ after a rotation of θ , we say that $\mathbf{V}_1$ leads $\mathbf{V}_2$ by θ . Alternatively, we could say that $\mathbf{V}_2$ lags $\mathbf{V}_1$ by θ . (Usually, we take θ as the smaller angle between the two phasors.)

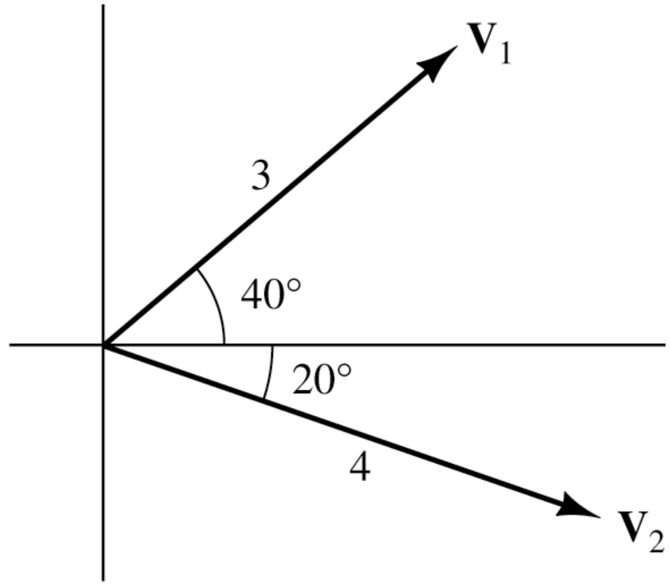


Figure 5.5 Because the vectors rotate counterclockwise, $\mathbf{v}_1$ leads $\mathbf{v}_2$ by 60° (or, equivalently, $\mathbf{v}_2$ lags $\mathbf{v}_1$ by 60°.)

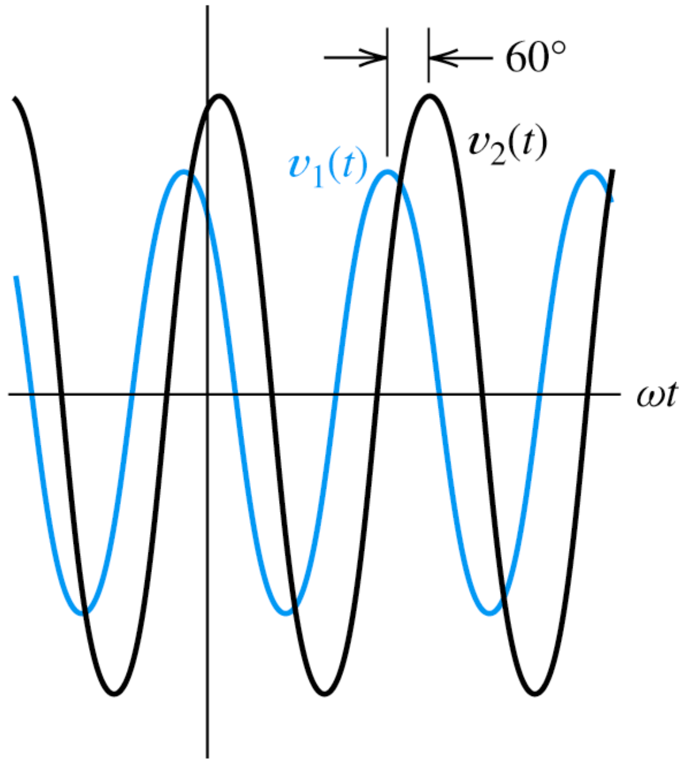
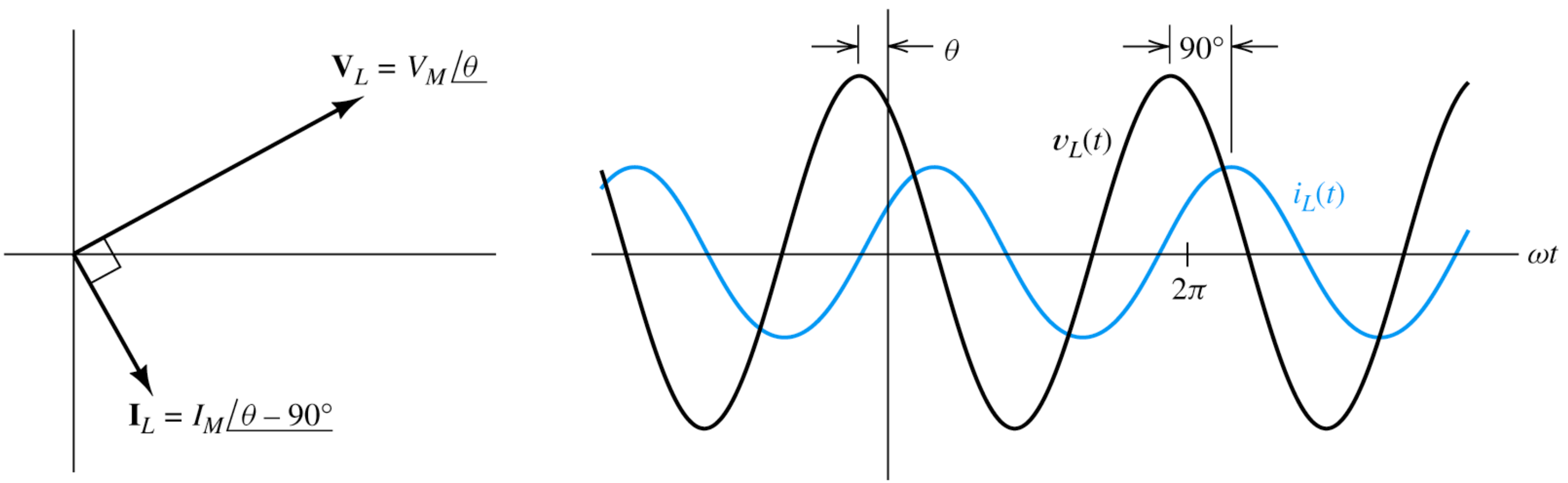


Figure 5.6 The peaks of $v_1(t)$ occur 60° before the peaks of $v_2(t)$. In other words, $v_1(t)$ leads $v_2(t)$ by 60°.

Complex Impedances

- Inductance
- Capacitance



(a) Phasor diagram

(b) Current and voltage versus time

Figure 5.7 Current lags voltage by 90° in a pure inductance.

Inductance

$$i_L(t) = I_m \sin(\omega t + \theta)$$

$$\mathbf{I}_L = I_m \angle \theta - 90^0$$

$$\mathbf{V}_L = j\omega L \times \mathbf{I}_L$$

$$Z_L = j\omega L = \omega L \angle 90^\circ$$

$$\mathbf{V}_L = Z_L \mathbf{I}_L$$

Capacitance

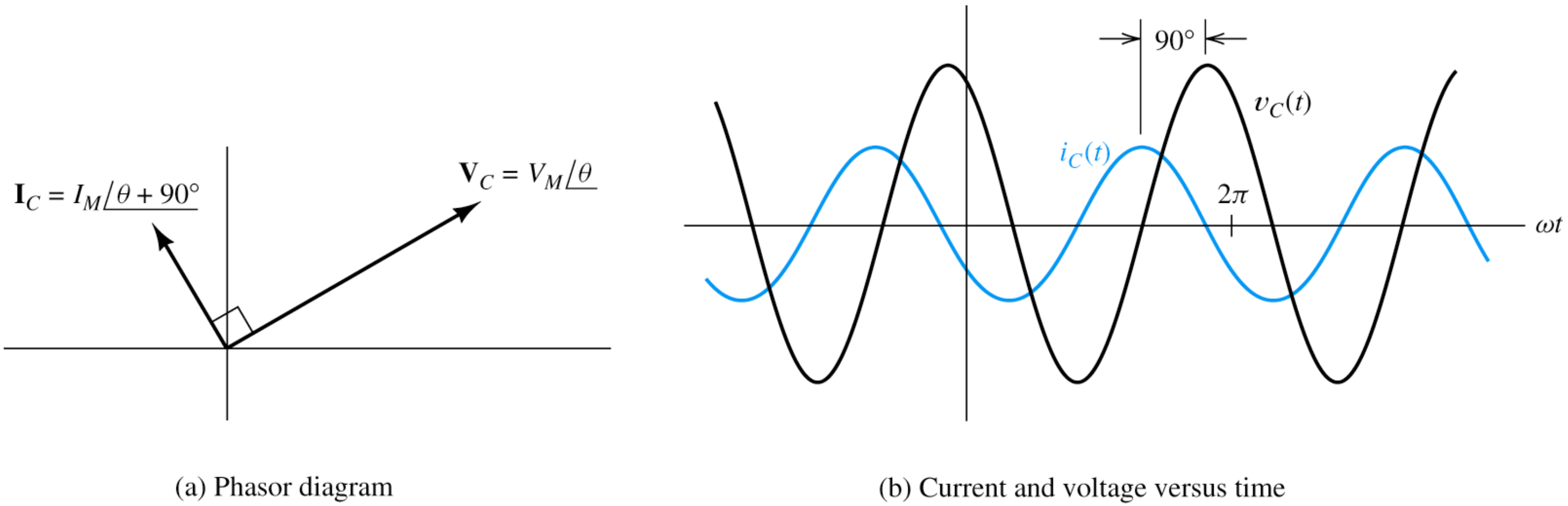
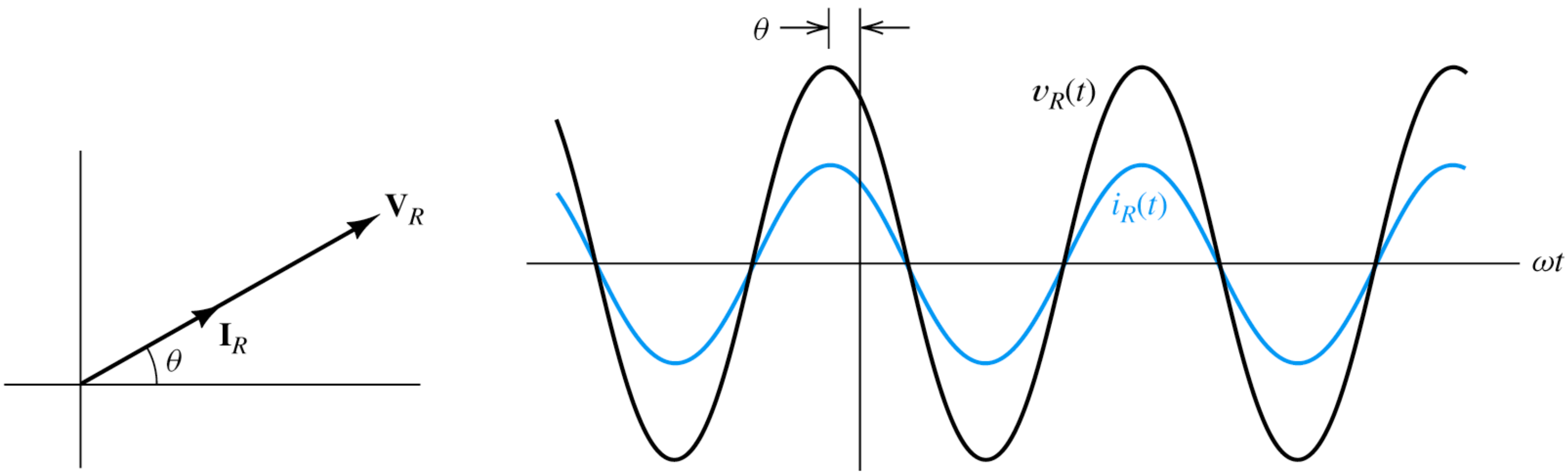


Figure 5.8 Current leads voltage by 90° in a pure capacitance.

$$\mathbf{V}_C = Z_C \mathbf{I}_C$$

$$Z_C = -j\frac{1}{\omega C} = \frac{1}{j\omega C} = \frac{1}{\omega C}\angle -90^\circ$$

Resistance



(a) Phasor diagram

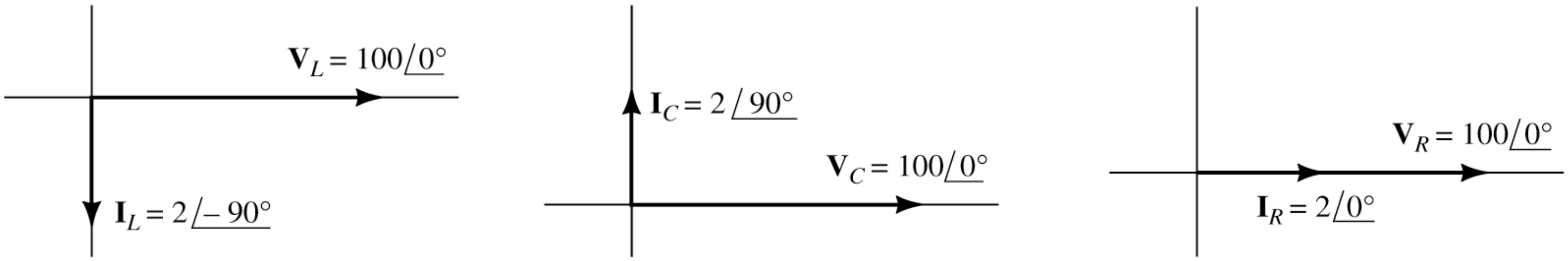
(b) Current and voltage versus time

Figure 5.9 For a pure resistance, current and voltage are in phase.

$$\mathbf{V}_R = R\mathbf{I}_R$$

Example Exercise

- V(t)=100cos(200t) to 0.25 H inductor
- V(t)=100cos(200t) to 100 microFarad capacitor
- V(t)=100cos(200t) to 50 Ohm resistance



(a) Exercise 5.6 (0.25 H inductance) (b) Exercise 5.7 (100 μF capacitance) (c) Exercise 5.8 (50 Ω resistance)

Circuit Analysis

We can apply KVL and KCL directly to phasors.

The sum of the phasor voltages equals zero for any closed path.

The sum of the phasor currents entering a node must equal the sum of the phasor currents leaving.

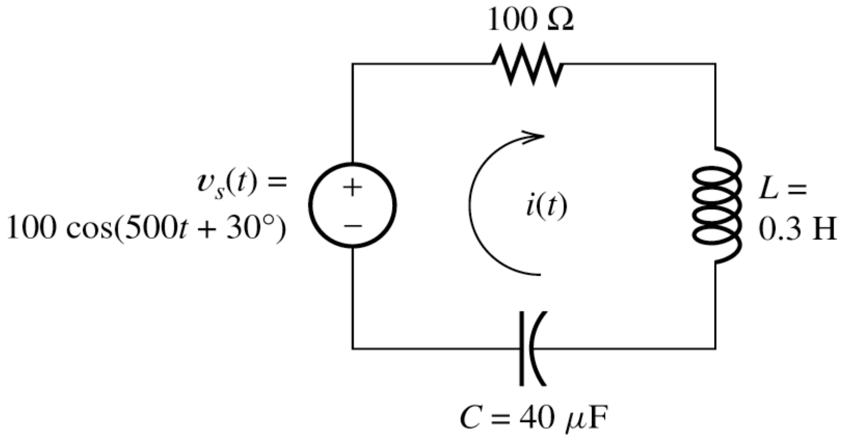
Circuit Analysis Using Phasors and Impedances

1. Replace the time descriptions of the voltage and current sources with the corresponding phasors. (All of the sources must have the same frequency.)

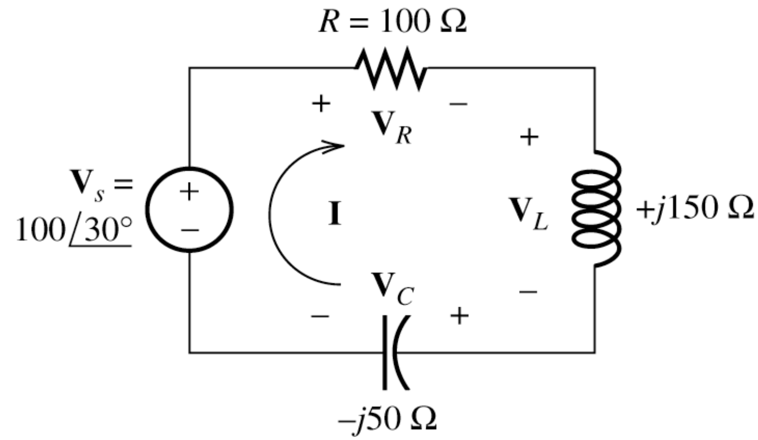
2. Replace inductances by their complex impedances $Z_L = j\omega L$. Replace capacitances by their complex impedances $Z_C = 1/(j\omega C)$. Resistances have impedances equal to their resistances.

3. Analyze the circuit using any of the techniques studied earlier, performing the calculations with complex arithmetic.

Example Exercise



(a)



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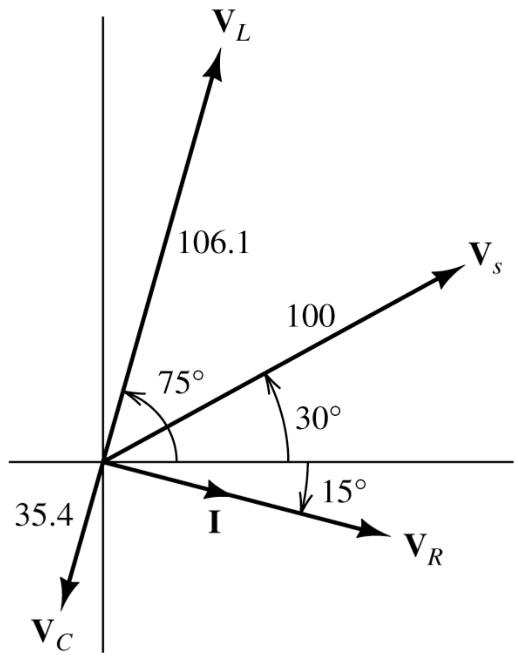


Example Exercise

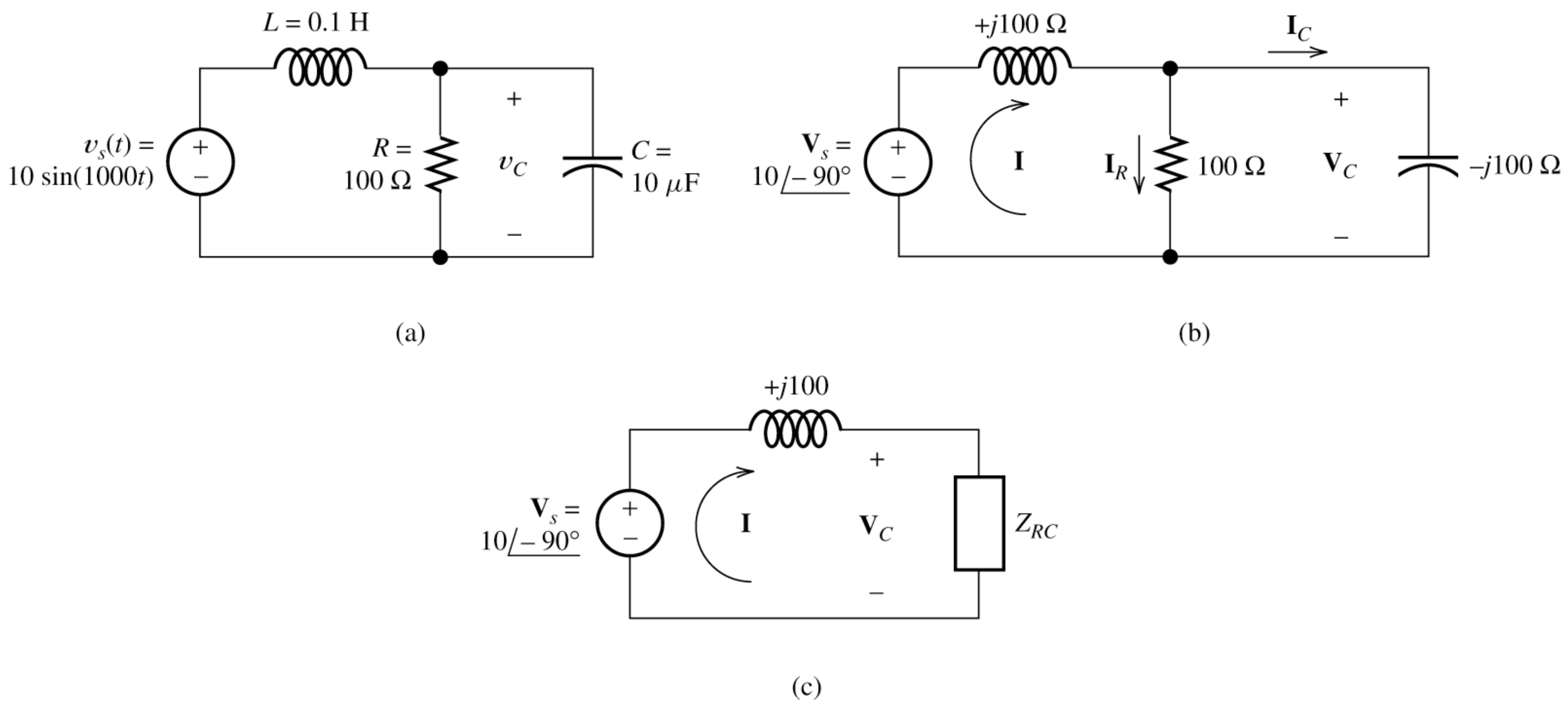


Figure 5.13 Circuit for Example 5.4.

Solution

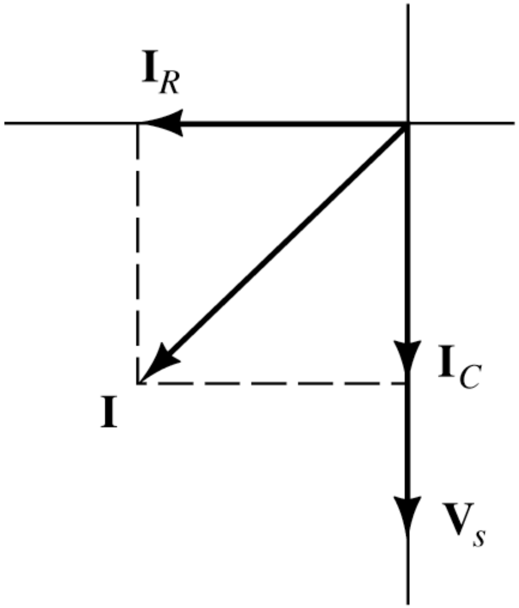


Figure 5.14 Phasor diagram for Example 5.4.

Node Voltage Analysis Exercise

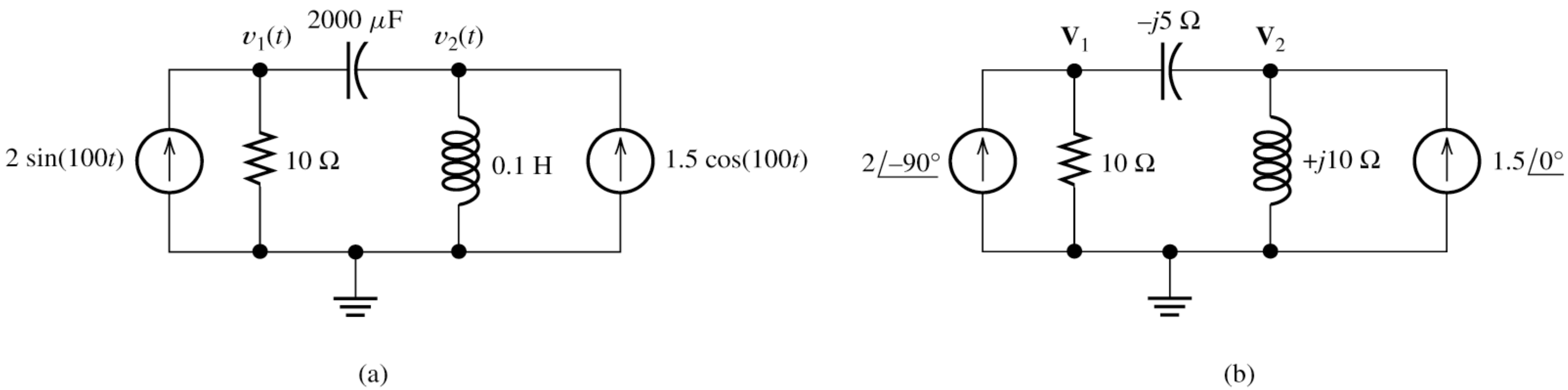
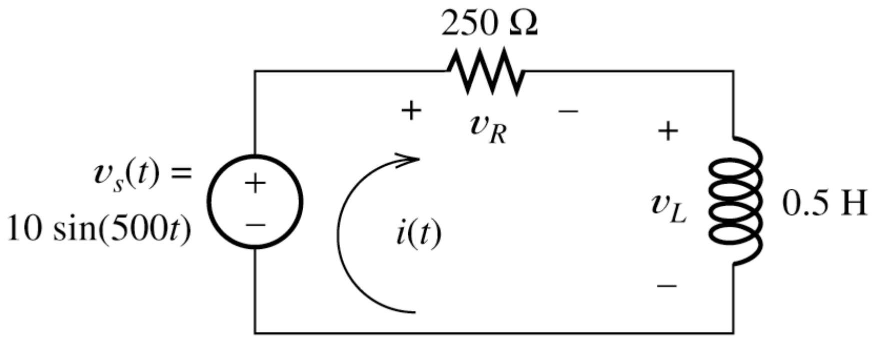
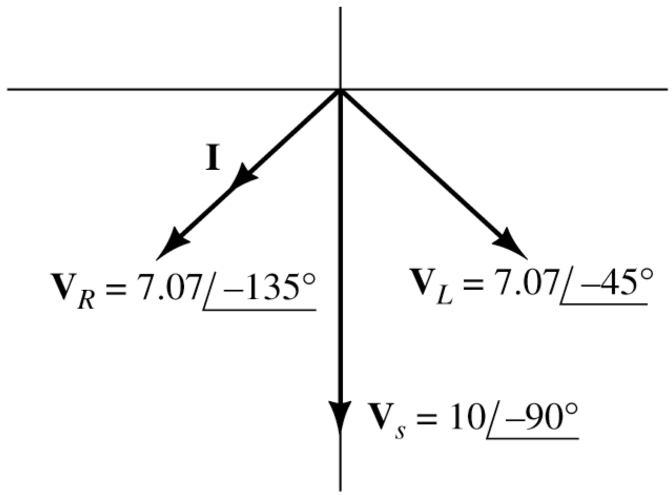


Figure 5.15 Circuit for Example 5.5.

Mesh Current Analysis Exercise





Power in AC Circuits

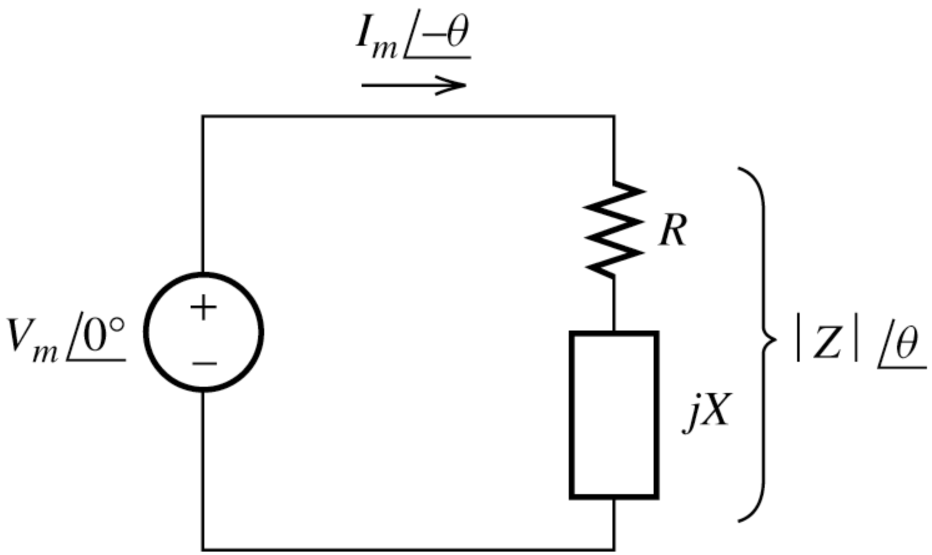


Figure 5.19 A voltage source delivering power to a load impedance $Z = R + jX$.

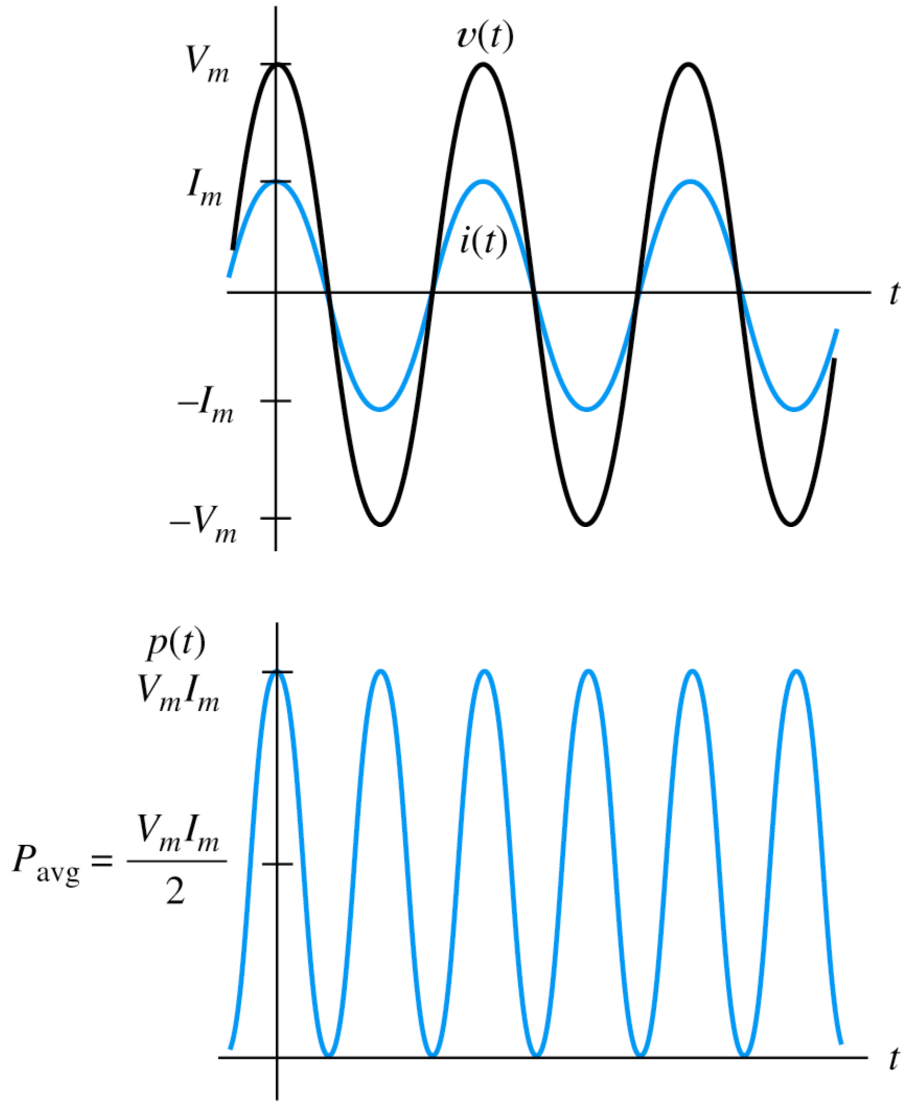
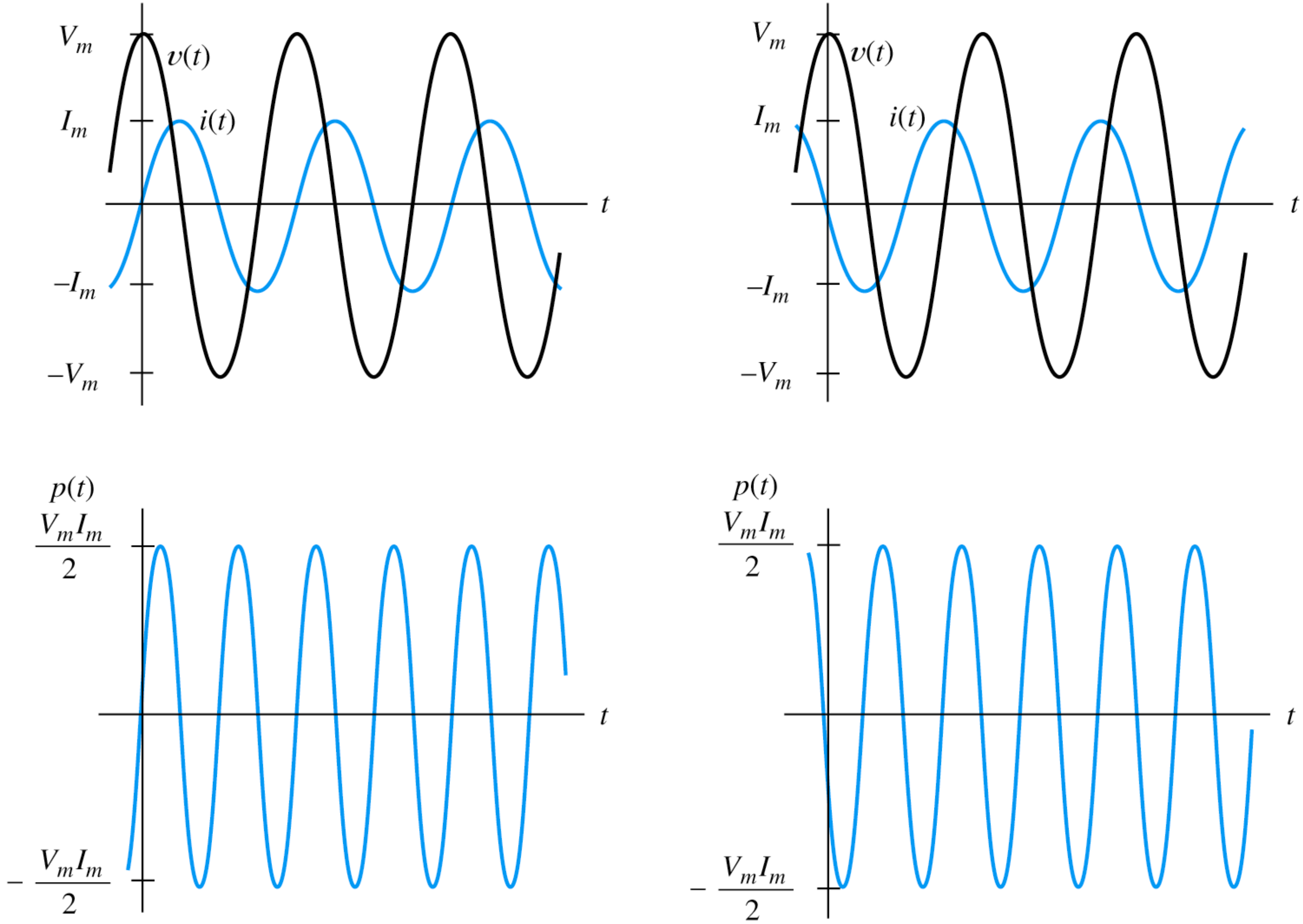


Figure 5.20 Current, voltage, and power versus time for a purely resistive load.



(a) Pure inductive load

(b) Pure capacitive load

Figure 5.21 Current, voltage, and power versus time for pure energy-storage elements.

AC Power Calculations

$$P = V_{\mathrm{rms}} I_{\mathrm{rms}} \cos(\theta)$$

$$\mathrm{PF} = \cos(\theta)$$

$$\theta = \theta_v - \theta_i$$

$$Q = V_{\mathrm{rms}} I_{\mathrm{rms}} \sin(\theta)$$

Power Factor

- "The power that continually bounces back and forth between the source and the load is known as reactive power (Q). Reactive power represents the energy that is first stored and then released in the magnetic field of an inductor, or in the electric field of a capacitor" [1].

  To add my own two cents: We generally try to keep the amount of reactive power present in a power system to a minimum. Since reactive power bounces back and forth between the load and source, it doesn't provide any energy to the load and requires more current on the lines to maintain that extra reactive power (more current, generally more resistive losses on lines).

  Inductive loads consume reactive power (positive Q), and capaciive loads produce reactive power (negative Q). Thus, we can cancel out the amount of reactive power on the lines by balancing the amount of reactive power consumed versus the amount produced (aka power factor correction).

Additional

- To explain PF is think about a horse pulling a barge along a canal.
  The horse must pull the barge from the shore; therefore it is pulling the barge at an angle to the direction of travel. Because the horse is pulling at an angle, not all of the horse's effort is used to move the barge along the canal. The effort of the horse is the total power or apparent power (kVA), the power used to move the barge is the working power or real power(kW), and the power that is trying to pull the barge to the side of the canal is nonworking power or reactive power(kVAr). The ratio of the real power to the apparent power it also known as the power factor. If the horse is led closer to the edge of the canal the angle of the rope will decrease and more of the apparent power will be used as real power, increasing the power factor.
  In the case of electricity the power factor is based on the phase difference between the current and the voltage sine waves. If the phase is zero then all the apparent power can be used as real power and the power factor is 1. This is also called "unity power factor". As the phase difference increases the power factor will decrease and more current needs to be supplied to give the same amount of real power.
  so Reactive power does no work.

$$\text{apparent power} = V_{\text{rms}} I_{\text{rms}}$$

$$P^2 + Q^2 = \left(V_{\text{rms}} I_{\text{rms}}\right)^2$$

Power Triangle

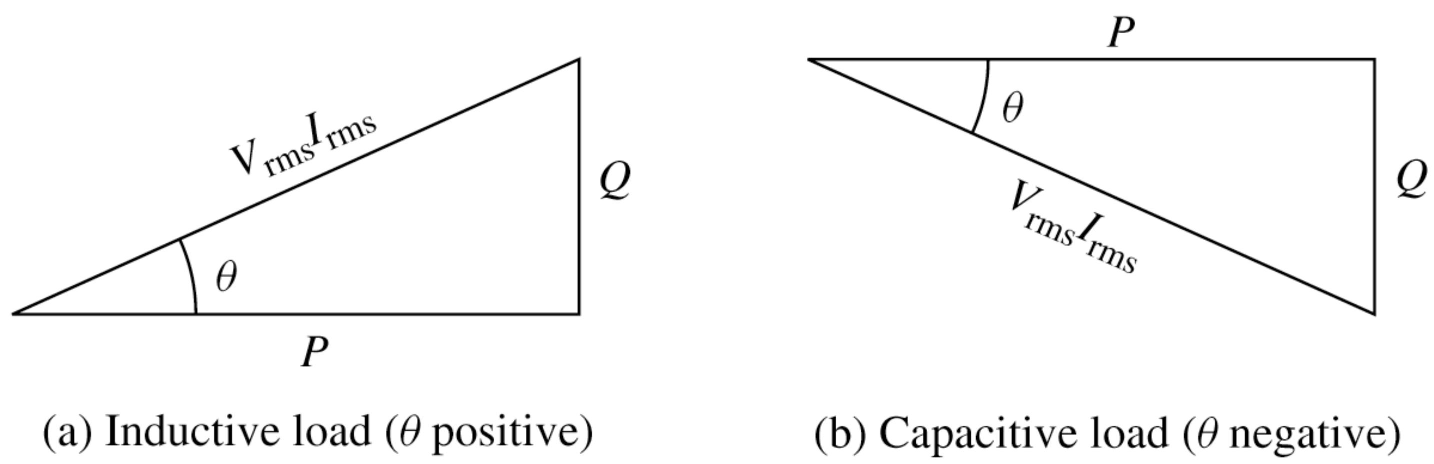


Figure 5.22 Power triangles for inductive and capacitive loads.

Additional Power Relationships

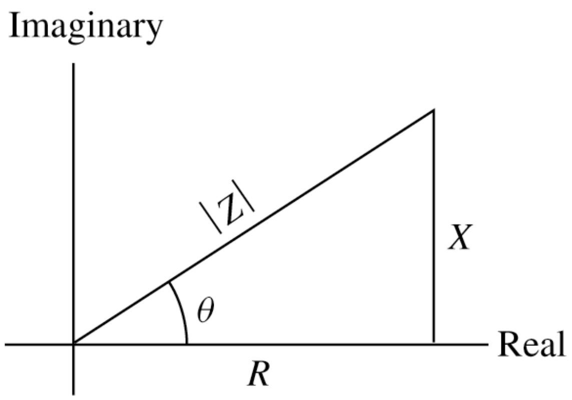


Figure 5.23 The load impedance in the complex plane.

$$P = I_{\text{rms}}^2 R$$

$$P = \frac{V_{R\text{rms}}^2}{R}$$

$$Q = I_{\text{rms}}^2 X$$

$$Q = \frac{V_{X\text{rms}}^2}{X}$$

Complex Power: $S = 1/2\,\mathbf{VI}^* = P + jQ$

Example Exercise

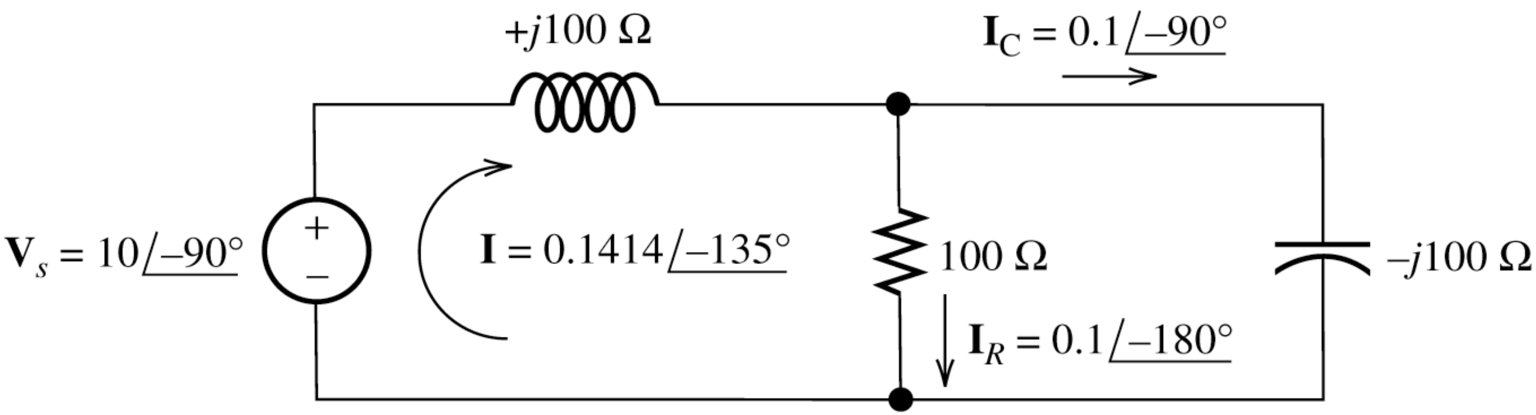


Figure 5.24 Circuit and currents for Example 5.6.

Example Exercise

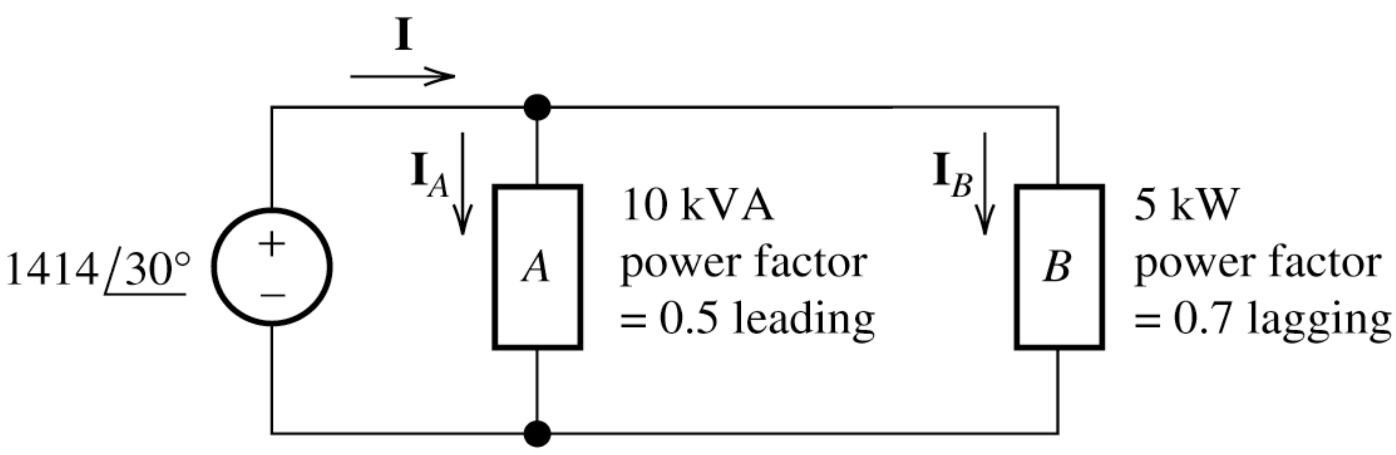


Figure 5.25 Circuit for Example 5.7.

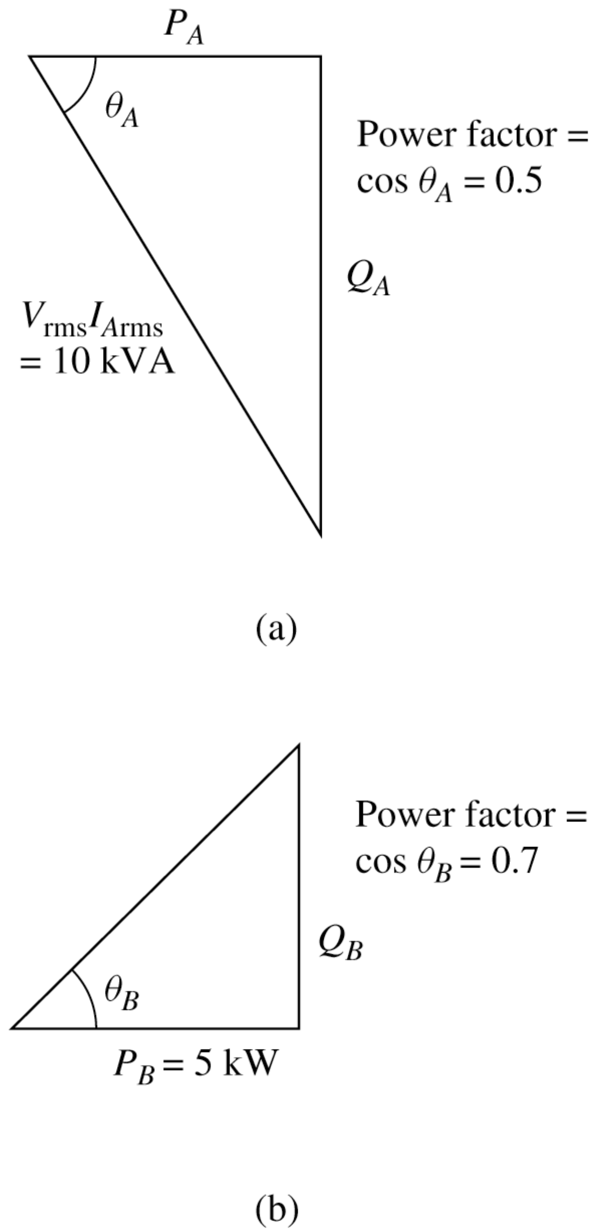


Figure 5.26 Power triangles for loads A and B of Example 5.7.

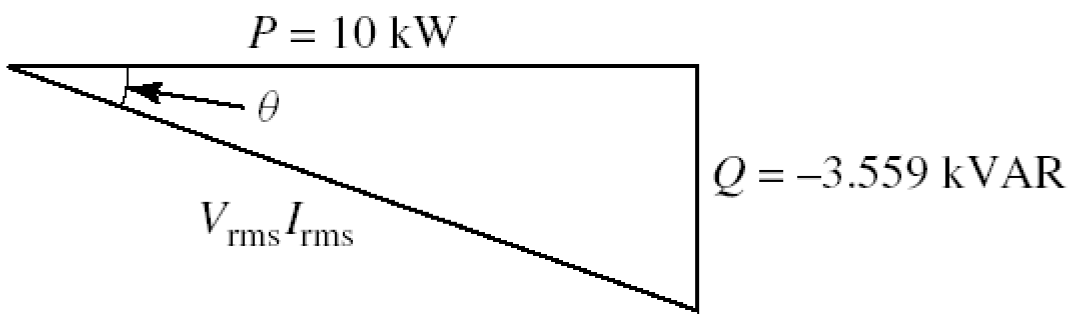


Figure 5.27 Power triangle for the source of Example 5.7.

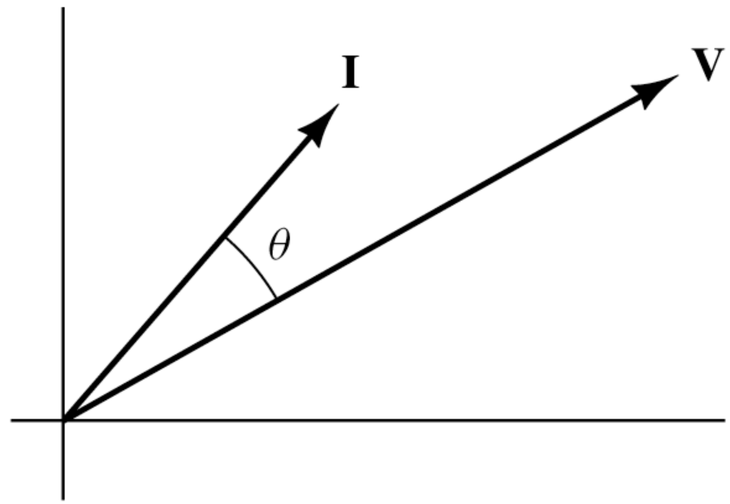


Figure 5.28 Phasor diagram for Example 5.7.

Thevenin Equivalent

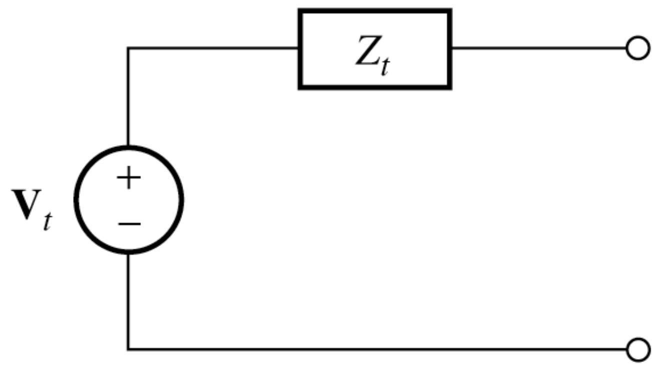


Figure 5.29 The Thévenin equivalent for an ac circuit consists of a phasor voltage source $\mathbf{v}_t$ in series with a complex impedance Z_t.

The Thévenin voltage is equal to the open-circuit phasor voltage of the original circuit.

$$\mathbf{V}_t = \mathbf{V}_{\text{oc}}$$

We can find the Thévenin impedance by zeroing the independent sources and determining the impedance looking into the circuit terminals.

The Thévenin impedance equals the open-circuit voltage divided by the short-circuit current.

$$Z_t = \frac{\mathbf{V}_{\mathrm{oc}}}{\mathbf{I}_{\mathrm{sc}}} = \frac{\mathbf{V}_t}{\mathbf{I}_{\mathrm{sc}}}$$

$$\mathbf{I}_n = \mathbf{I}_{\mathrm{sc}}$$

Norton Equivalent

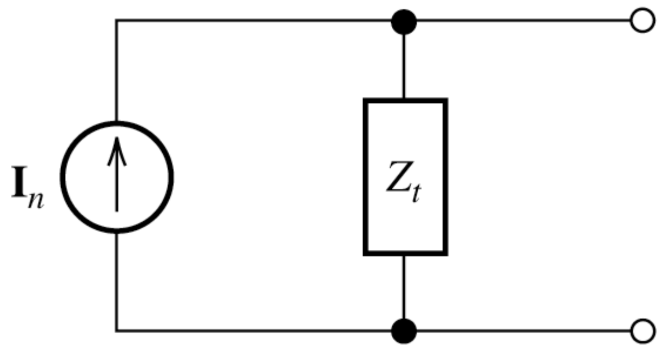


Figure 5.30 The Norton equivalent circuit consists of a phasor current source $\mathbf{I}_n$ in parallel with the complex impedance Z_t.

Example Exercise

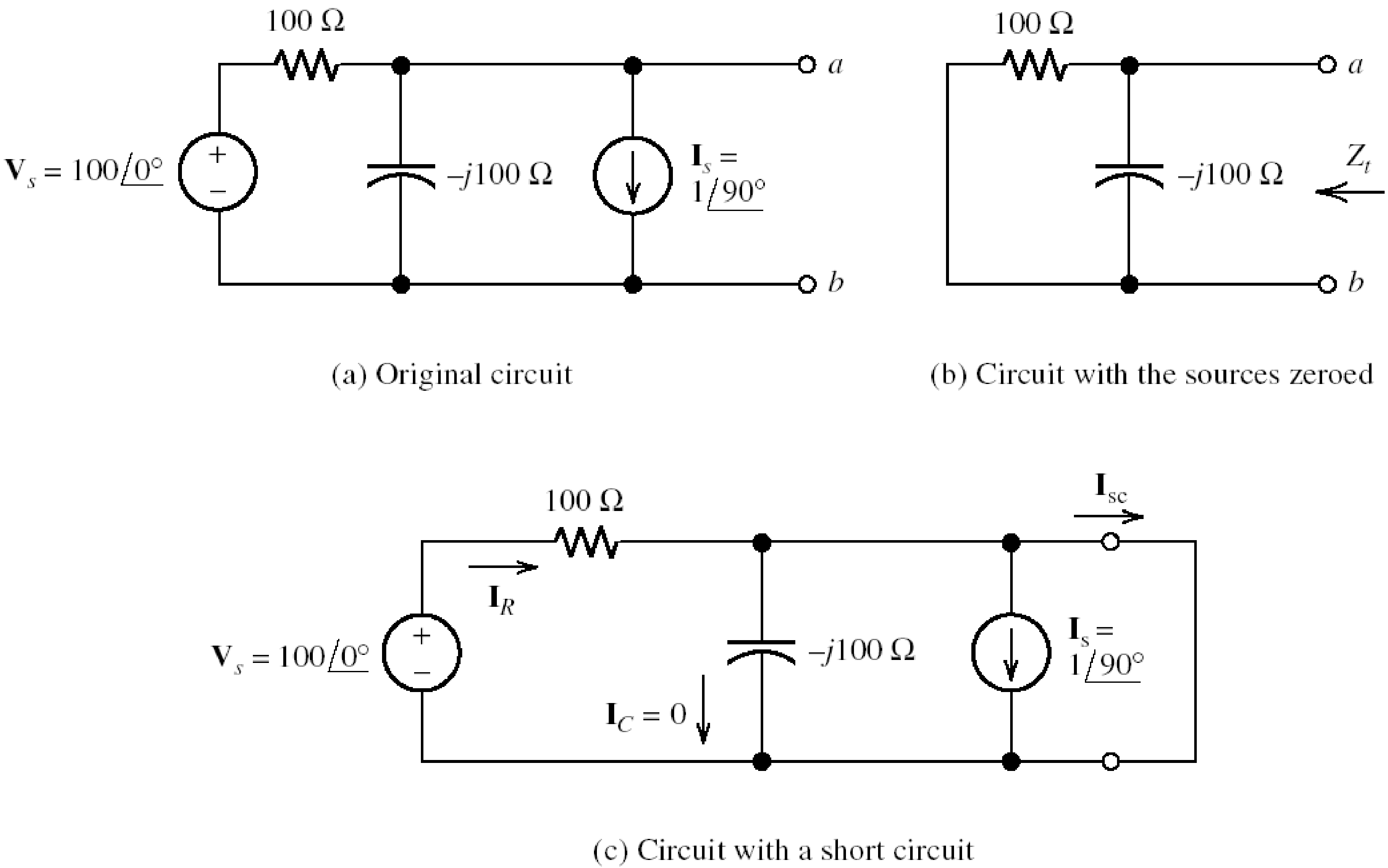


Figure 5.31 Circuit of Example 5.9.

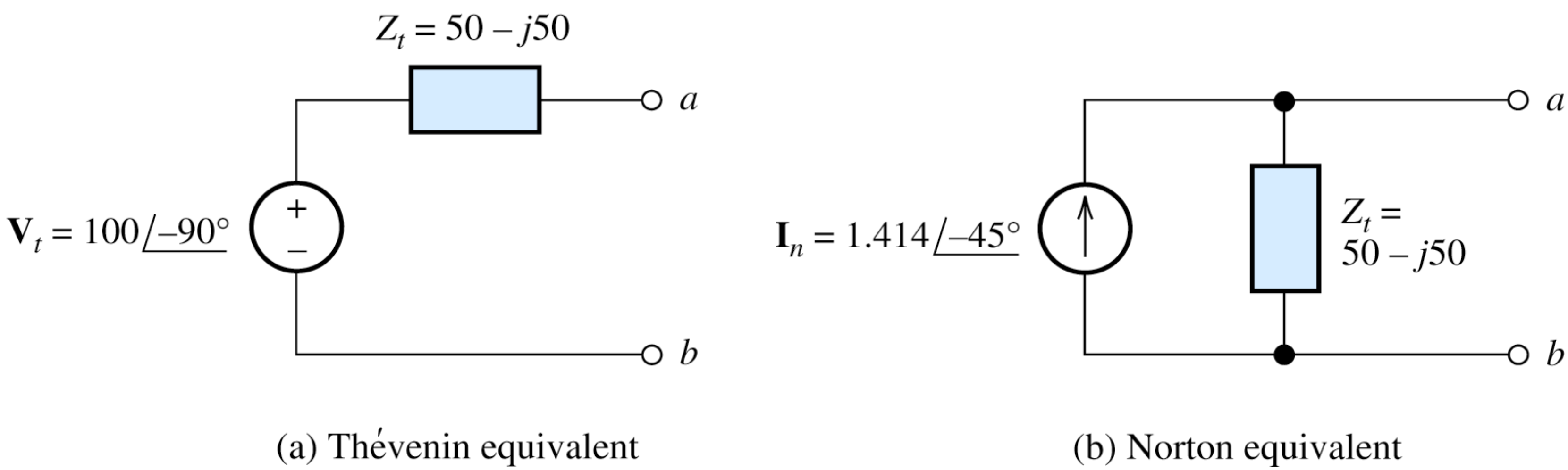


Figure 5.32 Thévenin and Norton equivalents for the circuit of Figure 5.31a.

Maximum Average Power Transfer

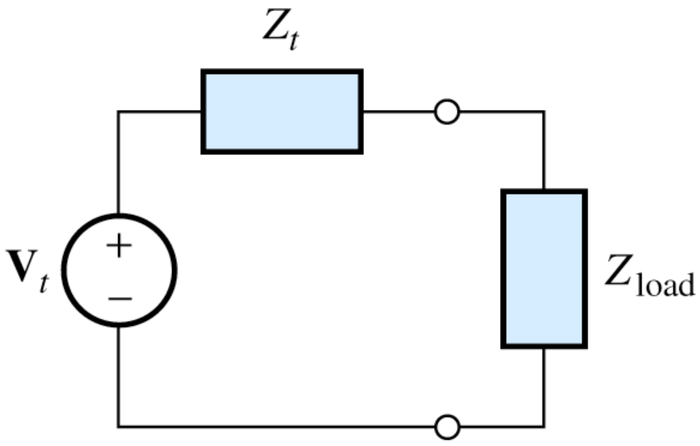


Figure 5.33 The Thévenin equivalent of a two-terminal circuit delivering power to a load impedance.

Maximum Average Power Transfer

- If the load can take on any complex value, maximum power transfer is attained for a load impedance equal to the complex conjugate of the Thévenin impedance.

- If the load is required to be a pure resistance, maximum power transfer is attained for a load resistance equal to the magnitude of the Thévenin impedance.

Example Exercise

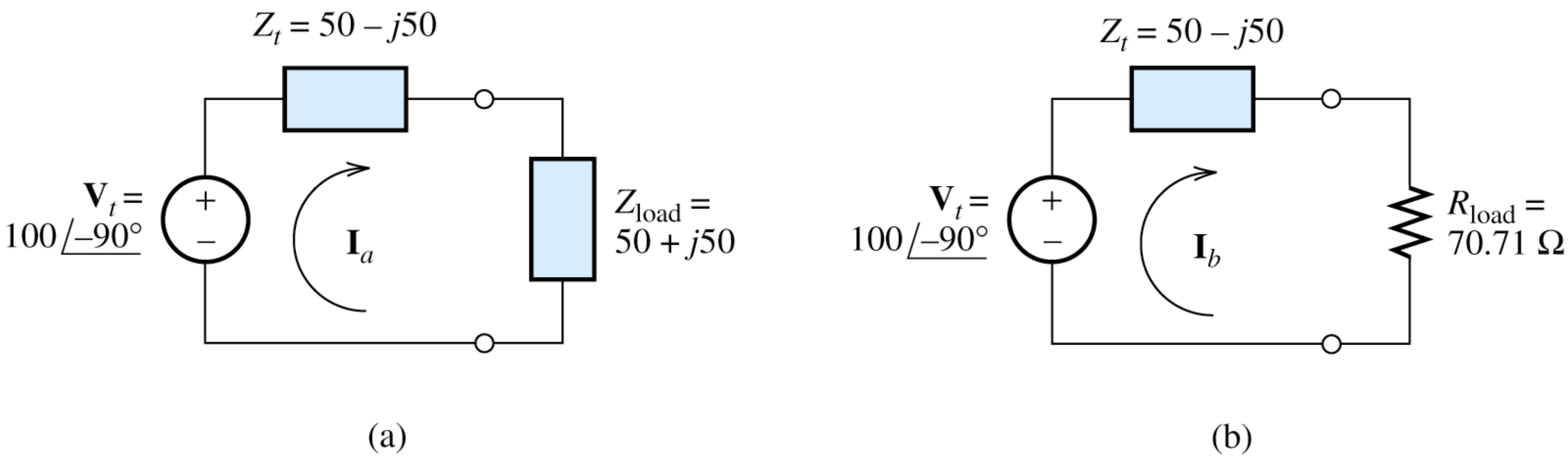


Figure 5.34 Thévenin equivalent circuit and loads of Example 5.10.

Example Exercise

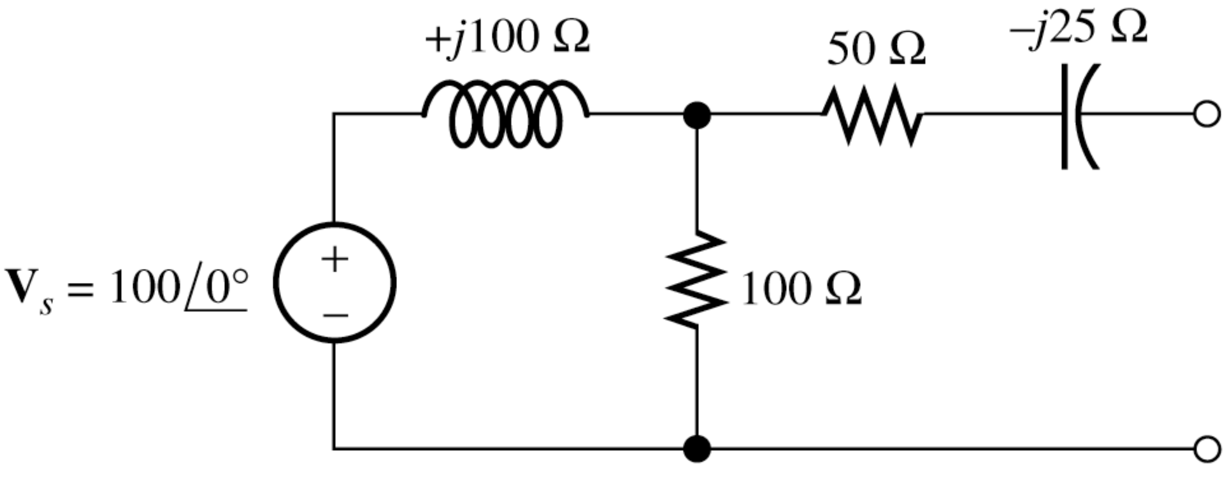
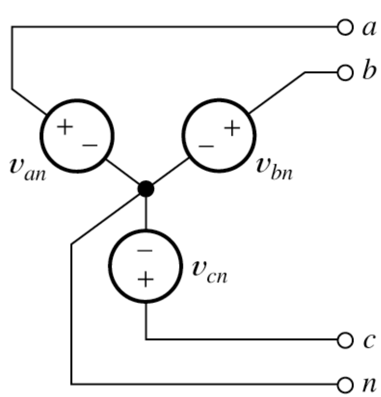


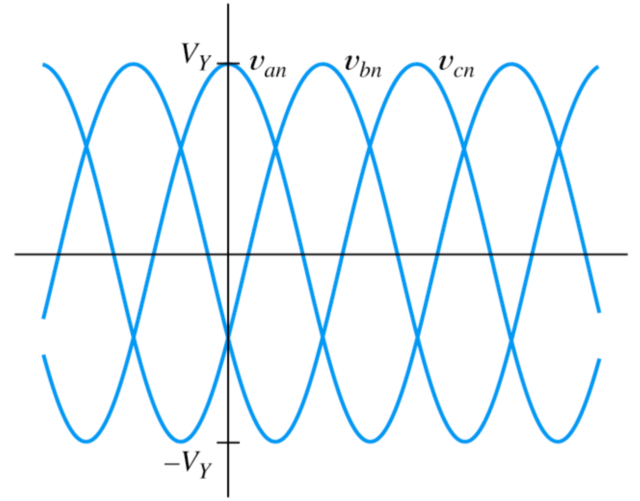
Figure 5.35 Circuit of Exercises 5.14 and 5.15.

BALANCED THREE-PHASE CIRCUITS

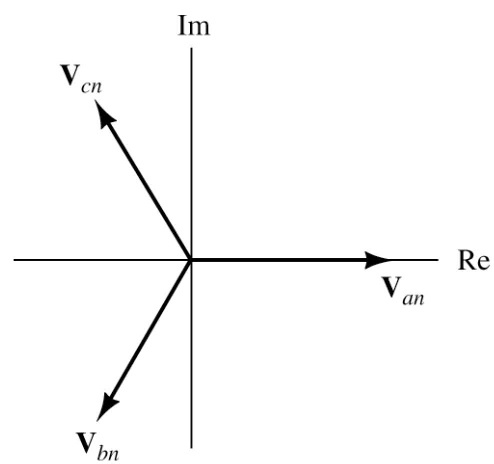
Much of the power used by business and industry is supplied by three-phase distribution systems. Plant engineers need to be familiar with three-phase power.



(a) Three-phase source



(b) Voltages versus time



(c) Phasor diagram

Figure 5.36 A balanced three-phase voltage source.

Phase Sequence

Three-phase sources can have either a positive or negative phase sequence.

The direction of rotation of certain three-phase motors can be reversed by changing the phase sequence.

Wye-Wye Connection

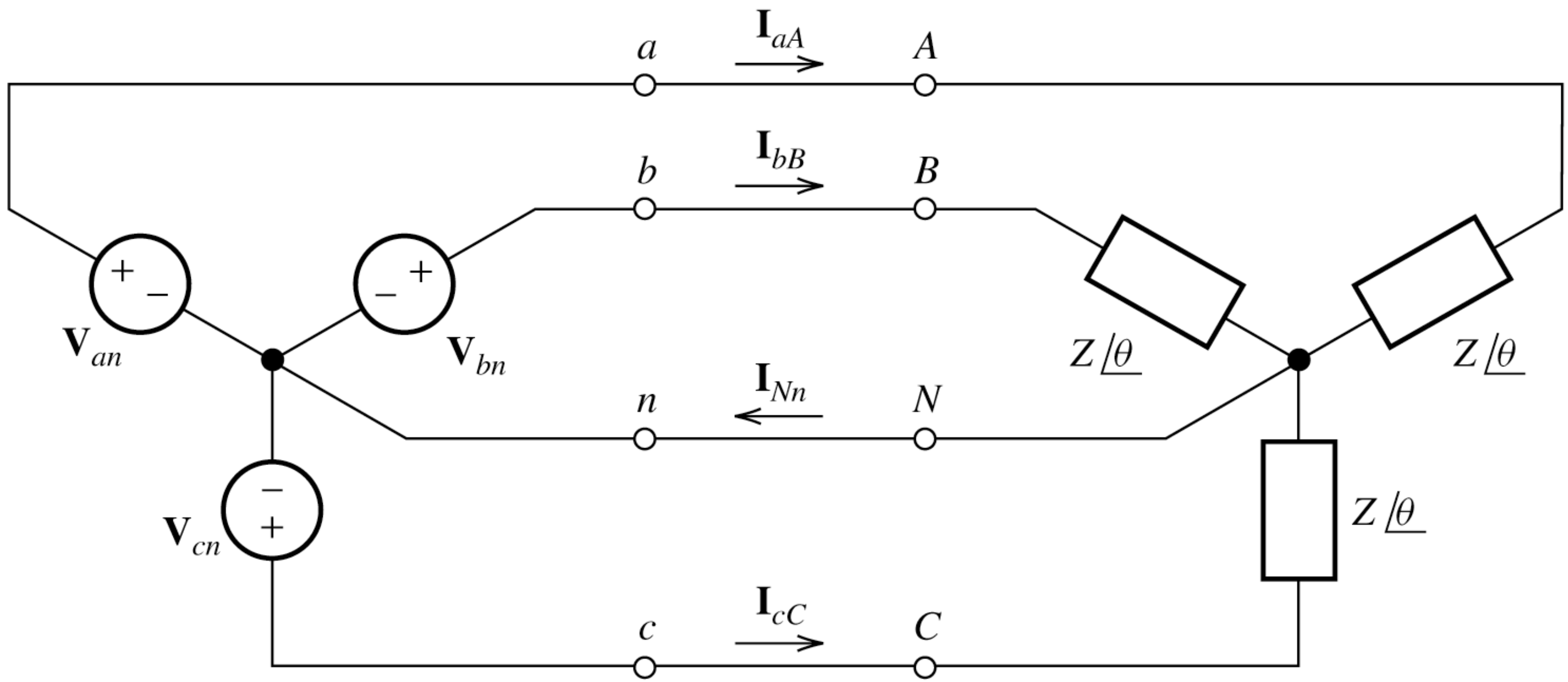


Figure 5.37 A three-phase wye–wye connection with neutral.

Wye-Wye Connection

Three-phase sources and loads can be connected either in a wye configuration or in a delta configuration.

The key to understanding the various three-phase
configurations is a careful examination of the wye-wye circuit.

Advantage

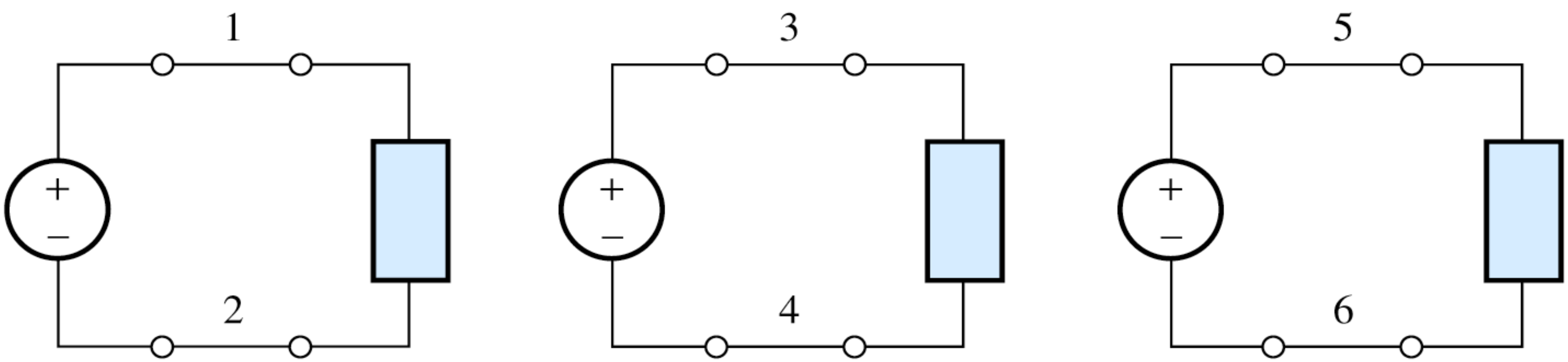


Figure 5.38 Six wires are needed to connect three single-phase sources to three loads. In a three-phase system, the same power transfer can be accomplished with three wires.

Advantage

$$P_{\text{avg}} = p(t) = 3V_{Y\text{rms}} I_{L\text{rms}} \cos(\theta)$$

$$Q = 3\frac{V_Y I_L}{2}\sin(\theta) = 3V_{Y\text{rms}} I_{L\text{rms}} \sin(\theta)$$

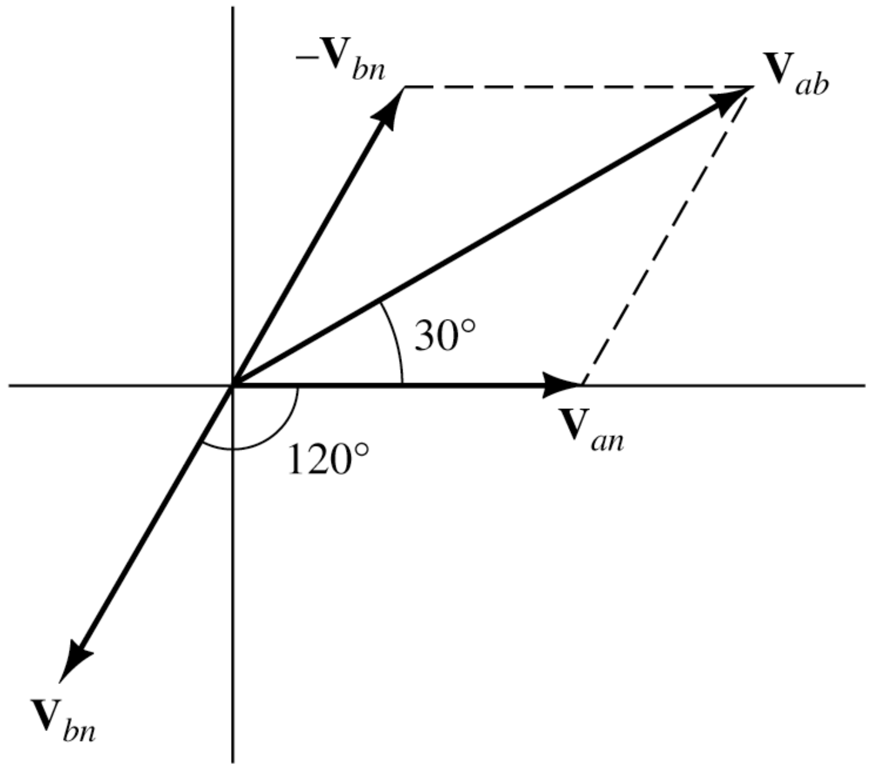
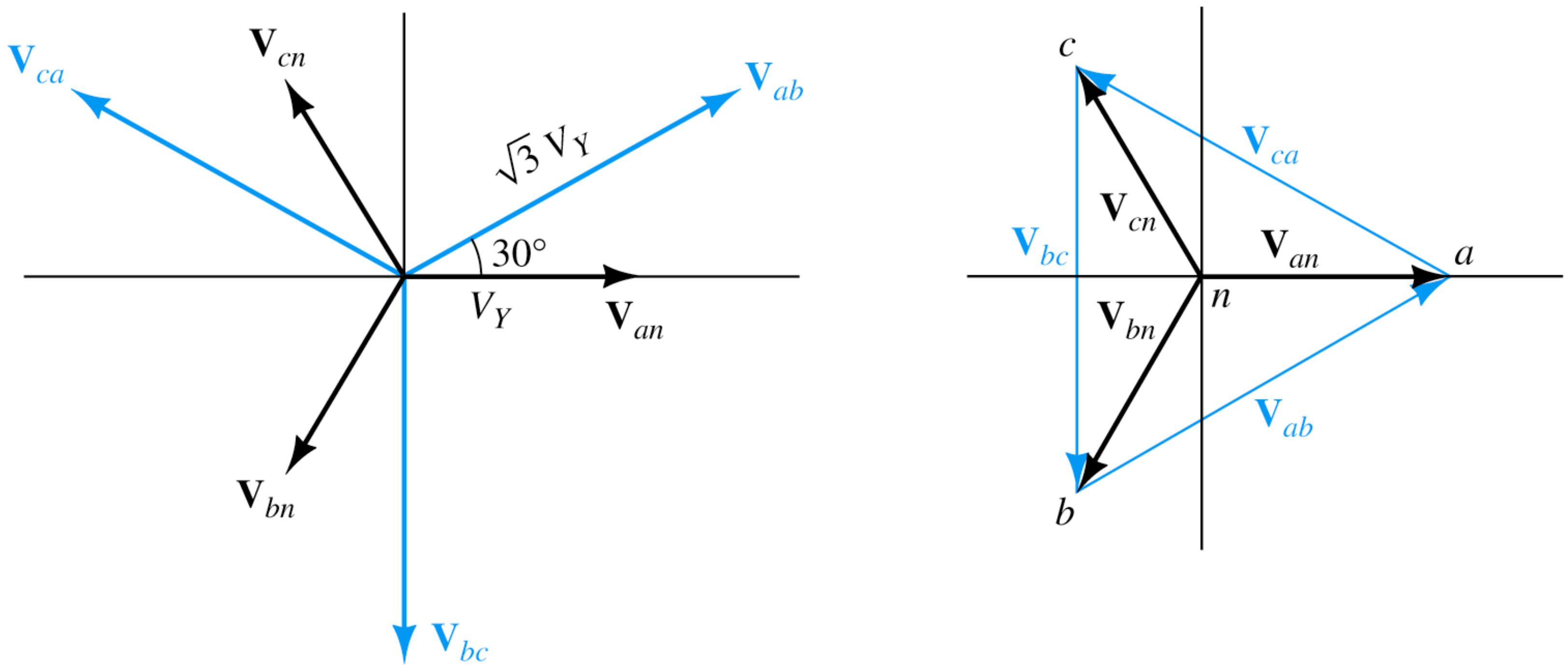


Figure 5.39 Phasor diagram showing the relationship between the line-to-line voltage $\mathbf{v}_{ab}$ and the line-to-neutral voltages $\mathbf{v}_{an}$ and $\mathbf{v}_{bn}$.

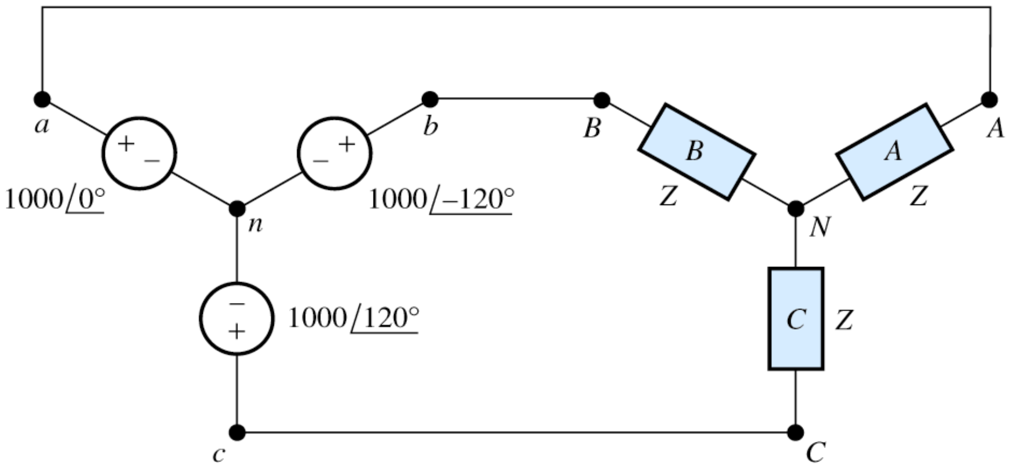


(a) All phasors starting from the origin

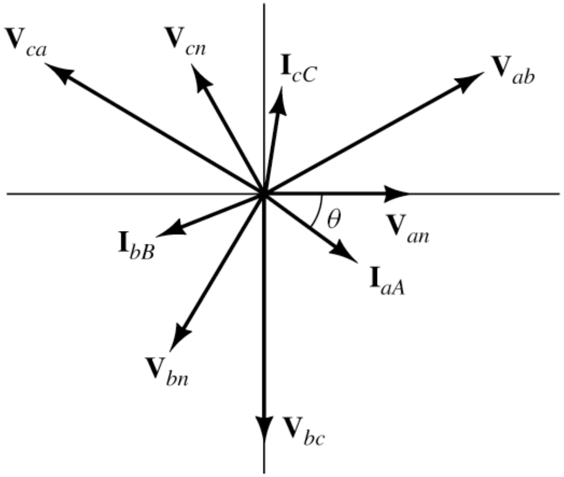
(b) A more intuitive way to draw the phasor diagram

Figure 5.40 Phasor diagram showing line-to-line voltages and line-to-neutral voltages.

Example Exercise



(a) Circuit diagram



(b) Phasor diagram

Figure 5.41 Circuit and phasor diagram for Example 5.11.

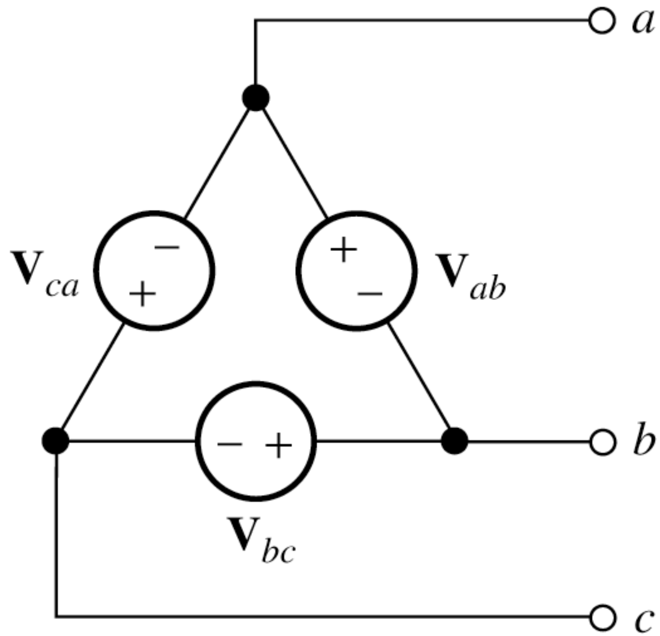


Figure 5.42 Delta-connected three-phase source.

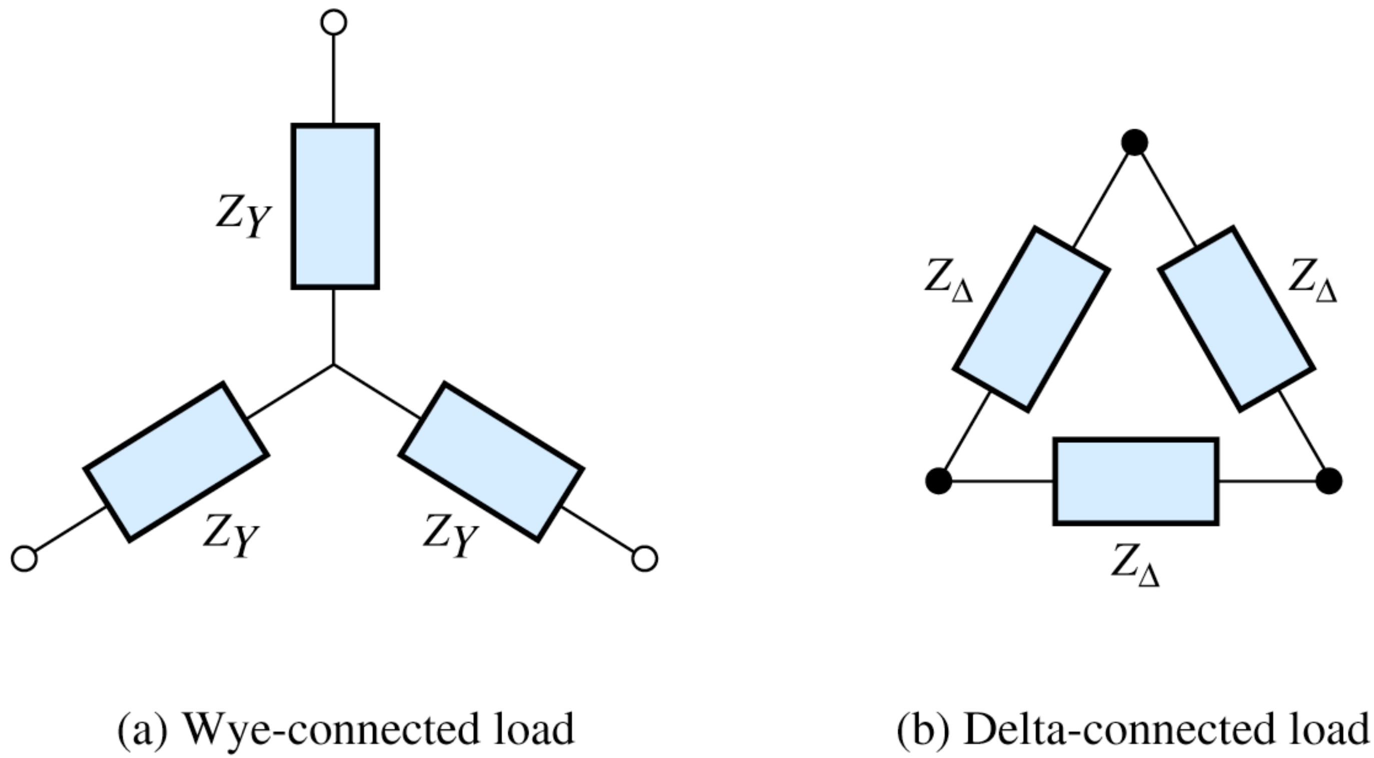
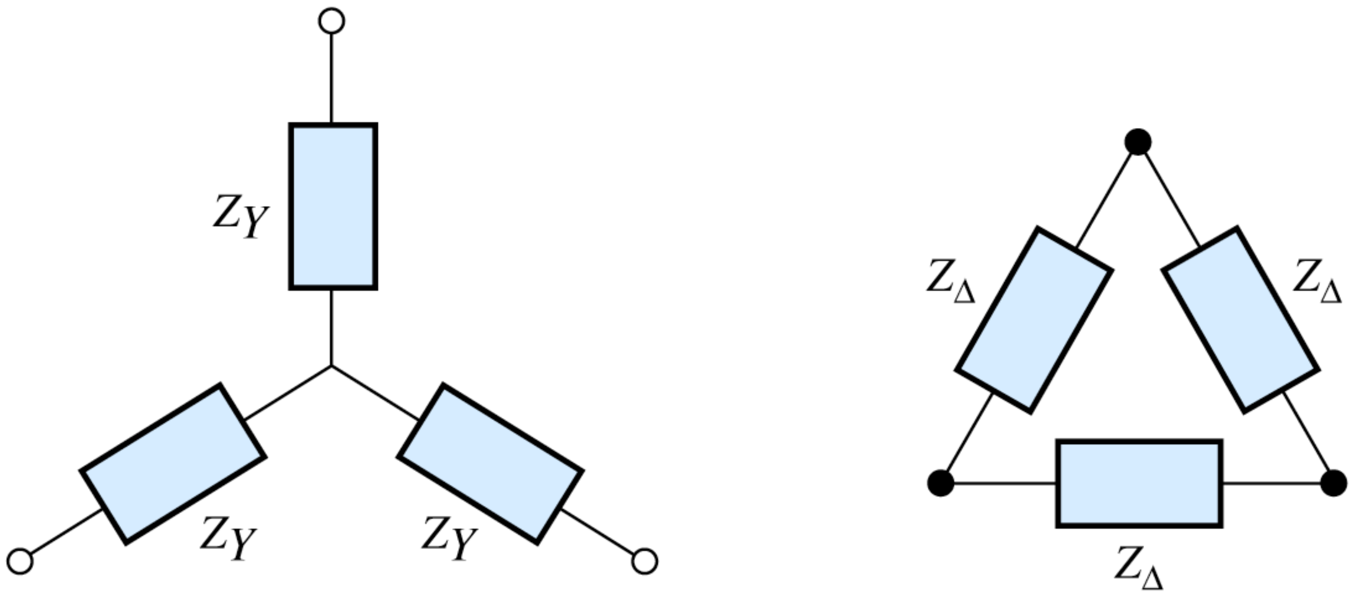


Figure 5.43 Loads can be either wye-connected or delta-connected.

$$Z_{\Delta} = 3Z_{Y}$$



(a) Wye-connected load   (b) Delta-connected load

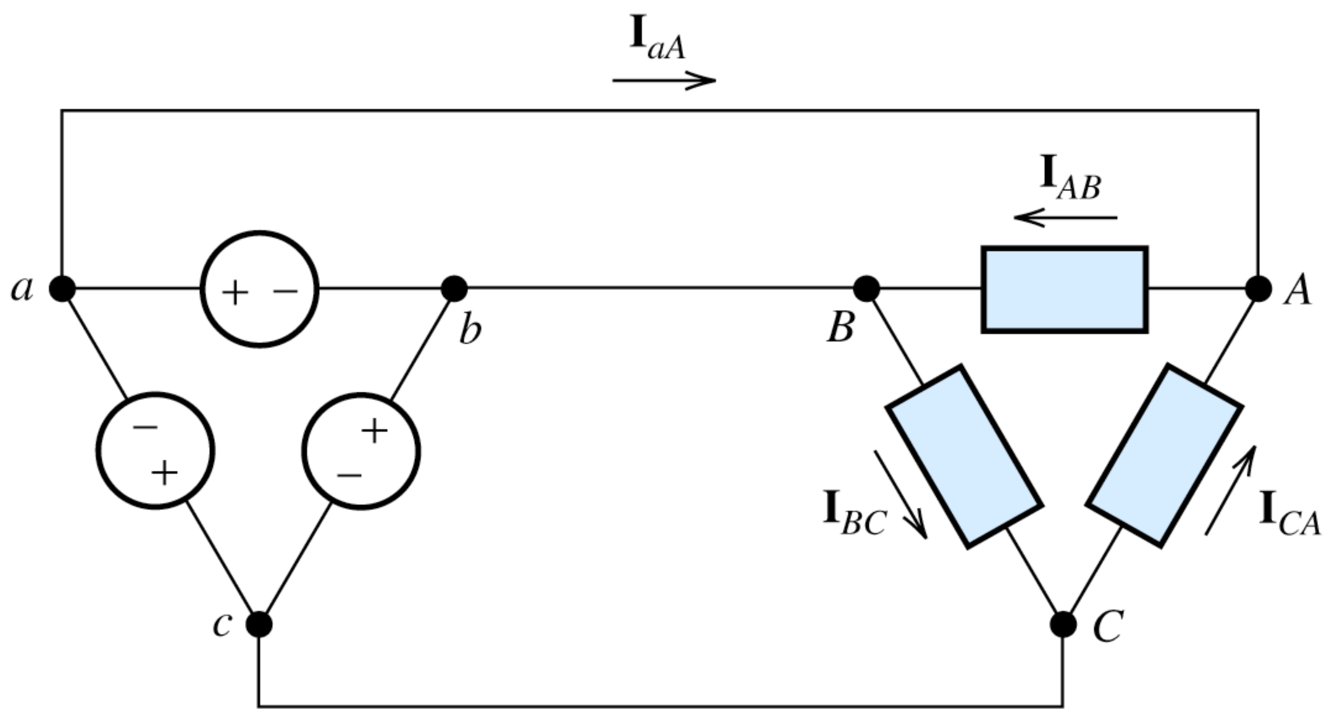
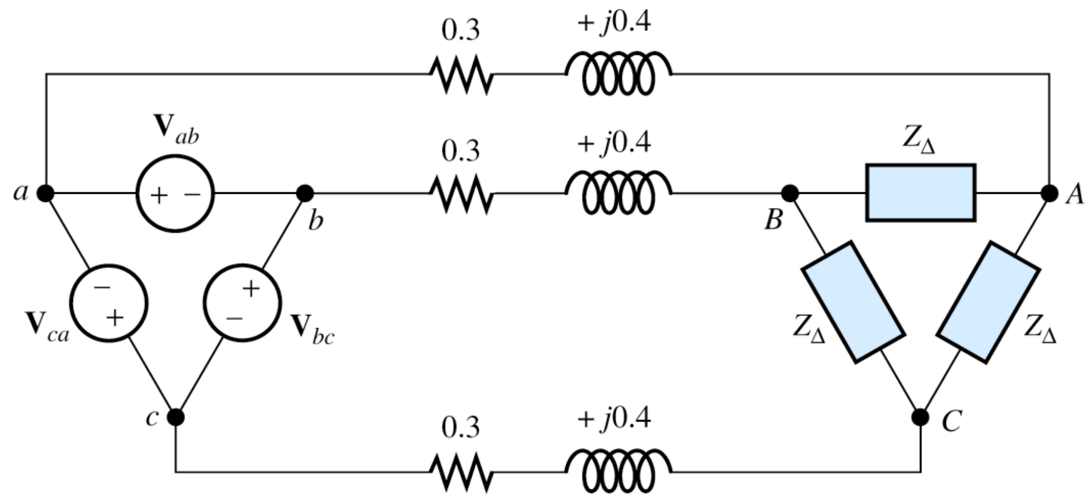
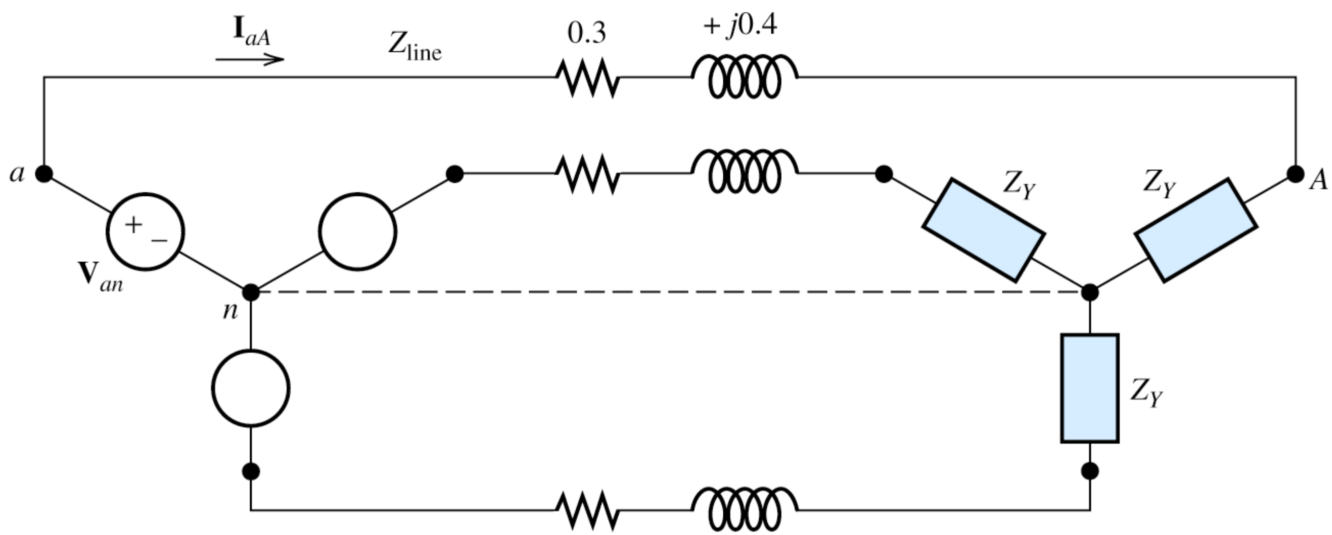


Figure 5.44 A delta-connected source delivering power to a delta-connected load.



(a) Original circuit



(b) Wye-connected equivalent circuit

Figure 5.45 Circuit of Example 5.12.