The Reduced Row Echelon Form of a Matrix Is Unique: A Simple Proof

THOMAS YUSTER
Middlebury College
Middlebury, VT 05753

One of the most simple and successful techniques for solving systems of linear equations is to reduce the coefficient matrix of the system to **reduced row echelon form**. This is accomplished by applying a sequence of elementary row operations (see, e.g., [**1**, p. 5]) to the coefficient matrix until a matrix B is obtained which satisfies the following description:

> If a row of B does not consist entirely of zeros then the first nonzero number in the row is a 1 (usually called a leading 1).
>
> If there are any rows that consist entirely of zeros, they are grouped together at the bottom of B.
>
> In any two successive non-zero rows of B, the leading 1 in the lower row occurs farther to the right than the leading 1 in the higher row.
>
> Each column of B that contains a leading 1 has zeros everywhere else.

The matrix B is said to be in reduced row echelon form.

It is well known that if A is an $m \times n$ matrix and x is an $n \times 1$ vector, then the systems $Ax = 0$ and $Bx = 0$ have the same solution set. However, the solution to the system $Bx = 0$ may be read off immediately from the matrix B. For example, consider the system:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ 2x_1 + 3x_2 + 7x_3 &= 0 \\ 3x_1 + 3x_2 + 9x_3 &= 0 \end{aligned}$$

The matrix of coefficients is

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & 7 \\ 3 & 3 & 9 \end{pmatrix}$$

and through use of elementary row operations we obtain

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which is seen to be in reduced row echelon form. Column 3 (unlike columns 1 and 2) does not contain a leading 1. We call such a column a **free** column, because in the solution to the system $Bx = 0$ (and hence $Ax = 0$), the variable corresponding to that column is a free parameter. Thus we can set $x_3 = t$, and read off the equations $x_1 + 2t = 0$ (row 1) and $x_2 + t = 0$ (row 2). The solution set is then $x_1 = -2t, x_2 = -t, x_3 = t$.

An important theoretical result is that the reduced row echelon form of a matrix is unique. Most texts either omit this result entirely or give a proof which is long and very technical (see [**2**, p. 56]). The following proof is somewhat clearer and less complicated than the standard proofs.

THEOREM. *The reduced row echelon form of a matrix is unique.*

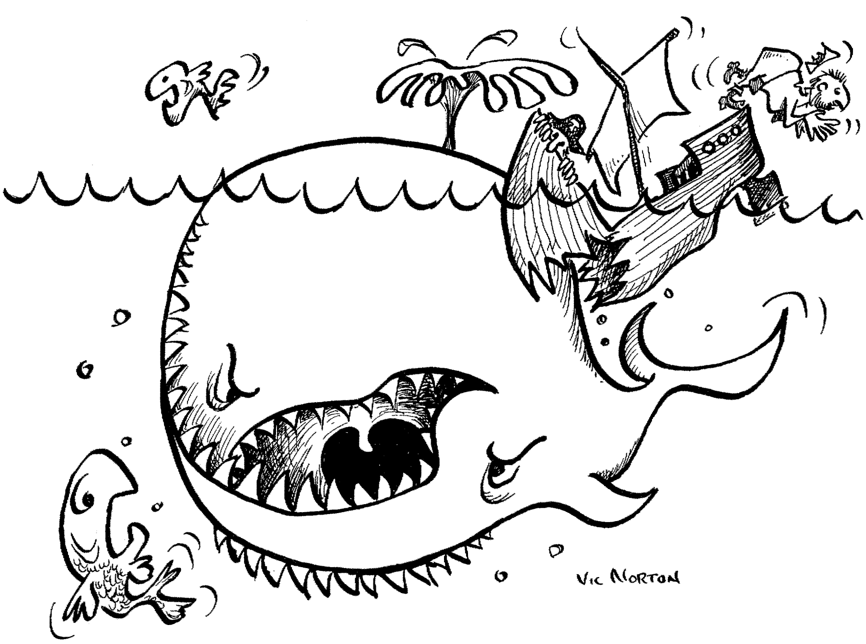
Proof. Let A be an $m \times n$ matrix. We will proceed by induction on n. For $n = 1$ the proof is obvious. Now suppose that $n > 1$. Let A' be the matrix obtained from A by deleting the nth column. We observe that any sequence of elementary row operations which places A in reduced row echelon form also places A' in reduced row echelon form. Thus by induction, if B and C are reduced row echelon forms of A, they can differ in the nth column only. Assume $B \neq C$. Then there is an integer j such that the jth row of B is not equal to the jth row of C. Let u be any column vector such that $Bu = 0$. Then $Cu = 0$ and hence $(B - C)u = 0$. We observe that the first $n - 1$ columns of $B - C$ are zero columns. Thus the jth coordinate of $(B - C)u$ is $(b_{jn} - c_{jn})u_n$. Since $b_{jn} \neq c_{jn}$ we must have $u_n = 0$. Thus any solution to $Bx = 0$ or $Cx = 0$ must have $x_n = 0$. It follows that both the nth columns of B and C must contain leading 1's, for otherwise those columns would be free columns and we could arbitrarily choose the value of x_n. But since the first $n - 1$ columns of B and C are identical, the row in which this leading 1 must appear must be the same for both B and C, namely the row which is the first zero row of the reduced row echelon form of A'. Because the remaining entries in the nth columns of B and C must all be zero, we have $B = C$, which is a contradiction. This establishes the theorem.

We remark that this proof easily generalizes to the following proposition: *Let A be an $m \times n$ matrix with row space W. Then there is a unique $m \times n$ matrix B in reduced row echelon form such that the row space of B is W.* (For another proof, see [**2**, p. 56].)

References

[1] H. Anton, Elementary Linear Algebra, Wiley, New York, 1977.

[2] K. Hoffman and R. Kunze, Linear Algebra, Prentice-Hall, Englewood Cliffs, New Jersey, 1961.



Riddle: What is non-orientable and lives in the sea?

—Robert Messer

(See News and Letters if you give up.—ed.)