8. Let $(V, \oplus, \odot)$ be a vector space over a field $(\mathbb{F}, \boldsymbol{\oplus}, \boldsymbol{\odot})$. Suppose $B = \{v_1, v_2, \ldots v_k\}$ spans V. Then show that following two conditions are equivalent:

   (a) B is an LI set

   (b) If $v \in V$ then the expression $v = \alpha_1 \odot v_1 \oplus \alpha_2 \odot v_2 \oplus \ldots, \alpha_n \odot v_n$ is unique. $\left[\mathbf{3}\frac{1}{2}\right]$

9. Define Range and Null space of a linear transformation. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $R(T) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 4x_1 - 3x_2 + x_3 = 0\}$ Fnd a basis and dimension of $R(T)$. Extend this basis of $R(T)$ to form a basis of $\mathbb{R}^3$ i.e. find a vector $v = (x, y, z)$ such that $\mathbb{R}^3 = R(T) \oplus_D [\{v\}]$. $\left[\mathbf{3}\frac{1}{2}\right]$

10. Let $(U, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ be two vector spaces over a field $(\mathbb{F}, \boldsymbol{\oplus}, \boldsymbol{\odot})$ and let $T : U \to W$ be a linear transformation then prove that T is one-one iff $N(T) = \{0_U\}$, where $N(T)$ is the null space of T. $\left[\mathbf{3}\frac{1}{2}\right]$

11. Let $(U, \oplus, \odot)$ and $(W, \boxplus, \boxdot)$ be two vector spaces over a field $(\mathbb{F}, \boldsymbol{\oplus}, \boldsymbol{\odot})$ and $T : U \to W$ be a linear transformation. If T is one-one and $u_1, u_2, \ldots, u_n$ are LI vectors of U then show that $T(u_1), T(u_2), \ldots, T(u_n)$ are LI vectors of W. If the dimension of U is equal to dimension of W then show that T is one-one **iff** T is onto. $\left[\mathbf{3}\frac{1}{2}\right]$

12. Prove that for a square matrix A of order n, the eigenvectors corresponding to distinct eigenvalues are linearly independent. $\left[\mathbf{3}\frac{1}{2}\right]$

13. Show that for a real symmetric matrix the eigenvalues are real and eigenvectors corresponding to distinct eigenvalues are orthogonal. $\left[\mathbf{3}\frac{1}{2}\right]$

14. Reduce the conic represented by the quadratic $10x^2 - 8xy + 4y^2 = 100$ to its principle axis. $\left[\mathbf{3}\frac{1}{2}\right]$

15. For the matrix A given below find the eigenvalues and their algebraic multiplicity. For what value of h the eigenspace corresponding to $\lambda = 5$ will be two dimensional. $\left[\mathbf{3}\frac{1}{2}\right]$

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 1 & h & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Without finding the eigenvectors what can you say about diagnoseability of A for this value of h?

Pure mathematics is, in its way, the poetry of logical ideas. –*Albert Einstein*