Indian Institute of Technology Mandi
भारतीय प्रौद्योगिकी संस्थान मण्डी

MA-201: B.Tech. II year

Odd Semester: 2011-12 **Exercise-8 Linear Algebra**

1. Which of the following map's are linear ?
   (a) $T:\mathbb{R}^1 \to \mathbb{R}^3$ defined by $T(x)=(x,x^2,x^3)$
   (b) $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z)=(x^2+xy,xy,yz)$
   (c) $T:\mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z)=(x+y+z,0),\alpha\neq 0$
   (d) $T:\mathscr{P}\to \mathbf{p}$ defined by $T(\mathbf{p})(x)=x\mathbf{p}(x)+\mathbf{p}(1)$
   (e) $T:\mathscr{P}[0,1]\to\mathbb{R}^2$ defined by $T(f)=(f(0),f(1))$
   (f) $T:P\to P$ defined by $T(p)=p(0)$
2. For the following Linear Transformations
   (a) Determine the range of the linear transformations. Also find the rank of T, where it exists.
   (b) Determine the kernel of the linear transformations. Also find the nullity of T, where it exists.
   (c) Pick out the maps that are one-one, onto, one-one and onto.
      i. $T:\mathbb{R}^2\to\mathbb{R}^3$ defined by $T(x_1,x_2)=(x_1,x_1+x_2,x_2)$.
      ii. $T:\mathbb{R}^3\to\mathbb{R}^3$ defined by $T(x_1,x_2,x_3)=\left(\frac{1}{2}x_1+x_2+x_3,x_1-\frac{1}{3}x_2,x_3\right)$.
      iii. $T:\mathbb{R}^3\to\mathbb{R}^4$ defined by $T(x_1,x_2,x_3)=(x_1,x_1+x_2,x_1+x_2+x_3,x_3)$.
      iv. $T:\mathscr{P}\to\mathscr{P}$ defined by $T(\mathbf{p})(x)=\mathbf{p}''(x)-2\mathbf{p}(x)$.
3. Find a linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ such that the set of all vectors (x_1,x_2,x_3) satisfying the equation $4x_1-3x_2+x_3=0$ is the kernel of T.
4. Find a trace and determinant of the following Linear Transformation on $\mathbb{R}^3$
   (a) $T(x_1,x_2,x_3)=(x_1+3x_2,3x_1-2x_3,x_1-4x_2-3x_3)$
   (b) $T(x_1,x_2,x_3)=(x_2+x_3,2x_1-4x_3,5x_1+7x_2)$
5. Find the change of basis matrix P from the usual basis $E=(e_1,e_2,e_3)$ to $\mathbb{R}^3$ to a basis S,the change of basis matrix Q from S back to E, and the coordinate of $v=(a,b,c)$ relative to S, for the following bases S:
   (a) $u_1=(1,0,0),u_2=(0,1,2),u_3=(0,0,1)$
   (b) $u_1=(1,0,1),u_2=(1,1,2),u_3=(1,2,4)$
   (c) $u_1=(1,2,1),u_2=(1,3,4),u_3=(2,5,6)$
6. Let $T:\mathbb{R}^2\to\mathbb{R}^2$ be defined by matrix $T=\begin{bmatrix}1 & -1\\ 3 & 2\end{bmatrix}$. Find the matrix B that represents the linear operator T relative to each of the following bases:
   (a) $B=\{(1,3),(2,5)\}$

(b) $B = \{(1,3),(2,4)\}$

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by matrix $T = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 7 & 4 \\ 1 & 4 & 3 \end{bmatrix}$. Find the matrix A that represents the linear operator T relative to the basis $T = \{(1,1,1),(0,1,1),(1,2,3)\}$

8. Find the matrix representation of each of the following linear maps relative to the usual basis for $\mathbb{R}^n$:

   (a) $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $F(x,y,z) = (2x - 4y + 9z, 5x + 3y - 2z)$.

   (b) $F : \mathbb{R}^2 \to \mathbb{R}^4$ defined by $F(x,y) = (3x + 4y, 5x - 2y, x + 7y, 4x)$.

   (c) $F : \mathbb{R}^4 \to \mathbb{R}$ defined by $F(x,y,z,w) = 2x + y - 7z - w$.

9. let $G : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by G(x,y,z) = (2x + 3y - z,4x - y + 2z)

   (a) Find the matrix A representing G relative to the bases $S = \{(1,1,0),(1,2,3),(1,3,5)\}$ and $S' = \{(1,2),(2,3)\}$

   (b) For any $v = (a,b,c)$ in $\mathbb{R}^3$,find $[v]_s$ and $[G(v)]_{s'}$

   (c) verify that $A[v]_s = [G(v)]_{s'}$

10. Consider the following linear operator G on $\mathbb{R}^2$ and basis S given by $G(x,y) = (2x - 7y, 4x + 3y)$ and $S = (1,3),(2,5)$.

    (a) Find the matrix representation of $[G]_s$ of G relative to S.

    (b) Verify $[G]_s[v]_s = [G(v)]_s$ for the vector $v = (4,-3)$ in $\mathbb{R}^2$.

11. let $H : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $H(x,y) = (2x + 7y, x - 3y)$ and consider the following bases of $\mathbb{R}^2$ **(i)** $S = \{(1,1),(1,2)\}$ and **(ii)** $S' = \{(1,4),(1,5)\}$

    (a) Find the matrix A representing H relative to the bases S and S'

    (b) Find the matrix B representing H relative to the bases S' and S

12. Find the characteristic and minimal polynomial of each of the following matrices if possible

$$A = \begin{bmatrix} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}, A = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}, A = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}, A = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & 7 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$