CS221: Digital Design

Boolean Algebra

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Outline

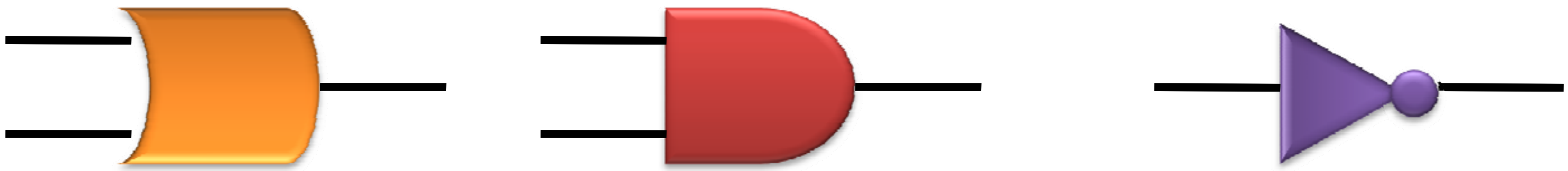
- Basic Gates in Digital Circuit
- Boolean Algebra : Definitions, Axioms
- Named, Simplification & Consensus Theorems
- Duality Principle, Shannon's Expansion
- Proof : Using Truth Table, Using Theorem
- Boolean function: Representation, Canonical form

Boolean Algebra

- Computer hardware using binary circuit greatly simply design
- Binary circuits: To have a conceptual framework to manipulate the circuits algebraically
- George Boole (1813-1864): developed a mathematical structure
  - To deal with binary operations with just two values.

Basic Gates in Binary Circuit

- Element 0 : “FALSE”. Element 1 : “TRUE”.
- ‘+’ operation “OR”,‘*’ operation “AND” and ’ operation “NOT”.



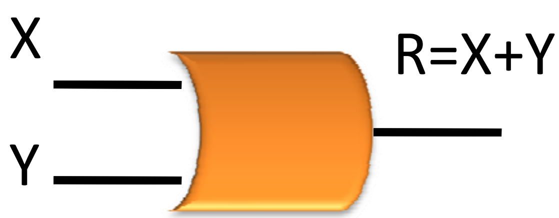
OR	0	1
0	0	1
1	1	1

AND	0	1
0	0	0
1	0	1

NOT	
0	1
1	0

OR Gate

- '+' operation "OR"



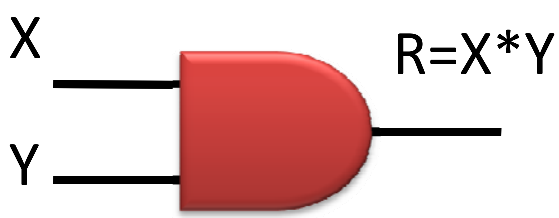
OR	0	1
0	0	1
1	1	1

X	Y	R=X OR Y R= X + Y
0	0	0
0	1	1
1	0	1
1	1	1

1 + Y = 1

AND Gate

- ‘*’ operation “AND”



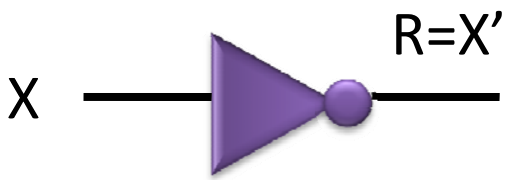
X	Y	R=X AND Y R= X * Y
0	**0**	**0**
0	**1**	**0**
1	0	0
1	1	1

AND	0	1
0	0	0
1	0	1

0 * Y = 0

NOT Gate

- ‘ operation “NOT”



X	R=X’ R= NOT X
0	1
1	0

Boolean Algebra Defined

- Boolean Algebra B : 5-tuple

{B, +, *, ’, 0, 1}

- **+** and ***** are *binary* operators,
- **‘** is a *unary* operator.

Boolean Algebra Defined

- ***Axiom #1: Closure***

  If a and b are Boolean

  (a + b) and (a * b) are Boolean.

- ***Axiom #2: Cardinality***

  if a is Boolean then a’ is Boolean

- ***Axiom #3: Commutative***

  (a + b) = (b + a)

  (a * b) = (b * a)

Boolean Algebra Defined

- ***Axiom #4: Associative :*** **If a and b are Boolean**

  $$(a + b) + c = a + (b + c)$$

  $$(a * b) * c = a * (b * c)$$

- ***Axiom #6: Distributive***

  $$a * (b + c) = (a * b) + (a * c)$$

  $$a + (b * c) = (a + b) * (a + c)$$

Boolean Algebra Defined

- ***Axiom #5: Identity Element :***
  - **B has identity to + and ***

0 is identity element for + : $a + 0 = a$

1 is identity element for * : $a * 1 = a$

- ***Axiom #7: Complement Element***

$$a + a' = 1$$

$$a * a' = 0$$

Terminology

- **Juxtaposition implies * operation:**

$$ab = a * b$$

- **Operator order of precedence is:**

$$(\,) > \,' > * > +$$

$$a+bc = a+(b*c) \neq (a+b)*c$$

$$ab' = a(b') \neq (a*b)'$$

Named Theorems

Idempotent	a + a = a	a * a = a
Boundedness	a + 1 = 1	a * 0 = 0
Absorption	**a + (a*b) = a**	**a*(a+b) = a**
Associative	(a+b)+c= a+(b+c)	(a*b)*c= a*(b*c)

Involution	(a’)’ = a	
DeMorgan’s	(a+b)’ = a’ * b’	(a*b)’=a’ + b’

Simplification Theorem

- Uniting :

$XY + XY' = X$ $\quad X(Y+Y')=X.1=X$

$(X + Y)(X + Y') = X$ $\quad XX+XY'+YX+YY'=X+X(\mathbf{Y+Y'})+0=X$

- Absorption:

$X + XY = X$ $\quad X(1+Y)=X.1=X$

$X(X + Y) = X$ $\quad XX+XY=X+XY=X$

- Adsorption

$(X + Y')Y = XY, \;\; XY' + Y = X + Y$ $\quad XY+YY'=XY+0=XY$

Principle of Duality

- Dual of a statement S is obtained
  - By interchanging * and +
  - By interchanging 0 and 1
  - ~~By interchanging for all x by x' also valid (for an expression)~~
- Dual of $(a*1)*(0+a') = 0$ is $(a+0)+(1*a') = 1$

$(a+b)' = a' * b'$

~~$(a+b) * 1 = (a'*b')+0$~~

Consensus Theorem

- $XY + X'Z + YZ = XY + X'Z$

$$
\begin{aligned}
& XY + X'Z + YZ \\
&= xy + x'z + (x + x')yz \\
&= xy + x'z + xyz + x'yz \\
&= xy + xyz + x'z + x'yz \\
&= xy(1 + z) + x'z(1 + y) \\
&= xy + x'z
\end{aligned}
$$

Duality

Consensus (collective opinion) of X.Y and X'.Z is Y.Z

- $(X + Y)(X' + Z)(Y + Z) = (X + Y)(X' + Z)$

Shannon Expansion

- **$F(X, Y, Z) = X \cdot F(1,Y,Z) + X' \cdot F(0, Y, Z)$**

Example:

$XY+X'Z+YZ$

$=X.(1.Y+0.Z+YZ) + X'(0.Y+1.Z+YZ)$

$=X.(Y+YZ)+X'(Z+YZ)$

$=X.(Y(1+Z))+X'(Z(1+Y))$

$= X(Y.1)+X'(Z.1)$

$=XY+X'Z$

N-bit Boolean Algebra

- Single bit to *n-bit* Boolean Algebra
- Let a = 1101010, b = 1011011
  - a + b = 1101010 + 1011011 = 1111011
  - a * b = 1101010 * 1011011 = 1001010
  - a’ = 1101010’ = 0010101

Proof by Truth Table

- Consider the distributive theorem:

  $$a + (b * c) = (a + b)*(a + c)$$

  Is it true for a two bit Boolean Algebra?
- Can prove using a truth table
  - How many possible combinations of *a*, *b*, and *c* are there?
- Three variables, each with two values
  - **$2*2*2 = 2^3 = 8$**

Proof by Truth Table

a	b	c	b*c	a+(b*c)	a+b	a+c	(a+b)*(a+c)
0	*0*	*0*	0	0	0	0	0
0	*0*	*1*	0	0	0	1	0
0	*1*	*0*	0	0	1	0	0
0	*1*	*1*	1	1	1	1	1
1	*0*	*0*	0	1	1	1	1
1	*0*	*1*	0	1	1	1	1
1	*1*	*0*	0	1	1	1	1
1	*1*	*1*	1	1	1	1	1

Proof using Theorems

- Use the properties of Boolean Algebra to proof

$$(x + y)(x + x) = x$$

- *Warning, make sure you use the laws precisely*

$(x+y)(x+x)$	**Given**
$(x+y)x$	**Idempotent**
$x(x+y)$	**Commutative**
x	**Absorption**

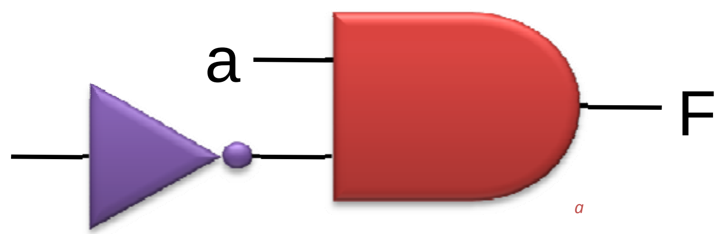
Converting to Boolean Equations

- Convert the following English statements to a Boolean equation
  - Q1. a is 1 and b is 1.
    - Answer: F = a AND b = ab
  - Q2. either of a or b is 1.
    - Answer: F = a OR b = a+b
  - Q3. both a and b are not 0.
    - Answer:
      - (a) Option 1: F = NOT(a) AND NOT(b) = a’b’
      - (b) Option 2: F = a OR b = a+b
  - Q4. a is 1 and b is 0.
    - Answer: F = a AND NOT(b) = ab’

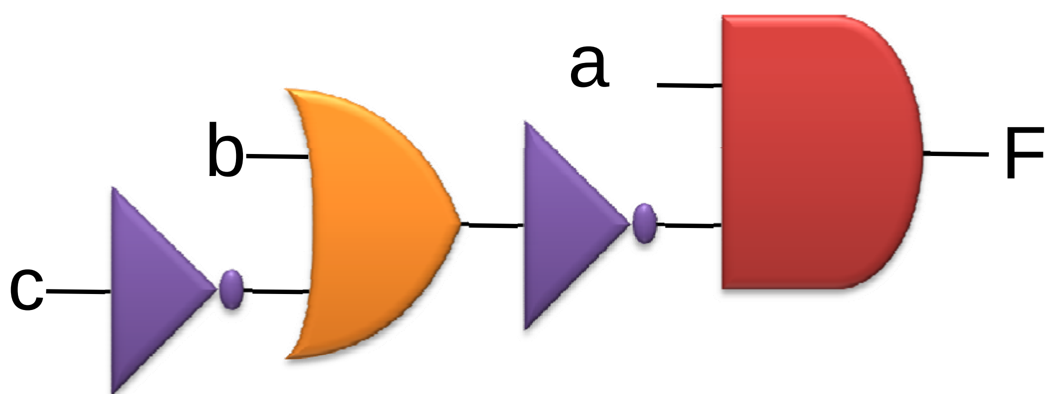
Example: Converting a Boolean Equation to a Circuit of Logic Gates

- Q: Convert the following equation to logic gates:

  F = a AND NOT(b OR NOT(c))

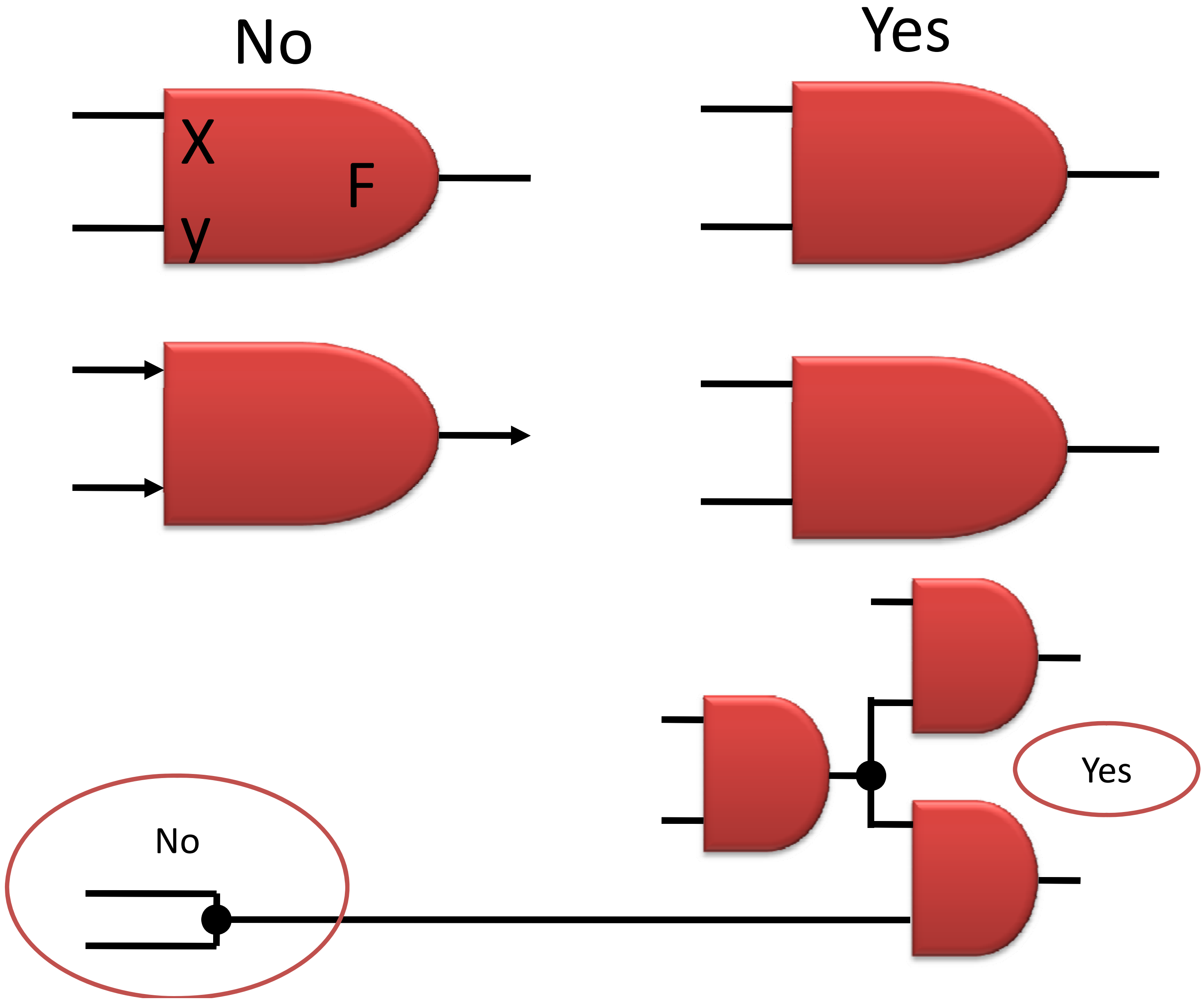


(a)



(b)

Some Circuit Drawing Conventions



Thanks