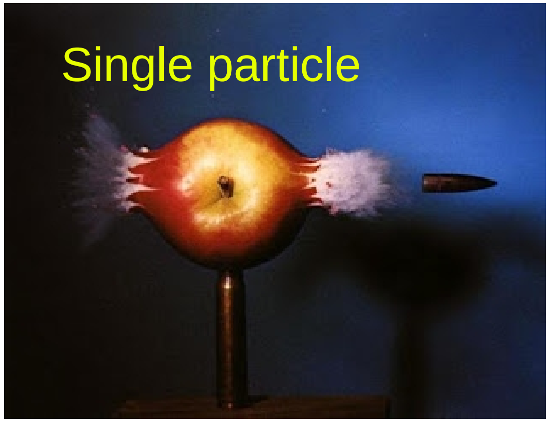
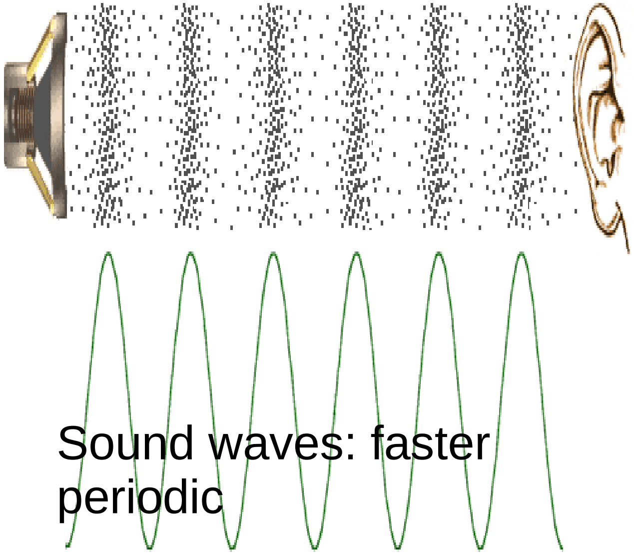
# IC121:Mechanics of particles and waves

## Part I: Classical Mechanics







# Learning goals

What Classical Physics deals with?

- Measurements of the physical quantities, accuracies, length scales, time scales, frames of reference, transformations.
- Prediction of the future state, nature of steady state or equilibrium
- Particle mechanics, theories of particle mechanics, limits of different theories, single and many particle mechanics
- Conservation and symmetries
- Many particles mechanics, vibrations, waves
- Relation between particle mechanics to many-body mechanics like continuum mechanics, rigid body mechanics......
- Limitation of classical mechanics for small particles , experimental observations, suggestion for a new formalism wave mechanics
- Schroedinger equation and basic principles of quantum mechanics application to simple models and consequences

Size of objects

Decreasing size of particles

Classical mechanics

Mechanics of planets, and larger objects,
Mechanics of balls, apples simple pendulum etc..

Limiting case mechanics of Cells and macro molecules

$$10^{-9}\, meters\,(nano)$$

Quantum mechanics

Mechanics of atoms and molecule as limiting case

Mechanics of electron and nucleus, protons and neutrons etc..

Any object that is smaller than atoms

# Syllabus

- Vectors and vector calculus - gradient, divergence and curl. line, surface and volume integrals - Helmholtz theorem. Gauss divergence, Stokes theorem - Generalized coordinates, Jacobian, Cartesian, cylindrical, and spherical coordinates. Introduction to Cartesian tensors. Vectors and vector spaces.

- Newtonian mechanics conservation laws – linear, angular momentum, energy- single and many particle systems.

- Oscillations as application of Newtonian mechanics, Driven damped and forced oscillations, generalized vector spaces, Fourier expansion and oscillations under periodic forces, coupled oscillations and normal modes. Nonlinear oscillations. Wave equation in a 1d string. LC circuit, simple pendulum, coupled pendulum.

- Potentials and fields, Fundamental interactions in nature. Gravitational and electrostatic potentials by point particles and extended objects. Multi-pole expansion. Poisson and Laplace equation in electrostatics

- Constraints and generalized coordinates - Lagrangian- Lagrange's equation of motion – relation to Newtonian mechanics - Two body problem - type of orbitals Variational principle of mechanics.
- Legendre transform - Hamiltonian mechanics -phase space representation Introduction to many body mechanics
- Inadequacy of classical mechanics. - Black body radiation- photo-electric effect. Classical unstable atoms. Bohr model of hydrogen atom. Frank-Hertz experiment.
- Uncertainty principle - Phase space and Hilbert space - Postulates of quantum mechanics-Schrödinger equation - observations and measurements – principle of superposition- operators and state functions- expectation value.
- Applications of Schrödinger equation - particle in a infinite square well potentials. Harmonic oscillator; rigid rotor and two body (Hydrogen atom) problem.

## Text

**Classical dynamics of particles and systems by S T Thornton and J B Marion**

## Reference

1) Mechanics : Berkley Physics course I by C. Kittel, W D Knight, M A Ruderman

2) Classical Mechanics by R Douglas Gregory

3) Classical Mechanics by T W B Kibble and F H Berkshire

4) The Feynman Lectures on Physics Vol I by R P Feynman

5) Introduction to Classical Mechanics by R G Takwale and PS Puranik

6) Introduction to classical mechanics by D. Morin

7) Classical mechanics point particles and relativity Walter Greiner

## Advanced References

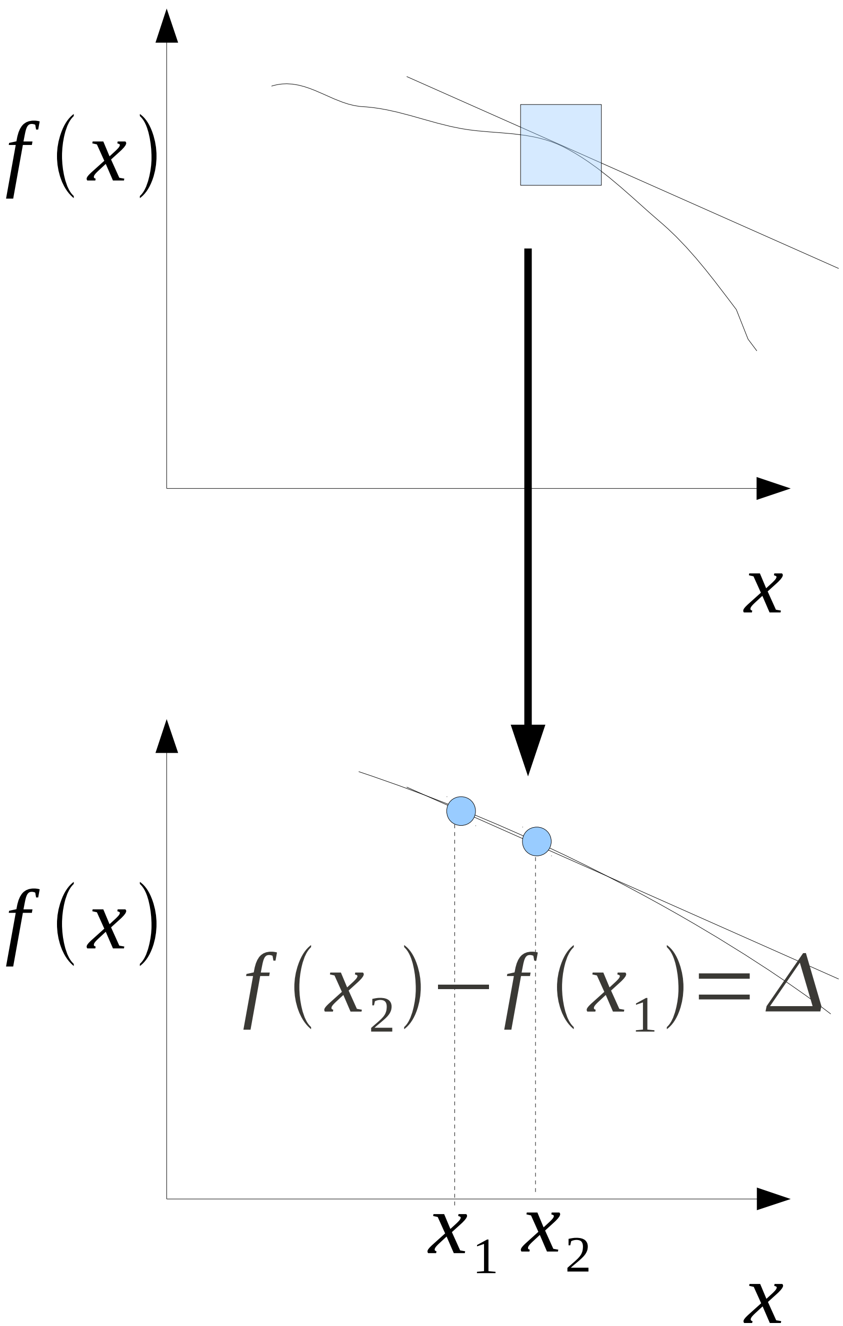
8) Classical Mechanics by J M Finn

9) Classical Mechanics by Rana and Joag

10) Classical Mechanics by H Goldstein

# Review of few mathematical principles

# Derivative of a function for a single variable




$$f(x_2)-f(x_1)=\Delta f(x)$$

The local slope of the function

$$\frac{f(x_2)-f(x_1)}{(x_2-x_1)}=m, \quad \frac{\Delta f(x)}{\Delta x}=m$$

As $\Delta x \to 0,$ $$\frac{d f(x)}{d x}=m$$

$$d f(x)=\frac{d f(x)}{d x} d x$$

For a straight line $f(x)=m x+c$, the first derivative can define the entire function with mention of starting point

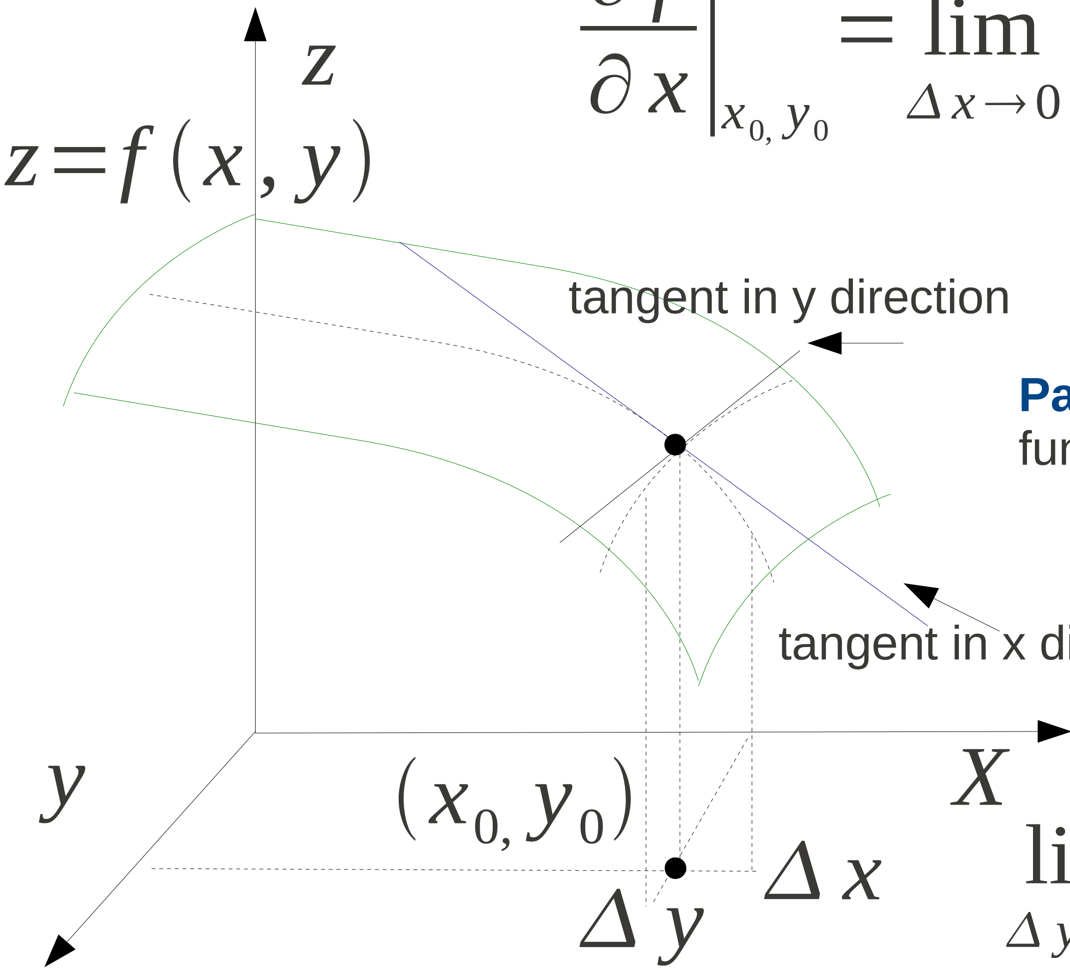
$$d f(x)=\frac{d f(x)}{d x} d x=m\, dx$$

# Derivative of a multi-variable function

**Partial derivative** with respect to $x$ for the function $f(x,y)$

$$\left.\frac{\partial f}{\partial x}\right|_{x_0, y_0} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

$$= \left.\frac{d f(x, y_0)}{d x}\right|_{y=y_0}$$




**Partial derivative** with respect to $y$ for the function $f(x,y)$

$$\left.\frac{\partial f}{\partial y}\right|_{x_0, y_0} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}$$

$$= \left.\frac{d f(x_0, y)}{d y}\right|_{x=x_0}$$

Thomas calculus chapter 14

The partial derivatives are extension of differentiation of function of single variable, the difference is while the differentiation take place in one variable other variables are considered a constant.

# Vectors and Coordinate systems

quantities useful in study of mechanics

velocity of a point particle

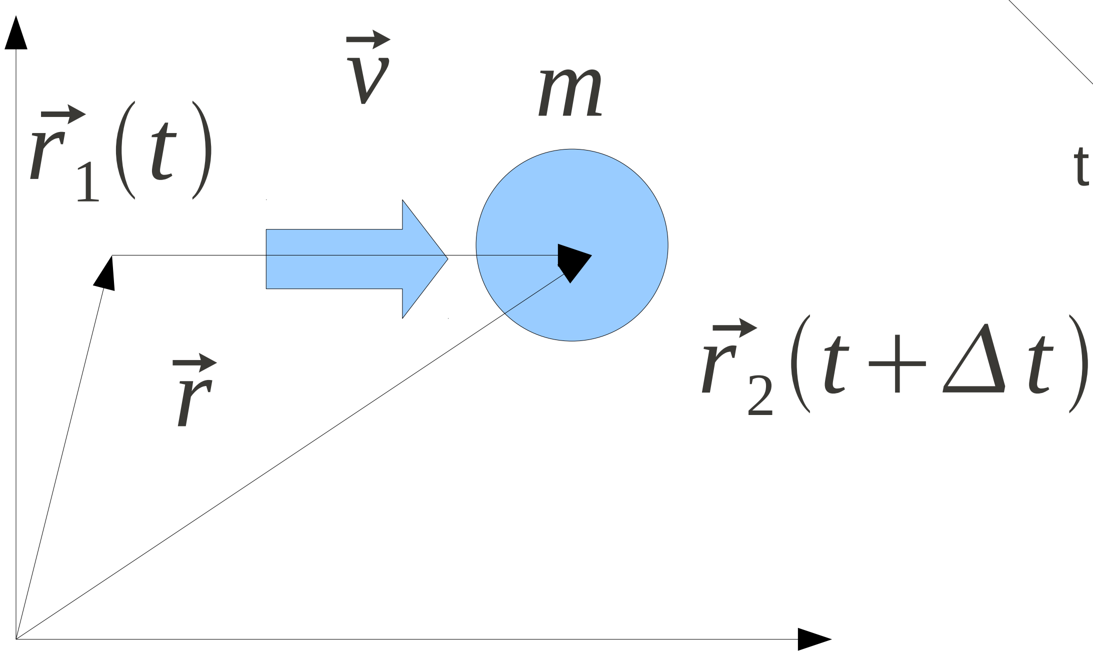
position of a point particle

$$\vec{\Delta} r = \vec{r}_2(t+\Delta t) - \vec{r}_1(t)$$

$$\vec{v} = \frac{d\,\vec{r}}{d\,t}$$

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\vec{\Delta} r}{\Delta t} = \frac{d\,\vec{r}}{dt}$$

time




The representation of various quantities position velocity, acceleration etc. use vectors for representation

# Definition of vectors

1)Vector equality;

$$\vec{x}=\vec{y}\Rightarrow x_i=y_i\,,i=1,2,3$$

2)Vector addition

$$\vec{x}+\vec{y}=\vec{z}\Rightarrow x_i+y_i=z_i\,,i=1,2,3$$

3)Scalar Multiplication

$$a\vec{x}\Rightarrow\left(a x_{1,}a x_{2,}a x_3\right),\left(with\, a\, real\right)$$

4)Negative of a vector – inverse element of addition

$$\left(-1\right)\vec{x}\Rightarrow\left(-x_{1,}-x_{2,}-x_3\right),\qquad \vec{x}+\left(-\vec{x}\right)=0$$

5)Null vector

$$0\vec{x}\Rightarrow\left(0,0,0\right)$$

- Addition of vectors is commutative

$$\vec{x}+\vec{y}=\vec{y}+\vec{x}$$

- Addition of vectors is associative

$$(\vec{x}+\vec{y})+\vec{z}=\vec{y}+(\vec{x}+\vec{z})$$

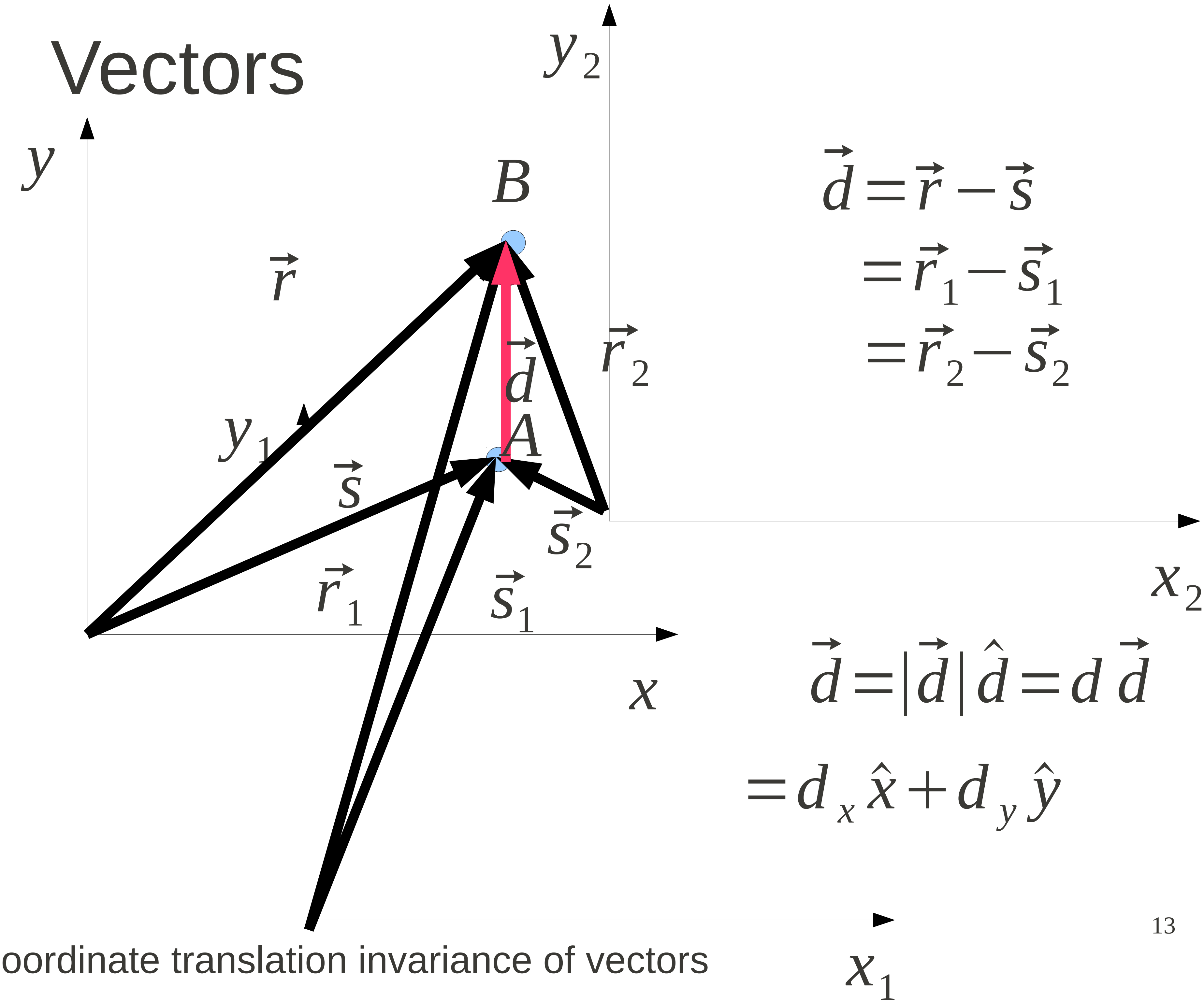
- Scalar multiplication is distributive

$$a(\vec{x}+\vec{y})=a\vec{x}+a\vec{y}$$

- Scalar multiplication is associative

$$(ab)\vec{x}=a(b\vec{x})$$

# Vectors




$$\vec{d} = \vec{r} - \vec{s}$$
$$= \vec{r}_1 - \vec{s}_1$$
$$= \vec{r}_2 - \vec{s}_2$$

$$\vec{d} = |\vec{d}|\hat{d} = d\,\vec{d}$$

$$= d_x \hat{x} + d_y \hat{y}$$

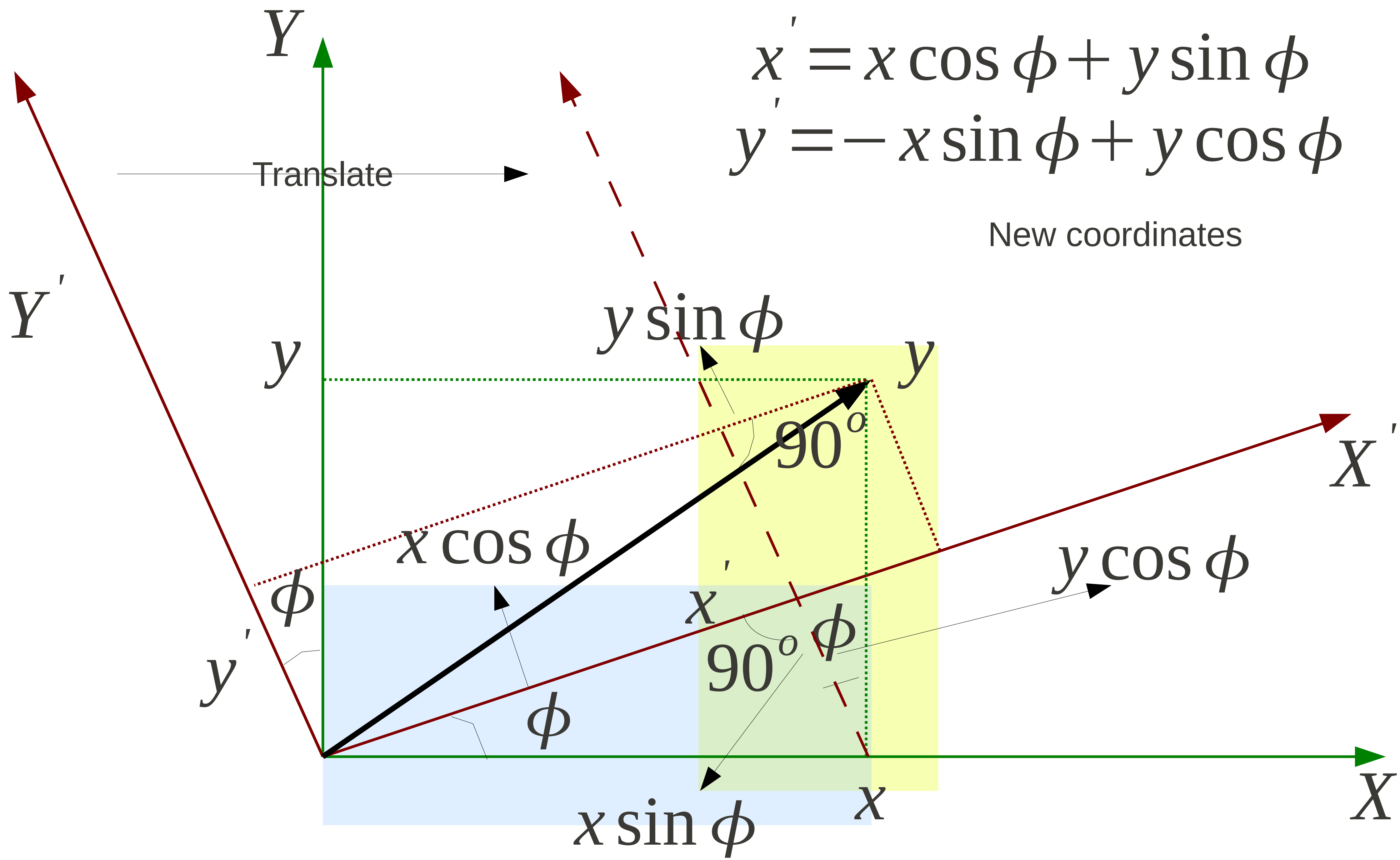
Coordinate translation invariance of vectors

Rotation of coordinate axis of vectors

Any three three dimensional rotation can be represented in two dimensions

$$x' = x\cos\phi + y\sin\phi$$
$$y' = -x\sin\phi + y\cos\phi$$

New coordinates




From the triangles highlighted by blue and yellow shaded regions

Whenever a pair of quantities transform as $(A_x, A_y) \rightarrow (A_x', A_y')$

$$A_x' = A_x \cos\phi + A_y \sin\phi$$

$$A_y' = -A_x \sin\phi + A_y \cos\phi$$

$\vec{A}$ can be an any arbitrary vector

Then we can call $(A_x, A_y)$ as components of vector $\vec{A}$

The magnitude of vector is invariant under rotation of the coordinate system

Converting system into a generalized and convenient form

$$x \rightarrow x_1 \qquad y \rightarrow x_2$$

$$a_{11} = \cos\phi \qquad a_{12} = \sin\phi$$

$$a_{21} = -\sin\phi \qquad a_{22} = \cos\phi$$

$$a_{ij}$$

$$x_1' = a_{11} x_1 + a_{12} x_2$$

$$x_2' = a_{21} x_1 + a_{22} x_2$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Matrix form

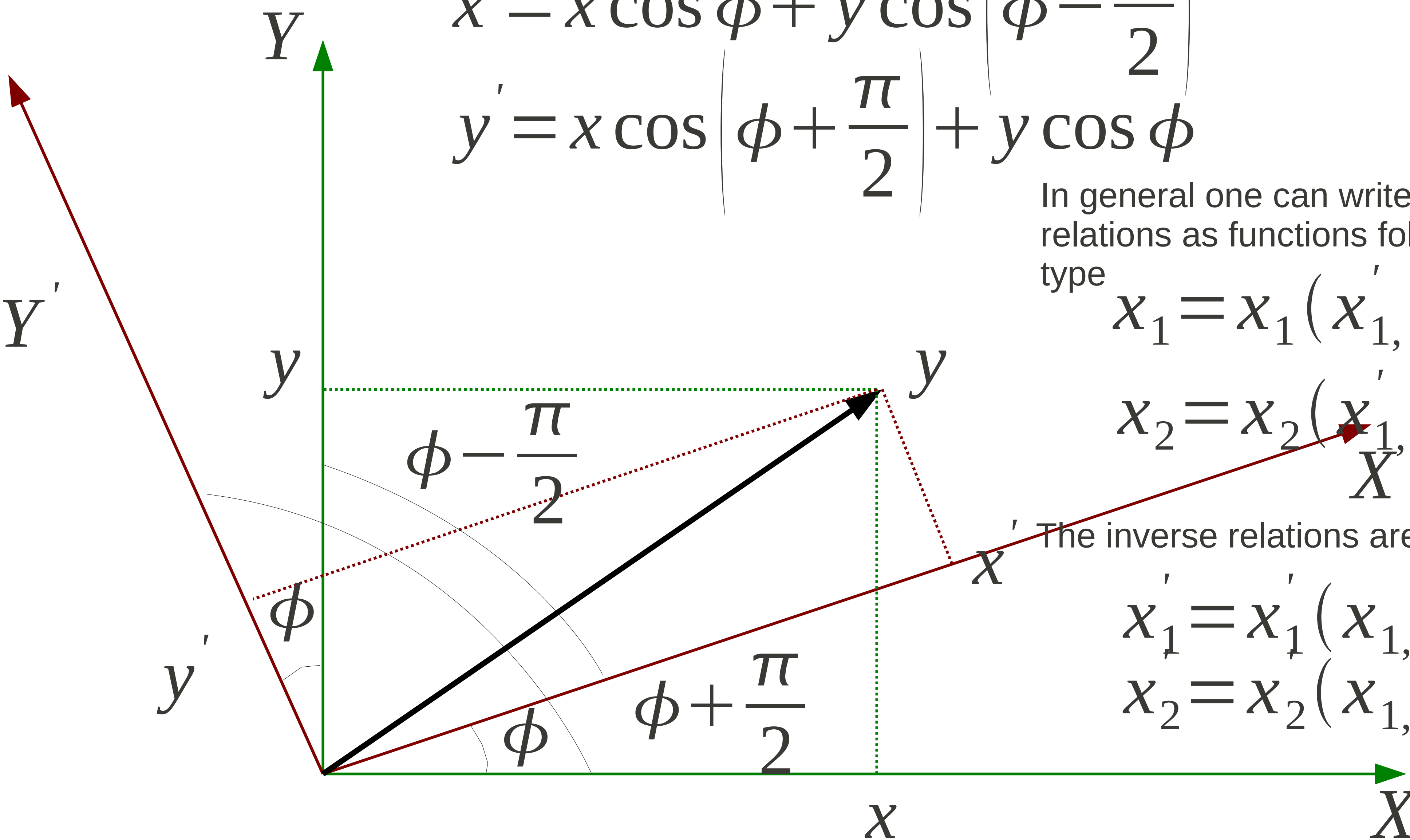
$a_{ij}$ is interpreted as **direction cosines** of angle between the axes

$$x' = x\cos\phi + y\sin\phi \qquad y' = -x\sin\phi + y\cos\phi$$

This relation may be re-written in terms of direction cosines

$$x' = x\cos\phi + y\cos\left(\phi - \frac{\pi}{2}\right)$$

$$y' = x\cos\left(\phi + \frac{\pi}{2}\right) + y\cos\phi$$




In general one can write the relations as functions following type

$$x_1 = x_1(x_1', x_2')$$

$$x_2 = x_2(x_1', x_2')$$

The inverse relations are

$$x_1' = x_1'(x_1, x_2)$$

$$x_2' = x_2'(x_1, x_2)$$

From the triangles highlighted by blue and yellow shade regions

$$a_{12} = \cos(x_1', x_2) = \sin\phi$$

$$a_{21} = \cos(\phi + \pi/2) = -\sin\phi$$

$$x_1' = a_{11} x_1 + a_{12} x_2 \qquad x_2' = a_{21} x_1 + a_{22} x_2$$

Hence the transformation rules can be written as summation

$$x_i' = \sum_{j=1}^{2} a_{ij} x_j \qquad i = 1,2.$$

The coefficient $a_{ij}$ can be expressed as

$$a_{ij} = \frac{\partial x_i'}{\partial x_j}$$

since

$$x_1 = x_1(x_1', x_2')$$

$$x_2 = x_2(x_1', x_2')$$

$$x_1' = x_1'(x_1, x_2)$$

$$x_2' = x_2'(x_1, x_2)$$

Using the inverse rotation $\phi \rightarrow -\phi$

$$x_j = \sum_{j=1}^{2} a_{ij} x_i' \qquad a_{ij} = \frac{\partial x_j}{\partial x_i'}$$

since the direction cosines are same

$$\Rightarrow a_{ij} = \frac{\partial x_j}{\partial x_i'} = \frac{\partial x_i'}{\partial x_j}$$

This leads to orthogonality relation

$$\sum_i a_{ij} a_{ik} = \sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_k}{\partial x_i'}$$

By inverting the second derivative using the relation $\dfrac{\partial x_j}{\partial x_i'} = \dfrac{\partial x_i'}{\partial x_j}$

$$= \sum_i \frac{\partial x_j}{\partial x_i'} \frac{\partial x_i'}{\partial x_k} = \delta_{jk}$$

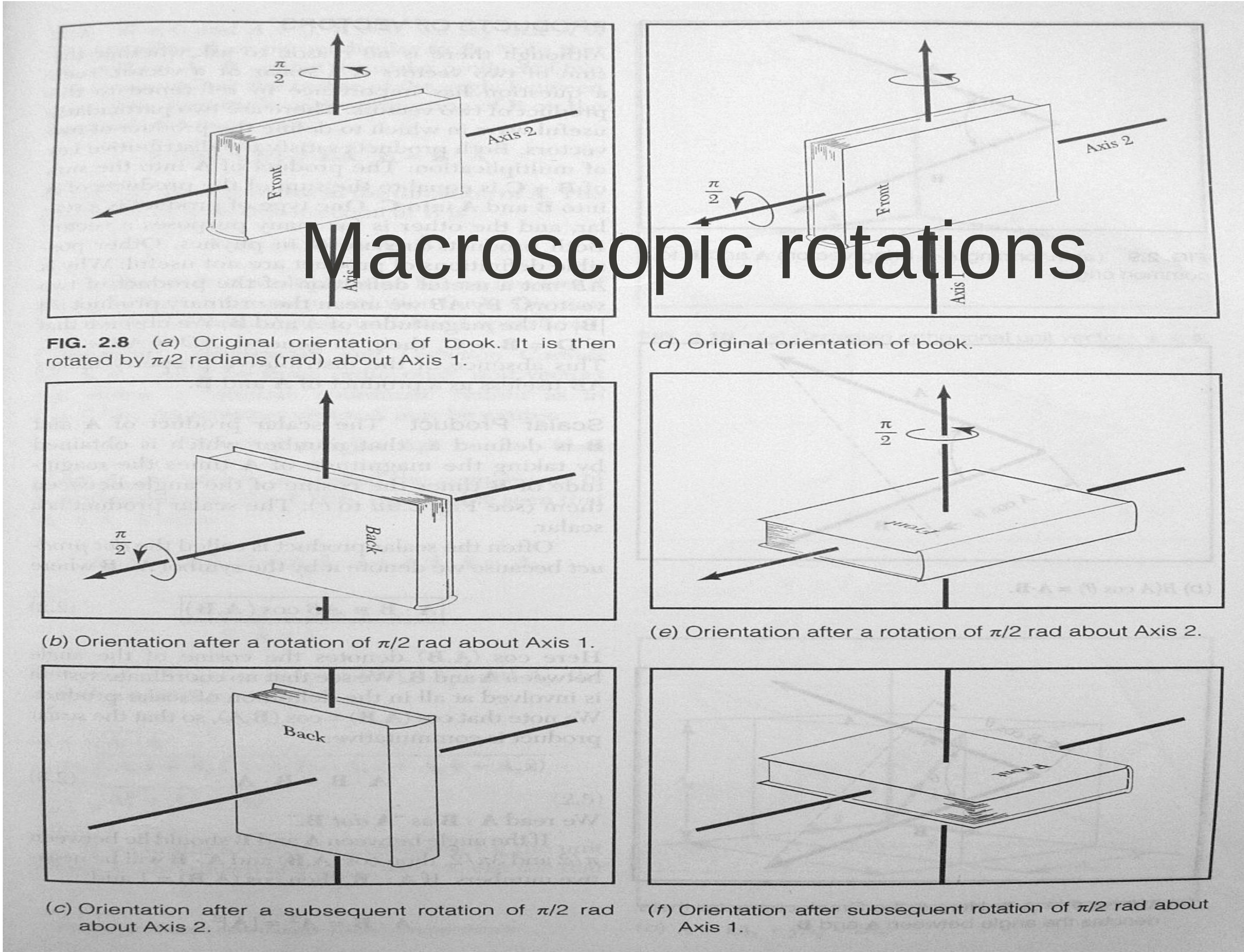
**Kronecker delta function** defined as

$\delta_{jk} = 0$ for $j \neq k$

$\delta_{jk} = 1$ for $j = k$

Every effect of the rotation should be canceled due to a reverse rotation in the opposite direction

# Macroscopic rotations

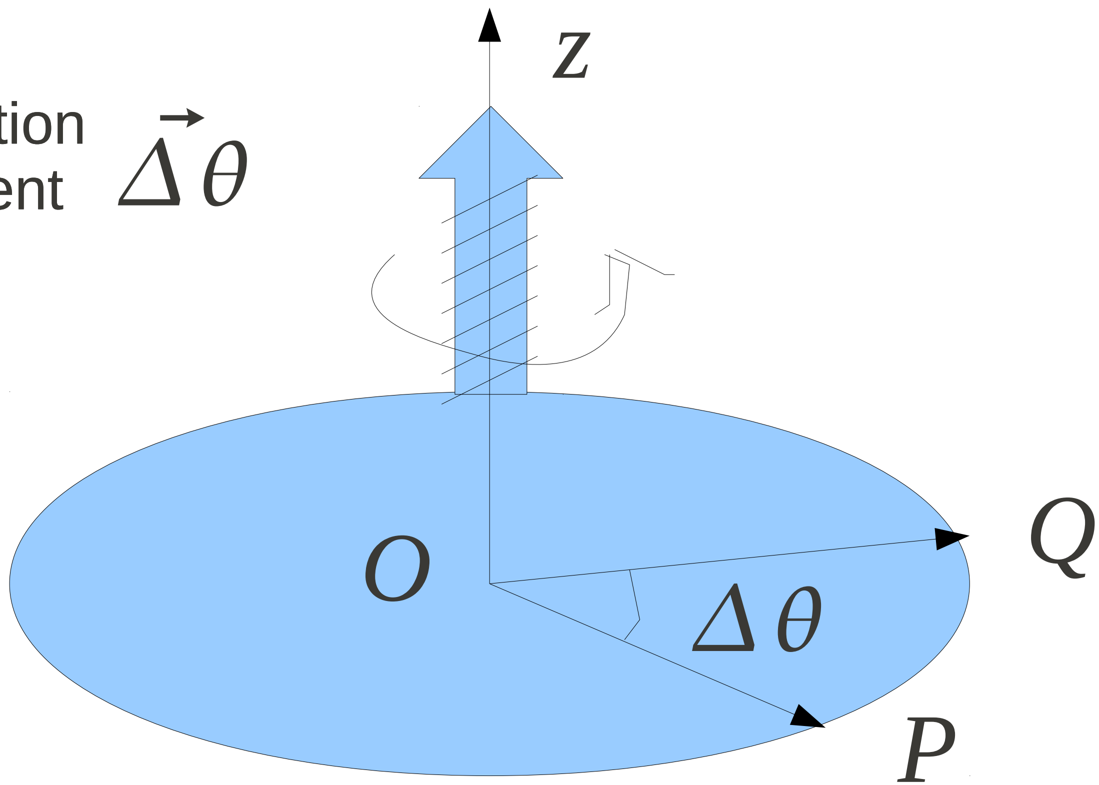



Quantities does not commute $\vec{x}+\vec{y}=\vec{y}+\vec{x}$

# Infinitesimal rotation as Vectors

Rotational quantities that can be represented as vectors by definition: it can only be confirmed by properties of vector addition, say commutation

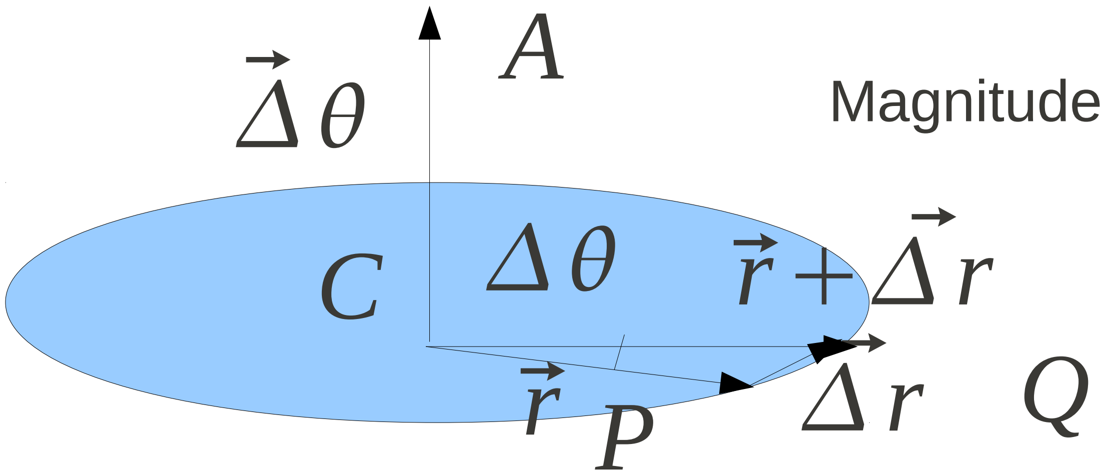
Consider a small rotation or angular displacement $\Delta\vec{\theta}$



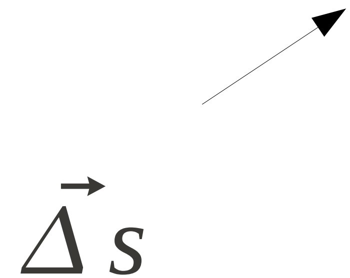

Textbook chapter 1

Length of $\Delta\vec{\theta}$ is proportional to magnitude of the rotation

The arrow head represent advancement of right handed screw




Magnitude proportional to the arc length




$$\vec{\Delta r} = \vec{\Delta\theta} \times \vec{r}$$

Let $\Delta\theta_1$ and $\Delta\theta_2$ be the two infinitesimal rotations

$$\vec{\Delta r}_1 = \vec{\Delta\theta}_1 \times \vec{r}$$

$$\vec{\Delta r}_2 = \vec{\Delta\theta}_2 \times \left(\vec{r} + \vec{\Delta r}_1\right)$$

The final position is given by

$$\vec{r} + \vec{\Delta r}_{12} = \vec{r} + \vec{\Delta r}_1 + \vec{\Delta r}_2$$

$$\vec{r}+\Delta\vec{r}_{12}=\vec{r}+\Delta\vec{\theta}_1\times\vec{r}+\Delta\vec{\theta}_2\times(\vec{r}+\Delta\vec{r}_1)$$

$$\simeq\vec{r}+(\Delta\vec{\theta}_1+\Delta\vec{\theta}_2)\times\vec{r}$$

neglecting the last term as infinitesimal displacement approaches zero

Now we reverse the order of the rotations

$$\vec{r}+\Delta\vec{r}_{21}=\vec{r}+\Delta\vec{r}_2+\Delta\vec{r}_1$$

$$=\vec{r}+\Delta\vec{\theta}_2\times\vec{r}+\Delta\vec{\theta}_1\times(\vec{r}+\Delta\vec{r}_2)$$

$$\simeq\vec{r}+(\Delta\vec{\theta}_2+\Delta\vec{\theta}_1)\times\vec{r}$$

neglecting the last term as in the previous case

therefore this relation is valid for very small rotations also vector addition of the of small rotation is commutative

This leads to definition of angular velocity $\vec{\omega}$

$$\vec{\omega} = \lim_{\Delta t \to 0} \frac{\vec{\Delta}\theta}{\Delta t} = \frac{d\vec{\theta}}{dt}$$

Now to find the relation between velocity and angular velocity

$$\vec{\Delta} r = \vec{\Delta}\theta \times \vec{r}$$

$$\lim_{\Delta t \to 0} \frac{\vec{\Delta} r}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\Delta}\theta}{\Delta t} \times \vec{r}$$

$$\Rightarrow \vec{v} = \vec{\omega} \times \vec{r}$$

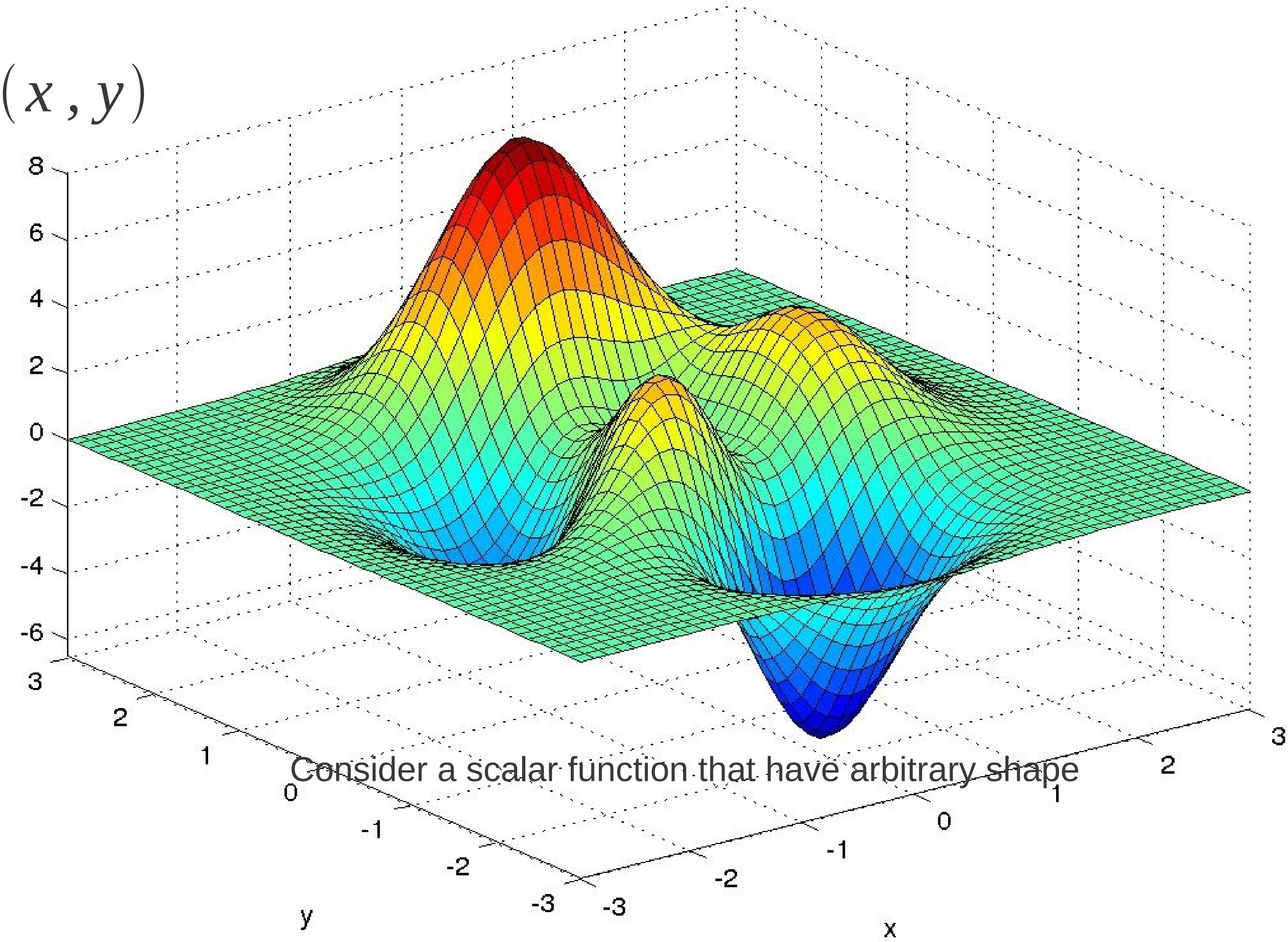
In a similar manner we can relate the angular momentum and

$$\vec{L} = \vec{r} \times \vec{p} \qquad \vec{N} = \vec{r} \times \vec{F}$$

# Vector calculus

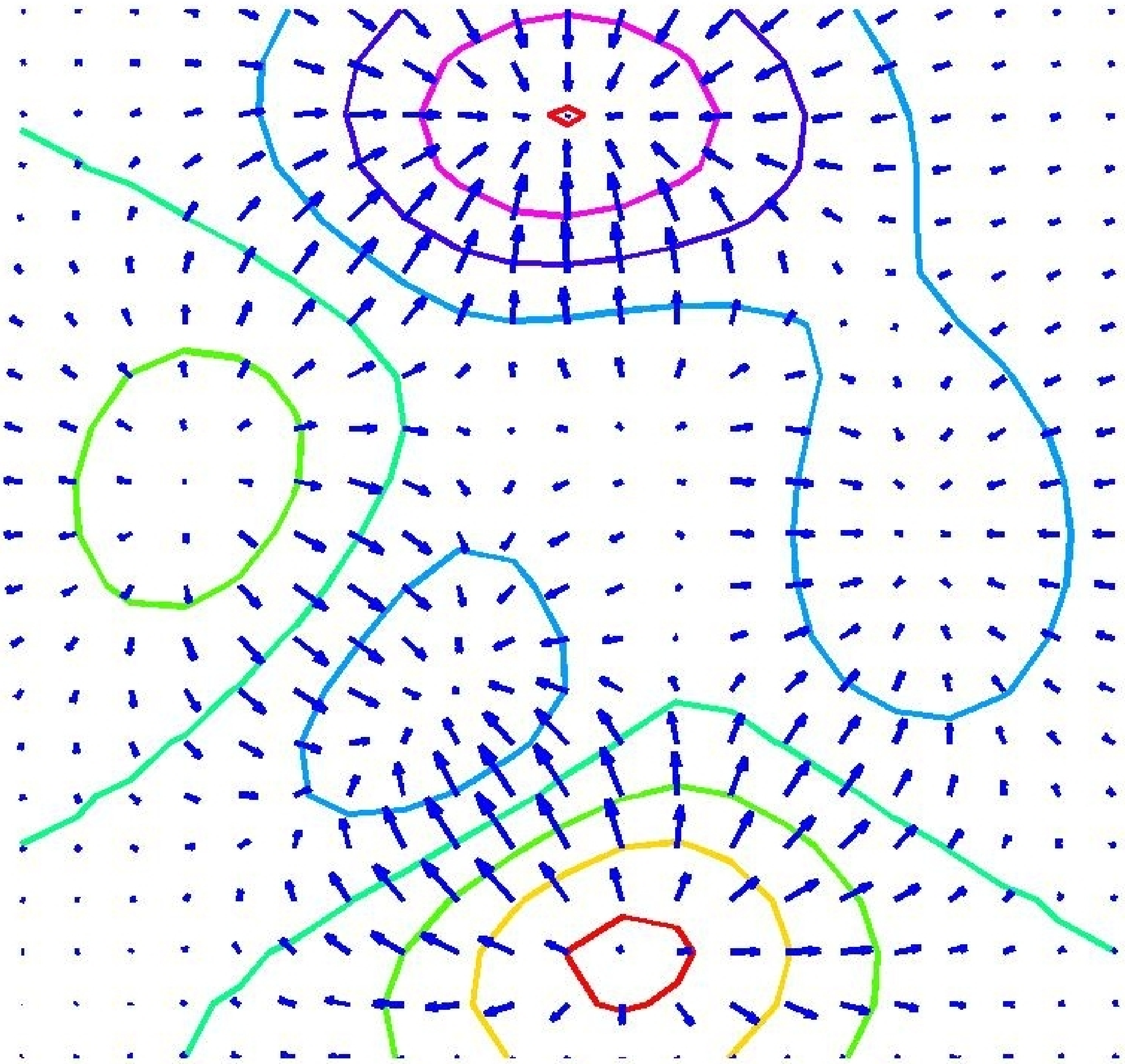
# Gradient

$F(x, y)$



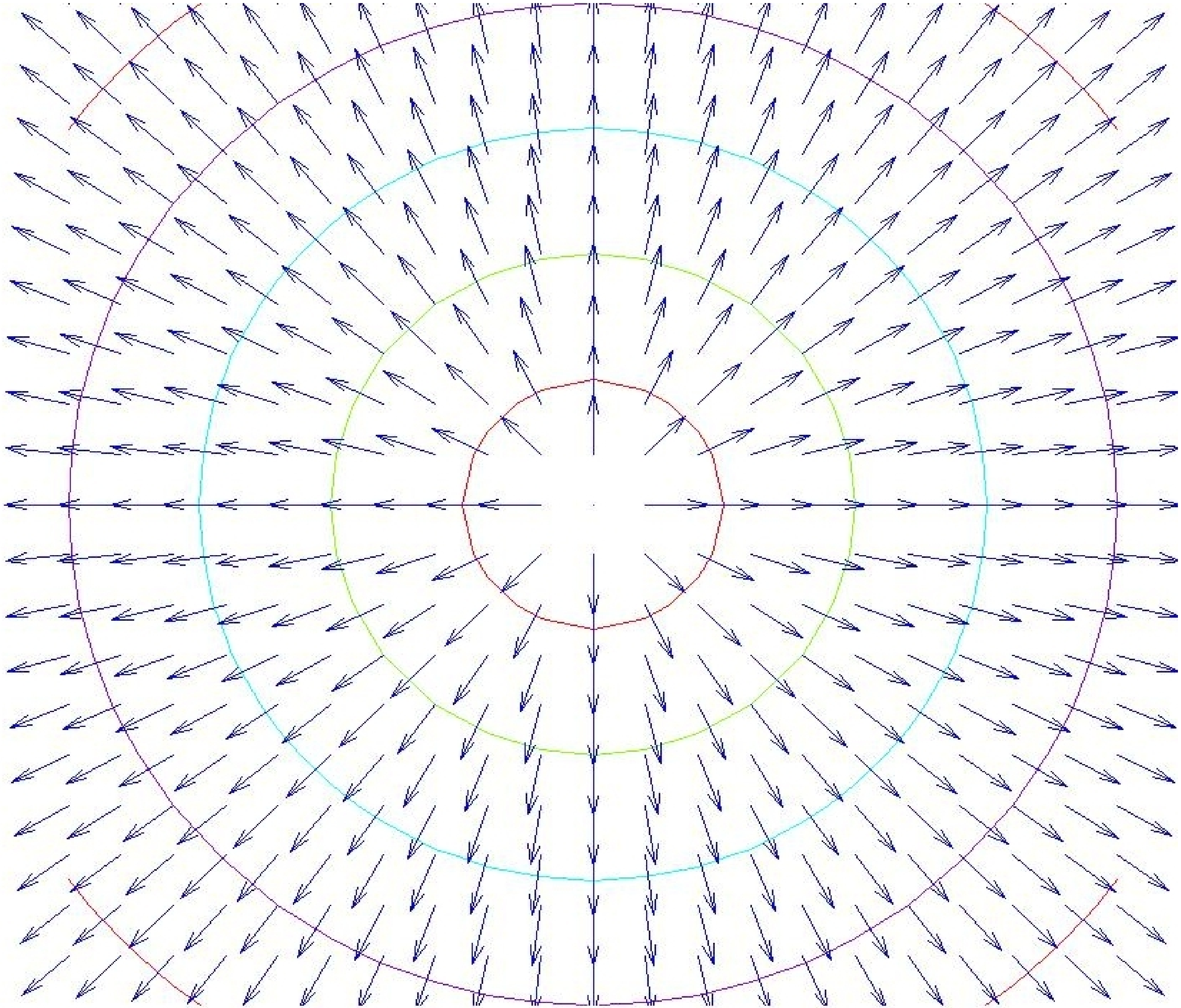

Consider a scalar function that have arbitrary shape

# Gradient of $F(x, y)$



The field that showing change of the function at any point. Arrow indicate direction of the rate of change and length of each arrow shows the magnitude

# Gradient of a function $f(r) = r$



$y$

$x$

Let $F(x, y)$ is a function of two variables $x, y$

The total variation of the function is

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

Total variation consists of two independent variations along $x$ and $y$

$$dF(x, y) = F(x+dx, y+dy) - F(x, y)$$

$$= F(x+dx, y) - F(x, y) +$$

$$F(x, y+dy) - F(x, y)$$

$$= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

This result is valid when the $F(x, y)$ is a continuous function

Chain rule of partial differentiation is given by

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \frac{dx}{dt} + \frac{\partial F}{\partial y} \frac{dy}{dt}$$

When $x, y$ and $F$ are functions of $t$

These results can be generalized to many dimensions, in 3 dimension

$$d\phi(x,y,z) = \left[\phi(x+dx, y+dy, z+dz) - \phi(x, y+dy, z+dz)\right]$$

$$+\left[\phi(x+dx, y+dy, z+dz) - \phi(x+dx, y, z+dz)\right]$$

$$+\left[\phi(x+dx, y+dy, z+dz) - \phi(x+dx, y+dy, z)\right]$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

This result can regarded algebraically as a scalar product of $d\vec{r}$ and direction dependent change in the function $\phi$ .

Suppose the $\phi(x, y, z)$ is scalar point function whose value depends on the coordinates $(x, y, z)$.

The transformation properties of the components can be used to identify the vectorial nature of the direction dependent change.

Using the fundamental properties according to which components of a vector transform under rotation of axis

$$\phi'(x_1', x_2', x_3') = \phi(x_1, x_2, x_3)$$

$$(x, y, z) = (x_1, x_2, x_3)$$

$$(x', y', z') = (x_1', x_2', x_3')$$

Differentiating with respect to $x'$. using the chain rule

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

$$\frac{\partial \phi'(x_1', x_2', x_3')}{\partial x_i'} = \frac{\partial \phi(x_1, x_2, x_3)}{\partial x_i'}$$

$$= \sum_j \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial x_i'} = \sum_j a_{ij}\frac{\partial \phi}{\partial x_j}$$

We have constructed a quantity that transforms like components of a vector(slide 17). Therefore we can define a vector from components as

$$\vec{\nabla}\phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$

$\vec{\nabla}\phi$ is the gradient of scalar field $\phi$

operator is called del or nabla

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$\vec{\nabla}$ Is a hybrid function both partial derivatives and of vectorial nature

## Application of gradient in the calculation of force – an Example

$$\vec{F} = -\vec{\nabla} V$$

The gradient of a potential

$$V(r) = V\left(\sqrt{x^2 + y^2 + z^2}\right)$$

$$\vec{\nabla} V(r) = \hat{x}\frac{\partial V(r)}{\partial x} + \hat{y}\frac{\partial V(r)}{\partial y} + \hat{z}\frac{\partial V(r)}{\partial z}$$

$$\frac{\partial V(r)}{\partial x} = \frac{\partial V(r)}{\partial r}\frac{\partial r}{\partial x}$$

$$\frac{\partial r}{\partial x}=\frac{\partial\left(x^2+y^2+z^2\right)^{1/2}}{\partial x}=\frac{x}{\left(x^2+y^2+z^2\right)^{1/2}}=\frac{x}{r}$$

$$\Rightarrow\frac{\partial V(r)}{\partial x}=\frac{\partial V(r)}{\partial r}\frac{x}{r}$$

Similarly

$$\frac{\partial r}{\partial y}=\frac{y}{r}\qquad\frac{\partial r}{\partial z}=\frac{z}{r}$$

$$\vec{\nabla}V(r)=\hat{x}\frac{\partial V(r)}{\partial x}+\hat{y}\frac{\partial V(r)}{\partial y}+\hat{z}\frac{\partial V(r)}{\partial z}$$

$$\Rightarrow\vec{\nabla}V(r)=\frac{\partial V(r)}{\partial r}\frac{1}{r}\left(\hat{x}\,x+\hat{y}\,y+\hat{z}\,z\right)$$

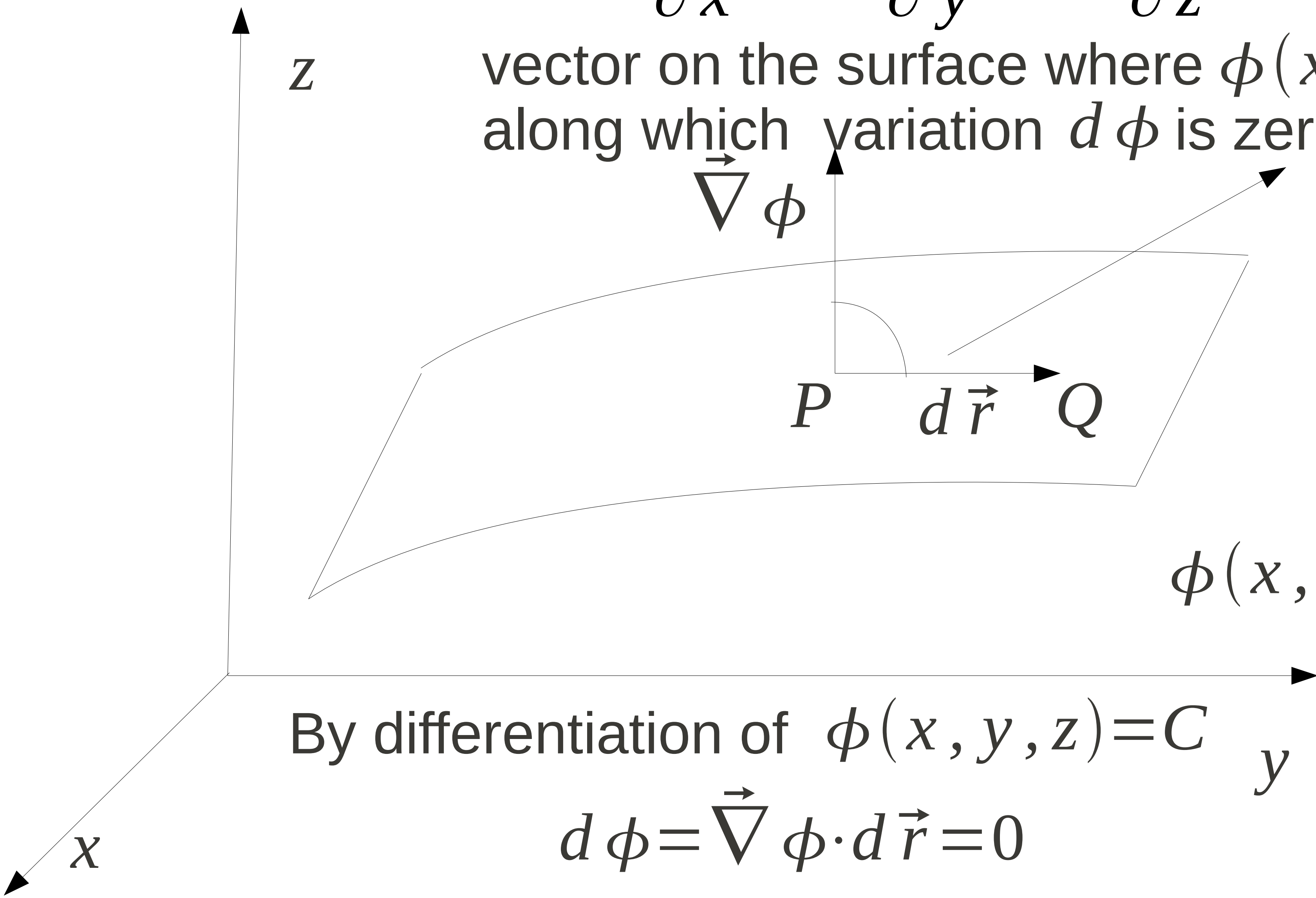
$$=\frac{\partial V(r)}{\partial r}\frac{\vec{r}}{r}=\hat{r}\frac{\partial V(r)}{\partial r}$$

# Geometrical interpretation or direction of a gradient

$$d\vec{r} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$$

$$\vec{\nabla}\phi \cdot d\vec{r} = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

vector on the surface where $\phi(x, y, z) = C$
along which variation $d\phi$ is zero

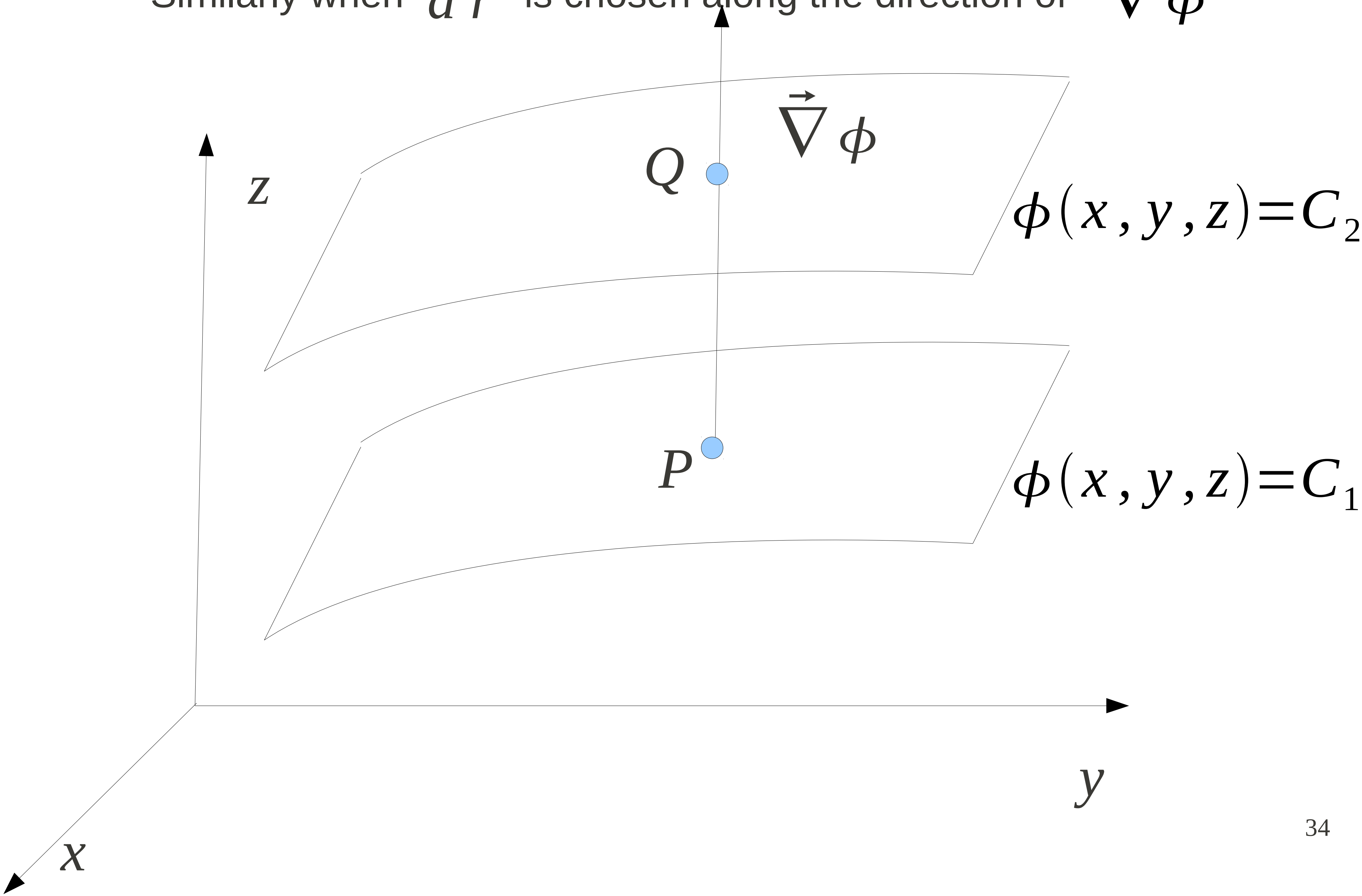



By differentiation of $\phi(x, y, z) = C$

$$d\phi = \vec{\nabla}\phi \cdot d\vec{r} = 0$$

$$d\phi = C_1 - C_2 = \Delta C = \vec{\nabla}\phi \cdot d\vec{r}$$

Similarly when $d\vec{r}$ is chosen along the direction of $\vec{\nabla}\phi$



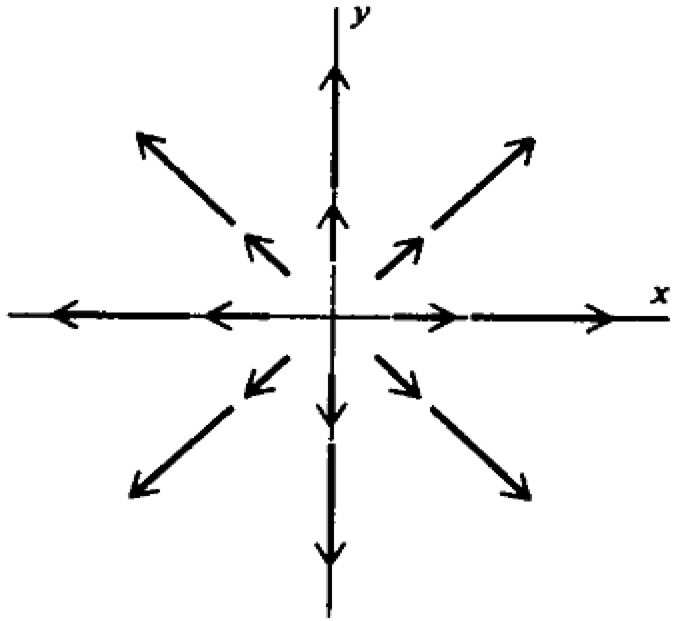
# Divergence $\vec{\nabla}\cdot$

$$\vec{\nabla}\cdot\vec{V}=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial y}$$

$$V_x=V_x(x,y,z)$$
$$V_y=V_y(x,y,z)$$
$$V_z=V_z(x,y,z)$$

Divergence of radial vector

$$\vec{\nabla}\cdot\vec{r}=3$$



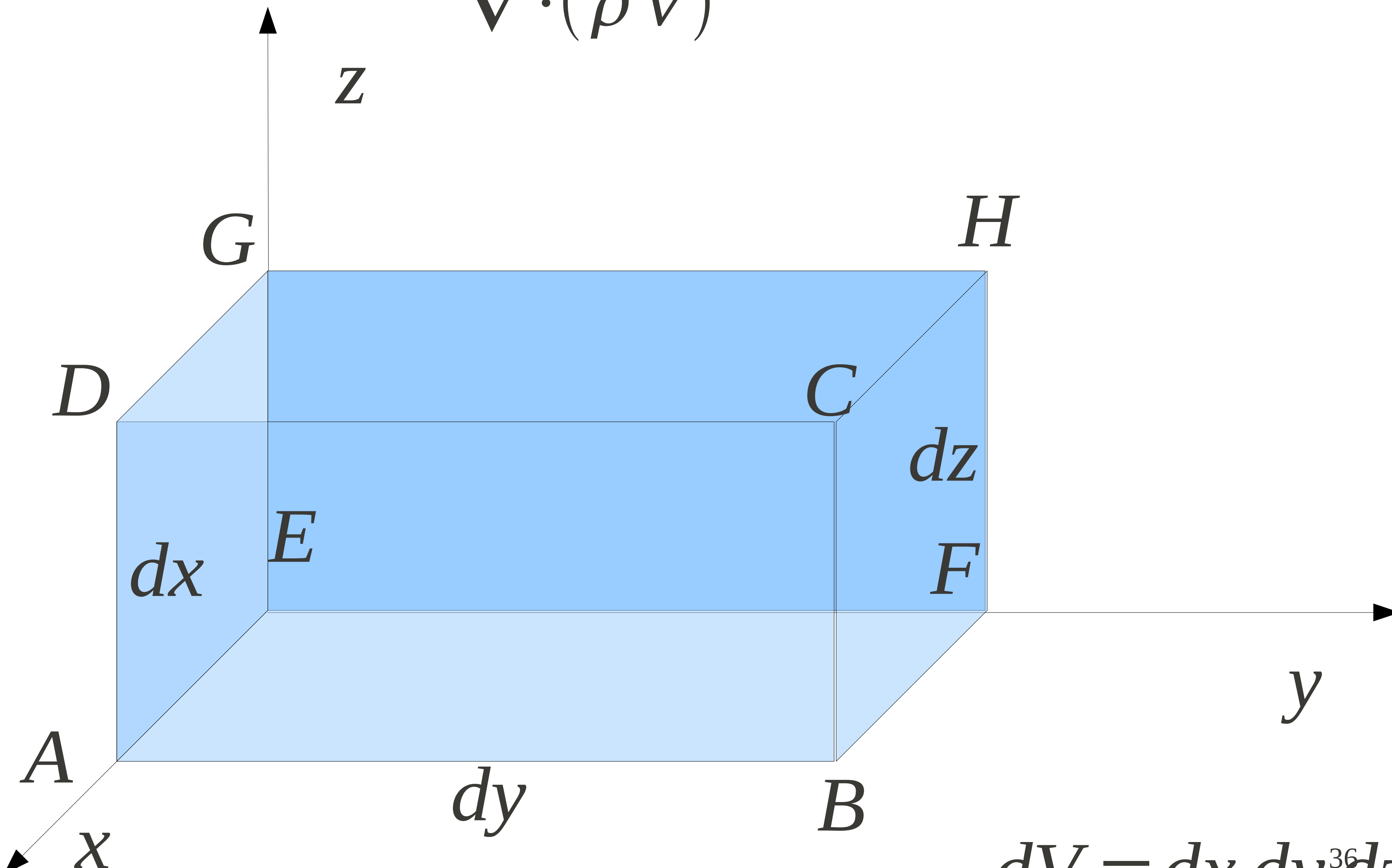

$$\vec{F}=x\,\hat{x}+y\,\hat{y}$$

$$\vec{\nabla}\cdot\vec{F}=2$$

Physical meaning of divergence

Velocity of a compressible fluid with density $\rho(x, y, z)$ is $\vec{v}$

$$\vec{\nabla}\cdot(\rho\vec{v})$$




$$dV = dx\,dy\,dz$$

Consider a small volume defined by the element

$$dV = dx\, dy\, dz$$

The component of the flow are $\left(\rho\, v_x, \rho\, v_y, \rho\, v_z\right)$

fluid flowing into this volume through the face EFGH per unit time

$$\rho\, v_x\Big|_{x=0} dy\, d\, z$$

other components $\rho\, v_y, \rho\, v_z$ have no contribution in this direction

fluid flowing into this volume through the face ABCD per unit time

$$\rho\, v_x\Big|_{x=dx} dy\, d\, z$$

This quantity is expanded as

$$\rho\, v_x\Big|_{x=dx} dy\, d\, z = \left[\rho\, v_x + \frac{\partial}{\partial\, x}(\rho\, v_x)\, dx\right]_{x=0} dy\, dz$$

Original flow    Correction in the flow

The net flow out is at face ABCD $= \frac{\partial}{\partial x}(\rho v_x)\, dx\, dy\, dz$

We can arrive at the result as

$$= \lim_{\Delta x \to 0} \frac{\rho v_x(\Delta x, 0, 0) - \rho v_x(0,0,0)}{\Delta x}$$

$$= \frac{\partial}{\partial x}(\rho v_x(x, y, z))\Big|_{0,0,0}$$

The formulation in the x direction can adapted into y and z direction also

By change  of the coordinates

$$x \rightarrow y \quad y \rightarrow z$$

Total net flow out the volume is

$$= \left( \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) \right) dx\, dy\, dz$$

$$= \vec{\nabla} \cdot (\rho \vec{v})\, dx\, dy\, dz$$

The net flow out the volume element $dx\,dy\,dz$ out of a compressible fluid is $=\vec{\nabla}\cdot(\rho\,\vec{v})$

Application of divergence: rate of change of density with respect time is equal to the divergence velocity field of that liquid element

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\,\vec{v})=0$$

Divergence of combinations of a scalar and and vector

$$\vec{\nabla}\cdot(f\,\vec{V})=\frac{\partial}{\partial x}(f\,V_x)+\frac{\partial}{\partial y}(f\,V_y)+\frac{\partial}{\partial z}(f\,V_z)$$

$$=\frac{\partial f}{\partial x}V_x+f\,\frac{\partial V_x}{\partial x}+\frac{\partial f}{\partial y}V_y+f\,\frac{\partial V_y}{\partial y}$$

$$+\frac{\partial f}{\partial z}V_z+f\,\frac{\partial V_z}{\partial z}$$

$$=\vec{\nabla}f\cdot\vec{V}+f\,\vec{\nabla}\cdot\vec{V}$$

$$\vec{V}\cdot\vec{\nabla}\neq\vec{\nabla}\cdot\vec{V}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

If bubble does not break inside volume, density of bubbles inside the volume obey this equation

rate at which bubbles density change

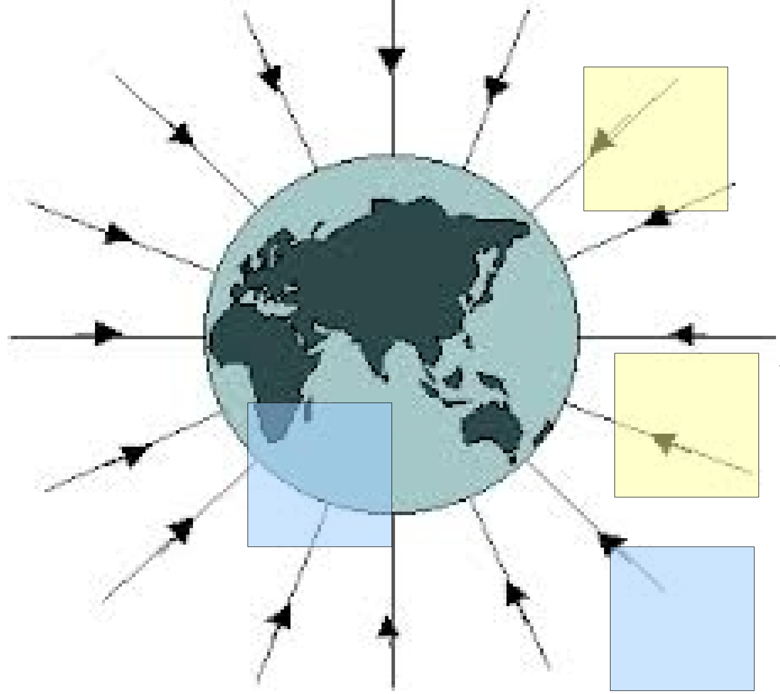
continuity equation

$$\vec{\nabla} \cdot (\rho \vec{v}) = 0$$

If bubbles do not break inside the volume

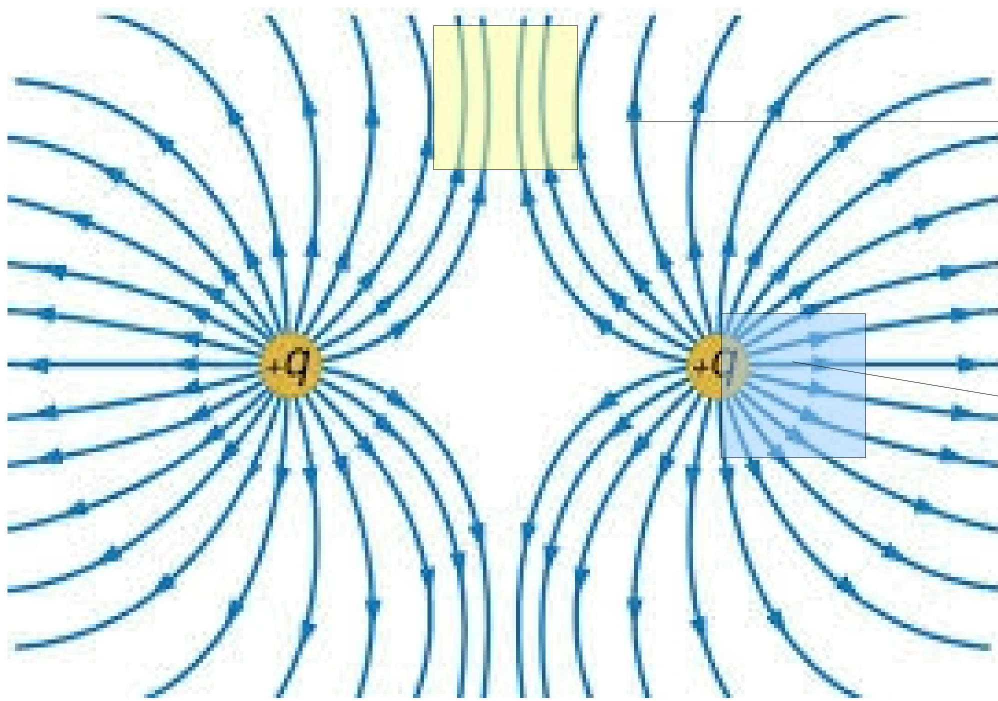
Usage of del operator over vector field

Gravitational field lines



$$\vec{\nabla}\cdot(\mathit{field\ vector})=0$$

$$\vec{\nabla}\cdot(\mathit{field\ vector})\neq 0$$




Electric field lines

$$\vec{\nabla}\cdot\vec{E}=0$$

$$\vec{\nabla}\cdot\vec{E}\neq 0$$

# Curl $\vec{\nabla}\times$

Another possible vector operator is the cross product

$$\vec{\nabla}\times\vec{V}=\hat{x}\left(\frac{\partial}{\partial y}(V_z)-\frac{\partial}{\partial z}(V_y)\right)+\hat{y}\left(\frac{\partial}{\partial z}(V_x)-\frac{\partial}{\partial x}(V_z)\right)$$

$$+\hat{z}\left(\frac{\partial}{\partial x}(V_y)-\frac{\partial}{\partial y}(V_x)\right)$$

$$V_x=V_x(x,y,z)$$

$$V_y=V_y(x,y,z)$$

$$V_z=V_z(x,y,z)$$

In matrix form

$$=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ V_x & V_y & V_z\end{vmatrix}$$

$$\vec{\nabla}\times\vec{V}\neq-\vec{V}\times\vec{\nabla}$$

When a scalar involved - Do it as an exercise

$$\vec{\nabla}\times(f\vec{V})=f\,\vec{\nabla}\times\vec{V}+(\vec{\nabla}f)\times\vec{V}$$

Curl of central force field

$$\vec{\nabla}\times(\vec{r}\,f(r))=f(r)\,\vec{\nabla}\times r+[\vec{\nabla}f(r)]\times\vec{r}$$

$$\vec{\nabla}\times r=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & y & z\end{vmatrix}=0$$

Using the relation $\vec{\nabla}f(r)=\hat{r}\dfrac{\partial f}{\partial r}$

We get $\vec{\nabla}f(r)\times\vec{r}=\dfrac{\partial f}{\partial r}\hat{r}\times\vec{r}=0$

$$\vec{\nabla}\times(\vec{r}\,f(r))=0$$

Consider the circulation of fluid around a differential loop in the x-y plane

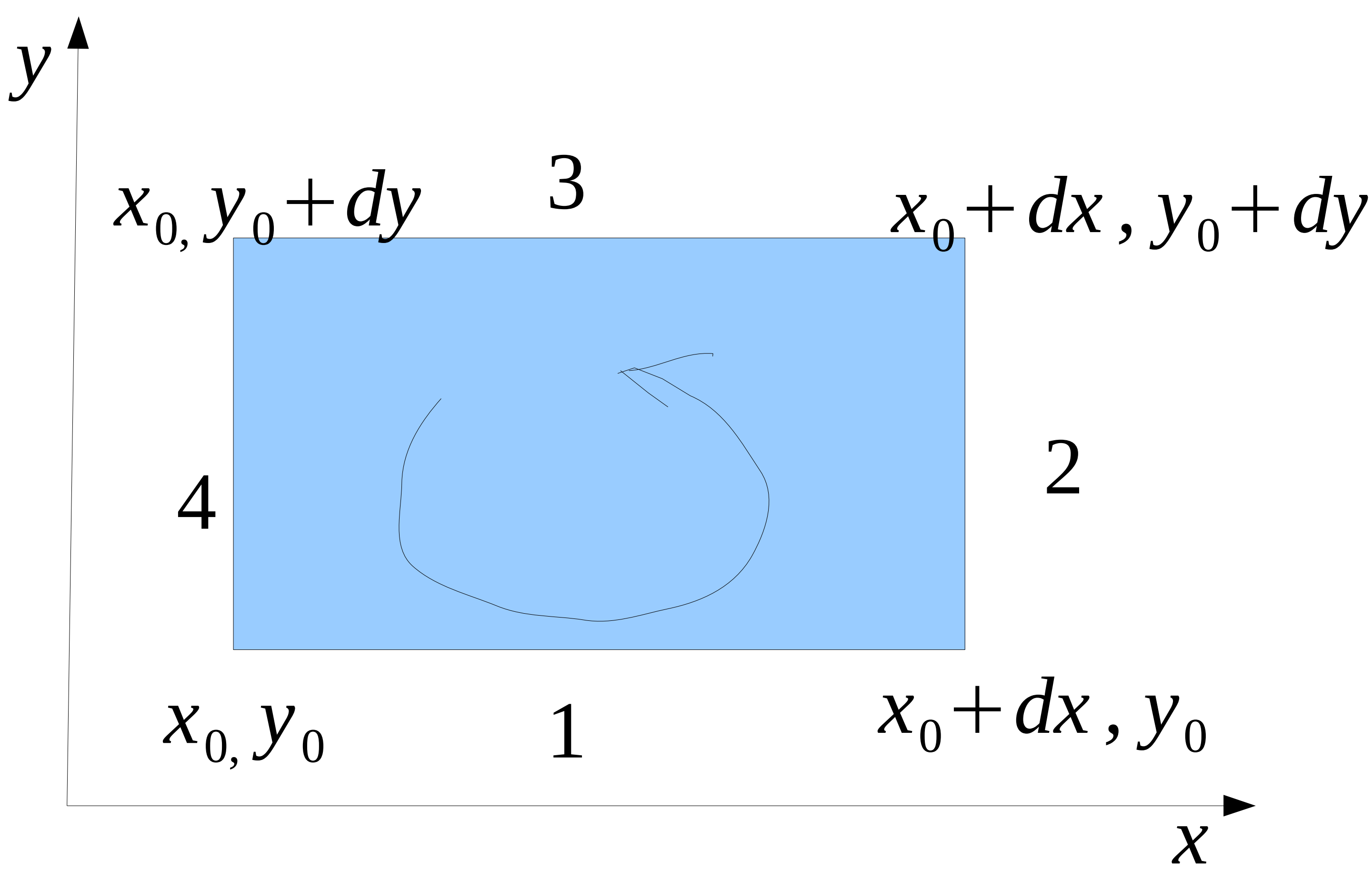
Let the vector function be defined as

$$\vec{V} = \hat{x}\, V_x(x, y) + \hat{y}\, V_y(x, y)$$

Circulation $C_{1234}$ over a loop is given by the vector integral $\int \vec{V} \cdot d\vec{\lambda}$
a scalar equivalent can be set up

$$C_{1234} = \int_1 V_x(x,y)\, d\lambda_x + \int_2 V_y(x,y)\, d\lambda_y + \int_3 V_x(x,y)\, d\lambda_x + \int_4 V_y(x,y)\, d\lambda_y$$

1,2,3,4 refer respectively to the line segments



$$C_{1234}=\int_1 V_x(x,y)\,dx+\int_2 V_y(x,y)\,dy-\int_3 V_x(x,y)\,dx$$
$$-\int_4 V_y(x,y)\,dy$$

For a very small line element

change of y component in x direction

$$C_{1234}=V_x(x_{0,}y_0)\,dx+\left[V_y(x_{0,}y_0)+\frac{\partial V_y}{\partial x}dx\right]dy-$$
$$\left[V_x(x_{0,}y_0)+\frac{\partial V_x}{\partial y}dy\right]dx\quad -V_y(x_{0,}y_0)\,dy$$

change of x component in y direction

$$C_{1234}=\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)dx\,dy$$

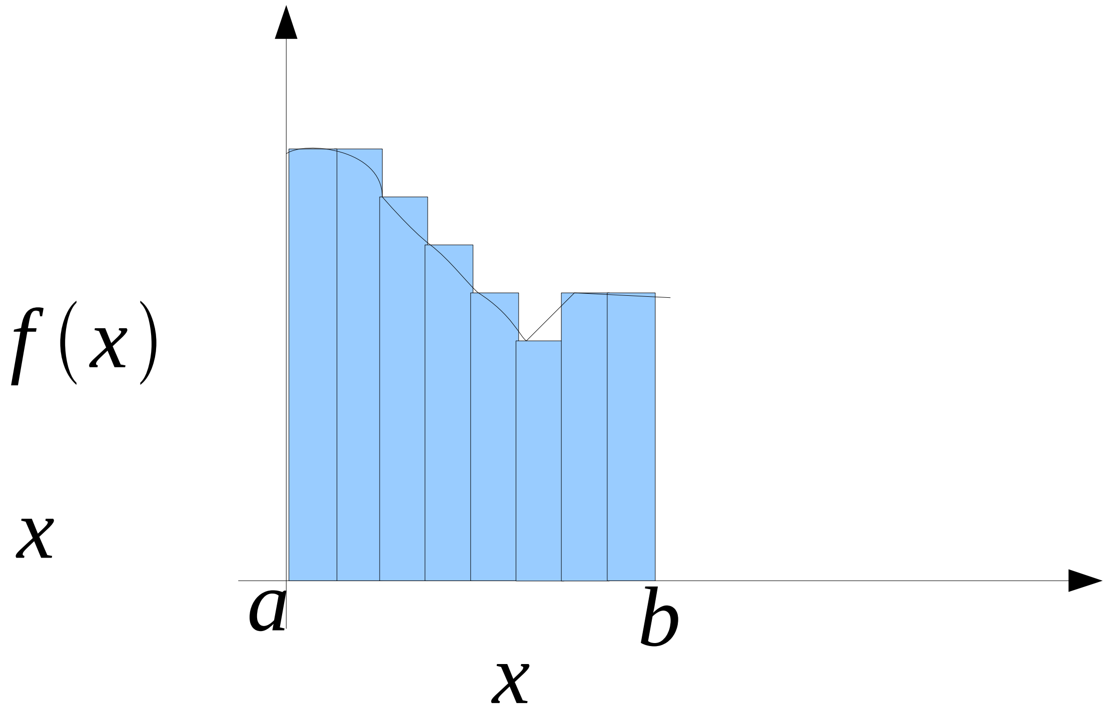
By comparing the term with standard component of a curl operator

The circulation per unit area

$$C_{1234}=\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)=\vec{\nabla}\times\vec{V}\Big|_z$$

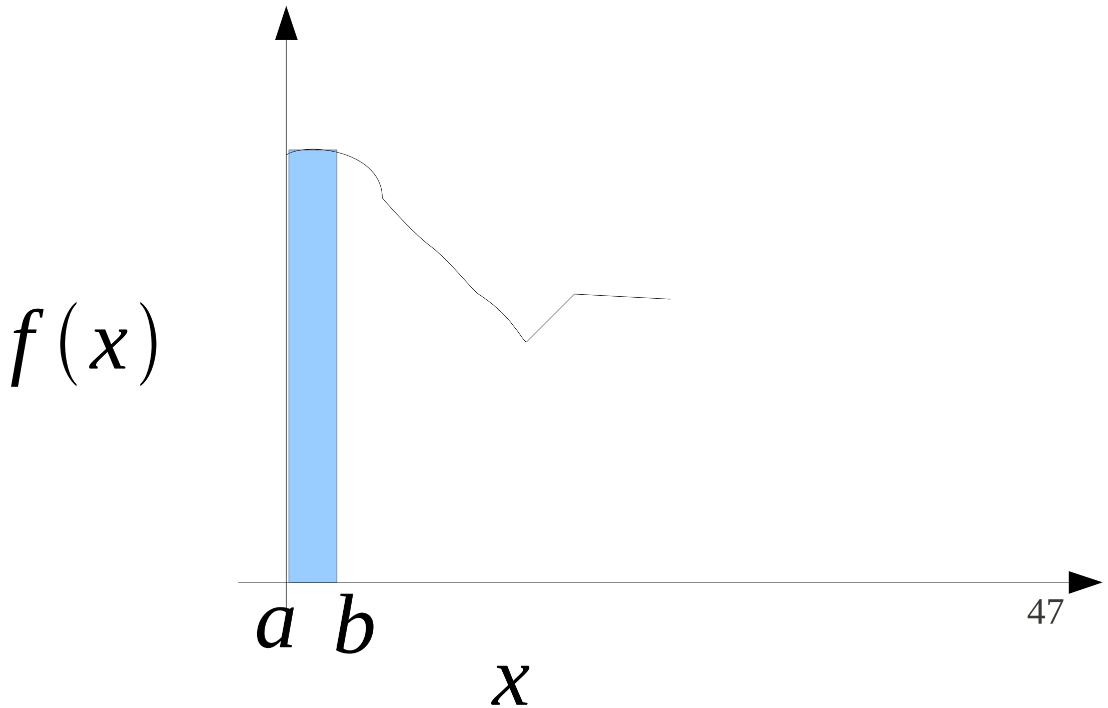
## Converting integral into sum

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$




As $b \to a$

$$\int_a^b f(x)\,dx = f(a)\,\Delta x$$



If we consider the small volume the total circulation is given by the curl of the velocity

$$\vec{\nabla} \times \vec{V}$$

When $\vec{\nabla} \times \vec{V} = 0$ the corresponding vector is called irrotational

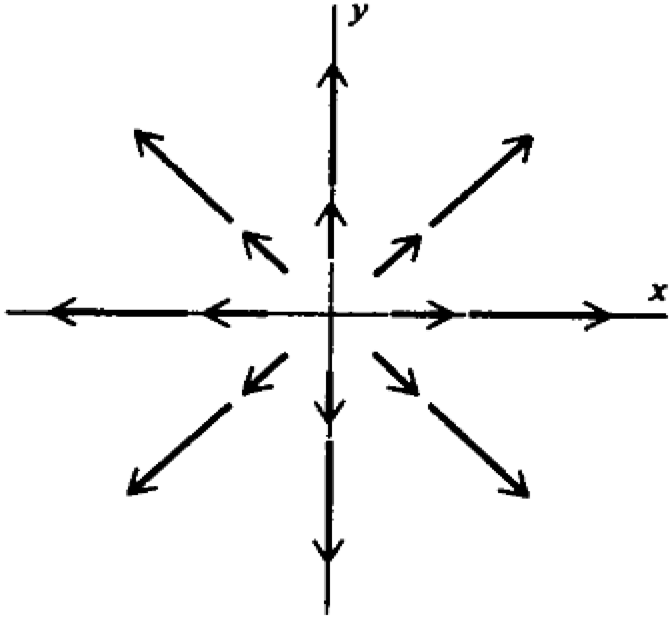
Gravitational and electrostatic forces represent **irrotational vectors**

Irrotational vectors can be represented by gradient of a scalar function

$$\vec{\nabla} \cdot \vec{V} = 0$$

Then the fields are called **solenoidal**

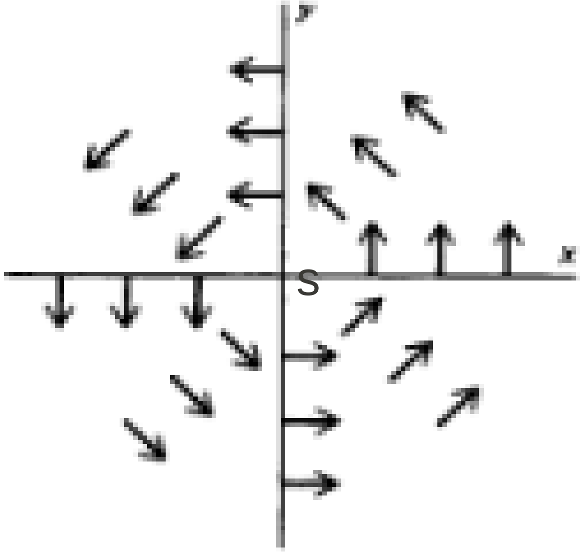
# Irrotational field




$$\vec{F} = x\,\hat{x} + y\,\hat{y}$$

$$\vec{\nabla} \times \vec{F} = 0$$

# Solenoidal field




$$\vec{G} = -x\,\hat{y} + y\,\hat{y}$$

# Successive application of $\vec{\nabla}$

$$\vec{\nabla}\cdot\vec{\nabla}\,\phi$$

$$\vec{\nabla}\cdot\vec{\nabla}\,\phi=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\phi$$

$$\vec{\nabla}\cdot\vec{\nabla}\,\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}$$

When $\phi$ is the electrostatic potential

$$\vec{\nabla}\cdot\vec{\nabla}\,\phi=0 \longrightarrow \text{Laplace equation}$$

$$\nabla^2\phi=0$$

Important equation in electromagnetism, fluid dynamics, astronomy etc..

# Vector integration

## Line integrals

Vectorial line element is $d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$

the possible forms of line integrals are

$$\int_c \phi\, d\vec{r} \qquad \int_c \vec{V}\cdot d\vec{r} \qquad \int_c \vec{V}\times d\vec{r}$$

The contour C may be of closed or open form

When the integrand is scalar

$$\int_c \phi\, d\vec{r} = \hat{x}\int_c \phi(x,y,z)\,dx + \hat{y}\int_c \phi(x,y,z)\,dy +$$

$$\hat{z}\int_c \phi(x,y,z)\,dz$$

We assume that the unit vectors are not variables –
this is true for Cartesian coordinates

$$\int_c \hat{x}\,\phi\,dx = \hat{x}\int \phi\,dx$$

When $\phi=1$ it gives the distance between two points

$$\hat{x}\int_c \phi(x,y,z)\,dx$$

To evaluate this integral y and z need to be expressed in terms of x. The implies evaluation requires specification of explicit path

$$\int_c \vec{V}\cdot d\vec{r}$$

This integral resembles to that of integral for calculating the work done by a force.

$$W=\int_c \vec{F}\cdot d\vec{r}=\int F_x(x,y,z)\,dx+\int F_y(x,y,z)\,dy$$
$$+\int F_z(x,y,z)\,dz$$

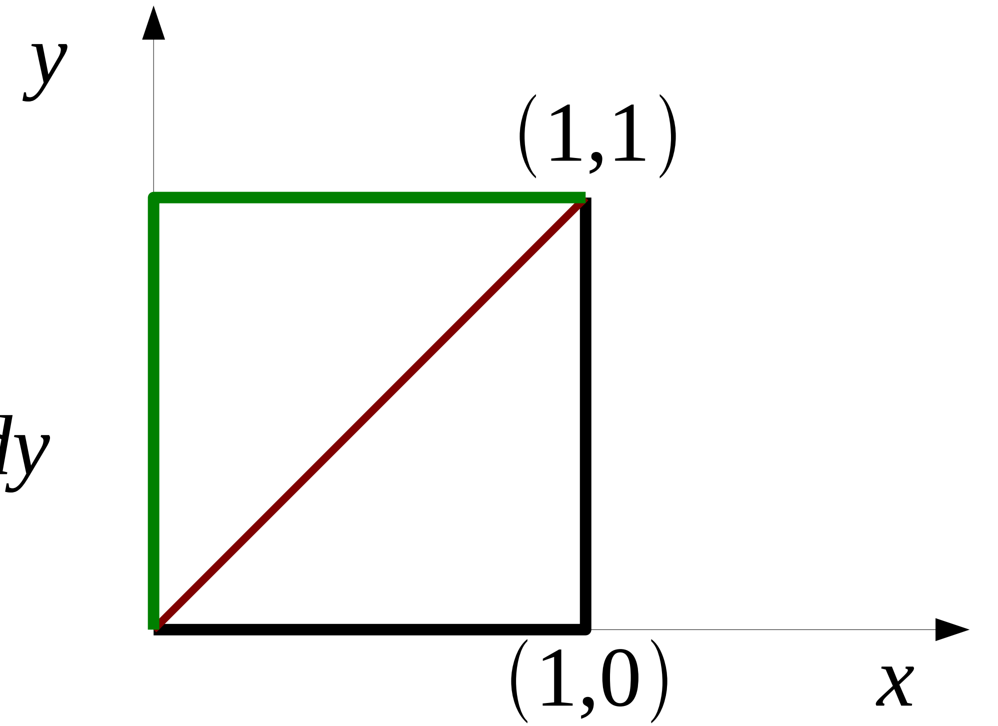
Force exerted on the particle

$$\vec{F} = -\hat{x}\,y + \hat{y}\,x$$

$$d\vec{r} = dx\,\hat{x} + dy\,\hat{y}$$

$$W = \int_{0,0}^{1,1} \vec{F}\cdot d\vec{r} = \int_{0,0}^{1,1} -y\,dx + x\,dy$$

$$W = -\int_0^1 y\,dx + \int_0^1 x\,dy$$




This requires specification of x in terms of y and y in terms of x

Now consider the path (black) shown in figure then evaluate integral along this path

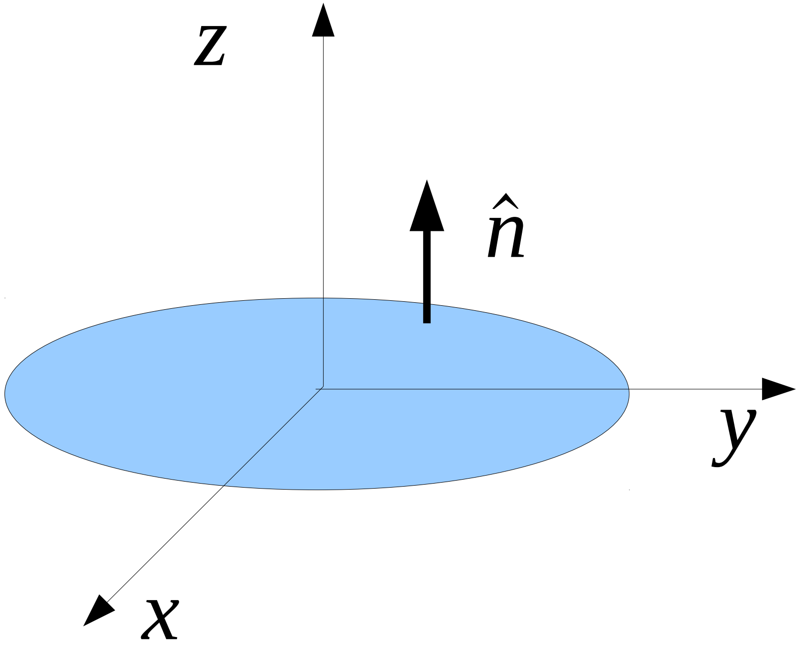
$$W = -\int_0^1 0\,dx + \int_0^1 1\,dy = 1$$

consider the path (green)

$$W = -\int_0^1 1\,dx + \int_0^1 0\,dy = -1$$

## Surface integral

The element of area is $d\vec{\sigma}$




Analogous to line integrals there three type of surface integrals

Analogous to line integrals there three type of surface integrals

$$\int \phi \, d\vec{\sigma} \qquad \int \vec{V} \cdot d\vec{\sigma} \qquad \int \vec{V} \times d\vec{\sigma}$$

Surface integral $\int \vec{V} \cdot d\vec{\sigma}$ is interpreted as flow of flux through the surface

## Volume integral

Volume elements are simple as volume element $d\tau$ is scalar quantity

$$\int_v \vec{V} \, d\tau = \hat{x} \int_v V_x \, d\tau + \hat{y} \int_v V_y \, d\tau + \hat{z} \int_v V_z \, d\tau$$

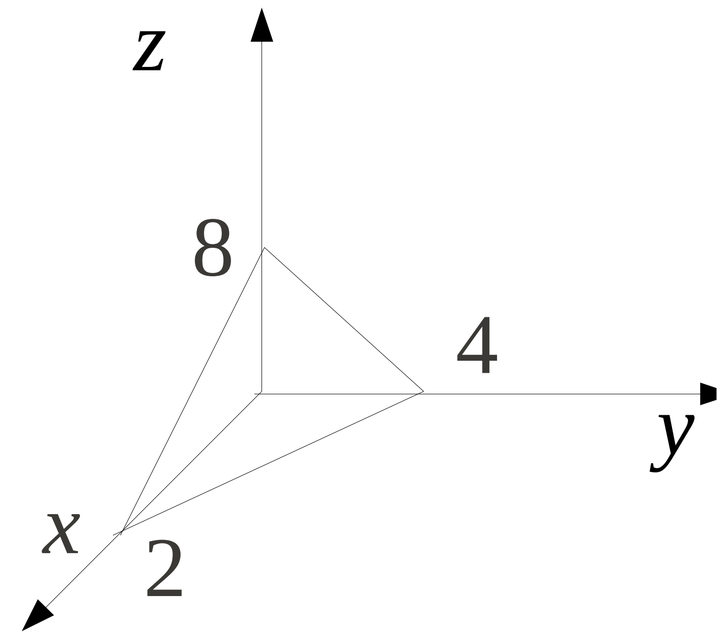
Vector sum of scalar integrals

## Volume integral example

Let $f = 45x^2 y$ be the function that give density in certain region. Find the total mass in the volume limited by the four planes $4x + 2y + z = 8,$

$$x = 0,\quad y = 0,\quad z = 0$$

Find the the integral over the volume (total mass)




The point at which the slanted plane touches the axes may be found by setting a pair of coordinates as zero

$$4x + 2y + z = 8,$$

The volume integral has to be performed in step by step – first choose the integration over $z$

Limits of the integration is between the plane $z = 0$ and second limit is $z = 8 - 4x - 2y$

Next integration may be performed over $y$

Limits of the integration is between the plane $y = 0,$ and second limit is $y = 4 - 2x$ by excluding $z$ dependency – (in the bottom plane )

Final integration is over $x$ in the limits between $x = 0,$ and $x = 2$

$$I = \int_v f\, d\tau = \int_V 45\, x^2\, y\, d\,x\, d\,y\, d\,z$$

$$I = \int_0^2 d\,x \int_0^{4-2x} dy \int_0^{8-4x-2y} 45\, x^2\, y\, d\,z$$

Performing first integration

$$I_1 = \int_0^{8-4x-2y} 45\, x^2\, y\, d\,z = 45x^2\, y\left(8 - 4\,x - 2\,y\right)$$

$$= 360\, x^2\, y - 180\, x^3\, y - 90\, x^2\, y^2$$

Performing second integration

$$I_2 = \int_0^{4-2x} \left(360\, x^2\, y - 180\, x^3\, y - 90\, x^2\, y^2\right) dy$$

$$= 960\, x^2 - 1440\, x^3 - 720\, x^4 - 120\, x^5$$

Final integration is over x

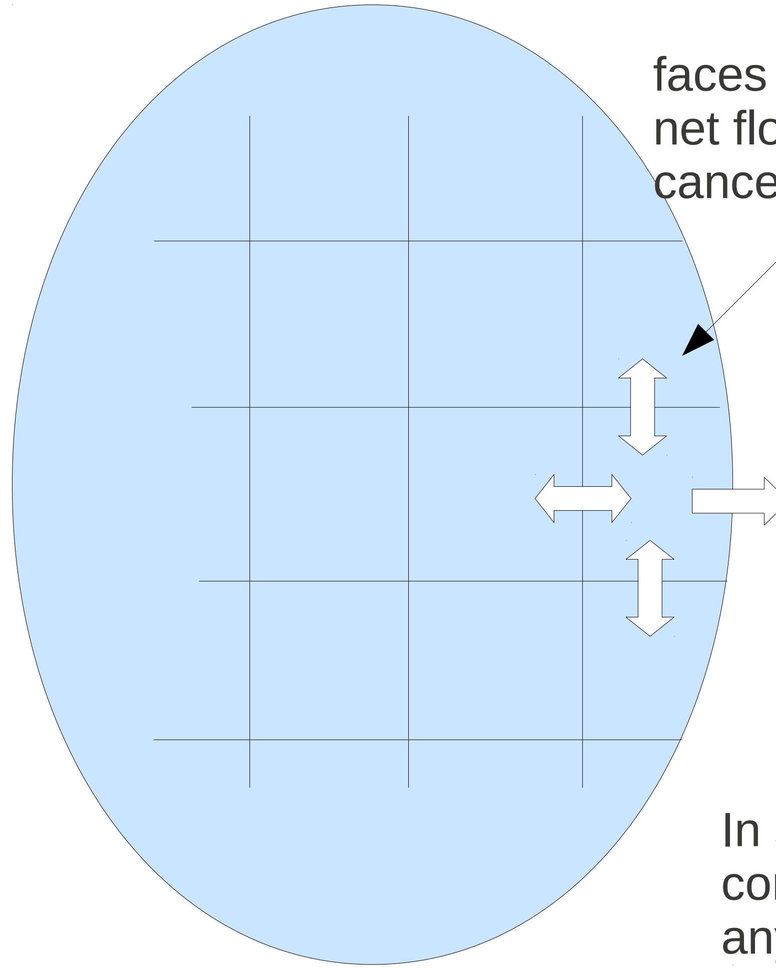
$$I = \int_0^2 d\,x \left(960\, x^2 - 1440\, x^3 - 720\, x^4 - 120\, x^5\right) = 128$$

## Gauss divergence theorem

This theorem connects the surface integral of vector and the volume integral of divergence of the same vector

$$\oiint_S \vec{V} \cdot d\vec{\sigma} = \iiint_V \vec{\nabla} \cdot \vec{V}\, d\tau$$

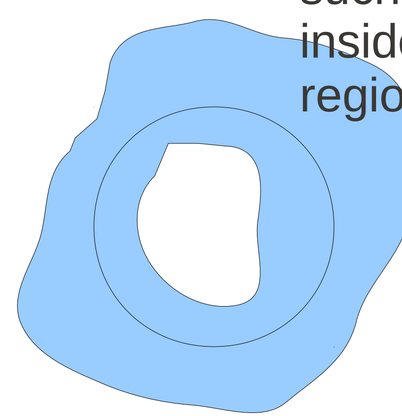
It is assumed that the vector $\vec{V}$ is continuous and with no discontinuities (simply connected)




Divide the volume into large number of parallelepipeds

$$\sum_{six\ faces} \vec{V} \cdot d\vec{\sigma} = \vec{\nabla} \cdot \vec{V}\, d\tau$$

In simply connected region any loop can reduce it size such that it falls inside the same region



multiply connected

$\vec{V} \cdot \vec{d}\,\vec{\sigma}$ Terms cancel at the interior faces

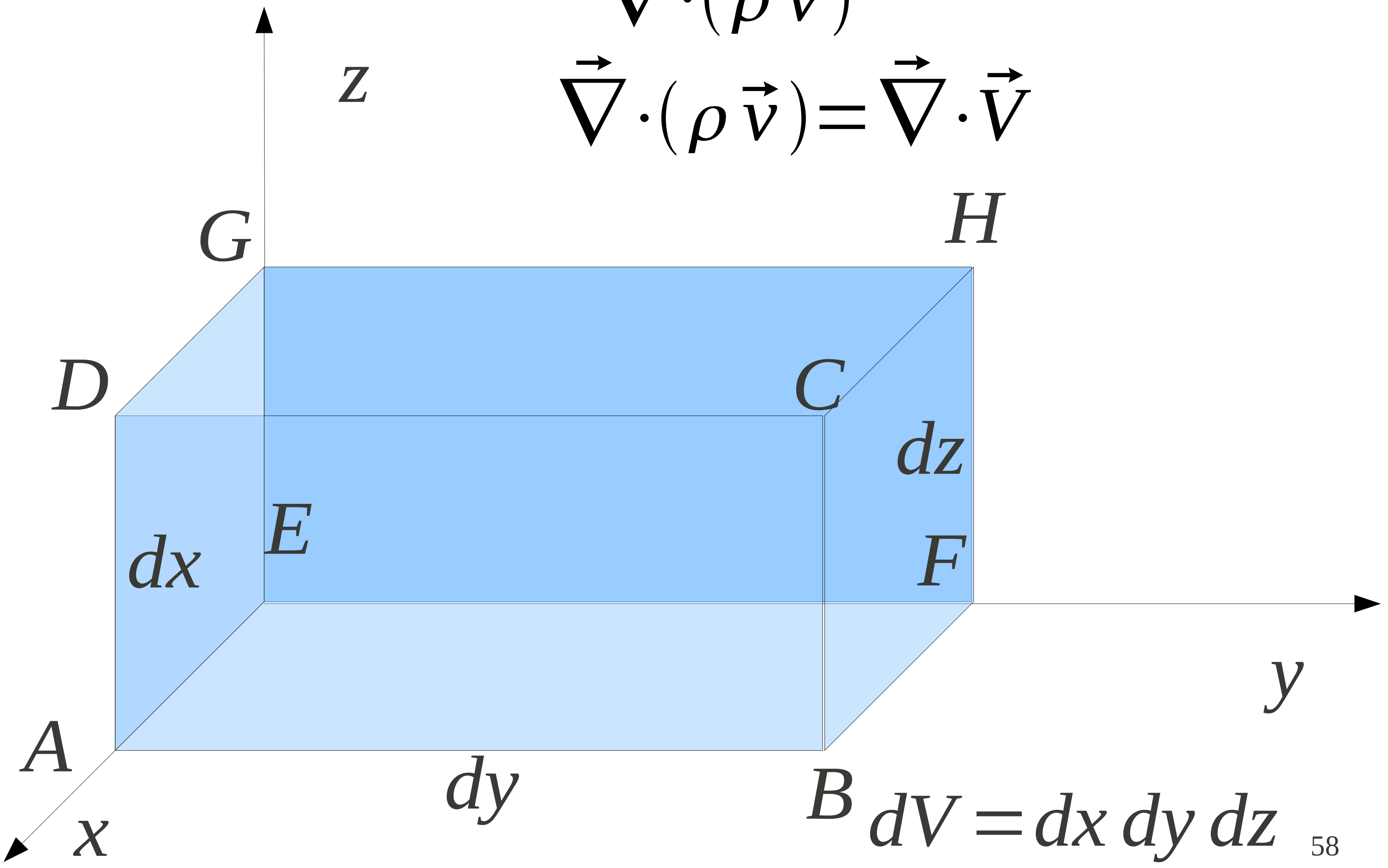
http://mathworld.wolfram.com/SimplyConnected.html

A small cell for proving the Gauss' theorem

Velocity of a compressible fluid with density $\rho(x, y, z)$ is $\vec{v}$

$$\vec{\nabla}\cdot(\rho\vec{v})$$

$$\vec{\nabla}\cdot(\rho\vec{v})=\vec{\nabla}\cdot\vec{V}$$




$$dV = dx\,dy\,dz$$

As integral is a limit of the sums we divide the volume into number of parallelepipeds as the number approaches infinity the dimension of each parallelepiped approaches zero

$$\sum_{exterior\,surfaces} \vec{V}\cdot d\vec{\sigma} = \sum_{volumes} \vec{\nabla}\cdot\vec{V}\,d\tau$$

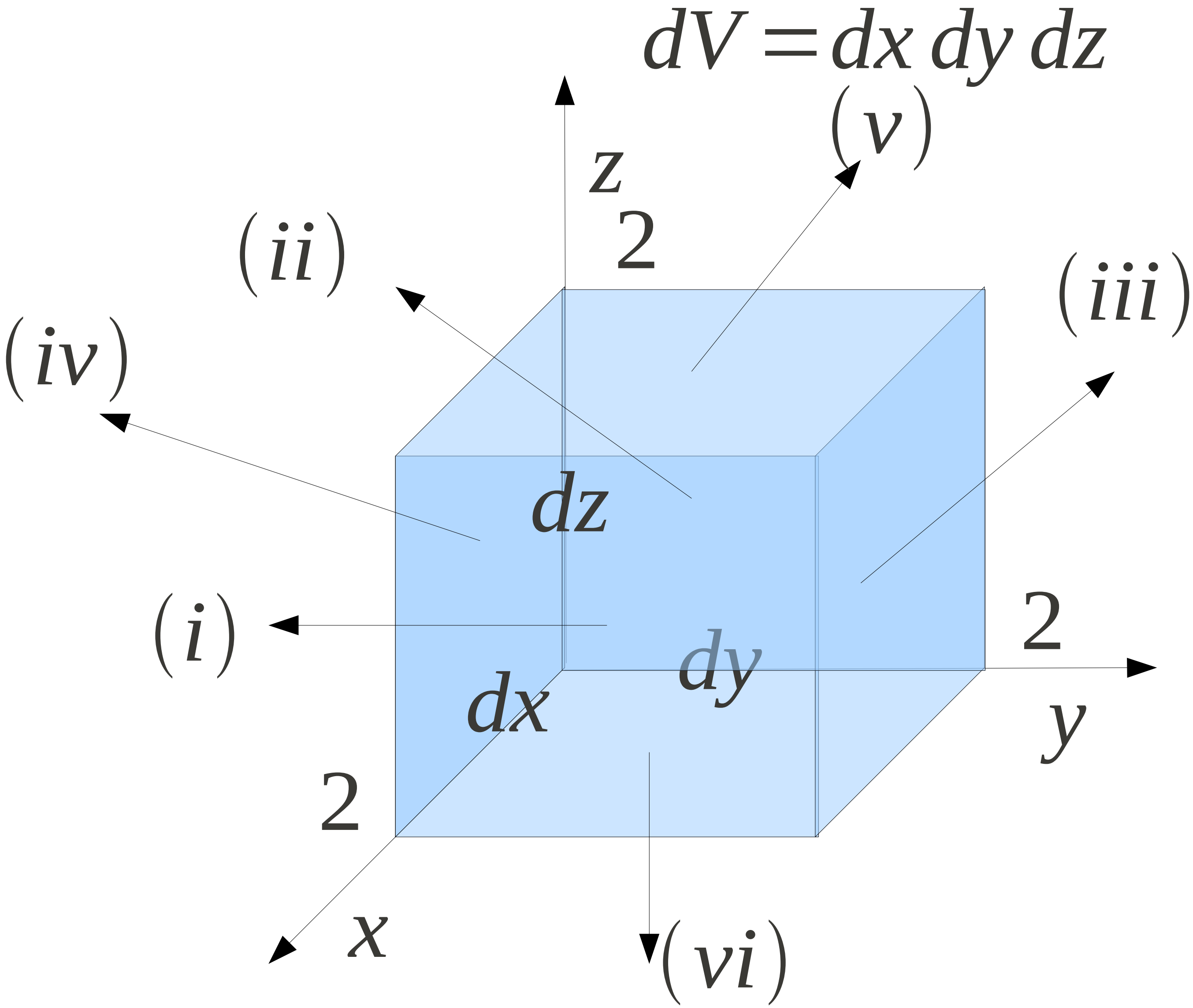
$$\downarrow \qquad\qquad \downarrow$$

$$\int_S \vec{V}\cdot d\vec{\sigma} = \int_V \vec{\nabla}\cdot\vec{V}\,d\tau$$

Net flow out of volume

Consider a cube with length of side 2 placed in the positive quadrant of the coordinate system. Test Gauss divergence theorem for the vector function given below for this volume

$$\vec{v} = x\,y\,\hat{x} + 2\,y\,z\,\hat{y} + 3\,z\,x\,\hat{z}$$



Computing the surface integrals for each surface for $\vec{v}=x\,y\,\hat{x}+2\,y\,z\,\hat{y}+3\,z\,x\,\hat{z}$

$$(i)\quad d\vec{a}=dy\,dz\,\hat{x}\qquad \vec{v}\cdot d\vec{a}=x\,y\,dy\,dz$$

$$\vec{v}\cdot d\vec{a}\Big|_{x=2}=2\,y\,dy\,dz$$

The surface integral is now given by

$$\int_S \vec{v}\cdot d\vec{a}=\int_0^2\int_0^2 2\,y\,dy\,dz=\int_0^2 4\,y\,dy=8$$

$$(ii)\quad d\vec{a}=dy\,dz\,(-\hat{x}),\ \vec{v}\cdot d\vec{a}=-x\,y\,dy\,dz$$

$$\vec{v}\cdot d\vec{a}\Big|_{x=0}=2(0)\,dy\,dz$$

The surface integral is now given by $\int_S \vec{v}\cdot d\vec{a}=0$

$$(iii)\quad d\vec{a}=dx\,dz\,\hat{y}\qquad \vec{v}\cdot d\vec{a}=2\,y\,z\,dx\,dz$$

$$\vec{v}\cdot d\vec{a}\Big|_{y=2}=4\,z\,dx\,dz$$

The surface integral is now given by

$$\int_S \vec{v}\cdot d\vec{a}=\int_0^2\int_0^2 4\,z\,dx\,dz=\int_0^2 8\,z\,dz=16$$

Computing the surface integrals for each surface for $\vec{v}=x\,y\,\hat{x}+2\,y\,z\,\hat{y}+3\,z\,x\,\hat{z}$

$$(iv)\quad d\vec{a}=dy\,dz(-\hat{x})\quad \vec{v}\cdot d\vec{a}=2\,y\,z\,dx\,dz$$

$$\vec{v}\cdot d\vec{a}\Big|_{y=0}=2(0)\,z\,dx\,dz$$

The surface integral is now given by $\int_S \vec{v}\cdot d\vec{a}=0$

$$(v)\quad d\vec{a}=dx\,dy\,\hat{z}\qquad \vec{v}\cdot d\vec{a}=3\,z\,x\,dx\,dy$$

$$\vec{v}\cdot d\vec{a}\Big|_{z=2}=6\,x\,dx\,dy$$

The surface integral is now given by

$$\int_S \vec{v}\cdot d\vec{a}=\int_0^2\int_0^2 6\,x\,dx\,dz=\int_0^2 12\,z\,dz=24$$

$$(vi)\quad d\vec{a}=dx\,dy(-\hat{z})\qquad \vec{v}\cdot d\vec{a}=-3\,z\,x\,dx\,dy$$

$$\vec{v}\cdot d\vec{a}\Big|_{z=0}=3(0)\,x\,dx\,dy$$

The surface integral is now given by $\int_S \vec{v}\cdot d\vec{a}=0$

$$\int_S \vec{v}\cdot d\vec{a}=i+ii+iii+iv+v+vi=8+16+24=48$$

$$\vec{\nabla}\cdot\vec{v}=\left(\hat{x}\frac{\partial}{\partial x}+\hat{y}\frac{\partial}{\partial y}+\hat{z}\frac{\partial}{\partial z}\right)\cdot\left(x\,y\,\hat{x}+2\,y\,z\,\hat{y}+3\,z\,x\,\hat{z}\right)$$

$$\vec{\nabla}\cdot\vec{v}=\left(\frac{\partial(x\,y)}{\partial x}+\frac{\partial(2\,y\,z)}{\partial y}+\frac{\partial(3\,z\,x)}{\partial z}\right)$$

$$\vec{\nabla}\cdot\vec{v}=\left(y+2\,z+3\,x\right)$$

$$\iiint\vec{\nabla}\cdot\vec{v}\,d\tau=\int_0^2\int_0^2\int_0^2\left(y+2\,z+3\,x\right)dx\,dy\,dz$$

$$=\int_0^2\int_0^2\left(x\,y+2\,x\,z+3\,x^2/2\right)_0^2 dy\,dz$$

$$=\int_0^2\int_0^2\left(2\,y+4\,z+6\right)dy\,dz$$

$$=\int_0^2\left(8\,z+16\right)dz=48\qquad=\int_S\vec{v}\cdot d\vec{a}$$

## Stokes' theorem

This theorem connects the surface integral of derivative of a function and the line integral of of the same function

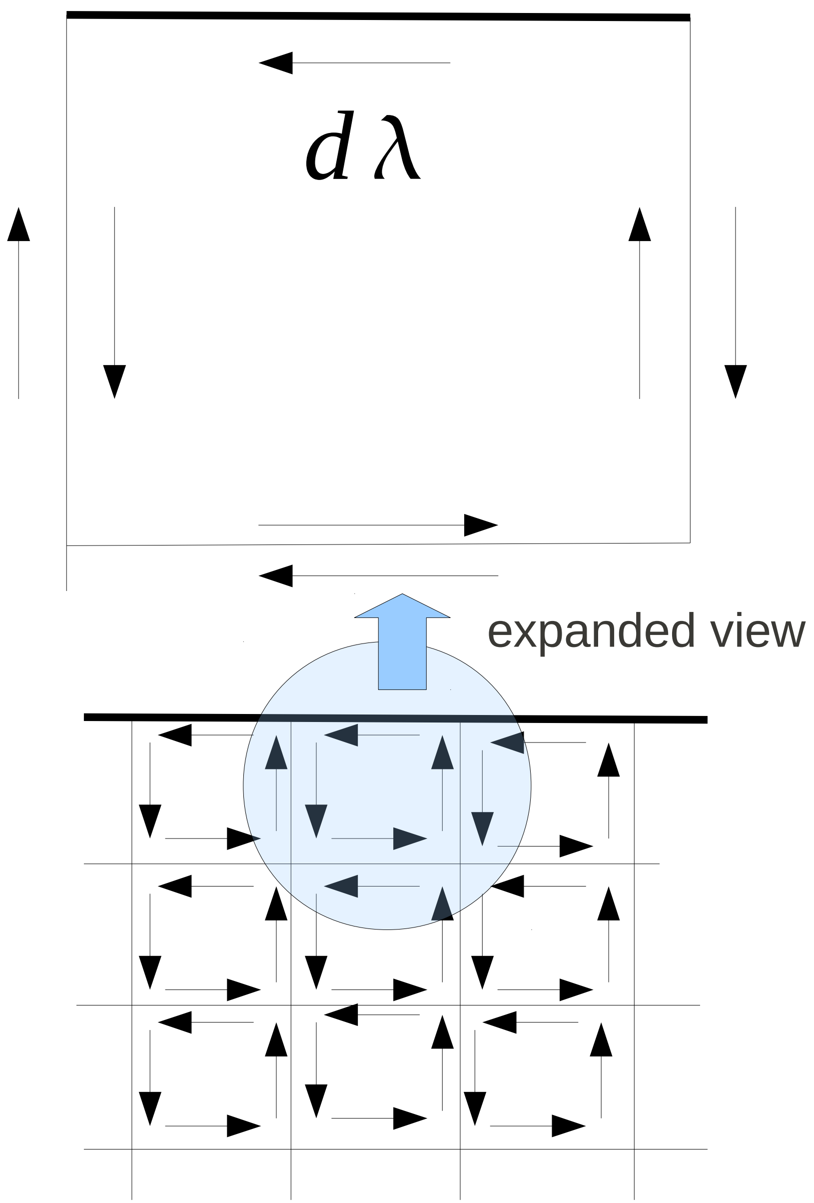
Divide the surface into network of arbitrary small rectangles

Circulation of one such arbitrary small rectangle is shown here, is given as

$$\left.\vec{\nabla}\times\vec{V}\right|_z dx\,dy$$

$$\sum_{four\ sides} \vec{V}\cdot d\vec{\lambda} = \vec{\nabla}\times\vec{V}\cdot d\vec{\sigma}$$

Circulation around the interior elements cancel each other



Taking the limit as size of rectangles approaches zero and number of rectangles approaches infinity

$$\sum_{\text{exterior line segments}} \vec{V}\cdot d\vec{\lambda} = \sum_{\text{rectangles}} \vec{\nabla}\times\vec{V}\cdot d\vec{\sigma}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\oint \vec{V}\cdot d\vec{\lambda} = \int_s \vec{\nabla}\times\vec{V}\cdot d\vec{\sigma}$$

# Conservative force field

Force generated by a scalar potential over a simply connected region of space is given by

$$\vec{F} = -\vec{\nabla}\,\phi$$

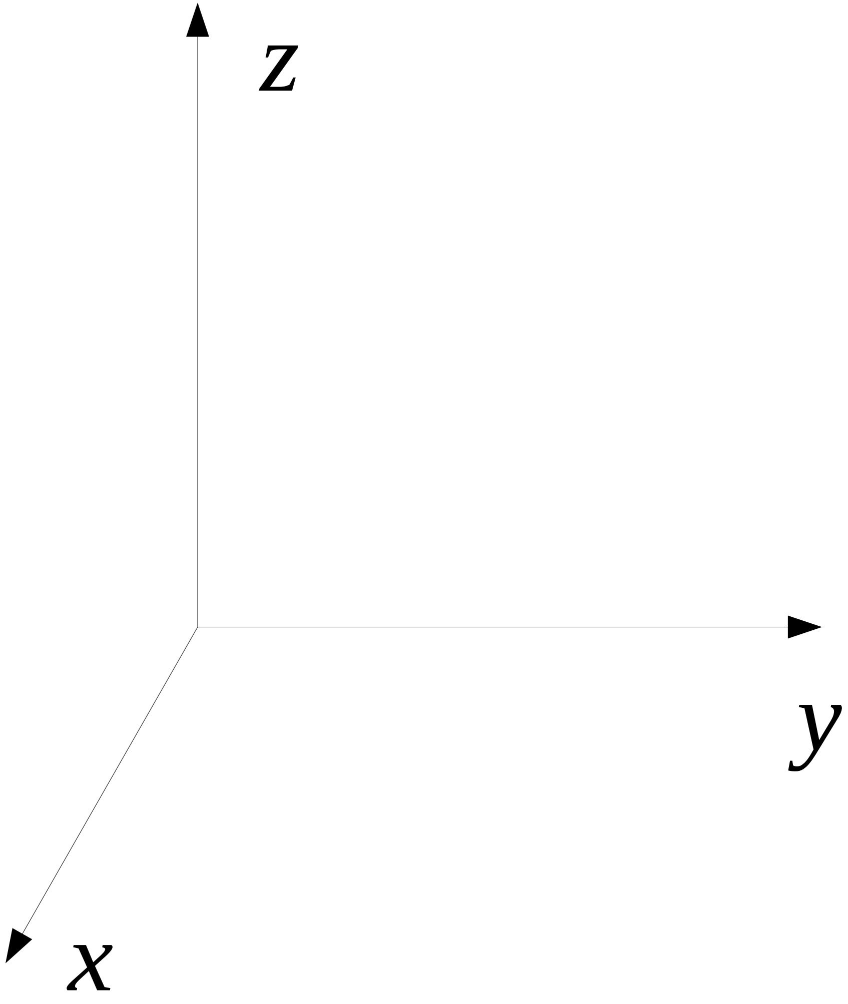
The force $\vec{F}$ appearing as a negative gradient single valued scalar potential is labeled as conservative force.

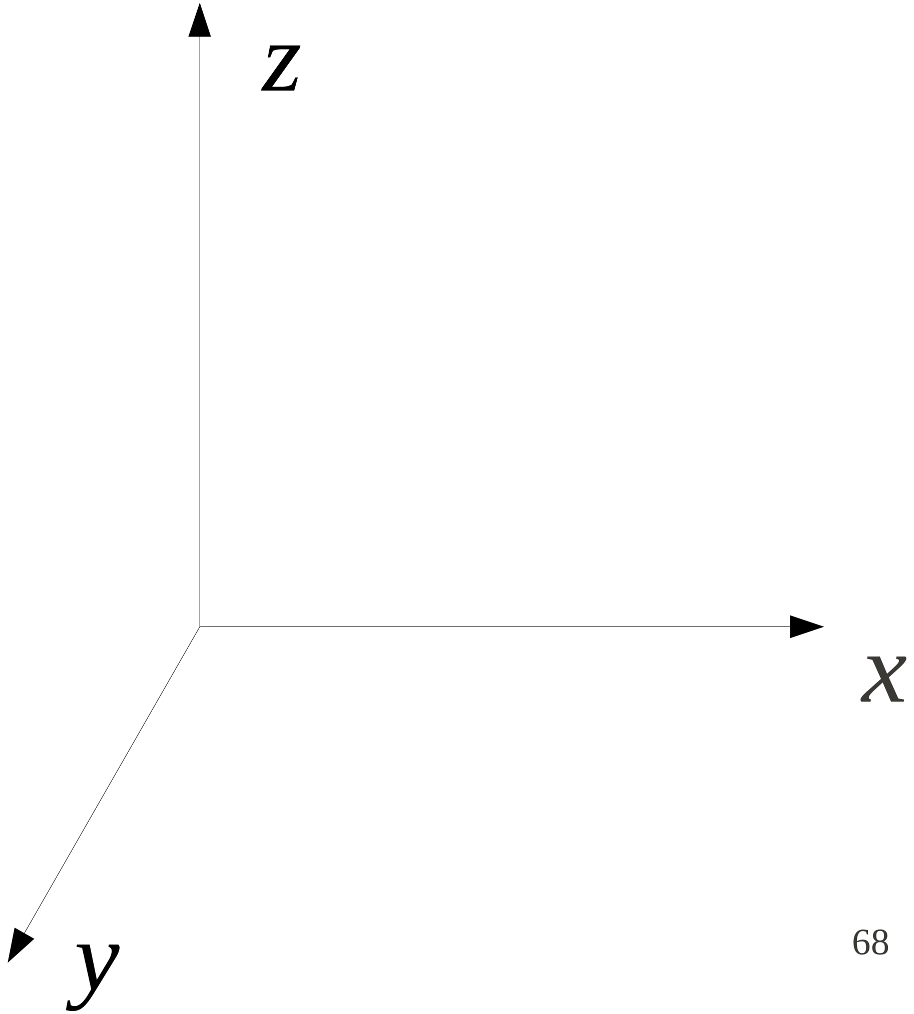
# Coordinate systems

- Displacement
- Area
- Volume
- Velocity
- Acceleration

# Rectangular Cartesian coordinate system

Right handed and left handed coordinate systems



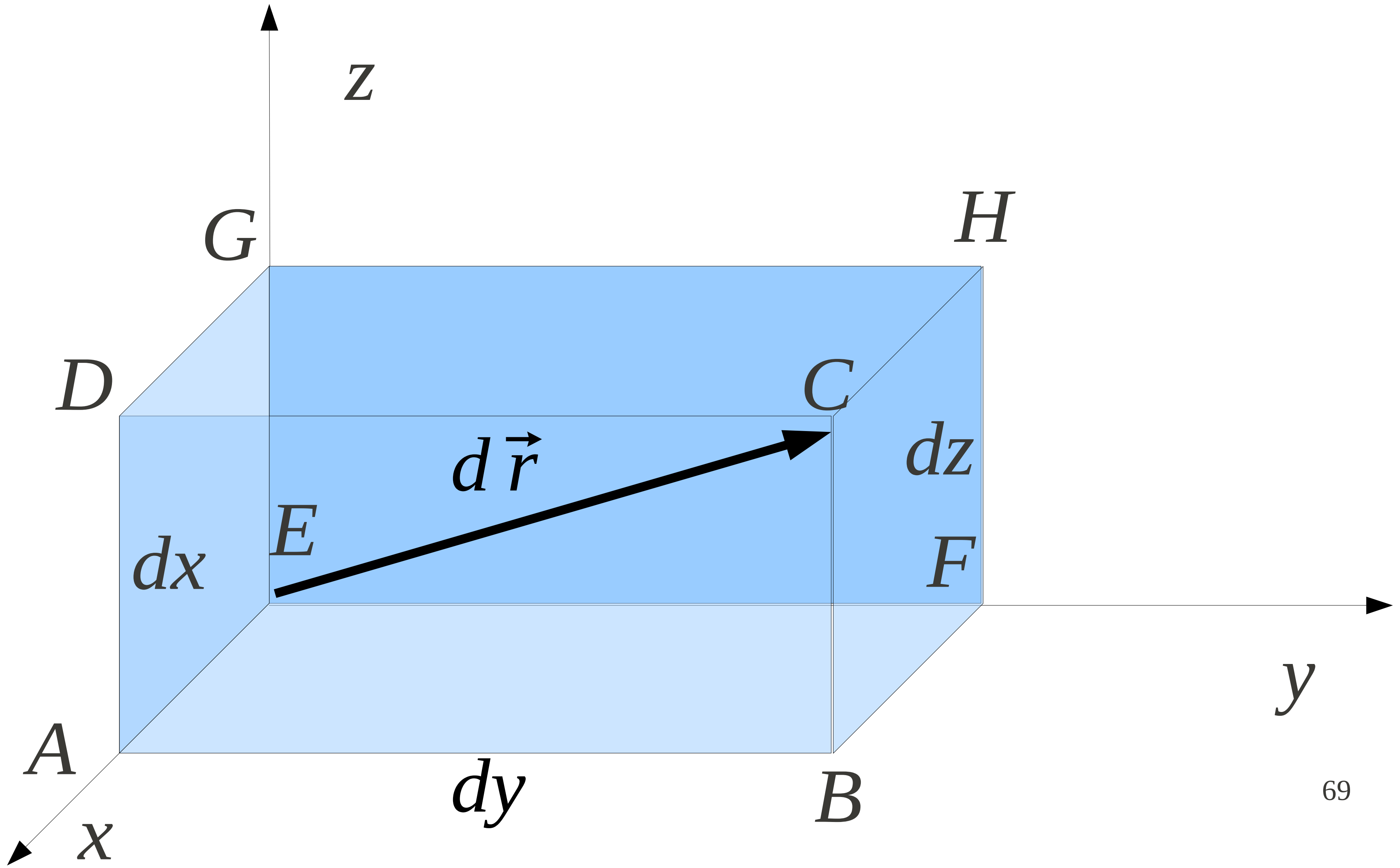



Displacement

$$\vec{r}=x\,\hat{x}+y\,\hat{y}+z\,\hat{z}$$

Vectorial increment of displacement

$$d\,\vec{r}=d\,x\,\hat{x}+d\,y\,\hat{y}+d\,z\,\hat{z}$$



Area

$$d\vec{A}_{xy} = d\vec{x} \times d\vec{y}$$

$$d\vec{A}_{xy} = \hat{x} \times \hat{y}\, dx\, dy$$

$$d\vec{A}_{xy} = \hat{z}\, dx\, dy$$

similarly $d\vec{A}_{zx} = \hat{y}\, dx\, dz$

$$d\vec{A}_{yz} = \hat{z}\, dx\, dy$$

Volume

$$dV = (d\vec{x} \times d\vec{y}) \cdot d\vec{z}$$

$$dV = (\hat{x} \times \hat{y})\, dx\, dy \cdot dz\, \hat{z}$$

$$dV = (\hat{z})\, dx\, dy \cdot dz\, \hat{z}$$

$$dV = dx\, dy\, dz$$

velocity

$$\vec{v} = \hat{x}\frac{dx}{dt} + \hat{y}\frac{dy}{dt} + \hat{z}\frac{dz}{dt}$$

$$\vec{v} = \hat{x}\,\dot{x} + \hat{y}\,\dot{y} + \hat{z}\,\dot{z}$$

acceleration

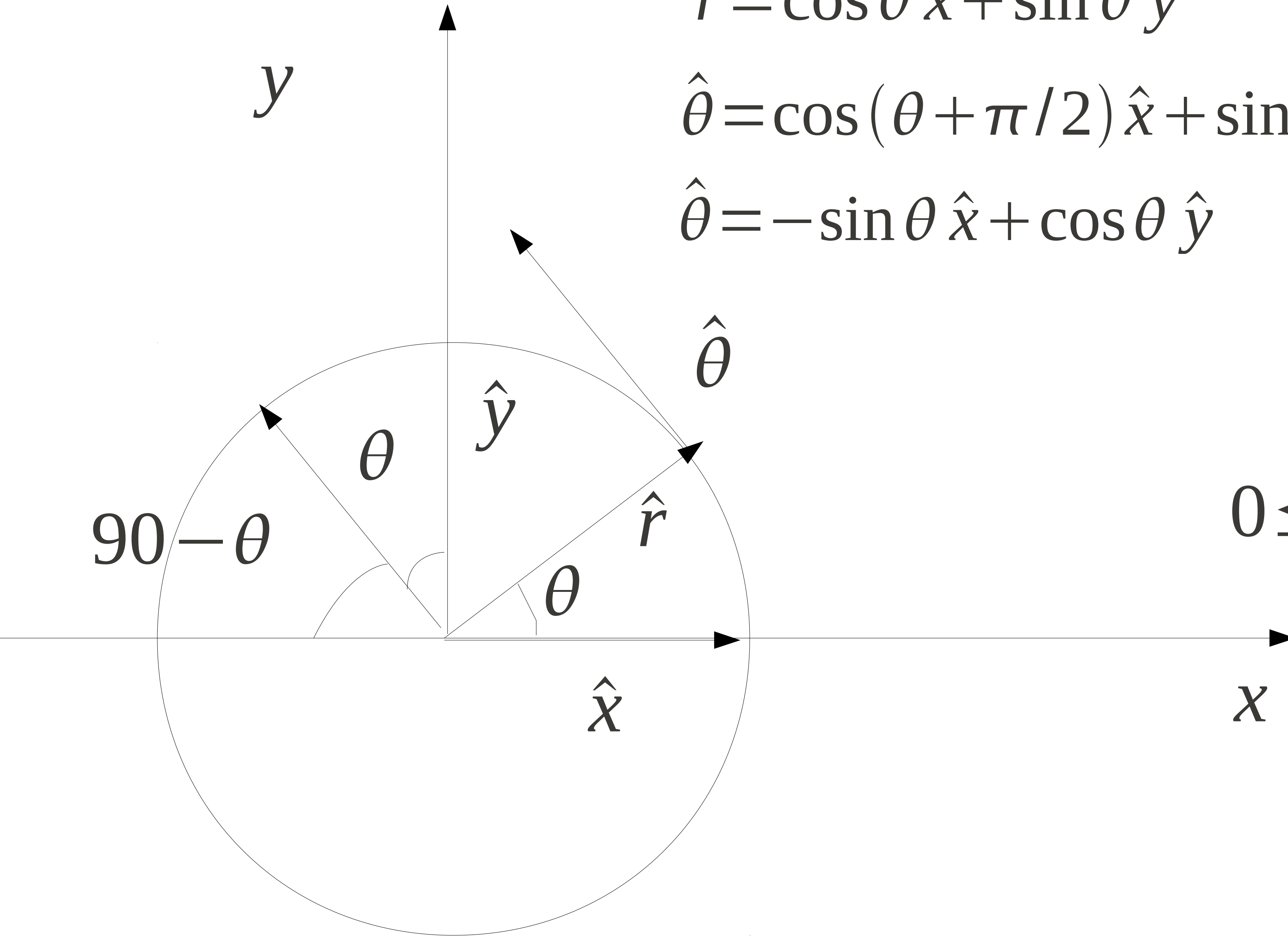
$$\vec{a} = \dot{\vec{v}} = \hat{x}\,\ddot{x} + \hat{y}\,\ddot{y} + \hat{z}\,\ddot{z}$$

# Polar coordinates

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{\theta} = \cos(\theta + \pi/2)\,\hat{x} + \sin(\theta + \pi/2)\,\hat{y}$$

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$




$$0 \leq \theta \leq 2\pi$$

In order to define an orthogonal coordinate system we need to define a set of orthogonal axes. This coordinate system the directions of the unit vectors are only defined locally

Since the unit vectors of this coordinate system has no unique direction they also vary as vectors change

$$\hat{r}=\cos\theta\,\hat{x}+\sin\theta\,\hat{y}$$

$$\dot{\hat{r}}=-\sin\theta\,\dot{\theta}\,\hat{x}+\cos\theta\,\dot{\theta}\,\hat{y}$$

$$\dot{\hat{r}}=\dot{\theta}\left(-\sin\theta\,\hat{x}+\cos\theta\,\hat{y}\right)$$

$$\dot{\hat{r}}=\dot{\theta}\,\hat{\theta}$$

Direction along theta

$$\hat{\theta}=-\sin\theta\,\hat{x}+\cos\theta\,\hat{y}$$

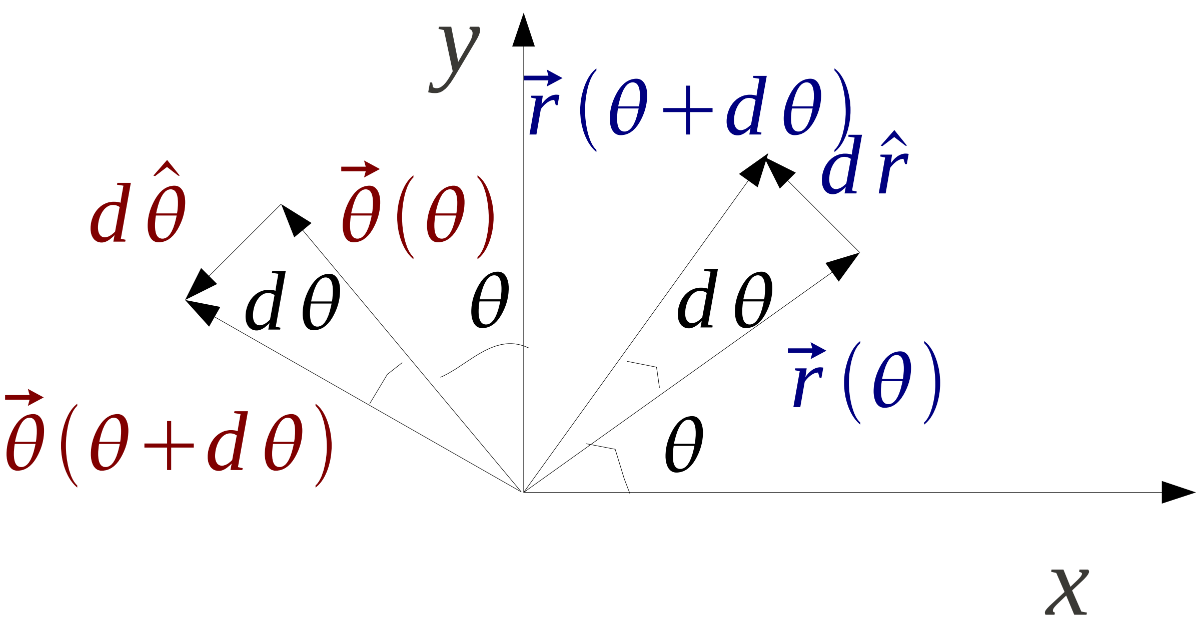
$$\dot{\hat{\theta}}=-\cos\theta\,\dot{\theta}\,\hat{x}-\sin\theta\,\dot{\theta}\,\hat{y}$$

$$\dot{\hat{\theta}}=-\dot{\theta}\left(\cos\theta\,\hat{x}+\sin\theta\,\hat{y}\right)$$

$$\dot{\hat{\theta}}=-\dot{\theta}\,\hat{r}$$

Direction along negative radial

Direction or unit vectors and derivative in new coordinate system




$$\hat{\theta}=-\sin\theta\,\hat{x}+\cos\theta\,\hat{y}$$

$$\hat{r}=\cos\theta\,\hat{x}+\sin\theta\,\hat{y}$$

Note on notation

$$\dot{\vec{r}}=\frac{d}{dt}(\vec{r})$$

$$\ddot{\vec{r}}=\frac{d^2}{dt^2}(\vec{r})$$

and so on...

# Velocity in polar coordinates

Derivative of unit vectors are used in the calculation of velocity of a particle in this coordinate system

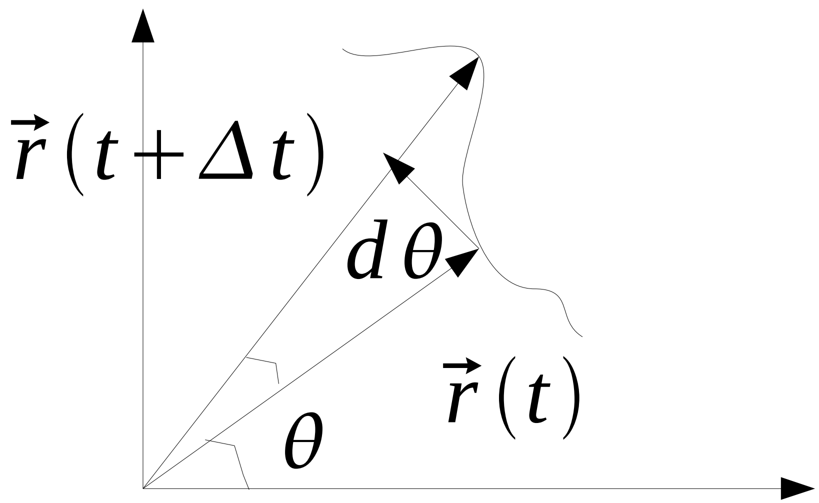
By product rule

$$\dot{\vec{r}} = \frac{d}{dt}(r\,\hat{r}) = \dot{r}\,\hat{r} + r\,\dot{\hat{r}}$$

Velocity in radial direction

Velocity in tangential direction

$$\dot{\vec{r}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$$




$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$$

Acceleration in polar coordinates

$$\dot{\hat{r}}=\dot{\theta}\,\hat{\theta}$$
$$\dot{\hat{\theta}}=-\dot{\theta}\,\hat{r}$$

$$\dot{\vec{r}}=\dot{r}\,\hat{r}+r\,\dot{\theta}\,\hat{\theta}$$

$$\ddot{\vec{r}}=\ddot{r}\,\hat{r}+\dot{r}\,\dot{\hat{r}}+\dot{r}\,\dot{\theta}\,\hat{\theta}+r\,\ddot{\theta}\,\hat{\theta}+r\,\dot{\theta}\,\dot{\hat{\theta}}$$

$$\ddot{\vec{r}}=\ddot{r}\,\hat{r}+\dot{r}\,(\dot{\theta}\,\hat{\theta})+\dot{r}\,\dot{\theta}\,\hat{\theta}+r\,\ddot{\theta}\,\hat{\theta}+r\,\dot{\theta}\,(-\dot{\theta}\,\hat{r})$$

Combining vector components of same type

$$\ddot{\vec{r}}=(\ddot{r}-r\,\dot{\theta}^2)\,\hat{r}+(r\,\ddot{\theta}+2\,\dot{r}\,\dot{\theta})\,\hat{\theta}$$

Therefore the force components acting on the particle is

$$\vec{F}_r=m(\ddot{r}-r\,\dot{\theta}^2)\,\hat{r}$$ radial

centrifugal term

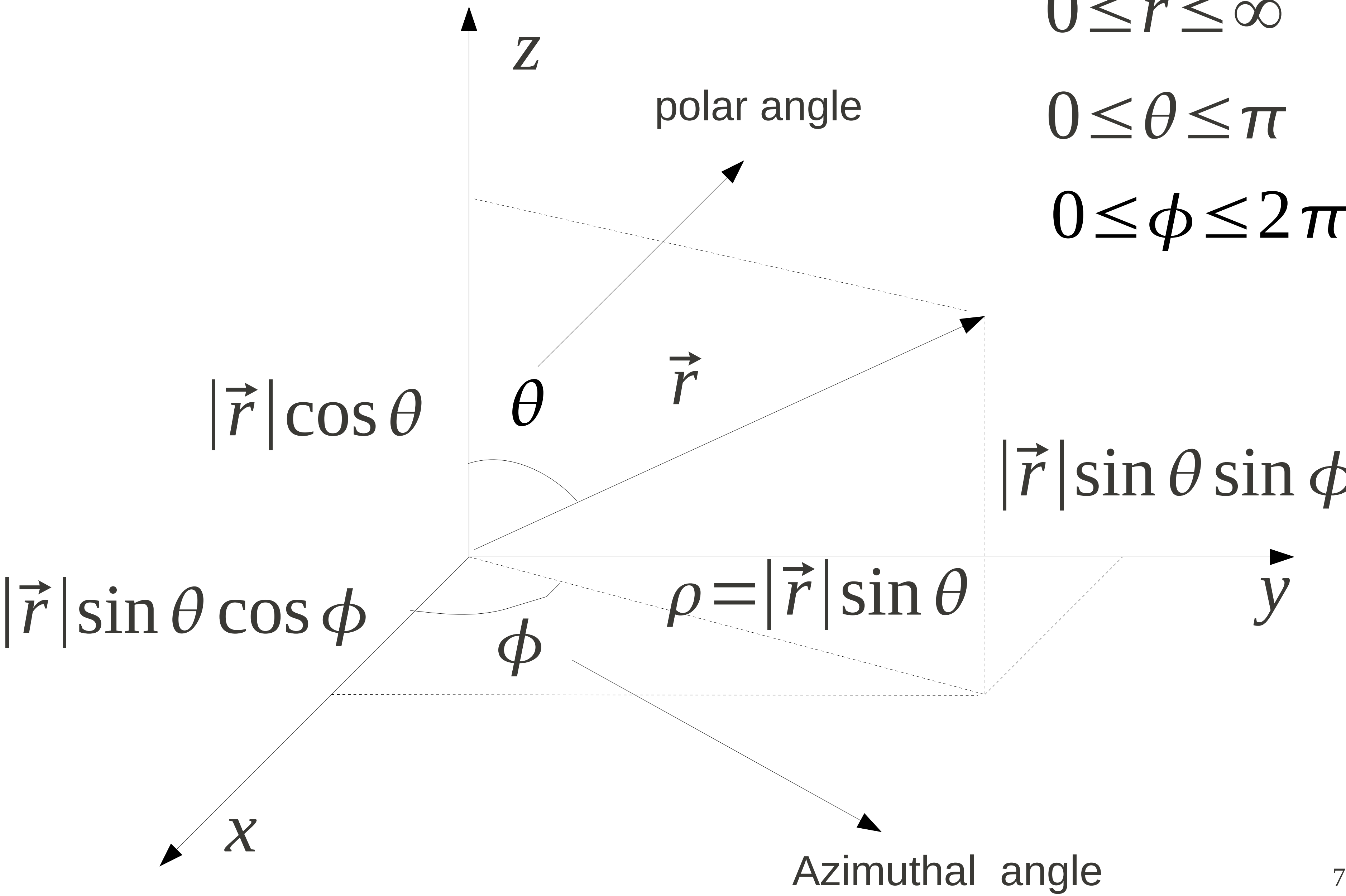
$$\vec{F}_\theta=m(r\,\ddot{\theta}+2\,\dot{r}\,\dot{\theta})\,\hat{\theta}$$ angular

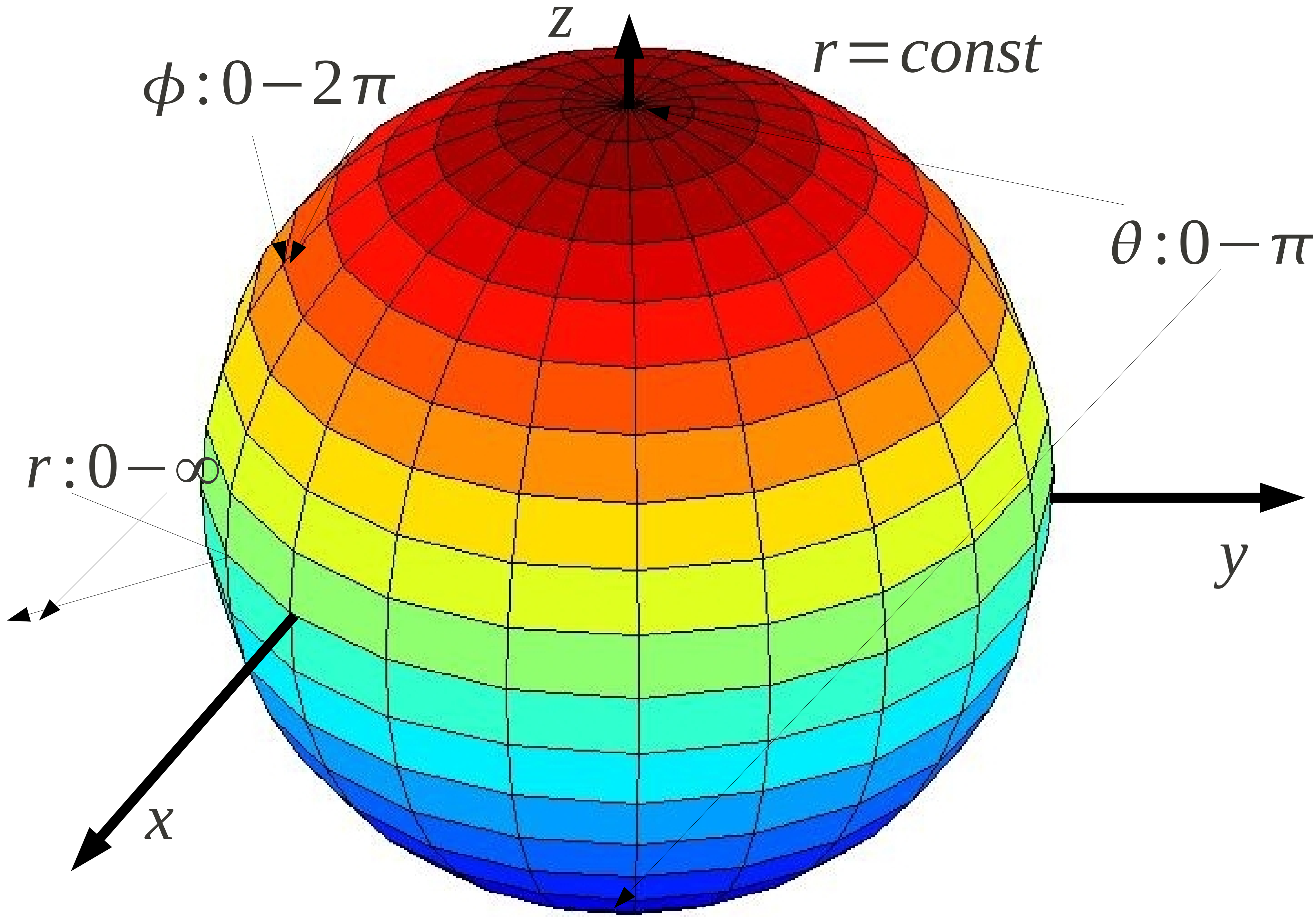
# Spherical coordinate system

$$0 \le r \le \infty$$

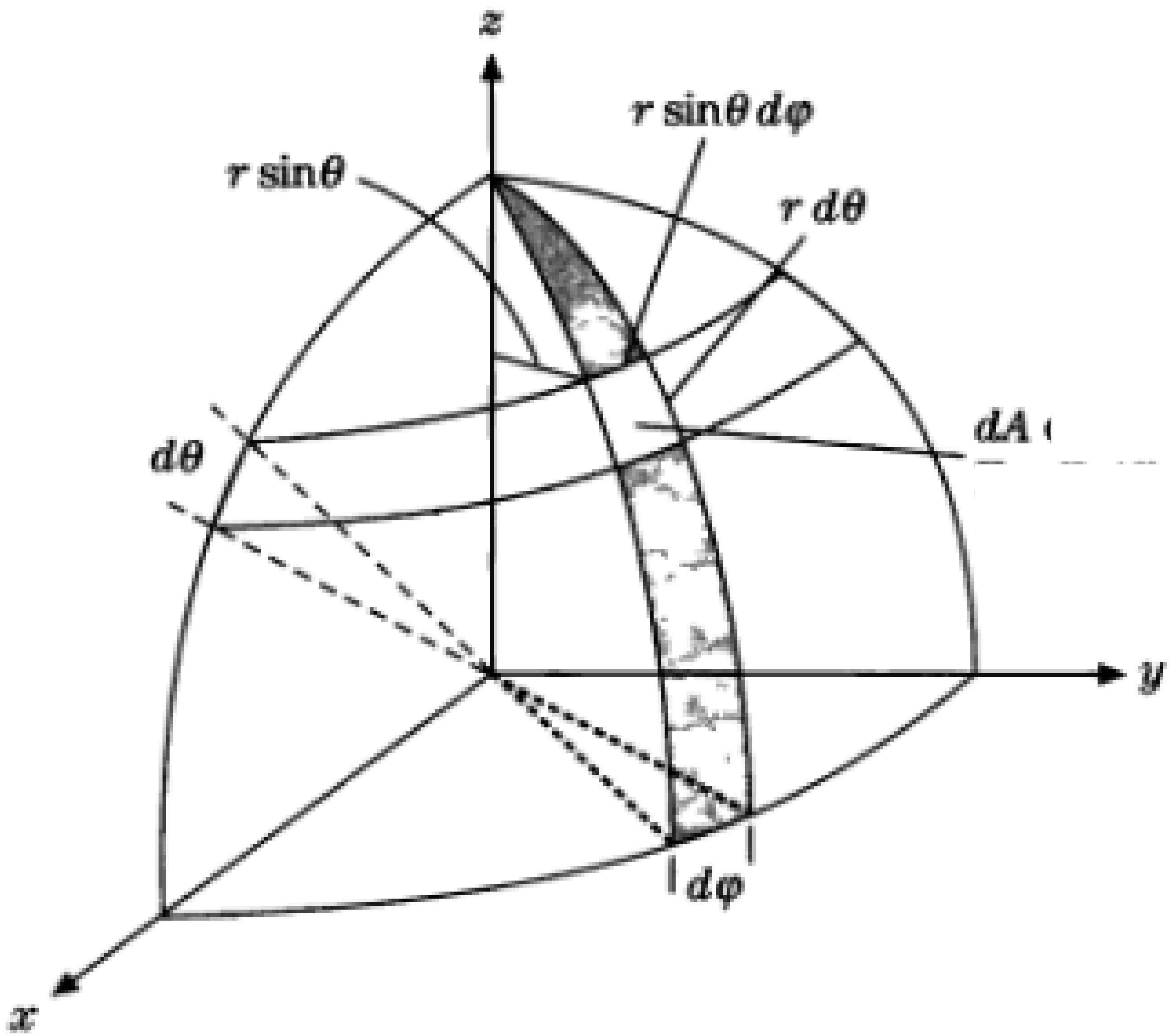
$$0 \le \theta \le \pi$$

$$0 \le \phi \le 2\pi$$




$z$
$r = const$
$\phi : 0 - 2\pi$
$\theta : 0 - \pi$
$r : 0 - \infty$
$y$
$x$

# Motion in spherical polar coordinates



line elements

$$dr\ , r\, d\,\theta\ , r \sin\theta\, d\,\phi$$

corresponding unit vectors are

$$dr\,\hat{r}\ , r\, d\,\theta\,\hat{\theta}\ , \rho\, d\,\phi\,\hat{\phi}$$ where $$\rho = r\sin\theta$$

increment in position

$$\vec{r}\,' = \vec{r} + d\,\vec{r}$$

$$d\,\vec{r} = d\,r\,\hat{r} + r\,d\,\theta\,\hat{\theta} + r\sin\theta\,d\,\phi\,\hat{\phi}$$

$$(d\,\vec{r}\,)^2 = d\,r^2 + (r\,d\,\theta)^2 + (r\sin\theta\,d\,\phi)^2$$

expressing the vector in Cartesian coordinates

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$\vec{r} = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$

radial unit vector

$$\hat{r} = \frac{\vec{r}}{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

unit vector of polar vector is perpendicular to $\hat{r}$ along $\hat{\theta}$ direction

$$\hat{\theta} = \sin(\pi/2+\theta)\cos\phi\,\hat{x} + \sin(\pi/2+\theta)\sin\phi\,\hat{y} + \cos(\pi/2+\theta)\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

azimuthal unit vector $\hat{\phi}$ is perpendicular to both $\hat{r}$ and $\hat{\theta}$

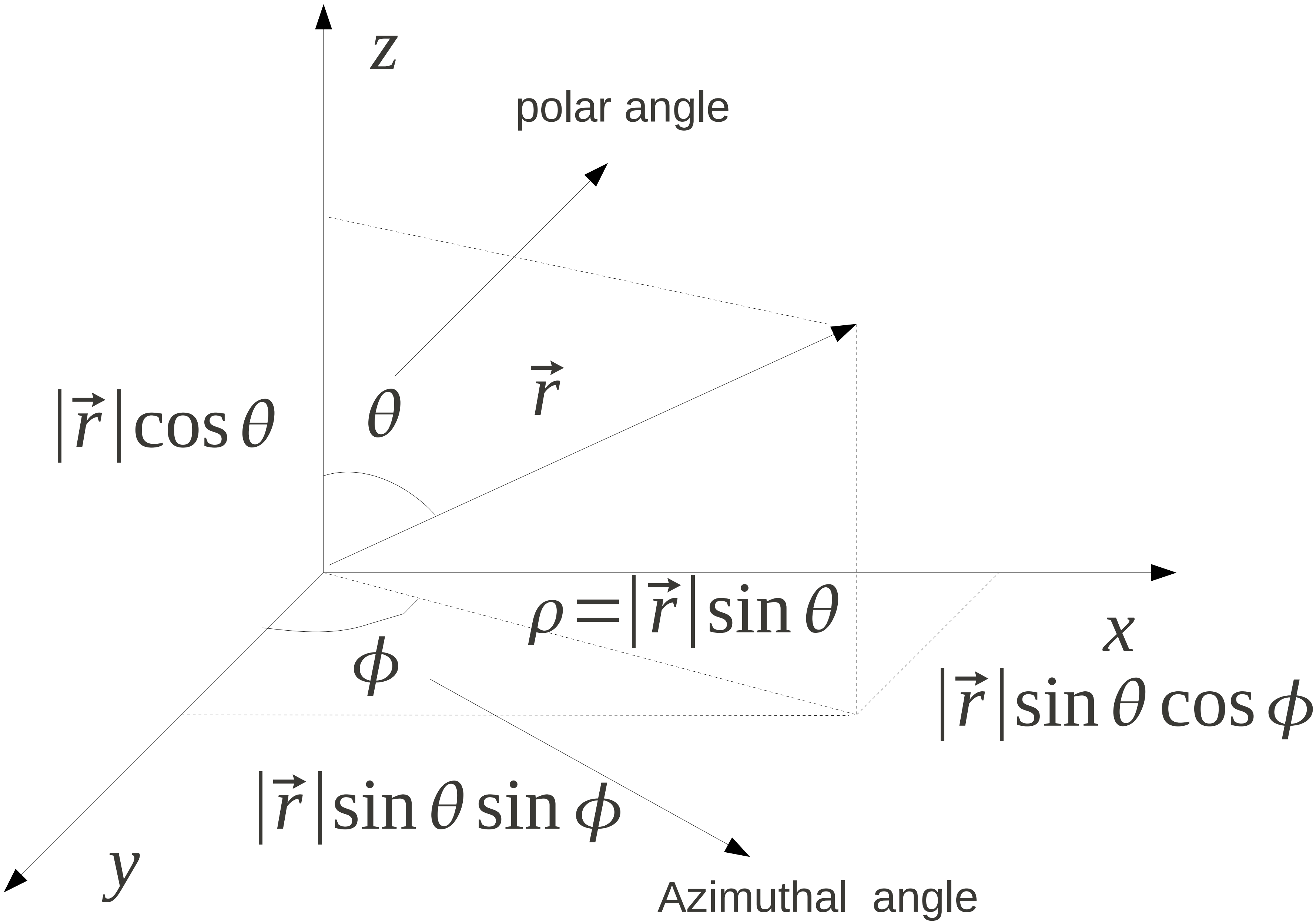
$$\vec{\rho} = \rho\cos\phi\,\hat{x} + \rho\sin\phi\,\hat{y}$$

Vector $\hat{\rho}$ represent the plane defined by $\hat{r}$ and $\hat{\theta}$

$$\hat{\rho} = \frac{\vec{\rho}}{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\rho} \perp \hat{\phi}$$

$$\hat{\phi} = \cos(\pi/2+\phi)\,\hat{x} + \sin(\pi/2+\phi)\,\hat{y}$$



$z$
polar angle
$\vec{r}$
$|\vec{r}|\cos\theta$
$\theta$
$\rho = |\vec{r}|\sin\theta$
$x$
$\phi$
$|\vec{r}|\sin\theta\cos\phi$
$|\vec{r}|\sin\theta\sin\phi$
$y$
Azimuthal angle

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

Now we compute the partial derivatives of these unit vectors, as they are functions that have multiple variable dependency

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\frac{\partial\hat{r}}{\partial\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} = \hat{\theta}$$

$$\frac{\partial\hat{r}}{\partial\phi} = -\sin\theta\sin\phi\,\hat{x} + \sin\theta\cos\phi\,\hat{y}$$
$$= \sin\theta\left(-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}\right) = \sin\theta\,\hat{\phi}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\frac{\partial\hat{\theta}}{\partial\theta} = -\sin\theta\cos\phi\,\hat{x} - \sin\theta\sin\phi\,\hat{y} - \cos\theta\,\hat{z} = -\hat{r}$$

$$\frac{\partial\hat{\theta}}{\partial\phi} = -\cos\theta\sin\phi\,\hat{x} + \cos\theta\cos\phi\,\hat{y} = \cos\theta\,\hat{\phi}$$

$$\hat{\rho} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\frac{\partial\hat{\phi}}{\partial\phi} = -\cos\phi\,\hat{x} - \sin\phi\,\hat{y} = -\hat{\rho}$$

$$\frac{\partial\hat{\phi}}{\partial\theta} = 0$$

## Velocity

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

$$\vec{v} = \dot{\vec{r}} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

orthogonal components of velocity in similar to Cartesian coordinates are $(\dot{r}, r\dot{\theta}, r\sin\theta\,\dot{\phi})$

## Acceleration

$$\dot{\vec{v}} = \ddot{\vec{r}} = \frac{d}{dt}(\dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi})$$

$$= \ddot{r}\,\hat{r} + \dot{r}\,\dot{\hat{r}} + \dot{r}\,\dot{\theta}\,\hat{\theta} + r\,\ddot{\theta}\,\hat{\theta} + r\,\dot{\theta}\,\dot{\hat{\theta}} + \dot{r}\sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$+ r\cos\theta\,\dot{\theta}\,\dot{\phi}\,\hat{\phi} + r\sin\theta\,\ddot{\phi}\,\hat{\phi} + r\sin\theta\,\dot{\phi}\,\dot{\hat{\phi}}$$

Now for defining the derivative of the unit vectors- that enable us to find various components of the acceleration vector

Since the unit vectors depends on many variables, chain rule of partial differentiation may be used

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$$

$$\dot{\hat{r}}(\theta,\phi) = \frac{\partial \hat{r}}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \hat{r}}{\partial \phi}\frac{\partial \phi}{\partial t}$$

$$\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\,\hat{\phi}$$

$$= \hat{\theta}\,\dot{\theta} + \sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} + \sin\theta\,\hat{z}$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$$

$$\dot{\hat{\theta}}(\theta,\phi) = \frac{\partial \hat{\theta}}{\partial \theta}\frac{\partial \theta}{\partial t} + \frac{\partial \hat{\theta}}{\partial \phi}\frac{\partial \phi}{\partial t}$$

$$\frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\,\hat{\phi}$$

$$= -\dot{\theta}\,\hat{r} + \dot{\phi}\cos\theta\,\hat{\phi}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\dot{\hat{\phi}}(\phi) = \frac{\partial \hat{\phi}}{\partial \phi}\frac{\partial \phi}{\partial t}$$

$$\frac{\partial \hat{\phi}}{\partial \phi} = -\hat{r}$$

$$= -\dot{\phi}\sin\theta\,\hat{r} - \dot{\phi}\cos\theta\,\hat{\theta}$$

Assuming that $\theta, \phi$ are explicit functions of time

$$\frac{\partial \theta}{\partial t}=\frac{d\theta}{dt}=\dot{\theta}, \qquad \frac{\partial \phi}{\partial t}=\frac{d\phi}{dt}=\dot{\phi}$$

By substituting these results in the equation

$$=\ddot{r}\,\hat{r}+\dot{r}\,\dot{\hat{r}}+\dot{r}\,\dot{\theta}\,\hat{\theta}+r\,\ddot{\theta}\,\hat{\theta}+r\,\dot{\theta}\,\dot{\hat{\theta}}+\dot{r}\sin\theta\,\dot{\phi}\,\hat{\phi}$$
$$+r\cos\theta\,\dot{\theta}\,\dot{\phi}\,\hat{\phi}+r\sin\theta\,\ddot{\phi}\,\hat{\phi}+r\sin\theta\,\dot{\phi}\,\dot{\hat{\phi}}$$

$$=\ddot{r}\,\hat{r}+\dot{r}\left(\hat{\theta}\,\dot{\theta}+\sin\theta\,\dot{\phi}\,\hat{\phi}\right)+\dot{r}\,\dot{\theta}\,\hat{\theta}+r\,\ddot{\theta}\,\hat{\theta}+$$
$$r\,\dot{\theta}\left(-\dot{\theta}\,\hat{r}+\dot{\phi}\cos\theta\,\hat{\phi}\right)+\dot{r}\sin\theta\,\dot{\phi}\,\hat{\phi}$$
$$+r\cos\theta\,\dot{\theta}\,\dot{\phi}\,\hat{\phi}+r\sin\theta\,\ddot{\phi}\,\hat{\phi}+r\sin\theta\,\dot{\phi}\left(-\dot{\phi}\sin\theta\,\hat{r}-\dot{\phi}\cos\theta\,\hat{\theta}\right)$$

Rearranging terms that have same unit vectors

$$\vec{a}=(\ddot{r}-r\dot{\theta}^2-r\sin^2\theta\,\dot{\phi}^2)\hat{r}+(2\dot{r}\,\dot{\theta}+r\,\ddot{\theta}-r\sin\theta\cos\theta\,\dot{\phi}^2)\hat{\theta}$$

$$+(2\dot{r}\sin\theta\,\dot{\phi}+2\mathrm{r}\cos\theta\,\dot{\phi}\,\dot{\theta}+r\sin\theta\,\ddot{\phi})\hat{\phi}$$

What are the common properties of orthogonal coordinates ?

Is there is general way to address them?

# Orthogonal coordinates

Three planes of Cartesian coordinates

$$x = Constant \quad y = Constant \qquad z = Constant$$

We can in general define another set of coordinates by families of surfaces defined by equation

$$q_1 = Constant \qquad q_2 = Constant \quad q_3 = Constant$$

These families are not necessarily planes, for simplicity they can be taken orthogonal to each other.

The unit vectors perpendicular to the surfaces that define coordinate system is given by

$$\hat{q}_1, \hat{q}_2, \hat{q}_3$$

A vector in this coordinate system can be expressed as

$$\vec{V} = V_1 \hat{q}_1 + V_2 \hat{q}_2 + V_3 \hat{q}_3$$

## Transformation between coordinates

$$x = x(q_1, q_2, q_3) \qquad q_1 = q_1(x, y, z)$$

$$y = y(q_1, q_2, q_3) \qquad q_2 = q_2(x, y, z)$$

$$z = z(q_1, q_{2,} q_3) \qquad q_3 = q_3(x, y, z)$$

Representation a vector in generalized coordinates

$$\vec{V} = \hat{q}_1 V_1 + \hat{q}_2 V_2 + \hat{q}_3 V_3$$

$$\hat{q}_i^2 = 1 \qquad \hat{q}_1 \cdot (\hat{q}_2 \times \hat{q}_3) > 0$$

Volume of the box product must be greater than zero

change from one representation to another

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$d\vec{r} = \left( \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3 \right) \hat{x} + \ldots\ldots = \sum_i \frac{\partial \vec{r}}{\partial q_i} dq_i$$

$$i \in \{1,2,3\}$$

in Cartesian coordinates the square of the distance between two neighboring points is

$$ds^2 = dx^2 + dy^2 + dz^2$$

In curvilinear coordinate system same distance is given by

$$ds^2 = d\vec{r} \cdot d\vec{r} = dr^2 = \sum_{ij} \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j} dq_i dq_j$$

$$= g_{11} dq_1^2 + g_{12} dq_1 dq_2 + g_{13} dq_1 dq_3 + g_{21} dq_2 dq_1$$

$$+ g_{22} dq_2^2 + g_{23} dq_2 dq_3 + g_{31} dq_3 dq_1 + g_{32} dq_3 dq_2 + g_{33} dq_3 dq_3$$

$$= \sum_{ij} g_{ij} dq_i dq_j \qquad \{i, j\} \in \{1,2,3\}$$

$$g_{ij}(q_{1,} q_{2,} q_3) = \frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j}$$

Appearance of cross terms indicates non-orthogonal coordinate system

Spaces for which expression $ds^2 = \sum_{ij} g_{ij}\, dq_i\, dq_j$ is true are called **Riemannian or metric**

$$g_{ij}(q_1, q_2, q_3) = \frac{\partial x}{\partial q_i}\frac{\partial x}{\partial q_j} + \frac{\partial y}{\partial q_i}\frac{\partial y}{\partial q_j} + \frac{\partial z}{\partial q_i}\frac{\partial z}{\partial q_j} = \frac{\partial \vec{r}}{\partial q_i}\cdot\frac{\partial \vec{r}}{\partial q_j}$$

$$\{i, j\} \in \{1,2,3\}$$

$g_{ij}(q_1, q_2, q_3)$ are the scalar product of the tangent vectors $\dfrac{\partial \vec{r}}{\partial q_i}$

to the curves $\vec{r}$ for $q_j =$ constant

Now we focus coordinate systems where $g_{ij}(q_1, q_2, q_3) = 0$ for $i \neq j$

here the unit vectors orthogonal to each other, therefore $\hat{q}_i \cdot \hat{q}_j = \delta_{ij}$

in short notation we may write $g_{ii} = h_i$

$$g_{ii}(q_{1,}q_{2,}q_3)=h_i^2>0$$

$$ds^2=(h_1 dq_1)^2+(h_2 dq_2)^2+(h_3 dq_3)^2=\sum_i (h_i dq_i)^2$$

Orthogonal coordinates are specified by the scale factors $(h_{1,}h_{2,}h_3)$

$$ds_i=h_i dq_i$$ (line element)

is the differential length along the vector $\hat{q}_i$ and always have the dimension of length, note that the elements $(q_{1,}q_{2,}q_2)$ alone may not have dimension of length

$$d\vec{r}=h_1 dq_1\hat{q}_1+h_2 d\,q_2\hat{q}_2+h_3 d\,q_3\hat{q}_3=\sum_i h_i dq_i\hat{q}_i$$

The curvilinear components of the line integral may be written as

$$\int\vec{V}\cdot d\vec{r}=\sum_i\int V_i h_i dq_i$$

The corresponding area and volume elements are

$$d\sigma_{ij} = ds_i\, ds_j = h_i h_j\, dq_i\, dq_j$$

$$d\tau = ds_1\, ds_2\, ds_3 = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$$

The surface element is explicitly written as

$$d\vec{\sigma} = ds_2\, ds_3\, \hat{q}_1 + ds_3\, ds_1\, \hat{q}_2 + ds_1\, ds_2\, \hat{q}_3$$

$$= h_2 h_3\, dq_2\, dq_3\, \hat{q}_1 + h_3 h_1\, dq_3\, dq_1\, \hat{q}_2 + h_1 h_2\, dq_1\, dq_2\, \hat{q}_3$$

The surface integral can be written as

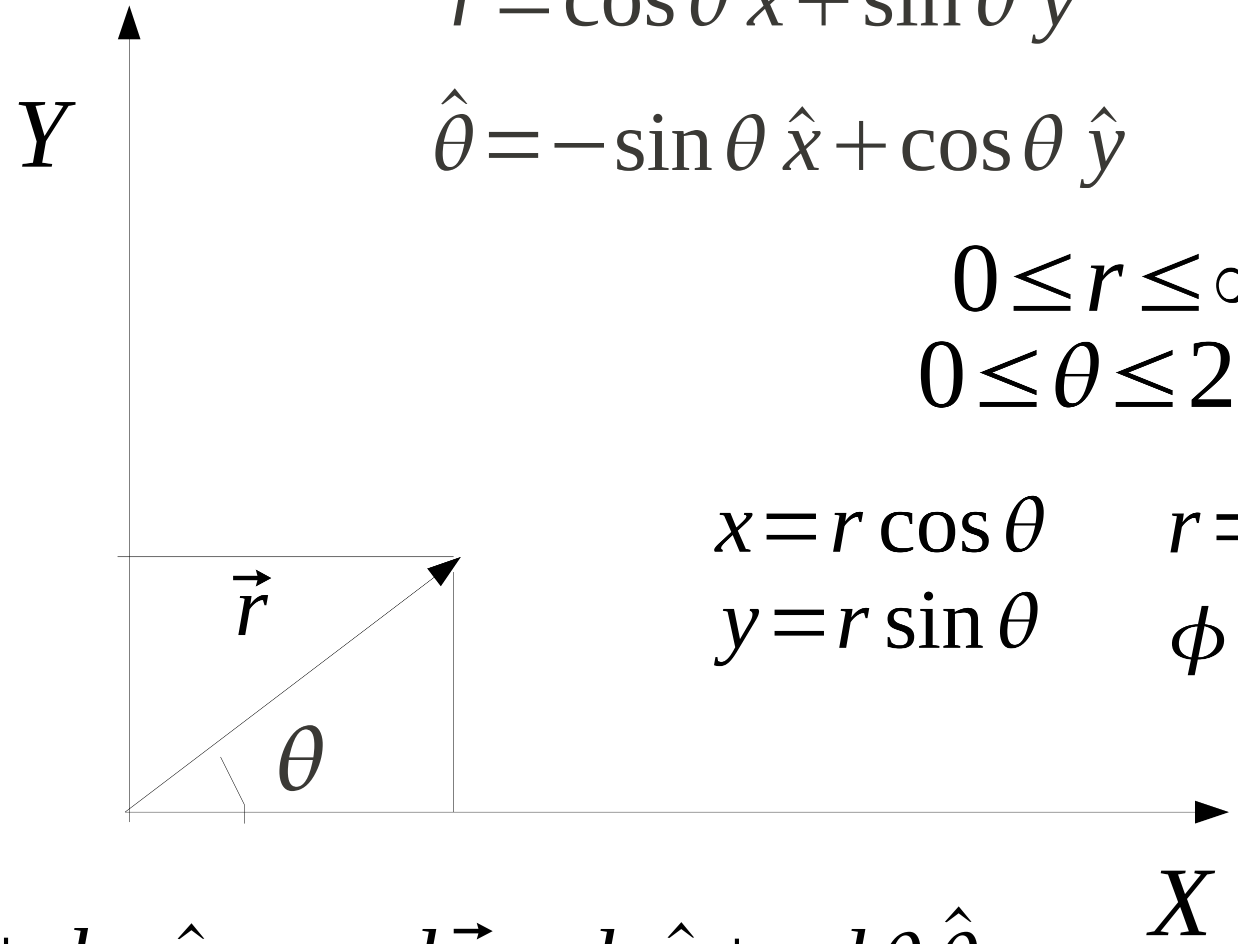
$$\int \vec{V} \cdot d\vec{\sigma} = \int V_1 h_2 h_3\, dq_2\, dq_3 + \int V_2 h_3 h_2\, dq_3\, dq_2$$

$$+ \int V_3 h_1 h_2\, dq_1\, dq_2$$

## Polar coordinates

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$




$$0 \le r \le \infty$$
$$0 \le \theta \le 2\pi$$

$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \phi = \tan^{-1} y/x$$

$$d\vec{r} = dx\,\hat{x} + d\,y\,\hat{y} \qquad d\vec{r} = dr\,\hat{r} + r\,d\,\theta\,\hat{\theta}$$

The distance in polar coordinates is

$$d\,s^2 = (d\,\vec{r}\,)^2 = dr^2 + r^2\,d\,\theta^2$$

Comparing this with general expression

$$ds^2 = (h_1\,dq_1)^2 + (h_2\,dq_2)^2$$

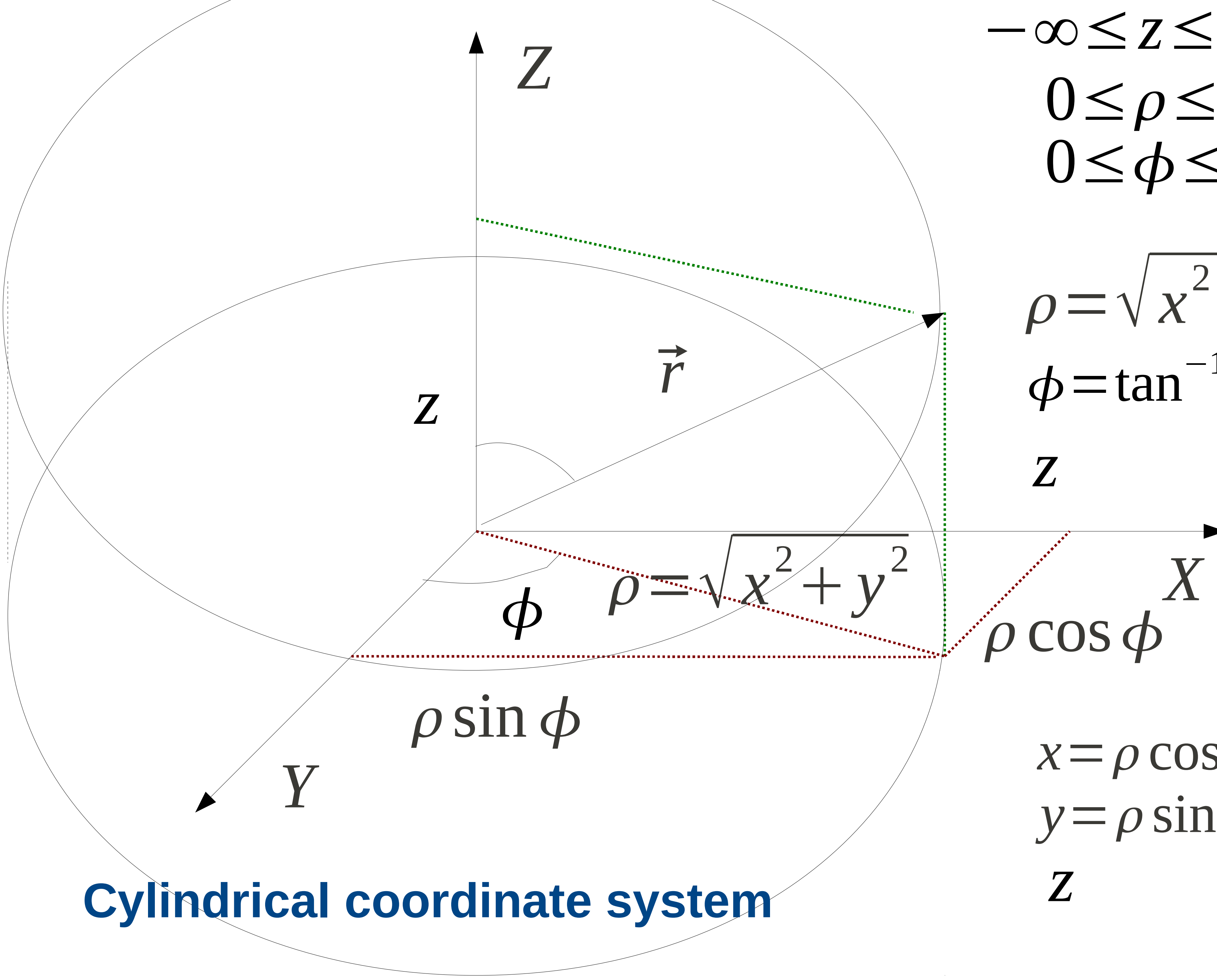
$$h_1 = 1 \qquad h_2 = r$$

Integrating over the area in polar coordinates using the generalized coordinate methods

$$A=\oint h_1 h_2\, dq_1\, dq_2=\int_0^{r_0}\int_0^{2\pi} r\, dr\, d\theta$$

$$=\int_0^{r_0} r\, dr\,[\theta]_0^{2\pi}$$

$$=2\pi\int_0^{r_0} r\, dr=\pi[r^2]_0^{r_0}=\pi r_0^2$$

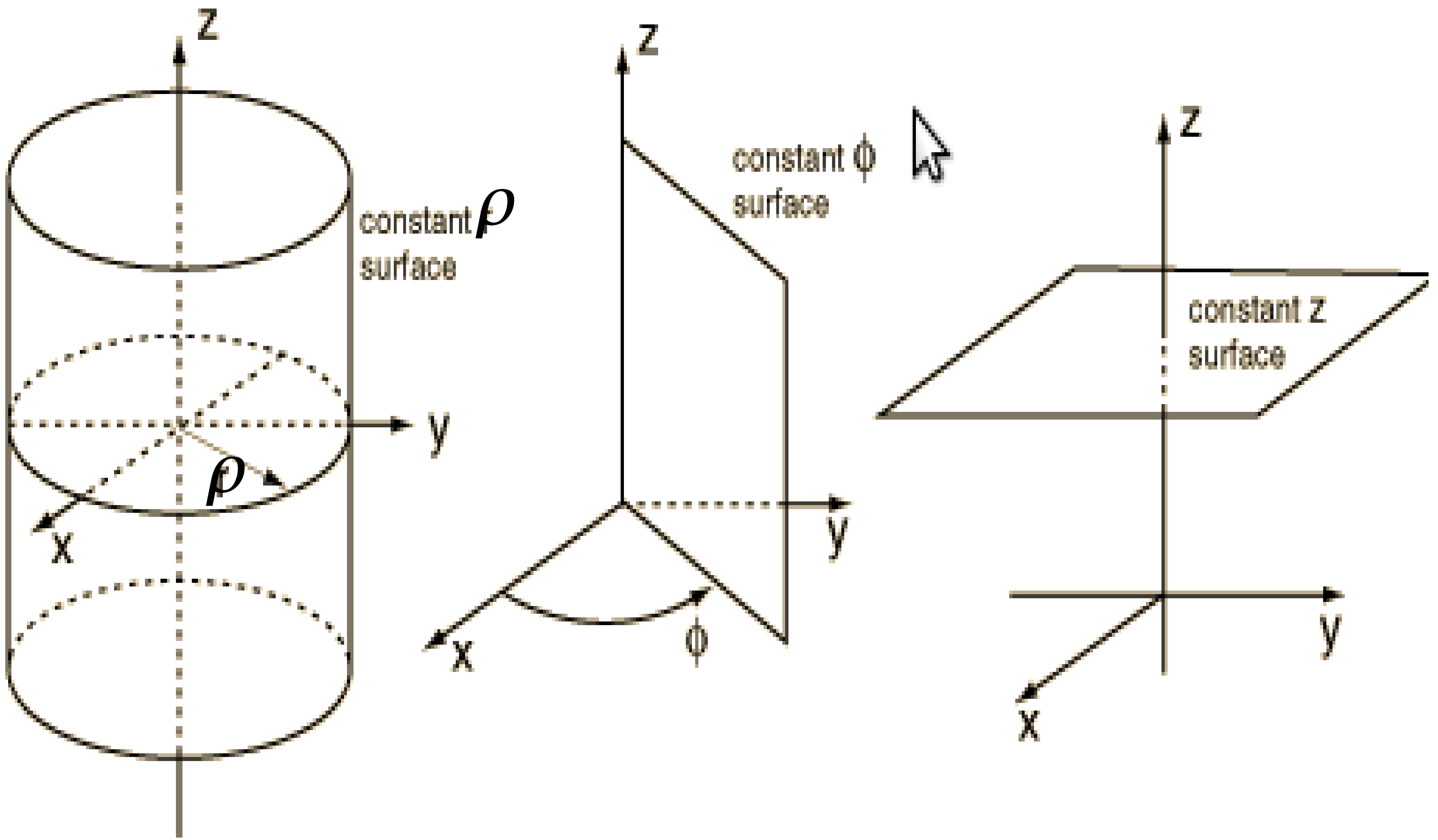



$$-\infty \le z \le \infty$$
$$0 \le \rho \le \infty$$
$$0 \le \phi \le 2\pi$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1} y / x$$
$$z$$

$$x = \rho \cos \theta$$
$$y = \rho \sin \theta$$
$$z$$

## Cylindrical coordinate system




$$\rho = \sqrt{x^2 + y^2} \qquad \phi = \tan^{-1} y/x \qquad z$$

$$\rho = constant \qquad \phi = constant \qquad z = constant$$

The unit vectors that defines this coordinate system are

$$(\hat{q}_{1,} \hat{q}_{2,} \hat{q}_3) = (\hat{\rho}, \hat{\phi}, \hat{z})$$

any vectors may be expressed in terms of these unit vectors

$$d\vec{r} = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}$$

the displacement  is now given by

$$ds^2 = (d\vec{r})^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2$$

By comparing it with standard expression of displacement in curvilinear coordinates

$$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$

we can obtain the scale factors of this coordinate system

$$h_1 = 1 \qquad h_2 = \rho \qquad h_3 = 1$$

$$r=(x^2+y^2+z^2)^{1/2}$$

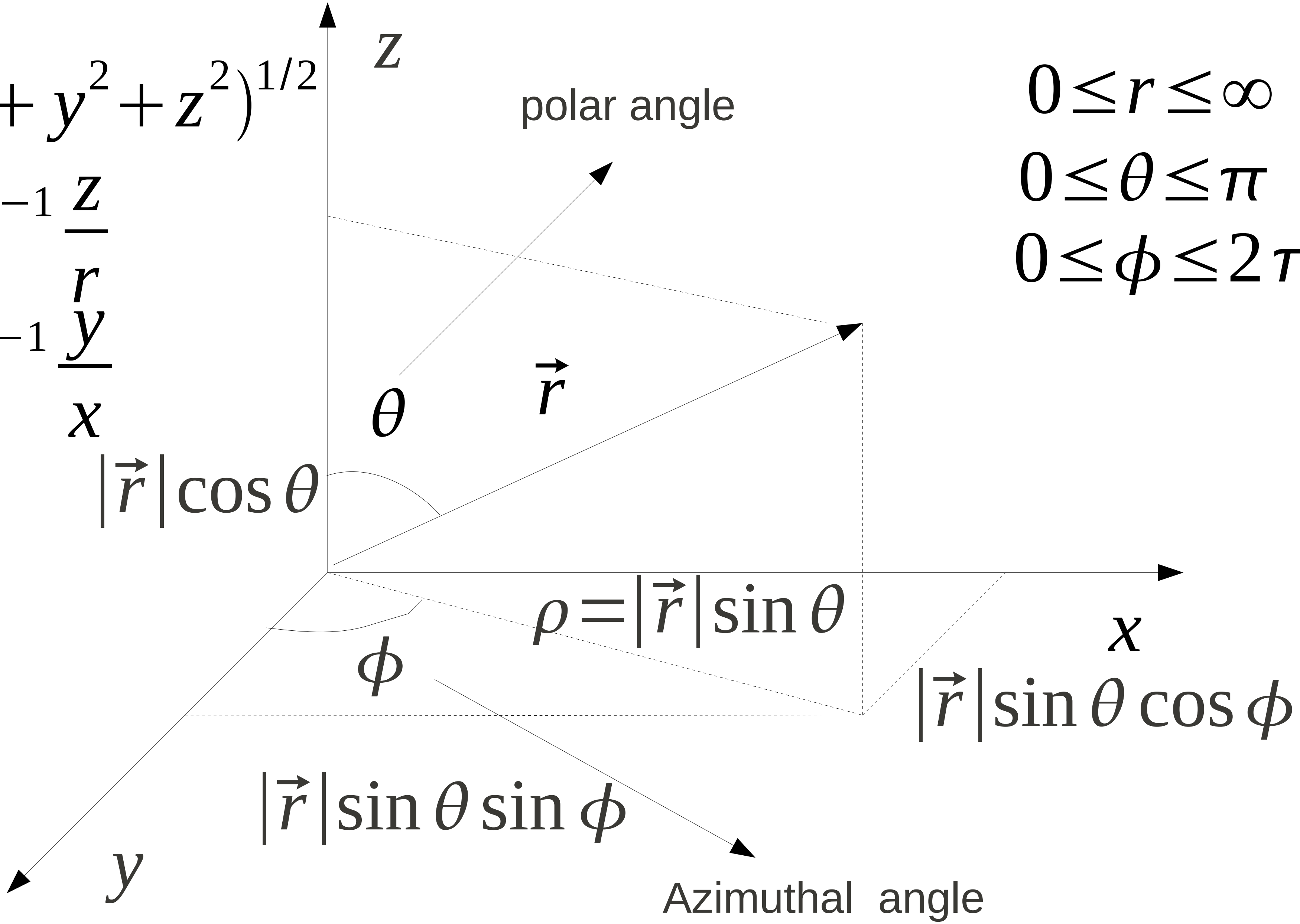
$$\theta=\cos^{-1}\frac{z}{r}$$

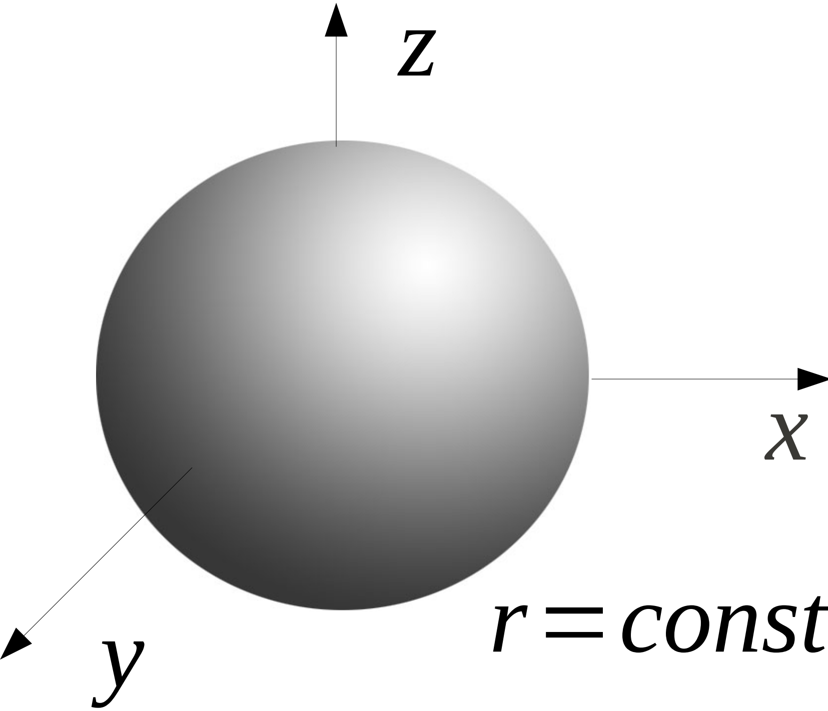
$$\phi=\tan^{-1}\frac{y}{x}$$

$$0\le r\le\infty$$

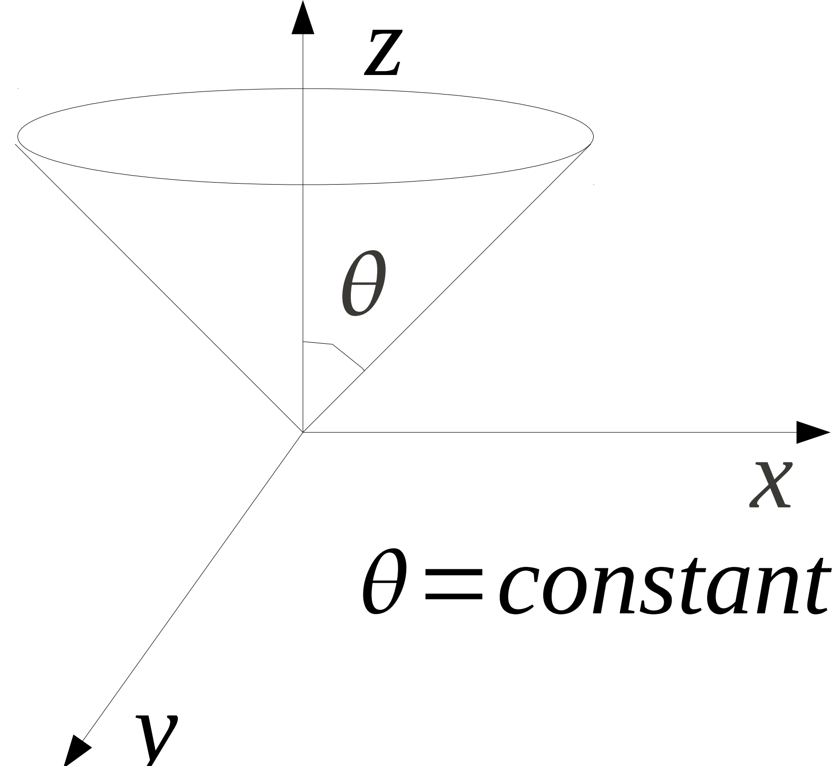
$$0\le\theta\le\pi$$

$$0\le\phi\le 2\pi$$

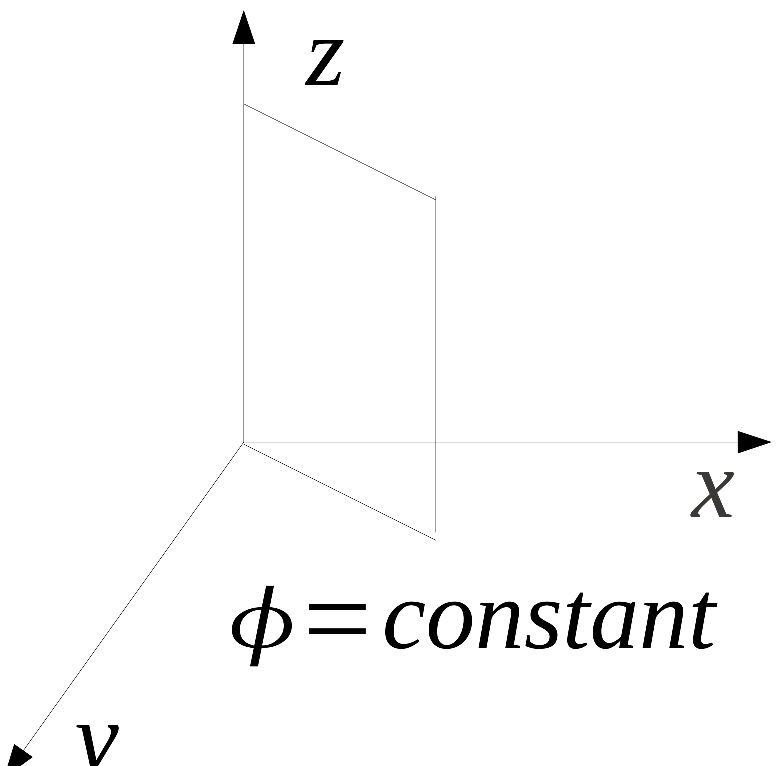





$r = constant$




$\theta = constant$




$\phi = constant$

$$r = \left(x^2 + y^2 + z^2\right)^{1/2}$$

$$\theta = \cos^{-1}\frac{z}{r}$$

$$\phi = \tan^{-1}\frac{y}{x}$$

The transformation equation between Cartesian and spherical polar coordinates

$$x = r\sin\theta\cos\phi \qquad r = (x^2 + y^2 + z^2)^{1/2}$$

$$y = r\sin\theta\sin\phi \qquad \theta = \cos^{-1}\frac{z}{r}$$

$$z = r\cos\theta \qquad \phi = \tan^{-1}\frac{y}{x}$$

The vectorial displacement in polar coordinate system is

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$$

The length element for the polar coordinates

$$ds^2 = (d\vec{r})^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\,d\phi^2$$

By comparing it with general equation of distance in curvilinear coordinates

$$ds^2 = (h_1 dq_1)^2 + (h_2 dq_2)^2 + (h_3 dq_3)^2$$

we can obtain the scale factors of this coordinate system

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = r\sin\theta$$

Surface area of a sphere can be calculated from the areal integration over surface $r = constant$

$$A = \oint h_1 h_2 dq_1 dq_2 = \int_0^{\pi} \int_0^{2\pi} r^2 \sin\theta \, d\theta \, d\phi$$

$$A = r^2 \int_0^{\pi} \int_0^{2\pi} \sin\theta \, d\theta \, d\phi = 4\pi r^2$$

Similarly an volume integration give volume of the sphere

$$V = \int_v h_1 h_2 h_3 dq_1 dq_2 dq_3 = \int_0^{r_0} \int_0^{\pi} \int_0^{2\pi} dr \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= 4\pi \int_0^{r_0} dr \, r^2 = \frac{4\pi}{3} r_0^3$$

The scalar products is defined in usual way, that is

$$\vec{A}\cdot\vec{B}=\sum_{ik} A_i \hat{q}_i \cdot \hat{q}_k B_k = \sum_{ik} A_i B_k \delta_{ik}$$

$$= A_1 B_1 + A_2 B_2 + A_3 B_3$$

The vector product is defined as

$$\vec{A}\times\vec{B}=\begin{vmatrix} \hat{q}_1 & \hat{q}_2 & \hat{q}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

We use for reference coordinates system as Cartesian for transforming between coordinate systems

## Coordinate transformation using Jacobian matrix

Consider a coordinate transformation between $(x, y, z) \rightarrow (q_{1,} q_{2,} q_3)$

$$x = x(q_{1,} q_{2,} q_{3,}) \quad y = y(q_{1,} q_{2,} q_{3,}) \quad z = z(q_{1,} q_{2,} q_{3,})$$

$$dx = \frac{\partial x}{\partial q_1} dq_1 + \frac{\partial x}{\partial q_2} dq_2 + \frac{\partial x}{\partial q_3} dq_3$$

$$dy = \frac{\partial y}{\partial q_1} dq_1 + \frac{\partial y}{\partial q_2} dq_2 + \frac{\partial y}{\partial q_3} dq_3$$

$$dz = \frac{\partial z}{\partial q_1} dq_1 + \frac{\partial z}{\partial q_2} dq_2 + \frac{\partial z}{\partial q_3} dq_3$$

This equation can be expressed in the matrix form called **Jacobian matrix**

$$\begin{bmatrix} dx \\ dy \\ dz \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{bmatrix} \begin{bmatrix} dq_1 \\ dq_2 \\ dq_3 \end{bmatrix}$$

Let us now see how area transform using Jacobian matrix – consider two differential vectorial lengths in a plane. The Cartesian surface elements $dx\,dy$ Become infinitesimal rectangle in new coordinates system formed by vectors

$$d\vec{r}_1=\vec{r}(q_1+dq_1,q_2)-\vec{r}(q_1,q_2)=\frac{\partial\vec{r}}{\partial q_1}dq_1$$

$$d\vec{r}_2=\vec{r}(q_1,q_2+dq_2)-\vec{r}(q_1,q_2)=\frac{\partial\vec{r}}{\partial q_2}dq_2$$

The area expressed in the direction of third vectorial element

$$dx\,dy=d\vec{r}_1\times d\vec{r}_2\Big|_z=\left|\frac{\partial x}{\partial q_1}\frac{\partial y}{\partial q_2}-\frac{\partial x}{\partial q_2}\frac{\partial y}{\partial q_1}\right|dq_1dq_2$$

This can be expressed as determinant of a matrix

$$=\begin{vmatrix}\dfrac{\partial x}{\partial q_1} & \dfrac{\partial x}{\partial q_2}\\[2ex] \dfrac{\partial y}{\partial q_1} & \dfrac{\partial y}{\partial q_2}\end{vmatrix}dq_1dq_2$$

The transformation coefficient matrix is called **Jacobian matrix.**

The transformation coefficient is given determinant of the Jacobian matrix, or **Jacobian**

Similarly the volume element can be expressed as triple scalar product of three infinitesimal displacement vectors

$$d\vec{r}_i = dq_i \frac{\partial \vec{r}}{\partial q_i}$$

Along $q_i$ direction of $\hat{q}_i$

The volume element can be expressed in matrix determinant form

$$dx\,dy\,dz = \begin{vmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} \end{vmatrix} dq_1\,dq_2\,dq_3$$

For orthogonal coordinates the Jacobian becomes



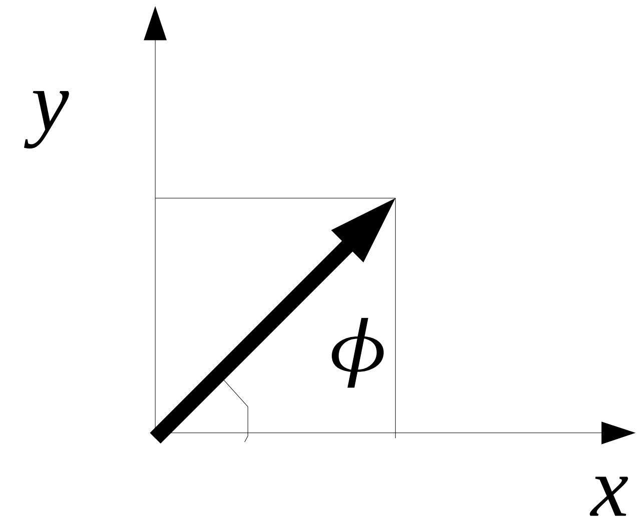
$$\Rightarrow h_1 h_2 h_3 (\hat{q}_1 \times \hat{q}_2) \cdot \hat{q}_3 = h_1 h_2 h_3$$

The Jacobian for the transformation between Cartesian and polar coordinates

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$dx\,dy = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \phi} \end{vmatrix} dr\,d\phi$$




$$dx\,dy = \begin{vmatrix} \cos\phi & -r\sin\phi \\ \sin\phi & r\cos\phi \end{vmatrix} dr\,d\phi$$

$$= r\left(\cos^2\phi + \sin^2\phi\right) dr\,d\phi$$

$$= r\,dr\,d\phi$$

The Jacobian for the spherical coordinates

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

complete the missing steps yourself

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2(\cos^2\theta\sin\theta + \sin^3\theta) = r^2\sin\theta$$

Hence transformation of the volume element using Jacobian is

$$dx\,dy\,dz = r^2\,dr\sin\theta\,d\theta\,d\phi$$

# Differential vector operators

**Gradient**

Using the geometrical interpretation of gradient as the vector having the maximum rate of change

$$\hat{q}_1 \cdot \vec{\nabla}\psi = \vec{\nabla}\psi\Big|_1 = \frac{\partial\psi}{\partial s_1} = \frac{\partial\psi}{\partial q_1}\frac{\partial q_1}{\partial s_1} = \frac{1}{h_1}\frac{\partial\psi}{\partial q_1}$$

Differential length along $\hat{q}_1$ is $ds_1 = h_1\, d\, q_1 = h_1 \partial q_1$

By repeating this operation in other directions and adding vectorial

$$\vec{\nabla}\psi(q_{1,}q_{2,}q_3) = \hat{q}_1\frac{\partial\psi}{\partial s_1} + \hat{q}_2\frac{\partial\psi}{\partial s_2} + \hat{q}_3\frac{\partial\psi}{\partial s_3}$$

$$= \hat{q}_1\frac{1}{h_1}\frac{\partial\psi}{\partial q_1} + \hat{q}_2\frac{1}{h_2}\frac{\partial\psi}{\partial q_2} + \hat{q}_3\frac{1}{h_3}\frac{\partial\psi}{\partial q_3}$$

$$= \sum_i \hat{q}_i\frac{1}{h_i}\frac{\partial\psi}{\partial q_i}$$

Gradient operator in cylindrical coordinate system

$$h_1 = 1 \qquad h_2 = \rho \qquad h_3 = 1$$

$$\vec{\nabla}\psi(q_1, q_2, q_3) = \hat{q}_1 \frac{1}{h_1}\frac{\partial\psi}{\partial q_1} + \hat{q}_2 \frac{1}{h_2}\frac{\partial\psi}{\partial q_2} + \hat{q}_3 \frac{1}{h_3}\frac{\partial\psi}{\partial q_3}$$

$$\vec{\nabla}\psi(\rho, \phi, z) = \hat{\rho}\frac{\partial\psi}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{z}\frac{\partial\psi}{\partial z}$$

Gradient operator in spherical coordinate system

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = r\sin\theta$$

$$\vec{\nabla}\psi(r, \theta, \phi) = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}$$

## Divergence

From Gauss' divergence theorem it is possible to arrive at definition of divergence

$$\int_V \vec{\nabla}\cdot\vec{V}(q_{1,}q_{2,}q_3)\,d\tau=\int_S \vec{V}\cdot d\vec{\sigma}$$

Therefore the divergence can be defined as

$$\vec{\nabla}\cdot\vec{V}(q_{1,}q_{2,}q_3)=\lim_{\int d\tau\to 0}\frac{\int \vec{V}\cdot d\vec{\sigma}}{\int d\tau}$$

The differential volume is given by $h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$

The coordinate system is chosen such that $(\hat{q}_{1,}\hat{q}_{2,}\hat{q}_3)$ form a right handed set and $(\hat{q}_3=\hat{q}_1\times\hat{q}_2)$

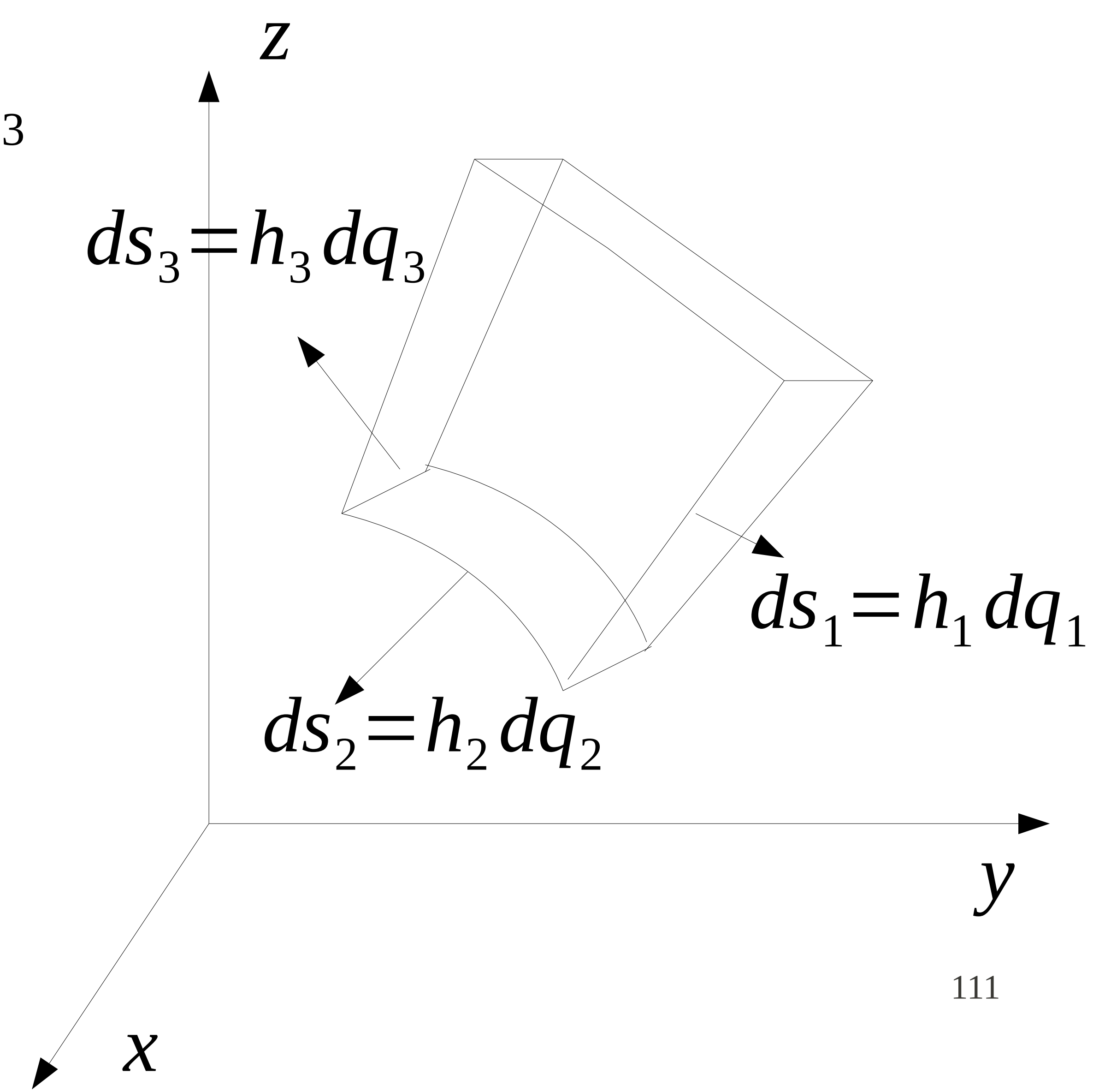
The difference in area integrals of flux for the two faces $q_1 = constant$

$$\left[ V_1 h_2 h_3 + \frac{\partial}{\partial q_1}(V_1 h_2 h_3)\, d\, q_1 \right] dq_2 dq_3 \;- V_1 h_2 h_3\, dq_2 dq_3$$

$$= \frac{\partial}{\partial q_1}(V_1 h_2 h_3)\, dq_1 dq_2 dq_3$$

Here $V_i = \vec{V} \cdot \hat{q}_i$

Adding similar result in each pair of faces



$$\int \vec{V}(q_{1,} q_{2,} q_3)\cdot\vec{d}\sigma \qquad d\tau = h_1 h_2 h_3 dq_1 dq_2 dq_3$$

$$= \left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right] dq_1 dq_2 dq_3$$

Using fundamental definition

$$\vec{\nabla}\cdot\vec{V}(q_{1,} q_{2,} q_3) = \lim_{\int d\tau \to 0} \frac{\int \vec{V}\cdot d\vec{\sigma}}{\int d\tau}$$

$$\vec{\nabla}\cdot\vec{V}(q_{1,} q_{2,} q_3)$$

$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3) + \frac{\partial}{\partial q_2}(V_2 h_3 h_1) + \frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right]$$

$$\vec{\nabla}\cdot\vec{V}(q_1, q_2, q_3)$$

$$=\frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(V_1 h_2 h_3)+\frac{\partial}{\partial q_2}(V_2 h_3 h_1)+\frac{\partial}{\partial q_3}(V_3 h_1 h_2)\right]$$

Divergence operator in cylindrical coordinate system

$$h_1=1 \qquad h_2=\rho \qquad h_3=1$$

$$\vec{\nabla}\cdot\vec{V}(\rho,\phi,z) \quad =\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho V_\rho)+\frac{1}{\rho}\frac{\partial}{\partial\phi}(V_\phi)+\frac{\partial}{\partial z}(V_z)$$

Divergence for spherical coordinate system

$$h_1=1 \qquad h_2=r \qquad h_3=r\sin\theta$$

$$\vec{\nabla}\cdot\vec{V}(r,\theta,\phi)$$

$$=\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2 V_r)+r\frac{\partial}{\partial\theta}(\sin\theta V_\theta)+r\frac{\partial}{\partial\phi}(V_\phi)\right]$$

**Laplacian**

Combining gradient operator

$$\vec{\nabla}\psi(q_{1,}q_{2,}q_3)=\hat{q}_1\frac{1}{h_1}\frac{\partial\psi}{\partial q_1}+\hat{q}_2\frac{1}{h_2}\frac{\partial\psi}{\partial q_2}+\hat{q}_3\frac{1}{h_3}\frac{\partial\psi}{\partial q_3}$$

with divergence

replace single function with component in divergence operator

$$\vec{\nabla}\cdot\vec{V}(q_{1,}q_{2,}q_3)=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(V_1h_2h_3)+\frac{\partial}{\partial q_2}(V_2h_3h_1)+\frac{\partial}{\partial q_3}(V_3h_1h_2)\right]$$

$$\vec{\nabla}\cdot\vec{\nabla}\psi(q_{1,}q_{2,}q_3)=\nabla^2\psi(q_{1,}q_{2,}q_3)$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}+\frac{\partial}{\partial q_2}\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial q_2}+\frac{\partial}{\partial q_3}\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right]$$

$$\nabla^2 \psi(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1} + \frac{\partial}{\partial q_2}\frac{h_3 h_1}{h_2}\frac{\partial \psi}{\partial q_2} + \frac{\partial}{\partial q_3}\frac{h_1 h_2}{h_3}\frac{\partial \psi}{\partial q_3}\right]$$

Laplacian for cylindrical coordinate system

$$h_1 = 1 \qquad h_2 = \rho \qquad h_3 = 1$$

$$\nabla^2 \psi(\rho, \phi, z) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \psi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Laplacian for spherical coordinate system

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = r \sin\theta$$

$$\nabla^2 \psi(r, \theta, \phi) = \frac{1}{r^2 \sin\theta}\left[\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial \phi^2}\right]$$

## Curl

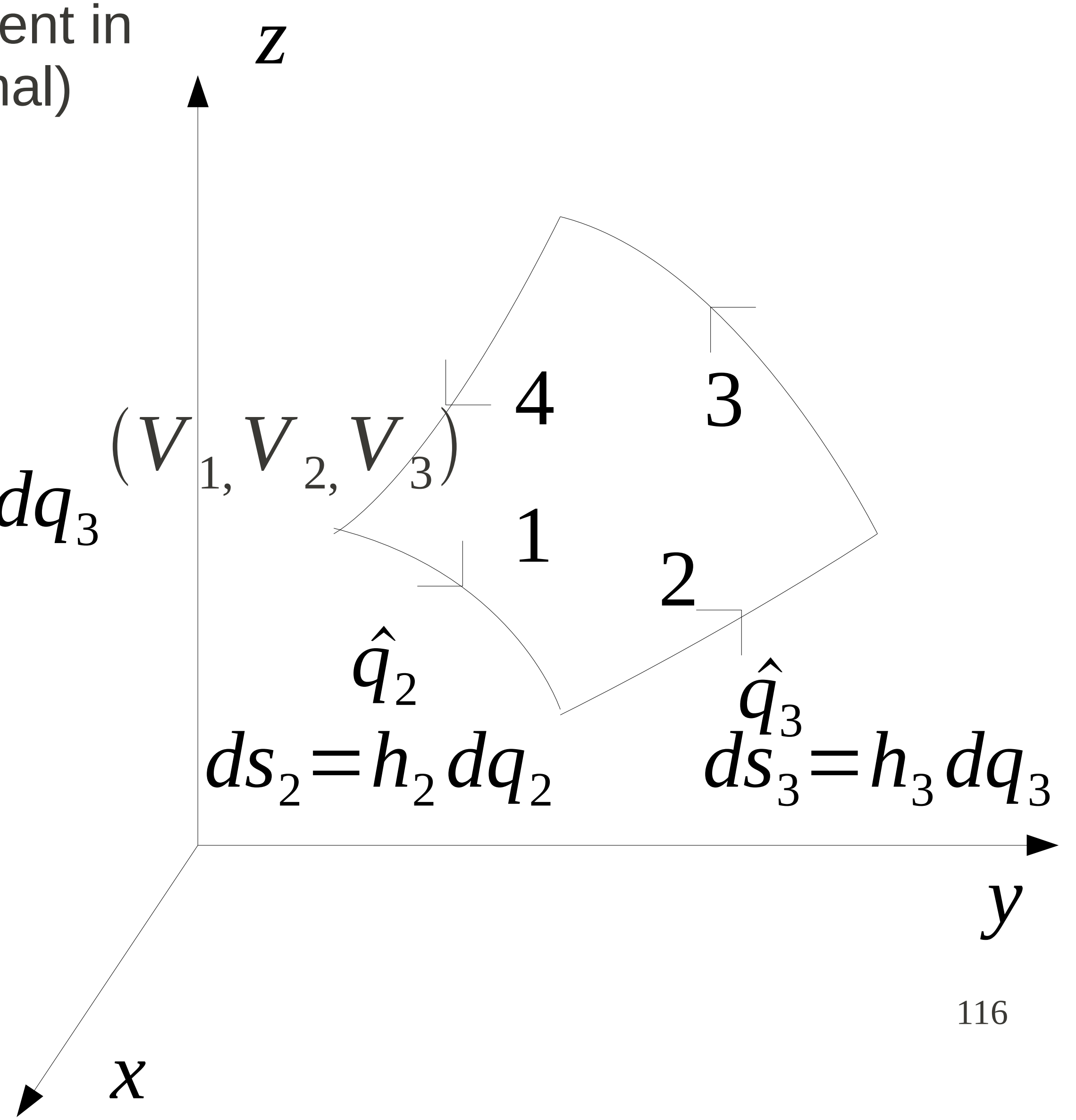
For applying the Stokes theorem to an infinitesimal element ,we consider two dimensional surface on a 3d system

Consider differential surface element in the curvilinear surface (infinitesimal)

$$q_1 = constant$$

$$\int_s \vec{\nabla}\times\vec{V}\cdot d\vec{\sigma} =$$

$$\hat{q}_1\cdot(\vec{\nabla}\times\vec{V})h_2 h_3 dq_2 dq_3$$




By Stokes' theorem

$$\hat{q}_1\cdot(\vec{\nabla}\times\vec{V})h_2 h_3 dq_2 dq_3$$

$$= \oint \vec{V}\cdot d\vec{r}$$

Following the loop (1,2,3,4)

$$\oint \vec{V}(q_{1,} q_{2,} q_3)\cdot d\vec{r} = V_2 h_2 d q_2 + \left[ V_3 h_3 + \frac{\partial}{\partial q_2}(V_3 h_3) dq_2 \right] dq_3$$

$$- \left[ V_2 h_2 + \frac{\partial}{\partial q_3}(V_2 h_2) dq_3 \right] dq_2 - V_3 h_3 dq_3$$

$$= \left[ \frac{\partial}{\partial q_2}(V_3 h_3) - \frac{\partial}{\partial q_3}(V_2 h_2) \right] dq_2 dq_3$$

Now using the relation $\hat{q}_1 \cdot (\vec{\nabla} \times \vec{V}) h_2 h_3 dq_2 dq_3 = \oint \vec{V} \cdot d\vec{r}$

$$\hat{q}_1 \cdot (\vec{\nabla} \times \vec{V}) h_2 h_3 dq_2 dq_3 = \left[ \frac{\partial}{\partial q_2}(V_3 h_3) - \frac{\partial}{\partial q_3}(V_2 h_2) \right] dq_2 dq_3$$

$$\vec{\nabla} \times \vec{V}\Big|_1 = \frac{1}{h_2 h_3} \left[ \frac{\partial}{\partial q_2}(V_3 h_3) - \frac{\partial}{\partial q_3}(V_2 h_2) \right]$$

Remaining components can be found by cyclic permutation. It is convenient to express in the matrix notation

$$\vec{\nabla}\times\vec{V}=\frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}$$

Curl for cylindrical coordinate system

$$h_1=1 \qquad h_2=\rho \qquad h_3=1$$

$$\vec{\nabla}\times\vec{V}=\frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ V_\rho & \rho V_\phi & V_z \end{vmatrix}$$

Curl for spherical coordinate system

$$h_1 = 1 \qquad h_2 = r \qquad h_3 = r\sin\theta$$

$$\vec{\nabla} \times \vec{V} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\,\hat{\theta} & r\sin\theta\,\hat{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ V_r & r\,V_\theta & r\sin\theta\,V_\phi \end{vmatrix}$$

# Representation of measurements in mechanics

- Position $\vec{r}$
- Velocity $\vec{v}$
- Angular velocity $\vec{\omega}$
- Force $\vec{F}$
- Torques $\vec{N}$
- Higher derivative of position

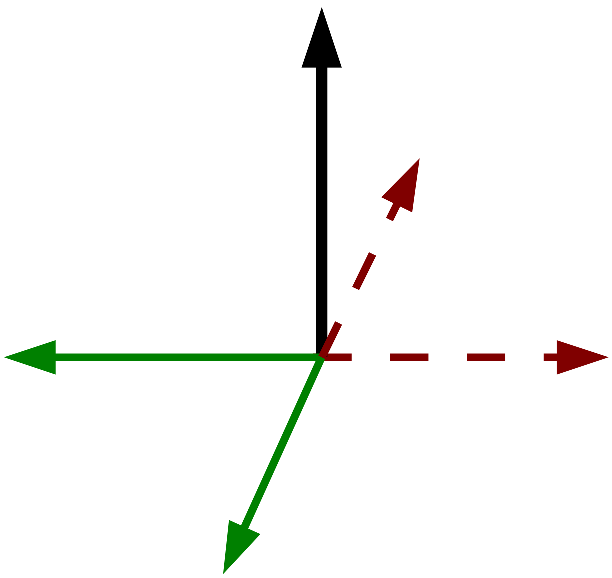
# Vectors and pseudo vectors

- Displacement
- Velocity
- Force
- Momentum
- Electric field

They directly point at the direction of the quantity. They are also called **POLAR** vectors

- Angle
- Angular velocity
- Torque
- Angular momentum
- Magnetic field

They point along the direction of the axis of rotation, also called **AXIAL** vectors



$$\vec{c} = \vec{a} \times \vec{b}$$

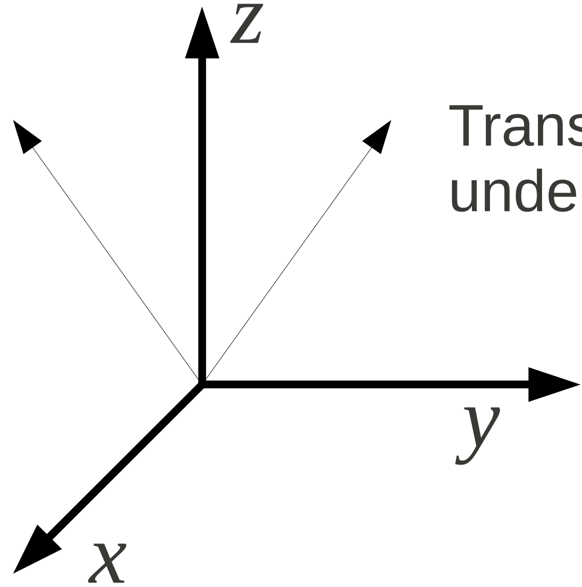
$$\vec{c} = -\vec{a} \times -\vec{b}$$

Under inversion the coordinate system pseudo vectors remain unchanged.

$$a=(a_x, a_y, a_z)$$

$$\downarrow$$

$$a=(a_x, -a_y, a_z)$$




Transformation of polar vector under reflection

Mirror location is x-z plane

Transformation of axial or pseudo vectors

Let $\mu=\vec{a}\times\vec{b}$

$$a=(a_x, a_y, a_z)$$

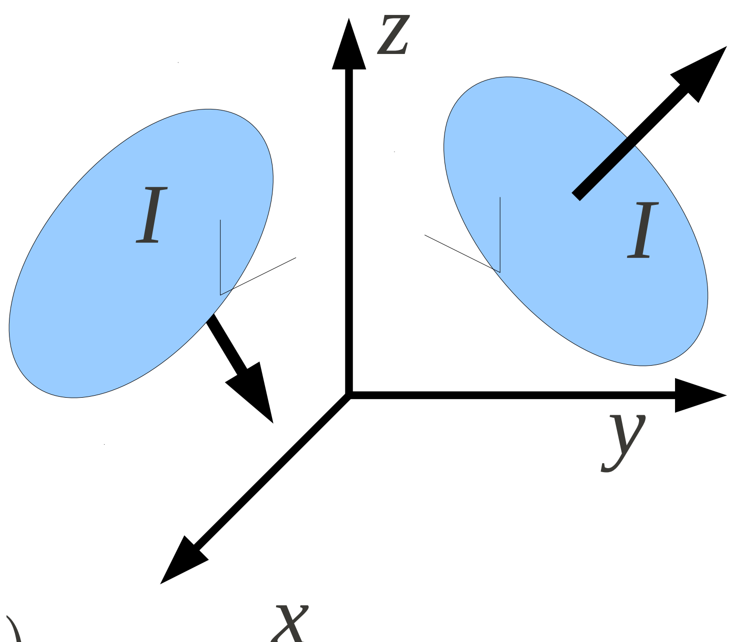
$$\downarrow$$

$$a=(a_x, -a_y, a_z)$$

$$b=(b_x, b_y, b_z)$$

$$\downarrow$$

$$b=(b_x, -b_y, b_z)$$




$$\mu=(\mu_x, \mu_y, \mu_z)$$

$$\downarrow$$

$$\mu'=(-\mu_x, \mu_y, -\mu_z)$$

# Basics of tensors

1) Tensors are quantities that does not depend on the coordinate system.

2) Scalars invariant  under coordinate transformation, they are tensors of rank 0, in $d$ dimensional space number of components of scaler is $d^0$

3) Vectors are invariant under coordinate rotation, in three dimensional space they have $3^1$ number of components, there for they are tensors of rank 1.

4) In a $d$ dimensional space a tensor of rank n has $d^n$ components.

Most of constants like diffusion, viscosity dielectric constant are tensors of rank 2.

## Reference for module “vectors and coordinate systems”

1) Introduction to electrodynamics: Third edition by D J Griffiths

Chapter 1 : Good source for problems and basic introduction

2) Mathematical methods for Physicists by Arfken and Weber 6$^{th}$ edition

Chapter 1 and 2 Good reference for theory

3) Advanced engineering mathematics by Erwin Kreyszig 8$^{th}$ edition

Chapter 8 and 9

4) Classical dynamics of particles and systems by S T Thornton and J B Marion

Chapter 1 Basic introduction

5) Classical mechanics point particles and relativity Walter Greiner

Part I