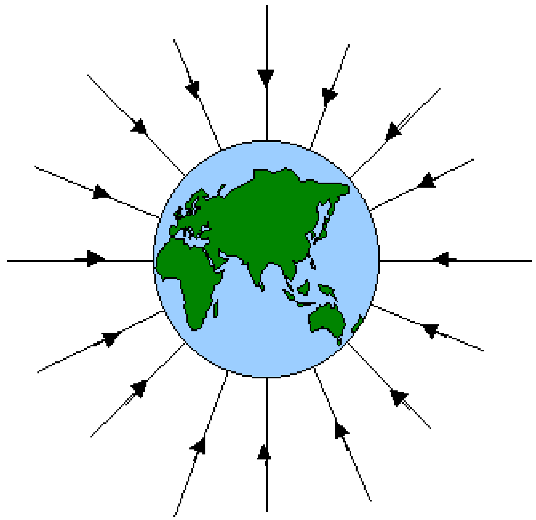
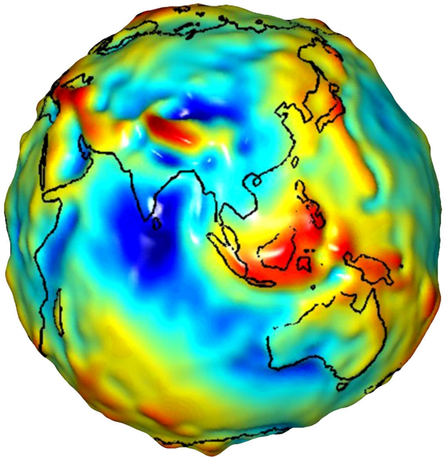
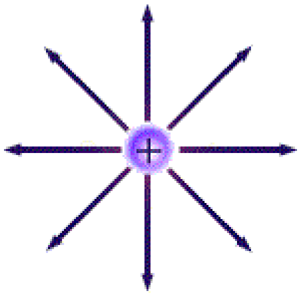
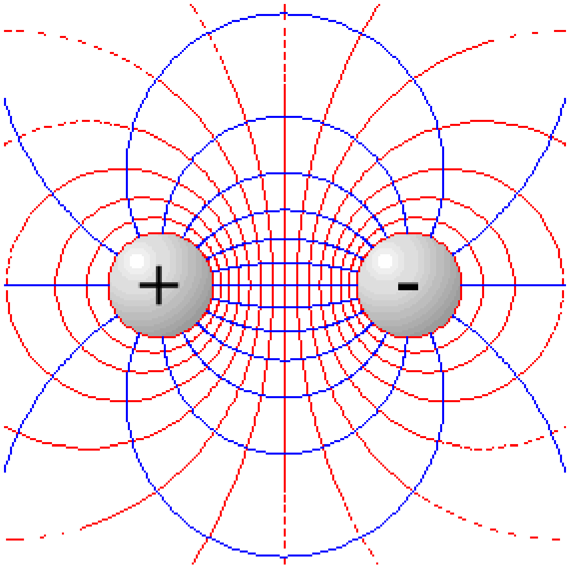
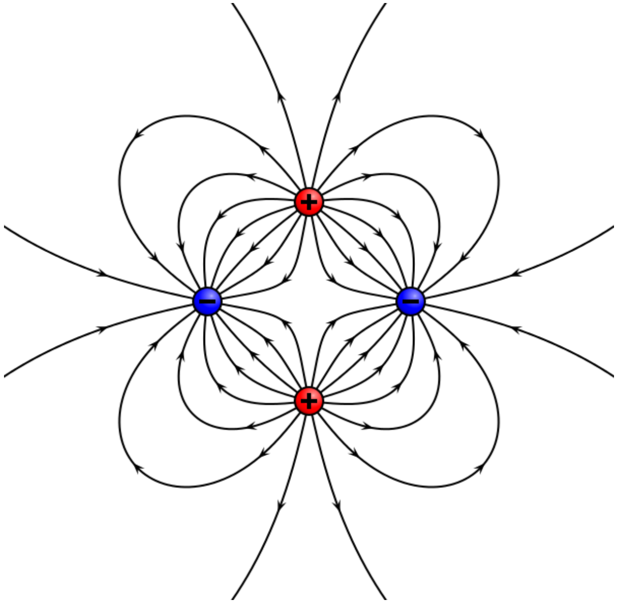
Potentials and fields







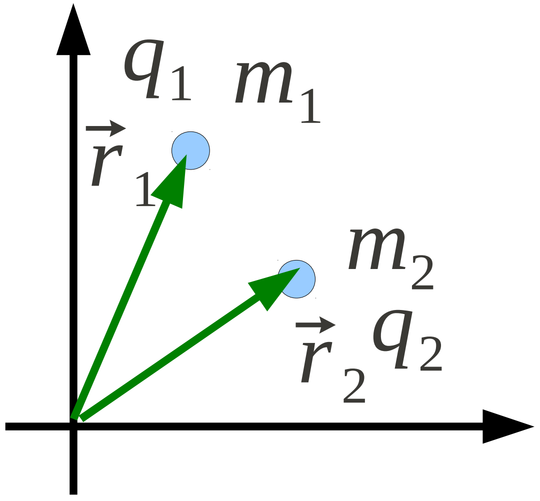




Fundamental interactions of nature

In the order of strength

1) Potential energy due to Gravitational interactions



$$V(\vec{r}) = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$$

2) weak

Interaction in nuclear beta particle decay

3) Potential energy due to electrostatic interaction

$$V(\vec{r}) = -\frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

4) nuclear or strong

Interaction that bind nucleons in a nucleus of an atom

Gravitational potential

The gravitational potential energy of a particle of mass m located at positioned at $\vec{r}$ and subjected to gravitational fields of several masses m_j located at $\vec{r}_j$.

$$V(\vec{r}) = -\sum_j \frac{G m m_j}{|\vec{r} - \vec{r}_j|}$$

We may rewrite the gravitational potential energy as

$$V(\vec{r}) = m\phi(\vec{r})$$

We may rewrite the gravitational potential energy as

$$\phi(\vec{r}) = -\sum_j \frac{G m_j}{|\vec{r} - \vec{r}_j|}$$

Under gravitational potential the equations of motion of the point mass

$$m\ddot{\vec{r}} = -\vec{\nabla} V(\vec{r}) = -m\vec{\nabla}\phi(\vec{r})$$

Therefore gravitational acceleration is independent of the mass

Then we may define the acceleration due to gravity $g(\vec{r})$ as

$$g(\vec{r}) = -\vec{\nabla}\phi(\vec{r})$$

This along with definition of $\phi(\vec{r})$ can determine the acceleration due to gravity

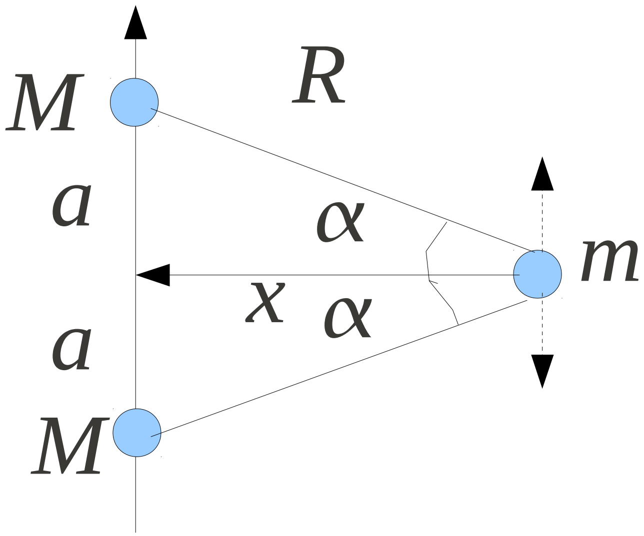
$$\phi(\vec{r}) = -\sum_j \frac{G m_j}{|\vec{r} - \vec{r}_j|}$$

Gravitational forces of extended distribution of masses

Gravitational forces on a mass that is symmetrically placed between another two masses

Each particle exerts a force of magnitude

$$F' = \frac{m M G}{R^2} \qquad R = (a^2 + x^2)^{1/2}$$



The the resultant force from the masses is given by

$$F = \frac{2 m M G \cos\alpha}{R^2} = \frac{2 m M G x}{R^3}$$

$$\cos\alpha = \frac{x}{R}$$

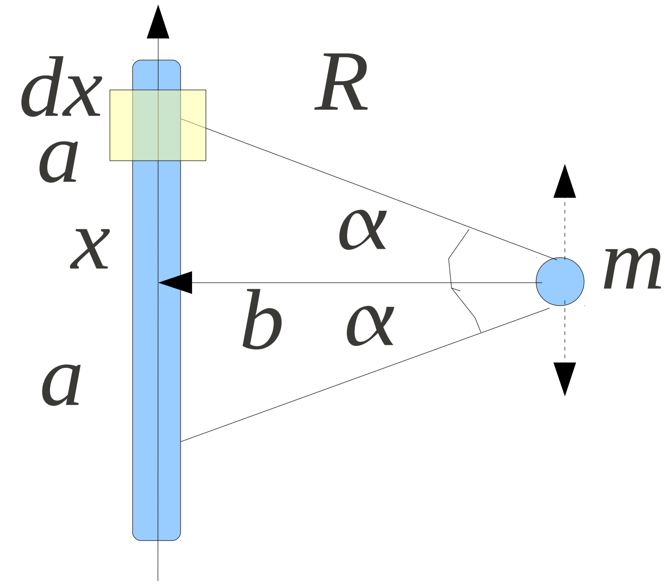
$$= \frac{2 m M G x}{(x^2 + a^2)^{3/2}}$$

$$F = \frac{2 m M G}{x^2}\left(1 + \frac{a^2}{x^2}\right)^{-3/2} \approx \frac{2 m M G}{x^2}$$

As x becomes large the factor in brackets becomes negligible

This is equivalent the attraction due to a point mass located at the center of mass

Gravitational forces by a rod of mass M



Consider a small element of the rod dx with mass $\frac{M\,dx}{2a}$

Gravitational force due to this distribution is

$$d\vec{F} = \frac{m\,M\,G\,dx}{2a\,R^2}\hat{R}$$

The gravitational attraction due to entire rod is given by integrating over the mas distribution, since the rod is symmetrically placed only one component survives

$$F = \frac{m\,M\,G}{2a}\int_{-a}^{a}\frac{\cos\alpha\,dx}{R^2} = \frac{m\,M\,G}{2a}\int_{-a}^{a}\frac{R\cos\alpha\,dx}{R^3}$$

$$= \frac{m\,M\,G}{2a}\int_{-a}^{a}\frac{b\,dx}{(x+b)^{3/2}}$$

$$F = \frac{m M G}{2a} \int_{-a}^{a} \frac{b\,dx}{(x^2+b^2)^{3/2}}$$

$$= \frac{m M G}{2a} [I]_{-a}^{a}$$ where $$I = \int \frac{b\,dx}{(x^2+b^2)^{3/2}}$$

Using the standard integral with substitution

$$x = b \tan\theta$$
$$dx = b\, sec^2\theta\, d\theta$$

$$I = \int \frac{b\,dx}{(x^2+b^2)^{3/2}}$$

$$= \int \frac{b^2 sec^2\theta\, d\theta}{((b\tan\theta)^2+b^2)^{3/2}}$$

$$= \int \frac{b^2 sec^2\theta\, d\theta}{b^3((\tan\theta)^2+1)^{3/2}}$$

$$I = \int \frac{b^2 sec^2 \theta \, d\theta}{b^3 sec^3 \theta}$$

$$= \int \frac{\cos\theta \, d\theta}{b}$$

$$= \frac{\sin\theta}{b} + c$$

$$\sin\theta = \left(\frac{\tan\theta}{\sqrt{1+\tan^2\theta}} \right)$$

$$= \frac{b \tan\theta}{b^2 \sqrt{1+\tan^2\theta}} + c$$

$$= \frac{x}{b\sqrt{b^2+x^2}} + c$$

$$I=\frac{x}{b\sqrt{b^2+x^2}}+c$$

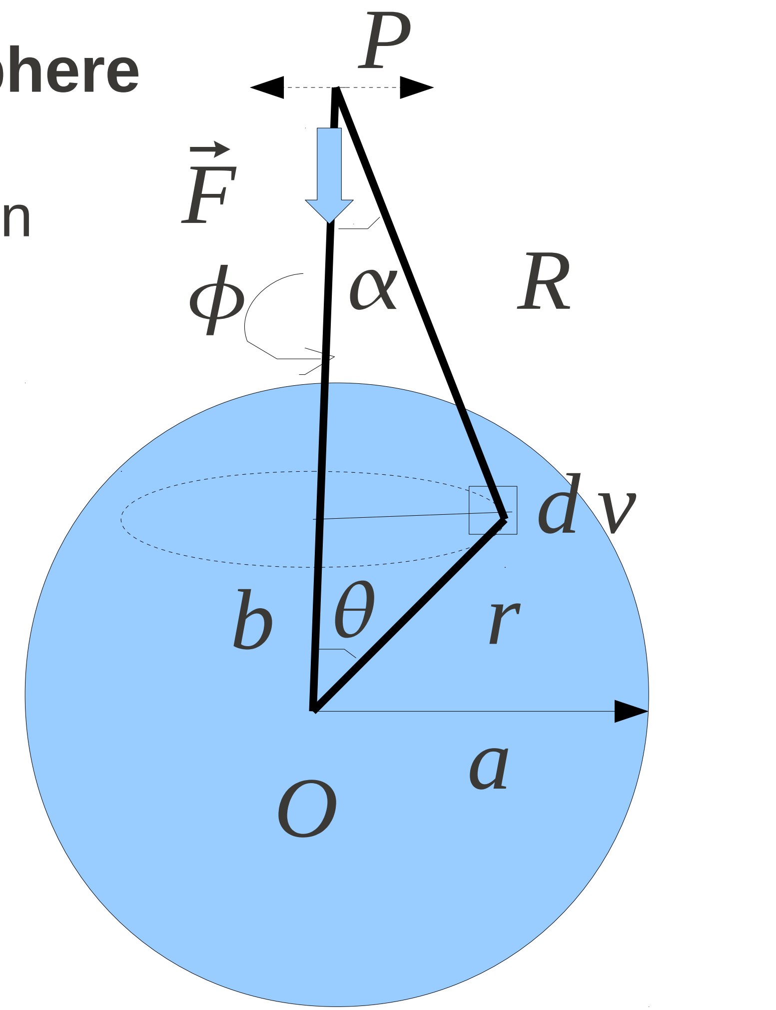
$$F=\frac{mMG}{2a}[I]_{-a}^{a}=\frac{mMG}{2a}\left[\frac{x}{b\sqrt{b^2+x^2}}\right]_{-a}^{a}$$

$$=\frac{mMG}{2a}\left(\frac{2a}{b(b^2+a^2)^{1/2}}\right)=\frac{mMG}{b(b^2+a^2)^{1/2}}$$

At long distances $a \ll b$ in this system also force behaves as it comes from the center of mass

$$F=\frac{mMG}{b^2}$$

Gravitational forces by a sphere



Gravitational force exerted by a sphere on an external mass m placed at P

$$d\,\vec{F} = \frac{m\,\rho\,d\,v\,G}{R^2}\,\hat{R}$$

Using symmetry consideration we can reduce the integral to

$$F = m\,G\int_V \frac{\rho\cos\alpha}{R^2}\,dv$$

The volume element in spherical polar coordinates

$$d\,v = r^2\sin\theta\;dr\,d\,\theta\,d\,\phi$$

$$F = m\,G\int_V \frac{\rho(r)\cos\alpha}{R^2}\,dv = m\,G\int_V \frac{\rho(r)\,R\cos\alpha}{R^3}\,dv$$

$$R\cos\alpha = b - r\cos\theta \qquad R^2 = r^2 + b^2 - 2\,r\,b\cos\theta$$

$$F = mG \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{\rho(r)(b - r\cos\theta)}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

After integration over ϕ

$$F = mG \int_{r=0}^{r=a} \int_{\theta=0}^{\theta=\pi} \frac{2\pi\rho(r)(b - r\cos\theta)}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}} r^2 \sin\theta \, dr \, d\theta$$

$$F = 2\pi mG \int_{r=0}^{r=a} r^2 \rho(r) \int_{\theta=0}^{\theta=\pi} \frac{(b - r\cos\theta)\sin\theta \, d\theta}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}} dr$$

$$F = 2\pi m G \int_{r=0}^{r=a} r^2 \rho(r) \int_{\theta=0}^{\theta=\pi} \frac{(b - r\cos\theta)\sin\theta\, d\theta}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}} dr$$

This integral is evaluated converting it over variable R

$$R^2 = r^2 + b^2 - 2rb\cos\theta$$

The range integration is over

$$b - r \leq R \leq b + r$$

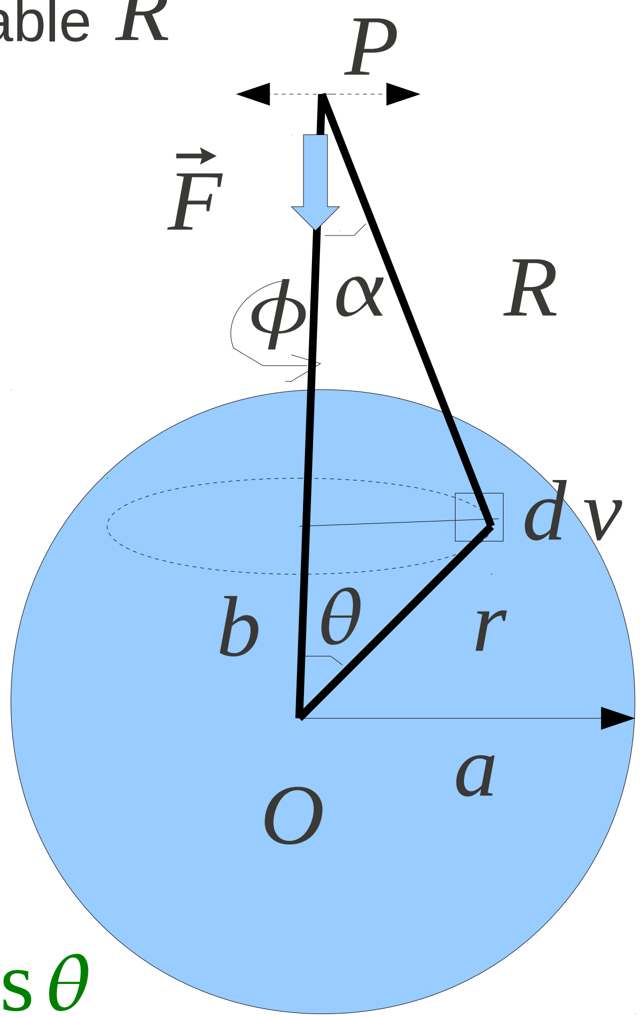
$$2R\,dR = 2rb\sin\theta\, d\theta$$

Multiplying by $2b$ on numerator and denominator

$$b - r\cos\theta = \frac{2b^2 - 2rb\cos\theta}{2b}$$

$$R^2 - r^2 = b^2 - 2rb\cos\theta$$

$$= \frac{b^2 + b^2 - 2rb\cos\theta}{2b} = \frac{b^2 + R^2 - r^2}{2b}$$



$$F = 2\pi m G \int_{r=0}^{r=a} r^2 \rho(r) \int_{\theta=0}^{\theta=\pi} \frac{(b - r\cos\theta)\sin\theta \, d\theta}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}} dr$$

$$b - r\cos\theta = \frac{R^2 + (b^2 - r^2)}{2b} \qquad 2R\,dR = 2rb\sin\theta\,d\theta$$

Substituting this the theta integral becomes

$$I = \int_{\theta=0}^{\theta=\pi} \frac{(b - r\cos\theta)\sin\theta \, d\theta}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}} = \int_{b-r}^{b+r} \left(\frac{R^2 + b^2 - r^2}{2bR^3} \right) \frac{R\,dR}{rb}$$

$$= \frac{1}{2rb^2} \int_{b-r}^{b+r} \left(1 + \frac{b^2 - r^2}{R^2} \right) dR$$

$$F = \frac{mG}{b^2} \left(4\pi \int_{r=0}^{r=a} r^2 \rho(r)\,dr \right)$$

$$I=\frac{1}{2rb^2}\int_{b-r}^{b+r}\left(1+\frac{b^2-r^2}{R^2}\right)dR$$

$$=\frac{1}{2rb^2}\left[\int_{b-r}^{b+r}dR+\int_{b-r}^{b+r}\frac{b^2-r^2}{R^2}dR\right]$$

$$=\frac{1}{2rb^2}\left[2r-\left[\frac{b^2-r^2}{R}\right]_{b-r}^{b+r}\right]$$

$$=\frac{1}{b^2}-\frac{1}{2rb^2}\left[\frac{b^2-r^2}{b+r}-\frac{b^2-r^2}{b-r}\right]$$

$$=\frac{1}{b^2}-\frac{1}{2rb^2}\left[b-r-b-r\right]$$

$$=\frac{2}{b^2}$$

$$F = 2\pi m G \int_{r=0}^{r=a} r^2 \rho(r) \int_{\theta=0}^{\theta=\pi} \frac{(b - r\cos\theta)\sin\theta \, d\theta}{(r^2 + b^2 - 2rb\cos\theta)^{3/2}} dr$$

$$= 2\pi m G \int_{r=0}^{r=a} r^2 \rho(r) I$$

$$= \frac{4\pi m G}{b^2} \int_{r=0}^{r=a} r^2 \rho(r)$$

$$I = \frac{2}{b^2}$$

$$F = \frac{mG}{b^2} \left(4\pi \int_{r=0}^{r=a} r^2 \rho(r) dr \right)$$

Explicit nature of the density function required to evaluate final integral when $\rho(r) = \rho$

$$F = \frac{4\pi m G \rho}{3b^2} a^3 \quad = \frac{M m G}{b^2}$$

where $M = \frac{4\pi\rho}{3} a^3$

mass of the sphere

The electrostatic interactions

$$V(\vec{r}) = q\,\phi(\vec{r})$$

$\phi(\vec{r})$ Is the potential due to the charges q_j placed at distances $\vec{r}_j$

$$\phi(\vec{r}) = \sum_j \frac{q_j}{4\pi\epsilon_0 \left|\vec{r} - \vec{r}_j\right|}$$

The acceleration of a charged particle is then given by

$$m\ddot{\vec{r}} = q\vec{E}$$

The electric field is then given by

$$\vec{E} = -\vec{\nabla}\,\phi(\vec{r})$$

Multipole expansion

A localized charge distribution appears as a point charge from a far distance

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

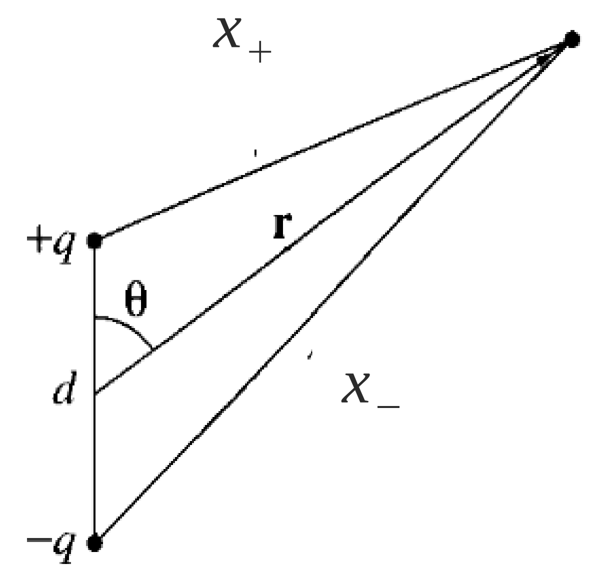
where Q is the total charge

However of the total charges are zero there is possibility of potential can be non-zero

A simple example of such a case is a dipole where charges are displaced

Potential at an arbitrary point

$$V=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{x_+}-\frac{q}{x_-}\right)$$



$$x_\pm^2=r^2+(d/2)^2\mp rd\cos\theta$$

$$x_\pm^2=r^2\left(1\mp\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}\right)$$

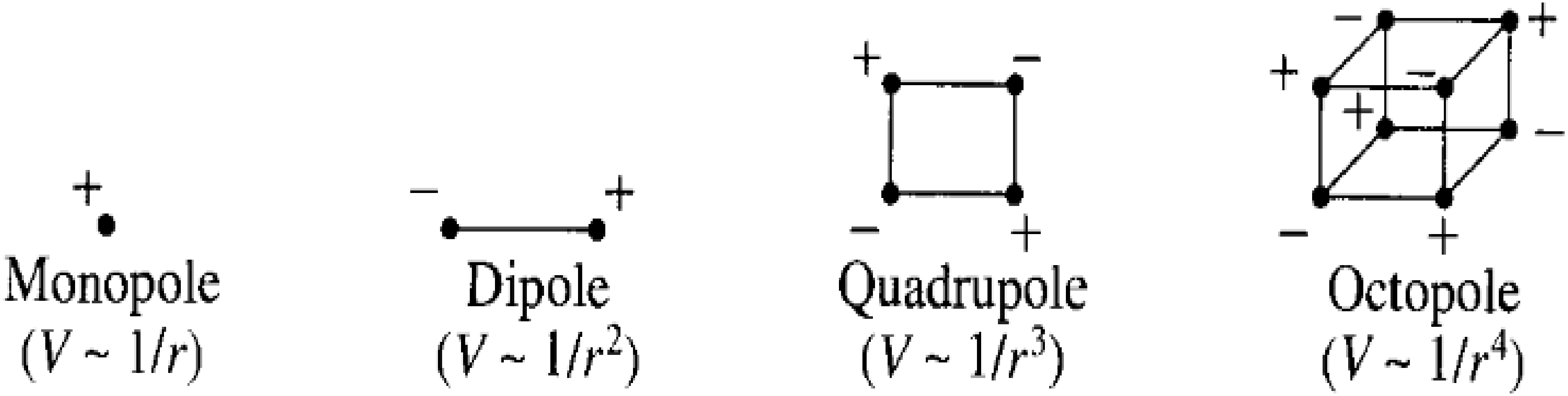
$$\cos(\pi-\theta)=-\cos(\theta)$$

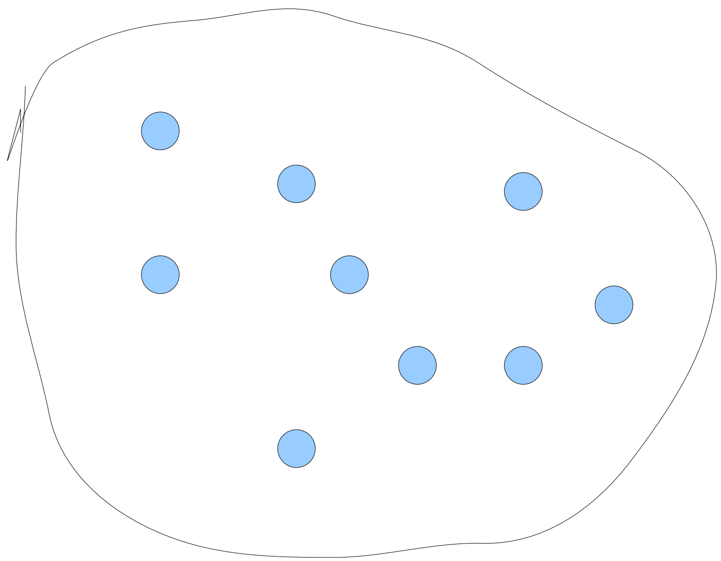
when $r\gg d$ the third term is negligible

$$\frac{1}{x_\pm}\simeq\frac{1}{r}\left(1\mp\frac{d}{r}\cos\theta\right)^{-1/2}\simeq\frac{1}{r}\left(1\pm\frac{d}{2r}\cos\theta\right)$$

$$\frac{1}{x_+}-\frac{1}{x_-}\simeq\frac{d}{r^2}\cos\theta$$

$$V(\vec{r}) \simeq \frac{1}{4\pi\epsilon_0}\frac{q\,d\cos\theta}{r^2}$$



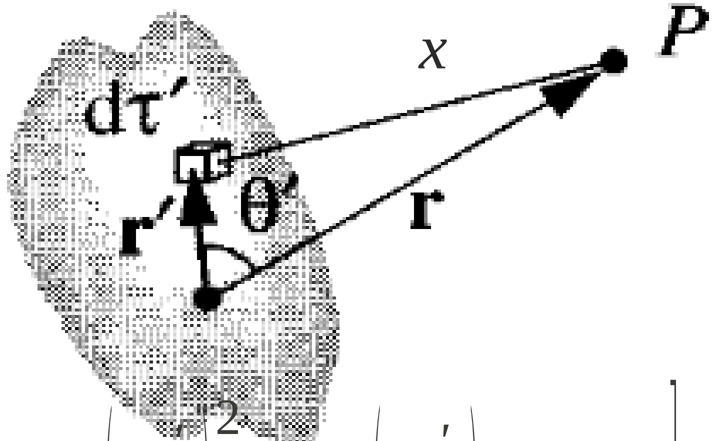


What is is the potential of a localized charge distribution

General method of obtaining multipole terms

for and arbitrary charge distribution

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{x}\rho(\vec{r}')\,d\tau'$$



Using law of cosines

$$x^2 = r^2 + (r')^2 - 2rr'\cos\theta' = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right]$$

$$x^2 = r\sqrt{1+\varepsilon}$$ where $$\varepsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right)$$

when $\varepsilon \ll 1$ binomial expansion can be used

$$\frac{1}{x}=\frac{1}{r}(1+\varepsilon)^{-1/2}=\frac{1}{r}\left(1-\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^{2}-\frac{5}{16}\varepsilon^{3}+.........\right)$$

substitute $\varepsilon=\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\theta'\right)$

$$\frac{1}{x}=\frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r}-2\cos\theta'\right)+\frac{3}{8}\left(\frac{r'}{r}\right)^{2}\left(\frac{r'}{r}-2\cos\theta'\right)^{2}\right.$$

$$\left.-\frac{5}{16}\left(\frac{r'}{r}\right)^{3}\left(\frac{r'}{r}-2\cos\theta'\right)^{3}+..........\right]$$

$$\frac{1}{x}=\frac{1}{r}\left(1-\frac{1}{2}\varepsilon+\frac{3}{8}\varepsilon^{2}-\frac{5}{16}\varepsilon^{3}+.........\right)$$

$$\varepsilon=\left(\left(\frac{r'}{r}\right)^{2}-2\left(\frac{r'}{r}\right)\cos\theta'\right)$$

$$-\frac{1}{2}\varepsilon=\left(\frac{1}{2}\left(\frac{r'}{r}\right)^{2}+\left(\frac{r'}{r}\right)\cos\theta'\right),$$

$$\frac{3}{8}\varepsilon^{2}=\left(.....-\frac{3}{2}\left(\frac{r'}{r}\right)^{3}\cos\theta+\frac{3}{2}\left(\frac{r'}{r}\right)^{2}\cos^{2}\theta'\right)$$

$$-\frac{5}{16}\varepsilon^{3}=\left(.....+\frac{5}{2}\left(\frac{r'}{r}\right)^{3}\cos^{3}\theta'\right)$$

Substitute and rearrange in powers of $\left(\frac{r'}{r}\right)$

$$\frac{1}{x}=\frac{1}{r}\left[1-\left(\frac{r'}{r}\right)\cos\theta'+\left(\frac{r'}{r}\right)^{2}\left(3/2\cos^{2}\theta'-1/2\right)+\left(\frac{r'}{r}\right)^{3}\left(5\cos^{3}\theta'-3\cos\theta'\right)/2\right]$$

This is a series of orthogonal Legendre polynomials

$$P_0(x)=1, \quad P_1(x)=x, \quad P_3(x)=(3x^2-1)/2, \ldots\ldots\ldots$$

The coefficients are Legendre polynomials therefore we can write the solution as

$$\frac{1}{x}=\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\int\frac{1}{x}\rho(\vec{r}')\,d\tau'$$

Substituting in equation

$$V(\vec{r})=\frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int(r')^n P_n(\cos\theta')\rho(\vec{r}')\,d\tau'$$

The various terms in the series obtained are explicitly written as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\vec{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}')\,d\tau' + \ldots\ldots\right]$$

monopole dipole

quadrupole

This is an approximation scheme for potential of a charge distribution

The monopole and dipole terms

A multipole expansion is dominated by the monopole term if there is residual charges

$$V_{mono}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r} \qquad \text{where} \qquad Q = \int \rho \, d\tau$$

when the charge is due to a point charge, this relation is exact

when net charge is zero the dipole term dominates

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r' \cos\theta' \rho(\vec{r}') \, d\tau'$$

$$r' \cos\theta' = \hat{r}\cdot\vec{r}'$$

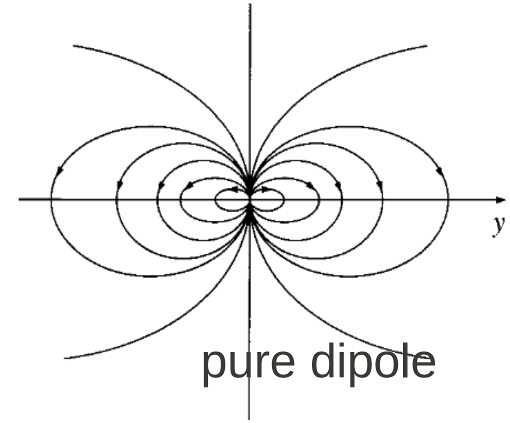
now the dipole potential can be defined as

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\int \vec{r}' \cos\theta' \rho(\vec{r}') \, d\tau'$$

$$\vec{p} = \int \vec{r}\,' \cos\theta' \rho(\vec{r}\,') d\tau'$$

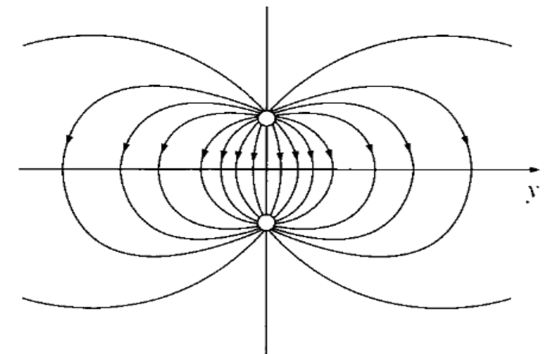
this is the dipole moment of the distribution

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r} \cdot \vec{p}}{r^2}$$



pure dipole

Simplified expression for the potential of a point dipo



real dipole

for a system of point charges

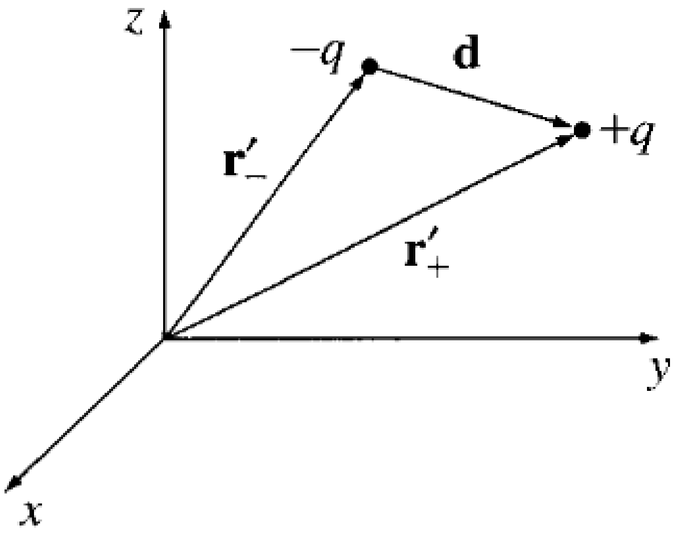
$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i$$

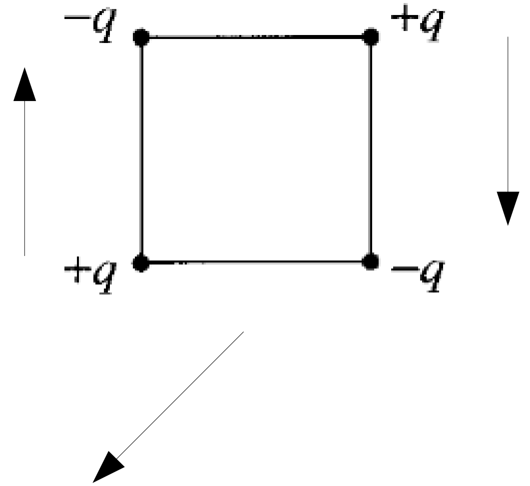
for a physical dipole

$$\vec{p} = q\vec{r}\,'_+ - q\vec{r}\,'_- = q\vec{d}$$

Physical dipole is approaches pure dipole in the limit of distance between dipoles approaches zero and charge approaches Infinity, keeping the quantity $\vec{p} = q\vec{d}$ a constant

Dipole moments add vectorialy





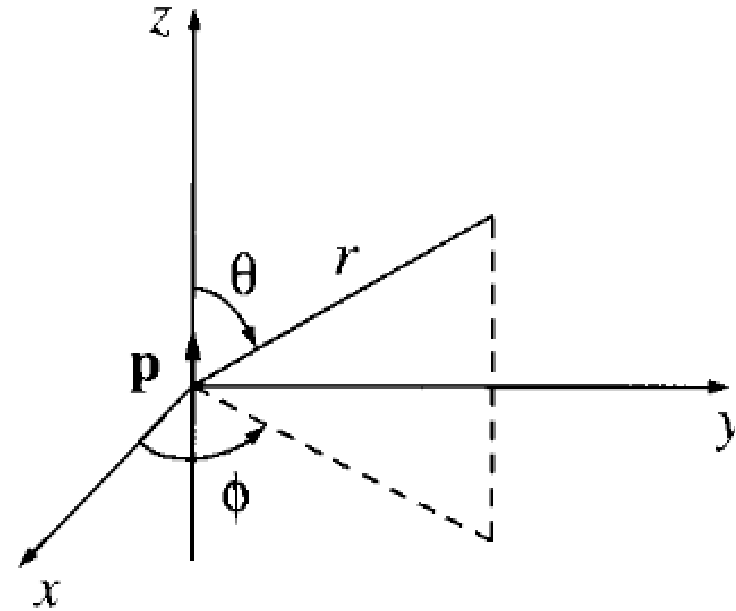
$$\vec{p} = \vec{p}_1 + \vec{p}_2$$

Quadrupole moments are dominant in a canceled dipole

The electric field of a dipole

$$V_{dip}(r,\theta)=\frac{\hat{r}\cdot\vec{p}}{4\pi\epsilon_0 r^2}$$

$$V_{dip}(r,\theta)=\frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

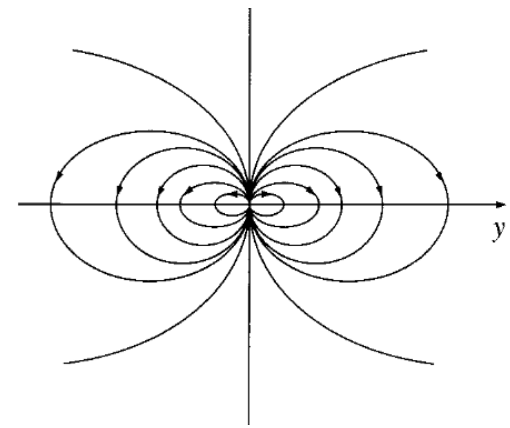


The problem has an angular symmetry

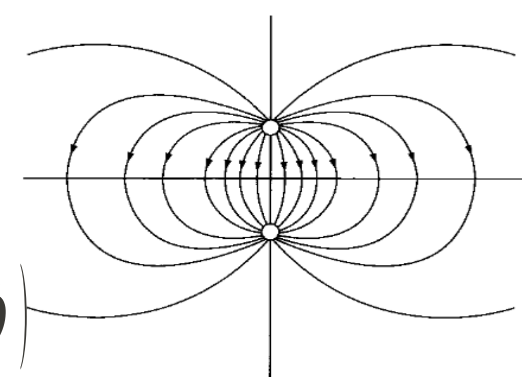
$$E_r=-\frac{\partial V}{\partial r}=\frac{2p\cos\theta}{4\pi\epsilon_0 r^3}$$

$$E_\theta=-\frac{1}{r}\frac{\partial V}{\partial\theta}=\frac{p\sin\theta}{4\pi\epsilon_0 r^3}$$

$$E_\phi=-\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}=0$$



pure dipole



real dipole

$$E_{dip}=\frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}\right)$$

The multipole expansion using tensors

$$V(\vec{r}) = \int_v \frac{\rho(\vec{r}\,')}{\vec{r} - \vec{r}\,'} d\tau' = \int_v \frac{\rho(\vec{r}\,')}{\vec{x}} d\tau'$$

To find the field at $\vec{r}$ far away from the source $\vec{r}\,'$

It is possible to expand $1/R$ in powers of r'/r

$$\frac{1}{\vec{x}} = \frac{1}{|\vec{r} - \vec{r}\,'|} = \left(\vec{r}^{\,2} - 2\vec{r}\cdot\vec{r}\,' + \vec{r}\,'^{2}\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 + \left(\frac{-2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}\right)\right)^{-1/2}$$

$$= \frac{1}{r}\left(1 - \frac{1}{2}\left(\frac{-2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}\right) + \frac{1}{2}\frac{3}{4}\left(\frac{-2\vec{r}\cdot\vec{r}\,'}{r^2} + \frac{r'^2}{r^2}\right)^2 \cdots\cdots\cdots\right)$$

$$=\frac{1}{r}\left(1-\frac{1}{2}\left(\frac{-2\vec{r}\cdot\vec{r}^{\,'}}{r^2}+\frac{r^{'2}}{r^2}\right)+\frac{1}{2}\frac{3}{4}\left(\frac{-2\vec{r}\cdot\vec{r}^{\,'}}{r^2}+\frac{r^{'2}}{r^2}\right)^2 \dots\dots\dots\right)$$

$$=\frac{1}{r}\left(1+\left(\frac{\vec{r}\cdot\vec{r}^{\,'}}{r^2}\right)+\frac{1}{2}\left(\frac{3(\vec{r}\cdot\vec{r}^{\,'})^2}{r^4}-\frac{r^{'2}}{r^2}\right)+\dots\dots\dots\right)$$

$$V(\vec{r})=\int_v \frac{\rho(\vec{r}^{\,'})}{\vec{r}-\vec{r}^{\,'}}\,d\tau^{'}=\int_v \frac{\rho(\vec{r}^{\,'})}{\vec{x}}\,d\tau^{'}$$

$$\frac{1}{\vec{x}}=\frac{1}{r}\left(1+\left(\frac{\vec{r}\cdot\vec{r}^{\,'}}{r^2}\right)+\frac{1}{2}\left(\frac{3(\vec{r}\cdot\vec{r}^{\,'})^2}{r^4}-\frac{r^{'2}}{r^2}\right)+\dots\dots\dots\right)$$

If the $\vec{r}\cdot\vec{r}^{\,'}$ is substituted with $rr^{'}\cos\theta$ it leads to Legendre polynomial expression of multipole expansion

Combining both terms

$$V(\vec{r}) = \frac{1}{r}\int_v \rho(\vec{r}^{\,'})\,d\tau^{'} + \frac{\vec{r}}{r^3}\cdot\int_v \vec{r}^{\,'}\rho(\vec{r}^{\,'})\,d\tau^{'}$$

$$+\frac{x_i x_j}{r^5}\int_v (3x_i^{'}x_j^{'} - \delta_{ij}r^{'2})\rho(\vec{r}^{\,'})\,d\tau^{'} + ...$$

$$x_i \in \{x, y, z\}$$

monopole $\int_v \rho(\vec{r}^{\,'})\,d\tau^{'}$ scalar (tensor rank 0)

dipole $\int_v \vec{r}^{\,'}\rho(\vec{r}^{\,'})\,d\tau^{'}$ vector (tensor rank 1)

quadrupole $\int_v (3x_i^{'}x_j^{'} - \delta_{ij}r^{'2})\rho(\vec{r}^{\,'})\,d\tau^{'}$ tensor of rank 2

Quadrupole moments are traceless tensors, that is, sum of diagonal terms is equal to zero

The Dirac delta function

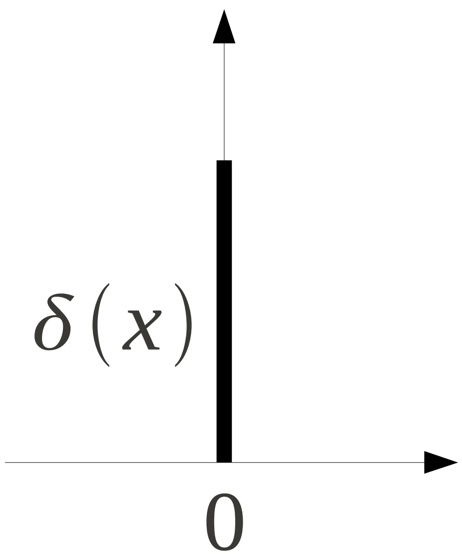
$$\delta(x)=\begin{cases}0, & if\ x\neq 0\\ \infty & if\ x=0\end{cases}$$

This function is called generalized function or distribution

By definition

$$\int_{-\infty}^{+\infty}\delta(x)\,dx=1$$

$$\int_{-\infty}^{+\infty}f(x)\,\delta(x)\,dx=f(0)$$



All other points except at the origin function contributes nothing

The position of the origin can also be shifted

$$\delta(x-a)=\begin{cases}0, & if\ x\neq a\\ \infty & if\ x=a\end{cases}$$

$$\int_{-\infty}^{+\infty}\delta(x-a)\,dx=1$$

$$\int_{-\infty}^{+\infty}f(x)\,\delta(x-a)\,dx=f(a)$$

$$\delta(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$$

In this case

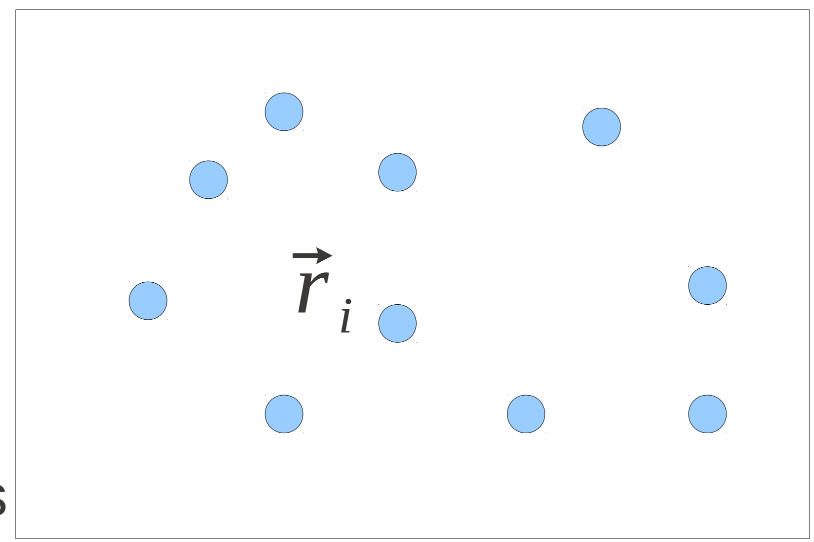
$$\int_v \delta(\vec{r})\,d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$$

Existence of a particle can be defined by Dirac delta function

$$\delta(\vec{r} - \vec{r}_i)$$

There for a system of particles can be represented by

$$\sum_{i=1}^{10} \delta(\vec{r} - \vec{r}_i)$$



Total number of the particle is obtained by now integrating over this volume

$$N = \int_v \sum_{i=1}^{10} \delta(\vec{r} - \vec{r}_i)\,d\vec{r} = 10$$

When Dirac delta function is multiplied by the corresponding mass density or charge density we can obtain total mass or charge

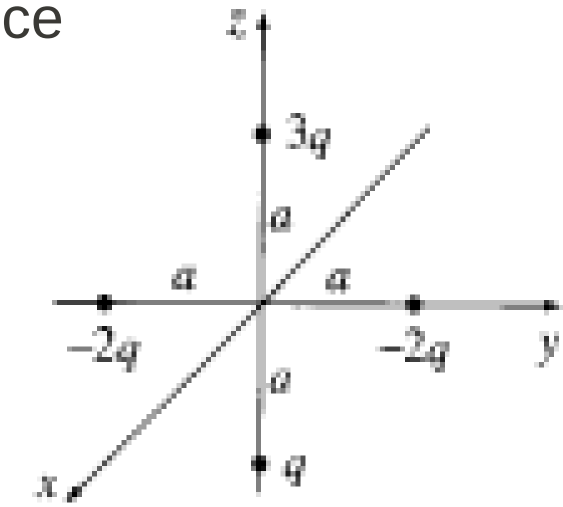
Let two charges be $q\,\delta(\vec{r}-\vec{r}_i)$ and $-q\,\delta(\vec{r}-\vec{r}_j)$

$$\rho(\vec{r}\,')=q(\delta(\vec{r}\,'-\vec{r}_i)-\delta(\vec{r}\,'-\vec{r}_j))$$

$$\vec{p}=\int_v \vec{r}\,'\rho(\vec{r}\,')d\tau'=\int_v \vec{r}\,' q(\delta(\vec{r}\,'-\vec{r}_i)-\delta(\vec{r}\,'-\vec{r}_j))d\tau'$$

$$=q(\vec{r}_j-\vec{r}_i)=q\vec{d}$$

Approximate value of the potential at far distance



$$\vec{p} = \sum q_i \vec{r}_i$$

$$\vec{p} = (3qa - qa)\hat{z} + (-2qa - 2q(-a))\hat{y} = 2qa\hat{z}$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$V = \frac{1}{4\pi\epsilon_0}\frac{2qa\cos\theta}{r^2}$$

Find the field due to the linear quadrupole

$$V=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}-\frac{2q}{r}+\frac{q}{r_2}\right)$$

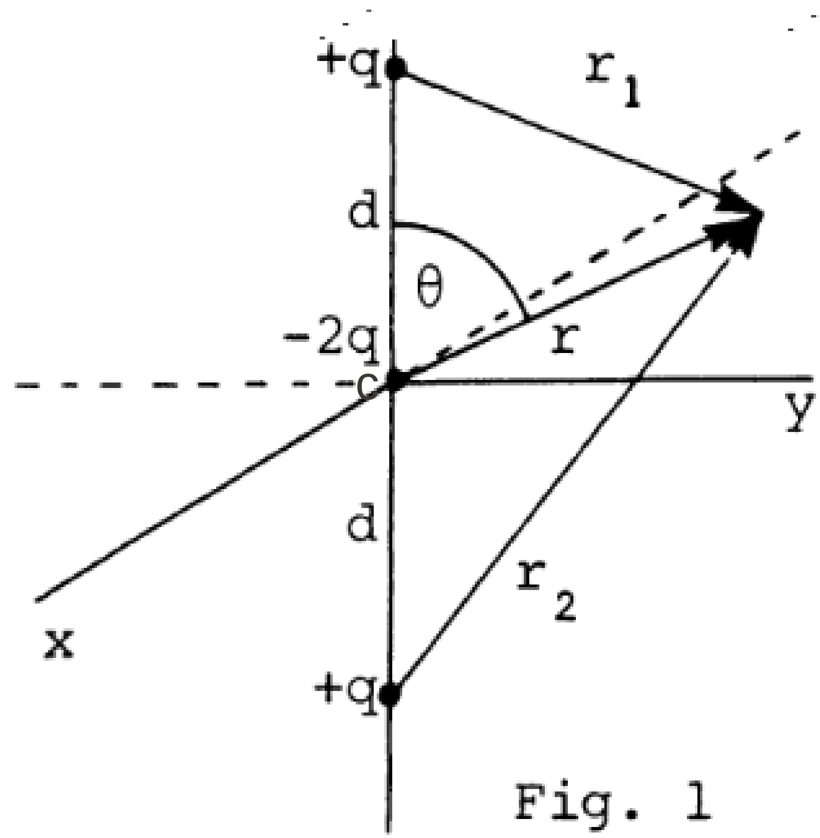


Fig. 1

Using law of cosines

$$r_1^2=r^2+d^2-2rd\cos\theta$$

$$r_1=r\left(1-2\frac{d}{r}\cos\theta+\frac{d^2}{r^2}\right)^{(1/2)}$$

$$\frac{1}{r_1}=\frac{1}{r}\left(1-2\frac{d}{r}\cos\theta+\frac{d^2}{r^2}\right)^{(-1/2)}$$

$d\ll r$ Using binomial expansion

$$\frac{1}{r_1}=\frac{1}{r}+\frac{d\cos\theta}{r^2}-\frac{d^2}{2r^3}+\frac{3d^2\cos^2\theta}{2r^3}$$

$$\frac{1}{r_1}=\frac{1}{r}+\frac{d\cos\theta}{r^2}+\frac{d^2}{2r^3}\left(3\cos^2\theta-1\right)$$

Similarly $$\frac{1}{r_2}=\frac{1}{r}-\frac{d\cos\theta}{r^2}+\frac{d^2}{2r^3}\left(3\cos^2\theta-1\right)$$

Potential is given by $$V=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_1}-\frac{2q}{r}+\frac{q}{r_2}\right)$$

$$V=\frac{q\,d^2\left(3\cos^2\theta-1\right)}{4\pi\epsilon_0 r^3}$$

To convert it into tensor form which is useful non standard geometries

$$e\overleftrightarrow{Q}=\sum q_i(3x_i'x_j'-\delta_{ij}r'^2)$$

$$eQ=q(3\mathrm{d}^2-d^2)-2\mathrm{q}(0-0)+q(3d^2-d^2)=2q(3d^2-d^2)$$

Effective charge

$$eQ=4\mathrm{qd}^2$$

$$V=\frac{qd^2(3\cos^2\theta-1)}{4\pi\epsilon_0 r^3}$$

$$V=\frac{eQ}{4}\frac{(3\cos^2\theta-1)}{4\pi\epsilon_0 r^3}$$

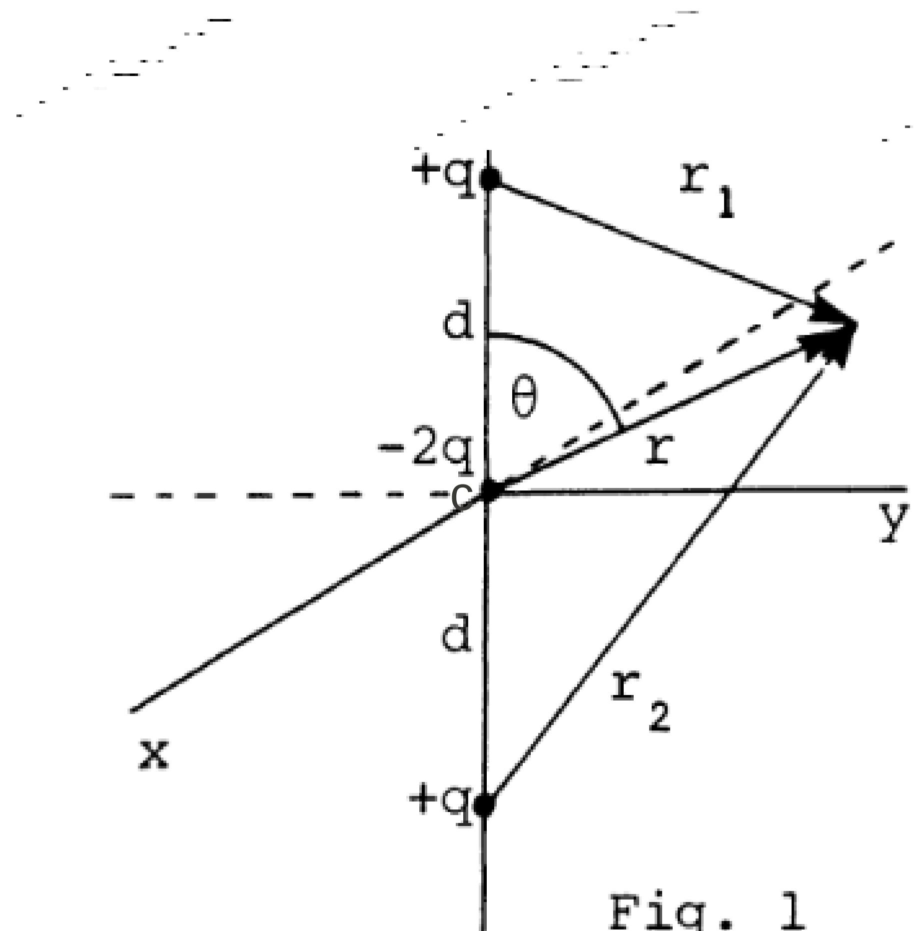


Fig. 1

Few applications of Delta function

Convert a volume integral to calculate surface area surface, surface to a line

For a cubical volume $$V = \int_0^a \int_0^a \int_0^a dx\, dy\, dz = a^3$$

Now $$V = \int_0^a \int_0^a \left(\int_0^a \delta(x)\, dx \right) dy\, dz = \int_0^a \int_0^a dy\, dz = a^2$$

Then the dimension of the volume reduces one less

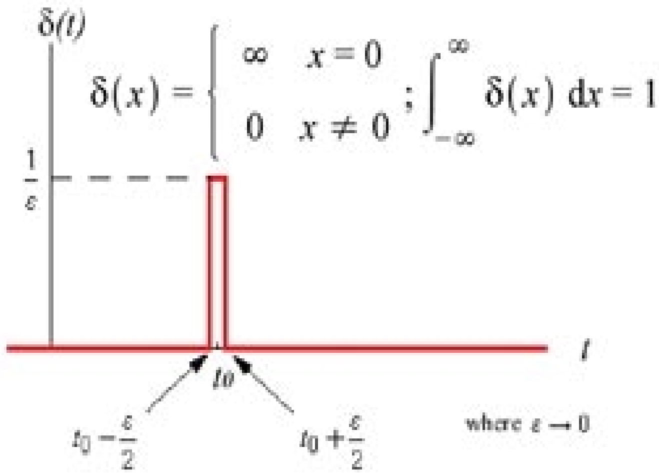
Similarly dimension of area reduces and become a line integral

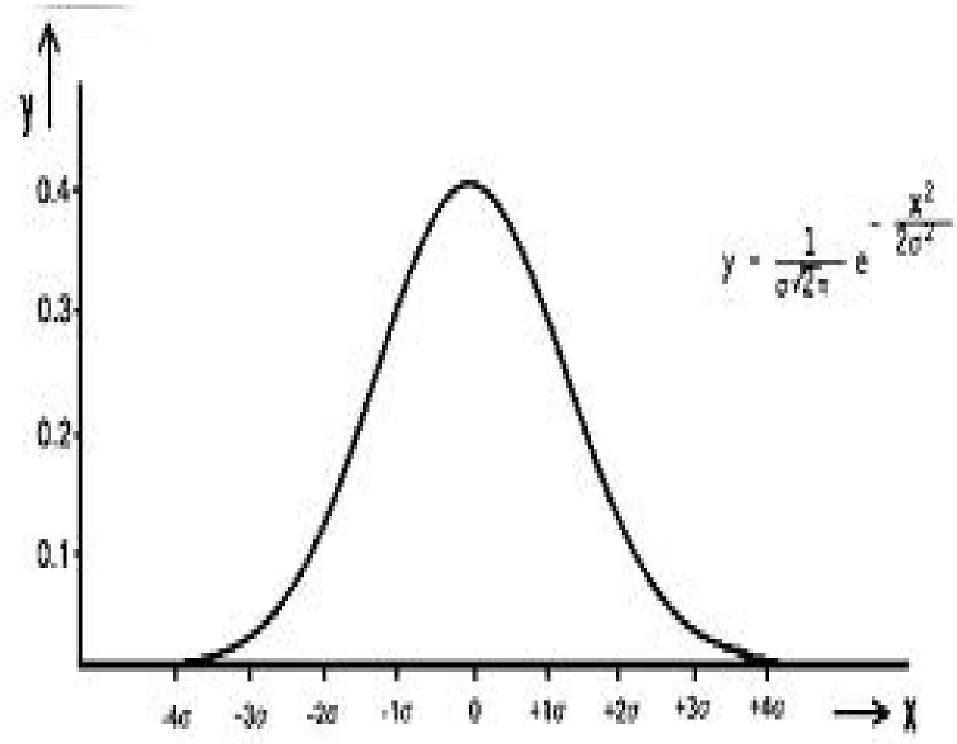
$$A = \int_0^a \int_0^a dx\, dy = a^2$$

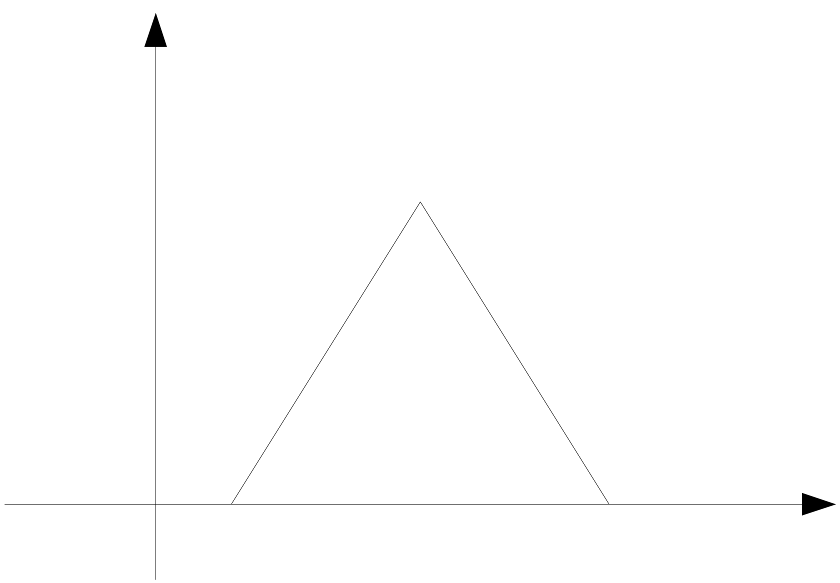
$$A = \int_0^a \left(\int_0^a \delta(x)\, dx \right) dy = \int_0^a dy = a$$

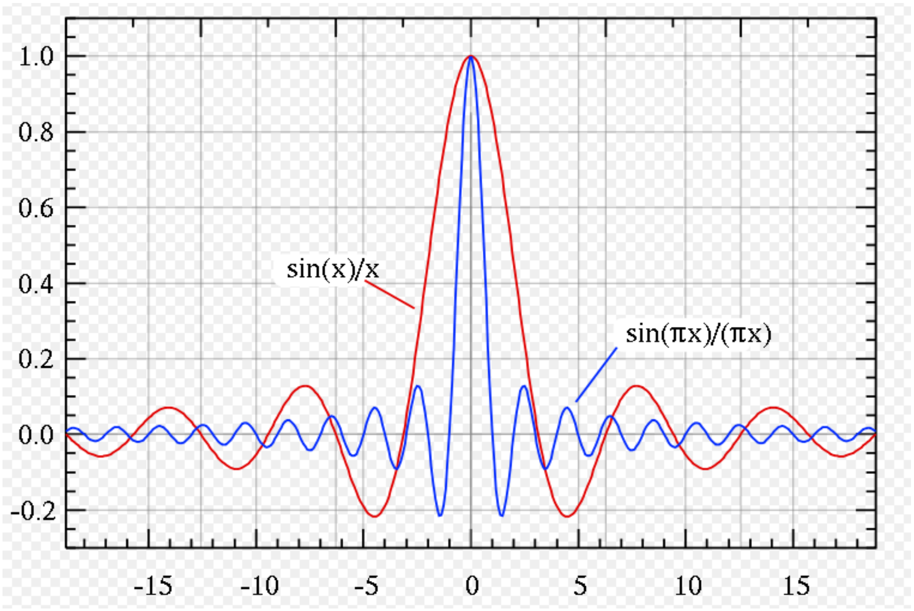
Hence the dimension of the delta function is inverse of the quantity that is being integrated

Representation of the delta function









http://mathworld.wolfram.com/DeltaFunction.html

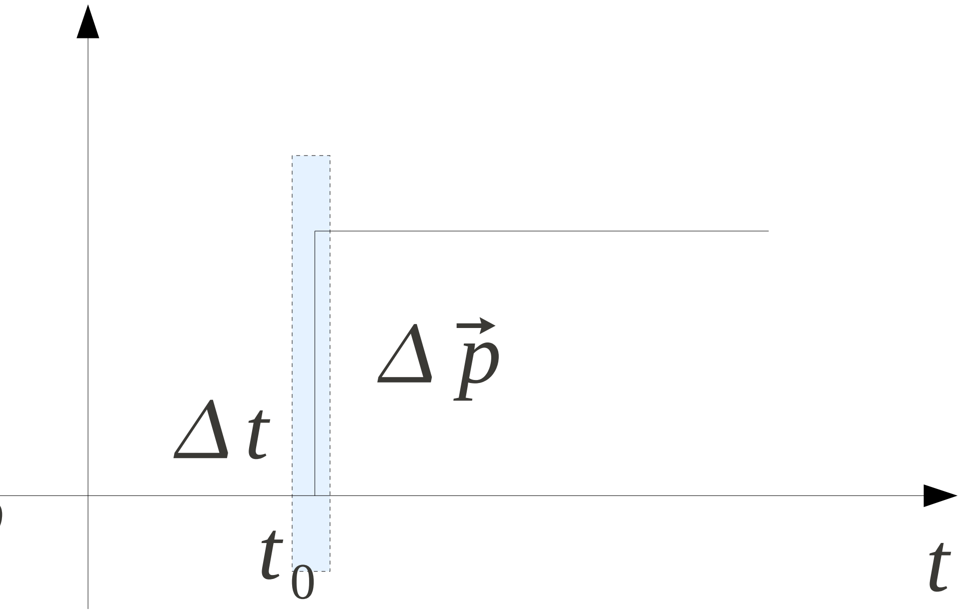
Impulse force and delta function

Impulse is another way to change the momentum of the particle

$$\int_{t_1}^{t_2} \vec{F}\, dt = \int_{t_1}^{t_2} \frac{d}{dt}(m\vec{v})\, dt$$

$$= \int_{t_1}^{t_2} d(m\vec{v})$$

$$= m\vec{v}_2 - m\vec{v}_1 = \vec{p}_2 - \vec{p}_1 = \Delta\vec{p}$$



Momentum change may be represented by a step function – Heaviside step function

$$H(t-t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t \geq t_0 \end{cases}$$

Momentum as a function of time $H(t-t_0)\,\Delta\vec{p}$

The nature of the force is not of great importance – only the duration must be short – then impulse force may be represented as

$$\int_{t_1}^{t_2} \vec{F}\, dt = \int_{t_1}^{t_2} \Delta \vec{p}\, \delta(t-t_0)\, dt = H(t-t_0)\, \Delta \vec{p}$$

Here delta function has the dimension of inverse of time

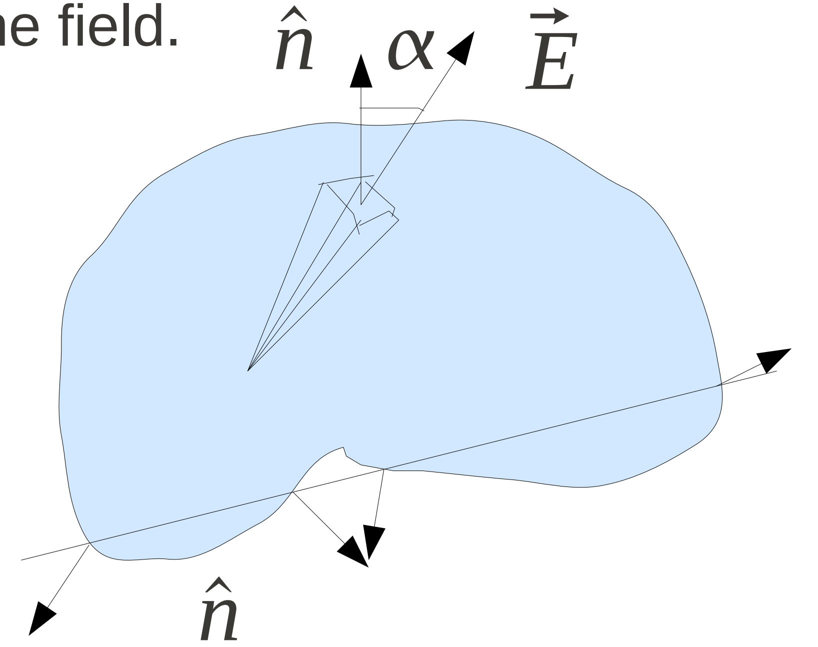
On integration we get a step function, therefore

$$\delta(t-t_0) = \frac{d\, H(t-t_0)}{d\, t}$$

This is one more representation of the Dirac delta function

The field equation

It is convenient to obtain the total electric field produced by an object from differential rather than integration of the field.



$$\vec{E}=\frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

Consider a closed surface S surrounding the charge

$$\iint \vec{E}\cdot\hat{n}\,dS=\frac{q}{4\pi\epsilon_0}\iint\frac{\hat{r}\cdot\hat{n}}{r^2}dS$$

$$=\frac{q}{4\pi\epsilon_0}\iint d\Omega=\frac{q}{\epsilon_0}$$

$$\Omega=\iint\frac{\hat{r}\cdot\hat{n}}{r^2}dS$$

solid angle defined as $$\Omega=\int_0^{\pi}\int_0^{2\pi}\hat{r}\cdot\hat{n}\sin\theta\,d\theta\,d\phi=4\pi$$

The contribution to electric field is only from the charges enclosed

http://mathworld.wolfram.com/SolidAngle.html

Let the volume contains continuous charge density $\rho(\vec{r})$

$$\iint \vec{E}\cdot\hat{n}\,dS=\frac{1}{\epsilon_0}\iiint \rho(\vec{r})\,d\tau$$

Gauss's law of electrostatics in integral form

By Gauss's divergence theorem

$$\iint \vec{E}\cdot\hat{n}\,dS=\iint \vec{\nabla}\cdot\vec{E}\,d\tau$$

$$\iint \vec{\nabla}\cdot\vec{E}\,d\tau=\frac{1}{\epsilon_0}\iiint \rho(\vec{r})\,d\tau$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E}=\frac{1}{\epsilon_0}\rho(\vec{r})$$

Differential form of Gauss's law

This condition along with fact that $\vec{\nabla}\times\vec{E}=(constant)\,\vec{\nabla}\times\frac{\hat{r}}{r^2}=0$

$$\Rightarrow \vec{E}=-\vec{\nabla}V$$

When charge is enclosed in the volume

$$\nabla^2 V=-\frac{1}{\epsilon_0}\rho(\vec{r})$$

Poisson equation of electrostatics

When the volume does not contain any charges

$$\nabla^2 V = 0$$

Laplace equation of electrostatics

The gravitational field equations are similar since both electrostatics and gravitational laws depends of inverse of length dependence.

$$\Rightarrow \vec{\nabla} \cdot \vec{g} = -4\pi G \rho(\vec{r})$$

Mass density

In case of gravitation also $\vec{\nabla} \times \vec{g} = 0$

The Poisson equation for gravitational potential is

$$\nabla^2 V(r) = -4\pi G \rho(\vec{r})$$

Potential at the origin and delta function

let $\vec{v}=\frac{\hat{r}}{r^2}$ $\qquad \vec{E}=\frac{q}{4\pi\epsilon_0 r^2}\hat{r}$

$$\vec{\nabla}\cdot\vec{v}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right)$$

when $r\neq 0$ $\quad \vec{\nabla}\cdot\vec{v}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}(1)=0$

when $r=0$ $\quad \vec{\nabla}\cdot\vec{v}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}(1)=not\ defined$

Now apply Gauss's divergence theorem over the surface of a sphere - in spherical coordinates

Now apply Gauss's divergence theorem over the surface of a sphere - in spherical coordinates

$$\oiint_S \vec{V}\cdot d\vec{\sigma} = \iiint_V \vec{\nabla}\cdot\vec{V}\, d\tau$$

The surface integral is given by

$$\oiint \vec{v}\cdot d\vec{a} = \int_s \frac{1}{R^2}\hat{r}\cdot\left(R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right)$$

$$\oiint \vec{v}\cdot d\vec{a} = \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi$$

but at origin the volume integral is given by

$$\iiint \vec{\nabla}\cdot\vec{v}\, d\tau = undefined$$

It appears that Gauss divergence theorem is not valid in this case as the value of the integrals do not match

If that is not the case for the contribution is coming from origin we need to assign specific values which is achieved through the **Dirac delta function**

Now re-assign the value of the integral as

$$\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \, \delta(\vec{r})$$

Now for any origin where $\vec{x} = \vec{r} - \vec{r}_0$

$$\vec{\nabla} \cdot \left(\frac{\hat{x}}{x^2} \right) = 4\pi \, \delta(\vec{x})$$

With the new definition we get the desirable value of the integral

$$\iiint \vec{\nabla} \cdot \vec{v} \, d\tau = \iiint 4\pi \, \delta(\vec{r}) \, d\tau = 4\pi$$

List of reference

1) Classical dynamics of particles and systems by S T Thornton and J B Marion (TM)

2) Classical mechanics by Kibble and Berkshire (KB)

3) Classical mechanics by Gregory(GY) (force generated at different geometries)

4) Introduction to electrodynamics by David J Griffiths (multipole expansion)