EIGENVALUES AND EIGENVECTORS

1 Introduction

The eigenvalue problem is a problem of considerable theoretical interest and wide-ranging application. For example, this problem is crucial in solving systems of differential equations, analyzing population growth models, and calculating powers of matrices (in order to define the exponential matrix). Other areas such as physics, sociology, biology, economics and statistics have focused considerable attention on "eigenvalues" and "eigenvectors"-their applications and their computations. Before we give the formal definition, let us introduce these concepts on an example.

Example 1 Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

Consider the three vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 6 \\ -13 \end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$$

We have

$$A\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad A\mathbf{x}_2 = \begin{bmatrix} 4 \\ -8 \\ -4 \end{bmatrix} \quad A\mathbf{x}_3 = \begin{bmatrix} 6 \\ 9 \\ -6 \end{bmatrix}$$

In Other words, we have

$$A\mathbf{x}_1 = 0\mathbf{x}_1, \quad A\mathbf{x}_2 = -4\mathbf{x}_2 \quad \text{and } A\mathbf{x}_3 = 3\mathbf{x}_3$$

In this case we say 0, -4 and 3 are eigenvalues of the matrix A, and $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{x}_3$ are eigenvectors of A.

Definition 1 *Let $A = (a_{ij})$ be any square matrix of order n. If there exists a non-zero column vector $\mathbf{x}$ and a scalar such λ that*

$$A\mathbf{x} = \lambda\mathbf{x}$$

then λ is called an **eigenvalue** *of the matrix A and $\mathbf{x}$ is called the* **eigenvector** *of the corresponding to the eigenvalue λ.*

To find the eigenvalues and the corresponding eigenvectors of a square matrix A of order n we proceed as follows. Let λ be an eigen value of A and $\mathbf{x}$ be the corresponding eigen vector. Then, by definition,

$$A\mathbf{x} = \lambda\mathbf{x} = \lambda I\mathbf{x},$$

where I is the unit matrix of order n. It follows that

$$(A - \lambda I)\mathbf{x} = O \tag{1}$$

Equation (1) is a system of homogeneous linear equations in unknown $x_1, \ldots, x_n$. Since $\mathbf{x} = [x_1, \ldots, x_n]^T$ is to be non-zero vector, $|A - \lambda I| \neq 0$, by Cremer's theorem. Thus, we solve this equation to get n values of λ.

Method of Finding Eigenvalues and Eigenvectors

To find eigenvalues and eigenvectors of a given matrix we proceed as follows:

1. Form the matrix $A - \lambda I$, that is, subtract λ from each diagonal element of A.
2. Solve the characteristic equation $|A - \lambda I| = 0$ for λ.
3. Take each value of λ in turn, substitute it into Equation (1) and solve the resulting homogeneous system for $\mathbf{x}$ using Gaussian elimination. Note that, since the determinant of the coeficient matrix is zero, row reduction of the augmented matrix must always lead to at least one row of zeros.

2 Examples

Example 2 Find the eigenvalue and eigenvectors of $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$

Solution: We have

$$\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

That is, $(5-\lambda)(2-\lambda) - 4 = 0$, which implies that $\lambda^2 - 7\lambda + 6 = 0$. Hence, the eigenvalue of A are $\lambda = 1, 6$.

The eigenvector corresponding to any λ is given by $(A - \lambda I) = 0$. That is,

$$\begin{bmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When $\lambda = 1$, the eigenvector is given by the system

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So the augumement of the system is

$$\begin{bmatrix} 4 & 4 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $x_1 = -x_2$. Taking $x_1 = 1$, $x_2 = -1$ Thus, the eigenvector corresponding to $\lambda_1 = 1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

When $\lambda_2 = 6$, the eigenvector is given by the system

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So the augumement of the system is

$$\begin{bmatrix} -1 & 4 & 0 \\ 1 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $x_1 = 4x_2$. Taking $x_2 = 1$, $x_1 = 41$. Therefore, the eigenvector corresponding to $\lambda_1 = 1$ is $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Example 3 Find the eigenvalue and eigenvectors of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Solution Let λ be an eigenvalue of A. It follows that

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0.$$

Then we have $(1-\lambda)\{\lambda^2 - 6\lambda + 4\} - (1 - \lambda - 3) + 3(1 - 15 + 3\lambda) = 0$. That is, $\lambda^3 - 7\lambda^2 + 36 = 0$ Then, eigenvalue of A are $\lambda = -2, 3, 6$.

Case $\lambda = -2$

The eigenvector is given by

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The augumentend matrix is

$$\left[\begin{array}{cccc} 3 & 1 & 3 & 0 \\ 1 & 7 & 1 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cccc} 3 & 1 & 3 & 0 \\ 1 & 7 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

That is,

$$\begin{aligned} x_1 + 7x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

Here we have three unknown with two equations.

Solve this problem by the rule of cross-multiplication, we have

$$\frac{x_1}{21-1} = \frac{x_2}{3-3} = \frac{x_3}{1-21}$$

That is,

$$\frac{x_1}{20} = \frac{x_2}{0} = \frac{x_3}{-20}$$

That is,

$$\frac{x_1}{1} = \frac{x_2}{0} = \frac{x_3}{-1}$$

Hence, eigenvector corresponding to $\lambda_1 = -2$ is $\left[\begin{array}{r} 1 \\ 0 \\ -1 \end{array}\right]$

Case $\lambda = 3$
The eigenvector is given by

$$\left[\begin{array}{rrr} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{array}\right] \left[\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ 0 \end{array}\right]$$

The augumentend matrix is

$$\left[\begin{array}{rrrr} -2 & 1 & 3 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr} 1 & 2 & 1 & 0 \\ -2 & 1 & 3 & 0 \\ 3 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & -5 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

By the rule of cross-multiplication, we have

$$\frac{x_1}{5} = \frac{x_2}{-5} = \frac{x_3}{5} \quad \text{or} \quad \frac{x_1}{1} = \frac{x_2}{-1} = \frac{x_3}{1}$$

Therefore, the eigenvector corresponding to $\lambda_1 = 3$ is $\left[\begin{array}{r} 1 \\ -1 \\ 1 \end{array}\right]$

Case $\lambda = 6$

Therefore, the eigenvector corresponding to $\lambda_1 = 6$ is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$,

Remarks

1. For given a matrix A of order n, the determinant of $A - \lambda I$ is a polynomial of degree n in λ.
2. The equation $|A - \lambda A|$ is called the **characteristic equation** of A.
3. Corresponding to an eigenvalue, the non-trivial solution of the system will be one value of the solutions. Hence the eigenvectors corresponding to an eigenvalue is not unique.
4. If all the eigenvalues $\lambda_1, \ldots, \lambda_n$ of a matrix A are distinct, then the corresponding eigenvectors are linearly independent.

3 Properties of Eigenvalues

1. A square matrix A and its transpose have the same eigenvalues
2. The sum of the eigenvalues of the a matrix is equal to the sum of the principal diagonal elements of A.
3. The product of the eigenvalues of a matrix A is equal to $|A|$.
4. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of a matrix A, then
   (a) $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$ are the eigenvalue of the matrix kA.
   (b) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ are the eigenvalue of the matrix A^{-1}.
5. The eigenvalues of the real symetric matrix are real.
6. The eignvectors corresponding to distict eigenvalues of real symetric matrix are orthogonal.

Example 4 Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution The characteristic of A is

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

This implies that $\lambda^3 - 3\lambda - 2 = 0$. That is, $(\lambda+1)^2(\lambda-2) = 0$. Hence, $\lambda = 1, -1, 2$

When eigenvector $\lambda = -1$. The eigenvector is given by

$$\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Thus, setting $\lambda = -1$, we obtain

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

All the tree equations reduce to one and the same equation $x_1 + x_2 + x_3 = 0$. There is one equation in three unknown. Therefore, two of the unknown, say, x_1 and x_2 are to be treated as free variables. Taking $x_1 = 1$ and $x_2 = 0$ we get $x_3 = -1$ and taking $x_1 = 0$ and $x_2 = 1$, we get $x_3 = -1$. Thus, the eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ correspond to the eigenvector $\lambda_1 = -1$.

When $\lambda = 2$, we have

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The eigenvector corresponding to $\lambda_2 = 2$ is $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Note that though two of the eigenvalues are equal, the eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are found to be linearly independent.

4 Cayley-Hamilton Theorem

This theorem provides an alternative method for finding the inverse of a matrix A. The theorem can be also to express any positive integer intergral power of A as a linear combination of those of lower degree.

Theorem 2 (Cayley-Hamilton Theorem) *Every square matrix satisfies its own characteristics equation.*

This means that, if $c_n\lambda^n + c_1\lambda^{n-1} + \ldots + c_{n-1}\lambda + c_n = 0$ is the characteristic equation of a square matrix A of order n, then

$$c_oA^n + c_1A^{n-1} + \ldots + c_{n-1} + A_{-1} + c_nI = 0 \tag{2}$$

Note that when λ is replaced by A in the characteristic equation, the constant term c_n is replaced by c_nI to get the result of Cayley-Hamilton theorem.

Corollary 3 *If A is non-singular matrix of oder n, then*

$$A^{-1} = \frac{1}{c_n}(c_oA^{n-1} + c_1A^{n-2} + \ldots + c_nI)$$

Example 5 Given that $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$. Use the Cayley-Hamilton to find A^{-1}.

Solution: The characteristic equation of A is $\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$. By Calley-Hamilton we have

$$A^3 - 4A^2 - 20A - 35I = 0 \tag{3}$$

Premultiplying the above equation to obtain

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

Thus,

$$A^{-1} = \frac{1}{35}(A^2 - 4A - 20I).$$

$$A^2 = A \times A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix}$$

Hence,

$$\begin{aligned} A^{-1} &= \frac{1}{35}\left(\begin{bmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{bmatrix} - \begin{bmatrix} 4 & 12 & 28 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}\right) \\ &= \frac{1}{35}\begin{bmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & 10 \end{bmatrix} \end{aligned}$$

FINDING EIGENVALUES AND EIGENVECTORS

EXAMPLE 1: Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

SOLUTION:

- In such problems, we first find the **eigenvalues** of the matrix.

FINDING EIGENVALUES

- To do this, we find the values of λ which satisfy the **characteristic equation** of the matrix A, namely those values of λ for which

$$\det(A - \lambda I) = 0,$$

where I is the 3×3 **identity matrix**.

- Form the matrix $A - \lambda I$:

$$A - \lambda I = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix}.$$

Notice that this matrix is just equal to A with λ subtracted from each entry on the main diagonal.

- Calculate $\det(A - \lambda I)$:

$$\begin{aligned} \det(A - \lambda I) &= (1-\lambda) \begin{vmatrix} -5-\lambda & 3 \\ -6 & 4-\lambda \end{vmatrix} - (-3) \begin{vmatrix} 3 & 3 \\ 6 & 4-\lambda \end{vmatrix} + 3 \begin{vmatrix} 3 & -5-\lambda \\ 6 & -6 \end{vmatrix} \\ &= (1-\lambda)\left((-5-\lambda)(4-\lambda) - (3)(-6)\right) + 3(3(4-\lambda) - 3\times 6) + 3(3\times(-6) - (-5-\lambda)6) \\ &= (1-\lambda)(-20 + 5\lambda - 4\lambda + \lambda^2 + 18) + 3(12 - 3\lambda - 18) + 3(-18 + 30 + 6\lambda) \\ &= (1-\lambda)(-2 + \lambda + \lambda^2) + 3(-6 - 3\lambda) + 3(12 + 6\lambda) \\ &= -2 + \lambda + \lambda^2 + 2\lambda - \lambda^2 - \lambda^3 - 18 - 9\lambda + 36 + 18\lambda \\ &= 16 + 12\lambda - \lambda^3. \end{aligned}$$

- Therefore

$$\det(A - \lambda I) = -\lambda^3 + 12\lambda + 16.$$

REQUIRED: To find solutions to $\det(A - \lambda I) = 0$ i.e., to solve

$$\lambda^3 - 12\lambda - 16 = 0. \tag{1}$$

  * Look for **integer** valued solutions.

* Such solutions **divide** the **constant** term (-16). The list of possible integer solutions is

$$\pm 1, \pm 2, \pm 4, \pm 8, \pm 16.$$

* Taking $\lambda = 4$, we find that $4^3 - 12.4 - 16 = 0$.
* Now factor out $\lambda - 4$:

$$(\lambda - 4)(\lambda^2 + 4\lambda + 4) = \lambda^3 - 12\lambda^2 + 16.$$

* Solving $\lambda^2 + 4\lambda + 4$ by formula[1] gives

$$\lambda = \frac{-4 \pm \sqrt{4^2 - 4.1.4}}{2} = \frac{-4 \pm 0}{2},$$

and so $\lambda = -2$ (a repeated root).

- Therefore, the eigenvalues of A are $\lambda = 4, -2$. ($\lambda = -2$ is a repeated root of the **characteristic equation**.)

FINDING EIGENVECTORS

- Once the **eigenvalues** of a matrix (A) have been found, we can find the **eigenvectors** by Gaussian Elimination.
- **STEP 1**: For each eigenvalue λ, we have

$$(A - \lambda I)\mathbf{x} = \mathbf{0},$$

where x is the **eigenvector** associated with **eigenvalue** λ.

- **STEP 2**: Find $\mathbf{x}$ by Gaussian elimination. That is, convert the augmented matrix

$$\left(A - \lambda I \,\vdots\, \mathbf{0} \right)$$

to row echelon form, and solve the resulting linear system by back substitution.

We find the **eigenvectors** associated with each of the **eigenvalues**

- **Case 1:** $\lambda = 4$

  - We must find vectors $\mathbf{x}$ which satisfy $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

[1] To find the roots of a quadratic equation of the form $ax^2 + bx + c = 0$ (with $a \neq 0$) first compute $\Delta = b^2 - 4ac$, then if $\Delta \geq 0$ the roots exist and are equal to $x = \frac{-b-\sqrt{\Delta}}{2a}$ and $x = \frac{-b+\sqrt{\Delta}}{2a}$.

- First, form the matrix $A-4I$:

$$A - 4I = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}.$$

- Construct the augmented matrix $\left(A - \lambda I \vdots \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{pmatrix} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \quad \xrightarrow{R1 \to -1/3 \times R3} \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow[R3 \to R3 - 6 \times R1]{R2 \to R2 - 3 \times R1} \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow{R2 \to -1/12 \times R2} \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & -12 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow{R3 \to R3 + 12 \times R2} \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow{R1 \to R1 - R2} \quad \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

- Rewriting this augmented matrix as a linear system gives

$$\begin{aligned} x_1 - 1/2x_3 &= 0 \\ x_2 - 1/2x_3 &= 0 \end{aligned}$$

So the eigenvector $\mathbf{x}$ is given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = \frac{x_3}{2} \\ x_2 = \frac{x_3}{2} \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$$

For any real number $x_3 \neq 0$. Those are the **eigenvectors of** A **associated with the eigenvalue** $\lambda = 4$.

- **Case 2**: $\lambda = -2$

  - We seek vectors $\mathbf{x}$ for which $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
  - Form the matrix $A - (-2)I = A + 2I$

$$A + 2I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}.$$

– Now we construct the augmented matrix $\left(A - \lambda I \,\vdots\, \mathbf{0}\right)$ and convert it to row echelon form

$$\begin{pmatrix} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix} \quad \xrightarrow{R1 \to 1/3 \times R3} \quad \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

$$\xrightarrow[R3 \to R3 - 6 \times R1]{R2 \to R2 - 3 \times R1} \quad \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \text{R1} \\ \text{R2} \\ \text{R3} \end{matrix}$$

– When this augmented matrix is rewritten as a linear system, we obtain

$$x_1 + x_2 - x_3 = 0,$$

so the eigenvectors $\mathbf{x}$ associated with the eigenvalue $\lambda = -2$ are given by:

$$\mathbf{x} = \begin{pmatrix} x_1 = x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix}$$

– Thus

$$\mathbf{x} = \begin{pmatrix} x_3 - x_2 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{for any } x_2, x_3 \in \mathbb{R}\backslash\{0\}$$

are the **eigenvectors of A associated with the eigenvalue** $\lambda = -2$.