Indian Institute of Technology Mandi
भारतीय प्रौद्योगिकी संस्थान मण्डी

MA-201: B.Tech. II year

Odd Semester: 2011-12 **Exercise-4 Linear Algebra**

1. For the following matrices
   (a) Find the characteristic polynomial.
   (b) Find the minimal polynomial.
   (c) Eigenvalues and corresponding eigenvectors.
   (d) Multiplicities of eigenvalues.

$$\begin{bmatrix}2&7\\7&2\end{bmatrix},\begin{bmatrix}3&-2\\1&-1\end{bmatrix},\begin{bmatrix}5&-3\\-4&3\end{bmatrix},\begin{bmatrix}2&1\\-1&4\end{bmatrix},\begin{bmatrix}3&-4\\4&8\end{bmatrix},\begin{bmatrix}5&3\\-4&4\end{bmatrix},\begin{bmatrix}7&-2\\2&3\end{bmatrix},\begin{bmatrix}1&1+i\\1-i&1\end{bmatrix},\begin{bmatrix}1&i\\2&1\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1\\2&3&1\\0&6&0\end{bmatrix},\begin{bmatrix}0&3&1\\3&0&2\\1&2&0\end{bmatrix},\begin{bmatrix}4&0&0\\5&3&2\\-2&0&2\end{bmatrix}\begin{bmatrix}-1&0&1\\-3&4&1\\0&0&2\end{bmatrix},\begin{bmatrix}6&-2&0\\-2&9&0\\5&8&3\end{bmatrix},\begin{bmatrix}5&-2&3\\0&1&0\\6&7&-2\end{bmatrix}$$

2. Find the value of h such that the the eigenspace corresponding to eigenvalue $\lambda = 5$ is two dimensional.

$$A=\begin{bmatrix}5&-2&6&-1\\0&1&h&0\\0&0&5&4\\0&0&0&1\end{bmatrix}$$

3. Prove that an $n\times n$ matrix A is diagonolizable if and only if A has n linearly independent eigenvectors.

4. Diagonolise the following matrices A_i if possible and find corresponding matrices P_i, $i = 1, 2, \ldots, 8$ such that $P_i^{-1}A_iP_i$ are all diagonal matrices.

$$A_1=\begin{bmatrix}-1&4&-2\\-3&4&0\\-3&1&3\end{bmatrix},A_2=\begin{bmatrix}4&2&2\\2&4&2\\2&2&4\end{bmatrix},A_3=\begin{bmatrix}2&2&-1\\1&3&-1\\-1&-1&2\end{bmatrix},A_4=\begin{bmatrix}4&0&-2\\2&5&4\\0&0&5\end{bmatrix}$$

$$A_5=\begin{bmatrix}7&4&16\\2&5&8\\-2&-2&-5\end{bmatrix},A_6=\begin{bmatrix}0&-4&-6\\-1&0&-3\\1&2&5\end{bmatrix},A_7=\begin{bmatrix}i&i+1\\-1+i&i\end{bmatrix},A_8=\begin{bmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{bmatrix}$$

5. Show that the following matrices are not diagonalizable.

   (a)
$$A=\begin{bmatrix}2&1\\-1&0\end{bmatrix}$$

   (b)
$$B=\begin{bmatrix}2&i\\i&0\end{bmatrix}$$

6. Let A, B and C be $n\times n$ be matrices. Prove that if A is similar to B and B is similar to C then A is similar to C.

7. Prove that the following matrices A and B are similar by showing that they are similar to the same diagonal matrix. Also ,find an invertible matrix P such that $P^{-1}AP = B$.

(a)

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 3 & 2 & -5 \\ 1 & 2 & -1 \\ 2 & 2 & -4 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

8. Let A be an $n \times n$ matrix with eigen values $\lambda_1, \lambda_2, ..., \lambda_n$,. Show that det(A)= $\lambda_1.\lambda_2.....\lambda_n$ and $tr(A) = \lambda_1 + \lambda_2 + ...\lambda_n$.

9. Prove that the eigen vectors corresponding to different eigenvalues are linearly independent.

10. Let A be an $n \times n$ matrix and let c $(\neq 0)$ be a constant. Show that λ is an eigen value of A iff $c\lambda$ is an eigen value of cA.

11. Let A be an $n \times n$ matrix. Show that A and A^t have same eigen values. Are their corresponding eigen spaces same?

12. Let A is invertible. Prove that if A is diagonalizable, then A^{-1} is diagonalisable.

13. Prove that if the matrix A is similar to B, then A^t is similar to B^t.

14. Prove that if A is diagonalizble, then A^t is also diagonolizable.

15. Let A and B be two $n \times n$ matrices. Prove that the sum of all the eigen values of $A + B$ is the sum of all the eigen values of A and B individually. Also, prove that the product of all the eigen values of AB is the product of all the eigen values of A and B individually.

16. Let A and B be two $n \times n$ matrices with eigen values λ and μ, respectively then

    (a) Give an example to show that $\lambda + \mu$ need not be an eigen value of $A + B$.

    (b) Give an example to show that $\lambda\mu$ need not be eigen value of AB.

    (c) Suppose that λ and μ, correspond to the same eigenvector x. Show that $\lambda + \mu$ is an eigenvalue of $A + B$ and $\lambda\mu$ is an eigen value of AB.

17. Prove that all the eigenvalues of a Hermitian matrix are real.

18. Prove that the eigen vectors (corresponding to distinct eigenvalues) of a Hermitian matrix are orthogonal.

19. The eigenvalues of a skew-Hermitian matrix are either purely imaginary, or zero.

20. The eigenvalues of a unitary matrix all have absolute value 1.

21. Find the symmetric matrix for the quadratic forms and determine the principal axis.

    (a) ${x_1}^2 - 2x_1x_2 + 4x_2x_3 - {x_2}^2 + {x_3}^2$

    (b) $3{x_1}^2 + 2x_1x_2 - 4x_1x_3 + 8x_2x_3 + {x_2}^2$

22. Given a polynomial $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + ... + a_{n-1}\lambda + a_n = 0$, construct a matrix A with $P(\lambda)$ as its characteristic polynomial. Is this matrix unique to the given polynomial.