Living Wallpaper

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1 Introduction

By leveraging the power of software, we can animate still mathematical art and bring it to life. Of particular interest are wallpaper patterns which are patterns that have translational symmetry in the horizontal and vertical directions; these patterns are explored extensively in Farris's book *Creating Symmetry* (Note: this book is the basis for the work here). These wallpaper patterns are the result of complex domain colorings where a photograph is used as the color domain. An image of a wallpaper pattern created from the photograph in Figure 1 is shown in Figure 2. In this project, we seek to bring these wallpaper patterns to life by advancing them forward through time in a way that is closely connected to physics. If the wallpaper were brought into three dimensions and displaced according to a gray-scale color mapping, the motion we seek is as if we were to release the wallpaper, allowing it to vibrate naturally. We seek to accomplish a similar motion with the complex wallpaper function on a plane.

2 The Case of a String

In order to understand the process for vibrating a wallpaper surface, we first turn to the simpler scenario of a vibrating horizontal string that is fixed at both ends. If we displace the center of the string upwards, there is a restorative force that wants to return the string to its original position; thus, the string accelerates downwards. Similarly, if we pull the string down, the string wants to accelerate upward to return to its original position. If we assume that the string is not bending dramatically (or that the slope at any point is near zero) and that the only forces acting on the string are vertical, then it is reasonable to assume that the acceleration is proportional to the

Figure 1: Photograph of Peppers provided with SymmetryWorks.

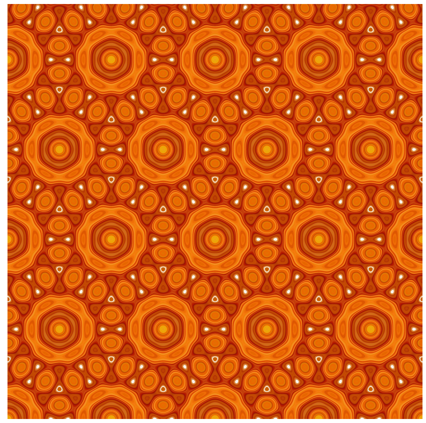

Figure 2: Wallpaper pattern created from the photograph on the left.

bending of the string. That is, if $f(x,t)$ is the strings displacement at a horizontal location x and time t, then we get the linear wave equation

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2}.$$

A function satisfying the linear wave equation behaves according to the physical situation of the vibrating string. Thus, if we started with a function $f(x)$ which gives the vertical displacement of a string at any location x, then we can find a function $f(x,t)$ which will move the string forward through time as if we had released it. Given a function in one dimension, we can find the Fourier sine series representation of the function. Moreover, since the wave equation is linear, we can break down any summation into individual functions. As a result, we only need to solve the linear wave equation for functions of the form $f(x) = a\sin(bx)$. Now, suppose we were to take two derivatives of $f(x)$ with respect to x. We would end up with

$$\frac{\partial^2 f}{\partial x^2} = -ab^2 \sin(bx),$$

which is almost what we started with. To find an $f(x,t)$ which satisfies the linear wave equation, we need a function which will kick out a $-b^2c^2$ after two derivatives. This works with $\cos(bct)$, so our family of solutions is

$$\begin{aligned} f(x,t) &= a\sin(bx)\cos(bct), \text{ since} \\ \frac{\partial^2 f}{\partial t^2} &= -ab^2c^2\sin(bx)\cos(bct) \\ &= c^2\frac{\partial^2 f}{\partial x^2}. \end{aligned}$$

Thus, given any function $f(x)$ given as a sum of sines, we can construct a solution to the linear wave equation for each addend and then sum these functions of x and t together. An example of a vibrating string with equation $f(x,t) = \sin(3x)\cos(3t/5) + 1/2\sin(7x)\cos(7t/5)$ was plotted in Maple and is pictured in Figure 3. Note that this function has $c = 1/5$; this is chosen to "slow" down the speed of the function in order to better observe the movement. Other values can be chosen for c to adjust the speed of the wave.

Additionally, the constant $\lambda = b$, the coefficient of ct has significance in the physical vibrating string situation. Here, λ gives a frequency and if λ is an integer, then the function is periodic and this number corresponds to harmonics, with $\lambda = 1$ corresponding to the fundamental frequency of the string. Now, our physical experiment with displacing a string has a remarkably close relationship to music—the plucking of a guitar string fixed at both ends. This gives rise to the possibility of making music from animated functions satisfying the linear wave equation.

3 The Case of a Real-Valued Surface

Next, we move onto a surface $f(x,y)$ which takes on real values. We interpret this surface to be as though it were a membrane of a drum. As we did with the string, we can imagine displacing the surface upwards and feeling the force trying to return the membrane to its original position. In this situation, our displacement causes the function to be concave down and the membrane wants to accelerate downwards. As with the string, if we assume that the surface is not very steep, we can reasonably assume that acceleration

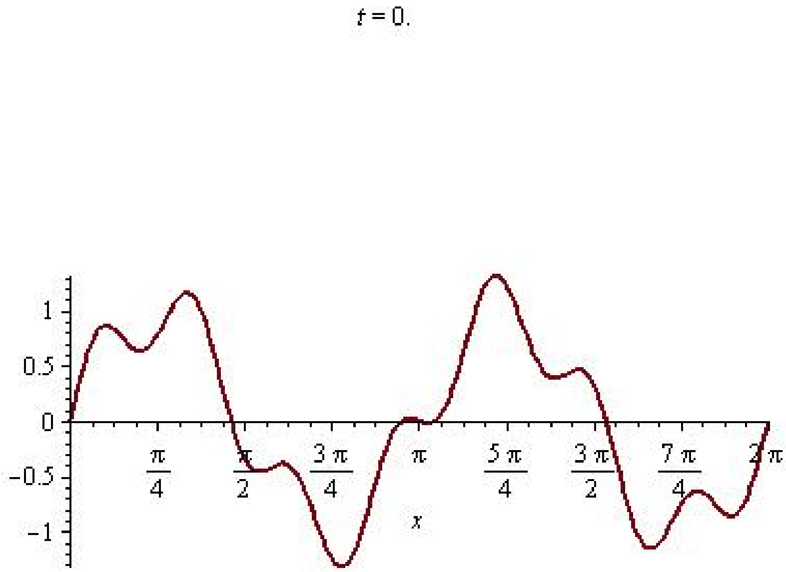

Figure 3: Vibrating string between two fixed ends.

is proportional to some notion related to concavity. In the surface case, we arrive at a similar linear wave equation for two dimensions,

$$\frac{\partial^2 f}{\partial t^2} = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

We can then apply the same methodology as with the case of a string to better understand how we can move a still, displaced surface through time. Given a function in two dimensions, we can still use a Fourier sine series to represent this function as a sum of sine functions. Since the two-dimensional wave equation is also linear, we need only solve this problem for a single term,

$f(x,y) = d\sin(ax)\sin(by)$. If we compute the sum of partial derivatives,

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= -a^2 d\sin(ax)\sin(by) - b^2 d\sin(ax)\sin(by)\\ &= -d(a^2+b^2)\sin(ax)\sin(by).\end{aligned}$$

Thus, a solution which satisfies the linear wave equation is

$$f(x,y,t) = d\sin(ax)\sin(by)\cos(c\sqrt{a^2+b^2}t).$$

From this solution, we can use the fact that the wave equation is linear to build the solution of any function $f(x,y)$ by decomposing its sine series into parts and solving each of them individually. An example of a vibrating surface with equation

$$f(x,y,t) = \frac{1}{2}\sin(3x)\sin(4y)\cos(\frac{5t}{5}).$$

was plotted in Maple and is pictured in Figure 4.

As with the string, we can extract information relating to periodicity and music from this solution. Now, we examine $\lambda = \sqrt{a^2+b^2}$. When λ is an integer, the solution is periodic and will return to its original state. However, this is only the case is a, b, λ form a perfect triple which is generally not the case. Thus, most of these real-valued functions and the complex-valued functions that we will examine in the next section will not be periodic. However, if we create a function such that λ is an integer, then the situation would relate to the vibration of a drum's membrane and we might be able to extract musical information from the function.

4 The Case of a Complex-Valued Surface

The case we are most interested in builds upon the previous case of a real-valued surface. The situation of a complex-valued surface matches the colored wallpaper functions which we wish to advance through time. To do this, we

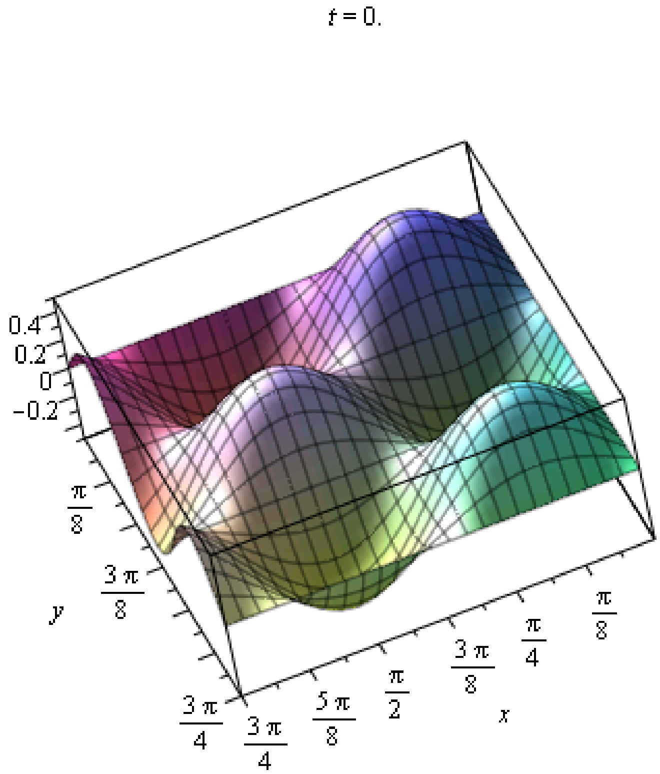

Figure 4: Vibrating surface satisfying the wave equation in two dimensions.

wish to solve the same linear wave equation in two dimensions which we used for real-valued surfaces,

$$\frac{\partial^2 f}{\partial t^2} = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

However, in the complex case, the Fourier series for our functions will represent them as a sum of complex exponential functions, rather than a sum of sines. As before, we can leverage the linearity of the wave equation and solve each term individually. Thus, we essentially must solve the wave equation for the following general term:

$$f(x, y) = de^{aix}e^{biy} = de^{aix+biy}.$$

Note: any constants in the exponent can be absorbed into the constant multiplier, d. If we compute the sum of second order partial derivatives, we see that

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= -a^2 d e^{aix+biy} - b^2 d e^{aix+biy} \\ &= -d(a^2+b^2)e^{aix+biy}.\end{aligned}$$

Thus, a solution which satisfies the two-dimensional wave equation is

$$f(x,y,t) = d e^{aix} e^{biy} e^{c\sqrt{a^2+b^2}it}.$$

Therefore, by linearity, we can solve the two-dimensional linear wave equation for any complex-valued surface (expressed in terms of complex exponential functions) by solving term-by-term. This is the process that we will use to bring once-still wallpaper functions to life by moving them forward through time in this fashion.

5 Application in *SymmetryWorks!*

SymmetryWorks! is an open source piece of software capable of creating still wallpaper functions created by Frank Farris and students at Bowdoin College. As an application of this mathematical research into animating wallpaper, I updated the *SymmetryWorks!* software to allow for animation of wallpaper patterns over time. To do this, I added time and waveVelocity variables which function as t and c, respectively, in the above equations. Then, I solved the linear wave equation for each of the mathematical functions within the software in order to advance the wallpaper properly through time. Some of the wallpaper functions were not written expressly in terms of exponential functions—for these functions, the linear wave equation was explicitly solved again where it could be done.

Moreover, work was done to add to the interface of the software. New widgets were created for varying the time and waveVelocity variables. Also, buttons were created for playing and stopping an animation as well as for exporting an animation. Currently, the animate and export functionality simply computes the wallpaper for many values of t and exports each frame as an image. Then, the user may use third-party software to stitch these frames together into a GIF animation. In the future, the GIF creation could be created and directly integrated into the software.

The resulting animations from this process are quite beautiful. The original wallpaper patters were already fascinating with their many symmetries. But then, when we move the wallpaper through time, we see this symmetry come to life. Over time, the wallpaper patterns twist and morph, all the while retaining the same translational symmetry of a wallpaper pattern. In Figure 5 below, four sequential frames are presented from left to right at times $t = 1, 4, 7, 10$ for the hex3 function with a waveSpeed of $c = 0.05$.

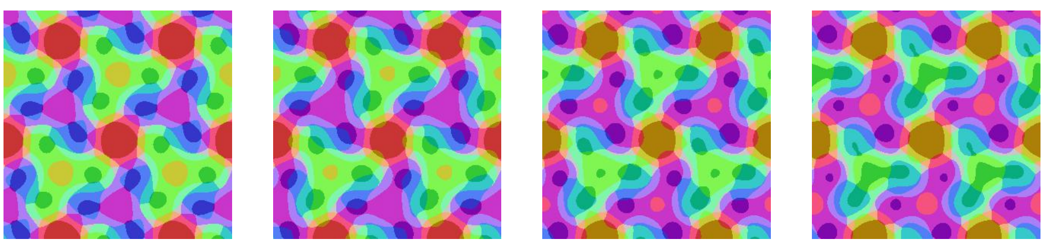

Figure 5: Multiple frames at times $t = 1, 4, 7, 10$ of the hex3 function.

6 Conclusion

We analyze still strings and surfaces and how they can be moved through time in a way that is closely connected to physical situations of strings and membranes vibrating. This concept can be extended to complex-valued functions so that we can animate complex wallpaper functions as if they were vibrating over time. This animation is rooted in physical vibrations through the linear wave equations. Any function can be advanced through time by finding its solution to the corresponding linear wave equation. In solving these differential equations, the linearity is leveraged to greatly simplify to essentially one case for each dimension (provided that a Fourier transformation can be performed to get the function in the desired form).

Future research can be done to determine the link between these animated wallpaper patterns and music. There is a close connection to music in the vibration of these functions. Moreover, when the differential equation is solved, the constant $\lambda = \sqrt{a^2 + b^2}$ corresponds to the frequency of the system. When λ is an integer, this corresponds to harmonic frequencies. This property could be leveraged to convert the animated wallpaper into musical notes.