QUANTUM ENTANGLEMENT

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A THESIS SUBMITTED

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

NATIONAL UNIVERSITY OF SINGAPORE

2019

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DECLARATION

I hereby declare that this thesis is my original work and it has been written by me in its entirety. I have duly acknowledged all the sources of information which have been used in the thesis.

This thesis has also not been submitted for any degree in any university previously.

Qian Long

July 6, 2019

ACKNOWLEDGEMENT

I would like to express the deepest gratitude to my supervisors Prof.

Besides my supervisors, I would like to express my sincere gratitude to Prof.

Qian Long

2019

CONTENTS

SUMMARY

This dissertation contains two parts. Part I dedicates to develop risk analysis tool.

LIST OF FIGURES

LIST OF TABLES

LIST OF SYMBOLS

LIST OF ALGORITHMS

LIST OF CODES

LIST OF PUBLICATIOINS

1. L. Qian. *Separability of multipartite quantum states with strong positive partial transpose*. Phys. Rev. A, **98**, (2018), p. 012307. http://dx.doi.org/10.1103/physreva.98.012307

2. L. Qian and D. Chu. *Decomposition of completely symmetric states*. Quantum Inf. Process., **18**, (2019), p. 208. http://dx.doi.org/10.1007/s11128-019-2318-2

3. L. Chen, D. Chu, L. Qian, and Y. Shen. *Separability of completely symmetric states in a multipartite system*. Phys. Rev. A, **99**, (2019), p. 032312. http://dx.doi.org/10.1103/physreva.99.032312

INTRODUCTION

Chapter I

SPPT STATES

1 Introduction

Entanglement lies at the heart of quantum computation and information theory, which is the resource of most applications in quantum information processing tasks. Since 1935, when the necessarily nonlocal nature of quantum mechanics was first highlighted by Einstein, Podolsky, and Rosen (EPR) [8], quantum entanglement has become a major quantum phenomenon which requires further understanding. One of the fundamental tasks about quantum entanglement is the separability problem, i.e., to check whether a given quantum state is separable. Given a density matrix ρ in a quantum bipartite system $A : B$, it is said to be separable if it can be written as a convex combination of product states [28], i.e.,

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \ \sum_i p_i = 1, p_i \geqslant 0, \tag{I.1}$$

where ρ_i^A and ρ_i^B are density matrices in subsystem A and B respectively. A quantum state is said to be entangled if it does not possess a decomposition of the form as Eq. (I.1). In this case, the state cannot be prepared by the information of its two subsystems [21].

Despite remarkable efforts over recent years, the operational necessary and sufficient condition for the separability remains unknown in general. It was found that the separable problem is NP-HARD even for the bipartite system [11].

Although it is hard to solve this problem in general, there are plenty of practical criteria which enable us to detect entanglement for some subclasses. One of the most famous criteria is called positive partial transpose (PPT) or Peres-Horodecki criterion [21]. It tells that if a state ρ is separable, then its partial transposed state $\rho^{\mathsf{T}_A} = (\mathsf{T} \otimes \mathbb{1})\rho$ must remain positive. Using positive maps, Horodecki *et al.* [15] showed that Peres-Horodecki criterion is also sufficient for $2 \otimes 2$ and $2 \otimes 3$ systems. However, it fails in higher dimensional spaces. Woronowicz [29] constructed a counterexample of a $2 \otimes 4$ entangled PPT state. See more entangle PPT states in Refs. [6, 26, 16]. Utilizing matrix analysis, Kraus *et al.* [20] showed that any $M \otimes N (M \leqslant N)$ PPT state of rank N is separable. Moreover, some generalized results are proposed in Refs. [17, 19, 9]. Since any $M \otimes N$ state of rank less than N is distillable [18], it suffices to consider such state whose rank is greater than its local ranks.

A subclass of PPT states, namely strong PPT (SPPT) states, were first considered by Chruściński *et al.* [7]. These states have a "strong PPT property". Based on several examples, it was conjectured that SPPT states are separable. Unfortunately, this conclusion fails for $M \otimes N$ PPT states when $NM \geqslant 9$. Actually, all $2 \otimes 4$ SPPT states are separable [13]. But, there exists a $2 \otimes 5$ SPPT which is entangled [13]. The separability of SPPT states become more complex in high dimensional spaces. The SPPT states encompass many previously known separable PPT states such as rank N states of $2 \otimes N$ system. Moreover, it is proved that SPPT states can be used to witness quantum discord (QD) in $2 \otimes N$ systems [1]. In addition, Bylicka *et al.* [2] constructed a special class of SPPT states, which were called super strong SPPT (SSPPT) states. In Ref. [10], the decomposition of SSPPT states was considered in both finite and infinite dimensional systems.

In a recent paper[30], the idea of SPPT states was generalized to the tripartite system $A_1 : A_2 : A_3$. However, these states are essentially bipartite SPPT with respect to the bi-partition $A_1A_2 : A_3$. As a result, some good properties may be lost in the tripartite sense. For instance, the SPPT cannot guarantee PPT, which is one of the most important features for SPPT states in the bipartite system. Also, the super SPPT cannot guarantee the separability in general. Therefore, it would be especially interesting to find a more appropriate generalization to tripartite or even multipartite systems. The purpose of this paper is to provide an alternative definition of SPPT states in the multipartite system. We begin with the simplest case when ρ is in $2 \otimes 2 \otimes N$, and eventually, extend to general many-body system.

The remainder of this paper is organized as follows. In Section 2, we present some preliminaries about the separability problem of the SPPT states. In Section 3, we recall the definition of tripartite SPPT states in Ref. [30]. We show that the defined SPPT and SSPPT cannot inherit some good properties as those in the bipartite case. In Section 4, we provide a new idea to define the SPPT and SSPPT state in $2 \otimes 2 \otimes N$ system. We extend this concept to $N_1 \otimes N_2 \otimes N_3$ case. Finally, we show the idea to the arbitrary multipartite system. In Section 5, we propose some sufficient conditions for separability of SPPT states. Some concluding remarks are given in Section 6.

2 preliminaries

We start this section with a formal definition of separability. Consider a d-particle state belonging to a Hilbert space $\mathcal{H}$. Denote by $A_1, A_2, \cdots, A_d$ the subsystems respectively. Each subsystem is a Hilbert space $\mathcal{H}_i$ with dimension N_i. By the postulate for composition of system in quantum computation theory, we have $\mathcal{H} = \otimes_{i=1}^{d}\mathcal{H}_i$ and $\dim(\mathcal{H}) = \prod_{i=1}^{d} N_i$.

To make more concise, we use vector based indexes in this paper especially when the number of parties is large. Let $\mathcal{I}$ be the set consisting of the d-tuples $(i_1, i_2, \ldots, i_d), 1 \leqslant i_k \leqslant N_k$. For any given index $\alpha = (i_1, i_2, \ldots, i_d) \in \mathcal{I}$, x_α represents $x_{i_1,i_2,\ldots,i_d}$. Furthermore, we assign an order for those indexes. Let

$$\pi(\alpha) = \prod_{k=1}^{d-1}(\prod_{l=k+1}^{d} N_l)(\alpha_k - 1) + \alpha_d.$$

For $\alpha, \beta \in \mathcal{I}$, $\alpha < \beta(\alpha \leqslant \beta)$ denotes $\pi(\alpha) < \pi(\beta)(\pi(\alpha) \leqslant \pi(\beta))$. For example, in the bipartite case, $(i,j) < (k,l)$ equals $(i-1)N_2 + j < (k-1)N_2 + l$. Moreover, $|\alpha\rangle$ represents the product vector $|i_1, i_2, \ldots, i_d\rangle$.

Hence a density matrix acting on the space $\mathcal{H}$ can be represented as

$$\rho = \sum_{\alpha,\beta \leqslant \alpha_0} \rho_{\alpha,\beta}|\alpha\rangle\langle\beta|, \tag{I.2}$$

where $\alpha, \beta \in \mathcal{I}$ and $\alpha_0 = (N_1, N_2, \ldots, N_d)$.

Now we recall the definition of separability of a quantum state. A density matrix ρ acting on $\mathcal{H}$ is said to be separable if it can be written as

$$\rho = \sum_{i=1}^{L} \lambda_i \left|x_i\right\rangle \left\langle x_i\right|, \tag{I.3}$$

where $\sum_i^L \lambda_i = 1, \lambda_i \geqslant 0$ and each $|x_i\rangle$ is a pure product vector in the space $\mathcal{H}$.

Peres-Horodecki criterion plays a crucial role in the separability problem, which is based on the partial transpose. Therefore it would be necessary to introduce the notations of partial transposes before going further. Denote by T the usual transpose operator. Then the composite operators $(\mathbb{1} \otimes \mathsf{T})$ and $(\mathsf{T} \otimes \mathbb{1})$ are called the partial transpose operators in the bipartite system. Furthermore, the partial transposed density matrices are denoted by $\rho^{\mathsf{T}_2} = (\mathbb{1} \otimes \mathsf{T})\rho$ and $\rho^{\mathsf{T}_1} = (\mathsf{T} \otimes \mathbb{1})\rho$ respectively. For general multipartite system $A_1 : A_2 \cdots : A_d$, denote by T_i ($i = 1, 2, \ldots, d$) the partial transpose with respect to i-th subsystem respectively. The corresponding partial transposed state is denoted by ρ^{T_i}. Generally, for any given index set $I = \{i_1, i_2, \ldots, i_k\}$, let T_I denote the partial transpose with respect to the subsystems in I, that is

$$\mathsf{T}_I = \circ_{k\in I}\mathsf{T}_k.$$

PPT criterion tells that if ρ is separable, then

$$\mathsf{T}_I \cdot \rho \geqslant 0,\ \forall I \subset \{1,2,\ldots,d\}. \tag{I.4}$$

Now we recall the definition of SPPT in the bipartite system.

Consider a density matrix ρ in $N_1 \otimes N_2$ system which has a block Cholesky decomposition $\rho = X^{\mathsf{H}}X$:

$$\begin{aligned} X &= \begin{pmatrix} S_{11}X_1 & \cdots & S_{1N_1}X_1 \\ \vdots & \ddots & \vdots \\ S_{N_1 1}X_{N_1} & \cdots & S_{N_1N_1}X_{N_1} \end{pmatrix} \\ &= \begin{pmatrix} X_1 & \cdots & S_{1N_1}X_1 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & X_{N_1} \end{pmatrix}, \end{aligned} \tag{I.5}$$

where S_{ij} and X_i are both $N_2 \times N_2$ matrices with

$$S_{ij} = \begin{cases} \mathbb{1}_{N_2}, & \text{if } i = j; \\ 0, & \text{if } i > j. \end{cases}$$

In this paper, $\mathbb{1}_n$ denotes the identity operator acting on the space $\mathbb{C}^n$. It is also written as $\mathbb{1}$ when there is no doubt about the dimension.

Definition 1. *Let ρ be a density matrix in $N_1 \otimes N_2$ system. And $\rho = X^{\mathsf{H}}X$, where X has the form as Eq.* (I.5)*. Then ρ is said to be SPPT if*

$$\rho^{\mathsf{T}_1} = Y^{\mathsf{H}}Y, \tag{I.6}$$

with

$$Y = \begin{pmatrix} S^{\dagger}_{11}X_1 & \cdots & S^{\dagger}_{1N_1}X_1 \\ \vdots & \ddots & \vdots \\ S^{\dagger}_{N_1 1}X_1 & \cdots & S^{\dagger}_{N_1N_1}X_1 \end{pmatrix},$$

or equivalently,

$$\sum_{k=1}^{N_1} X_k^{\mathsf{H}}[S^{\mathsf{H}}_{kj}, S_{ki}]X_k = 0,\ 1 \leqslant i \leqslant j \leqslant N_1. \tag{I.7}$$

Here the commutator $[A, B]$ is defined by $[A, B] = AB - BA$.

In particular, Eq. (I.7) is naturally satisfied if

$$[S^{\mathsf{H}}_{kj}, S_{ki}] = 0, 1 \leqslant i, j \leqslant N_1. \tag{I.8}$$

These states are named super SPPT (SSPPT) states [7], which are proved to be separable [2].

3 Previous definition of tripartite SPPT states

In this section, firstly we will introduce the definition of tripartite SPPT states in Ref. [30]. After that, we will show that these SPPT states will not preserve some good properties as that in the bipartite system. For example, the SPPT state may not be PPT. Besides, pure or super SPPT states may not be separable.

Suppose ρ is a density matrix in the tripartite system $A_1 : A_2 : A_3$, which has a decomposition $\rho = X^{\mathsf{H}}X$. Under the bi-partition $A_1 : A_2A_3$, X can be written as an $N_1 \times N_1$ block matrix:

$$X = \begin{pmatrix} Z_{11} & \cdots & Z_{1N_1} \\ \vdots & \ddots & \vdots \\ Z_{N_1 1} & \cdots & Z_{N_1N_1} \end{pmatrix}, Z_{ij} \in \mathbb{C}^{N_2N_3 \times N_2N_3}. \tag{I.9}$$

Again, each Z_{ij} can be written as an $N_2 \times N_2$ block matrix:

$$Z_{ij} = \begin{pmatrix} S_{ij11}X_{i1} & \cdots & S_{ij1N_2}X_{i1} \\ \vdots & \ddots & \vdots \\ S_{ijN_21}X_{iN_2} & \cdots & S_{ijN_2N_2}X_{iN_2} \end{pmatrix}, \tag{I.10}$$

where

$$S_{ijkl} = \begin{cases} \mathbb{1}_{N_3}, & (i,k) = (j,l); \\ 0, & (i,k) > (j,l). \end{cases} \tag{I.11}$$

Let $\alpha = (i,k), \beta = (j,l)$, then

$$\rho = X^{\mathsf{H}} X, \tag{I.12}$$

in which the (α,β)-th entry of X is $S_{\alpha,\beta}X_\alpha$. For simplicity, let (α,β)-th entry denote the element in $\pi(\alpha)$-th row and $\pi(\beta)$-th column of a block matrix.

Recall the definition of SPPT state in Ref. [30]:

Definition 2. *Let $\rho = X^{\mathsf{H}}X$ be a density matrix in the tripartite system $A_1 : A_2 : A_3$ with X being of the form as Eq. (I.12). Then ρ is said to be SPPT if*

$$\rho^{\mathsf{T}_{12}} = Y^{\mathsf{H}} Y, \tag{I.13}$$

where the (α,β)-th entry of Y is $S^\dagger_{\alpha,\beta}X_\alpha$.

Note that in the above definition, the condition (I.13) is equivalent to

$$\sum_{\gamma_0 \leqslant \alpha \leqslant \gamma_1} X^{\mathsf{H}}_\alpha \left[S_{\alpha,\beta}, S^{\mathsf{H}}_{\alpha,\beta'}\right] X_\alpha = 0, \tag{I.14}$$

where $\gamma_0 = (1,1)$, $\gamma_1 = (N_1, N_2)$, and $\gamma_0 \leqslant \beta \leqslant \beta' \leqslant \gamma_1$.

Similarly, super SPPT (SSPPT) are also defined for tripartite system in Ref. [30].

Definition 3. *Let ρ be an SPPT state with a decomposition as in Eq. (I.12), then ρ is said to be SSPPT if*

$$[S_{\alpha,\beta}, S^{\mathsf{H}}_{\alpha,\beta'}] = 0, \ \gamma_0 \leqslant \alpha \leqslant \beta \leqslant \beta' \leqslant \gamma_1. \tag{I.15}$$

Note that α refers to the $n(\alpha)$-th row and β refers to the $n(\beta)$-th column. Therefore, if we reorder the 2-tuple α, β by $n(\alpha)$ and $n(\beta)$ respectively, then the definition of SPPT will be identical to that in the bipartite system $A_1A_2 : A_3$. Consequently, the same properties may not be kept in tripartite system.

PPT, for instance, is one of the most important features for SPPT states. However, the tripartite SPPT states defined this way may lose it. Here we construct an example to show this defect.

Let

$$\rho = v^{\mathsf{H}}v,\ v = (1,0,0,0,0,0,1,0),$$

which is an entangled state in $2\otimes 2\otimes 2$ system.

In order to prove that it is an SPPT state, let

$$X_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, X = \begin{pmatrix} X_{11} & 0 & 0 & X_{11} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{I.16}$$

Then ρ can be written as

$$\rho = X^{\mathsf{H}}X. \tag{I.17}$$

Put

$$S_{\alpha,\beta} = \begin{cases} \mathbb{1}, & \text{if } \alpha = \beta = (1,1), \\ 0, & \text{otherwise}. \end{cases} \tag{I.18}$$

By checking the conditions (I.15), ρ is SSPPT. However, it is not PPT since any pure PPT state is separable. Moreover, this example implies that super and pure SPPT states may not be separable.

In Theorem 1 of Ref. [30], the authors pointed out that any SSPPT state is bi-separable. It should be noted that the tripartite SSPPT states are bi-separable only with respect to the bi-partition $A_1A_2 : A_3$. Consider the example $\rho = v^{\mathsf{H}}v$ again. It is SSPPT by definition but not separable under either the bi-partition $A_1 : A_2A_3$ or $A_1A_3 : A_2$.

In the next section, we will define our SPPT and SSPPT states in another way. It turns out that many desired properties will be preserved.

4 SPPT states in multipartite case

In this section, we provide an alternative way to define the tripartite SPPT states. The main idea is to require the states to satisfy SPPT conditions under a series of bi-partitions. As a result, these states will remain positive under any partial transpose. Moreover, some other properties are also preserved compared with the previous definition in [30].

We begin with the simplest case $2 \otimes 2 \otimes N$, then we extend the idea to general $N_1 \otimes N_2 \otimes N_3$ tripartite system. Lastly, we give the definition of SPPT in the arbitrary multipartite system. Correspondingly, the SSPPT states are also introduced, which turn out to be separable. In addition, we give some examples of SPPT states, which may be helpful to shed new lights on understanding the structure of PPT states in multipartite system.

4.1 SPPT states in $2 \otimes 2 \otimes N$ system

We begin with the simplest case when ρ is a density matrix in $2 \otimes 2 \otimes N$ system. This can serve as an example to show the intuition of generalization for many body system.

Let ρ be a density matrix with $\rho = X^{\mathsf{H}} X$. In order to keep PPT, we require ρ to be SPPT under either the bi-partition $A_1 : A_2A_3$ or $A_1A_2 : A_3$. For $A_1 : A_2A_3$, X should be an upper triangular block matrix as

$$X = \begin{pmatrix} X_1 & SX_1 \\ 0 & X_2 \end{pmatrix}. \tag{I.19}$$

Under bi-partition $A_1 : A_2A_3$, X should have the following upper triangular block representation:

$$X = \begin{pmatrix} X_{11} & R_{1112}X_{11} & R_{1121}X_{11} & R_{1122}X_{11} \\ 0 & X_{12} & R_{1221}X_{12} & R_{1222}X_{12} \\ 0 & 0 & X_{11} & R_{2122}X_{21} \\ 0 & 0 & 0 & X_{22} \end{pmatrix}$$

$$= \begin{pmatrix} R_{1111}X_{11} & R_{1112}X_{11} & R_{1121}X_{11} & R_{1122}X_{11} \\ R_{1211}X_{12} & R_{1212}X_{12} & R_{1221}X_{12} & R_{1222}X_{12} \\ R_{2111}X_{21} & R_{2112}X_{21} & R_{2121}X_{11} & R_{2122}X_{21} \\ R_{2211}X_{22} & R_{2212}X_{22} & R_{2221}X_{22} & R_{2222}X_{22} \end{pmatrix},$$

where

$$R_{ijkl} = \begin{cases} \mathbb{1}_N, & \text{if } (i,j) = (k,l), \\ 0, & \text{if } (i,j) > (k,l). \end{cases}$$

In order to have the above form , we require

$$X_1 = \begin{pmatrix} X_{11} & T_1 X_{11} \\ 0 & X_{12} \end{pmatrix}, X_2 = \begin{pmatrix} X_{21} & T_2 X_{21} \\ 0 & X_{22} \end{pmatrix}.$$

And S should be a block diagonal matrix, i.e.,

$$S = \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix}.$$

Hence, X can be written as a 4×4 upper triangular block matrix with each block being of size $N \times N$:

$$X = \begin{pmatrix} X_{11} & T_1 X_{11} & S_1 X_{11} & S_1 T_1 X_{11} \\ 0 & X_{12} & 0 & S_2 X_{12} \\ 0 & 0 & X_{21} & T_2 X_{21} \\ 0 & 0 & 0 & X_{22} \end{pmatrix}. \tag{I.20}$$

Let

$$Y_1 = \begin{pmatrix} X_1 & S^{\mathsf{H}} X_1 \\ 0 & X_2 \end{pmatrix}, \tag{I.21}$$

and

$$Y_2 = \begin{pmatrix} X_{11} & T_1^{\mathsf{H}} X_{11} & S_1^{\mathsf{H}} X_{11} & (S_1 T_1)^{\mathsf{H}} X_{11} \\ 0 & X_{12} & 0 & S_2^{\mathsf{H}} X_{12} \\ 0 & 0 & X_{21} & T_2^{\mathsf{H}} X_{21} \\ 0 & 0 & 0 & X_{22} \end{pmatrix}.$$

Here we give the definition of SPPT states in the $2 \otimes 2 \otimes N$ system formally.

Definition 4. *Let $\rho = X^{\mathsf{H}} X$ be a density matrix in $2 \otimes 2 \otimes N$ system where X has the form as in Eq. (I.20). Then ρ is said to be SPPT if*

$$\rho^{\mathsf{T}_1} = Y_1^{\mathsf{H}} Y_1, \tag{I.22}$$

and

$$\rho^{\mathsf{T}_{12}} = Y_2^{\mathsf{H}} Y_2. \tag{I.23}$$

Alternatively, the above two conditions in the definition of SPPT can be reformulated as

$$\begin{gathered} X_1^{\mathsf{H}} [S, S^{\mathsf{H}}] X_1 = 0, \\ \sum_{i,k=1}^{2} X_{ik}^{\dagger} [R_{ijkl}, R_{ij'kl'}^{\dagger}] X_{ik} = 0. \end{gathered} \tag{I.24}$$

For this simple case, Eq. (I.24) can be written explicitly as

$$\begin{cases} X_{11}^{\mathsf{H}}[T_1, T_1^{\mathsf{H}}]X_{11} & = 0, \\ X_{11}^{\mathsf{H}}[S_1, S_1^{\mathsf{H}}]X_{11} & = 0, \\ X_{11}^{\mathsf{H}}[T_1, S_1^{\mathsf{H}}]X_{11} & = 0, \\ X_{11}^{\mathsf{H}}[S_1, S_1^{\mathsf{H}}]T_1X_{11} & = 0, \\ X_{11}^{\mathsf{H}}[S_1, T_1^{\mathsf{H}}S_1^{\mathsf{H}}]X_{11} & = 0, \\ X_{11}^{\mathsf{H}}[T_1, T_1^{\mathsf{H}}S_1^{\mathsf{H}}]X_{11} & = 0, \\ X_{11}^{\mathsf{H}}T_1^{\mathsf{H}}[S_1, S_1^{\mathsf{H}}]T_1X_{11} + X_{12}^{\mathsf{H}}[S_2, S_2^{\mathsf{H}}]X_{12} & = 0, \\ X_{11}^{\mathsf{H}}[S_1T_1, (S_1T_1)^{\mathsf{H}}]X_{11} & \\ \quad + X_{12}^{\mathsf{H}}[S_2, S_2^{\mathsf{H}}]X_{12} + X_{21}^{\mathsf{H}}[T_2, T_2^{\mathsf{H}}]X_{21} & = 0. \end{cases} \tag{I.25}$$

Note that conditions (I.22) and (I.23) guarantee a "strong PPT property". This is one of the most different aspects compared with definition in previous section. Consider Eq. (I.24), if all the commutators vanishes, the SPPT conditions will be satisfied automatically. This class of SPPT states are thus said to be super SPPT (SSPPT).

Definition 5. *Suppose $\rho = X^{\mathsf{H}}X$ is an SPPT state where X has form as in Eq. (I.20). Then ρ is called a SSPPT state if*

$$\begin{aligned} [S, S^{\mathsf{H}}] &= 0, \\ [R_{ijkl}, R_{ij'kl'}^{\dagger}] &= 0. \end{aligned} \tag{I.26}$$

In detail, Eq. (I.26) can be written as

$$\begin{cases} [S_i, S_i^{\mathsf{H}}] & = 0, i = 1, 2, \\ [T_i, T_i^{\mathsf{H}}] & = 0, i = 1, 2, \\ [S_1, T_1^{\mathsf{H}}] & = 0. \end{cases} \tag{I.27}$$

In the following theorem, we show that SSPPT can guarantee the separability.

Theorem 1. *All SSPPT states in the tripartite system $2 \otimes 2 \otimes N$ are separable.*

Proof. According to Eq. (I.27), T_1, T_2, S_1, S_2 are normal and T_1 commutes with S_1.

Therefore, we have the following diagonalizations:

$$\begin{aligned} S_1 &= U\Sigma_1 U^{\mathsf{H}}, T_1 = U\Sigma_2 U^{\mathsf{H}}, \\ S_2 &= V_1\Lambda_1 V_1^{\mathsf{H}}, T_2 = V_2\Lambda_2 V_2^{\mathsf{H}}, \end{aligned} \tag{I.28}$$

where Σ_i, Λ_i are the diagonal matrices and U, V_1 and V_2 are all unitary matrices. Then

$$\begin{aligned} X &= \begin{pmatrix} UU^{\mathsf{H}}X_{11} & U\Sigma_2 U^{\mathsf{H}}X_{11} & U\Sigma_1 U^{\mathsf{H}}X_{11} & U\Sigma_1\Sigma_2 U^{\mathsf{H}}X_{11} \\ 0 & V_1V_1^{\mathsf{H}}X_{12} & 0 & V_1\Lambda_1V_1^{\mathsf{H}}X_{12} \\ 0 & 0 & V_2V_2^{\mathsf{H}}X_{21} & V_2\Lambda_2V_2^{\mathsf{H}}X_{21} \\ 0 & 0 & 0 & X_{22} \end{pmatrix} \\ &= \begin{pmatrix} U & 0 & 0 & 0 \\ 0 & V_2 & 0 & 0 \\ 0 & 0 & V_1 & 0 \\ 0 & 0 & 0 & \mathbb{1}_N \end{pmatrix} \begin{pmatrix} \widetilde{X}_{11} & \Sigma_2\,\widetilde{X}_{11} & \Sigma_1\,\widetilde{X}_{11} & \Sigma_1\Sigma_2\,\widetilde{X}_{11} \\ 0 & \widetilde{X}_{12} & 0 & \Lambda_1\,\widetilde{X}_{12} \\ 0 & 0 & \widetilde{X}_{21} & \Lambda_2\,\widetilde{X}_{21} \\ 0 & 0 & 0 & X_{22} \end{pmatrix} \\ &= G\,\widetilde{X}, \end{aligned} \tag{I.29}$$

where

$$\widetilde{X}_{11}= U^{\mathsf{H}}X_{11}, \widetilde{X}_{12}= V_1^{\mathsf{H}}X_{12}, \widetilde{X}_{21}= V_2^{\mathsf{H}}X_{21}. \tag{I.30}$$

Let

$$\begin{aligned} C_1 &= \begin{pmatrix} \widetilde{X}_{11} & \Sigma_2\,\widetilde{X}_{11} & \Sigma_1\,\widetilde{X}_{11} & \Sigma_1\Sigma_2\,\widetilde{X}_{11} \end{pmatrix}, \\ C_2 &= \begin{pmatrix} 0 & \widetilde{X}_{12} & 0 & \Lambda_1\,\widetilde{X}_{12} \end{pmatrix}, \\ C_3 &= \begin{pmatrix} 0 & 0 & \widetilde{X}_{21} & \Lambda_2\,\widetilde{X}_{21} \end{pmatrix}, \\ C_4 &= \begin{pmatrix} 0 & 0 & 0 & X_{22} \end{pmatrix}. \end{aligned}$$

Note that $\rho = X^{\mathsf{H}}X$ and G is a unitary matrix, then we obtain,

$$\rho = \sum_{i=1}^{4} C_i^{\mathsf{H}}C_i.$$

On the other hand,

$$
\begin{aligned}
C_1 &= (\mathbb{1}, \Sigma_1) \otimes (\mathbb{1}, \Sigma_2) \otimes \widetilde{X}_{11}, \\
C_2 &= (\mathbb{1}, \Lambda_1) \otimes (\,0\,, \mathbb{1}\,) \otimes \widetilde{X}_{12}, \\
C_3 &= (\,0\,, \mathbb{1}\,) \otimes (\mathbb{1}, \Lambda_2) \otimes \widetilde{X}_{21}, \\
C_4 &= (\,0\,, \mathbb{1}\,) \otimes (\,0\,, \mathbb{1}\,) \otimes \widetilde{X}_{22}\ .
\end{aligned}
$$

It follows that each $C_i^{\mathsf{H}} C_i$ is separable, which implies the separability of ρ. ∎

Note that this proof can also be regarded as a method to find the separability decomposition of $2 \otimes 2 \otimes N$ SSPPT states.

Here we give an example of SPPT states in our definition.

Example 1. *It was proved that if a PPT state ρ supported on $2 \otimes 2 \otimes N (N \geqslant 2)$ has rank N, then it is separable and has the canonical form [19]*

$$
\rho = \begin{pmatrix} \mathbb{1}_N \\ \\ B^{\mathsf{H}} \\ \\ C^{\mathsf{H}} \\ \\ B^{\mathsf{H}} C^{\mathsf{H}} \end{pmatrix} \begin{pmatrix} \mathbb{1}_N & B & C & CB \end{pmatrix}, \tag{I.31}
$$

where B, C are normal commuting operators in the third subsystem.

It is easy to check that this canonical form is SPPT by definition. Forward, it is also an SSPPT state.

4.2 SPPT states in $N_1 \otimes N_2 \otimes N_3$ tripartite system

In this subsection we will extend the SPPT states to general tripartite system $N_1 \otimes N_2 \otimes N_3$. The basic idea is to require ρ to be SPPT under the bi-partition $A_1 : A_2A_3$ and $A_1A_2 : A_3$ simultaneously.

Let ρ be the density matrix with a decomposition $\rho = X^{\mathsf{H}} X$ in the tripartite system $N_1 \otimes N_2 \otimes N_3$.

Under the bipartite partition $A_1 : A_2A_3$, X can be written as an $N_1 \times N_1$ block matrix:

$$X = \begin{pmatrix} X_1 & S_{12}X_1 & \cdots & S_{1N_1}X_1 \\ 0 & X_2 & \cdots & S_{2N_1}X_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{N_1} \end{pmatrix}. \tag{I.32}$$

For consistent, let

$$S_{ij} = \begin{cases} 0, & \text{if } i > j, \\ \mathbb{1}_{N_2 N_3}, & \text{if } i = j. \end{cases} \tag{I.33}$$

Similarly, X_i can be written as an $N_2 \times N_2$ block matrix:

$$X_i = \begin{pmatrix} X_{i1} & S^2_{i,12}X_{i1} & \cdots & S^2_{i,1N_2}X_{i1} \\ 0 & X_{i2} & \cdots & S^2_{i,2N_2}X_{i2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{iN_2} \end{pmatrix}$$

Here the superscript 2 in the matrices $S^2_{i,kl}$ indicates the subsystem A_2 and let

$$S^2_{i,kl} = \begin{cases} \mathbb{1}_{N_3}, & \text{if } k = l; \\ 0, & \text{if } k > l. \end{cases}$$

In order to be compatible with the SPPT structure in the bipartite system $A_1A_2 : A_3$, we require S_{ij} to be a block diagonal matrix,

$$S_{ij} = \begin{pmatrix} S^1_{ij,1} & 0 & \cdots & 0 \\ 0 & S^1_{ij,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S^1_{ij,N_2} \end{pmatrix}. \tag{I.34}$$

Hence

$$\rho = X^{\mathsf{H}} X, \tag{I.35}$$

where the $(N_2(i-1)+j, N_2(k-1)+l)$-th entry of X is

$$S^1_{ij,k} S^2_{i,kl} X_{ik}.$$

Let

$$Y_1 = \begin{pmatrix} X_1 & S_{12}^{\mathsf{H}} X_1 & \cdots & S_{1N_1}^{\mathsf{H}} X_1 \\ 0 & X_2 & \cdots & S_{2N_1}^{\mathsf{H}} X_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_{N_1} \end{pmatrix}, \tag{I.36}$$

and Y_2 be an $N_1N_2 \times N_1N_2$ block matrix whose $(N_2(i-1)+j, N_2(k-1)+l)$-entry is

$$(S^2_{i,kl})^{\mathsf{H}} \, (S^1_{ij,k})^{\mathsf{H}} \, X_{ik}. \tag{I.37}$$

Now we are ready to define the SPPT state in general tripartite system with the matrices introduced above.

Definition 6. *Let ρ be the density matrix in the tripartite system $A_1 : A_2 : A_3$, and has a decomposition of the form as in Eq. (I.35). Then ρ is said to be SPPT if*

$$\rho^{\mathsf{T}_1} = Y_1^{\mathsf{H}} Y_1, \tag{I.38}$$

and

$$\rho^{\mathsf{T}_{1,2}} = Y_2^{\mathsf{H}} Y_2. \tag{I.39}$$

Note that the above conditions required for SPPT are equivalent to the following matrix equations:

$$\begin{aligned} \sum_{i=1}^{N_1} X_i^{\mathsf{H}}[S_{ip}, S_{iq}^{\mathsf{H}}]X_i = 0, \\ \sum_{i=1}^{N_1}\sum_{k=1}^{N_2} X_{ik}^{\mathsf{H}}[S_{ij,k}^1 S_{i,kl}^2, (S_{ij',k}^1 S_{i,kl'}^2)^{\mathsf{H}}]X_{ik} = 0. \end{aligned} \tag{I.40}$$

It is clear from the definition that SPPT states defined here are indeed PPT, i.e. positive under any partial transpose. Moreover, Eq. (I.40) will be satisfied if all the commutators vanishes. Similarly to the previous subsection, these states are called SSPPT.

Definition 7. *Let ρ be a state in tripartite system $N_1 \otimes N_2 \otimes N_3$ with a decomposition $\rho = X^{\mathsf{H}}X$ of the form as in Eq. (I.35). Then ρ is said to be SSPPT if*

$$\begin{aligned} [S_{ip}, S_{iq}^{\mathsf{H}}] = 0, \\ [S_{ij,k}^1 S_{i,kl}^2, (S_{ij',k}^1 S_{i,kl'}^2)^{\mathsf{H}}] = 0. \end{aligned} \tag{I.41}$$

Now we can show that this SSPPT can guarantee the separability just as in the bipartite system.

Theorem 2. *Every SSPPT state in tripartite system is separable.*

Proof. Let ρ be an SPPT in the tripartite system $N_1 \otimes N_2 \otimes N_3$, which possesses a decomposition as in Eq. (I.35). It follows from the condition (I.41) that S_{ij} and $S_{ij'}$ are mutually commuting normal for any given i.

In particular, for any i, k,

$$[S_{ij,k}^1, S_{ij',k}^1] = 0. \tag{I.42}$$

Note that if we put $l = k$ and $j' = i$ in Eq. (I.41), then for any given i, k we have

$$[S_{ij,k}^1, S_{i,kl'}^2] = 0. \tag{I.43}$$

In the similar way, let $j = i$ and $j' = i$ in Eq. (I.41), we obtain,

$$[S_{i,kl'}^2, S_{i,kl}^2] = 0. \tag{I.44}$$

Therefore we have a simultaneous diagonalizations:

$$\begin{aligned} S^1_{ij,k} &= U_{ik}\Lambda^1_{ijk}U^{\mathsf{H}}_{ik}, \\ S^2_{i,kl} &= U_{ik}\Lambda^2_{ikl}U^{\mathsf{H}}_{ik}, \end{aligned} \tag{I.45}$$

where U_{ik} are unitary matrices and $\Lambda^1_{ijk}, \Lambda^2_{ikl}$ are diagonal matrices with

$$\begin{aligned} \Lambda^1_{ijk} &= \mathrm{diag}(\lambda^1_{ijk1}, \lambda^1_{ijk2}, \ldots, \lambda^1_{ijkN_3}), \\ \Lambda^2_{ikl} &= \mathrm{diag}(\lambda^2_{ikl1}\,, \lambda^2_{ikl2}, \ldots,\, \lambda^2_{iklN_3}). \end{aligned}$$

Let

$$\begin{aligned} U &= \begin{pmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{N_1} \end{pmatrix}, \\ \widetilde{X} &= \begin{pmatrix} X_{11} & X_{12} & \cdots & Y_{1N_1} \\ X_{21} & X_{22} & \cdots & Y_{2N_1} \\ \vdots & \vdots & \ddots & \vdots \\ Y_{N_1,1} & Y_{N_1,2} & \cdots & Y_{N_1,N_1} \end{pmatrix}, \end{aligned} \tag{I.46}$$

where

$$U_i = \begin{pmatrix} U_{i1}^{\mathsf{H}} & 0 & \cdots & 0 \\ 0 & U_{i2}^{\mathsf{H}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_{i,N_2}^{\mathsf{H}} \end{pmatrix},$$

$$Y_{ij} = \begin{pmatrix} \Lambda_{ij1}^1 \Lambda_{i11}^2 X_{i1} & \cdots & \Lambda_{ij1}^1 \Lambda_{i1N_2}^2 X_{i1} \\ \vdots & \ddots & \vdots \\ \Lambda_{ijN_2}^1 \Lambda_{iN_21}^2 X_{iN_2} & & \Lambda_{ijN_2}^1 \Lambda_{iN_2N_2}^2 X_{iN_2} \end{pmatrix},$$

$$\widetilde{X}_{ik} = U_{ik}^{\mathsf{H}} X_{ik}.$$

Since U is unitary and $X = U\,\widetilde{X}$, we have $\rho = \widetilde{X}^{\mathsf{H}} \widetilde{X}$.

Suppose $\widetilde{X}_{ik} = (a_{ik1}, a_{ik2}, \ldots, a_{ikN_3})^{\mathsf{T}}$ where each a_{ikl} is a row vectors in $\mathbb{C}^{N_3}$ space. Now consider the $n(i,k,p)$-th row of $\widetilde{X}$, which is denoted by v_{ikp}. Then we have

$$v_{ikp} = w_{ikp} \otimes a_{ikp}, \tag{I.47}$$

where

$$\begin{aligned} w_{ikp} &= (y_{ikp1}, y_{ikp2}, \ldots, y_{ikpN_2}), \\ y_{ikpj} &= (\lambda_{ijkp}^1 \lambda_{ik1p}^2, \lambda_{ijkp}^1 \lambda_{ik2p}^2, \ldots, \lambda_{ijkp}^1 \lambda_{ikN_2p}^2) \\ &= \lambda_{ijkp}^1 (\lambda_{ik1p}^2, \lambda_{ik2p}^2, \ldots, \lambda_{ikN_2p}^2). \end{aligned}$$

It follows that each v_{ikp} is a product vector,

$$\begin{aligned} v_{ikp} =& (\lambda_{i1kp}^1, \ldots, \lambda_{iN_1kp}^1) \\ & \otimes (\lambda_{ik1p}^2, \ldots, \lambda_{ikN_2p}^2) \otimes a_{ikp}. \end{aligned} \tag{I.48}$$

Therefore ρ is separable. ∎

This proof can also be utilized as a method to find the separability decomposition of SSPPT states in tripartite system. Now we end this subsection by giving some examples of tripartite SPPT states.

Example 2. *Recall that a state ρ on $N_1 \otimes N_2$ is said to be a CQ state [22] if it has the form*

$$\rho = \sum_{i=1}^{N_1} p_i |i\rangle \langle i| \otimes \rho_i^{A_2}, \tag{I.49}$$

Where $\rho_i^{A_2}$ are density matrices in A_2 subsystem. It was proved that any CQ state is in fact SSPPT [22]. Similarly, we construct a class of SPPT states in tripartite system $A_1 : A_2 : A_3$ as follows,

$$\rho = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} p_{ij} |ij\rangle \langle ij| \otimes \rho_{ij}^{A_3}, \tag{I.50}$$

where $\rho_{ij}^{A_3}$ are density matrices in subsystem A_3. This is in fact an SSPPT states with $S^1_{ij,k} = \delta_{ij} I$, $S^2_{i,kl} = \delta_{kl} I$ and $X_{ik} = p_{ik}\rho_{ik}^{A_3}$.

Example 3. *Let ρ be a PPT state in $N_1 \otimes N_2 \otimes N_3$. And $\mathrm{rank}(\langle 00|\rho\rangle\, 00) = \mathrm{rank}(\rho) = N_3$. It was prove that ρ can be transformed into the following canonical form by using a reversible local operator [27]:*

$$\rho = T^{\mathsf{H}} T, \tag{I.51}$$

where

$$T = (\mathbb{1}_{N_3}, A_2, \dots, A_{N_1}) \otimes (\mathbb{1}_{N_3}, B_2, \dots, B_{N_2}). \tag{I.52}$$

Moreover A_i, B_i are a set of mutually commuting normal matrices.

Now we show this canonical form is actually an SSPPT state.

Assume $A_1 = B_1 = \mathbb{1}_{N_3}$ and $X_{ij} = \mathbb{1}_{N_3}, \forall i, j$. Let $S^1_{1,j,1} = A_j$ and $S^2_{1,1l} = B_l$ and all the other $S^1_{ij,k}, S^2_{i,kl}$ are zero matrices. Then T coincides with X in Eq. (I.35). Since all the $S^1_{ij,k}$ and $S^2_{i,kl}$ are mutually commuting normal, ρ is SSPPT.

4.3 SPPT states in multipartite system

In this subsection, we will finally give the definition of SPPT in $(d+1)$-particle system $N_1 \otimes N_2 \otimes \cdots \otimes N_d \otimes N_{d+1}$ where $d > 1$. To begin with, we will fix some notations for representing matrices in the multipartite system.

Let $\alpha_n = (i_1, i_2, \dots, i_n)$ be the n-tuple where $i_k \in \{1, 2, \dots, N_k\}, k \in \{1, 2, \dots, n\}, n \in \{1, 2 \dots, d\}$. Similarly, let $\beta_n = (j_1, j_2, \dots, j_n)$. And i_n, j_n correspond to the n-th subsystem unless no otherwise stated. For simplicity, vector α_n is used to represent the multiple indexes $(i_1, i_2, \ldots, i_n)$. For example,

$$
\begin{aligned}
X_{\alpha_n} &= X_{i_1,i_2,\ldots,i_n}, \\
S_{\alpha_n,j_m} &= S_{i_1,i_2,\ldots,i_m,j_m,i_{m+1},\ldots,i_n}, m \leqslant n.
\end{aligned}
\tag{I.53}
$$

Recall the mapping π defined in the preliminary section, which is used to represent the entries of a block matrix. For instance, (α_n, β_n)-th entry of a matrix denotes the element in the $\pi(\alpha_n)$-th row and $\pi(\beta_n)$-th column of a matrix.

Let $\rho = X^{\mathsf{H}} X$ be a density matrix in the $N_1 \otimes N_2 \otimes \cdots \otimes N_d \otimes N_{d+1}$ system. Consider the following upper triangular block matrix X, whose (α, β)-th is

$$
\prod_{p=1}^{d} S^p_{\alpha,j_p} X_\alpha, \quad S^p_{\alpha,j_p}, X_\alpha \in \mathbb{C}^{N_{d+1} \times N_{d+1}}, \tag{I.54}
$$

where

$$
\begin{aligned}
S^p_{\alpha_n,j_p} &= \operatorname{diag}(S^p_{\alpha_n,1,j_p}, \ldots, S^p_{\alpha_n,N_{n+1},j_p}), \\
X_{\alpha_n} &= \begin{pmatrix}
S^{n+1}_{\alpha_n,1,1} X_{\alpha_n,1} & \cdots & S^{n+1}_{\alpha_n,1,N_{n+1}} X_{\alpha_n,1} \\
\vdots & \ddots & \vdots \\
S^{n+1}_{\alpha_n,N_{n+1},1} X_{\alpha_n,N_{n+1}} & \cdots & S^{n+1}_{\alpha_n,N_{n+1},N_{n+1}} X_{\alpha_n,N_{n+1}}
\end{pmatrix}.
\end{aligned}
$$

Note that $S^{n+1}_{\alpha_n,i_{n+1},j_{n+1}}$ and $X_{\alpha_n,i_{n+1}}$ can be written as $S^{n+1}_{\alpha_{n+1},j_{n+1}}$ and $X_{\alpha_{n+1}}$ respectively. Let

$$
S^p_{\alpha_n,j_p} = \begin{cases}
\mathbb{1}, & \text{if } j_p = i_p, \\
0, & \text{if } j_p < i_p.
\end{cases}
\tag{I.55}
$$

Moreover, let

$$
Y_n = \begin{pmatrix}
(S^{n+1}_{\alpha_n,1,1})^{\mathsf{H}} X_{\alpha_n,1} & \cdots & (S^{n+1}_{\alpha_n,1,N_{n+1}})^{\mathsf{H}} X_{\alpha_n,1} \\
\vdots & \ddots & \vdots \\
0 & \cdots & (S^{n+1}_{\alpha_n,N_{n+1},N_{n+1}})^{\mathsf{H}} X_{\alpha_n,N_{n+1}}
\end{pmatrix},
$$

for $n = 1, 2, \ldots, d$. Hence the (α_n, β_n)-th entry of Y_n is

$$\left(\prod_{p=1}^{n} S^p_{\alpha_n, j_p}\right)^{\mathsf{H}} X_{\alpha_n}.$$

If ρ is SPPT under any bi-partition $A_1 \cdots A_n : A_n \ldots A_{d+1}$, then it must be a PPT state. Following this idea, we give the definition of SPPT in multipartite case.

Definition 8. *Let ρ be the density matrix in the $(d+1)$-body system $N_1 \otimes \cdots \otimes N_d \otimes N_{d+1}$ with the decomposition $\rho = X^{\mathsf{H}} X$ of the form as in Eq. (I.54). Then ρ is said to be SPPT if*

$$\rho^{\mathsf{T}_{1,2,\ldots,n}} = Y_n^{\mathsf{H}} Y_n, \quad n = 1, 2, \ldots, d. \tag{I.56}$$

Alternatively, the conditions required above can be reformulated as

$$\sum_{\alpha_n} X^{\mathsf{H}}_{\alpha_n} \left[\prod_{p=1}^{n} S^p_{\alpha_n, j_p}, \left(\prod_{q=1}^{n} S^q_{\alpha_n, j'_q}\right)^{\mathsf{H}} \right] X_{\alpha_n} = 0, \tag{I.57}$$

for any $\beta_n = (j_1, j_2, \ldots, j_n)$, $\beta'_n = (j'_1, j'_2, \ldots, j'_n)$ and $n = 1, 2, \ldots, d$.

The following theorem shows that this generalization of SPPT preserves the PPT property.

Theorem 3. *Any SPPT state is PPT.*

Proof. Since any partial transpose can be written as the combination of $\mathsf{T}_{1,2,\ldots,n}, n = 1, 2, \ldots, n$ and T, the proof completes by Eq. (I.56). ∎

In a similar way, the SSPPT states in the multipartite system are introduced as follows.

Definition 9. *Let $\rho = X^{\mathsf{H}} X$ be an SPPT state, where X has the form as in Eq. (I.54). Then ρ is said to be SSPPT if*

$$\left[\prod_{p=1}^{n} S^p_{\alpha_n, j_p}, \left(\prod_{q=1}^{n} S^q_{\alpha_n, j'_q}\right)^{\mathsf{H}} \right] = 0, \tag{I.58}$$

for any $\alpha_n, \beta_n, \beta'_n$ and $n = 1, 2, \ldots, d$.

The following theorem shows that SSPPT guarantees the separability in the multipartite system.

Theorem 4. *Any SSPPT state is separable.*

Proof. Let ρ be the density matrix in $N_1 \otimes \cdots N_d \otimes N_{d+1}$. Suppose $\rho = X^{\mathsf{H}} X$ with X being of the form as Eq. (I.54).

Fix α, for any k and l, choose β_n and β_n' such that

$$j_p = i_p, p = 1, \ldots, n, p \neq k;$$
$$j_q' = i_q, q = 1, \ldots, n, q \neq l.$$

It follows from Eq. (I.55) that

$$S^p_{\alpha_n, j_p} = S^q_{\alpha_n, j_q'} = \mathbb{1}, p \neq k, q \neq l. \tag{I.59}$$

Consider the condition (I.58), we have

$$\left[S^k_{\alpha_n, j_k} S^l_{\alpha_n, j_l'} \right] = 0. \tag{I.60}$$

Forward by the structure of $S^p_{\alpha_n, j_p}$ as in Eq. (I.54), we have

$$\left[S^k_{\alpha_d, j_k} S^l_{\alpha_d, j_l'} \right] = 0. \tag{I.61}$$

Note that there is no requirement for j_k and j_l' when choosing β and β', hence $\{S^p_{\alpha_d, j_p}\}_{p, j_p}$ is a set of mutually commuting normal matrices for fixed α. Consequently, we have the simultaneous diagonalizations

$$S^p_{\alpha, j_p} = U_\alpha \Lambda^p_{\alpha, j_p} U^{\mathsf{H}}_\alpha,$$

where U_α is an $N_{d+1} \times N_{d+1}$ unitary matrix and Λ^p_{α, j_p} are diagonal matrices:

$$\Lambda^p_{\alpha, j_p} = \mathrm{diag}(\lambda^p_{\alpha, 1, j_p}, \lambda^p_{\alpha, 2, j_p}, \ldots, \lambda^p_{\alpha, N_{d+1}, j_p}).$$

Let U be the block matrix whose (α, α)-th entry is U_α and all other entries are zeros. Put $\widetilde{X} = UX$ and $\widetilde{X}_\alpha = U_\alpha X_\alpha$, then $\rho = \widetilde{X}^{\mathsf{H}} \widetilde{X}$.

Let

$$\widetilde{X}_\alpha = \begin{pmatrix} a_{\alpha,1} \\ a_{\alpha,2} \\ \vdots \\ a_{\alpha,N_{d+1}} \end{pmatrix},$$

where each a_{α,i_0} is a row vector in $\mathbb{C}^{N_{d+1}}$. Note that (α,β)-th entry of $\widetilde{X}$ is

$$\left(\prod_{p=1}^{n} \Lambda_{\alpha,j_p}\right) \widetilde{X}_\alpha \ .$$

Hence we have

$$v_{\alpha,i_0} = w_{\alpha,i_0} \otimes a_{\alpha,i_0}, \tag{I.62}$$

where

$$\begin{aligned} w_{\alpha,i_0} &= \otimes_{p=1}^{d} y^p_{\alpha,i_0}, \\ y^p_{\alpha,i_0} &= (\lambda^p_{\alpha,i_0,1}, \lambda^p_{\alpha,i_0,2}, \ldots, \lambda^p_{\alpha,i_0,N_p}) \in \mathbb{C}^{N_p}. \end{aligned}$$

Now that each row of $\widetilde{X}$ is a product vector, it follows that ρ is separable. ■

The following lemma shows an example of SPPT state in the general multipartite system.

Lemma 5. *A pure state is separable if and only if it is SPPT.*

Proof. Since any pure PPT state is separable, it suffices to prove that pure product state is indeed SPPT states. Let ρ be a pure state in $N_1 \otimes \cdots N_d \otimes N_{d+1}$ system. Then ρ can be written as

$$\begin{aligned} \rho &= vv^{\mathsf{H}}, v = w^{\mathsf{H}}, \\ w &= (\otimes_{i=1}^{d} w_i) \otimes w_0, \\ w_p &= (w_{p,1}, w_{p,2}, \ldots, w_{p,N_p}), 1 \leqslant p \leqslant d. \end{aligned} \tag{I.63}$$

Let $\alpha_1 = (1,1,\ldots,1)$ and X_{α_1} be an $N_{d+1} \times N_{d+1}$ matrix whose first row is w_0 and all other entries are zeros. Consider w as a block vector with each block being an N_{d+1} dimensional row vector, then the α-entry of w is $\prod_{p=1}^{d} w_{p,j_p} w_0$.

Let $\beta = (j_1, j_2, \ldots, j_d)$ and

$$S^p_{\alpha_1, j_p} = \mathrm{diag}(w_{p,j_p}, 0, \ldots, 0). \tag{I.64}$$

For any other $\alpha \neq \alpha_1$, let $S^p_{\alpha, j_p} = 0$ and $X_\alpha = 0$. Then $\rho = X^{\mathsf{H}} X$, where the (α, β)-th entry of X is

$$\left(\prod_{p=1}^{d} S^p_{\alpha, j_p} \right) X_\alpha, \tag{I.65}$$

which has the same structure as that in the definition of SPPT. Moreover, $S^p_{\alpha,p}$ are normal commuting matrices, which implies ρ is SSPPT. ■

We end this subsection by giving another example in the multipartite system.

Example 4. *It was proved that any PPT state supported in $N_1 \otimes N_2 \otimes \cdots \otimes N_d \otimes N_{d+1}$ is separable [27] if*

$$\begin{aligned} \mathrm{rank}(\rho) &= \mathrm{rank}(\langle 0_1, 0_2, \ldots, 0_d | \, \rho \, | 0_1, 0_2, \ldots, 0_d \rangle) \\ &= N_{d+1}. \end{aligned} \tag{I.66}$$

And it has a canonical form by using a reversal local operator:

$$\rho = T^{\mathsf{H}} T, \tag{I.67}$$

where

$$\begin{aligned} T =& (D^1_1, D^1_2, \ldots, D^1_{N_1}) \otimes (D^2_1, D^2_2, \ldots, D^2_{N_1}) \cdots \\ & \otimes (D^d_1, D^d_2, \ldots, D^d_{N_d}). \end{aligned}$$

Here $D^i_1 = \mathbb{1}$ and D^q_p are a set of mutually commuting normal matrices.

Suppose that X has the form as in Eq. (I.54). Let $\alpha_0 = (1, 1, \ldots, 1)$ be a d-tuple. Put $S^p_{\alpha_0, j_p} = D^p_{j_p}$ for any p, $S^p_{\alpha, j_p} = 0$ for all other $\alpha \neq \alpha_0$, and $X_{\alpha_0} = \mathbb{1}$. Simple calculation follows that $\rho = X^{\mathsf{H}} X$. Note that S^p_{α, j_p} are all mutually normal commuting, hence it is SSPPT.

5 Sufficient separability conditions of SPPT states

In this section, we will consider the separability conditions for SPPT states.

Let ρ be a density matrix in $2 \otimes d$ system with a block Cholesky decomposition,

$$\rho = X^{\mathsf{H}} X,$$
$$X = \begin{pmatrix} X_1 & SX_1 \\ 0 & X_2 \end{pmatrix}. \tag{I.68}$$

Note that ρ is SPPT if

$$X_1^{\mathsf{H}}(S_1 S_1^{\mathsf{H}} - S_1^{\mathsf{H}} S_1) X_1 = 0. \tag{I.69}$$

It has been proved that SPPT states in $2 \otimes 4$ system is separable [14]. In fact we have the following conclusion.

Lemma 6. *Let ρ be an SPPT state of the form as in Eq. (I.68). Then ρ is separable in either of the following cases:*

1. $d \leqslant 4$;
2. $\operatorname{rank}(X_1) = d$.

The second condition can be further improved as follows.

Lemma 7. *Let ρ be an SPPT state of the form as in Eq. (I.68). Then ρ is separable if* $\operatorname{Im}(S) \subset \operatorname{Im}(X_1)$ *or* $\operatorname{Im}(S^{\mathsf{H}}) \subset \operatorname{Im}(X_1)$.

Proof. It suffices to prove for the case $\operatorname{Im}(S) \subset \operatorname{Im}(X_1)$ since otherwise we can consider the partial transposed state ρ^{T_1}.

Suppose $\operatorname{rank}(X) < d$, then it has an SVD decomposition:

$$X_1 = U \Lambda V^{\mathsf{H}} = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^{\mathsf{H}}, \tag{I.70}$$

where U, V are unitary matrices and Σ is a diagonal matrix with dimension less than d. Let

$$\sigma = Y^{\mathsf{H}} Y,$$
$$Y = \begin{pmatrix} \Gamma & U^{\mathsf{H}} S U \Gamma \\ 0 & X_2 V \end{pmatrix}. \tag{I.71}$$

Then $\rho = (\mathbb{1} \otimes V)\sigma(\mathbb{1} \otimes V^{\mathsf{H}})$. Simple calculation gives that σ is also SPPT. Write S in block matrix form according to that of Λ:

$$S = \begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix}. \tag{I.72}$$

Note that

$$\mathrm{Im}(S) \subset \mathrm{Im}(X_1) \Leftrightarrow \mathrm{Im}(U^{\mathsf{H}}SU) \subset \mathrm{Im}(\Lambda).$$

It follows that $S_3 = 0, S_4 = 0$. By condition (I.69), $S_4 = 0$ and S_1 is normal. Therefore S is normal, which implies ρ is separable. ■

Another sufficient condition for separability of $2 \otimes d$ SPPT state was given in Ref. [10].

Lemma 8. *Let*

$$\rho = \begin{pmatrix} A & B \\ B^{\mathsf{H}} & D \end{pmatrix} \tag{I.73}$$

be a density matrix in the $2 \otimes d$ system. If $A > D$, then ρ is SSPPT and thus separable.

Noted that, when ρ is written in Eq. (I.68), the sufficient condition in the above lemma is equivalent to

$$X_1^{\mathsf{H}} X_1 > X_1^{\mathsf{H}} S^{\mathsf{H}} S X_1 + X_2^{\mathsf{H}} X_2.$$

We can further relax the condition by:

Lemma 9. *Let $\rho = X^{\mathsf{H}} X$ with X being of the form as in Eq. (I.68) in $2 \otimes d$ system. Then ρ is separable if*

$$X_1^{\mathsf{H}} X_1 > X_1^{\mathsf{H}} S^{\mathsf{H}} S X_1.$$

Proof. Now ρ can be written as

$$\rho = \begin{pmatrix} X_1^{\mathsf{H}} S^{\mathsf{H}} S X_1 & X_1^{\mathsf{H}} S X_1 \\ X_1^{\mathsf{H}} S^{\mathsf{H}} X_1 & X_1^{\mathsf{H}} S^{\mathsf{H}} S X_1 \end{pmatrix} + \begin{pmatrix} X_1 X_1^{\mathsf{H}} - X_1^{\mathsf{H}} S^{\mathsf{H}} S X_1 & 0 \\ 0 & X_2^{\mathsf{H}} X_2 \end{pmatrix}.$$

The former term is a positive Toeplitz block matrix which is separable by the Proposition 1 in Ref. [12]. Since the latter term is separable, ρ is separable. ■

To sum up the conditions in term of S, we have the following corollary.

Corollary 10. *Let ρ be an SPPT state of the form as in Eq. (I.68). Then ρ is separable if S is in any of the following cases:*

1. *S is contractive,*
2. *S is normal,*
3. $\mathrm{Im}(S) \subset \mathrm{Im}(X_1)$,
4. *Dimension of S is less than or equal to 4.*

Kil-Chan Ha constructed a $2 \otimes 5$ SPPT state which is entangled [13]. Here we study further about $2 \otimes 5$ SPPT states. Before going further, we introduce the definition of the edge state.

Definition 10. *Let σ be a bipartite state. It is said to be an edge state if there does not exist $|x, y\rangle$ such that*

$$\begin{aligned} &|x, y\rangle \in \mathcal{R}(\sigma), \\ &|\overline{x}, y\rangle \in \mathcal{R}(\sigma^{\mathsf{T}_1}). \end{aligned} \tag{I.74}$$

Theorem 11. *ρ is an SPPT state in $2 \otimes 5$ system of the form as in Eq. (I.68), then ρ is separable except the following case:*

$$\begin{cases} \mathrm{rank}(X_1) = 4, \\ \mathrm{rank}(\sigma) = \mathrm{rank}(\sigma^{\mathsf{T}_1}) = 5, \\ \sigma \text{ is an edge state,} \end{cases} \tag{I.75}$$

where

$$\sigma = W^{\mathsf{H}} W, \quad W = \begin{pmatrix} X_1 & SX_1 \end{pmatrix}.$$

Proof. Let $r = \mathrm{rank}(X_1)$. By Lemma 6, ρ is separable if it has full rank. Hence we assume $r \leqslant 4$. Consider σ, an SPPT state supported in $2 \otimes r$ subspace. If $r < 4$, then it is separable by the Peres-Horodecki criterion. Note that if σ is separable, so is ρ. We are thus able to assume $r = 4$.

By the PPT property, we have $\operatorname{rank}(\sigma) \geqslant r$. And σ is separable if $\operatorname{rank}(\sigma) = 4$ or $\operatorname{rank}(\sigma^{\mathsf{T}_1}) = 4$. Consider the size of W, σ is not separable only if $\operatorname{rank}(\sigma) = \operatorname{rank}(\sigma^{\mathsf{T}_1}) = 5$.

Consider the case $\operatorname{rank}(X_1) = 4$ and $\operatorname{rank}(\sigma) = \operatorname{rank}(\sigma^{\mathsf{T}_1}) = 5$. Since any $2 \otimes 4$ birank$(5,5)$ state is entangled if and only if it is an edge state. Hence ρ is separable when σ is not an edge state, which completes our proof. ■

Recall the $2 \otimes 5$ SPPT entangled state in Ha's paper [14],

$$X_1 = \begin{pmatrix} \mathbb{1}_4 & 0 \\ 0 & 0 \end{pmatrix}, S = \begin{pmatrix} 0 & 1 & 0 & 0 & \beta_1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \beta_2 \\ \beta_2 & 0 & 0 & \beta_1 & 0 \end{pmatrix}, \tag{I.76}$$

where $\beta_1 = \sqrt{(1-b)/2b}$ and $\beta_2 = \sqrt{(1+b)/2b}$ with $0 < b < 1$. Put $X_2 = 0$. Then the defined σ in Eq. (I.75) is

$$\sigma = \left(\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & \gamma_1 & 0 & 0 & \gamma_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \gamma_2 & 0 & 0 & \gamma_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), \tag{I.77}$$

where $\gamma_1 = (b+1)/2b, \gamma_2 = \sqrt{b^2-1}$. It is easy to check that σ is supported on $2 \otimes 4$ subspace and has birank $(5,5)$. By computing all the product vectors in the range of σ and σ^{T_1} respectively, σ is an edge state, which coincides with our theorem.

Furthermore, we have studied the rank 4 SPPT state.

Theorem 12. *Any SPPT state of rank less than or equal to 4 is separable.*

Proof. Since all the rank $1, 2,$ and 3 PPT states are separable, it only remains to consider the rank 4 states. It was proved in Refs. [4, 5] that any rank four PPT state is separable except in the $2 \otimes 2 \otimes 2$ and $3 \otimes 3$ systems. And in these cases, the state is separable if and only if its range contains a product vector. Therefore, it suffices to prove that $\mathcal{R}(\rho)$ contains a product vector in $2 \otimes 2 \otimes 2$ and $3 \otimes 3$ systems respectively.

Firstly, we consider the $2 \otimes 2 \otimes 2$ case. Let $\rho = X^{\mathsf{H}} X$ be an SPPT state in $2 \otimes 2 \otimes 2$ system, where X satisfies conditions (I.20) and (I.25). Note that ρ has a product vector in its range is equivalent to that X has a product vector in its row range.

If $X_{22} \neq 0$, then $\mathcal{R}(X^{\mathsf{T}})$ contains a product vector.

Let $\rho' = (X_{21}, T_2 X_{21})^{\mathsf{H}} (X_{21}, T_2 X_{21})$, then ρ' is a $2 \otimes 2$ state. It is known from Ref. [25] that any two dimensional subspace of $2 \otimes 2$ system always contains a product vector. If $\operatorname{rank}(X_{21}, T_2 X_{21}) = 2$, then ρ' contains a product vector in its range, namely u. Moreover, $(0,1) \otimes u$ is a product vector in the range of ρ. On the other hand, if $\operatorname{rank}(X_{21}, T_2 X_{21}) = 1$, we claim that ρ also contains a product vector in its range. Consider the SVD decomposition of X_{21}, denoted by $X_{21} = U \Sigma V^{\mathsf{H}}$. Then the $5,6$-th rows of X is

$$U \begin{pmatrix} 0 & 0 & 0 & 0 & \sigma & 0 & t_1\sigma & 0 \\ \\ 0 & 0 & 0 & 0 & 0 & 0 & t_2\sigma & 0 \end{pmatrix} V^{\mathsf{H}},$$

where

$$\Sigma = \begin{pmatrix} \sigma & 0 \\ \\ 0 & 0 \end{pmatrix}, U^{\mathsf{H}} T_2 U = \begin{pmatrix} t_1 & t_3 \\ \\ t_2 & t_4 \end{pmatrix}.$$

Note that $(0,0,0,0,\sigma,0,t_1\sigma,0)$ is always a product vector for any t_1.

Similar way, we can show that ρ contains a product vector in its range if $X_{12} \neq 0$.

However, $\operatorname{rank}(\rho) = 4$ contradicts with $X_{12} = 0, X_{21} = 0,$ and $X_{22} = 0$. It follows than ρ is separable.

Next, we consider the case when ρ is in $3 \otimes 3$ bipartite system. Let ρ be an SPPT state with

$$\rho = X^{\mathsf{H}} X,$$
$$X = \begin{pmatrix} X_1 & S_{12}X_1 & S_{13}X_1 \\ 0 & X_2 & S_{23}X_2 \\ 0 & 0 & X_3 \end{pmatrix}.$$

Here X is a block matrix with its entries being of size 3×3. If $X_3 \neq 0$, then ρ contains a product in its range. We can then assume $X_3 = 0$.

If $\operatorname{rank}(X_1) = 0$, it will contradicts with the condition $\operatorname{rank}(\rho) = 4$.

If $\operatorname{rank}(X_1) = 1$, we have

$$X_1 = \begin{pmatrix} \lambda_1 a \\ \lambda_2 a \\ \lambda_3 a \end{pmatrix}, \tag{I.78}$$

where a is a row vector in $\mathbb{C}^3$, $\lambda_i \in \mathbb{C}, i = 1, 2, 3$. Note that the row ranges of $S_{12}X_1$ and $S_{13}X_1$ are all contained in that of X_1, therefore $S_{12}X_1$ and $S_{13}X_1$ can be written as

$$S_{12}X_1 = \begin{pmatrix} \sigma_1 a \\ \sigma_2 a \\ \sigma_3 a \end{pmatrix}, S_{13}X_1 = \begin{pmatrix} \delta_1 a \\ \delta_2 a \\ \delta_3 a \end{pmatrix},$$

for some $\sigma_i, \delta_i \in \mathbb{C}$, $i = 1, 2, 3$. The first to third rows of X are

$$(\lambda_i a, \sigma_i a, \delta_i a) = (\lambda_i, \sigma_i, \delta_i) \otimes a, i = 1, 2, 3. \tag{I.79}$$

Note that $\lambda_i, \sigma_i, \delta_i$ cannot vanish at the same time, therefore ρ contains a product vector in its range.

Consider the case when $\operatorname{rank}(X_1) = 2$. Note that the rank of ρ is 4, then $1 \leqslant \operatorname{rank}(X_2) \leqslant 2$. If $\operatorname{rank}(X_2) = 1$, in the similar way to the case $\operatorname{rank}(X_1) = 1$, we can prove ρ is separable. Now we need only to consider the case when $\operatorname{rank}(X_2) = 2$. Let $\sigma = (0, X_2, S_{23}X_2)^{\mathsf{H}}(0, X_2, S_{23}X_2)$ which is supported in $2 \otimes 2$ subspace. Since any two dimensional subspace in $2 \otimes 2$ system contains at least one product vector [25], ρ must contain a product vector in its range.

If X_1 has full rank, then X_2 must be rank one. Similar to the discussion in the case when $\text{rank}(X_1) = 1$, we have that ρ contains a product vector in its range.

Above all, we conclude that any rank 4 SPPT state is separable. ■

6 Conclusion

We extend the concept of well-known SPPT states to the arbitrary multipartite system. We compare the difference between the definition of SPPT in Ref. [30] and ours. It turns out that our states can inherit the structure of PPT and many good properties as those in the bipartite system. For example, any SPPT states are separable, pure states are separable if and only if they are SPPT, and any SPPT state of rank 4 is separable. Besides, we also give some sufficient conditions for separability of SPPT states. In particular, for the $2 \otimes 5$ SPPT states, we showed that most of the states are separable except a special subclass. We hope our work will be helpful for investigating the structure of multipartite PPT states.

Chapter II

FIGURES AND TABLES

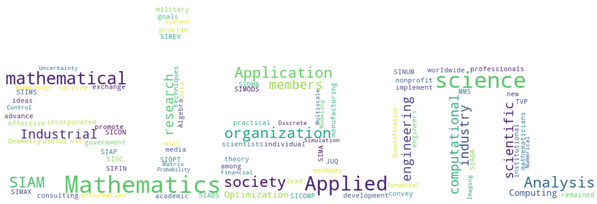

Figure 2.1: fig

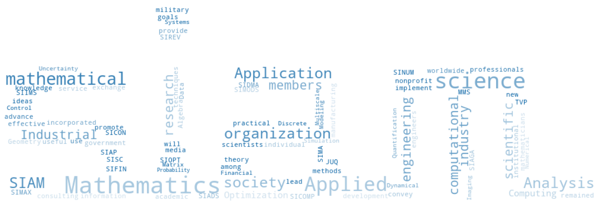

Figure 2.2: sfd

a	ads

Table 2.1: as

a	ads

Table 2.2: as

a	ads

Table 2.3: asasd

Algorithm 2.1: A example of the algorithms

Input : Parameters: i.
Output: Result: s.

$I \leftarrow \{1, 2, 3, \ldots, 100\}$
foreach $i \in I$ **do**
    $L_i \leftarrow i$
    $M_i \leftarrow e^i$
end
for $i = 1$ **to** 100 **do**
    $p^i \leftarrow \texttt{sum}(l_i, M_i)$
    if i *is even* **then**
        $p_i \leftarrow p_i - 1$
    end
end
for $i = 1 : 100$ **do**
    $q^i \leftarrow \texttt{sum}(e^{l_i}, e^{M_i})$
    if i *is even* **then**
        $q_i \leftarrow p_i - 1$
    end
end
while $i \neq 2$ **do**
    $i = i + 1$
end
$s \leftarrow p^i + q^i$

Code 1: matlab code

```matlab
%%        function to get the neareat product vector
%         input : w in tensor space of N times N space
%         output: x in N dimensional  real sapce
%                 y in N dimensional real space

function [x,y,e] = nearest_product(w,N_A)
    if isrow(w) % change to the column vector
        w = w';
    end

    n = length(w);
    N_B = n/N_A;
    if abs(round(N_B)-N_B)  > eps
        fprintf('the product of the sub dimensions must equal to dimension of w\n');
        return
    end

    w = reshape(w,N_B,N_A);
    w = w';
    [u,e,v] = svds(w,1);
    x = u;
    y = conj(v);
end
```

Chapter III

FIGURES AND TABLES

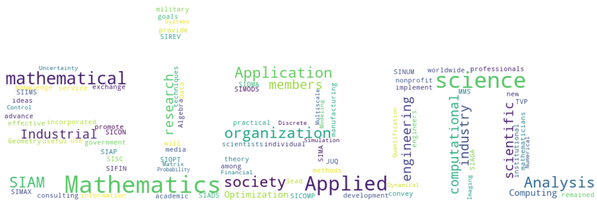

Figure 3.1: fig

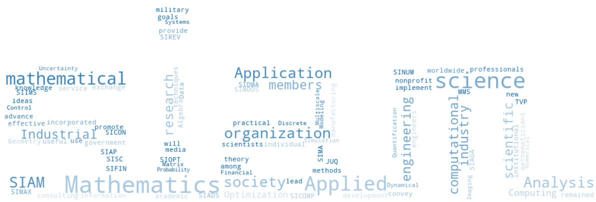

Figure 3.2: sfd

a	ads

Table 3.1: as

a	ads

Table 3.2: as

a	ads

Table 3.3: asasd

Algorithm 3.1: A example of the algorithms

Input : Parameters: i.
Output: Result: s.

$I \leftarrow \{1, 2, 3, \ldots, 100\}$
foreach $i \in I$ **do**
  $L_i \leftarrow i$
  $M_i \leftarrow e^i$
end
for $i = 1$ **to** 100 **do**
  $p^i \leftarrow \texttt{sum}(l_i, M_i)$
  if *i is even* **then**
    $p_i \leftarrow p_i - 1$
  end
end
for $i = 1 : 100$ **do**
  $q^i \leftarrow \texttt{sum}(e^{l_i}, e^{M_i})$
  if *i is even* **then**
    $q_i \leftarrow p_i - 1$
  end
end
while $i \neq 2$ **do**
  $i = i + 1$
end
$s \leftarrow p^i + q^i$

Code 2: matlab code

```matlab
%%       function to get the neareat product vector
%        input : w in tensor space of N times N space
%        output: x in N dimensional  real sapce
%                y in N dimensional real space

function [x,y,e] = nearest_product(w,N_A)
    if isrow(w) % change to the column vector
        w = w';
    end

    n = length(w);
    N_B = n/N_A;
    if abs(round(N_B)-N_B)  > eps
        fprintf('the product of the sub dimensions must equal to dimension
     of w\n');
        return
    end

    w = reshape(w,N_B,N_A);
    w = w';
    [u,e,v] = svds(w,1);
    x = u;
    y = conj(v);
end
```

Chapter IV

SAMPLE

Reference symbols: x, v, a, t, F,xx.

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