GEH1036/GEK1505 Tutorial 10 (week 12)

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    Go to the website, and scroll down, see the **Work** entry, in the "Tutor" project, click the link for more information. I will provide this slide and the source .tex file on that site.
- Course information
  - EVERY WEEK MONDAY 10:00-11:00 S16-0431

1. Lottery

Lottery

Question

10000 people each has 4-digit number. We will generator randomly 3 4-digit numbers for the first, second and third prize. The numbers for the three prize are independent, which may be all the same.

(i) at least one win first prize

(ii) at least one will win the any of the three prizes.

- (i) Consider one person, he/she win the first prize with probability $q = \frac{1}{10^4} = 10^{-4}$. He/She will not win with the probability $p = 1 - q = 1 - 10^{-4}$.

  All the 10000 people will not win the first prize with probability

  $$p^{10000} = (1-q)^{10000} \approx e^{10^4 q} = e^{-1} \approx 0.36787944117$$

  where we using the approximation $(1+x)^N \approx e^{Nx}$ provided x is very small. Answer is 0.63212055882.

Lottery

(ii) At least one will win any of the prizes.

- Note that the three kind of prizes are independent, then the probability that no one win any prizes is p^3.
- Thus the probability at list one win some prizes is $1-\left(p^3\right)^{10000} = 1-e^{-3} \approx 0.95021293163$.

2. Lottery continue

Lottery continue

Question

Same problem as previous when we have 20,000 and 30,000 persons.

- For 20,000 persons, the probability at least one win the first prize is
$$1 - p^{20000} \approx 1 - e^{-2} \approx 0.86466471676.$$
the probability at least one win any of the three prizes is
$$1 - p^{3\,2000} \approx 1 - e^{-6} \approx 0.99752124782.$$
- The respective answers for 30000 persons are
$$1 - e^{-3} \approx 0.95021293163 \quad 1 - e^{-9} \approx 0.99987659019.$$

3. Treasure boxes

Treasure boxes

Question

In a TV game show, Mr Brown has the chance to win the grand prize which is hidden in one of three treasure boxes. Mr Brown chooses one box. The host, who knows where the grand prize is, then opens one of the remaining boxes which is empty. She then gives Mr Brown the choice of sticking to his box or switching to the other box. Should Mr Brown switch?

- If he decide not to switch, then the probability he win the prize is $\frac{1}{3}$ since what the host do is after Brown's decision and won't affect the probability he choose the right one from three boxes.
- If he decide to switch after the host opening a empty box, it is equivalent that he choose two of the three boxes (Host help to open one), the probability is $\frac{2}{3}$.
- So Mr Brown should choose switching.

4. Winning number

Winning number

Question

In a certain game of chance, 7 winning numbers are drawn one by one without replacement from the set $A = \{1, 2, \dots 45\}$. A player picks 9 numbers from A. Find the probability that the player picks
(i) exactly 6 of the winning numbers.
(ii) exactly 5 of the winning numbers.

- For question (i), the sample space S is the set of all 9-element subsets of A, $|S| = \binom{45}{9}$.
- The number of elements in S which contain exactly 6 winning numbers is $\binom{7}{6}\binom{38}{3}$.
- The probability is then $\binom{7}{6}\binom{38}{3}/\binom{45}{9} \approx 0.000066638$.
- Similar for (ii), the probability is $\binom{7}{5}\binom{38}{4}/\binom{45}{9} \approx 0.001749243$.

Alternative way

- First we have chosen 9 numbers from A ahead, denoted by B, then we choose the 7 winning numbers. The question becomes to calculate the probability that exactly 6 of the 7 winning numbers are from B.
- Consider the sample space S, the set of 7 winning numbers. $|S| = \binom{45}{7}$.
- For the set of 7 winning numbers. 6 are from B and 1 from the other 36 numbers.
- Probability is $\binom{9}{6}\binom{36}{1}/\binom{45}{7}$.
- Similarly for (ii), the probability is $\binom{7}{6}\binom{36}{2}/\binom{45}{9}$.

5. Sending letters

Question

There are 4 letters to be sent to 4 different persons and there are 4 envelopes each bearing the name and address of the respective recipient. You put the letters into the envelopes at random, one letter into one envelope. Find the probability that
(i) exactly one letter is put into the correct envelope,
(ii) at least one letter is put into the correct envelope,
(iii) no letters are put into the correct envelopes.

- We assume that the correct envelope for the i^{th} letter is the i^{th} envelope, $i = 1, 2, 3, 4$. The sample space S is all the ways for putting the letters into the envelopes. Thus $|S| = 4!$.

Sending letters

- (i) only one letter is put into the correct envelop. We can choose this letter 4 ways: 1,2 ,3,4. Assume letter 1 is correct, then for (2,3,4), there are only two ways to put them into envelopes: (3,4,2) and (4,2,3). Similar for the other letters. Hence the probability is $4 \times 2/4! = 1/3$.
- (ii) At least one letter is put correctly.
  - One is correct: $4 \times 2 = 8$.
  - Two are correct: First having $\binom{4}{2}$ ways to choose the two letters. And assume $1, 2$ is correct, then there are only 1 way to put (3,4) wrongly. Totally : $\binom{4}{2} \times 1 = 6$.
  - Three is correct: this will imply that all of the letters are put correctly. Only 1 way.
  - Probability is $(8 + 6 + 1)/4! = 5/8$.
- (iii) No letter is put correctly. The probability is $1 - 5/8 = 3/8$.

6. Opening the Door

Opening the Door

Question

To open a door with n keys. Only one is the correct key. Try them randomly and discard the wrong keys after trying. Find the probability you will open the door in the

(i) 1st attempt

(ii) 2nd attempt

(iii) 3rd attempt.

- (i) Only need consider the first key. It is correct with probability of $1/n$.
- (ii) 1st key is wrong, 2nd key is correct. On the condition (i) is false, probability $p_1 = \frac{n-1}{n}$, 2nd key is correct with probability $p_2 = \frac{1}{n-1}$. So the final probability is $p_1 p_2 = \frac{n-1}{n}\frac{1}{n-1} = \frac{1}{n}$.
- (iii) Similarly we can calculate it on the condition (i) and (ii) all fail. The probability is $(1 - \frac{1}{n} - \frac{1}{n})\frac{1}{n-2} = \frac{1}{n}$.

Opening the Door

For the general case: calculate the probability If it takes k attempts to get the right key.

- The sample space is the set of sequences of distinct keys $x_1 x_2 \ldots x_k$. Thus $|S| = n(n-1)\cdots(n-k+1)$.
- This required event E consists of all sequences S of the form $x_1 x_2 \cdots x_{k-1} X$, where X is the unique correct key. Then $|E| = (n-1)(n-2)\cdots(n-k+1)$.
- Probability is $|E|/|S| = \frac{1}{n}$.

Or you can think in a easier way: we try the keys with a order, the probability that the correct key lies at the k^{th} place is $\frac{1}{n}$.

7. Birthday

Question

Assuming that there are 365 equally probable birthdays, find the probability that in a group of k persons, all of them will have different birthdays for $k = 3, 4, \dots 10$.

- Suppose k persons' birthdays are $(x_1, x_2, \dots, x_k)$, probable birthday of each person is 365 days. The sample space S has the size 365^k.
- The event E that the birthdays of k persons are all different is the set of sequences in which the x_i is are distinct. $|E| = \binom{365}{k} k!$.
- $P(E) = \frac{|E|}{|S|} = \frac{\binom{365}{k} k!}{365^k}$. Calculate we have the following table.

k	3	4	5	6	7	8	9	10
P(E)	0.9918	0.9836	0.9729	0.9595	0.9438	0.9257	0.9054	0.8831
P(E^c)	0.0082	0.0164	0.0271	0.0405	0.0562	0.0743	0.0946	0.1169

Birthday

Question

(i) Hence find the probability that in a group of k persons, at least two persons will have the same birthday for $k = 2, 3, \ldots 10$.
(ii) Find the smallest value of k for which the corresponding probability is more than 0.5.

(i) It is exactly $P(E^c)$.
(ii) The smallest value of k for which the corresponding probability $P(E^c)$ is greater that 0.5 is 23 and the probability is 0.508.

8. Independent event

Question

Let A be the event that John and Jane have different birthdays and B be the event that John and Jason have different birthdays while C be the event that Jason and Jane have different birthdays. (Assume that there are 365 days in a year.)

(a) Calculate P (A).

(b) Are the events A,B independent?

(c) Are the events A,B,C independent?

- (a) Arrange two days for John and Jane. Totally 365^2 choices for two days, and $A_{365}^2 = 365 \times 364$ for two different days. Hence $P(A) = \frac{364}{365}$.

Independent events

- It is easy to see that $P(A) = P(B) = P(C) = \frac{364}{365}$.
- (b) Check whether A, B are independent.
  - If $P(A \cap B) = P(A)P(B)$ then A,B are independent, otherwise, they are not independent.
  - $A \cap B$ means the event that John has different birthday with Jane and Jason. For the three days, first choose for John, it has 365 choices, then for Jane, cannot be same as John, it has 364 choices, same for Jason it has 364 choices. then $P(A \cap B) = \frac{365 \times 364 \times 364}{365^3} = P(A)P(B)$.
  - Therefore A,B are independent.
- Check whether A,B and C are independent.
  - Note that $A \cap B \cap C$ means that these three person have different birthdays. Then $P(A \cap B \cap C) = A_{365}^3 / 365^3$, which is not equal to $P(A)P(B)P(C)$.
  - Hence A,B and C are not independent.

9. Coins

Coins

Question

Five fair coins are tossed. If you are told that at least one head turned up, what is the conditional probability that there will be
(i) exactly 2 heads,
(ii) at least 2 heads.

- Let X be the number of heads in a toss of 5 coins. Then for $0 \leqslant k \leqslant 5$, $P(X = k) = \binom{5}{k}/2^5 = \binom{5}{k}/32$.
- The probability of the condition at least one is head is $P(X \geqslant 1) = 1 - P(0) = 1 - 1/32 = 31/32$.
- Exactly two is head: $P(X = 2) = \binom{5}{2}/32 = 10/32$.
- The conditional probability of (i) id $P(X = 2|X \geqslant 1) = \frac{P(X=2 \cap X \geqslant 1)}{P(X \geqslant 1)} = \frac{P(X=2)}{P(X \geqslant 1)} = \frac{10/32}{31/32} = \frac{10}{31}$.

Coins

The conditional probability of at least two heads under the condition of $X \geqslant 1$.

- Note that $\{X \geqslant 1\} \cap \{X \geqslant 2\} = \{X \geqslant 1\}$.
- $P(X \geqslant 2) = 1 - P(X = 0) - P(X = 1) = 1 - \frac{1}{32} - \frac{5}{32} = \frac{26}{32}$.
- Then the conditional probability of (ii) is

$$
\begin{aligned}
P(X \geqslant 2 | X \geqslant 1) &= \frac{P(\{X \geqslant 1\} \cap \{X \geqslant 2\})}{P(X \geqslant 1)} \\
&= \frac{P(X \geqslant 2)}{P(X \geqslant 1)} \\
&= \frac{26/32}{31/32} = \frac{26}{32}.
\end{aligned}
$$

10. Gender of children

Gender of children

Question

Assume that it is equally probable that a child is a girl or a boy. For Mr Lim's 6 children, find the probability that
(i) the last child is a boy if it is known that he has exactly one son.
(ii) he has exactly one son if it is known that the last child is a boy.

- Let A be the event that Mr Lim's last child is a boy and B be the event that Mr Lim has exactly one son.
- Then $A|B$ corresponds (i) and $B|A$ corresponds (ii).
- $P(A) = \frac{1}{2}$ and $P(B) = \binom{6}{1}/2^6 = \frac{6}{64}$.
- $A \cap B$ is the event that only the last child is boy. Then the gender of all the children is determined, i.e. 1 choice, $P(A \cap B) = \frac{1}{64}$.
- $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/64}{6/64} = \frac{1}{6}$.
- $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/64}{1/2} = \frac{1}{32}$.

11. Vowels in a text

Vowels in a a text

Question

It is found that the frequencies of occurrence of the vowels a, e, i, o, u in the English are 8.2, 12.7, 7, 7.5 2.8 % respectively. What is the expected number of vowels in a text consisting of 10,000 letters?

- Probability for vowels occurring in a letter is $p = 8.2\% + 12.7\% + 7\% + 7.5\% + 2.8\% = 38.2\%$.
- The expected number for 10000 letters is $E = 10000p = 3820.$

12. Expected number of tosses

Expected number of tosses

Question

A die is tossed repeatedly. What is the expected number of tosses in order to get a “6” ?

- Let X be the number if tosses required. In each toss the probability for “6” is $q = \frac{1}{6}$, different from “6” is $p = \frac{5}{6}$.
- For each integer k, $P(X = k) = p^{k-1}q$.
- Then the expectation of X is

$$
\begin{aligned}
E(X) = \sum_{k=0}^{\infty} kP(X = k) &= \sum_{k=0}^{\infty} kp^{k-1}q = \frac{q}{p}\sum_{k=1}^{\infty} kp^k \\
&= \frac{q}{p}\frac{p}{(1-p)^2} = \frac{q}{(1-p)^2} \\
&= 6.
\end{aligned}
$$

Expected number of tosses

Calculate $S_n = \sum_{k=1}^{n} p^k$.

$$
\begin{aligned}
pS_n &= \sum_{k=1}^{n} p^{k+1} = \sum_{l=2}^{n+1} p^l \text{(let l=k+1)} \\
&= \sum_{l=2}^{n} p^l + p^{n+1} \\
&= \sum_{k=1}^{n} p^k - p^1 + p^{n+1} \\
&= S_n - p^1 + p^{n+1}.
\end{aligned}
$$

Hence $S_n = \frac{p(1-p^n)}{1-p}$.

And $S = \sum_{k=1}^{\infty} = \lim_{n\to\infty} S_n = \frac{p}{1-p} \lim_{n\to\infty} (1-p^n) = \frac{p}{1-p}$.

Expected number of tosses

Calculate $G = \sum_{k=1}^{\infty} kp^k$.

$$pG = \sum_{k=1}^{\infty} kp^{k+1} = \sum_{l=2}^{\infty}(l-1)p^l \text{(let l=k+1)}$$
$$= \sum_{k=2}^{\infty}(k-1)p^k.$$

Hence

$$G - pG = \sum_{k=1}^{\infty} kp^k - \sum_{k=2}^{\infty}(k-1)p^k = \sum_{k=2}^{\infty} p^k + p$$
$$= \sum_{k=1}^{\infty} p^k = S = \frac{p}{1-p}.$$

Therefore, $G = \frac{p}{(1-p)^2}$.

13. Bet for car registration number

Bet for car registration number

Question

For 10 randomly chosen 2-digit numbers from 00 to 99, if at least two are the same, you win \$1, otherwise loss \$ 1. Is the game acceptable? What if you win \$ 2 or loss \$ 1 with same conditions as above?

- Let A be the event at least two numbers are the same. A^c is the event every two numbers are different.
- $|A^c| = A_{100}^{10}$. (Choosing 10 numbers and permuting the order).
- $P(A^c) = \frac{A_{100}^{10}}{100^{10}} \approx 0.628$. $P(A) = 1 - P(A^c) = 0.372$.
- Let X be the random variable of wining \$ 1 or -\$1 (loss).
- The expectation for X is $E(X) = 1 \times 0.372 + (-1) \times 0.628 = -0.256$.
- Thus you should not accept the game.

Bet for car registration number

If you win \$ 2 for that at least two numbers are the same and loss \$ 1 otherwise.

As we calculated already, the expectation of the dollars you win is $E(X) = 2 \times 0.372 + (-1) \times 0.628 = 0.120$, then you can accept the game since the expectation is positive, which means you will approximately win \$0.120 $\times n$ if betting for n times provided n is large enough.

14. Game of keys

Game of keys

Question

Two boxes, one containing \$10, other \$1. Two bunches of keys:

- A contains one key opens the \$10-box and one key can open the \$1-box and 4 keys cannot open neither;
- B contains two keys can open the \$10-box, two keys can open the \$1-box and 5 keys cannot open neither.

Which one of the following games would you choose to play? Why?

- Game A: You pay \$1. You select one key from Bunch A and one of two boxes. If the selected key opens the selected box, you keep whatever is in the box. Otherwise, you get nothing.
- Game B: You pay \$1.35. You select one key from Bunch B and one of the two boxes. If the selected key opens the selected box, you keep whatever is in the box. Otherwise, you get nothing.

Game of keys

We should choose the game having the smaller expected loss (or larger winning value).

- For game A, 6 keys, sample size is $2 \times 6 = 12$. The probabilities for opening \$1 and \$10 are both $\frac{1}{12}$. The probability opening nothing is $1 - \frac{2}{12} = \frac{5}{6}$.
- The expectation of the loss is
$$E_A = (1-1) \times \frac{1}{12} + (1-10) \times \frac{1}{12} + 1 \times \frac{5}{6} = \frac{1}{12} \approx 0.083.$$
- For game B, there are 9 keys, and sample size is $2 \times 9 = 18$.
- Probability for opening either box is the same, which is $\frac{2}{18} = \frac{1}{9}$. Probability for opening nothing is $1 - 2 \times \frac{1}{9} = \frac{7}{9}$.
- The expectation of the loss is
$$E_B = (1.35-1) \times \frac{1}{9} + (1.35-10) \times \frac{1}{9} + 1.35 \times \frac{7}{9} \approx 0.1278.$$
- There for we should choose game A.

Thank you for listening!