GEH1036/GEK1505 Tutorial 7(week 9)

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    Go to the website, and scroll down, see the **Work** entry, in the "Tutor" project, click the link for more information. I will provide this slide and the source .tex file on that site.
- Course information
  - EVERY WEEK MONDAY 10:00-11:00 S16-0431

1. Counting by fingers

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Question

A child counts from 1 to 500 on the fingers of her left hand by starting on the thumb, then moving to the next finger until she reaches the last finger, after which she reverses the direction of movement of the counting until she reaches the thumb, and so on. On which finger will the counting stop? What if she starts counting from the last finger?

- Denote the five fingers by 1,2,..., 5.

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- Denote the five fingers by 1,2,..., 5.
- The rule of counting is

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- We see that every 8 counting is a cycle. From 1(thumb) to 1(thumb).

1. Counting by fingers (Contd.)

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- And that $500 = 8 \times 62 + 4$, hence the counting will stop at finger 4, i.e. the ring finger.

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- If we count starting from the last finger. It is same as the case that we have counted 4 already. And we continue to count 500. Totally we are going to count 504 start from the 1 (thumb) finger.
- $504 = 500 + 4 = 8 \times 63$, so the counting stop at finger 2(index finger).

2. Particle moving

Question

A particle moves along the edges of the following graph starting from vertex 0 and moves along the path

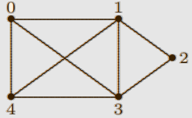
$$0 \to 1 \to 2 \to 3 \to 4 \to 0 \to 3 \to 1 \to 4 \to 0 \tag{1}$$

and subsequently moves along the path that is the reverse of the above path

$$0 \to 4 \to 1 \to 3 \to 0 \to 4 \to 3 \to 2 \to 1 \to 0 \tag{2}$$

and thereupon repeats the path (1) and so on.

If it takes one second to move from 1 vertex to another, what is the period of the motion? At which vertex will the particle be 5 minutes after it started?



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- Now 5 minutes equal 300 seconds, and $300 \equiv 12 \pmod{18}$,i.e. $300 = 18 \times 16 + 12$.
- The particle ended at the 12^{th} vertex of a period,

Time	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Path	0	1	2	3	4	0	3	1	4	0	4	1	3	0	4	3	2	1	0

which is 3 according to the table.

3. Desk calendar

Question

For any month of a desk calendar, draw a square around 9 dates.

(a) Find the sum in terms of the number in the center of the square.

(b) Find the sum in terms of the number at a corner of the square.

JANUARY						2017
SUN	MON	TUE	WED	THU	FRI	SAT
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

3. Desk calendar (Contd.)

- Suppose the 9 dates are

d	d+1	d+2
d+7	d+8	d+9
d+14	d+15	d+16

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- Suppose C is the date in the center, then $C = d + 8$, and $S = 9C$.
- Suppose the dates at the corners are

$$X = d, Y = d + 2, Z = d + 14, W = d + 16$$

then, $S = 9(X + 8) = 9(Y + 6) = 9(Z - 6) = 9(W - 8)$.

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- Recall some properties we have

Theorem

If $a \equiv b(\mod n)$ and $c \equiv d(\mod n)$, then

$$a + c \equiv b + d(\mod n) \tag{1}$$

$$ac \equiv bd(\mod n) \tag{2}$$

$$a^k \equiv b^k(\mod n) \tag{3}$$

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- Using equation (1), we can calculate each term of the sum respectively.
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- Calculate $2^{2003}(\mod 10)$ using $6^k \equiv 6(\mod 10)$,

$$\begin{aligned} 2^{2003} &\equiv 2^{2000}2^3 \equiv (2^4)^{50}8 \equiv 16^{50} \times 8 \text{ using (2)} \\ &\equiv (16\%10)^{50} \times 8 \equiv 6^{50} \times 8 \equiv 6 \times 8 \\ &= 8(\mod 10). \end{aligned}$$

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- Finally, the remainder is 9(1+8), which is the last digit of S.

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- `If her birthday is 29 Feb.`, she would not have celebrated her birthday in 2014 since 2014 is not a leap year.
- `If it is before 29 Feb.`, then the number of days between her two birthdays (3 leap years) is $x = 365 \times 10 + 3 \equiv 6 \pmod 7$. Hence her birthday this year would be on a Saturday.

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- `If it is before 29 Feb.`, then the number of days between her two birthdays (3 leap years) is $x = 365 \times 10 + 3 \equiv 6 \pmod 7$. Hence her birthday this year would be on a Saturday.
- `If it is after 29 Feb.`, then similarly, $x = 365 \times 10 + 2 \equiv 5 \pmod 7$ (two leap years). Hence her birthday lies on a Friday.

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- First, find the distance between these to data.
  - Number of leap years from 2000 to 2999.
    (a) Years of multiple of 4, $[2999/4] - [1999/4] = 750 - 500 = 250$.
    (b) Years of multiple of 100, $[2999/100] - [1999/100] = 10$.
    (C) Years of multiple of 100, $[2999/400] - [1999/400] = 3$.
    Hence the number of leap year is $250 + 3 - 10 = 243$.

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    Hence the number of leap year is $250 + 3 - 10 = 243$.
  - Distance between the two dates is $x = 365 \times 1000 + 243$.
- Calculate the remainder of x divided by 7, which is 4. Hence 01-01-3000 is 4 days after Saturday, i.e. Wednesday.

7. SG100

Question

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- Distance between 1-1-2000 and 9-8-2065.
  - Leap years: $[2064/4] - [1999/4] = 17$.
  - Total years (2000–2064): $2064 - 2000 + 1 = 65$.
  - Distance: $x = 365 \times 65 + 17 + (31 + 28 + 31 + 30 + 31 + 30 + 31 + 8) \equiv 1 \pmod 7$.

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  - Distance: $x = 365 \times 65 + 17 + (31 + 28 + 31 + 30 + 31 + 30 + 31 + 8) \equiv 1 \pmod 7$.
- Hence SG100 is on Sunday (one day after Saturday).

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- One useful fact. Note that 365   1 (mod 7). Thus if the National Day in year y falls on day k, then in year y + 1, if it is not a leap year, the National Day will fall on day k + 1; Otherwise it will fall on day k + 2.

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- Another useful fact. Note that between 1965 and 2065, every multiple of 4 is a leap year.

8. Times of Sundays

- For convenience, we start with the National Day which falls on a k^{th} day in a leap year.

L	N	N	N	L	N	N	N
0	1	2	3	5	6	0	1
3	4	5	6	1	2	3	4
6	0	1	2	4	5	6	0
2	3	4	5	0	1	2	3
5	6	0	1	3	4	5	6
1	2	3	4	6	0	1	2
4	5	6	0	2	3	4	5

where 0 indicates k^{th} day, 1 indicates $k+1^{th}$ day, and so on. For example, if $k=3$, i.e. Wednesday, then 0 is Wednesday, 1 is Thursday, 2 is Friday,…, 6 is Tuesday.

8. Times of Sundays

- From the table, we can see that every 28 years, a new National day will fall on a k^{th} day again. Similarly, if started from any non-leap year, we can also return to k^{th} day after 28 years. That is 28 is a period.
- This is because after each 4 years, the remainder of the additional days divided by 7 will increase by 5 (1+1+1+3). So after 28 years, it will increase by $5 \times 7 = 0 \pmod 7$.

- For example we start with the 2015's National Day which falls on a Sunday.

N	L	N	N
0^{2015}	2	3	4
5	0^{2020}	1	2
3	5	6	0^{2026}
1	3	4	5
6	1	2	3
4	6	0^{2037}	1
2	4	5	6
0^{2043}	2	3	4

- Here 0 means Sunday, 1 means Monday, 2 means Tuesday, ..., 6 means Saturday.
- So 0s are these years when National day falls on Sunday, which is what we need to find.
- In a period, we have 4 years whose National day falls on Sunday.(2015 2042: 2015,2020,2026,2037).

8. Times of Sundays

- From 1965 to 2065, according to the period, we can divided it by several pieces: 1965–1986; 1987–2014; 2015–2042; 2043–2065

- So we have two complete period, 1987–2014, 2015–2042, having 8 years whose National days are Sunday.
- By checking in the previous table, 2043–2065 have 4 years and 1965–1986 have 2 years.
- Totally have 14 years whose National days is Sunday.

Checking 1965–1986.

N	L	N	N
0^{1959}	2	3	4
5	0	1^{1965}	2
3	5	6	0^{2070}
1	3	4	5
6	1	2	3
4	6	0^{2081}	1
2	4	5	6^{1986}

Checking 2043–2065.

N	L	N	N
0^{2043}	2	3	4
5	0^{2048}	1	2
3	5	6	0^{2054}
1	3	4	5
6	1	2	3
4	6	0^{2065}	1
2	4	5	6^{2070}

Thank you for listening!