# GEH1036/GEK1505 Tutorial 10 (week 12)

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# Personal Information

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    Go to the website, and scroll down, see the **Work** entry, in the "Tutor" project, click the link for more information. I will provide this slide and the source .tex file on that site.
- Course information
  - EVERY WEEK MONDAY 10:00-11:00 S16-0431

# 1. Lottery

# Lottery

## Question

10000 people each has 4-digit number. We will generator randomly 3 4-digit numbers for the first, second and third prize. The numbers for the three prize are independent, which may be all the same.
(i) at least one win first prize
(ii) at least one will win the any of the three prizes.

- (i) Consider one person, he/she win the first prize with probability $q = \frac{1}{10^4} = 10^{-4}$. He/She will not win with the probability $p = 1 - q = 1 - 10^{-4}$.
  All the 10000 people will not win the first prize with probability
  $$p^{10000} = (1-q)^{10000} \approx e^{10^4 q} = e^{-1} \approx 0.36787944117$$
  where we using the approximation $(1+x)^N \approx e^{Nx}$ provided $N$ is very large. Answer is 0.63212055882.

(ii) At least one will win any of the prizes.

- Note that the three kind of prizes are independent, then the probability that no one win any prizes is $p^3$.
- Thus the probability at list one win some prizes is $1 - p^3 = 1 - e^{-3} \approx 0.95021293163$.

# 2. Lottery continue

# Lottery continue

## Question

Same problem as previous when we have 20,000 and 30,000 persons.

- For 20,000 persons, the probability at least one win the first prize is

$$1 - p^{20000} \approx 1 - e^{-2} \approx 0.86466471676.$$

the probability at least one win any of the three prizes is

$$1 - p^{3^{2000}} \approx 1 - e^{-6} \approx 0.99752124782.$$

- The respective answers for 30000 persons are

$$1 - e^{-3} \approx 0.95021293163 \quad 1 - e^{-9} \approx 0.99987659019.$$

# 3. Treasure boxes

# Treasure boxes

## Question

In a TV game show, Mr Brown has the chance to win the grand prize which is hidden in one of three treasure boxes. Mr Brown chooses one box. The host, who knows where the grand prize is, then opens one of the remaining boxes which is empty. She then gives Mr Brown the choice of sticking to his box or switching to the other box. Should Mr Brown switch?

- If he decide not to switch, then the probability he win the prize is $\frac{1}{3}$ since what the host do is after Brown's decision and won't affect the probability he choose the right one from three boxes.
- If he decide to switch after the host opening a empty box, it is equivalent that he choose two of the three boxes (Host help to open one), the probability is $\frac{2}{3}$.
- So Mr Brown should choose switching.

# 4. Winning number

# Winning number

## Question

In a certain game of chance, 7 winning numbers are drawn one by one without replacement from the set $A = \{1, 2, \ldots 45\}$. A player picks 9 numbers from A. Find the probability that the player picks
(i) exactly 6 of the winning numbers.
(ii) exactly 5 of the winning numbers.

- For question (i), the sample space S is the set of all 9-element subsets of A, $|S| = \binom{45}{9}$.
- The number of elements in S which contain exactly 6 winning numbers is $\binom{7}{6}\binom{38}{3}$.
- The probability is then $\binom{7}{6}\binom{38}{3}/\binom{45}{9} \approx 0.000066638$.
- Similar for (ii), the probability is $\binom{7}{5}\binom{38}{4}/\binom{45}{9} \approx 0.001749243$.

# Alternative way

- First we have chosen 9 numbers from A ahead, denoted by B, then we choose the 7 winning numbers. The question becomes to calculate the probability that exactly 6 of the 7 winning numbers are from B.
- Consider the sample space S, the set of 7 winning numbers. $|S| = \binom{45}{7}$.
- For the set of 7 winning numbers. 6 are from B and 1 from the other 36 numbers.
- Probability is $\binom{9}{6}\binom{36}{1}/\binom{45}{7}$.
- Similarly for (ii), the probability is $\binom{7}{6}\binom{36}{2}/\binom{45}{9}$.

# 5. Sending letters

# Sending letters

## Question

There are 4 letters to be sent to 4 different persons and there are 4 envelopes each bearing the name and address of the respective recipient. You put the letters into the envelopes at random, one letter into one envelope. Find the probability that
(i) exactly one letter is put into the correct envelope,
(ii) at least one letter is put into the correct envelope,
(iii) no letters are put into the correct envelopes.

- We assume that the correct envelope for the $i^{th}$ letter is the $i^{th}$ envelope, $i = 1, 2, 3, 4$. The sample space S is all the ways for putting the letters into the envelopes. Thus $|S| = 4!$.

# Sending letters

- (i) only one letter is put into the correct envelop. We can choose this letter 4 ways: 1,2 ,3,4. Assume letter 1 is correct, then for (2,3,4), there are only two ways to put them into envelopes: (3,4,2) and (4,2,3). Similar for the other letters. Hence the probability is $4 \times 2/4! = 1/3$.
- (ii) At least one letter is put correctly.
  - One is correct: $4 \times 2 = 8$.
  - Two are correct: First having $\binom{4}{2}$ ways to choose the two letters. And assume $1, 2$ is correct, then there are only 1 way to put (3,4) wrongly. Totally : $\binom{4}{2} \times 1 = 6$.
  - Three is correct: this will imply that all of the letters are put correctly. Only 1 way.
  - Probability is $(8 + 6 + 1)/4! = 5/8$.
- (iii) No letter is put correctly. The probability is $1 - 5/8 = 3/8$.

# 6. Opening the Door

# Opening the Door

## Question

To open a door with $n$ keys. Only one is the correct key. Try them randomly and discard the wrong keys after trying. Find the probability you will open the door in the

(i) 1st attempt

(ii) 2nd attempt

(iii) 3rd attempt.

- (i) Only need consider the first key. It is correct with probability of $1/n$.
- (ii) 1st key is wrong, 2nd key is correct. On the condition (i) is false, probability $p_1 = \frac{n-1}{n}$, 2nd key is correct with probability $p_2 = \frac{1}{n-1}$. So the final probability is $p_1 p_2 = \frac{n-1}{n}\frac{1}{n-1} = \frac{1}{n}$.
- (iii) Similarly we can calculate it on the condition (i) and (ii) all fail. The probability is $(1 - \frac{1}{n} - \frac{1}{n})\frac{1}{n-2} = \frac{1}{n}$.

# Opening the Door

For the general case: calculate the probability If it takes k attempts to get the right key.

- The sample space is the set of sequences of distinct keys $x_1 x_2 \ldots x_k$. Thus $|S| = n(n-1)\cdots(n-k+1)$.
- This required event E consists of all sequences S of the form $x_1 x_2 \cdots x_{k-1} X$, where $X$ is the unique correct key. Then $|E| = (n-1)(n-2)\cdots(n-k+1)$.
- Probability is $|E|/|S| = \frac{1}{n}$.

Or you can think in a easier way: we try the keys with a order, the probability that the correct key lies at the $k^{th}$ place is $\frac{1}{n}$.

# 7. Birthday

# Birthday

## Question

Assuming that there are 365 equally probable birthdays, find the probability that in a group of k persons, all of them will have different birthdays for $k = 3, 4, \dots 10$.

- Suppose $k$ persons' birthdays are $(x_1, x_2, \dots, x_k)$, probable birthday of each person is 365 days. The sample space $S$ has the size $365^k$.
- The event E that the birthdays of k persons are all different is the set of sequences in which the $x_i$ is are distinct. $|E| = \binom{365}{k} k!$.
- $P(E) = \frac{|E|}{|S|} = \frac{\binom{365}{k} k!}{365^k}$. Calculate we have the following table.

| $k$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| P(E) | 0.9918 | 0.9836 | 0.9729 | 0.9595 | 0.9438 | 0.9257 | 0.9054 | 0.8831 |
| P($E^c$) | 0.0082 | 0.0164 | 0.0271 | 0.0405 | 0.0562 | 0.0743 | 0.0946 | 0.1169 |

# Birthday

**Question**

(i) Hence find the probability that in a group of k persons, at least two persons will have the same birthday for $k = 2, 3, \dots 10$.
(ii) Find the smallest value of k for which the corresponding probability is more than 0.5.

(i) It is exactly $P(E^c)$.
(ii) The smallest value of k for which the corresponding probability $P(E^c)$ is greater that 0:5 is 23 and the probability is 0.508.

# Thank you for listening!