GEH1036/GEK1505 Tutorial 8(week 10)

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    Go to the website, and scroll down, see the **Work** entry, in the "Tutor" project, click the link for more information. I will provide this slide and the source .tex file on that site.
- Course information
  - EVERY WEEK MONDAY 10:00-11:00 S16-0431

1. Different representation

Question

Find

1. the binary representation
2. the octal representation
3. the hexadecimal

representation of the (decimal) numbers 123 and 2006.

1. Different representation

- Finding a representation of a number a based on n, the way is to dived a by n repeatedly, and then get the remainder for each dividing. The remainder collected together (the opposite direction) is the base-n representation.
- Finding the binary representation of 123.

$$
\begin{aligned}
123 \div 2 &= 61 \cdots 1 \\
61 \div 2 &= 30 \cdots 1 \\
30 \div 2 &= 15 \cdots 0 \\
15 \div 2 &= 7 \cdots 1 \\
7 \div 2 &= 3 \cdots 1 \\
3 \div 2 &= 1 \cdots 1
\end{aligned}
$$

Hence the binary representation is 111011_2.

1. Different representation

- Using the same way, we get the binary, octal, hexadecimal representation of 125 and 2006 respectively.
  - 123=1111011_2.
  - 123=173_8.
  - 123=7B$_{16}$.
  - 2006=11111010110_2.
  - 2006=3726_8.
  - 2006=7D6$_{16}$.

2. Octal representation

Question

Find the last digit of the octal representation of the binary number $N = (111\dots1)_2$ with 100 ones.

- To make it clearly, the question equals finding the remainder of $N = 2^{99} + 2^{98} + \cdots + 2^1 + 2^0$.
- Note that $2^3 = 8$, and hence $2^k (\mod 8) = 0, \forall k \geqslant 3$.
- Therefor,

$$\begin{aligned} N(\mod 8) =& \{2^{99} + 2^{98} + \cdots + 2^4 + 2^3\}(\mod 8) \\ &+ (2^2 + 2^1 + 2^0)(\mod 8) \\ =& 7(\mod 8) = 7. \end{aligned}$$

- The last digit of the octal representation of N is 7.

3. Encoding

Question

Encode B3_8 and B23F modulo 37 with the table.

Symbol	0	1	2	3	4	5	6	7	8	9
Numerical	0	1	2	3	4	5	6	7	8	9
Symbol	A	B	C	D	E	F	G	H	I	J
Numerical	10	11	12	13	14	15	16	17	18	19
Symbol	K	L	M	N	O	P	Q	R	S	T
Numerical	20	21	22	23	24	25	26	27	28	29
Symbol	U	V	W	X	Y	Z	blank			
Numerical	30	31	32	33	34	35	36			

3. Encoding

- According to the procedure of encoding. First put the check digit as x.
- Then x should satisfy

$$w(e_n e_{n-1} \cdots e_2 x) \equiv 0 \pmod{37}$$

- Encoding B3_8:
  - The equation is

$$0 \equiv 5(11)+4(3)+3(36)+2(8)+x \equiv 191+x \equiv 6+x \pmod{8}$$

  - we get $x = 31$, corresponds to V, encoded word is “B3_8V”.
- Encoding B23F:

$$0 \equiv 5(11)+4(2)+3(3)+2(15)+x \equiv 102+x \equiv 28+x \pmod{37}$$

then $x = 9$, encoded word is “B23F9”.

4. ISBN code 9810N11392

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Question

Find the value of the digit N in the ISBN code 981 – 0N – 11392.

- ISBN code has the property that the weighted sum is a multiple of 11.
- Calculate the weighted sum:

$$
\begin{aligned}
S &= 10 \cdot 9 + 9 \cdot 8 + 8 \cdot 1 + 7 \cdot 0 + 6N + 5 \cdot 1 + 4 \cdot 1 + 3 \cdot 3 \\
&\quad + 2 \cdot 9 + 1 \cdot 2 \\
&= 208 + 6N \\
&= 6N - 1 \pmod{11}
\end{aligned}
$$

- Where N can be choose from 0–9, 2 satisfies the congruence equation.
- N=2.

5. Another ISBN

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Question

Show, by means of two different examples, that the last 2 digits of the ISBN 1–886544–45–X can be altered to yield another valid ISBN.

- The difference of two weighted sum of ISBNs must be a multiple of 11.
- If only last two digits altered, the difference of the last two digits is also a multiple 0f 11.
- Original is $2 \cdot 5 + 1 \cdot 10 = 20$, hence another sum could be 9, 20, 31…, here 31 is impossible. The largest sum of the last two is $28 = 2 \cdot 9 + 1 \cdot 10$.
- For 9, the last two digits could be 09,17,25,33,41.
- For 20, the last two digits could be 5X,68,76,84,92.

6. Hanmming coding

Question

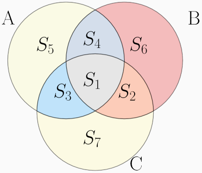
Encode the messages 1011, 1111 and 0111 using the Hamming (7,4) and (8,4) codes.

- Denote the code by $s_1 s_2 s_3 s_4 s_5 s_6 s_7$, with condition

A: $s_1 + s_3 + s_4 + s_5 \equiv 0 \pmod 2$

B: $s_1 + s_2 + s_4 + s_6 \equiv 0 \pmod 2$

C: $s_1 + s_2 + s_3 + s_7 \equiv 0 \pmod 2$



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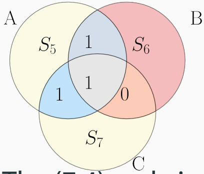

- Coding 1011. We have $s_5 = 1, s_6 = 0, s_7 = 0$. The (7,4) code is "1011100".

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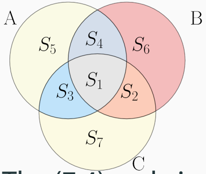

- Coding 1011. We have $s_5 = 1, s_6 = 0, s_7 = 0$. The (7,4) code is "1011100".
- Similarly, the (7,4) codes of "1111" and "0111" are 1111111" and " 0111000" respectively.

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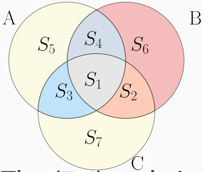

- Coding 1011. We have $s_5 = 1, s_6 = 0, s_7 = 0$. The (7,4) code is "1011100".
- Similarly, the (7,4) codes of "1111" and "0111" are 1111111" and " 0111000" respectively.
- The (8,4) codes are 10111000, 11111111, 01110001. (Adding a last digit to make they have even 1's).

7. Decoding Hamming (8,4) codes

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Question

Decode, if possible, the following message received assuming that each word has been encoded using the Hamming (8, 4) code and that each has at most two errors:
a= 11101111; b= 11010100; c= 10011100.

- First, overall parity check. Failure means 1 (odd) error. Passing means 0 or 2 (even) error. then a has exactly one error. b, c has 0 or 2 errors.
- Second, parity check. For single error, it can detect and corrected. For two, only can detect but not correct.

7. Decoding Hamming (8,4) codes

By checking

A: $s_1 + s_3 + s_4 + s_5 \equiv 0 (\mod 2)$

B: $s_1 + s_2 + s_4 + s_6 \equiv 0 (\mod 2)$

C: $s_1 + s_2 + s_3 + s_7 \equiv 0 (\mod 2)$

We can correct (a) and (b) according to the following table

Error	bit Parity check
s1	All circles A, B and C fail the parity check
s2	Only B and C fail the parity check
s3	Only A and C fail the parity check
s4	Only A and B fail the parity check
s5	Only A fails the parity check
s6	Only B fails the parity check
s7	Only C fails the parity check
no	error All A, B and C pass the parity check

7. Decoding Hamming (8,4) codes

- It turns out that a fail both A and B while pass C.
- Hence 4^{th} bit is wrong, a is “11111111”, original code is “1111”.
- While b passes all parity checks. So it is correct.
- And c fails all A,B,C. It has two errors. The errors could be “1,8”, “2,5”, “3,6” and “4,7”. Cannot be corrected.

8. Errors of c

Question

Find all the the possible errors in c of the preceding question.

- From previous question, we know exactly two error occur in the code.
- A,B,C all fail shows that each circle has odd errors, in fact one error.
- If one error occurs at 8^{th}, then another error can only at 1^{th}.
- No error occurs at 8^{th}, then no error at 1^{th}, otherwise some circle has two errors.
  - One is in $A\backslash\{B \cup C\}$, another must belong to $B \cap C$. i.e. "2,5".
  - One is in $B\backslash\{A \cup C\}$, another must belong to $A \cap C$. i.e. "3,6".
  - One is in $C\backslash\{A \cup B\}$, another must belong to $B \cap C$. i.e. "4,7".
- All the possible errors could be "1,8", "2,5", "3,6" and "4,7"

9. Transmitted word

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Question

Transmit a Hamming (7,4) code from a channel. If ?0?0111 is received, what is the transmitted word, assuming that the recognizable bits are correct?

- Denote the original word by $x0y0111$.
- Applying parity check:

$$\text{A: } x + y + 1 \equiv 0 \pmod 2$$

$$\text{B: } x + 1 \equiv 0 \pmod 2$$

$$\text{C: } x + y + 1 \equiv 0 \pmod 2$$

- The solution is $x = 1, y = 0$, the message is "1000111"

Question

Show that if ??11001 is received, then the assumption that the recognizable bits are correct cannot be valid.

- Assume the recognizable bits are correct.
- Denote the code by $xy11001$.
- Applying parity check:

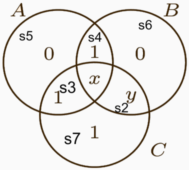
$$\begin{aligned} \mathsf{A} &: x+1+1 \equiv 0 \pmod 2; \\ \mathsf{B} &: x+y+1 \equiv 0 \pmod 2; \\ \mathsf{C} &: x+y+1+1 \equiv 0 \pmod 2 \end{aligned}$$

- The equations have no solution, hence the assumption fails. The recognizable bits are not correct.

Question

If ?0?0111 is received, can the correct word be recovered if at most one error in the recognizable bits.

- The parity check: Applying parity check:



$$\text{A} : x + 1 + 1 \equiv 0 \pmod 2,$$
$$\text{B} : x + y + 1 \equiv 0 \pmod 2,$$
$$\text{C} : x + y + 1 + 1 \equiv 0 \pmod 2.$$

- Since B and C are inconsistent, the error must occur at 3^{rd}, 4^{th}, 6^{th} or 7^{th} bit.
- If it is 3^{rd}, then x=1,y=0.
- If it is 4^{th}, then x=1,y=1. If it is 6^{th}, then x=0,y=0. If it is 7^{th}, then x=0,y=0.
- So we cannot correct the error.

10. Number of strings

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Question

How many binary strings of length 7 (i.e. words with 7 bits) are there? How many of these are codewords of the Hamming (7,4) code?

- For the number of strings with length 7.
  Each bit has two choice 0 or 1. Then totally we have $2^7 = 128$ strings.
- For strings of Hamming (7,4) codes.
  Note that first 4 bits will determine the rest 3 bits. Hence we have totally $2^4 = 16$ strings.

Thank you for listening!