GEH1306 Tutorial 1(week 3)

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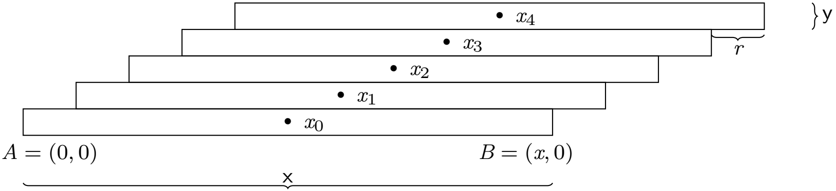
1.Gravity center of slabs



- Since we just consider the horizontal component of gravity center, vertical properties can be ignored.

1.Gravity center of slabs (contd.)

- For each slab $i = 1, 2, \ldots, n$, the gravity center

$$x_i = x + (i - 1)r. \tag{1}$$

1.Gravity center of slabs (contd.)

- The gravity center for the entire configuration is

$$
\begin{aligned}
x_c &= \frac{1}{n}\sum_{i=1}^{n} x_i \\
&= \frac{1}{n}\sum_{i=1}^{n}(x+(i-1)r) \\
&= \frac{1}{n}\left[\sum_{i=1}^{n} x + (\sum_{i=1}^{n}(i-1))r\right] \\
&= \frac{1}{n}\left[nx + \frac{n(n-1)}{2}r\right] \\
&= x + \frac{(n-1)r}{2}.
\end{aligned}
$$

1. Gravity center of slabs

- The gravity center formula comes from

$$\frac{x_c = \sum_{i=1}^{n} m_i x_i}{\sum m_i.} \tag{2}$$

for n objects with mass m_i, gravity center x_i.

1. Gravity center of slabs (contd.)

For each slab, the mass is the same, we can denote it as m. Then the gravity center of the entire configuration is

$$x_c = \frac{\sum_{i=1}^{n} m x_i}{\sum_{i=1}^{n} m} \tag{3}$$

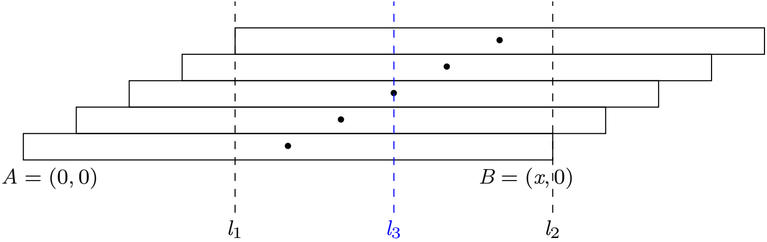
$$= \frac{\sum_{i=1}^{n} m x_i}{mn} \tag{3}$$

$$= \frac{\sum_{i=1}^{n} x_i}{n}. \tag{4}$$

http://hyperphysics.phy-astr.gsu.edu/hbase/cm.html

1. Gravity center of slabs (contd.)

- Another way



1. Gravity center of slabs (contd.)

You may note that the gravity center lies at where the left and the right sides have the equal mass. From the figure, the mass on the left of line l_1 and the right of l_3 equals. Hence the gravity center lie on the line l_2, the position of which is
$x_c = nr + \frac{2x-(n-1)r}{2} = x + \frac{(n-1)r}{2}$.

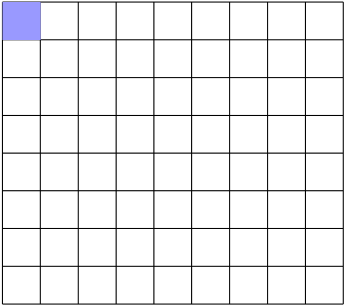
1. Gravity center of slabs (b)

Check when $x_c > |AB| = 2x$. For the computation:

$$x + \frac{(n-1)r}{2} > 2x \Rightarrow n > \frac{2x}{r} - 1.$$

- for $r = 5$, n>21;
- for $r = 8$, n>13.5.

2.Number of squares



- To counter the squares, start with a square at the top left corner of the board and move it vertically down and then horizontally across to account for all squares of that shape.

2.Number of squares (contd.)

- 1×1 squares, number: 8×9
- $2 \times 2, \ldots, 8 \times 8$ squares, number: $7 \times 8, \ldots, 1 \times 2$
- Total number: $8 \times 9 + 7 \times 8 + \cdots + 1 \times 2 = 240$

- **Alternative way** A square, when projected to the sides of the grid, yield two segments of equal length. Conversely, a horizontal and a vertical segment of equal length,corresponds to a square. There are 8 vertical segments and 9 horizontal segments of length 1. Thus there are 8×9 number of 1×1 squares. Likewise, there are 7×8 number of 2×2 squares, etc.

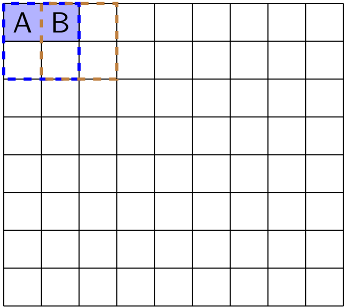
2.Number of squares (contd.)

- Number of rectangles
  Note that each rectangle corresponds uniquely to a vertical and horizontal segment.
  - Number of different horizontal segments
    Among the 10 points, any different pair of two points corresponds a different segment, do we have $\binom{10}{2}$ different choices.
  - As above, choose two points from the 8 points. We have $\binom{9}{2}$ different choices.
  - Totally we have $\binom{10}{2} \times \binom{9}{2} = 45 \times 36 = 240$ different choices.

3. Number of squares (two removed)

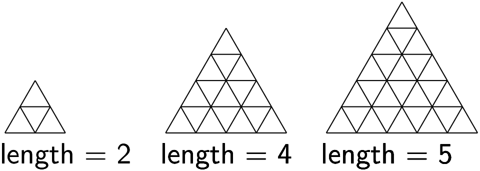
Main idea: Check how many squares in Problem 2,(a) contain the removed two squares A, B.

- Number of squares contain A or B,
  - 1×1: 2;
  - 2×2: 2;
  - $\ldots$
  - 8×8: 2.
- Required number of squares in incomplete chequered: 240 -16=224.



4. Number of triangles

Find the number of triangles in each of the following figures.



Two type of triangles

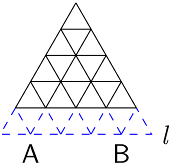
- Up triangles $\triangle$, denote by $\triangle_k$ the up triangles of side length k.
- Down triangles $\bigtriangledown$,denote by $\bigtriangledown_k$ the up triangles of side length k.

4. Number of triangles (contd.)

- In the first figure
  - Number of up triangles: $3 \times \triangle_1 + \triangle_2$;
  - Number of down triangles: $\bigtriangledown_1$;
  - Total 5 triangles.
- In the second case
  - $\triangle_1$:10, $\triangle_2$:6, $\triangle_3$:3, $\triangle_4$:1;
  - $\bigtriangledown_1$: 6, $\bigtriangledown_2$:1, $\bigtriangledown_3$:0;
  - Total 27 triangles.
- In the second case
  - $\triangle_1$:15, $\triangle_2$:10, $\triangle_3$:6, $\triangle_4$:3, $\triangle_5$:1;
  - $\bigtriangledown_1$:10, $\bigtriangledown_2$:3, $\bigtriangledown_3$:0;
  - Total 48 triangles.

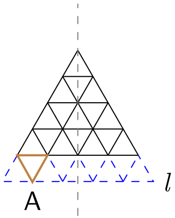
4. Number of triangles (contd.)

- Deduction for the general case We consider for the numbers of extra triangles of $\triangle_{n+1}$ compared to thath of $\triangle_n$
  - Extra up triangles $\triangle$
    Add one line l beneath the lowest line. Intersect the non horizontal lines with l giving $n+2$ points. For each type $\triangle$ triangle, extend the two non horizontal lines to intersect l at two points, say A, B, with A on the left. We can see that the bottom side of the triangle must lies on the line l. Then each pair of (A, B) can determine an extra triangle. We have $\binom{n+2}{2}$ different pairs of (A, B), thus $\binom{n+2}{2}$ extra up triangles.



4. Number of triangles (contd.)

- Extra down triangles $\bigtriangledown$
  - $n = 2k$ is even
    We can see that each extra down triangle must satisfy that its lowest vertex A lie on the line l.
    With different A, we have
    $2(1+2+\cdots+k) = k(k+1) = \frac{n^2}{4} + \frac{n}{2}$ extra triangles.
  - $n = 2k+1$ is odd
    Move A from left to center point (showing as the dashed line and the right side is symmetric with left), we have
    $2(1+2+\cdots+k)+(k+1) = (k+1)^2 = (\frac{n+1}{4})^2$
    extra triangles.



4. Number of triangles (contd.)

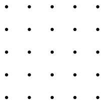
- The total number of the extra triangles is $\frac{3}{4}n^2 + 2n + 1$ when n is even and $\frac{3}{4}n^2 + 2n + \frac{5}{4}$ when n is odd, which is $\frac{3}{4}n^2 + 2n + 1 + \frac{1}{8} - \frac{(-1)^n}{8}$.
- The general formula of number of triangles of $\triangle_n$ is

$$\frac{n^3}{4} + \frac{5n^2}{8} + \frac{n}{4} - \frac{1}{16} + \frac{(-1)^n}{16}.$$

(We compute it by add extra numbers from $\triangle_1$ with the extra number formula deduced above)

5. Number of squares formed by joining dots

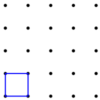
Find the number of squares that can be formed by joining dots in the following diagram.



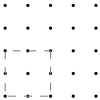
5. Number of squares formed by joining dots (contd.)

Type of squares:we denote (x, y) by the length of square.

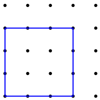
- (0,1),(0,2),(0,3),(0,4) i.e. Squares whose sides are vertical or horizontal. Showing as



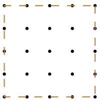
(0,1), 4^2 squares



(0,2), 3^2 squares



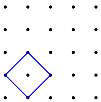
(0,3), 2^2 squares



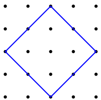
(0,4), 1 square

5. Number of squares formed by joining dots (contd.)

- (1,1),(2,2), i.e. Squares with side length equal to $\sqrt{2}, \sqrt{8}$.



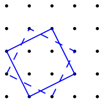
(1,1), 3^2 squares



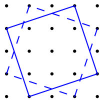
(2,2),1 square

5. Number of squares formed by joining dots (contd.)

- (1,2),(2,1),(1,3),(3,1), i.e. Squares with side length equal to $\sqrt{1^2+2^2}, \sqrt{1^2+3^2}$.



(1,2)or (2,1) 2×2^2 squares



(1,3)or (3,1), 2 squares

- So we have $1^2+2^2+3^2+4^2+9+1+8+2=50$ squares in all.

6.Number of Regions

Question:

- Draw a triangle on the plane. The plane is divided into 2 regions.
- Draw two triangles in the plane, what is the maximum number of regions that the plane is divided into?
- What is the answer if there are n triangles, where $n \geq 1$?

6.Number of Regions

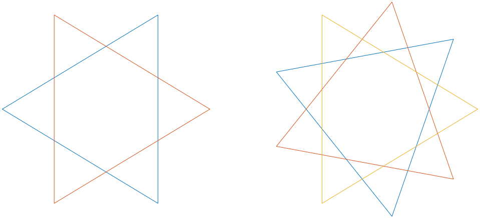


Figure: n=2,3

6.Number of Regions (contd.)

Every line can intersect with most 2 edges of a trangle. Therefore, when n=2, the maximum number of regions is 8.

6.Number of Regions (contd.)

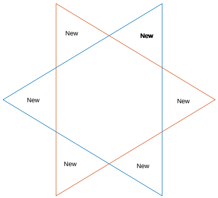


Figure: Counting new regions for n=2

6.Number of Regions (contd.)

- n=1, the blue triangle cut the plane into 2 regions;
- n=2, the orange one interscet the blue one in 6 points, Its perimeter is cut into 6 portions;
- Every poition of the orange curve cuts a new region form the old one. (Note: "new" is w.r.t. counting.)

6.Number of Regions

In n^{th} step:

- If the new triangle intersects an existing triangle in 6 points, dividing the perimeter of the new triangle into 6 portions, then each portion divides an existing region into 2 and therefore we get 6 additional regions;
- If the new triangle intersects every existing triangle in 6 points, then the total number of additional regions is $6(n-1)$ since there are $n-1$ existing triangles.

The two above condition can be satisifed:
If you rotate a regualr triangle less than $2\pi/3$ with its center fixed, it will interscet with a triangle in the original position in 6 points.

6.Number of Regions

# Triangle: n	0	1	2	3	$\cdots$	n
Plus	-	1	6	12	$\cdots$	6(n-1)
Regions	1	2	8	20	$\cdots$	2+3n(n-1)

$$2 + 6(1 + 2 + 3 + \cdots + (n-1)) = 2 + 3n(n-1).$$

6.Number of Regions

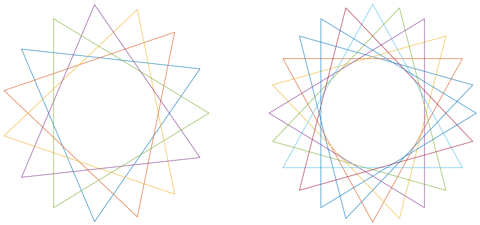


Figure: n=5,8

Thank you for your listening