Multivariate Models

The goal of this project is to study a concrete example of multivariate stock returns using `R` packages and functions.

- Perform graphical and formal tests of normality
- Fit multivariate models to the data
- test the goodness-of-fit of the model using the log-likelihood ratio test.

0) Setup

The first step is to load the necessary libraries.

```
#install.packages(c('xts','qrmtools','qrmdata','ghyp'))

library(xts)
library(qrmtools)
library(qrmdata)
library(ghyp)
```

1) Data preparation

The data consists of an `xts` object containing adjusted close prices of the constituents of the Dow Jones index. The data hs been obtained from Yahoo Finance as of 2016-01-03, via the function `get_data()` from the package **qrmtools**. The constituents data ranges from the first date at least one of the constituents is available (with missing data if not available) to 2015-12-31.

```
## Dow Jones constituents
data("DJ_const")
str(DJ_const)
```

```
## An 'xts' object on 1962-01-02/2015-12-31 containing:
##   Data: num [1:13595, 1:30] NA NA NA NA NA NA NA NA NA NA ...
##  - attr(*, "dimnames")=List of 2
##   ..$ : NULL
##   ..$ : chr [1:30] "AAPL" "AXP" "BA" "CAT" ...
##   Indexed by objects of class: [Date] TZ: UTC
##   xts Attributes:
## List of 2
##  $ src    : chr "yahoo"
##  $ updated: POSIXct[1:1], format: "2016-01-03 03:51:40"
```

```
head(DJ_const[,1:10],5)
```

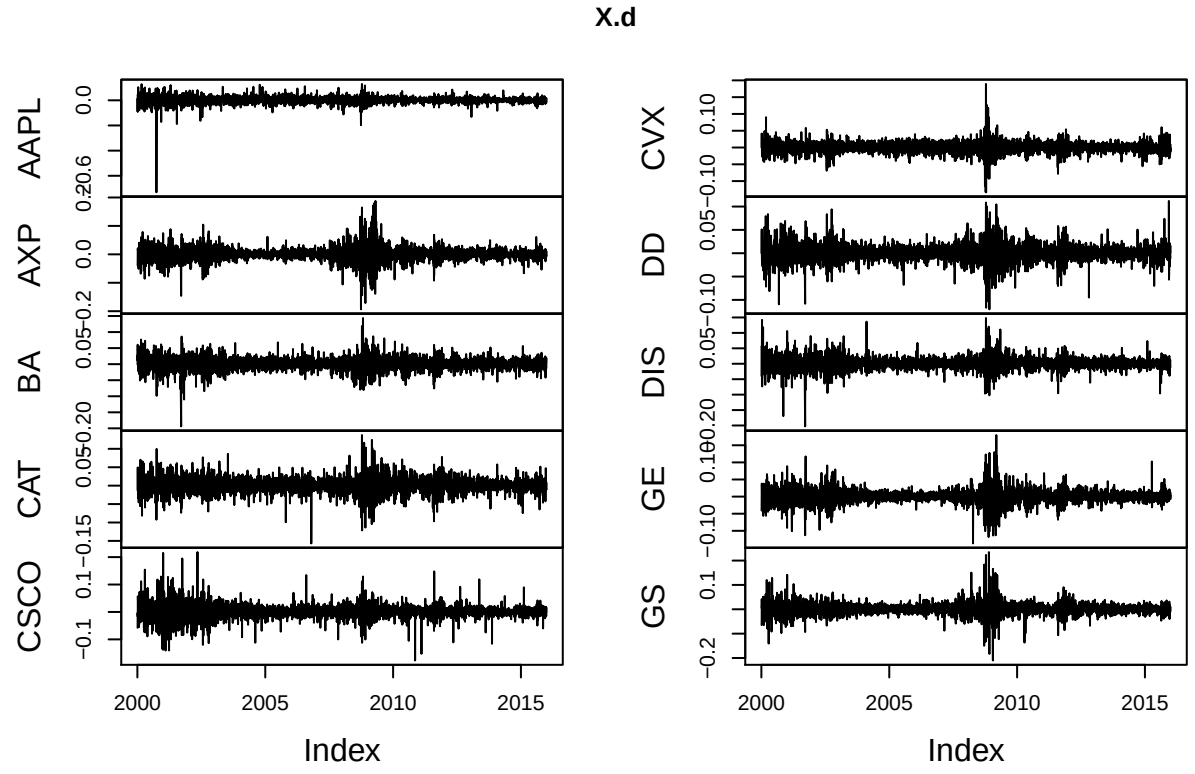
```
##            AAPL AXP       BA      CAT CSCO CVX       DD      DIS       GE GS
## 1962-01-02   NA  NA 0.212905 0.593184   NA  NA 1.227958 0.061014 0.145967 NA
## 1962-01-03   NA  NA 0.217163 0.598964   NA  NA 1.229230 0.061832 0.144501 NA
## 1962-01-04   NA  NA 0.215034 0.614372   NA  NA 1.220331 0.061832 0.142794 NA
## 1962-01-05   NA  NA 0.210773 0.620152   NA  NA 1.187280 0.062039 0.139132 NA
## 1962-01-08   NA  NA 0.211306 0.624001   NA  NA 1.169484 0.061832 0.138889 NA
```

We will only focus on the 10 first constituents, starting from January 1^{st}, 2000.

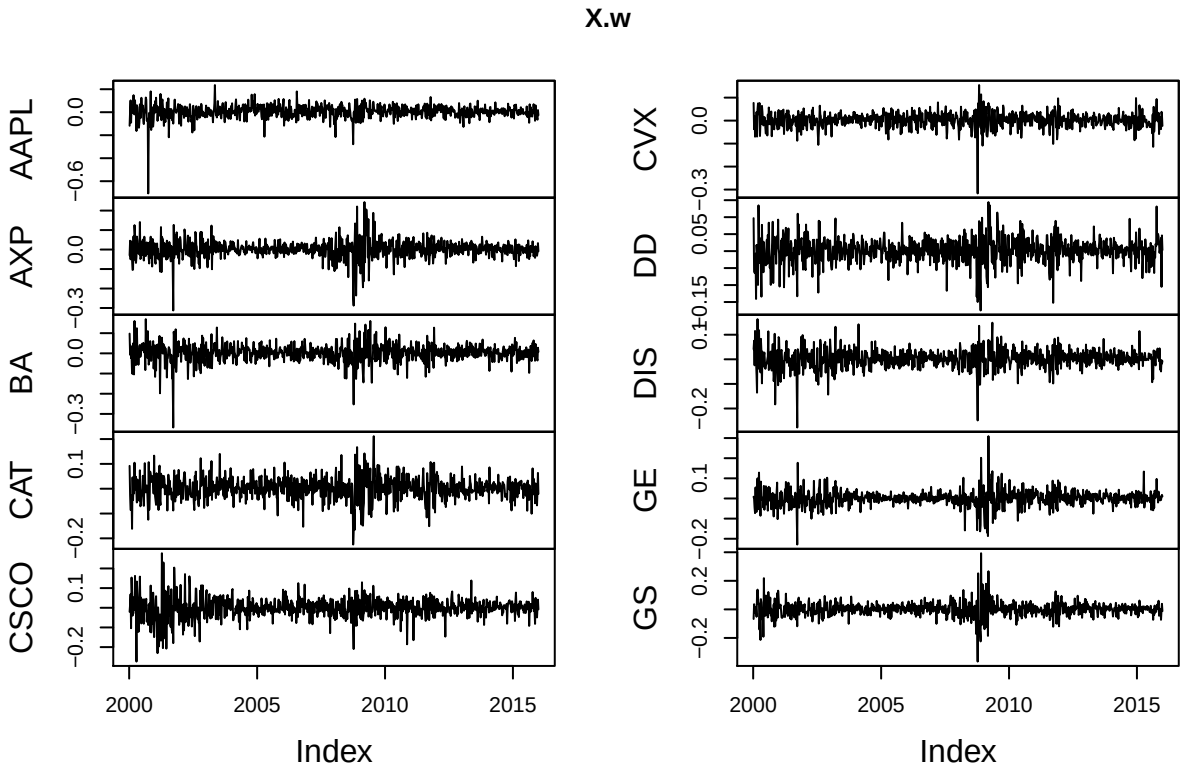
- Compute the daily log-returns using the function `returns` in the package **qrmtools**.
- Compute the weekly log-returns. (*hint*: Use the daily log-returns series and `apply.weekly` from **xts**).
- Plot the daily (and weekly) log-returns of each constituent.

```
d <- 10 # for the number of components we pick out
S <- DJ_const['2000-01-01/',1:d] # first d components from January 1st 2000 onwards

## Daily log-returns
X.d <- returns(S)
plot.zoo(X.d)
```




```
## Weekly log-returns
X.w <- apply.weekly(X.d, FUN = colSums)
plot.zoo(X.w)
```



2) Is the return data jointly normal?

Our first step is to test the normality of the return data (first daily then weekly),

$$\mathcal{H}_0 : \mathbf{X}_1, \ldots, \mathbf{X}_n \overset{iid}{\sim} N(\mu, \sigma)$$

2.1) Visual tests of normality

- We create scatterplots for each pair of constituants, to examine any possible resemblance to bivariate Gaussian distributions?
- Then, for each individual constituant, we draw QQ-plots against a standard Gaussain reference distribution.
- We define the squared Mahalanobis distance

$$D_i^2 = (\mathbf{X}_i - \bar{\mathbf{X}})' S^{-1} (\mathbf{X}_i - \bar{\mathbf{X}});\ i = 1, \ldots, n$$

and the Mahalanobis angle between $(\mathbf{X}_i - \bar{\mathbf{X}})$ and $(\mathbf{X}_j - \bar{\mathbf{X}})$ is

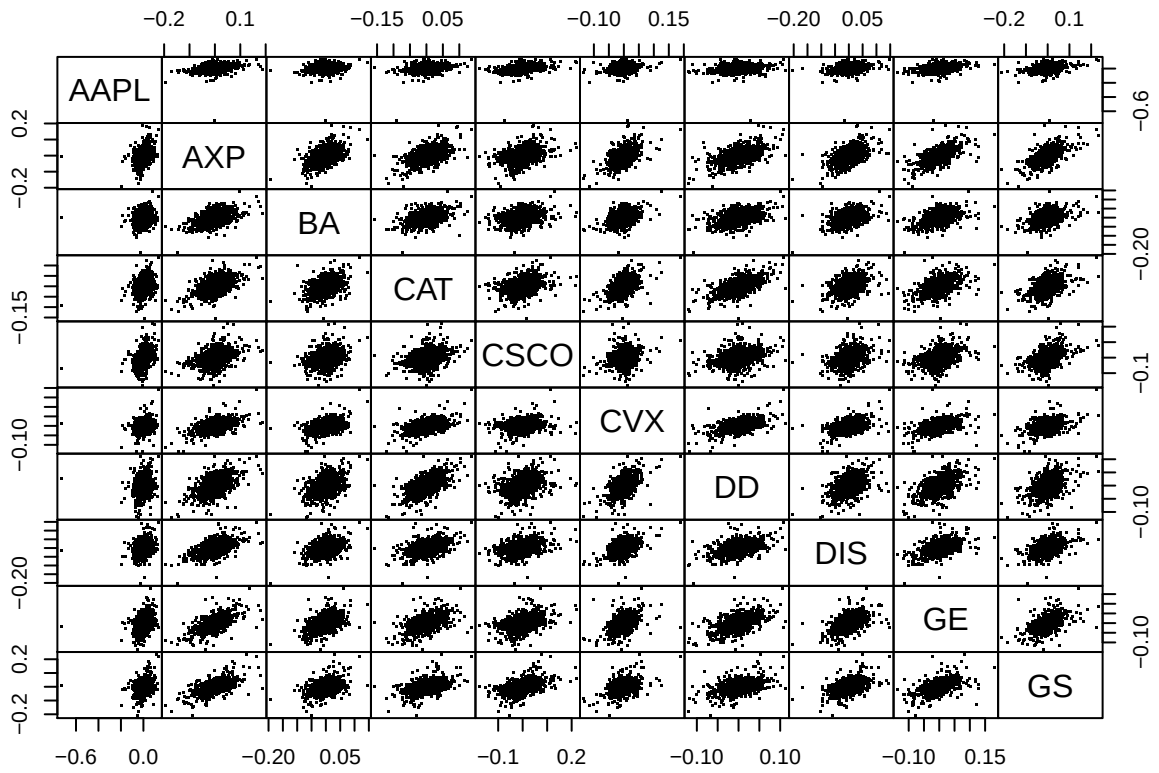
$$D_{ij}^2 = (\mathbf{X}_i - \bar{\mathbf{X}})' S^{-1} (\mathbf{X}_j - \bar{\mathbf{X}});\ i, j = 1, \ldots, n$$

The marginal distribution of D_i^2 under the null hypothesis arne not exactly chi-squared, although they converge to it for large values of n. We expect $D_1^2, \ldots, D_n^2$ to behave roughly like an iid sample form the χ_d^2 distribution, so we test them against this reference distribution to check whether the original data $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are multivariate Gaussian.

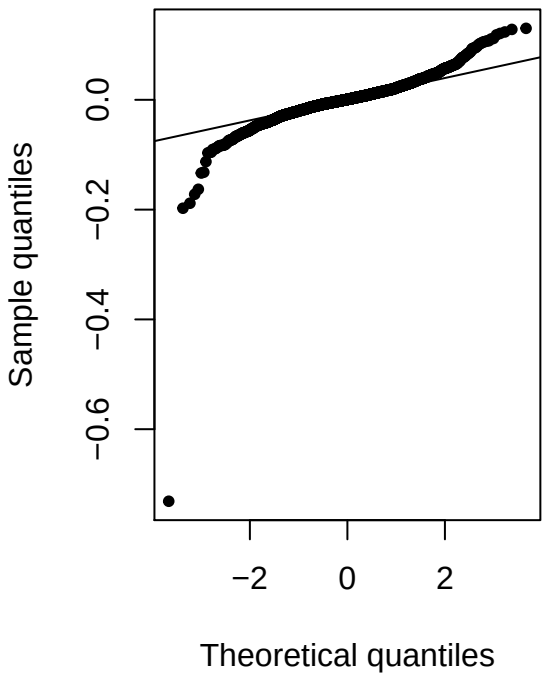
Finally, we compute the squared Mahalanobis distances $D_1^2, \ldots, D_n^2$ and draw the corresponding QQ-plot.

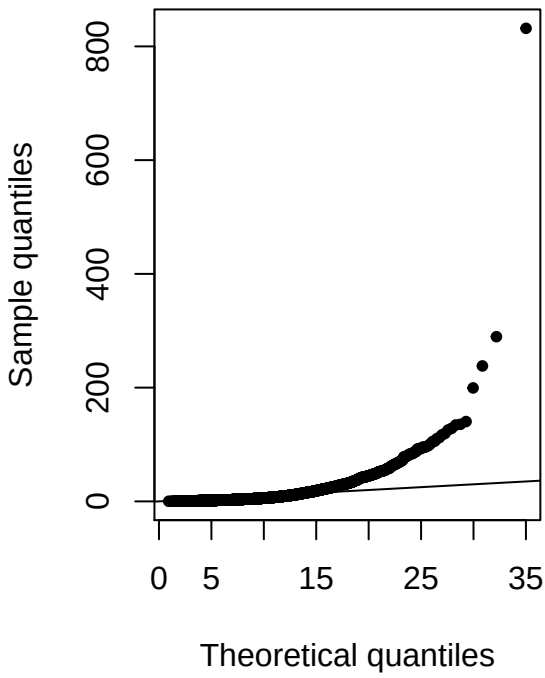
Daily log-returns

```r
pairs(as.matrix(X.d), gap = 0, pch = ".",lower=NULL) # visual assessment
```



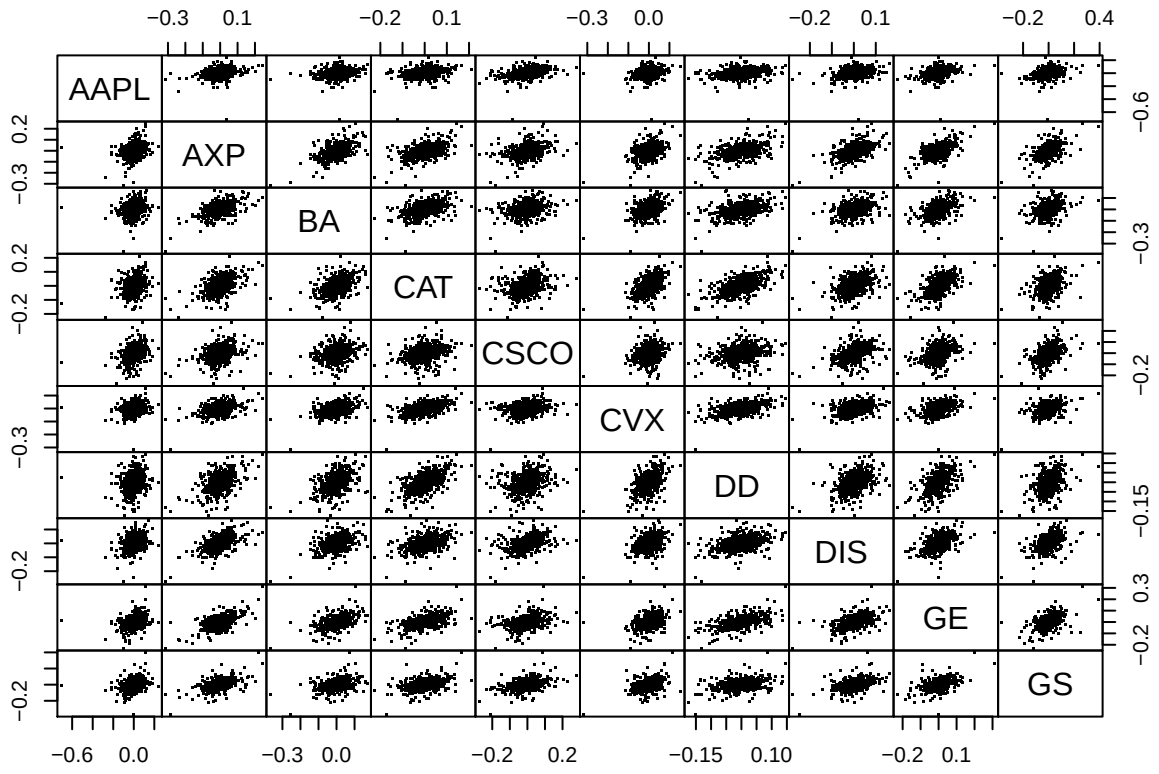

```r
par(mfrow=c(1,2))
qq_plot(X.d[,1], FUN = qnorm, method = "empirical",pch=20) # first margin only => already not normal
D2.d <- mahalanobis(X.d, center = colMeans(X.d), cov = cov(X.d)) # squared Mahalanobis distances
qq_plot(D2.d, FUN = function(p) qchisq(p, df = d),pch=20) # => departure clearly visible
```



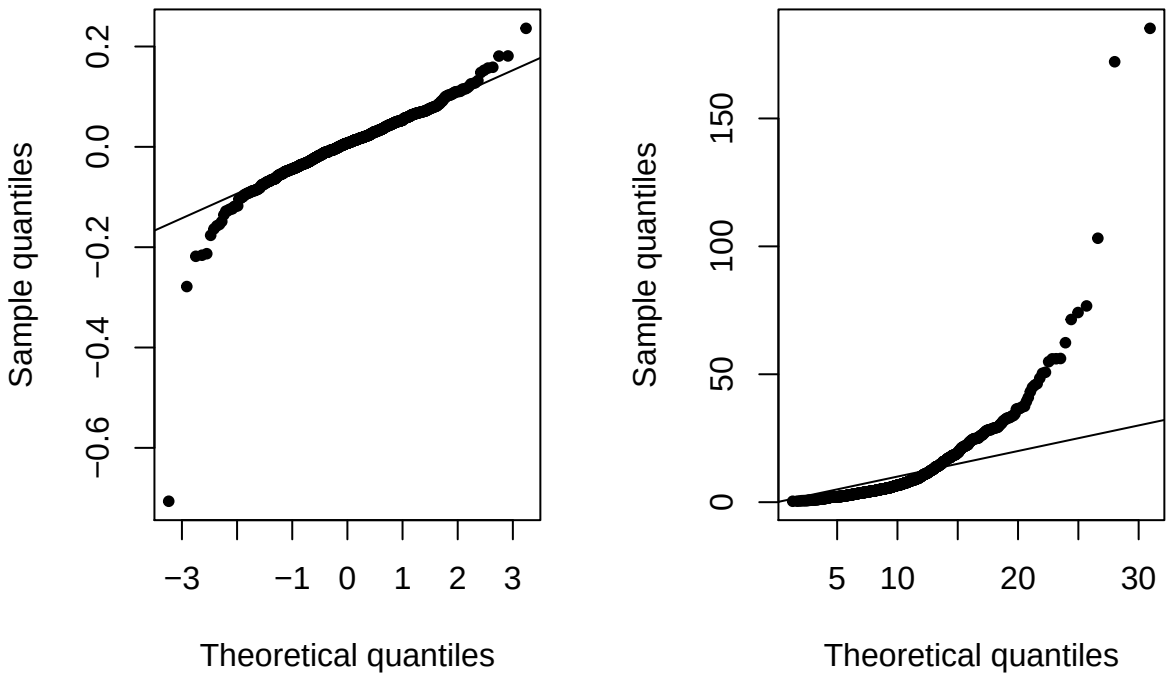


Weekly log-returns

```
pairs(as.matrix(X.w), gap = 0, pch = ".") # visual assessment
```




```
par(mfrow=c(1,2))
qq_plot(X.w[,1], FUN = qnorm, method = "empirical",pch=20) # better but still not too good
D2.w <- mahalanobis(X.w, center = colMeans(X.w), cov = cov(X.w)) # squared Mahalanobis distances
qq_plot(D2.w, FUN = function(p) qchisq(p, df = d),pch=20) # => departure clearly visible
```



2.2) Formal tests of normality We will now conduct formal tests based on the return data themselves and the corresponding squared Mahalanobis distances and angles by:

- Checking the marginal normality of each constituant using the shapiro test.
- Checking the joint normality of the data:

1. Testing whether $D_1^2, \ldots, D_n^2$ are χ_d^2 using the Anderson-Darling test.
2. Using the Mardia test of skewness and kurtosis (defined as functions of D_i^2 and D_{ij}^2)

Based on the plots and the tests $p-$values, we can see that the multivariate normal is not a particularly suitable model for the data, which of example shows fatter tails.

Daily log-returns

```
apply(X.d, 2, function(x) shapiro.test(x)$p.value) # tests of marginal normality
```

```
##         AAPL          AXP           BA          CAT         CSCO          CVX
## 1.123225e-53 7.557251e-47 1.923304e-34 5.628375e-34 2.114241e-45 2.082424e-41
##           DD          DIS           GE           GS
## 6.720522e-39 2.922888e-42 1.628754e-45 6.515510e-47
```

```
maha2_test(X.d) # Anderson--Darling test of squared Mahalanobis distances being chi_d^2
```

```
##
##  Anderson-Darling GoF Test
##
## data:  D2  and  pchisq
## AD = Inf, p-value = 1.491e-07
## alternative hypothesis: NA
```

```
mardia_test(X.d) # Mardia's kurtosis test of joint normality based on squared Mahalanobis distances
```

```
##
##  Mardia's kurtosis test (computed with method = 'direct')
##
## data:  X.d
## statistic = 756.63, p-value < 2.2e-16
## alternative hypothesis: two-sided
```

```
mardia_test(X.d, type = "skewness") # Mardia's skewness test of joint normality based on Mahalanobis an
```

```
##
##  Mardia's skewness test (computed with method = 'direct')
##
## data:  X.d
## statistic = 26782, p-value < 2.2e-16
## alternative hypothesis: one-sided
```

Weekly log-returns

```
apply(X.w, 2, function(x) shapiro.test(x)$p.value) # tests of marginal normality
```

```
##         AAPL          AXP           BA          CAT         CSCO          CVX
## 2.523832e-24 2.175050e-21 2.404761e-19 4.184531e-10 4.774746e-17 7.576028e-18
##           DD          DIS           GE           GS
## 4.337361e-13 6.625413e-17 6.408195e-21 6.903104e-22
```

```
maha2_test(X.w) # Anderson--Darling test of squared Mahalanobis distances being chi_d^2
```

```
##
##  Anderson-Darling GoF Test
##
## data:  D2  and  pchisq
## AD = Inf, p-value = 7.186e-07
## alternative hypothesis: NA
```

```
mardia_test(X.w) # Mardia's kurtosis test of joint normality based on squared Mahalanobis distances
```

```
##
##  Mardia's kurtosis test (computed with method = 'direct')
##
## data:  X.w
## statistic = 150.52, p-value < 2.2e-16
## alternative hypothesis: two-sided
```

```
mardia_test(X.w, type = "skewness") # Mardia's skewness test of joint normality based on Mahalanobis an
```

```
##
##  Mardia's skewness test (computed with method = 'direct')
##
## data:  X.w
## statistic = 3238.7, p-value < 2.2e-16
## alternative hypothesis: one-sided
```

3) Fit various multivariate models

Now, we fit generalized hyperbolic models to the (daily then weekly) return data.

We will consider different subclasses - t, hyperbolic nad NIG distributions- with a symmetry constraint (meaning the skewness parameter γ is set to 0) then in the generalized form (where the skewness parameter γ varies).

```
max.iter <- 1e4 # maximal number of iterations for the fitting procedures
```

3.1) Symmetric models Daily log-returns

```
fit.t.sym.d   <- fit.tmv   (X.d, symmetric = TRUE, nit = max.iter, silent = TRUE) # t
fit.NIG.sym.d <- fit.NIGmv (X.d, symmetric = TRUE, nit = max.iter, silent = TRUE) # normal inverse Gaus
fit.GH.sym.d  <- fit.ghypmv(X.d, symmetric = TRUE, nit = max.iter, silent = TRUE) # generalized hyperbo
```

Weekly log-returns

```
fit.t.sym.w   <- fit.tmv   (X.w, symmetric = TRUE, nit = max.iter, silent = TRUE) # t
fit.NIG.sym.w <- fit.NIGmv (X.w, symmetric = TRUE, nit = max.iter, silent = TRUE) # normal inverse Gaus
fit.GH.sym.w <- fit.ghypmv(X.w, symmetric = TRUE, nit = max.iter, silent = TRUE) # generalized hyperbol
```

3.2) Skewed models Daily log-returns

```
fit.t.skw.d   <- fit.tmv   (X.d, symmetric = FALSE, nit = max.iter, silent = TRUE) # t
fit.NIG.skw.d <- fit.NIGmv (X.d, symmetric = FALSE, nit = max.iter, silent = TRUE) # normal inverse Gau
fit.GH.skw.d  <- fit.ghypmv(X.d, symmetric = FALSE, nit = max.iter, silent = TRUE) # generalized hyperb
## Note: Warnings are due to deprecated recycling of an array of length 1 ('ghyp' problem)

#fit.GH.skw.d@gamma # skewness parameters
```

Weekly log-returns

```
fit.t.skw.w   <- fit.tmv   (X.w, symmetric = FALSE, nit = max.iter, silent = TRUE) # t
fit.NIG.skw.w <- fit.NIGmv (X.w, symmetric = FALSE, nit = max.iter, silent = TRUE) # normal inverse Gau
fit.GH.skw.w  <- fit.ghypmv(X.w, symmetric = FALSE, nit = max.iter, silent = TRUE) # generalized hyperb

## Note: Warnings are due to deprecated recycling of an array of length 1 ('ghyp' problem)
#fit.GH.skw.d@gamma # skewness parameters
```

4) Likelihood-ratio tests for comparing the goodness-of-fit of two models

In this section, we will compare the goodness-of-fit of the models via the log-likelihood ratio test,

$$\mathcal{H}_0 : \text{Model 1 is no better than Model 2.}$$

We compute the log-likelihood ratio statistics to compare

- Symmetric versus skewed t models
- Symmetric versus skewed hyperbolic models
- Symmetric t versus skewed NIG models

Compute the corresponding $p-$values and conclude.

Daily log-returns

```
likelihoods.d <- c(fit.t.sym.d@llh, fit.NIG.sym.d@llh, fit.GH.sym.d@llh,
                   fit.t.skw.d@llh, fit.NIG.skw.d@llh, fit.GH.skw.d@llh)
which.max(likelihoods.d) # => skewed generalized hyperbolic
```

```
## [1] 6
```

```
LRstat <- 2 * (fit.t.skw.d@llh - fit.t.sym.d@llh) # test statistic
1 - pchisq(LRstat, 10) # => H0 not rejected at 5% level
```

```
## [1] 0.7547205
```

```
LRstat <- 2 * (fit.GH.skw.d@llh - fit.GH.sym.d@llh) # test statistic
1 - pchisq(LRstat, 10) # => H0 not rejected at 5% level
```

```
## [1] 0.7284726
```

```
LRstat <- 2 * (fit.GH.sym.d@llh - fit.t.sym.d@llh) # test statistic
1 - pchisq(LRstat, 1) # => H0 rejected at 5% level => symmetric generalized hyperbolic preferred
```

```
## [1] 8.056387e-05
```

Weekly log-returns

```
likelihoods.w <- c(fit.t.sym.w@llh, fit.NIG.sym.w@llh, fit.GH.sym.w@llh,
                   fit.t.skw.w@llh, fit.NIG.skw.w@llh, fit.GH.skw.w@llh)
which.max(likelihoods.w) # => also skewed generalized hyperbolic
```

```
## [1] 6
```

```
LRstat <- 2 * (fit.t.skw.w@llh - fit.t.sym.w@llh) # test statistic
1 - pchisq(LRstat, 10) # => H0 not rejected at 5% level
```

```
## [1] 0.2992959
```

```
LRstat <- 2 * (fit.GH.skw.w@llh - fit.GH.sym.w@llh) # test statistic
1 - pchisq(LRstat, 10) # => H0 not rejected at 5% level
```

```
## [1] 0.2712829
```

```
LRstat <- 2 * (fit.GH.sym.w@llh - fit.t.sym.w@llh) # test statistic
1 - pchisq(LRstat, 1) # => H0 rejected at 5% level => symmetric generalized hyperbolic preferred
```

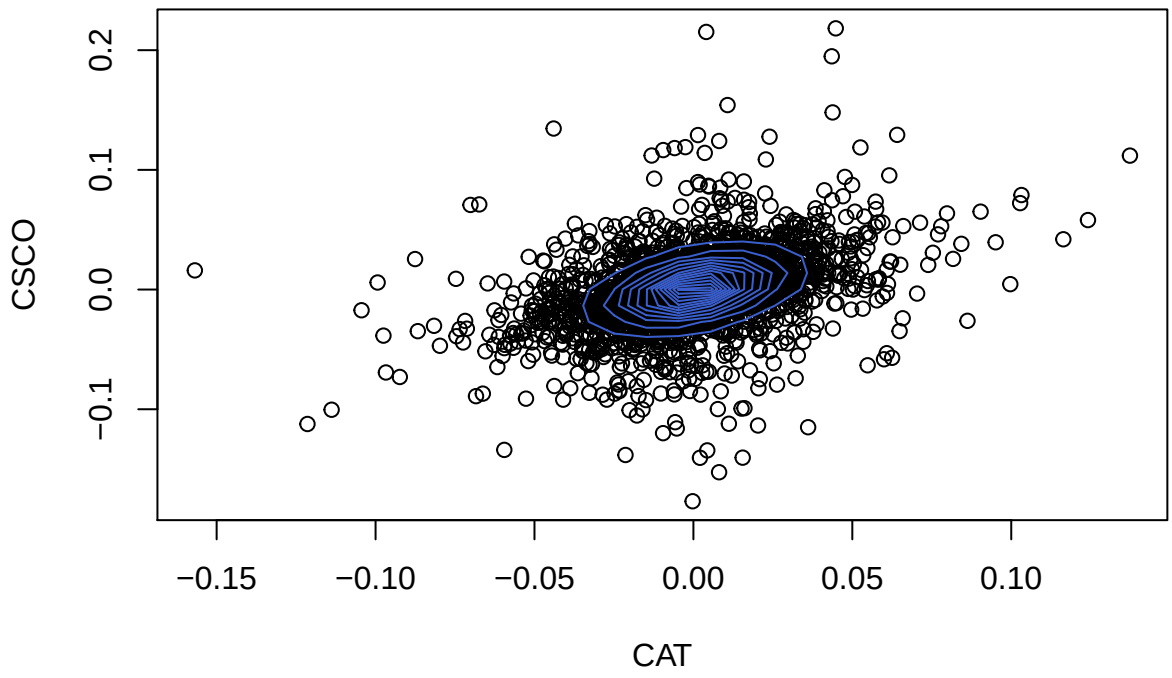
```
## [1] 0.02429803
```

5) Visual assessment

We now visually compare two constituents against the corresponding pairwise implied contour lines

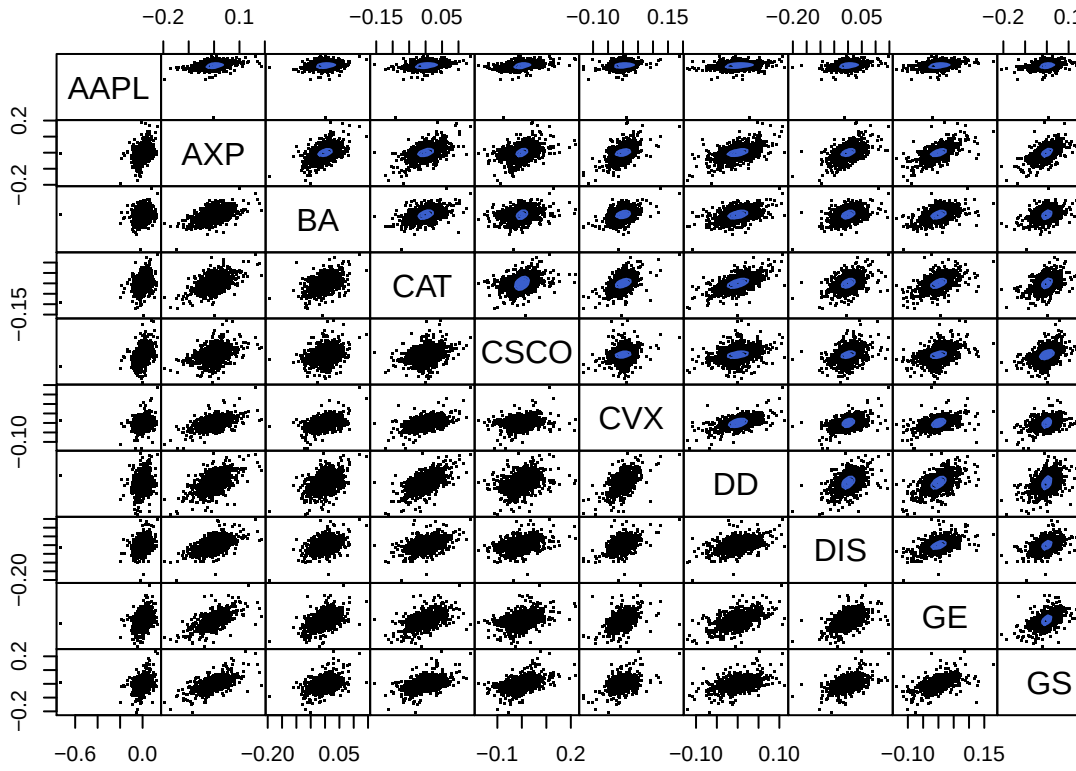
```
X <- as.matrix(X.d) # data we consider
```

```
ind <- c(4, 5) # "CAT", "CSCO"
X. <- X[,ind]
plot(X.)
obj <- fit.GH.sym.d # get d-dimensional object
obj@mu <- obj@mu[ind]
obj@sigma <- obj@sigma[ind, ind]
obj@gamma <- obj@gamma[ind[2]] # (0 in symmetric case)
x. <- seq(min(X.[,1]), max(X.[,1]), length.out = 30) # x-locations of density evaluation points
y. <- seq(min(X.[,2]), max(X.[,2]), length.out = 30) # y-locations of density evaluation points
z. <- outer(x., y., function(xx, yy)
  dghyp(cbind(xx, yy), object = obj)) # implied bivariate density
contour(x., y., z = z., drawlabels = FALSE, axes = FALSE, col = "royalblue3", add = TRUE)
```




```
row <- 1 # row number in upper-right panel
col <- 1 # column number in upper-right panel
pairs(X, gap = 0, pch = ".",
      upper.panel = function(x, y, ...) {
        ## Figure out in which row and column we are in the plot (needed later)
        row <<- row + 1
        if(row >= col) {
          col <<- col + 1
          row <<- 1
        }
        ## Plot the points with indices (i, j)
        points(x, y, ...)
        ## Extract the (i, j)th 'margin' of the object of class 'ghyp'
        obj <- fit.GH.sym.d # get d-dimensional object
        ind <- c(row, col) # pairwise indices to consider
        obj@mu <- obj@mu[ind]
        obj@sigma <- obj@sigma[ind, ind]
        obj@gamma <- obj@gamma[col] # (0 in symmetric case)
        ## Compute contours based on
        x. <- seq(min(x), max(x), length.out = 30) # x-locations of density evaluation points
        y. <- seq(min(y), max(y), length.out = 30) # y-locations of density evaluation points
        z. <- outer(x., y., function(xx, yy)
          dghyp(cbind(xx, yy), object = obj)) # implied bivariate density
        ## Plot contours
```

```
    contour(x., y., z = z., nlevels = 5,
            drawlabels = FALSE, axes = FALSE, col = "royalblue3", add = TRUE)
})
```



Similarly for all d pairs

References This project is basesd on R code from QRMtutorials.org