NAVIER STOKES FOR CAVITY FLOW

PANAGIOTIS TSILIFIS

ABSTRACT. Source: http://nbviewer.jupyter.org/github/barbagroup/CFDPython/blob/master/lessons/15_Step_11.ipynb

1. EQUATIONS

The final two steps in this interactive module teaching beginning [CFD with Python] (https://bitbucket.org/cfdpython/cfd-python-class) will both solve the Navier-Stokes equations in two dimensions, but with different boundary conditions. The momentum equation in vector form for a velocity field $\vec{v}$ is:,

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v}. \tag{1}$$

This represents three scalar equations, one for each velocity component (u, v, w). But we will solve it in two dimensions, so there will be two scalar equations. Remember the continuity equation? This is where the Poisson equation for pressure comes in!

1.1. **Cavity flow in 2d.**

Here is the system of differential equations: two equations for the velocity components u, v and one equation for pressure:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{3}$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial v}{\partial y}\right). \tag{4}$$

From the previous steps, we already know how to discretize all these terms. Only the last equation is a little unfamiliar. But with a little patience, it will not be hard!

2. Discretization

First, let's discretize the u-momentum equation, as follows:

$$\frac{u_{i,j}^{n+1}-u_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i-1,j}^{n}}{\Delta x}+v_{i,j}^{n}\frac{u_{i,j}^{n}-u_{i,j-1}^{n}}{\Delta y} \tag{5}$$

$$=-\frac{1}{\rho}\frac{p_{i+1,j}^{n}-p_{i-1,j}^{n}}{2\Delta x}+\nu\left(\frac{u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n}}{\Delta x^2}+\frac{u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n}}{\Delta y^2}\right). \tag{6}$$

Similarly for the v-momentum equation:

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta t}+u_{i,j}^{n}\frac{v_{i,j}^{n}-v_{i-1,j}^{n}}{\Delta x}+v_{i,j}^{n}\frac{v_{i,j}^{n}-v_{i,j-1}^{n}}{\Delta y} \tag{7}$$

$$=-\frac{1}{\rho}\frac{p_{i,j+1}^{n}-p_{i,j-1}^{n}}{2\Delta y}+\nu\left(\frac{v_{i+1,j}^{n}-2v_{i,j}^{n}+v_{i-1,j}^{n}}{\Delta x^2}+\frac{v_{i,j+1}^{n}-2v_{i,j}^{n}+v_{i,j-1}^{n}}{\Delta y^2}\right) \tag{8}$$

Finally, the discretized pressure-Poisson equation can be written thus:

$$\frac{p_{i+1,j}^{n}-2p_{i,j}^{n}+p_{i-1,j}^{n}}{\Delta x^2}+\frac{p_{i,j+1}^{n}-2p_{i,j}^{n}+p_{i,j-1}^{n}}{\Delta y^2} \tag{9}$$

$$=\rho\left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}+\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\right)\right. \tag{10}$$

$$-\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}\frac{u_{i+1,j}-u_{i-1,j}}{2\Delta x}-2\frac{u_{i,j+1}-u_{i,j-1}}{2\Delta y}\frac{v_{i+1,j}-v_{i-1,j}}{2\Delta x} \tag{11}$$

$$\left.-\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\frac{v_{i,j+1}-v_{i,j-1}}{2\Delta y}\right]. \tag{12}$$

You should write these equations down on your own notes, by hand, following each term mentally as you write it. As before, let's rearrange the equations in the way that the iterations need to proceed in the code. First, the momentum equations for the velocity at the next time step.

The momentum equation in the u direction:

$$u_{i,j}^{n+1}=u_{i,j}^{n}-u_{i,j}^{n}\frac{\Delta t}{\Delta x}(u_{i,j}^{n}-u_{i-1,j}^{n})-v_{i,j}^{n}\frac{\Delta t}{\Delta y}(u_{i,j}^{n}-u_{i,j-1}^{n})-\frac{\Delta t}{\rho 2\Delta x}(p_{i+1,j}^{n}-p_{i-1,j}^{n}) \tag{13}$$

$$+\nu\left(\frac{\Delta t}{\Delta x^2}(u_{i+1,j}^{n}-2u_{i,j}^{n}+u_{i-1,j}^{n})+\frac{\Delta t}{\Delta y^2}(u_{i,j+1}^{n}-2u_{i,j}^{n}+u_{i,j-1}^{n})\right). \tag{14}$$

The momentum equation in the v direction:

$$v_{i,j}^{n+1}=v_{i,j}^{n}-u_{i,j}^{n}\frac{\Delta t}{\Delta x}(v_{i,j}^{n}-v_{i-1,j}^{n})-v_{i,j}^{n}\frac{\Delta t}{\Delta y}(v_{i,j}^{n}-v_{i,j-1}^{n})-\frac{\Delta t}{\rho 2\Delta y}(p_{i,j+1}^{n}-p_{i,j-1}^{n}) \tag{15}$$

$$+\nu\left(\frac{\Delta t}{\Delta x^2}(v_{i+1,j}^{n}-2v_{i,j}^{n}+v_{i-1,j}^{n})+\frac{\Delta t}{\Delta y^2}(v_{i,j+1}^{n}-2v_{i,j}^{n}+v_{i,j-1}^{n})\right). \tag{16}$$

Almost there! Now, we rearrange the pressure-Poisson equation:

$$p_{i,j}^n = \frac{(p_{i+1,j}^n + p_{i-1,j}^n)\Delta y^2 + (p_{i,j+1}^n + p_{i,j-1}^n)\Delta x^2}{2(\Delta x^2 + \Delta y^2)} - \frac{\rho \Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)} \times \tag{17}$$

$$\left[\frac{1}{\Delta t}\left(\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right) - \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x}\right. \tag{18}$$

$$\left. -2\frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y}\frac{v_{i+1,j} - v_{i-1,j}}{2\Delta x} - \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\frac{v_{i,j+1} - v_{i,j-1}}{2\Delta y}\right] \tag{19}$$

Department of Civil Engineering, University of Southern California, Los Angeles, CA 90089

E-mail address: tsilifis@usc.edu