

Chapter 12: Sequences and Series

12.1 Sequence

A sequence is a set of numbers written down in a specific order. For example: 1,3,5,7,8 or -1,-2,-3,-4

Each number in the sequence is called a **term** of the sequence. The number of terms in the first sequence above is five, and the number of terms in the second is four.

Sometimes we use the symbol '**...**' to indicate that the sequence continues.

All of the sequences given above have a finite number of terms. They are known as **finite sequences**. Some sequences go on for ever, and these are called **infinite sequences**. To indicate that a sequence might go on for ever we can use the '**...**' notation. So when we write 1,3,5,7,9 ... it can be assumed that this sequence continues indefinitely.

Notation used for sequences

We use a subscript notation to refer to different terms in a sequence. For example, suppose we denote (indicate) the sequence 1,3,5,7,9,11 by x . Then the first term can be called x_1 , the second term x_2 and so on. That is, $x_1 = 1$, $x_2 = 3$, $x_3 = 5$, $x_4 = 7$, and so on

12.2 Arithmetic progressions

A particularly simple way of forming a sequence is to calculate each new term by **adding** a fixed amount to the previous term. For example, suppose the first term is 1 and we find the subsequent terms by repeatedly adding 6. We obtain 1,7,13,19 Such a sequence is called **arithmetic progression** or **arithmetic sequence**. The fixed amount that is added each time is a constant called the **common difference**.

Key points

An arithmetic progression can be written

$a, a + d, a + 2d, a + 3d \dots$

a is the **first term**, d is the **common difference**.

Study the following pattern:

- The first term is a
- The second term is $a + d$
- The third term is $a + 2d$
- The forth term is $a + 3d$

And so on. This lead to the following formula for the **n th term**:

$a + (n-1)d$

12.3 Geometric Progression

Another simple way of forming a sequence is to calculate each new term by multiplying the previous term by a fixed amount. For example, suppose the first term of a sequence is 2, and we find the second, third, and forth terms by repeatedly multiplying by 5. We obtain the sequence 2, 10, 50, 250. Such a sequence is called a geometric sequence or geometric progression. The fixed amount by which term is multiplied is called the common ratio

Key points

A geometric progression can be written: $a, ar, ar^2, ar^3 \dots$ a is the first term, r is the common ratio

Study the following pattern:

- The first term is a
- The second term is ar
- The third term is ar^2
- The fourth term is ar^3

and so on. This leads us to the following formula for the n^{th} term:

The n^{th} term of a geometric progression is given by ar^{n-1}

12.4 Infinite sequences

Some sequences continue indefinitely, and these are called infinite sequences

12.5 Series and Sigma notation

If the terms of a sequence are added the result is known as a series. For example, if we add the terms of the sequence 1, 2, 3, 4, 5 we obtain the series $1 + 2 + 3 + 4 + 5$. Clearly a series is a sum.

Sigma notation $\sum_{k=1}^{12}$, provides a concise and convenient way of writing long sums. The sum $1+2+3+\dots+10+11+12$ can be written very concisely using the Greek letter \sum as

$\sum_{k=1}^{12}$

The \sum stands for sum, in this case the sum of all the values of k as k ranges through all whole numbers from 1 to 12.

12.6 Arithmetic series

If the terms of an arithmetic sequence are added, the result is known as an arithmetic series. For example, the arithmetic progression with five terms having first term 4 and common difference 5 is 4, 9, 14, 19, 24. It is easily verified that this has sum 70. If the series has a large number of terms then finding its sum by directly adding all the terms will be laborious. Fortunately there is a formula that enables us to find the sum of an arithmetic series.

Key points

The sum of the first n terms of an arithmetic series with first term a and common difference d is denoted by S_n and given by:

$$S_n = \frac{n}{2} (2a + (n - 1)d)$$

12.7 Geometric series

If the terms of a geometric sequence are added, the result is known as a **geometric series**. For example, the geometric progression with five terms having first term 2 and common ratio 3 is 2, 6, 18, 54, 162. If the terms are added we obtain the geometric series 2+6+18+54+162

Key points

The sum of the first n terms of a geometric series with first term a and common ratio r is denoted by S_n and given by

$$S_n = \frac{a(1-r^n)}{1-r}$$
 provided r is not equal to 1

12.8 Infinite geometric series

When the terms of an infinite sequence are added we obtain an infinite series. It may seem strange to try to add an infinite number of terms but under some circumstances their sum is finite and can be found.

Consider the special case of an infinite geometric series for which the common ratio r lies between -1 and 1.

Key points

The sum of an infinite number of terms of a geometric series is denoted by $S_\infty = \frac{a}{1-r}$ provided $-1 < r < 1$

Recursion (not in book)

This lecture introduces recursion, a powerful concept in mathematics and programming where a function or sequence is defined in terms of itself. To work with a recursive definition, it must always be grounded with one or more starting terms.

1. Factorial Sequence ($n!$)

This is the most well-known recursive function.

- Grounding Terms: $u_1 = 1$
- Recursive Formula: $u_n = n \times u_{n-1}$

Term (n)	Calculation	Value
v_2	$2 \times v_1 = 2 \times 1$	2

Term (n)	Calculation	Value
v_3	$3 * v_2 = 3 * 2$	6
v_4	$4 * v_3 = 4 * 6$	24
v_5	$5 * v_4 = 6 * 24$	120

This formula generates the factorials ($n!$), which are used extensively in probability and combinatorics.

2. The Fibonacci Sequence

This sequence is defined by adding the two immediately preceding terms.

- Grounding Terms: $u_1 = 0$ and $u_2 = 1$
- Recursive Formula: $u_{n+2} = u_{n+1} + u_n$ (The next term is the sum of the two before it.)

Term (n)	Calculation	Value
v_3	$v_2 + v_1 = 1 + 0$	1
v_4	$v_3 + v_2 = 1 + 2$	2
v_5	$v_4 + v_3 = 2 + 1$	3
v_6	$v_5 + v_4 = 3 + 2$	5
v_7	$v_6 + v_5 = 5 + 3$	8

3. The Golden Ratio

(ϕ) The Fibonacci sequence has a beautiful mathematical property: the ratio of consecutive terms approaches the Golden Ratio

(ϕ) as n approaches infinity. $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \phi$

The Golden Ratio is an irrational number (cannot be represented as a simple fraction p/q) and has historical significance in architecture and design.