Question 1

(a)

$$\text{Value}_{10} = (a \cdot x^2) + (b \cdot x^1) + (c \cdot x^0) + (d \cdot x^{-1}) + (e \cdot x^{-2})$$

(b)

(i.)

$$10.0011_2 = (1 \times 2^1) + (0 \times 2^0) + (0 \times 2^{-1}) + (0 \times 2^{-2}) + (1 \times 2^{-3}) + (1 \times 2^{-4})$$

- $(1 \times 2) = 2$
- $(0 \times 1) = 0$
- $(0 \times 0.5) = 0$
- $(0 \times 0.25) = 0$
- $(1 \times 0.125) = 0.125$
- $(1 \times 0.0625) = 0.0625$

$2 + 0.125 + 0.0625 = 2.1875_{10}$

(ii.)

- The first 1 is in the third position after the point: 2^{-3} (which is $\frac{1}{8}$ or 0.125).
- The second 1 is in the fourth position after the point: 2^{-4} (which is $\frac{1}{16}$ or 0.0625).

(iii.)

- $1 + 1 = 10_2$ (which is 2 in decimal)
- $10_2 + 1 = 11_2$ (which is 3 in decimal)
- $11_2 + 1 = 100_2$ (which is 4 in decimal)Final Result: 100_2

Result: 100_2

(iv.)

Result is 10.1_2

(c)

Convert binary to deciaml

- 11_2: $(1 \times 2^1) + (1 \times 2^0) = 2 + 1 = \mathbf{3}$
- 1000_2: $(1 \times 2^3) + (0 \times 2^2) + (0 \times 2^1) + (0 \times 2^0) = \mathbf{8}$
- 101_2: $(1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0) = 4 + 1 = \mathbf{5}$

Use decimal to solve the equation

$$\frac{3}{x} = \frac{8}{x + 5}$$

Now, cross-multiply to solve for x:

- $3(x + 5) = 8x$
- $3x + 15 = 8x$
- $15 = 8x - 3x$
- $15 = 5x$
- $x = 3$

Converting to base 2

- $3_{10} = (1 \times 2^1) + (1 \times 2^0) = \mathbf{11_2}$

(d)

Step 1: Expand the Base b Expressions

According to the expansion method, a two-digit number XY_b is equal to $X \cdot b^1 + Y \cdot b^0$.

- $AB_b = A \cdot b + B$
- $BA_b = B \cdot b + A$

Now, substitute these into the original equation:

$$(Ab + B) + (Bb + A) = 121_{10}$$

Step 2: Simplify the Equation

Combine like terms (the A terms and the B terms):

$$Ab + A + Bb + B = 121$$

$$A(b + 1) + B(b + 1) = 121$$

$$(A + B)(b + 1) = 121$$

Step 3: Analyze the Factors of 121

We now have two factors, $(A + B)$ and $(b + 1)$, which when multiplied equal 121. The integer factors of 121 are:

- 1×121
- 11×11
- 121×1

We can evaluate these possibilities based on the constraints of number bases:

1. Possibility 1: $(A + B) = 1$, $(b + 1) = 121$

- This implies $b = 120$. However, if $A + B = 1$ and $A \neq B$, one digit must be 0 and the other 1. Since A and B are lead digits in AB_b and BA_b, neither can be 0. This is ruled out.

2. Possibility 2: $(A + B) = 11$ and $(b + 1) = 11$

   - This implies $b = 10$.

- If the base is 10, then $A + B = 11$. Distinct digits that sum to 11 works (e.g., $A = 5, B = 6$ or $A = 2, B = 9$).
- Verification: $56_{10} + 65_{10} = 121_{10}$. This holds true.

3. Possibility 3: $(A + B) = 121$ and $(b + 1) = 1T$

   - his implies $b = 0$, which is impossible as a base must be greater than 1.

Final SolutionBased on the factor analysis:

- Base (b): 10
- Digits (A and B): Any two distinct digits that sum to 11. Examples include $A = 5, B = 6$ or $A = 4, B = 7$.

Question 2

(a)

To find the number of bacteria present immediately after the second sterilization, we calculate the population changes through each growth and sterilization phase step-by-step.

Step 1: First Growth Phase (0 to 12 hours)

The bacteria population follows a geometric growth pattern where the population doubles every 3 hours.

- Initial population (a): 200
- Doubling time: 3 hours
- Total time: 12 hours
- Number of doublings (n): $\frac{12}{3} = 4$ doublings.

Using the geometric sequence logic $a_{n+1} = a \cdot r^n$ where $r = 2$:

$$N_{12} = 200 \cdot 2^4 = 200 \cdot 16 = 3,200 \text{ bacteria}$$

Step 2: First Sterilization (at 12 hours)

The agent kills 70 of the bacteria instantly, leaving 30 remaining.

Remaining population: $3,200 \cdot 0.30 = 960$ bacteria

Step 3: Second Growth Phase (Additional 36 hours)

The remaining bacteria continue to double every 3 hours.

- Initial population for this phase: 960
- Total time: 36 hours
- Number of doublings: $\frac{36}{3} = 12$ doublings.

The population after 36 hours of growth is:

$$N_{48} = 960 \cdot 2^{12} = 960 \cdot 4,096 = 3,932,160 \text{ bacteria}$$

Step 4: Second Sterilization (at 48 hours)

A second identical agent is applied, again killing 70 (30 remaining).

- Final population: $3,932,160 \cdot 0.30 = 1,179,648$

(b)

(i.)

The equation for the sum of the first n terms of any geometric sequence is $S_n = \frac{a(1-r^n)}{1-r}$.

1. Identify the given values:

- First term (a): 4
- Common ratio (r): 3
- Sum (S_n): 364

2. Substitute the values into the formula:

$$364 = \frac{4(1-3^n)}{1-3}$$

3. Solve for n:

- Simplify the denominator: $364 = \frac{4(1-3^n)}{-2}$
- Divide 4 by -2: $364 = -2(1-3^n)$
- Divide both sides by -2: $-182 = 1 - 3^n$
- Rearrange the equation: $3^n = 182 + 1 = 183$

Since $3^4 = 81$ and $3^5 = 243$, the value $n = 6$ would require a sum of 1456. If we assume the target was $3^n - 1 = 242$ ($S_n = 484$), n would be 5. Given the exact prompt $3^n = 183$, n is approximately 4.75. However, in discrete sequence problems, n must be an integer. If the sum was 484, $n = 5$. If the sum was 120, $n = 4$.

(ii.)

The formula for the n^{th} (last) term of a geometric sequence is $a_n = a \cdot r^{(n-1)}$.

If we use the calculated value (assuming $n = 5$ for a standard integer result):

- $a_5 = 4 \cdot 3^4$
- $a_5 = 4 \cdot 3^{(5-1)}$
- $a_5 = 4 \cdot 81$
- $a_5 = 324$

Result:

- Number of terms: $n \approx 4.75$ (or 5 if the sum was intended to be 484). Last term: If $n = 5$, the last term is 324.

(c)

To find any possible values of common ratio, we can translate the given information into algebraic equations.

1. Define the Terms

Let's assume the first term of the geometric progression (GP) be a and the common ratio be r. The terms of the GP are defined as $T_n = ar^{n-1}$.

- The 2nd term is ar.
- The 4th term is ar^3.
- The 6th term is ar^5.

2.Set up the Arithmetic Progression (AP) Condition

In an arithmetic progression, the difference between consecutive terms is constant. Therefore, the middle term is the average of the two surrounding terms:

$$2 \times (\text{4th term}) = (\text{2nd term}) + (\text{6th term})$$

Substituting our GP terms:

$$2(ar^3) = ar + ar^5$$

3. Simplify and Solve for r

Since we are told the first term a is positive ($a > 0$), we can safely divide both sides by a. Additionally, if $r = 0$, the sequence becomes $a, 0, 0, 0...$, which is technically an AP, but typically $r \neq 0$ in these problems.

Dividing by ar:

$$2r^2 = 1 + r^4$$

Rearrange this into a quadratic form by moving all terms to one side:

$$r^4 - 2r^2 + 1 = 0$$

This expression is a perfect square trinomial:

$$(r^2 - 1)^2 = 0$$

Taking the square root of both sides:

$$r^2 - 1 = 0$$

$$r^2 = 1$$

$$r = \pm 1$$

(d)

1. The First 6 Books (Full Price)The prices follow a geometric progression where the first term $a = 3$ and the common ratio $r = 2$.

The prices are:

- Book 1: £3
- Book 2: £6
- Book 3: £12
- Book 4: £24
- Book 5: £48
- Book 6: £96

Sum of first 6 books:$3 + 6 + 12 + 24 + 48 + 96 = \mathbf{£189}$

2. The Last 4 Books (Discounted)

First, let's find what these books would have cost at full price by continuing the doubling pattern:

- Book 7: $96 \times 2 = £192$
- Book 8: $192 \times 2 = £384$
- Book 9: $384 \times 2 = £768$
- Book 10: $768 \times 2 = £1,536$

Total "Full Price" for books 7–10:$192 + 384 + 768 + 1,536 = £2,880$

Apply the 15% discount:If the discount is 15%, Ben pays 85% of the price (1100).

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$$2,880 \times 0.85 = \mathbf{£2,448}$$

3. Total Expenditure

Now, add the two totals together:

$$£189 \text{ (Books 1-6)} + £2,448 \text{ (Books 7-10)} = \mathbf{£2,637}$$

Total Amount Paid: £2,637

Question 3

(a)

We have two requirements for x:

$x \equiv 5 \pmod{6}$ $x \equiv 3 \pmod{7}$

List Multiples (The "Simple" Method)

Since the moduli (6 and 7) are small, we can list the numbers that satisfy the second (larger) modulus and check them against the first. Numbers satisfying $x \equiv 3 \pmod{7}$:

- $x = 3$ ($3 \div 6$ leaves remainder 3) — No
- $x = 10$ ($10 \div 6$ leaves remainder 4) — No
- $x = 17$ ($17 \div 6$ leaves remainder 5) — Yes!

(b)

1. Calculate the Remainder

We need to find $224 \pmod{24}$. We know that $24 \times 10 = 240$. 240 is exactly 10 full days. Subtracting 24 from 240 gives us 216 ($24 \times 9 = 216$). The difference between the total hours (224) and the last full day (216) is:

$$224 - 216 = 8 \text{ hours}$$

So, 224 hours is equivalent to 9 full days and 8 hours.

2. Apply the Elapsed Time

Now, add the remaining 8 hours to the current time:

- Current Time: 14:00
- Elapsed Hours: + 8 hours
- Calculation: $14 + 8 = 22$

ConclusionThe clock will show 22:00.

(c)

Find the Least Common Multiple (LCM)

Prime	Highest power
2	2^3=8
3	3^2=9
5	5^1=5
7	7^1=7

Multiply these highest powers together:

$$\text{LCM} = 8 \times 9 \times 5 \times 7$$

$$\text{LCM} = 72 \times 35$$

$$\text{LCM} = 2,520$$

Then we solve for x

We know that $x + 1 = 2,520$.Therefore:

$$x = 2,520 - 1$$

$$x = 2,519$$

AnswerThe smallest positive integer is 2,519.

(d)

Check if its solvable

A linear congruence of the form $ax \equiv b \pmod{n}$ has solutions if and only if the greatest common divisor (GCD) of a and n divides b.

- $a = 6$
- $n = 21$
- $b = 15$

Calculate the GCD of 6 and 21:

$$\text{GCD}(6, 21) = 3$$

Check if 3 divides 15:

$$15 \div 3 = 5$$

Since it divides evenly, solutions exist. Specifically, there are exactly 3 distinct solutions modulo 21.

Simplify for congruence

We can divide the entire congruence (including the modulus) by the GCD (3) to make it easier to solve.

Original: $6x \equiv 15 \pmod{21}$Divide by 3:

$$2x \equiv 5 \quad \pmod{7}$$

Solve the simplified congruence

Now we need to find an x such that $2x$ leaves a remainder of 5 when divided by 7. We can test small integer values for x:

- If $x = 1$, $2(1) = 2$
- If $x = 2$, $2(2) = 4$
- If $x = 3$, $2(3) = 6$
- If $x = 4$, $2(4) = 8 \equiv 1 \pmod{7}$
- If $x = 5$, $2(5) = 10 \equiv 3 \pmod{7}$
- If $x = 6$, $2(6) = 12 \equiv 5 \pmod{7}$ $\rightarrow$ This is the solution.

So, $x \equiv 6 \pmod{7}$.

Find all solutions modulo 21

The solution $x \equiv 6 \pmod{7}$ means that x can be written as:

$$x = 6 + 7k$$

where k is an integer.

To find the specific solutions modulo 21, we substitute $k = 0, 1, 2$ (since there are 3 solutions):

- k = 0: $x = 6 + 7(0) = \mathbf{6}$
- k = 1: $x = 6 + 7(1) = \mathbf{13}$
- k = 2: $x = 6 + 7(2) = \mathbf{20}$

The integers x satisfying the congruence are:

$$x \equiv 6, 13, 20 \pmod{21}$$

Question 4

(a)

(i.)

The Law of Cosines states that for a triangle with sides a, b, and c, and angle C opposite side c:

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$

Substitute the given values ($a = 7$, $b = 8$, and $C = x$):

$$c^2 = 7^2 + 8^2 - 2(7)(8)\cos(x)$$

Simplifying the squares and the product:

$$c^2 = 49 + 64 - 112\cos(x)$$

$$c^2 = 113 - 112\cos(x)$$

(ii.)

Now we are given that side $c = 9$. We substitute this into our equation from step (i):

$$9^2 = 113 - 112\cos(x)$$

$$81 = 113 - 112\cos(x)$$

Rearrange the equation to isolate the term with x:

$$112\cos(x) = 113 - 81$$

$$112\cos(x) = 32$$

Divide by 112:

$$\cos(x) = \frac{32}{112}$$

This fraction simplifies (divide numerator and denominator by 16):

$$\cos(x) = \frac{2}{7}$$

(iii.)

To find x, we take the inverse cosine ($\arccos$) of the fraction we found.

$$x = \cos^{-1}\left(\frac{2}{7}\right)$$

Using a calculator:

$$x \approx 73.39845...^\circ$$

Rounding to 2 decimal places:

$$C \approx 73.40^\circ$$

(b)

To find the angle A, we can adapt the area formula provided. The general formula for the area of a triangle given two sides and the included angle is:

$$ Area = \frac{1}{2} \times \text{side}_1 \times \text{side}_2 \times \sin(\text{Included Angle})

$$In this problem, the sides AB and AC meet at angle A.

1. Set up the Equation

Substitute the given values ($AB = 10$, $AC = 13$, $Area = 30$) into the formula:

$$30 = \frac{1}{2} \times 10 \times 13 \times \sin(A)$$

2. Solve for $\sin(A)$

First, multiply the constants on the right side:

$$30 = 5 \times 13 \times \sin(A)$$

$$30 = 65\sin(A)$$

Now, isolate $\sin(A)$:

$$\sin(A) = \frac{30}{65}$$

Simplifying the fraction by dividing top and bottom by 5:

$$\sin(A) = \frac{6}{13}$$

3. Calculate Angle A

To find the angle, we take the inverse sine ($\sin^{-1}$) of the fraction. Note that there are two possible angles between 0° and 180° that have a positive sine value: one acute and one obtuse.

Case 1: Acute Angle

$$A = \sin^{-1}\left(\frac{6}{13}\right)$$

$$A \approx 27.49^\circ$$

Case 2: Obtuse Angle

$$A = 180^\circ - 27.49^\circ$$

$$A \approx 152.51^{\circ}$$

(c)

To solve the equation $\sin(3x) = \sin(x)$ for $0^{\circ} \leq x \leq 360^{\circ}$, we use the general property of sine.

If $\sin(A) = \sin(B)$, there are two possible cases for the angles:

$A = B + 360^{\circ}n$ (The angles are coterminal) $A = 180^{\circ} - B + 360^{\circ}n$ (The angles are supplementary)

Here, $A = 3x$ and $B = x$. We will solve for x in both cases.

Case 1: The angles are equal (plus full rotations)

$$3x = x + 360^{\circ}n$$

Subtract x from both sides:

$$2x = 360^{\circ}n$$

Divide by 2:

$$x = 180^{\circ}n$$

Now, substitute integer values for n to find solutions within $0^{\circ} \leq x \leq 360^{\circ}$:

If $n = 0 \rightarrow x = \mathbf{0}^{\circ}$ If $n = 1 \rightarrow x = \mathbf{180}^{\circ}$ If $n = 2 \rightarrow x = \mathbf{360}^{\circ}$

Case 2: The angles are supplementary

$$3x = 180^{\circ} - x + 360^{\circ}n$$

Add x to both sides:

$$4x = 180^{\circ} + 360^{\circ}n$$

Divide by 4:

$$x = 45^{\circ} + 90^{\circ}n$$

Substitute integer values for n:

- If $n = 0 \rightarrow x = \mathbf{45}^{\circ}$
- If $n = 1 \rightarrow x = 45^{\circ} + 90^{\circ} = \mathbf{135}^{\circ}$
- If $n = 2 \rightarrow x = 135^{\circ} + 90^{\circ} = \mathbf{225}^{\circ}$
- If $n = 3 \rightarrow x = 225^{\circ} + 90^{\circ} = \mathbf{315}^{\circ}$
- (If $n = 4$, $x = 405^{\circ}$, which is outside our range.)

Final Answer

Combining the results from both cases and ordering them from smallest to largest, the solutions are:

$x \in 0^{\circ}, 45^{\circ}, 135^{\circ}, 180^{\circ}, 225^{\circ}, 315^{\circ}, 360^{\circ}$

(d)

Group Terms

It is often helpful to group the first and last terms

($\sin x$ and $\sin 3x$) because their average angle is $2x$, which matches the middle term.

$$(\sin 3x + \sin x) + \sin 2x = 0$$

Apply Sum-to-Product Formulas

Use the identity $\sin A + \sin B = 2\sin(\frac{A+B}{2})\cos(\frac{A-B}{2})$.Here, $A = 3x$ and $B = x$.

verage angle: $\frac{3x+x}{2} = 2x$ Difference angle: $\frac{3x-x}{2} = x$

So, $(\sin 3x + \sin x)$ becomes $2\sin(2x)\cos(x)$.

Substitute this back into the equation:

$$2\sin(2x)\cos(x) + \sin(2x) = 0$$

Factor the Equation

Now, we can factor out the common term $\sin(2x)$:

$$\sin(2x)[2\cos(x) + 1] = 0$$

This gives us two separate equations to solve:

$\sin(2x) = 0$ $2\cos(x) + 1 = 0$

Solve the First Part: $\sin(2x) = 0$

We know $\sin(\theta) = 0$ when θ is a multiple of 180° $(0^\circ, 180^\circ, 360^\circ, \ldots)$.Since our range for x is 0° to 360°, the range for $2x$ is 0° to 720°.

Possible values for $2x$:

$$2x = 0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ$$

Dividing by 2 gives the values for x:

$$x = 0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$$

Solve the Second Part: $2\cos(x) + 1 = 0$

Rearrange to solve for $\cos(x)$:

$$2\cos(x) = -1$$

$$\cos(x) = -\frac{1}{2}$$

Cosine is negative in Quadrants II and III. The reference angle where $\cos(\theta) = 1/2$ is 60°.

- Quadrant II: $180^\circ - 60^\circ = \mathbf{120}^\circ$
- Quadrant III: $180^\circ + 60^\circ = \mathbf{240}^\circ$

Combining all solutions and listing them in order:

$$x \in 0^{\circ}, 90^{\circ}, 120^{\circ}, 180^{\circ}, 240^{\circ}, 270^{\circ}, 360^{\circ}$$

Question 5

(a)

(i.)

Since an object can change direction while maintaining a constant speed , none of the first three statements MUST be true. Therefore, the answer is None of the above.

(ii.)

Velocity: 0 Acceleration: $\approx 9.8, \mathrm{m/s}^2$ (downwards)

Therefore, Statement 2 is the truth.

(iii.)

1. $$v = u + at$$

Substitute the known values:

$$0 = 30 + (-10)t$$

$$10t = 30$$

$$t = 3, s$$

2. We can use the equation relating displacement, initial velocity, and time:

$$s = ut + \frac{1}{2}at^2$$

Substitute the values:

$$s = (30)(3) + \frac{1}{2}(-10)(3)^2$$

$$s = 90 + (-5)(9)$$

$$s = 90 - 45$$

$$s = 45, m$$

Conclusion: 45m, 3s

(b)

(i.)

The domain is the set of all real numbers x for which the function is defined. The function is undefined where the denominator is zero.Set the denominator to zero:

$$x^2 + 9x - 22 = 0$$

Factor the quadratic:

$$(x + 11)(x - 2) = 0$$

Identify excluded values:

$$x = -11 \quad \text{and} \quad x = 2$$

Answer:The domain is $x \in \mathbb{R}, x \neq -11, x \neq 2$.Interval notation: $(-\infty, -11) \cup (-11, 2) \cup (2, \infty)$.

(ii.)

To find the range, we determine which values y can take. We set $y = f(x)$ and rearrange to form a quadratic equation in terms of x.

Set up the equation:

$$y = \frac{x - 3}{x^2 + 9x - 22}$$

$$y(x^2 + 9x - 22) = x - 3$$

$$yx^2 + 9yx - 22y - x + 3 = 0$$

$$yx^2 + (9y - 1)x + (3 - 22y) = 0$$

Analyze the Discriminant (Δ):For x to be a real number, the discriminant of this quadratic must be non-negative ($\Delta \geq 0$).

$$\Delta = b^2 - 4ac$$

$$\Delta = (9y - 1)^2 - 4(y)(3 - 22y)$$

$$\Delta = (81y^2 - 18y + 1) - (12y - 88y^2)$$

$$\Delta = 169y^2 - 30y + 1$$

Solve the inequality $169y^2 - 30y + 1 \geq 0$:

First, find the roots of $169y^2 - 30y + 1 = 0$:

$$y = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(169)(1)}}{2(169)}$$

$$y = \frac{30 \pm \sqrt{900 - 676}}{338} = \frac{30 \pm \sqrt{224}}{338}$$

$$y = \frac{30 \pm 4\sqrt{14}}{338} = \frac{15 \pm 2\sqrt{14}}{169}$$

Approximate values: $y_1 \approx 0.0445$

$y_2 \approx 0.1330$

Since the quadratic in y opens upwards, the inequality holds outside the roots.Answer:The range is:

$$y \in \left(-\infty, \frac{15 - 2\sqrt{14}}{169}\right] \cup \left[\frac{15 + 2\sqrt{14}}{169}, \infty\right)$$

Approximate Range: $y \in (-\infty, 0.044] \cup [0.133, \infty)$(There is a "gap" in the y-values between roughly 0.044 and 0.133) $$