First assignment in ordinary differential equations

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S1. General solution of first order equations

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{e^x}{3+6e^x}$

x is the only variable on the hand right side of the equation. Then the solution is given by integrating this side.

$$\int \mathrm{d}y = \int \frac{1}{6}\frac{6e^x}{(3+6e^x)}\,\mathrm{d}x$$
$$y(x) = \frac{1}{6}\ln|3+6e^x| + \mathrm{C}$$

But $3+6e^x > 0\ \forall\ x$ then,

$$\boxed{y(x) = \frac{1}{6}\ln(3+6e^x) + \mathrm{C}}$$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x^2}{y}$

This is a *separable equation*. Then, separate all the variable y on the left hand side and all the variable x on the right hand side of the equation. After integrate it.

$$y\,\mathrm{d}y = x^2\,\mathrm{d}x$$
$$\int y\,\mathrm{d}y = \int x^2\,\mathrm{d}x$$
$$\frac{y^2}{2} = \frac{x^3}{3} + \mathrm{C}$$
$$y^2 = \frac{2x^3}{3} + \mathrm{K} \quad \text{where } K = 2C$$

Thus $y(x)$ exists only if $\dfrac{2x^3}{3} + \mathrm{K} \geq 0$ which is equivalent to $x^3 \geq -\dfrac{3K}{2}$

$$\boxed{y(x) =}$$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + 3y = x + e^{-2x}$

This a *linear first order differential equations.*

$$a(x) = 3 \quad \text{and} \quad b(x) = x + e^{-2x}$$

The integrating factor is ,

$$I(x) = e^{\int 3dx} = e^{3x}$$

$\dfrac{d(e^{3x}y(x))}{dx} = e^{3x}(x + e^{-2x}) = xe^{3x} + e^{x}$

$$e^{3x}y(x) = \int xe^{3x}dx + \int e^{x}dx$$

Using integration by parts,
$\int xe^{3x}dx = [\frac{x}{3}e^{3x}] - \int \frac{1}{3}e^{3x}dx = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}$
Then,
$e^{3x}y(x) = \frac{x}{3}e^{3x} - \frac{1}{9}e^{3x} + C + e^{x}$
Finally,

$$\boxed{y(x) = \frac{x}{3} - \frac{1}{9} + Ce^{-3x} + e^{-2x}}$$

(d) $x\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\cos(2x) - y$

$$x\frac{dy(x)}{dx} - [x\cos(2x) - y] = 0$$

This is an *exact equation.*

$$a(x,y) = x \quad \text{and} \quad b(x,y) = -x\cos(2x) + y$$
$$\frac{\partial a(x,y)}{\partial x} = 1 \quad \text{and} \quad \frac{\partial b(x,y)}{\partial y} = 1$$

Then , there exists a twice-differentiable function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with

$$\frac{\partial \psi(x,y)}{\partial y} = a(x,y) \quad \text{and} \quad \frac{\partial \psi(x,y)}{\partial x} = b(x,y)$$

$$\frac{\partial \psi(x,y)}{\partial y} = x \tag{1}$$
$$\frac{\partial \psi(x,y)}{\partial x} = -x\cos(2x) + y \tag{2}$$

From (3), $\psi(x,y) = xy + C(x)$, bring it to (4), we have

$$y + C'(x) = -x\cos(2x) + y$$
$$C'(x) = -x\cos(2x)$$
$$C(x) = -\int x\cos(2x)dx + C_0$$

Using integration by parts,

$$C(x) = -\left(\frac{x}{2}\sin(2x) - \frac{1}{2}\int \sin(2x)dx\right)$$
$$= -\frac{x}{2}\sin(2x) - \frac{1}{4}\cos(2x) + C_1$$

So,
$\psi(x,y) = xy - \frac{x}{2}\sin(2x) - \frac{1}{4}\cos(2x) + C_1$

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y^2+2xy}{x^2}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{y^2+2xy}{x^2}\\ &= \frac{y^2}{x^2} + \frac{2xy}{x^2}\\ &= \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)\end{aligned}$$

This is *an homogeneous equation*, by *changing variable* $u = \dfrac{y}{x}$, then

$$\begin{aligned}y &= xu\\ \frac{dy}{dx} &= u + x\frac{du}{dx}\end{aligned}$$

and,

$$\frac{dy}{dx} = u^2 + 2u$$

So,

$$\begin{aligned}x\frac{du}{dx} &= \frac{dy}{dx} - u\\ &= u^2+2u-u\\ &= u^2+u\end{aligned}$$

Separating variables,

$$\begin{aligned}\frac{1}{u^2+u}du &= \frac{1}{x}dx\\ \frac{1}{u(u+1)}du &= \frac{1}{x}dx \quad \text{,provided} \quad x\neq 0, u\neq 0 \text{ and } u\neq -1\\ \left(\frac{1}{u} - \frac{1}{u+1}\right)du &= \frac{1}{x}dx \quad \text{,integrate each side}\\ \int\frac{1}{u}du - \int\frac{1}{u+1}du &= \int\frac{1}{x}dx\\ \ln|u| - \ln|u+1| &= \ln|x| + C_0\\ \ln\left|\frac{u}{u+1}\right| &= \ln|x| + C_0 \qquad \text{take the exponential,}\\ \left|\frac{u}{u+1}\right| &= C_1|x| \quad \text{with} \quad C_1 = e^{C_0} > 0\\ \left|\frac{u}{u+1}\right|\frac{1}{|x|} &= C_1\\ \text{then} \quad \frac{u}{u+1}\frac{1}{x} &= C_2\end{aligned}$$

(f) $xy^2 - x + (x^2y+y)\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

This is an *exact equation.*

$$a(x,y) = x^2y + y \quad \text{and} \quad b(x,y) = xy^2 - x$$

$$\frac{\partial a(x,y)}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial b(x,y)}{\partial y} = 2xy$$

Then , there exists a twice-differentiable function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with

$$\frac{\partial \psi(x,y)}{\partial y} = a(x,y) \quad \text{and} \quad \frac{\partial \psi(x,y)}{\partial x} = b(x,y)$$

$$\frac{\partial \psi(x,y)}{\partial y} = x^2 y + y \tag{3}$$

$$\frac{\partial \psi(x,y)}{\partial x} = xy^2 - x \tag{4}$$

From (3), $\psi(x,y) = \frac{y^2}{2}\left(x^2+1\right) + C(x)$, bring it to (4), we have

$$\begin{aligned} xy^2 + C'(x) &= x(y^2-1) \\ C'(x) &= x[(y^2-1) - y^2] \\ &= -x \quad \text{, then} \\ C(x) &= -\frac{x^2}{2} + K_0 \end{aligned}$$

So,

$\psi(x,y) = \frac{x^2}{2}\left(y^2-1\right) + \frac{y^2}{2} + K_0$

$$y^2\left(\frac{x^2}{2} + \frac{1}{2}\right) = \frac{x^2}{2} - K_0$$

$$\boxed{y(t) = \sqrt{\frac{x^2-K}{x^2+1}}}$$

S2. Initial value problems

(a) $\quad (\sin x + x^2 e^y - 1)\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x + 2xe^y = 0, \quad y(0) = 0$

This is an *exact equation.*

$$a(x,y) = \sin x + x^2 e^y - 1 \quad \text{and} \quad b(x,y) = y\cos x + 2xe^y$$

$$\frac{\partial a(x,y)}{\partial x} = \cos x + 2xe^y \quad \text{and} \quad \frac{\partial b(x,y)}{\partial y} = \cos x + 2xe^y$$

Then , there exists a twice-differentiable function $\psi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with

$$\frac{\partial \psi(x,y)}{\partial y} = a(x,y) \quad \text{and} \quad \frac{\partial \psi(x,y)}{\partial x} = b(x,y)$$

$$\frac{\partial \psi(x,y)}{\partial y} = \sin x + x^2 e^y - 1 \tag{5}$$

$$\frac{\partial \psi(x,y)}{\partial x} = y\cos x + 2xe^y \tag{6}$$

From (5), $\psi(x,y) = y\sin x + x^2 e^y - y + C(x)$, bring it to (6), we have

$$\begin{aligned} y\cos x + 2xe^y + C'(x) &= y\cos x + 2xe^y \\ C'(x) &= 0 \\ \text{Then,} \quad C(x) &= K \end{aligned}$$

$$\psi(x,y) = y\sin x + x^2 e^y - y + K$$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = y^4, \quad y(0) = 1$

This a *Bernoulli equation.*

$$\begin{aligned} \alpha &= 4 \\ a(t) &= 1 \\ b(t) &= 1 \end{aligned}$$

Let be $u = y^{1-\alpha} = y^{-3}$
So, the equation becomes:

$$\frac{du}{dt} - 3u = -3$$

This is a *linear first order differential equation.*
Using *integrating factor* $I(t) = e^{\int -3dt} = e^{-3t}$,

$$\frac{d}{dt}(e^{-3t}u(t)) = -3e^{-3t}$$

Integrate it,

$$\begin{aligned} e^{-3t}u(t) &= -3\int e^{-3t}dt \\ &= e^{-3t} + C \\ \text{then,} \quad u(t) &= 1 + Ce^{3t} \end{aligned}$$

But , $u(t) = y(t)^{-3}$, then $\quad 1 + Ce^{3t} = y(t)^{-3}$
$y(0) = 1$, then $y(0)^{-3} = 1 = 1 + C$, so $C = 0$
Finally , $y(t)^{-3} = 1$

$$\boxed{y(t) = 1}$$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = y^4, \quad y(0) = 2$

This is the same *Bernoulli equation* as before, but with the initial value problem $y(0) = 2$.
The same process as before,

$$1 + Ce^{3t} = y(t)^{-3}$$
$$y(0)^{-3} = 1 + C = 2^{-3}$$

So,

$$C = \frac{1}{8} - 1 = \frac{-7}{8}$$

And ,

$$y(t)^{-3} = 1 - \frac{7}{8}e^{3t}$$
$$\boxed{y(t) = 2\sqrt[3]{\frac{1}{8 - 7e^{3t}}}}$$

S5. Mathematical theory of epidemics

$$N = \text{the number of individuals in the community}$$
$$I = \text{the number of infected individuals in the community}$$
$$U = \text{the number of uninfected individuals in the community}$$

$$x = \frac{I}{N} \quad \text{and} \quad y = \frac{U}{N} \quad \text{with} \quad x, y \in [0,1] \quad \text{and} \quad x + y = 1$$

t is the time ,

$$\frac{dx}{dt} = \beta xy$$

where β is a real and positive constant of proportionality.

(a) Differential equation for $x(t)$:

$$x + y = 1 \tag{7}$$
$$\frac{dx}{dt} = \beta xy \tag{8}$$

From (7), $y = 1 - x$, then (8) becomes

$$\frac{dx}{dt} = \beta x(1-x)$$

(b) The solution of this differential equation for $x(0) = x_0$

Separate the two variables,

$$\frac{dx}{x(1-x)} = \beta dt$$
$$\left(\frac{1}{x} + \frac{1}{(1-x)}\right) dx = \beta dt$$
$$\text{integrate each side,} \quad \int \left(\frac{1}{x} + \frac{1}{(1-x)}\right) dx = \int \beta dt$$
$$\ln|x| - \ln|1-x| = \beta t + K$$
$$\ln\left|\frac{x}{1-x}\right| = \beta t + K$$
$$\ln\left(\frac{x}{1-x}\right) = \beta t + K \quad \text{because} \quad x \geq 0 \quad \text{and} \quad 1 - x \geq 0$$
$$\text{taking exponential,} \quad \frac{x}{1-x} = K' e^{\beta t}$$
$$x = (1-x) K' e^{\beta t}$$
$$x + x K' e^{\beta t} = K' e^{\beta t}$$
$$\text{so,} \quad x(t) = \frac{K' e^{\beta t}}{1 + K' e^{\beta t}}$$
$$\text{then,} \quad x(0) = \frac{K'}{1 + K'} = x_0$$
$$\text{we obtain} \quad K' = \frac{x_0}{1 - x_0}$$

Finally,

$$\boxed{x(t) = \frac{x_0}{e^{-\beta t}(1 - x_0) + x_0}}$$

(c) $\lim_{t \longrightarrow +\infty} x(t)$ **if** $x_0 > 0$**:**

$$\lim_{t \longrightarrow +\infty} x(t) = \lim_{t \longrightarrow +\infty} \frac{x_0}{e^{-\beta t}(1 - x_0) + x_0} = 1 \quad \text{if} \quad x_0 > 0$$

This result means that after a long time , $\frac{I}{N} = 1$, precisely $I = N$, all the individuals in the community will be infected.

(d) Critics of this model: