Nonlinear Waves: Exercise Sheet 1

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1. (iii) Consider the partial differential equation

$$u_t + \alpha u_x + \beta u_{xxx} = 0 \quad \text{where} \quad \alpha \quad \text{and} \quad \beta \quad \text{are constants.} \tag{1}$$

Suppose we seek a solution in the form

$$u(x,t) = e^{i(kx-wt)}$$

Then,

$$\begin{aligned} u_t &= -iwu \\ u_x &= iku \\ u_{xxx} &= (ik)^3 u = -ik^3 u \end{aligned}$$

From the equation (1), we obtain

$$(-iw + i\alpha k - \beta k^3)u = 0$$

$$-w + \alpha k - \beta k^3 = 0 \tag{2}$$

The dispersion relation $D(w,k)$ is

$$\boxed{D(w,k) = w - \alpha k + \beta k^3}$$

From the equation (2),

$$w(k) = \alpha k - \beta k^3$$

Then the phase velocity c is

$$\boxed{c = \frac{w(k)}{k} = \alpha - \beta k^2}$$

The group velocity v_g is

$$\boxed{v_g = \frac{\mathrm{d}w}{\mathrm{d}k} = \alpha - 3\beta k^2}$$

(v) Consider the partial differential equation

$$u_{tt} = \alpha u_{xx} + \beta u_{xxxx} \quad \text{where} \quad \alpha \quad \text{and} \quad \beta \quad \text{are constants.} \tag{1}$$

Suppose we seek a solution in the form

$$u(x,t) = e^{i(kx - wt)}$$

Then,

$$\begin{aligned} u_{tt} &= -w^2 u \\ u_{xx} &= -k^2 u \\ u_{xxxx} &= k^4 u \end{aligned}$$

From the equation (1), we obtain

$$-w^2 u = -\alpha k^2 u + \beta k^4 u$$

$$-(w^2 + \alpha k^2 - \beta k^4) u = 0$$

$$-w^2 + \alpha k^2 - \beta k^4 = 0 \tag{2}$$

The dispersion relation $D(w,k)$ is

$$\boxed{D(w,k) = -w^2 + \alpha k^2 - \beta k^4}$$

From the relation (2)

$$w = \pm\sqrt{-\beta k^4 + \alpha k^2}$$

The phase velocity c is

$$\boxed{c = \frac{w(k)}{k} = \pm\sqrt{-\beta k^2 + \alpha}}$$

The group velocity v_g is

$$\begin{aligned} v_g &= \frac{\mathrm{d}w}{dk} \\ &= \pm\frac{1}{2}\frac{4\beta k^3 - 2\alpha k}{\sqrt{\beta k^4 - \alpha k^2}} \\ &= \pm\frac{2\beta k^3 - \alpha k}{\sqrt{\beta k^4 - \alpha k^2}} \\ &= \boxed{\pm\frac{k(2\beta k^2 - \alpha)}{\sqrt{-\beta k^4 + \alpha k^2}}} \end{aligned}$$

2. Fourier transform.

(ii)

$$f(x) = \begin{cases} \mu & ,\text{if} \quad a < x < b \\ \\ 0 & ,\text{otherwise} \end{cases}$$

First, check if the function behaves well. It is clear here that $f(x) = 0$ if $x > b$ and $f(x) = 0$ if $x < a$ then $f(x) \to 0$ as $|x| \to \infty$.

$$\begin{aligned} \mathcal{F}\{f(x)\}(k) &= \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x \\ &= \int_a^b \mu e^{-ikx}\mathrm{d}x \\ &= \mu \left[\frac{-1}{ik}e^{-ikx}\right]_a^b \\ \mathcal{F}\{f(x)\}(k) &= \mu\frac{i}{k}[e^{-ikb} - e^{-ika}] \end{aligned}$$

(ii)

$$f(x) = \begin{cases} 1 - x^2 & ,\text{if} \quad -1 < x < 1 \\ \\ 0 & ,\text{otherwise.} \end{cases}$$

Clearly $f(x) \to 0$ as $|x| \to \infty$.

$$\begin{aligned} \mathcal{F}\{f(x)\} &= \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x \\ &= \int_{-1}^{1} (1 - x^2)e^{-ikx}\mathrm{d}x \\ &= \int_{-1}^{1} e^{-ikx}\mathrm{d}x - \int_{-1}^{1} x^2 e^{-ikx}\mathrm{d}x \\ &= \left[-\frac{1}{ik}e^{-ikx}\right]_{-1}^{1} - \int_{-1}^{1} x^2 e^{-ikx}\mathrm{d}x \end{aligned}$$

applying part integration to

$$\int_{-1}^{1} x^2 e^{-ikx}\mathrm{d}x$$

we obtain,

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \left[-\frac{1}{ik}e^{-ikx}\right]_{-1}^{1} - \left[x^2(-\frac{1}{ik})e^{-ikx}\right]_{-1}^{1} + \int_{-1}^{1}\frac{i}{k}2xe^{-ikx}\mathrm{d}x\\ &= \frac{i}{k}\int_{-1}^{1}2xe^{-ikx}\mathrm{d}x\end{aligned}$$

By part integration again, we get

$$\begin{aligned}\mathcal{F}\{f(x)\} &= \frac{i}{k}\left[2x\frac{i}{k}e^{-ikx}\right]_{-1}^{1} - \frac{i}{k}\int_{-1}^{1}2\frac{i}{k}e^{-ikx}\mathrm{d}x\\ &= \frac{i}{k}\left[2x\frac{i}{k}e^{-ikx}\right]_{-1}^{1} + \frac{2}{k^2}\int_{-1}^{1}e^{-ikx}\mathrm{d}x\\ &= -\frac{2}{k^2}(e^{-ik}+e^{ik}) + \frac{2}{k^2}\left[\frac{i}{k}e^{-ikx}\right]_{-1}^{1}\\ &= -\frac{2}{k^2}(2\cos k) + \frac{2}{k^2}\frac{i}{k}(2i\sin k)\\ \mathcal{F}\{f(x)\} &= -\frac{4}{k^2}\cos k + \frac{4}{k^3}\sin k\end{aligned}$$

3. Suppose that $F(k) = \mathcal{F}\{f(x)\}$, the Fourier transform of $f(x)$, and $f(x) \longrightarrow 0$ as $|x| \longrightarrow \infty$.
Then, according to the definition,We have

$$\begin{aligned}\mathcal{F}\{xf(x)\} &= \int_{-\infty}^{\infty}xf(x)e^{-ikx}\mathrm{d}x\\ &= \int_{-\infty}^{\infty}\frac{-i}{-i}xf(x)e^{-ikx}\mathrm{d}x\\ &= i\int_{-\infty}^{\infty}f(x)(-ix)e^{-ikx}\mathrm{d}x\\ &= i\int_{-\infty}^{\infty}f(x)\frac{\partial}{\partial k}[e^{-ikx}]\mathrm{d}x\end{aligned}$$

$f(x)$ is a constant compared to the variable k then,

$$\begin{aligned}\mathcal{F}\{xf(x)\} &= i\int_{-\infty}^{\infty}\frac{\partial}{\partial k}[f(x)e^{-ikx}]\mathrm{d}x\\ &= i\frac{\partial}{\partial k}\int_{-\infty}^{\infty}[f(x)e^{-ikx}]\mathrm{d}x\end{aligned}$$

$F(k) = \int_{-\infty}^{\infty}[f(x)e^{-ikx}]\mathrm{d}x$ is a function of k

$$\mathcal{F}\{xf(x)\} = i\frac{\mathrm{d}F}{\mathrm{d}k}$$

5. Fourier transforms.

(i) Consider the heat equation

$$u_t = u_{xx}, \tag{3}$$

with initial condition $u(x,0) = f(x)$.

Let denote $U(k,t) = \mathcal{F}\{u\}$,
(3) is equivalent to $u_t - u_{xx} = 0$
Take the Fourier transforms,

$$\begin{aligned}\mathcal{F}\{u_t - u_{xx}\} &= \mathcal{F}\{0\}\\ \mathcal{F}\{u_t\} - \mathcal{F}\{u_{xx}\} &= 0\end{aligned}$$

because the transformation is linear.
We have,

$$\begin{aligned}\mathcal{F}\{u_t\} &= \int_{-\infty}^{\infty} u_t(x,t)e^{-ikx} = \int_{-\infty}^{\infty} \frac{\partial}{\partial t}(u(x,t)e^{-ikx})\mathrm{d}x\\ &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\mathrm{d}x = \frac{\partial U}{\partial t}(k,t).\end{aligned}$$

And (3) becomes,

$$\frac{\partial U}{\partial t} + k^2 U = 0.$$

Then we get,

$$U(k,t) = C(k)e^{(-k^2 t)}$$

where C is an arbitrary function.
Let find $C(k)$,
$U(k,0) = C(k)e^{(0)} = C(k)$
$U(k,0) = \mathcal{F}\{u(x,0)\} = F(k)$, so we obtain the equality

$$U(k,t) = F(k)e^{(-k^2 t)}$$

by the inverse Fourier transforms of U, we get

$$\begin{aligned}u(x,t) &= \mathcal{F}^{-1}\{F(k)e^{(-k^2 t)}\}\\ &= \int_{-\infty}^{\infty} F(k)e^{(-k^2 t)}e^{ikx}\mathrm{d}k\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\zeta)e^{(-ik\zeta)}\mathrm{d}\zeta\right)e^{(-k^2 t)}e^{ikx}\mathrm{d}k\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{-k^2 t + ik(x-\zeta)}\mathrm{d}k\right)f(\zeta)\mathrm{d}\zeta\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sqrt{\pi}}{\sqrt{t}}e^{\frac{-(x-\zeta)^2}{4t}}f(\zeta)\mathrm{d}\zeta\\ u(x,t) &= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{\frac{-(x-\zeta)^2}{4t}}f(\zeta)\mathrm{d}\zeta\end{aligned}$$

In the case of the exercise,

$u(x,0) = f(x) = \begin{cases} 1, & \text{if } -1 < x < 2, \\ 0, & \text{otherwise} \end{cases}$

Then,

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-1}^{2} e^{\frac{-(x-\zeta)^2}{4t}} \mathrm{d}\zeta$$

Let ask

$$v = \frac{x-\zeta}{2\sqrt{t}}, \quad \text{thus} \quad \mathrm{d}v = -\frac{\mathrm{d}\zeta}{2\sqrt{t}},$$

for $-1 < \zeta < 2$, we have $\dfrac{x+1}{2\sqrt{t}} \geq \dfrac{x-\zeta}{2\sqrt{t}} \geq \dfrac{x-2}{2\sqrt{t}}$

so,

$$\begin{aligned} u(x,t) &= -\frac{2\sqrt{t}}{\sqrt{4\pi t}} \int_{\frac{x+1}{2\sqrt{t}}}^{\frac{x-2}{2\sqrt{t}}} e^{(-v^2)} \mathrm{d}v \\ &= \frac{1}{2}\left(\frac{2}{\sqrt{\pi}} \int_{0}^{\frac{x+1}{2\sqrt{t}}} e^{(-v^2)} \mathrm{d}v + \frac{2}{\sqrt{\pi}} \int_{\frac{x-2}{2\sqrt{t}}}^{0} e^{(-v^2)} \mathrm{d}v\right) \\ u(x,t) &= \frac{1}{2}\left[erf\left(\frac{x+1}{\sqrt{2t}}\right) - erf\left(\frac{x-2}{\sqrt{2t}}\right)\right] \end{aligned}$$

```
var('x,t')
u(x,t) = (1/2)*(erf((x+1)/(sqrt(2*t)))-erf((x-2)/(sqrt(2*t))))
P1 = plot(lambda x: u(x,0.1),-10,10)
P2 = plot(lambda x: u(x,1),-10,10)
P3 = plot(lambda x: u(x,5),-10,10)
P4 = plot(lambda x: u(x,15),-10,10)
P5 = plot(lambda x: u(x,10),-10,10)
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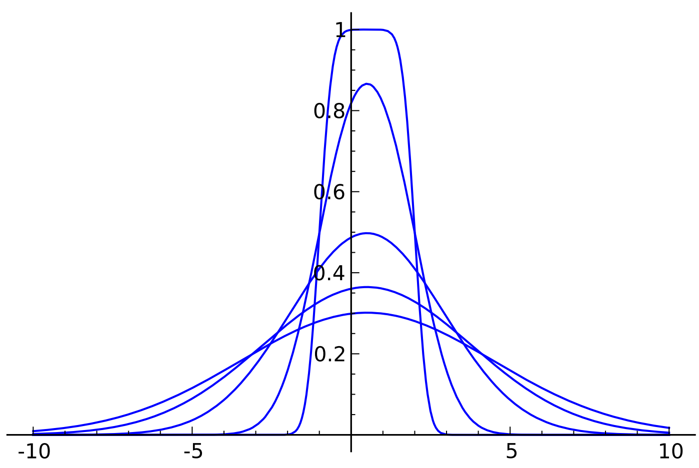

8. Suppose that $u(x,t)$ satisfies the initial value problem

$$u_{tt} + u_{xxxx} = 0, \quad u(x,0) = e^{(-\frac{1}{2}x^2)}, \quad u_t(x,0) = 0$$

Take the Fourier transforms of the partial derivative equation above

$$\mathcal{F}\{u_{tt} + u_{xxxx}\} = \mathcal{F}\{u_{tt}\} + \mathcal{F}\{u_{xxxx}\}$$

$$\begin{aligned}
\mathcal{F}\{u_{tt}(x,t)\} &= \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2}\{u(x,t)\}e^{-ikx}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2}\{u(x,t)e^{-ikx}\}\mathrm{d}x \\
&= \frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} u(x,t)e^{-ikx}\mathrm{d}x \\
\mathcal{F}\{u_{tt}(x,t)\} &= \frac{\partial^2}{\partial t^2}U(k,t),
\end{aligned}$$

and

$$\mathcal{F}\{u_{xxxx}(x,t)\} = (ik)^4U(k,t) = k^4U(k,t).$$

Then,

$$\mathcal{F}\{u_{tt} + u_{xxxx}\} = \frac{\partial^2}{\partial t^2}U + k^4U = 0,$$

the solution of is given by,

$$U(k,t) = A(k)\cos(k^2t) + B(k)\sin(k^2t)$$

where A and B are determined by the initial conditions.

$$\begin{aligned}
U(x,0) = A(k) &= \int_{-\infty}^{\infty} u(x,0)e^{-ikx}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{(-\frac{1}{2}x^2)}e^{-ikx}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x^2+2ikx)}\mathrm{d}x \\
&= \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x+ik)^2-\frac{1}{2}k^2}\mathrm{d}x \\
&= e^{-\frac{1}{2}k^2)}\int_{-\infty}^{\infty} e^{-\left(\frac{1(x+ik)}{\sqrt{2}}\right)^2}\mathrm{d}x \\
&= e^{-\frac{1}{2}k^2}\int_{-\infty}^{\infty} \sqrt{2}e^{-v^2}\mathrm{d}v \qquad \text{where } v = \frac{1(x+ik)}{\sqrt{2}} A(k) \quad = e^{-\frac{1}{2}k^2}\sqrt{2\pi}
\end{aligned}$$

And,

$$u_t(x,t) = -k^2A(k)sin(k^2t) + k^2B(k)\cos(k^2t)$$

$$u_t(x,0) = k^2 B(k) = 0 \quad \text{then} \quad B(k) = 0$$

So

$$U(k,t) = \sqrt{2\pi} e^{(-\frac{1}{2}k^2)} \cos(k^2 t)$$

But $\cos(k^2 t) = \mathrm{Re}\{e^(ik^2 t)\}$, thus

$$\begin{aligned}
\sqrt{2\pi} e^{(-\frac{1}{2}k^2)} \cos(k^2 t) &= \mathrm{Re}\left\{ \sqrt{2\pi} e^{(-\frac{1}{2}k^2)} e^{(}ik^2 t) \right\} \\
&= \mathrm{Re}\left\{ \sqrt{2\pi} e^{(-\frac{1}{2}k^2} + ik^2 t) \right\} \\
&= \mathrm{Re}\left\{ \sqrt{2\pi} e^{-\left(\frac{1}{2} - it\right)k^2} \right\}
\end{aligned}$$

Now we can compute $u(x,t)$ by using the inverse of Fourier Transform

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} U(k,t) e^{ikx} \mathrm{d}k \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\left(\sqrt{2\pi} e^{-\left(\frac{1}{2} - it\right)k^2} e^{ikx} \right) \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{Re}\left(\sqrt{2\pi} e^{-\left(\frac{1}{2} - it\right)k^2 + ixk} \right)
\end{aligned}$$

If $f = \mathrm{Re}(f) + i\mathrm{Im}(f)$, then $\int f = \int \mathrm{Re}(f) + i \int \mathrm{Im}(f)$, so $\mathrm{Re} \int f = \int \mathrm{Re}(f)$, thus we have

$$\begin{aligned}
u(x,t) &= \frac{1}{2\pi} \mathrm{Re} \int_{-\infty}^{\infty} \left(\sqrt{2\pi} e^{-\left(\frac{1}{2} - it\right)k^2 + ixk} \right) \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{Re} \int_{-\infty}^{\infty} \left(e^{-\left(\frac{1}{2} - it\right)k^2 + ixk} \right)
\end{aligned}$$

then,

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{2\pi}} \mathrm{Re} \left(\frac{\sqrt{\pi}}{\sqrt{\frac{1}{2} - it}} e^{-\frac{x^2}{2(1-2it)}} \right) \\
&= \frac{1}{\sqrt{2\pi}} \mathrm{Re} \left(\frac{\sqrt{2\pi}}{\sqrt{1 - 2it}} e^{-\frac{x^2}{2(1-2it)}} \right)
\end{aligned}$$

finally,

$$u(x,t) = \mathrm{Re} \left(\frac{1}{\sqrt{1 - 2it}} e^{-\frac{x^2}{2(1-2it)}} \right)$$

Now we can plot the solution

```
var('x,t')
u(x,t) = real((1/sqrt(1-2*I*t))*exp(-(x^2/(2*(1-2*I*t)))))
var('x,t')
u2(x,t) = real((1/sqrt(1-2*I*t))*exp(-(x^2/(2*(1-2*I*t)))))
A1 = plot(lambda x: u2(x,0),-20,20,axes_labels=['$x$','$u$'])
A2= plot(lambda x: u2(x,1),-20,20)
A3 = plot(lambda x: u2(x,5),-20,20)
A4 = plot(lambda x: u2(x,10),-20,20)
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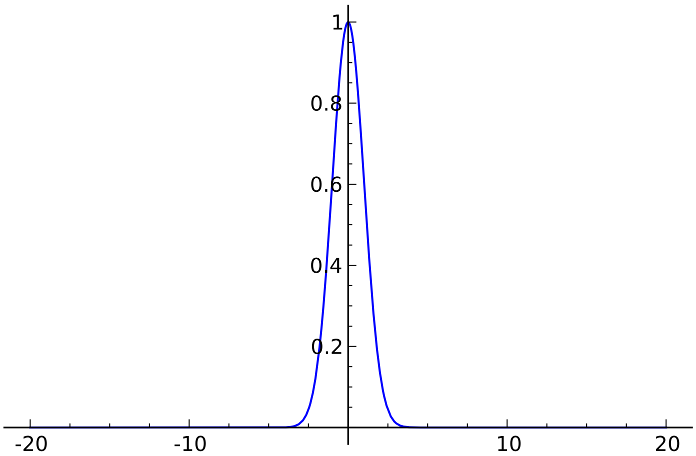

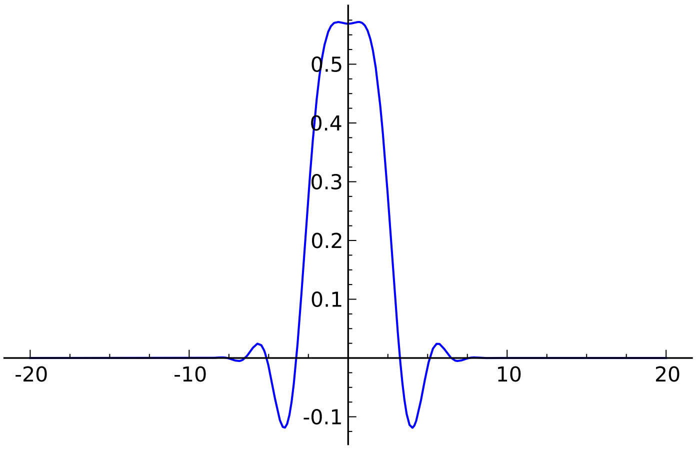

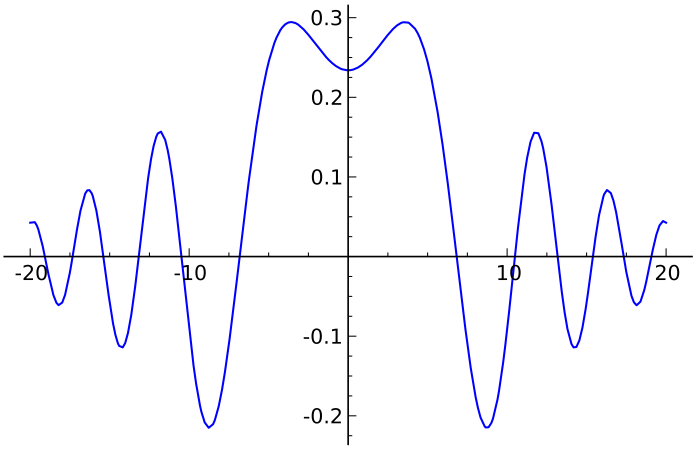
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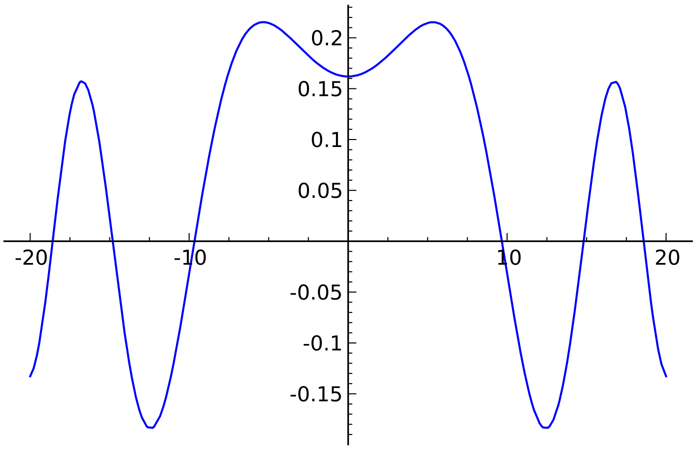
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