Systems of **ODEs** assignment

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S1

Write the 4^{th} equation $x^{(4)} - 2x^{(2)} - 8x = 0$ as the first order system .

Let introduce a four-component vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$ such that ,

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} = \dot{x}_1 \\ x_3 &= \ddot{x} = \dot{x}_2 \\ x_4 &= \dddot{x} = \dot{x}_3 \end{aligned}$$

Then, the required equation is:

$$\begin{array}{ll} \dot{x}_1 & = x_2 \\ \dot{x}_2 & = x_3 \\ \dot{x}_3 & = x_4 \\ \dot{x}_4 & = 2x_3 + 8x_1 \end{array} \iff \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

This is a system of linear equations with constant coefficients.
Let be,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 0 & 2 & 0 \end{pmatrix}$$

λ the eigenvalue of A, λ satisfies $det(A - \lambda I) = 0$

$$det(A-\lambda I) = \begin{vmatrix} 0-\lambda & 1 & 0 & 0 \\ 0 & 0-\lambda & 1 & 0 \\ 0 & 0 & 0-\lambda & 1 \\ 8 & 0 & 2 & 0-\lambda \end{vmatrix} = (\lambda-2)(\lambda+2)(\lambda-i\sqrt{2})(\lambda+i\sqrt{2})$$

Then, the general solution of this equation is

$$x(t) = C_1 e^{-2t} + C_2 e^{2t} + C_3 e^{-i\sqrt{2}} + C_4 e^{i\sqrt{2}} = C_1 e^{-2t} + C_2 e^{2t} + C_3 \cos(-t\sqrt{2}) + C_4 \cos(t\sqrt{2})$$

S2

Consider a function $f : \mathbb{R} \times \mathbb{C} \longrightarrow \mathbb{C}$ and the complex initial value problem

$$\frac{dz}{dt} = f(t, z), \quad z(0) = z_0 \tag{1}$$

for a complex valued function $z : \mathbb{R} \longrightarrow \mathbb{C}$ and a complex number z_0.

$$\begin{aligned} z \quad \text{can be written} \quad z(t) &= x(t) + iy(t) \quad \text{then,} \\ \frac{dz(t)}{dt} &= \frac{dx(t)}{dt} + i\frac{dy(t)}{dt} \\ \text{and} \quad z(0) &= x(0) + iy(0) = z_0 = x_0 + iy_0 \end{aligned}$$

So, from this last equation, after identifying each side we have $x(0) = x_0$ and $y(0) = y_0$
Then,

$$\begin{aligned} f : \quad \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (t, x, y) &\longmapsto f(t, x, y) = \alpha(t, x, y) + i\beta(t, x, y) \end{aligned}$$

() becomes,

$$\frac{dx(t)}{dt} + i\frac{dy(t)}{dt} = \alpha(t, x, y) + i\beta(t, x, y)$$

Identify each side,

$$\begin{cases} \dfrac{dx(t)}{dt} & = \alpha(t, x, y) \\ \\ \dfrac{dy(t)}{dt} & = \beta(t, x, y) \end{cases} \quad \text{with initial conditions} \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \tag{2}$$

$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is continuous in some region $I \times U$, where $I = (t_1, t_2)$ is an open interval and $U \in \mathbb{R}^2$ is an open set, and that the partial derivative matrix

$$Df = \begin{pmatrix} \dfrac{\partial \alpha}{\partial x} & \dfrac{\partial \alpha}{\partial y} \\ \\ \dfrac{\partial \beta}{\partial x} & \dfrac{\partial \beta}{\partial y} \end{pmatrix}$$

are also continuous there. Then, every $t_0 \in I$ and $x_0 \in U$, the initial value problem (2) has a unique solution in some open interval containing t_0.
From equation (), for $z(t) = e^{it}$

$$\frac{dz(t)}{dt} = \frac{d(e^{it})}{dt} = ie^{it} = iz(t) \quad \text{and}$$

$$z(0) = e^0 = 1$$

then $z(t) = e^{it}$ satisfies the equation ()
For $z(t) = \cos t + i \sin t$

$$\frac{dz(t)}{dt} = \frac{d(\cos t + i \sin t)}{dt} = -\sin t + i \cos t = i^2 \sin t + i \cos t = iz(t) \quad \text{and}$$

$$z(0) = \cos 0 + i \sin 0 = 1$$

then $z(t) = \cos t + i \sin t$ satisfies also the equation ()
We have,

$$\begin{aligned} f(t,z) &= iz \quad \text{then,} \\ f(t,x,y) &= i(x+iy) = ix - i^2 y = -y + ix \end{aligned}$$

f is linear, then f is continuous in $\mathbb{R}^3$.

$$Df = \begin{pmatrix} \dfrac{\partial(-y)}{\partial x} & \dfrac{\partial(-y)}{\partial y} \\ \\ \dfrac{\partial x}{\partial x} & \dfrac{\partial x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Df are constant in $\mathbb{R}^3$, then Df are continuous there, then for $0 \in \mathbb{R}$ and $z(0) = 1 \in \mathbb{R}^2$, the initial value problem has a unique solution in $\mathbb{R}$.
Then e^{it} must be equal to $\cos t + i \sin t$. Therefore, we deduce the Euler's formula $e^{it} = \cos t + i \sin t$.

S3

We have the real system

$$\dot{x} = ax - by, \quad \dot{y} = bx + ay \tag{3}$$

Let be $z = x + iy$, then $\quad \dot{z} = \dot{x} + i\dot{y} = (ax - by) + i(bx + ay)$ because of the relation (3)
Let combine all the coefficients of x and all the coefficients of y

$$\begin{aligned} \dot{z} &= (a+ib)x + (-b+ia)y \\ &= (a+ib)x + (i^2 b + ia)y \quad \text{because } i^2 = -1 \\ &= (a+ib)x + i(ib+a)y \\ &= (a+ib)x + iy(a+ib) \\ (a+ib) \text{ is a common factor,} \quad \dot{z} &= (a+ib)(x+iy) \end{aligned}$$

Let be $c = a + ib$, then

$$\dot{z} = cz \tag{4}$$

Now, let be $z = re^{i\phi}$, $\dot{z} = \dfrac{dz}{dt} = \dot{r}e^{i\phi} + i\dot{\phi}re^{i\phi}$ because r and ϕ are function of t.
(4) becomes

$$\dot{r}e^{i\phi} + i\dot{\phi}re^{i\phi} = (a + ib)re^{i\phi}$$
$$\dot{r}e^{i\phi} + i\dot{\phi}re^{i\phi} = are^{i\phi} + ibre^{i\phi}$$

Simplify each side by $e^{i\phi}$,

$$\dot{r} + i\dot{\phi}r = ar + ibr$$

Identify each side, the real part and the imaginary part of each side,

$$\dot{r} = ar$$
$$\dot{\phi}r = br$$

$$\text{equivalent to} \quad \dot{r} = ar, \dot{\phi} = b \tag{5}$$