Nonlinear Waves assignment Sheet 2

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1. **Method of characteristics**

(i) $u_t + \alpha t u_x = \beta x$

The characteristics equations are

$$\frac{\mathrm{d}x}{\alpha t} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}u}{\beta x}$$

So,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha t \Longrightarrow x = \frac{1}{2}\alpha t^2 + z \tag{1}$$

where z is an arbitrary constant.

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \beta x \Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}t} = \beta(\frac{1}{2}\alpha t^2 + z) \quad \text{from equation} \quad (1)$$
$$\Longrightarrow u = \beta(\frac{1}{6}\alpha t^3 + zt) + w(z)$$

According to equation (1), $z = x - \frac{1}{2}\alpha t^2$

$$u(x,t) = \beta(\frac{1}{6}\alpha t^3 + xt - \frac{1}{2}\alpha t^3) + w(z)$$

Finally,

$$\boxed{u(x,t) = \beta(xt - \frac{1}{2}\alpha t^3) + w(z) \quad \text{,where} \quad z = x - \frac{1}{2}\alpha t^2}$$

2. **General solution of the following equation**

(iv) $(y + xu)u_x - (x + yu)u_y = x^2 - y^2$

Let us use the method of characteristics

$$\frac{\mathrm{d}x}{y + xu} = \frac{\mathrm{d}y}{-(x + yu)} = \frac{\mathrm{d}u}{x^2 - y^2} = \mathrm{d}s$$

Then we get the following equations

$$\frac{\mathrm{d}x}{\mathrm{d}s} = y + xu \qquad \frac{\mathrm{d}y}{\mathrm{d}s} = -(x + yu) \qquad \frac{\mathrm{d}u}{\mathrm{d}s} = x^2 - y^2$$

Because of the relation

$$y\frac{\mathrm{d}x}{\mathrm{d}s} + x\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\mathrm{d}u}{\mathrm{d}s} = y(y + xu) - x(x + yu) + x^2 - y^2 = 0$$

we can find, by integrating

$$xy + u = z$$

where z is an arbitrary constant, the equation of the characteristic curves.
We have found also,

$$-x\frac{\mathrm{d}x}{\mathrm{d}s} - y\frac{\mathrm{d}y}{\mathrm{d}s} + u\frac{\mathrm{d}u}{\mathrm{d}s} = -x(y + xu) + y(x + yu) + ux^2 - uy^2 = 0$$

by integrating we get,

$$u^2 - x^2 - y^2 = w(z)$$

Therefore,

$$\boxed{u(x, y) = \pm\sqrt{x^2 + y^2 + w(z)} \quad , \quad \text{where} \quad z = u + xy}$$

3. **Initial value problems**

(ii)

$$\begin{cases} u_t + cu_x & = 1 + u^2 \\ u(x, 0) & = \exp(-x^2) = f(x) \end{cases}$$

The characteristics equations are

$$\frac{\mathrm{d}x}{c} = \frac{\mathrm{d}t}{1} = \frac{\mathrm{d}u}{1 + u^2}$$

So,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c \Longrightarrow x = ct + z$$

where z is an arbitrary constant.

$$\mathrm{d}t = \frac{\mathrm{d}u}{1 + u^2}$$

By integrating each side,

$$\begin{aligned}
&\Longrightarrow t + w(z) = \arctan u \\
&\tan(t + w(z)) = u \\
&\tan(t + w(z)) = u(x,t) \quad \text{where} \quad z = x - ct
\end{aligned}$$

Considering the initial condition,

$$\begin{aligned}
u(x,0) &= \tan(w(x)) = f(x) \\
\text{Then} \quad w(x) &= \arctan(f(x)) \\
&= \arctan\left(exp(-x^2)\right) \\
u(x,t) &= \tan\left(t + \arctan(exp(-z^2))\right)
\end{aligned}$$

$$\boxed{u(x,t) = \frac{\exp(-z^2) + \tan t}{1 - \exp(-z^2)\tan t} \quad , z = x - ct}$$

(iv)

$$\begin{cases} u_t + uu_x = -u^2 \\ u(x,0) = 1 = f(x) \end{cases}$$

The characteristics equations are

$$\frac{dx}{u} = \frac{dt}{1} = -\frac{du}{u^2}$$

Then,

$\dfrac{dx}{dt} = u \Longrightarrow x = ut + z$,where $z = x - ut$ is an arbitrary constant.

$-\dfrac{du}{u^2} = dt \Longrightarrow \dfrac{1}{u} = t + w(z)$ by integrating each side.

Thus,

$$\begin{aligned}
u &= \frac{1}{t + w(z)} \\
u(x,t) &= \frac{1}{t + w(x - ut)}
\end{aligned}$$

Considering the initial condition,

$$\begin{aligned}
u(x,0) = \frac{1}{w(x)} = 1 &\Longrightarrow w(x) = 1 \\
&\Longrightarrow w(z) = 1
\end{aligned}$$

Finally,

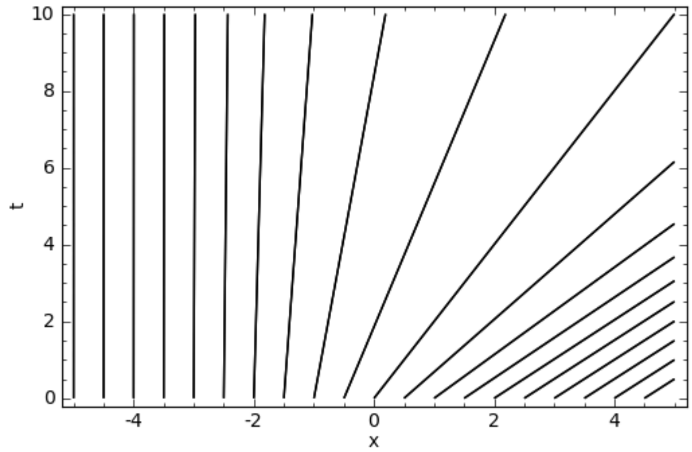
$$\boxed{u(x,t) = \frac{1}{1+t}}$$

5. **Initial value problem**

(i) $f(x) = \frac{1}{2}\{1 + \tanh(x)\}$

As we can see in the figure, the characteristics don't overlap

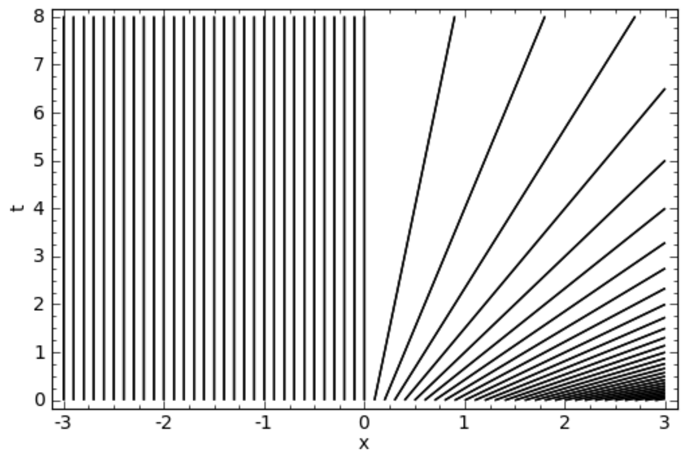
$$\text{Characteristics:} \quad x = \frac{1}{2}\{1 + \tanh(\xi)\}t + \xi$$



(ii) $f(x) = xH(x)$

$$\text{Characteristics:} \quad x = \xi H(\xi)t + \xi$$

As we can see in the figure, the characteristics don't overlap

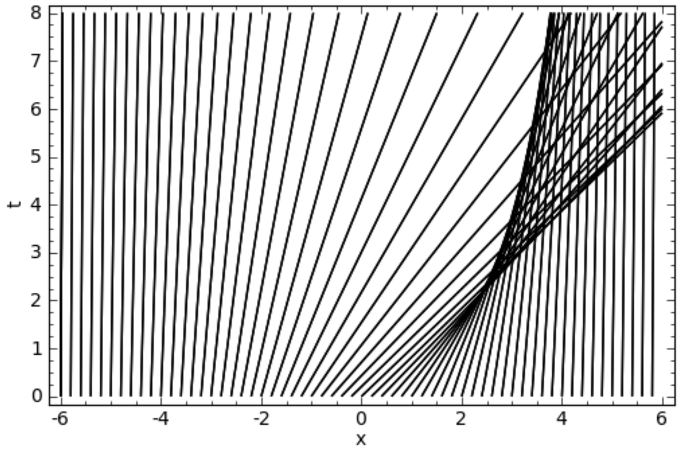


(iii)

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = f(x) \end{cases}$$

The characteristics cross in the cases $f(x) = \operatorname{sech}(x)$ and

$f(x) = \dfrac{1}{1+x^2}$



Breaking time:
We have

$$\begin{aligned} F'(\xi) &= f'(\xi) = -\tanh(\xi)\operatorname{sech}(\xi) \\ F''(\xi) &= \tanh(\xi)^2 \operatorname{sech}(\xi) + \left(\tanh(\xi)^2 - 1\right)\operatorname{sech}(\xi) \end{aligned}$$

$$\begin{aligned} F''(\xi) = 0 &\Leftrightarrow \tanh(\xi)^2 \operatorname{sech}(\xi) + \left(\tanh(\xi)^2 - 1\right)\operatorname{sech}(\xi) = 0 \\ &\Leftrightarrow 2\tanh(\xi) = 1 \\ &\Leftrightarrow \xi = \operatorname{arctanh}\left(\frac{1}{2}\sqrt{2}\right) \end{aligned}$$

Therefore the breaking time is:

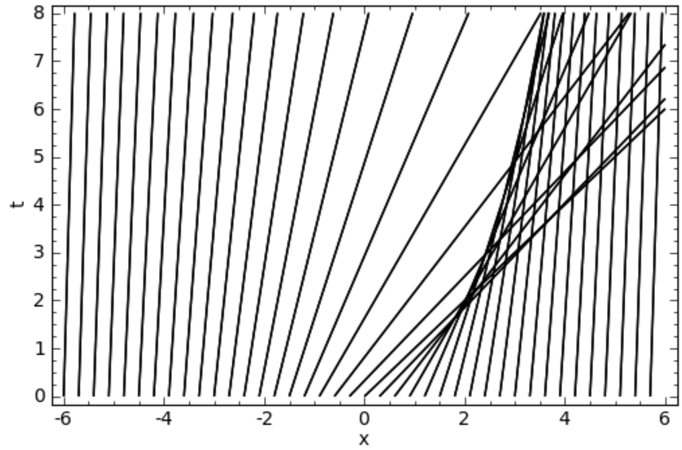
$$\begin{aligned} t_b &= -\frac{1}{f'(\operatorname{arctanh}\left(\frac{1}{2}\sqrt{2}\right))} \\ &= \frac{\sqrt{2}}{\operatorname{sech}\left(\operatorname{arctanh}\left(\frac{1}{2}\sqrt{2}\right)\right)} \end{aligned}$$

Thus

$$\boxed{t_b = 2}$$

(iv)

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \dfrac{1}{1+x^2} \end{cases}$$



$$F(\xi) = f(\xi) = \frac{1}{1+\xi^2}$$

$$F'(\xi) = -\frac{2\xi}{(1+\xi^2)^2}$$

$$F''(\xi) = -\left[\frac{2(1+\xi^2)^2 - 8\xi^2(1+\xi^2)}{(1+\xi^2)^4}\right] \quad = -\left[\frac{2(1+\xi^2) - 8\xi^2}{(1+\xi^2)^3}\right]$$

$$F''(\xi) = 0 \Longleftrightarrow (1+\xi^2) - 4\xi^2 = 0$$

$$\Longleftrightarrow -3\xi^2 + 1 = 0$$

$$\Longleftrightarrow \xi = \pm\frac{\sqrt{3}}{3}$$

Then the breaking time is given by

$$\begin{aligned} t_b &= -\frac{1}{F'(\xi)} \\ &= \frac{(1+\xi^2)^2}{2\xi} \\ &= \frac{(1+\frac{1}{3}^2)^2}{2\frac{\sqrt{3}}{3}} \\ &= \frac{8\sqrt{3}}{9} \end{aligned}$$

Thus

$$\boxed{t_b = \frac{8\sqrt{3}}{9}}$$

7. Consider the initial value problem

$$u_t + uu_x = 0, \qquad u(x,0) = \begin{cases} U_1, & if \quad x > 0 \\ U_2, & if \quad x < 0 \end{cases}$$

The characteristic equations are:

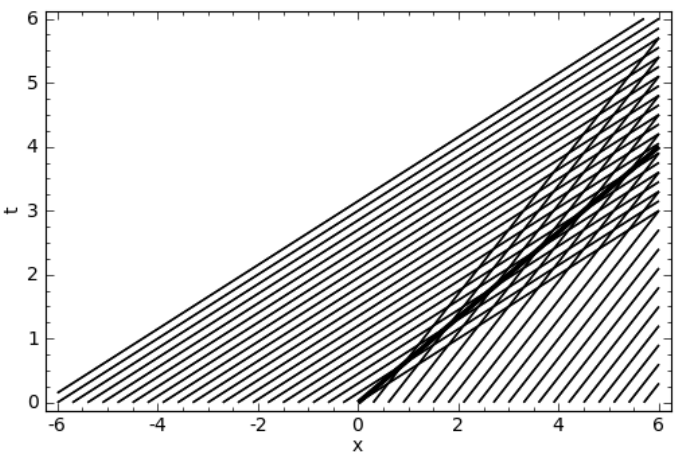
$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{u} = \frac{\mathrm{d}u}{-u}$$

So the equation of the characteristic curves of this PDE,

$$x = u(x,t)t + z \quad \text{where z is an arbitrary constant} \tag{2}$$

$$x = u(\xi,0)t + \xi = \begin{cases} U_1 t + \xi & \text{if} \quad \xi > 0 \\ U_2 t + \xi & \text{if} \quad \xi < 0 \end{cases}$$

If $U_1 < U_2$ The characteristics overlap immediately



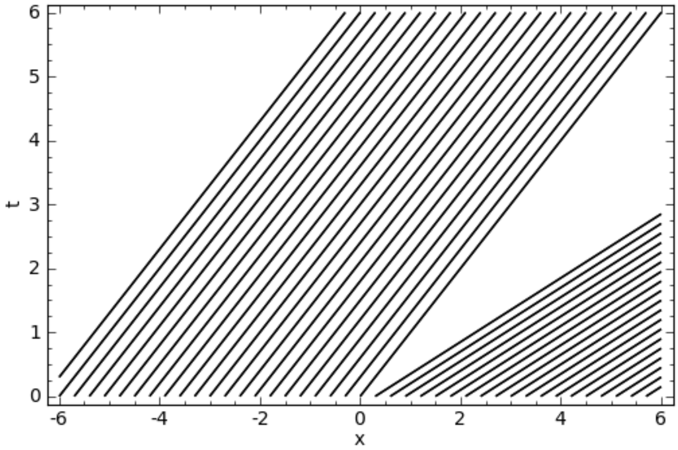
The shock path $x = s(t)$ is given by

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{2}(u_+ + u_-) = \frac{1}{2}(U_1 + U_2)$$

Thus

$$s(t) = \frac{1}{2}(U_1 + U_2)t + s(0) = \frac{1}{2}(U_1 + U_2)t$$

If $U_2 < U_1$, we see from the figure that the characteristics never over-



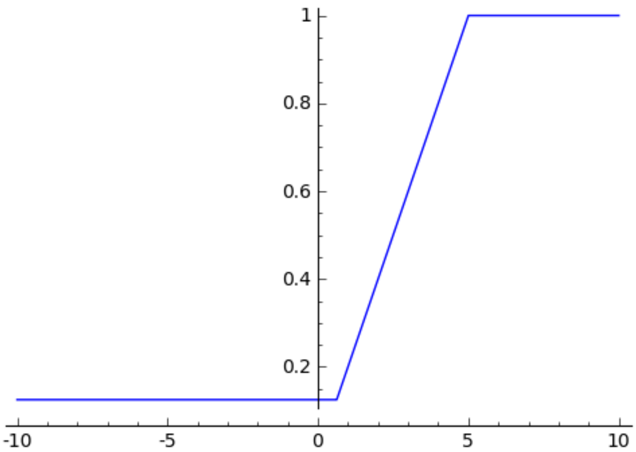
lap.

We have a rarefaction wave, we have to fill the region in the middle of the plane by straight lines which joins the origin, and satisfy the equation, the function $u(x,t) = x/t$ satisfy that, because

$$u_t + uu_x = -\frac{x}{t^2} + \frac{x}{t}\frac{1}{t} = 0,$$

and $u(t, U_1 t) = U_1$ and $u(t, U_2 t) = U_2$, so we can say that the following function is a solution $u(x,t) = \begin{cases} U_1, & \text{if} \quad x > U_1 t, \\ \frac{x}{t}, & \text{if} \quad U_2 t < x < U_1 t, \\ U_2, & \text{if} \quad x < U_2 t \end{cases}$

This is the sketch of such a solution.



9. **Initial value problem**

(i)

$$\begin{cases} u_x^2 + u_y^2 = 4u \\ u(x,0) = x^2 + 1 = f(x) \end{cases}$$

9 Solve the initial value problem

(i) $u_x^2 + u_y^2 = 4u$ with $u(x,0) = x^2 + 1$

Let $p = u_x$ and $q = u_y$. So, the equation can be written in the form

$$F(x, y, p, q, u) = p^2 + q^2 - 4u = 0.$$

The *Charpit's equations* are

$$\frac{dx}{dt} = 2p$$
$$\frac{dy}{dt} = 2q$$
$$\frac{dp}{dt} = 4p$$
$$\frac{dq}{dt} = 4q$$
$$\frac{du}{dt} = 2p^2 + 2q^2$$

$$\frac{dp}{dt} = 4p \Rightarrow p = Be^{4t}$$
$$\frac{dq}{dt} = 4q \Rightarrow q = Ae^{4t}$$

So substituting p and q in the first equation, we get

$$u = \frac{1}{4}e^{8t}(A^2 + B^2)$$

Also we substitute p and q in the equations in x and y

$$\frac{dx}{dt} = 2p \quad \Rightarrow \frac{dx}{dt} = 2Be^{4t} \quad \Rightarrow \quad x = \frac{B}{2}e^{4t} + C$$

and

$$\frac{dy}{dt} = 2q \quad \Rightarrow \frac{dy}{dt} = 2Ae^{4t} \quad \Rightarrow \quad y = \frac{A}{2}e^{4t} + D$$

We need to parametrize the initial conditions

$$x_o(s) = s,\ y_o(s) = 0,\ u_o(s, 0) = 1 + s^2.$$

We need to compute $p_o(s)$ and $q_o(s)$.
Since $p_o(s) = u_x(s)$ and $u_o(s) = 1 + s^2$,

$$p_o(s) = 2s$$

The initial conditions implies

$$(2s^2) + q^2 = 4(s^2 + 1) \ \Rightarrow\ q = \pm 2$$

at $t = 0$, we get $A = \pm 2$, we get also $B = 2s$,$C = 0$ and $D = -1$

From the equation in x and y above

$$x = se^{4t},\ y = e^{4t} - 1,\ u(x, t) = s^2 e^{4t} + e^{4t}$$

This implies

$$x^2 = s^2 e^{8t}, y^2 = e^{8t} - 2e^{4t} + 1, u = e^{4t}(s^2 + 1)$$

therefore So our solution is

$$u(x, y) = x^2 + y^2 + 2y + 1$$