PETROV–GALERKIN METHODS ON MULTIPLY CONNECTED DOMAINS FOR THE VORTICITY–STREAM FUNCTION FORMULATION OF THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS

T. E. TEZDUYAR

Department of Aerospace Engineering and Mechanics and Minnesota Supercomputer Institute, University of Minnesota, Minneapolis, MN 55455, U.S.A.

R. GLOWINSKI

Departments of Mathematics and Mechanical Engineering, University of Houston, Houston, TX 77004, U.S.A.

AND

J. LIOU

Department of Aerospace Engineering and Mechanics and Minnesota Supercomputer Institute, University of Minnesota, Minneapolis, MN 55455, U.S.A.

SUMMARY

In this paper we present streamline-upwind/Petrov–Galerkin finite element procedures for two-dimensional fluid dynamics computations based on the vorticity–stream function formulation of the incompressible Navier–Stokes equations. We address the difficulties associated with the convection term in the vorticity transport equation, lack of boundary condition for the vorticity at no-slip boundaries, and determination of the value of the stream function at the internal boundaries for multiply connected domains. The proposed techniques, implemented within the framework of block-iteration methods, have successfully been applied to various problems involving simply and multiply connected domains.

KEY WORDS   Petrov–Galerkin   Multiply connected domains   Vorticity–stream function

1. INTRODUCTION

There are some advantages in using the vorticity–stream function formulation of the incompressible Navier–Stokes equations for two-dimensional computations. Compared to the velocity–pressure formulation, the vorticity–stream function formulation leads to computed flow fields which satisfy the incompressibility condition automatically; also the number of unknown functions is reduced from three to two and the vorticity field is computed directly instead of being obtained by differentiation of the velocity field. The last advantage becomes important if one needs to study the vorticity field and therefore wants that this field be represented as accurately as possible.

We propose suitable finite element procedures for the solution of the time-dependent vorticity transport equation and the Poisson's equation which relates the stream function to the vorticity. The difficulties associated with the convection term in the vorticity transport equation, lack of

boundary conditions for the vorticity at no-slip boundaries, and determination of the value of the stream function at the internal boundaries for multiply connected domains are addressed.

The vorticity transport equation involves a convection term which becomes more and more dominant as the Reynolds number of the flow increases. It is well known that, due to such dominant convection terms, especially in the presence of the sharp layers in the solution, regular (Bubnov–) Galerkin finite element and classical centred finite difference methods lead to spurious oscillations in the solution. To minimize such oscillations, we employ streamline-upwind/Petrov–Galerkin (SUPG) schemes. These schemes, which introduce minimal numerical diffusion, have been successfully applied to various fluid dynamics and convection–diffusion–reaction problems.[1–6] The weighting function which would otherwise lead to a regular Galerkin formulation is perturbed by a term which senses the temporal and spatial discretizations of a problem.[5] Certain choices lead to symmetric, positive-definite coefficient matrices, and this is an advantage when one uses iterative solution techniques such as conjugate gradient methods[7] and element-by-element approximate factorization schemes.[8]

At no-slip boundaries corresponding to solid surfaces, while there are two boundary conditions available (normal and tangential derivatives) for the stream function one can find none for the vorticity. By employing proper function spaces, from the Poisson's equation we extract variational equations for the vorticity at such boundaries and for the stream function in the interior regions of the domain. In our block-iteration scheme we keep the solution of the vorticity transport and Poisson's equations uncoupled. This enables us to leave the matrices for the equation systems corresponding to the Poisson's equation as symmetric and positive-definite. Alternative approaches to the numerical treatment of the vorticity at no-slip boundaries are discussed in References 9 and 10.

For simply connected domains the stream function at a no-slip boundary can be obtained by integrating its tangential derivative. For such an integration the integration constant is either known or can be arbitrarily set equal to a reference stream function value. However, for problems with multiply connected domains the integration constant for each of the internal boundaries is also an unknown and needs to be determined as part of the solution. The additional equation required for each internal boundary can be obtained by integrating the equation of motion in the velocity–pressure formulation along that boundary. The resulting equation together with the vorticity transport equation can be used with the aid of a suitable function space to obtain the additional variational equation needed. Discussion of problems with multiply connected domains can also be found in References 11 and 12. It is proper to note that for computations based on the velocity–pressure formulation of the Navier–Stokes equations (see e.g. References 12–15) one does not have to deal with the technical difficulty related to multiply connected domains.

In Section 2 the problem statement is given. The finite element formulation is presented in Section 3. We describe our time integration scheme in Section 4 and the selection of the Petrov–Galerkin weighting function in Section 5. Numerical examples and conclusions are presented in Sections 6 and 7 respectively.

2. FORMULATION OF THE PROBLEM

The vorticity–stream function formulation of the incompressible Navier–Stokes equations in two dimensions consists of two coupled scalar equations: a time-dependent transport equation for the unknown vorticity function ω and an equation which relates the unknown stream function ψ to the vorticity.

Let Ω be a domain in R^2 and let T be a positive real number. The spatial and temporal coordinates are denoted by $x \in \bar{\Omega}$ and $t \in [0, T]$, where a superposed bar indicates the set closure. The

equations for $\omega(\mathbf{x}, t)$ and $\psi(\mathbf{x}, t)$ are given as follows:

$$\partial\omega/\partial t + \mathbf{u}\cdot\nabla\omega = v\nabla^2\omega \quad \text{on } \Omega\times(0, T), \tag{1}$$

$$\nabla^2\psi = -\omega \quad \text{on } \Omega\times(0, T), \tag{2}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity and v is the kinematic viscosity. The vorticity and the stream function are related to the velocity field by the following equations:

$$\omega = \partial u_2/\partial x_1 - \partial u_1/\partial x_2, \tag{3}$$

$$u_1 = \partial\psi/\partial x_2, \tag{4}$$

$$u_2 = -\partial\psi/\partial x_1. \tag{5}$$

We assume first that the domain is simply connected and that the boundary Γ of the domain Ω admits the following decompositions with respect to the types of boundary conditions specified for ψ and ω:

$$\Gamma = \overline{\Gamma_g \cup \Gamma_G \cup \Gamma_h}, \tag{6}$$

$$\Gamma_g \cap \Gamma_G = \varnothing, \qquad \Gamma_g \cap \Gamma_h = \varnothing, \qquad \Gamma_G \cap \Gamma_h = \varnothing, \tag{7}$$

$$\Gamma = \overline{\Gamma_{\tilde{g}} \cup \Gamma_G \cup \Gamma_{\tilde{h}}}, \tag{8}$$

$$\Gamma_{\tilde{g}} \cap \Gamma_G = \varnothing, \qquad \Gamma_{\tilde{g}} \cap \Gamma_{\tilde{h}} = \varnothing, \qquad \Gamma_G \cap \Gamma_{\tilde{h}} = \varnothing. \tag{9}$$

Here Γ_g and Γ_h denote the subsets of Γ where the boundary condition specified for ψ is of Dirichlet and Neumann type respectively. Similarly, $\Gamma_{\tilde{g}}$ and $\Gamma_{\tilde{h}}$ represent the Dirichlet- and Neumann-type boundaries for ω. Both Dirichlet- and Neumann-type boundary conditions for ψ are specified on Γ_G where neither ω nor its normal derivative is known. It should be noted that all solid surfaces will be subsets of Γ_G (unless the flow is inviscid and therefore the surface permits slip).

Consider the following set of boundary conditions for ψ and ω:

$$\psi(\mathbf{x}, t) = g(\mathbf{x}, t) \quad \text{on } \Gamma_g\times(0, T), \tag{10}$$

$$\psi(\mathbf{x}, t) = G(\mathbf{x}, t) \quad \text{on } \Gamma_G\times(0, T), \tag{11}$$

$$\mathbf{n}\cdot\nabla\psi(\mathbf{x}, t) = u_\tau(\mathbf{x}, t) = \mathbf{u}\cdot\tau \quad \text{on } \Gamma_G\times(0, T), \tag{12}$$

$$\mathbf{n}\cdot\nabla\psi(\mathbf{x}, t) = h(\mathbf{x}, t) \qquad \text{on } \Gamma_h\times(0, T), \tag{13}$$

$$\omega(\mathbf{x}, t) = \tilde{g}(\mathbf{x}, t) \qquad \text{on } \Gamma_{\tilde{g}}\times(0, T), \tag{14}$$

$$v\mathbf{n}\cdot\nabla\omega(\mathbf{x}, t) = \tilde{h}(\mathbf{x}, t) \qquad \text{on } \Gamma_{\tilde{h}}\times(0, T), \tag{15}$$

where $\mathbf{n}$ and τ are the unit normal and tangential vectors to the boundary Γ (see Figure 1), and u_τ is the tangential component of the velocity on the boundary. The functions $g(\mathbf{x}, t)$, $G(\mathbf{x}, t)$, $u_\tau(\mathbf{x}, t)$, $h(\mathbf{x}, t)$, $\tilde{g}(\mathbf{x}, t)$ and $\tilde{h}(\mathbf{x}, t)$ are assumed to be given.

The initial condition corresponding to the vorticity transport equation (1) is given as

$$\omega(\mathbf{x}, 0) = \omega_0(\mathbf{x}) \quad \text{on } \Omega. \tag{16}$$

Remarks

1. The vorticity transport equation (1) belongs to a general class of convection–diffusion equations. In this case the transported quantity is the vorticity ω.

2. The flow field obtained from the solution of (1) and (2) is inherently divergence-free.
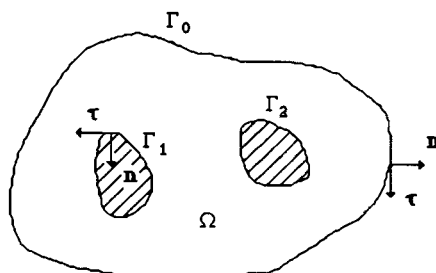
Figure 1. Convention for the normal and tangential vectors

3. The boundary condition (11) originates from the following condition on the normal velocity:

$$-\boldsymbol{\tau} \cdot \nabla \psi(\mathbf{x}, t) = u_n(\mathbf{x}, t) = \mathbf{u} \cdot \mathbf{n} \quad \text{on } \Gamma_G \times (0, T), \tag{17}$$

where $u_n(\mathbf{x}, t)$ is a given function (which is usually zero). The trace of the stream function G can be obtained by integration:

$$G(\mathbf{x}, t) = -\int u_n(\mathbf{x}, t) \, d\tau, \tag{18}$$

which necessitates an integration constant. If the domain is *not* multiply connected, this integration constant is known or can arbitrarily be set equal to a reference stream function value. However, for problems with multiply connected domains where there is at least one internal boundary, the integration constants for the internal boundaries become unknowns themselves to be determined as part of the solution.

Multiply connected domains

Let Γ_0 denote the external boundary of a multiply connected domain Ω and let Γ_k denote the boundary corresponding to the kth hole (obstacle) in the domain Ω with $k = 1, 2, \ldots, q$, where q is the number of holes in the domain. The entire boundary Γ can be expressed as

$$\Gamma = \bigcup_{k=0}^{q} \Gamma_k. \tag{19}$$

The decompositions expressed by (6)–(9) now apply to the external boundary Γ_0. To the boundary conditions given by (10)–(15) we add the following conditions for the internal boundaries:

$$\mathbf{n} \cdot \nabla \psi(\mathbf{x}, t) = u_\tau(\mathbf{x}, t) \quad \text{on } \Gamma_k \times (0, T), \quad k = 1, 2, \ldots, q, \tag{20}$$

$$-\boldsymbol{\tau} \cdot \nabla \psi(\mathbf{x}, t) = u_n(\mathbf{x}, t) \quad \text{on } \Gamma_k \times (0, T), \quad k = 1, 2, \ldots, q. \tag{21}$$

These conditions are identical to those given by (12) and (17) for the boundary Γ_G.

Assuming that $u_n = 0$, we can deduce from (21) that ψ is constant (an unknown constant) along each Γ_k. Additional equations needed to determine these constants can be obtained by writing the velocity–pressure formulation of the equation of motion along each Γ_k; that is,

$$\partial u_\tau / \partial t + u_\tau \, \partial u_\tau / \partial \tau + (1/\rho) \partial p / \partial \tau + \nu \partial \omega / \partial n = 0$$
$$\text{on } \Gamma_k \times (0, T), \quad k = 1, 2, \ldots, q, \tag{22}$$

or

$$\partial u_\tau/\partial t + \partial(\tfrac{1}{2}u_\tau^2 + p/\rho)/\partial\tau + v\partial\omega/\partial n = 0$$

$$\text{on } \Gamma_k \times (0, T), \quad k = 1, 2, \ldots, q. \quad (23)$$

Assuming $u_\tau = u_\tau(t)$ (no stretching on the surface of the obstacle) and integrating both sides of (23) along Γ_k, we obtain

$$-v\int_{\Gamma_k} (\partial\omega/\partial n)\mathrm{d}\Gamma = (\partial u_\tau/\partial t)\int_{\Gamma_k} \mathrm{d}\Gamma + \int_{\Gamma_k} [\partial(\tfrac{1}{2}u_\tau^2 + p/\rho)/\partial\tau]\mathrm{d}\Gamma. \quad (24)$$

Since $(\tfrac{1}{2}u_\tau^2 + p/\rho)$ is single-valued, the second integral on the right-hand side of equation (24) is identically zero; therefore

$$v\int_{\Gamma_k} (\partial\omega/\partial n)\mathrm{d}\Gamma = -(\partial u_\tau/\partial t)\int_{\Gamma_k} \mathrm{d}\Gamma = -S_k\partial u_\tau/\partial t \quad \text{on} \quad \Gamma_k \times (0, T), \quad k = 1, 2, \ldots, q, \quad (25)$$

where S_k is the length of the boundary Γ_k. Equation (25) represents the additional constraint that the vorticity field needs to satisfy for each internal boundary Γ_k.

3. FINITE ELEMENT FORMULATION

Let Q^h denote a finite element discretization of the computational domain Ω into subdomains Ω^e, $e = 1, 2, \ldots, N_{el}$, where N_{el} is the number of elements. Let Γ^e denote the boundary of Ω^e. We assume

$$\bar{\Omega} = \bigcup_{e=1}^{N_{el}} \bar{\Omega}^e, \quad (26)$$

$$\varnothing = \bigcap_{e=1}^{N_{el}} \Omega^e. \quad (27)$$

The internal boundary is defined as

$$\Gamma_{\mathrm{int}} = \bigcup_{e=1}^{N_{el}} \Gamma^e - \Gamma. \quad (28)$$

We associate to Q^h the following finite-dimensional spaces:

$$H^{1h} = \{\phi^h | \phi^h \in C^0(\bar{\Omega}), \phi^h|_{\Omega^e} \in P^1 \ \forall\Omega^e \in Q^h\}, \quad (29)$$

where P^1 is the space of first-order polynomials in x_1, x_2, and then

$$S^h = \{\psi^h | \psi^h \in H^{1h}, \psi^h \doteq g \text{ on } \Gamma_g, \psi^h \doteq G \text{ on } \Gamma_G\}, \quad (30)$$

$$\tilde{S}^h = \{\omega^h | \omega^h \in H^{1h}, \omega^h \doteq \tilde{g} \text{ on } \Gamma_{\tilde{g}}\}, \quad (31)$$

$$V_*^h = \{w^h | w^h \in H^{1h}, w^h \doteq 0 \text{ on } \Gamma_g \cup \Gamma_G\}, \quad (32)$$

$$\tilde{V}_*^h = \{w^h | w^h \in H^{1h}, w^h \doteq 0 \text{ on } \Gamma_{\tilde{g}} \cup \Gamma_G\}, \quad (33)$$

$$V_G^h = \{w^h | w^h \in H^{1h}, w^h|_{\Omega^e} = 0 \ \forall\Omega^e \notin Q_G^h\}, \quad (34)$$

where

$$Q_G^h = \{\Omega^e | \Omega^e \in Q^h, \Gamma^e \cap \Gamma_G \neq \varnothing\}. \quad (35)$$

The finite element formulation associated with equation (1) is given as follows: Find $\psi^h \in S^h$ and $\omega^h \in \tilde{S}^h$ such that for all $w^h \in \tilde{V}^h_*$,

$$\int_\Omega w^h(\partial\omega^h/\partial t + \mathbf{u}\cdot\nabla\omega^h)\,d\Omega + \int_\Omega \nu\nabla w^h\cdot\nabla\omega^h\,d\Omega$$

$$+ \sum_{e=1}^{N_{el}} \int_{\Omega^e} p^h(\partial\omega^h/\partial t + \mathbf{u}\cdot\nabla\omega^h - \nu\nabla^2\omega^h)\,d\Omega = \int_{\Gamma_{\tilde{h}}} w^h\tilde{h}\,d\Gamma, \tag{36}$$

where p^h is a $C^{-1}(\Omega)$ perturbation to the weighting function w^h. Selection of this Petrov–Galerkin perturbation term will be discussed further in Section 5. The Euler–Lagrange equations corresponding to (36) can be obtained by integration by parts:

$$\sum_{e=1}^{N_{el}} \int_{\Omega^e} \tilde{w}^h(\partial\omega^h/\partial t + \mathbf{u}\cdot\nabla\omega^h - \nu\nabla^2\omega^h)\,d\Omega$$

$$+ \int_{\Gamma_{int}} w^h[\nu\mathbf{n}\cdot\nabla\omega^h]\,d\Gamma + \int_{\Gamma_{\tilde{h}}} w^h(\nu\mathbf{n}\cdot\nabla\omega^h - \tilde{h})\,d\Gamma = 0, \tag{37}$$

where $[\cdot]$ is the 'jump' operator.

Remarks

1. If the perturbation term p^h is set to zero, then the formulation reduces to a (Bubnov–) *Galerkin* one; otherwise it becomes a Petrov–Galerkin formulation. Note that the modified weighting function, i.e.

$$\tilde{w}^h = w^h + p^h, \tag{38}$$

acts only in the element interiors and therefore is allowed to be discontinuous across element boundaries.

2. Various formulations for p^h within the framework of streamline-upwind, sigma-weighting and transport-weighting Petrov–Galerkin procedures have been tested successfully in References 1–6.

The discrete variational formulation associated with (2) can be stated as follows: Find $\psi^h \in S^h$ and $\omega^h \in \tilde{S}^h$ such that

$$\int_\Omega \nabla w^h\cdot\nabla\psi^h\,d\Omega - \int_\Omega w^h\omega^h\,d\Omega = \int_{\Gamma_h} w^h h\,d\Gamma \quad \forall w^h \in V^h_*, \tag{39}$$

and

$$\int_\Omega \nabla w^h\cdot\nabla\psi^h\,d\Omega - \int_\Omega w^h\omega^h\,d\Omega = \int_{\Gamma_h} w^h h\,d\Gamma + \int_{\Gamma_G} w^h u_\tau\,d\Gamma \quad \forall w^h \in V^h_G. \tag{40}$$

We can write ψ^h and ω^h as sums of their component functions as follows:

$$\psi^h = \psi^h_* + \psi^h_g + \psi^h_G, \tag{41}$$

$$\omega^h = \omega^h_* + \omega^h_g + \omega^h_G. \tag{42}$$

The unknowns are ψ_*^h, ω_*^h and ω_G^h. Note that

$$\psi_G^h \doteq G \quad \text{on } \Gamma_G, \tag{43}$$

$$\psi_g^h \doteq g \quad \text{on } \Gamma_g, \tag{44}$$

$$\omega_{\tilde{g}}^h \doteq \tilde{g} \quad \text{on } \Gamma_{\tilde{g}}, \tag{45}$$

$$\psi_*^h \in V_*^h, \tag{46}$$

$$\omega_*^h \in \tilde{V}_*^h, \tag{47}$$

$$\omega_G^h \in V_G^h. \tag{48}$$

In our implementation we employ a block-iteration scheme to solve equations (36), (39) and (40). During each iteration associated with (36), only ω_*^h is treated as unknown while the previous iteration values of ψ_*^h and ω_G^h are used. Equations (39) and (40) are taken as a single block; that is, during each iteration corresponding to (39) and (40), ψ_*^h and ω_G^h are treated simultaneously as unknowns.

Alternatively, we can assign a block to each of the equations (36), (39) and (40). That is, during each iteration associated with (36), (39) and (40), we can treat respectively ω_*^h, ψ_*^h and ω_G^h as unknown. As a result, the coefficient matrices for the second and third blocks would be symmetric, positive-definite; and this would provide an advantage for employing iterative solution techniques such as conjugate gradient methods[7] and element-by-element approximate factorization schemes.[8]

Multiply connected domains

To accommodate the boundary conditions associated with the multiply connected domains, we need to modify and add new ones to the finite-dimensional spaces defined previously. We modify the definitions of S^h, V_*^h and $\tilde{V}_*^h$ as follows:

$$S^h = \{\psi^h | \psi^h \in H^{1h}, \psi^h \doteq g \text{ on } \Gamma_g, \psi^h \doteq G \text{ on } \Gamma_G, \partial \psi^h / \partial \tau \doteq 0 \text{ on } \Gamma_k, k = 1, 2, \ldots, q\}, \tag{49}$$

$$V_*^h = \left\{ w^h | w^h \in H^{1h}, w^h \doteq 0 \text{ on } \Gamma_g \cup \Gamma_G \bigcup_{k=1}^q \Gamma_k \right\}, \tag{50}$$

$$\tilde{V}_*^h = \left\{ w^h | w^h \in H^{1h}, w^h \doteq 0 \text{ on } \Gamma_{\tilde{g}} \cup \Gamma_G \bigcup_{k=1}^q \Gamma_k \right\}. \tag{51}$$

Note that in (49) we assumed that the normal velocity is zero. Furthermore, we define two new spaces for $k = 1, 2, \ldots, q$:

$$V_k^h = \{w^h | w^h \in H^{1h}, w^h|_{\Omega^e} = 0 \ \forall \Omega^e \notin Q_k^h\}, \tag{52}$$

$$V_{kR}^h = \{w^h | w^h \in V_k^h, \partial w^h / \partial \tau \doteq 0 \text{ on } \Gamma_k\} \quad \text{('Ring function')}, \tag{53}$$

where

$$Q_k^h = \{\Omega^e | \Omega^e \in Q^h, \Gamma^e \cap \Gamma_k \neq \varnothing\}. \tag{54}$$

The discrete variational formulations given by (36) and (39) remain valid with the modified/new definitions of the finite-dimensional spaces. The variational formulation given by (40) also remains valid with the provision that the variational space V_G^h is replaced by $V_G^h \bigcup_{k=1}^q V_k^h$ and the boundary Γ_G by $\Gamma_G \bigcup_{k=1}^q \Gamma_k$.

The variational formulation associated with (25) can be obtained from the vorticity transport equation (1) with the aid of the 'Ring function' space given by (53). This new variational formulation reads as follows: Find $\psi^h \in S^h$ and $\omega^h \in \bar{S}^h$ such that for all $w^h \in V_{kR}^h$, $k = 1, 2, \ldots, q$,

$$\int_\Omega w^h (\partial \omega^h / \partial t) \, d\Omega + \int_\Omega w^h \mathbf{u} \cdot \nabla \omega^h \, d\Omega + v \int_\Omega \nabla w^h \cdot \nabla \omega^h \, d\Omega = v \int_{\Gamma_k} w^h (\partial \omega^h / \partial n) \, d\Gamma. \tag{55}$$

Since w^h is constant along Γ_k, the right-hand side of (55) can be obtained from (25):

$$\int_\Omega w^h (\partial \omega^h / \partial t) \, d\Omega + \int_\Omega w^h \mathbf{u} \cdot \nabla \omega^h \, d\Omega + v \int_\Omega \nabla w^h \cdot \nabla \omega^h \, d\Omega = - w^h (\Gamma_k) S_k \partial u_\tau / \partial t. \tag{56}$$

These equations are the additional variational formulations needed to solve the problem. Note that the term on the right-hand side of (56) is a known function of time and represents a source term.

The decompositions of ψ^h and ω^h given by (41) and (42) are modified as follows:

$$\psi^h = \psi_*^h + \psi_g^h + \psi_G^h + \sum_{k=1}^q \psi_k^h, \tag{57}$$

$$\omega^h = \omega_*^h + \omega_g^h + \omega_G^h + \sum_{k=1}^q \omega_k^h, \tag{58}$$

where

$$\psi_k^k \in V_{kR}^h, \quad k = 1, 2, \ldots, q, \tag{59}$$

$$\omega_k^h = V_k^h, \quad k = 1, 2, \ldots, q. \tag{60}$$

Remark. Another variational form can be obtained by replacing V_G^h in (40) with V_{kR}^h (the 'Ring function' space): find $\psi^h \in S^h$ and $\omega^h \in \bar{S}^h$ such that for all $w^h \in V_{kR}^h$, $k = 1, 2, \ldots, q$,

$$\int_\Omega \nabla w^h \cdot \nabla \psi^h \, d\Omega - \int_\Omega w^h \omega^h \, d\Omega = \int_{\Gamma_k} w^h u_\tau \, d\Gamma. \tag{61}$$

Note that since $V_{kR}^h \subset V_k^h$, this equation provides no more information than the multiply connected domain version of (40) already does. However, this equation can be substituted for one of the equations of (40) corresponding to any one of the basis functions in V_k^h.

In the remaining part of this section, as we describe the block iteration schemes employed, whenever we refer to a certain variational formulation we mean to refer to its multiply connected version.

The first block corresponds to (36) in which ω_*^h is treated as unknown. Equations (39) and (61) are taken as the second block; during each iteration associated with (39) and (61), ψ_*^h and ψ_k^h, $k = 1, 2, \ldots, q$, are treated simultaneously as unknowns. In the third block, associated with (40), ω_G^h and ω_k^h, $k = 1, 2, \ldots, q$, are taken as unknowns. However, for each interior boundary Γ_k, one of the equations of the third block (corresponding to any one of the basis functions in V_k^h) needs to be eliminated in favour of each equation given by (56) associated with that boundary. The fourth block consists of these equations corresponding to (56) and treats one of the nodal vorticity values on each Γ_k as unknown. Note that these nodal vorticity values need to be removed from the list of unknowns for the third block. In other words, for every internal boundary Γ_k, the third block loses the equation and the unknown vorticity associated with a certain node; whereas the fourth block, which always has the same set of equations, gains the unknown lost by the third block.

Remarks

1. The matrices involved in the second and third blocks are symmetric and positive-definite.

2. With proper implementation, the matrix in the third block can be of tridiagonal form. This makes the solution of this block essentially as easy as the solution of a one-dimensional problem.

3. Of course one can always couple the third and fourth blocks but at the expense of the nice properties mentioned in the first two remarks.

4. Another approach would involve a local iteration procedure between the third and fourth blocks until a local convergence is reached. This procedure may allow selecting a different node to be 'lost' by the third block for each local iteration.

4. TEMPORAL DISCRETIZATION

We describe the temporal integration procedures employed within the framework of our block-iteration schemes. For problems with simply connected domains, the following system of ordinary differential equations is obtained from equations (36), (39) and (40):

$$\tilde{\mathbf{M}}\mathbf{a}_* + \tilde{\mathbf{C}}\mathbf{v}_* = \tilde{\mathbf{F}}, \tag{62}$$

$$\mathbf{v}_*(0) = (\mathbf{v}_*)_0, \tag{63}$$

$$\mathbf{K}\mathbf{d}_* - \mathbf{M}\mathbf{v}_G = \mathbf{F}, \tag{64}$$

where $\mathbf{v}_*$, $\mathbf{a}_*$, $\mathbf{d}_*$ and $\mathbf{v}_G$ are the vectors of the nodal values of ω_*^h, $\partial\omega_*^h/\partial t$, ψ_*^h and ω_G^h respectively. The vectors $\tilde{\mathbf{F}}$ and $\mathbf{F}$ as well as the matrices $\tilde{\mathbf{M}}$, $\tilde{\mathbf{C}}$, $\mathbf{K}$ and $\mathbf{M}$ are derived from equations (36), (39) and (40). These vectors and matrices are updated at the end of every iteration. The procedure, which uses a predictor/multi-corrector algorithm, is described as follows:

Given:

$$(\mathbf{v}_*)_n, (\mathbf{a}_*)_n, (\mathbf{d}_*)_n \text{ and } (\mathbf{v}_G)_n.$$

Prediction:

$$(\mathbf{v}_*)_{n+1}^0 = (\mathbf{v}_*)_n + (1 - \alpha\,\Delta t)(\mathbf{a}_*)_n, \tag{65}$$

$$(\mathbf{a}_*)_{n+1}^0 = \mathbf{0}, \tag{66}$$

$$\mathbf{K}(\mathbf{d}_*)_{n+1}^0 - \mathbf{M}(\mathbf{v}_G)_{n+1}^0 = \mathbf{F}_{n+1}^0, \tag{67}$$

$$(\mathbf{u})_{n+1}^0 = (\partial\psi^h/\partial x_2, -\partial\psi^h/\partial x_1)_{n+1}^0. \tag{68}$$

Correction:

$$(\tilde{\mathbf{M}} + \alpha\Delta t\tilde{\mathbf{C}})_{n+1}^i \Delta(\mathbf{a}_*)_{n+1}^i = \tilde{\mathbf{F}}_{n+1}^i - (\tilde{\mathbf{M}}\mathbf{a}_* + \tilde{\mathbf{C}}\mathbf{v}_*)_{n+1}^i = \mathbf{R}_{n+1}^i, \tag{69}$$

$$(\mathbf{v}_*)_{n+1}^{i+1} = (\mathbf{v}_*)_{n+1}^i + \alpha\,\Delta t\,\Delta(\mathbf{a}_*)_{n+1}^i, \tag{70}$$

$$(\mathbf{a}_*)_{n+1}^{i+1} = (\mathbf{a}_*)_{n+1}^i + \Delta(\mathbf{a}_*)_{n+1}^i, \tag{71}$$

$$\mathbf{K}(\mathbf{d}_*)_{n+1}^{i+1} - \mathbf{M}(\mathbf{v}_G)_{n+1}^{i+1} = \mathbf{F}_{n+1}^{i+1}, \tag{72}$$

$$(\mathbf{u})_{n+1}^{i+1} = (\partial\psi^h/\partial x_2, -\partial\psi^h/\partial x_1)_{n+1}^{i+1}. \tag{73}$$

The corrections continue until the L_2^h-norm of the residual vector $\mathbf{R}_{n+1}^{i+1}$ becomes smaller than a

predetermined error toll. Here Δt is the time step and α is a parameter which controls the stability and accuracy of the time integration.

In the procedure described above, equations (39) and (40) are taken as a single block represented by (64). As was mentioned in Section 3, an alternative approach would involve the decoupling of these two blocks. In that case, each of the equations given by (64), (67) and (72) needs to be replaced by a pair of equations, respectively, as follows:

In the differential equations:

$$\mathbf{K}_{II}\mathbf{d}_* = \mathbf{F}_{II},\tag{74}$$

$$\mathbf{M}_{III}\mathbf{v}_G = \mathbf{F}_{III}.\tag{75}$$

In the prediction phase:

$$\mathbf{K}_{II}(\mathbf{d}_*)^0_{n+1} = (\mathbf{F}_{II})^0_{n+1},\tag{76}$$

$$\mathbf{M}_{III}(\mathbf{v}_G)^0_{n+1} = (\mathbf{F}_{III})^0_{n+1}.\tag{77}$$

In the correction phase:

$$\mathbf{K}_{II}(\mathbf{d}_*)^{i+1}_{n+1} = (\mathbf{F}_{II})^{i+1}_{n+1},\tag{78}$$

$$\mathbf{M}_{III}(\mathbf{v}_G)^{i+1}_{n+1} = (\mathbf{F}_{III})^{i+1}_{n+1}.\tag{79}$$

Note that the matrices $\mathbf{K}_{II}$ and $\mathbf{M}_{III}$ are symmetric and positive-definite. With proper implementation $\mathbf{M}_{III}$ can be of tridiagonal form.

Multiply connected domains

In this case the procedure is similar to the alternative approach described above. The block given by (62) remains as it is. The block of (74) now has the unknown vectors $\mathbf{d}_*$ and $\mathbf{d}_k$ (vector of nodal values of ψ^h_k), $k = 1, 2, \ldots, q$. The third block, given by (75), has the unknown vectors $\mathbf{v}_G$ and $\mathbf{v}_k$ (vector of nodal values of ω^h_k), $k = 1, 2, \ldots, q$. Note that for every internal boundary, the third block loses the equation and the unknown vorticity associated with one node on that boundary. The fourth block replaces those equations and gains the unknowns lost by the third block.

Note that the matrices of the second and third blocks are symmetric and positive-definite. Again, with proper implementation the matrix in the third block can be of tridiagonal form.

5. SELECTION OF THE PETROV–GALERKIN WEIGHTING FUNCTION

In the streamline-upwind/Petrov–Galerkin (SUPG) formulation employed, the perturbation to the weighting function can depend upon temporal as well as spatial discretizations. The weighting function associated with element node a is expressed, analogous to (38), as follows:

$$\tilde{N}_a = N_a + P_a,\tag{80}$$

where N_a is the bilinear weighting function (associated with node a) leading to a Galerkin formulation. The corresponding Petrov–Galerkin perturbation function P_a is defined by the following expression:[5]

$$P_a = z C_{2\tau}(h/2)\mathbf{s} \cdot \nabla N_a,\tag{81}$$

where

$$\mathbf{s} = \mathbf{u}/\|\mathbf{u}\|, \tag{82}$$

$$h = 2\left(\sum_{a=1}^{N_{en}} \left|\mathbf{s} \cdot \nabla N_a\right|\right)^{-1} \quad \text{('element length')}, \tag{83}$$

and N_{en} is the number of element nodes. The parameter z depends on the element Reynolds number Re_h and is given as follows:[2]

$$z = \min(Re_h/3, 1) \quad \text{(doubly asymptotic approximation)}. \tag{84}$$

The element Reynolds number is defined as

$$Re_h = \|\mathbf{u}\|h/(2v) \in [0, \infty). \tag{85}$$

Of the three expressions given below for the 'algorithmic Courant number' $C_{2\tau}$,[5] Only the first one does not depend upon the temporal discretization. They are

$$C_{2\tau} = 1, \tag{86}$$

$$C_{2\tau} = \min(C_{\Delta t}, 1), \tag{87}$$

$$C_{2\tau} = \min(2/\sqrt{15} + (1 - 2/\sqrt{15})C_{\Delta t}, 1), \tag{88}$$

where $C_{\Delta t} = \Delta t \|\mathbf{u}\|/h$ is the element Courant number. A justification and a detailed stability and accuracy analysis for these choices can be found in Reference 5.

Remark. The expression given by (87) leads to a symmetric and positive-definite coefficient matrix for the block corresponding to (36); for a proof see Reference 8.

6. NUMERICAL EXAMPLES

The methods described were tested on several problems including flow past a circular cylinder and plane jet impinging on a wedge (edgetone problem). In both cases the computed flow field starts out symmetric and then becomes unsymmetric due to a perturbation in the flow field introduced directly or by machine truncation error.

Flow past a circular cylinder

In this problem the Reynolds number based on the freestream velocity and the cylinder diameter is 100. The dimensions of the computational domain, normalized by the cylinder diameter, are 16 and 8 in the flow and cross-flow directions respectively. The finite element mesh, shown in Figure 2, consists of 1940 elements and 2037 nodes. Around the cylinder there are 20 elements in the radial and 80 elements in the circumferential directions. As initial condition the value of the vorticity is set to zero everywhere. At the upstream boundary we specify the stream function and its normal derivative to have a uniform velocity field with magnitude 0·125. The upper and lower computational boundaries are taken as streamlines with zero vorticity. At the downstream boundary the normal derivative of both the vorticity and the stream function is set to zero. The time step of the computation is taken to be 1·0.

We allowed the value of the stream function on the cylinder surface to be unknown and therefore employed our procedure for the multiply connected domains. However, since the computational domain is symmetric and the vortex shedding (non-symmetrical solution) occurs mainly behind
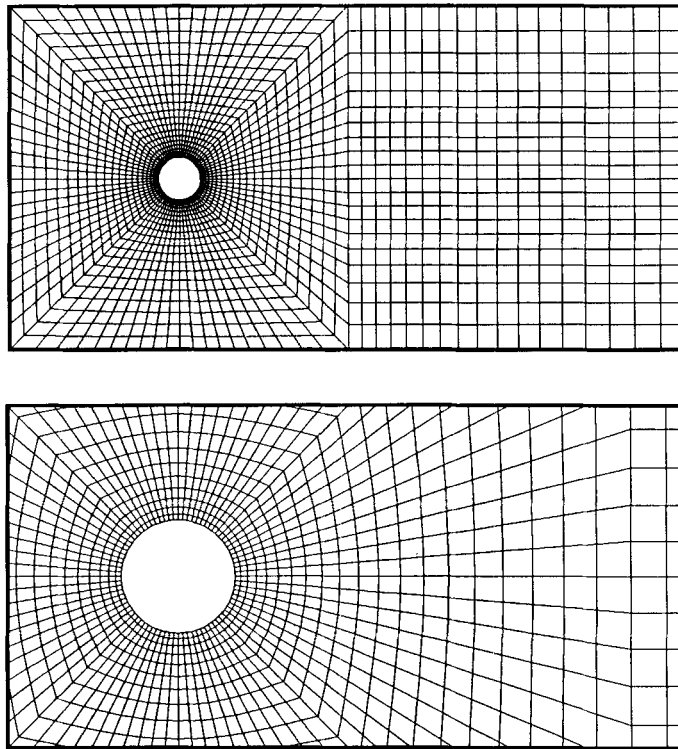
Figure 2. Flow past a circular cylinder at $Re = 100$; finite element mesh: 1940 elements, 2037 nodes

the cylinder, it is also reasonable to assume that the stream function on the cylinder is known. Solutions obtained by such an assumption[16] compare well with our solutions.

Initially the flow field develops in a symmetrical pattern with two attached eddies growing in the wake of the cylinder leading to a steady-state solution (see Figure 3). Upon introduction of a perturbation in the flow field, the symmetry breaks; and after a transient phase in which the sizes of the standing vortices oscillate, vortex shedding begins. The vortices are formed alternately at the upper and lower downstream vicinity of the cylinder and are carried downstream along the so-called Karman vortex street (see Figures 4 and 5).

Plane jet impinging on a wedge (edgetone problem)

This problem involves the interaction between a plane jet and a sharp leading edge. Simulations of this kind were first performed by Ohring[17] based on the finite difference method.

In our application the jet inlet consists of a parabolic velocity profile with both the width and the mean value set to unity; and the Reynolds number based on these values is 250. The distance between the jet inlet and the leading tip of the wedge is 7·5. The dimensions of the computational domain are 35 and 30 in the flow and cross-flow directions respectively. The mesh has 2392 elements and 2519 nodes; it is shown in Figure 6. The initial vorticity values are set to zero everywhere. At the jet inlet we specify the stream function and its normal derivative to have a parabolic velocity profile with mean value 1·0. The left external boundary and the upper and lower
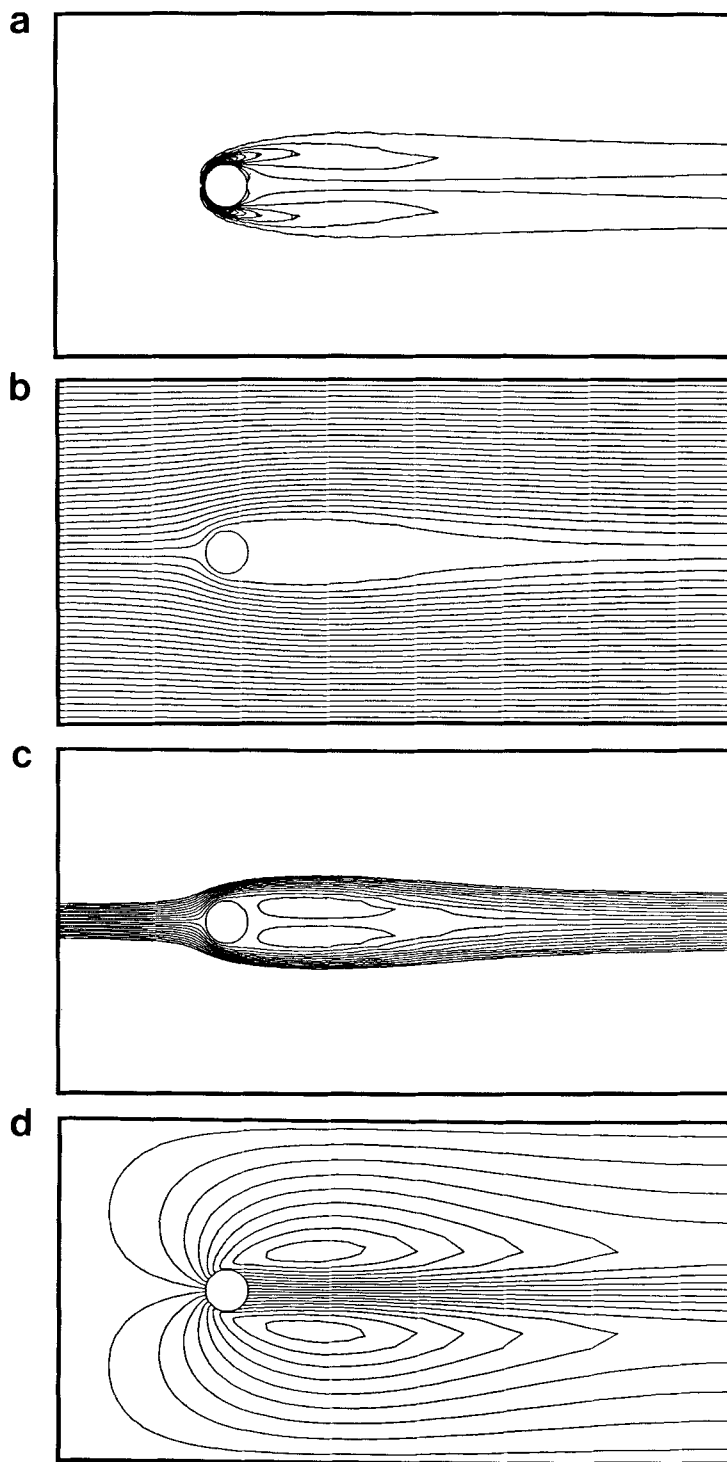
Figure 3. Flow past a circular cylinder at Re = 100, time step = 716; (a) vorticity contours, (b) (c) streamlines, (d) stationary streamlines
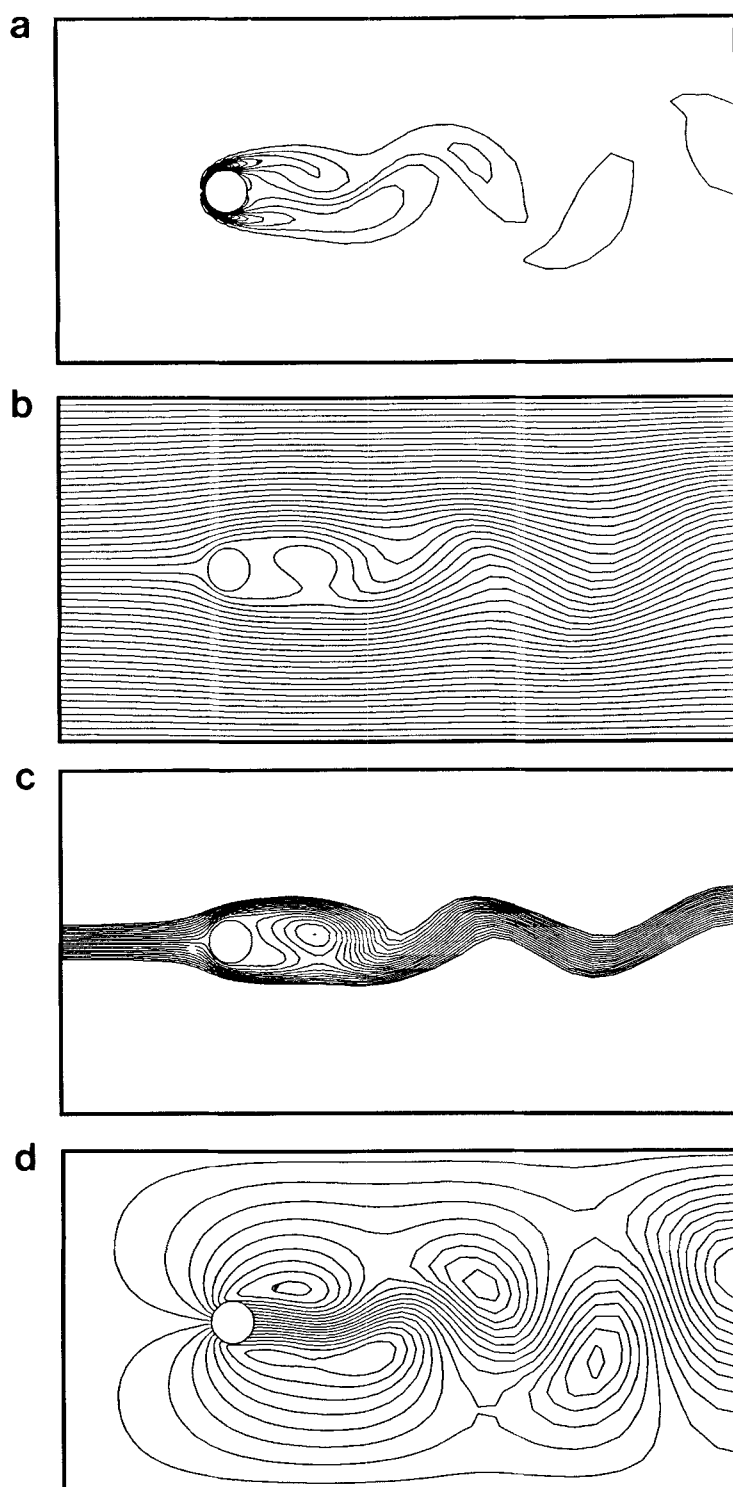
Figure 4. Flow past a circular cylinder at Re = 100, time step = 1200; (a) vorticity contours, (b) (c) streamlines,
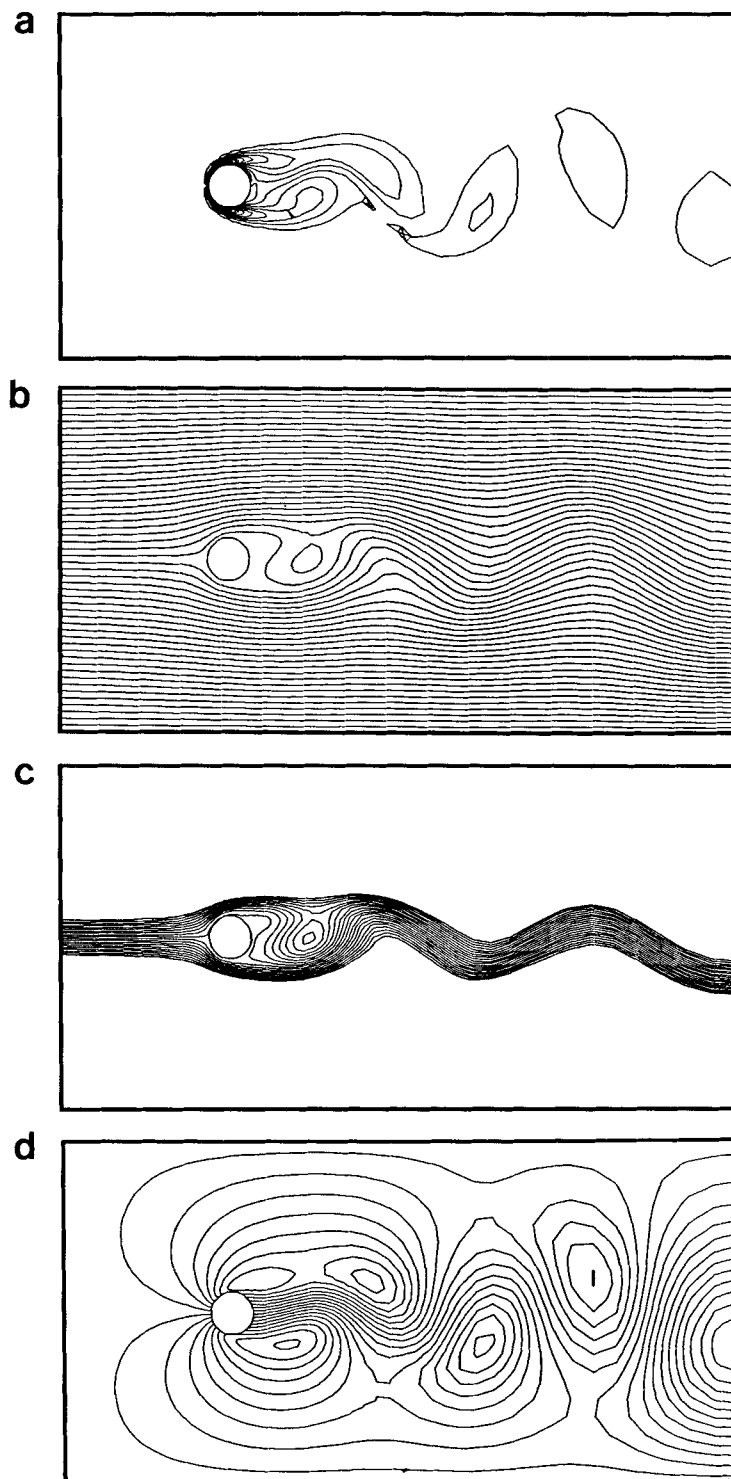(d) stationary streamlines

Figure 5. Flow past a circular cylinder at Re = 100, time step = 1800; (a) vorticity contours, (b) (c) streamlines, (d) stationary streamlines
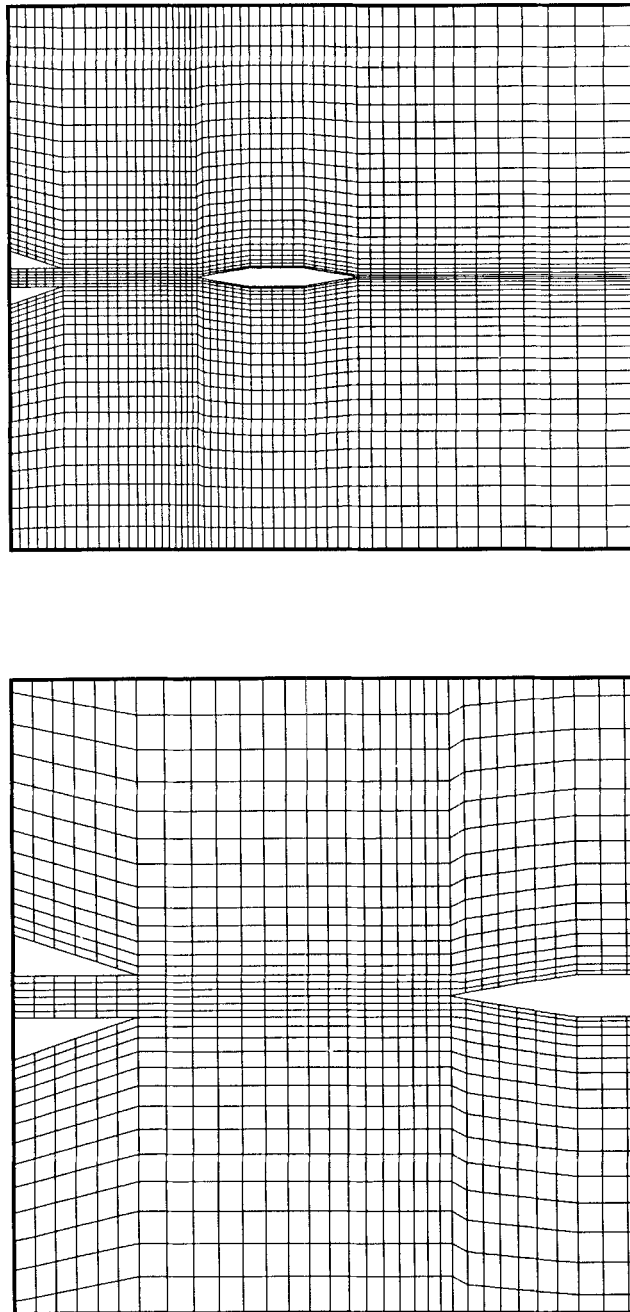
Figure 6. Edgetone problem at Re = 250; finite element mesh: 2392 elements, 2519 nodes

computational boundaries are taken as streamlines with zero vorticity. At the downstream boundary the normal derivative of both the vorticity and the stream function is set to zero. Computations are performed with a time step of 0·05. In this problem it would be grossly unjustified to assume that the value of the stream function on the wedge is known. It is therefore imperative to use a multiply connected domain approach.
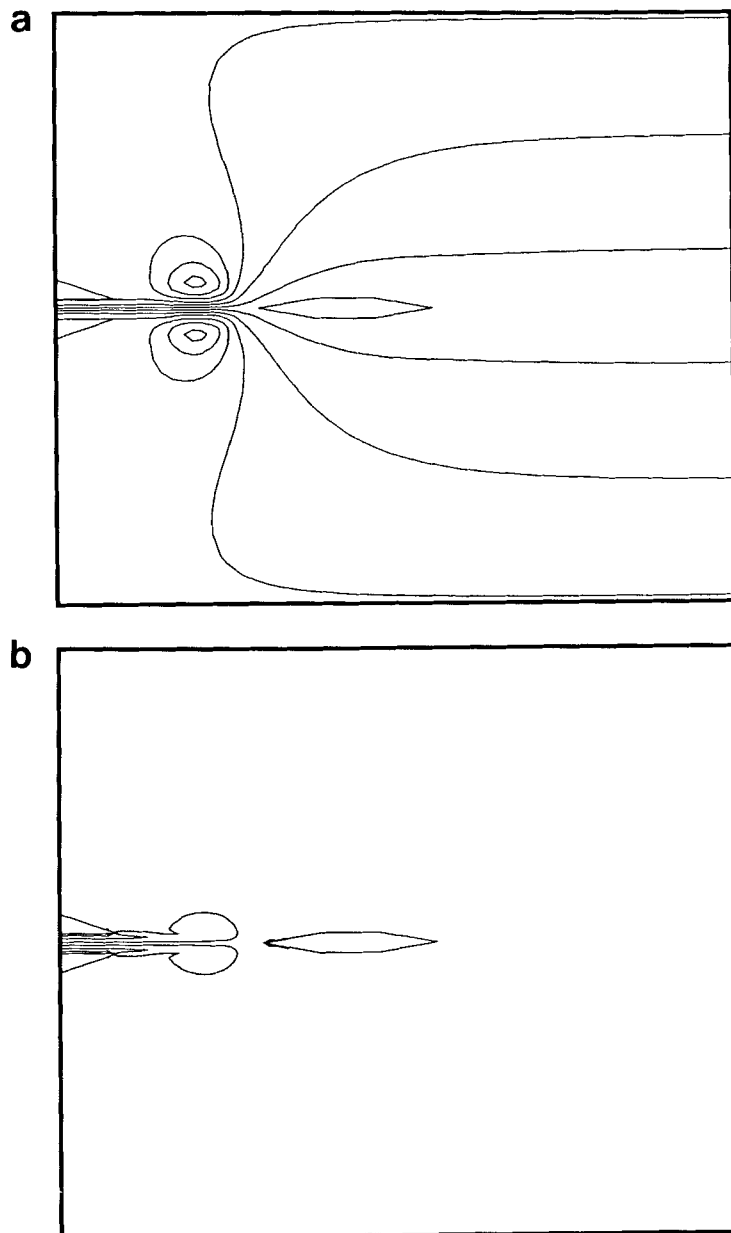
Figure 7. Edgetone problem at Re = 250, time step = 250; (a) streamlines, (b) vorticity contours

Our solutions compare qualitatively well with those obtained by Ohring,[17] who used a computational grid with 39481 nodes. Initially the plane jet develops symmetrically as shown in Figure 7. An asymmetric disturbance is introduced at time step 250. The perturbed flow field then loses its symmetry. As it can be seen in Figures 8–11, the jet oscillates transversely and has strong interaction with the wedge.
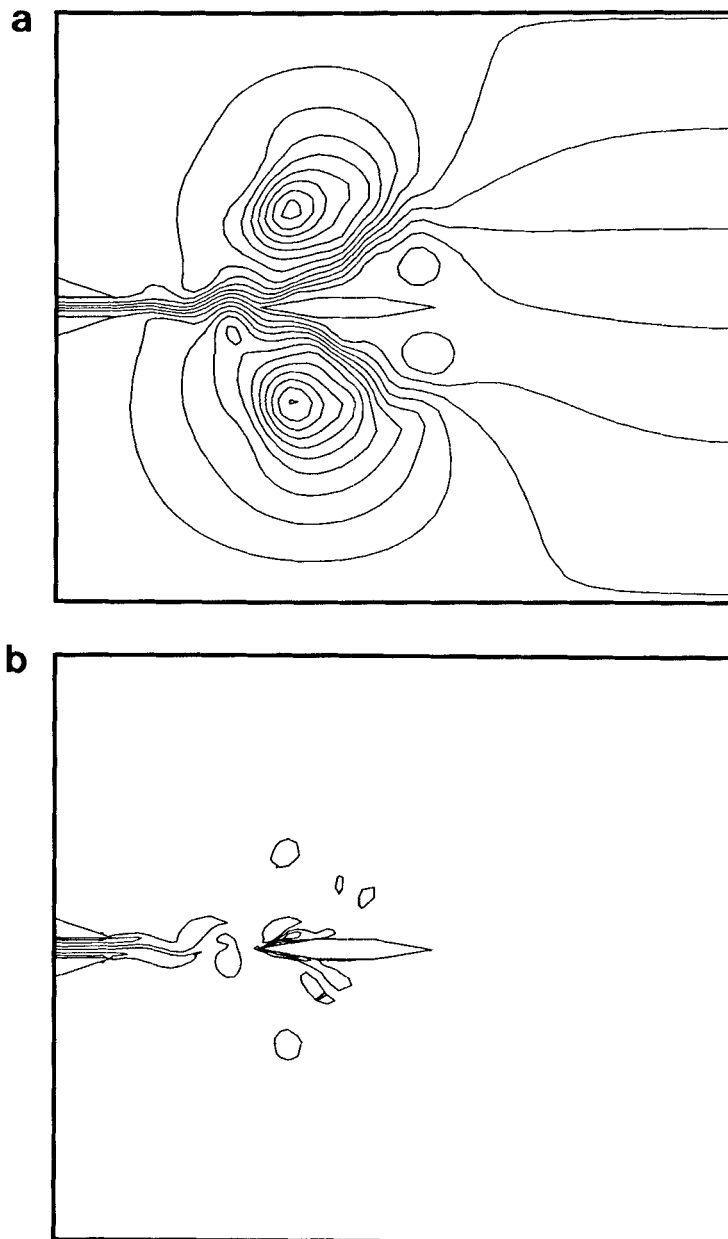

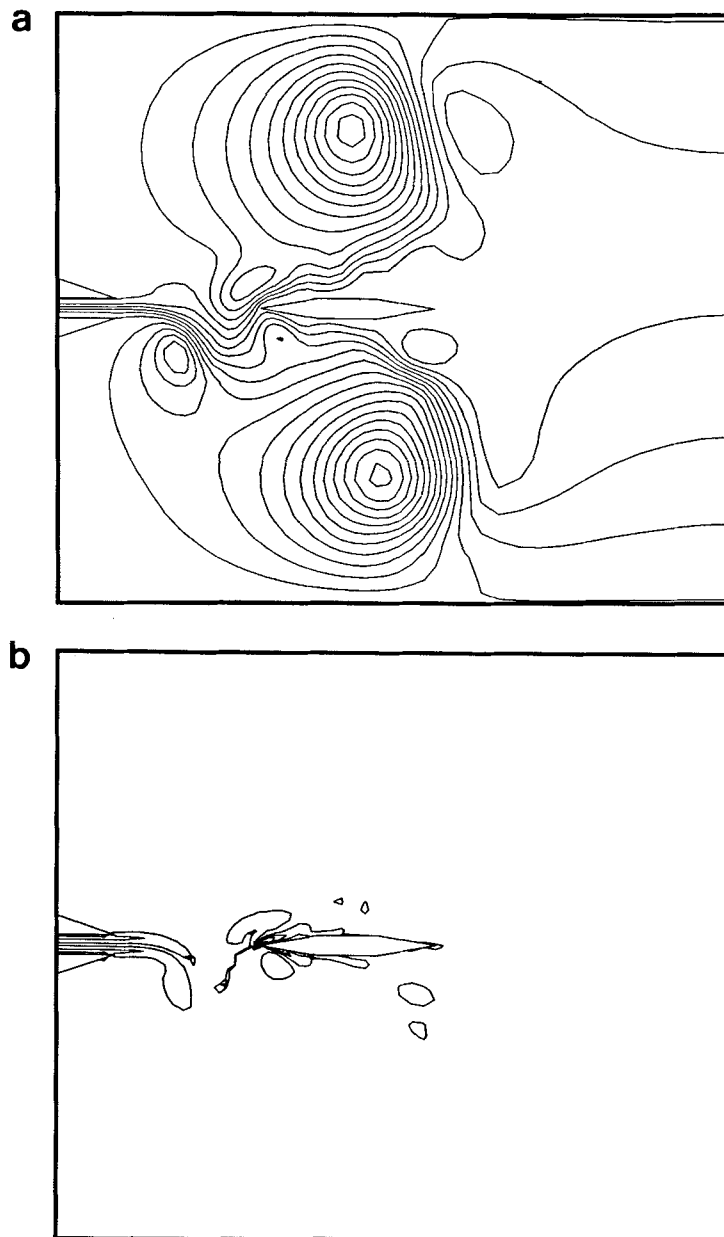Figure 8. Edgetone problem at Re = 250, time step = 1052; (a) streamlines, (b) vorticity contours

Figure 9. Edgetone problem at Re = 250, time step = 1752; (a) streamlines, (b) vorticity contours

7. CONCLUSIONS

In this work we have addressed some of the difficulties associated with the finite element approximation of the vorticity–stream function formulation of the incompressible Navier–Stokes equations in two dimensions. These difficulties are related to the presence of a convection term in
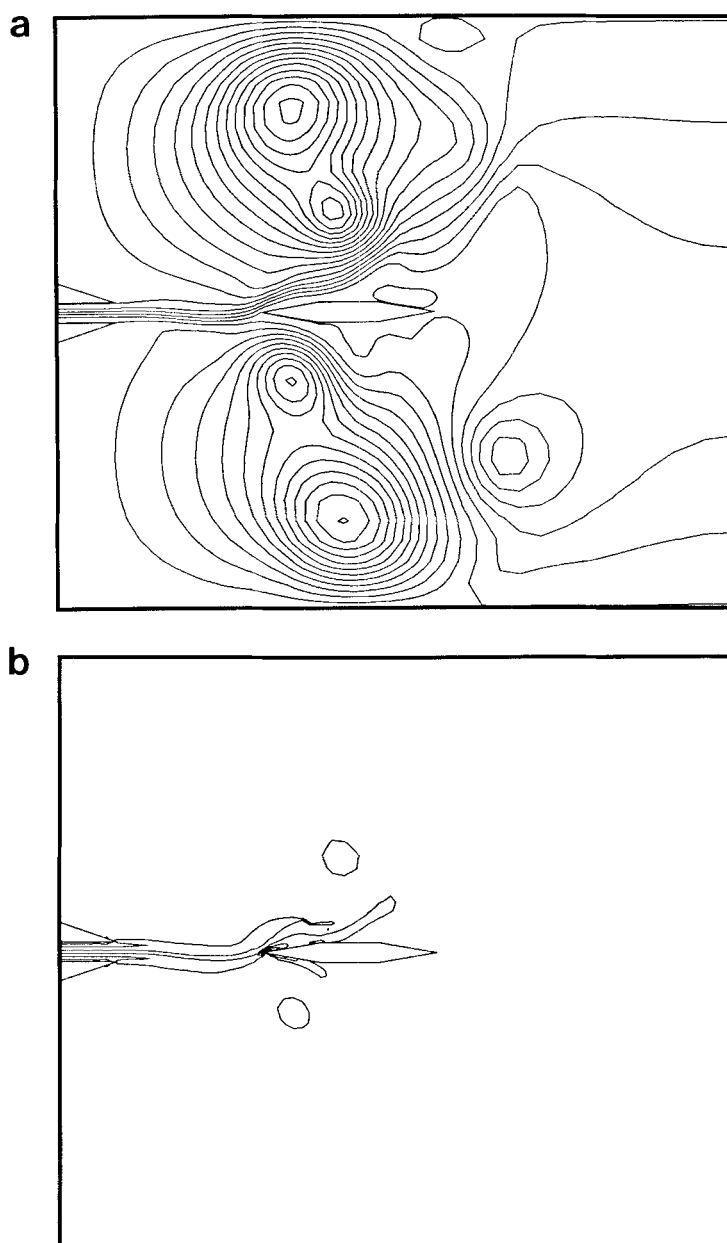
Figure 10. Edgetone problem at Re = 250, time step = 2352; (a) streamlines, (b) vorticity contours

the vorticity transport equation, lack of boundary condition for vorticity on no-slip surfaces, and computation of the stream function at the internal boundaries.

We have employed streamline-upwind/Petrov–Galerkin procedures to minimize the numerical oscillations which may appear in convection-dominated problems, especially in the presence of sharp layers in the flow field. We have developed proper variational formulations to compute the
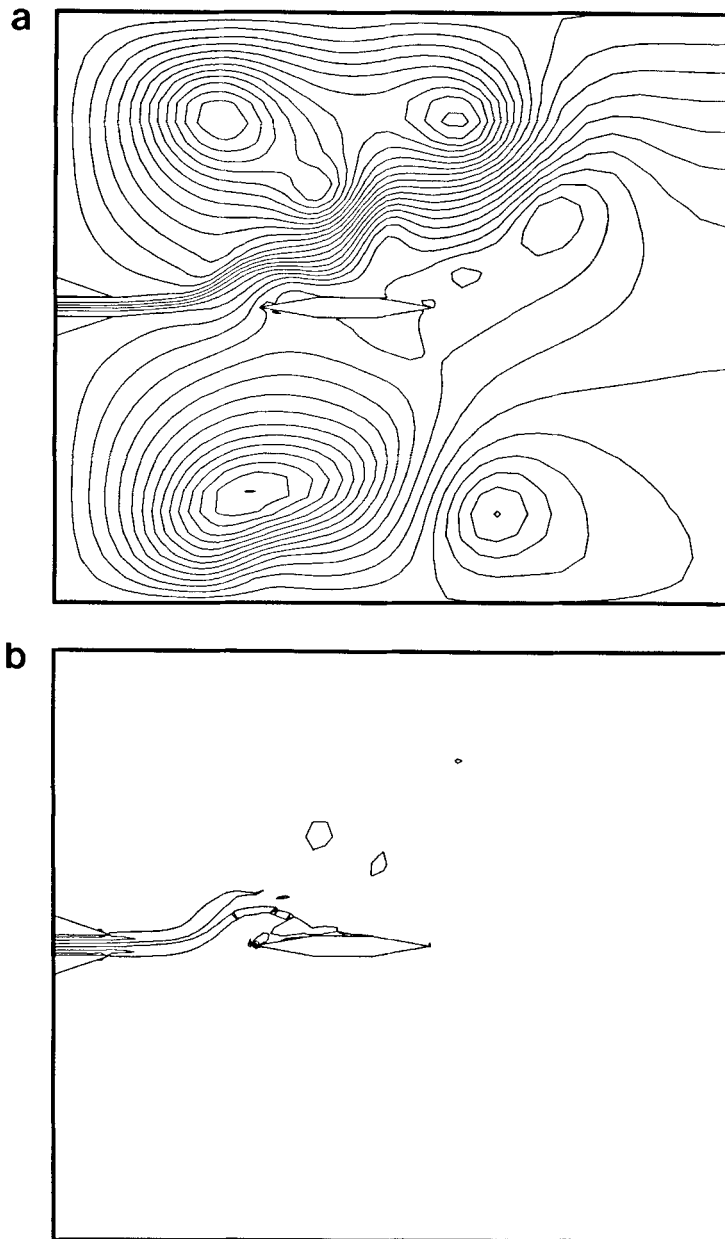
Figure 11. Edgetone problem at Re = 250, time step = 2952; (a) streamlines, (b) vorticity contours

vorticity on no-slip surfaces and the stream function on internal surfaces, and implemented these formulations within the framework of block-iteration schemes. These block-iteration schemes favourably permit the utilization of various sophisticated iterative solution techniques such as conjugate gradient methods and element-by-element approximate factorization schemes.

The methods developed were tested on several problems with simply and multiply connected

domains. In all cases the results obtained compared qualitatively well with the previously published results. It is our belief that these methods, when implemented together with proper iterative schemes, will provide a good capability to perform large-scale two-dimensional computational studies of problems related to both basic sciences and technological applications.

ACKNOWLEDGEMENTS

REFERENCES

1. T. J. R. Hughes and A. N. Brooks, 'A theoretical framework for Petrov–Galerkin methods with discontinuous weighting functions: application to the streamline-upwind procedure', in R. H. Gallagher *et al.* (eds), *Finite Elements in Fluids*, Wiley, London, 1982, pp. 46–65.
2. A. N. Brooks and T. J. R. Hughes, 'Streamline-upwind/Petrov–Galerkin formulations for convection dominated flows with particular emphasis on incompressible Navier–Stokes equation', *Comput. Methods Appl. Mech. Eng.*, **32**, 199–259 (1982).
3. T. E. Tezduyar and T. J. R. Hughes, 'Finite element formulations for convection-dominated flows with particular emphasis on the compressible Euler equations', *AIAA Paper 83-0125*, Reno, NV, 1983.
4. T. J. R. Hughes, M. Mallet annd L. P. Franca, 'Entropy-stable finite element methods for compressible fluids: application for high Mach number flows with shocks', in *Finite Element Methods for Nonlinear Problems*, Springer, Berlin, 1986.
5. T. E. Tezduyar and D. K. Ganjoo, 'Petrov–Galerkin formulations with weighting functions dependent upon spatial and temporal discretization: application to transient convection–diffusion problems', *Comput. Methods Appl. Mech. Eng.*, **59**, 49–71 (1986).
6. T. E. Tezduyar and Y. J. Park, 'Discontinuity-capturing finite element formulations for nonlinear convection–diffusion–reaction equations', *Comput. Methods Appl. Mech. Eng.*, **59**, 307–325 (1986).
7. R. Glowinski, *Numerical Methods for Nonlinear Variational Problems*, Springer-Verlag, New York, 1984.
8. T. E. Tezduyar and J. Liou, 'Element-by-element and implicit–explicit finite element formulations for computational fluid dynamics', *First Int. Symposium on Domain Decomposition Methods for Partial Differential Equations*, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Periaux (eds.), SIAM, Philadelphia, 1988.
9. A. Campion-Renson and M. J. Crochet, 'On the stream function–vorticity finite element solutions of Navier–Stokes equations', *Int. j. numer. methods eng.*, **12**, 1809–1818 (1978).
10. M. F. Peeters, W. G. Habashi and E. G. Dueck, 'Finite element stream function–vorticity solutions of the incompressible Navier–Stokes equations', *Int. j. numer. methods fluids*, **7**, 17–27 (1987).
11. R. Glowinski and O. Pironneau, 'Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem', *SIAM Rev.*, **21**(2), 167–212 (1979).
12. M. O. Bristeau, R. Glowinski and J. Periaux, 'Numerical methods for the Navier–Stokes equations; applications to the simulation of compressible and incompressible viscous flows', *Comput. Phys. Rep.*, **6**, 73–187 (1987).
13. C. Taylor, C. E. Thomas and K. Morgan, 'Turbulent heat transfer via the finite element method for heat transfer in rod bundles', *Int. j. numer. methods fluids*, **3**, 363–375 (1983).
14. P. M. Gresho, S. T. Chan, R. L. Lee and C. D. Upson, 'A modified finite element method for solving the incompressible Navier–Stokes equations, part 1: theory', *Int. j. numer. methods fluids*, **4**, 557–598 (1984).
15. P. M. Gresho, S. T. Chan, R. L. Lee and C. D. Upson, 'A modified finite element method for solving the incompressible Navier–Stokes equations, part 2: applications', *Int. j. numer. methods fluids*, **4**, 619–640 (1984).
16. T. E. Tezduyar, R. Glowinski and F. Glaisner, 'Streamline-upwind/Petrov–Galerkin procedures for the vorticity–stream function form of the Navier–Stokes equations, in C. Taylor *et al.* (eds), *Numerical Methods in Laminar and Turbulent Flow*, *Vol. 5*, Pineridge Press, Swansea, 1987, pp. 197–209.
17. S. Ohring, 'Calculations of self-excited impinging jet flow', *J. Fluid Mech.*, **163**, 69–89 (1986).