State Equations - Motor Position Control

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June 2022

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1 System Model

We calculate the parameters of a DC motor system, as described in Figure 1.

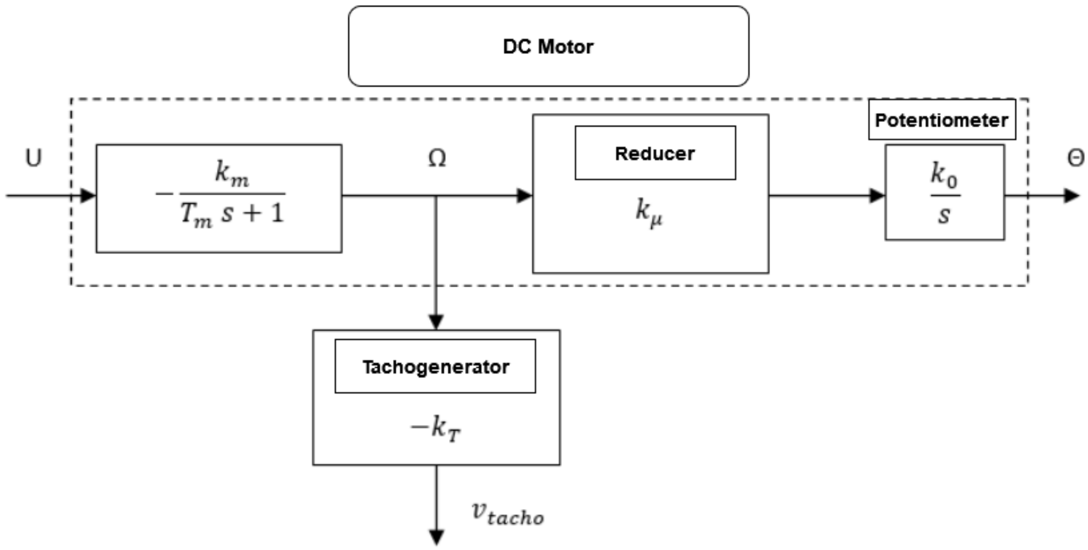

Figure 1: DC Motor

U is the input voltage, Ω is the rotational speed of the tachogenerator in rpm (revolutions per minute), Θ is the position-voltage of the motor's axis, and V_{tacho} is the voltage across the tachogenerator.

1.1 Parameter Calculation

1.1.1 $k_m k_T$ amd T_m

Initially, we excite the system with a step input.

$$U = 10V \tag{1}$$

and measure the voltage at the output of the tachogenerator in the steady-state.

$$V_{tacho,ss} = 9V. \tag{2}$$

As shown in Figure 1, we approximate the transfer function of the input voltage-tachogenerator voltage relationship with the following equation

$$\frac{V_{tacho}(s)}{U(s)} = \frac{k_m k_T}{T_m s + 1}. \tag{3}$$

In the steady-state, it's evident that

$$\frac{V_{tacho,ss}}{U} = k_m k_T. \tag{4}$$

By substituting the measurements from 1 and 2 into 4, we have:

$$k_m k_T = \frac{9}{10}. \tag{5}$$

For the calculation of T_m, it is sufficient to compute the time at which the waveform reaches 63.3% of its maximum value. We calculate

$$V_{tacho,63.3\%} = 0.633 V_{tacho,ss} = 5.697V \tag{6}$$

and using an oscilloscope, we determine that

$$T_m = 550ms \tag{7}$$

1.1.2 k_μ

The parameter k_μ of the reducer corresponds to the ratio of the angle of rotation of the "output axis" to the angle of rotation of the motor axis. By manually turning the disk brake on the motor axis, we observe how much the "output axis" rotates in one complete rotation of the motor axis, and we have

$$k_\mu = \frac{1}{36}. \tag{8}$$

1.1.3 k_m, k_T and k_0

As shown in Figure 1, we approximate the transfer function of the rotational speed of the tachogenerator to the position of the motor's axis with the following equation.

$$\frac{\Theta(s)}{\Omega(s)} = \frac{k_\mu k_0}{s} \tag{9}$$

and in the time domain:

$$\dot{\theta}(t) = k_\mu k_0 \omega. \tag{10}$$

We set the motor in motion and monitor its position using an oscilloscope. The position appears as a sawtooth waveform, as shown in Figure 2, due to the potentiometer wiper jumping from the edge of its rotation back to its initial position. Using the oscilloscope, we calculate

$$\Delta\theta = 15V \tag{11}$$

$$\Delta t = 0.867s \tag{12}$$

and by using 10

$$k_\mu k_0 \omega = 17.3. \tag{13}$$

The time Δt is the period of one complete rotation of the output axis; therefore, its rotational speed in rpm is

$$\omega_{out} = \frac{60s}{\Delta t} = 69.2rpm. \tag{14}$$

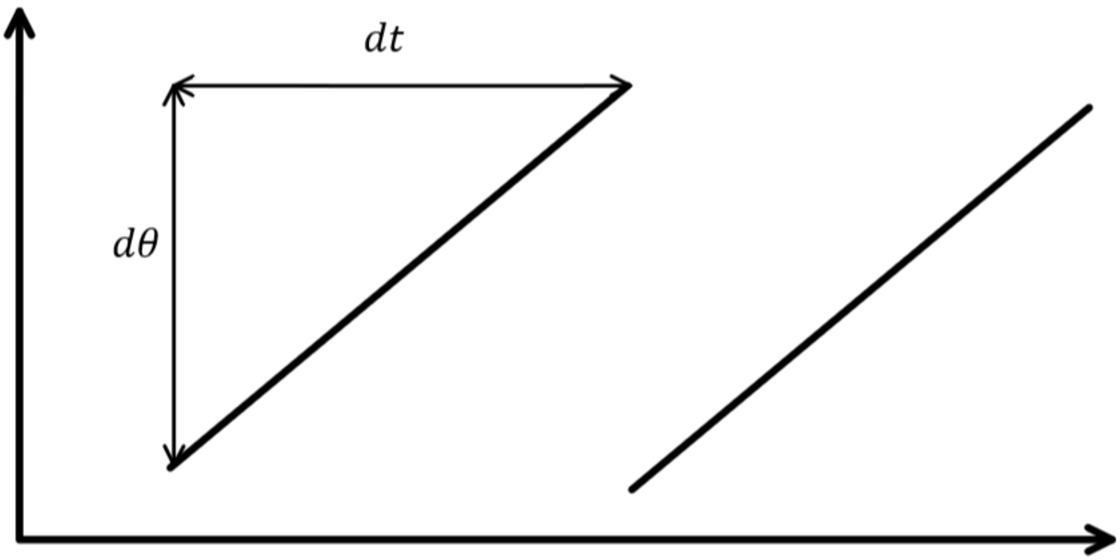

Figure 2: Position of the Motor on the Oscilloscope.

For the reducer ratio k_μ, the following holds:

$$k_\mu = \frac{\omega_{out}}{\omega}. \tag{15}$$

From 8 and 15 we get

$$\omega = 2491.35rpm \tag{16}$$

From 1:

$$\begin{gathered} V_{tacho,ss} = k_T\omega \\ k_T = 3.61 \times 10^{-3} \end{gathered} \tag{17}$$

Finally, from 13 and 5 we calculate k_0 and k_m respectively

$$k_0 = \frac{\Delta\theta}{k_\mu \omega \Delta t} = 0.25 \tag{18}$$

$$k_m = 249.3. \tag{19}$$

1.2 State Equations

In order to design the required controllers, it is necessary to describe the system with state equations. From Figure 1, we extract the transfer functions, assuming a constant input voltage.

$$u(t) = u \Rightarrow U(s) = u/s \tag{20}$$

We end up with the following differential equations

$$\begin{aligned}\frac{\Theta(s)}{\Omega(s)} &= \frac{k_\mu k_0}{s} \Rightarrow \\ \omega(t) &= \frac{1}{k_\mu k_0}\dot{\theta}(t)\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{V_{tacho}(s)}{\Omega(s)} &= k_T \Rightarrow \\ \omega(t) &= \frac{V_{tacho}(t)}{k_T}\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{\Omega(s)}{U(s)} &= \frac{k_m}{T_m s + 1} \Rightarrow \\ \Omega(s) &= \frac{km}{T_m}\frac{1}{s + \frac{1}{T_M}}\frac{u}{s} \Rightarrow \\ \omega(t) &= \int_0^t \frac{k_m u}{T_m} e^{-\frac{\tau}{T_m}} d\tau \\ &= k_m u(1 - e^{-\frac{t}{T_m}})\end{aligned} \tag{23}$$

We define the state variables $x_1 = \theta$ and $x_2 = V_{tacho}$. So we get:

$$\begin{aligned}\dot{x_1} = \dot{\theta}(t) &= k_\mu k_0 \omega(t) \Rightarrow \\ \dot{x_1} &= \frac{k_\mu k_0}{k_T} x_2\end{aligned} \tag{24}$$

$$\begin{aligned}\dot{x_2} = \dot{V}_{tacho}(t) &= k_T \dot{\omega}(t) \\ &= k_T k_m u \frac{d(1 - e^{-\frac{t}{T_m}})}{dt} \\ &= \frac{k_T k_m}{T_m} u e^{-\frac{t}{T_m}} \\ = -\frac{k_T k_m}{T_m} u(1 - e^{-\frac{t}{T_m}}) &+ \frac{k_T k_m}{T_m} u \\ &= -\frac{1}{T_m} x_2 + \frac{k_T k_m}{T_m} u\end{aligned} \tag{25}$$

and in matrix form:

$$\begin{aligned}\dot{\boldsymbol{x}} &= A\boldsymbol{x} + B\boldsymbol{u} \\ y &= C\boldsymbol{x}\end{aligned} \tag{26}$$

with

$$A = \begin{bmatrix} 0 & \frac{k_\mu k_0}{k_T} \\ 0 & -\frac{1}{T_m} \end{bmatrix}, \tag{27}$$

$$B = \begin{bmatrix} 0 \\ \frac{k_T k_m}{T_m} \end{bmatrix} \tag{28}$$

and

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}. \tag{29}$$

2 State Linear Feedback

We design a state feedback controller so that the position of the axis converges to a desired value. Additionally, it is required that the closed-loop system's response does not exhibit overshoot, and the settling time is minimized.

2.1 Controller Design

Before we begin designing a controller, we need to confirm that the system is controllable. We calculate the controllability matrix.

$$M = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_\mu k_0 k_m}{T_m} \\ \frac{k_T k_m}{T_m} & -\frac{k_T k_m}{T_m^2} \end{bmatrix} \tag{30}$$

and we observe that its columns are linearly independent, so the system is observable since $rank(M) = 2$.

Let the controller be $u = -k_1 x_1 - k_2 x_2 + k_r r$. Substituting this into Equation 26, we obtain

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & \frac{k_\mu k_0}{k_T} \\ -\frac{k_T k_\mu}{T_m} k_1 & -\frac{1}{T_m}(1 + k_2 k_T k_m) \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} 0 \\ \frac{k_T k_m}{T_m} k_r \end{bmatrix} r. \tag{31}$$

We find the characteristic polynomial of the system.

$$\begin{aligned} det(sI - \tilde{A}) &= \begin{vmatrix} s & -\frac{k_\mu k_0}{k_T} \\ \frac{k_T k_m}{T_m} k_1 & s + \frac{1}{T_m}(1 + k_2 k_T k_m) \end{vmatrix} \\ &= s^2 + \frac{1}{T_m}(1 + k_2 k_T k_m)s + \frac{k_m k_\mu k_0}{T_m} k_1 \\ &= s^2 + P_1 s + P_2 = s^2 + 2\zeta\omega_n s + \omega_n^2. \end{aligned} \tag{32}$$

So for $k1$ and $k2$ we get

$$P2 = \frac{k_m k_\mu k_0}{T_m} k_1 \Rightarrow k1 = \frac{T_m P2}{k_m k_\mu k_0}, \tag{33}$$

$$P1 = \frac{1}{T_m}(1 + k_2 k_T k_m) \Rightarrow k2 = \frac{P1Tm - 1}{k_T k_m} \tag{34}$$

and the gain ratio is

$$\frac{k1}{k2} = \frac{T_m P_2 k_t}{k_\mu k_0 (k_1 P1 - 1)}. \tag{35}$$

Furthermore, from linear systems theory, we know that in order to avoid overshoot, the damping coefficient should satisfy $\zeta = 1$. Therefore,

$$\begin{aligned} P_1 &= 2\zeta\omega_n \\ P_2 = \omega_n^2 &\Rightarrow P_2 = \frac{P_1^2}{4}. \end{aligned} \tag{36}$$

Lastly, for the value of k_r we have

$$k_r = -\frac{1}{C(A - Bk)^{-1}B} \tag{37}$$

where

$$k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}. \tag{38}$$

After some calculations, we obtain

$$k_r = k_1. \tag{39}$$

2.2 Application

Using the above results, we implement the controller in MATLAB and apply it to the system.

2.2.1 Finding k_2

The motor was required to start from the position $\theta_0 = 2V$ and reach the position $\theta_{ref} = 5V$. We assume an arbitrary value for k_2 and, using the relationships 35 and 36, calculate the corresponding k_1 to avoid system overshoot. In Figure 3 (left), the results for $k_2 = 2.1$ (and correspondingly $k_1 = 2.2$) are shown, which were found to be very small and resulted in a position error in steady-state.

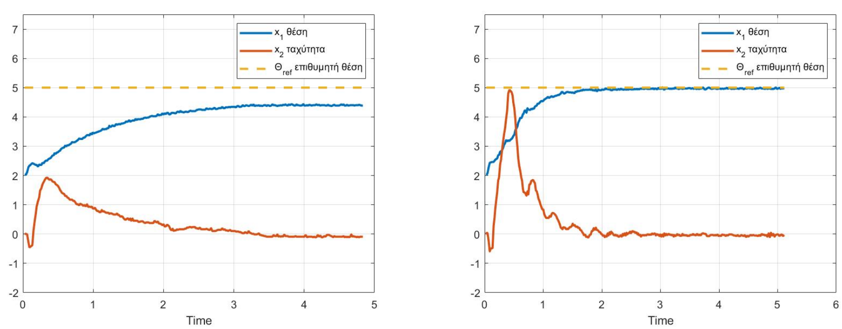

Figure 3: State linear feedback with: $k_1 = 2.2$ and $k_2 = 2.1$ (left) and $k_1 = 19.6$ and $k_2 = 8.5$ (right).

By changing the value of k_2, we find that the ideal value for minimizing the settling time is $k_2 = 8.5$ (and correspondingly $k_1 = 19.6$), which results in the outcome in Figure 3 (right).

We observe that initially the system accelerates much faster than before, and in just $1.4s$, it reaches the steady state where the value of x_1 becomes equal, without error, to the desired value, and the velocity becomes zero.

In Figure 4, the control input u is shown for both cases.

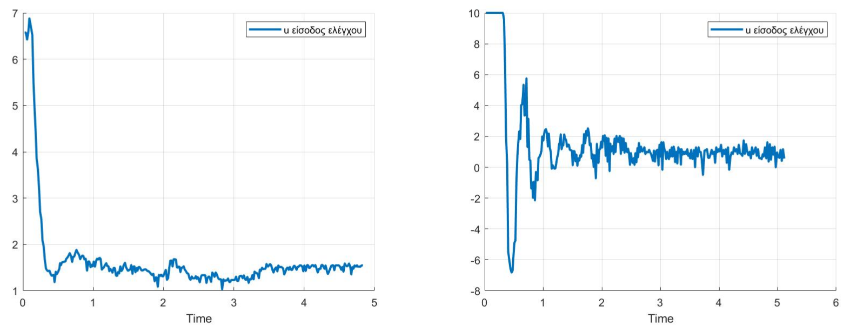

Figure 4: State linear feedback with: $k_1 = 2.2$ and $k_2 = 2.1$ (left) and $k_1 = 19.6$ and $k_2 = 8.5$ (right). Control input u.

2.2.2 Magnetic Brake

By releasing the magnetic brake included in the motor setup, we introduce a disturbance to the system, which, for $k_2 = 8.5$ (as calculated before), results in a position error in steady-state of $e_{ss} = 0.56$. Additionally, the system's response time is longer, as it now reaches steady state in 1.8 seconds. The system's response after introducing the disturbance is shown in Figure 5.

2.2.3 Sinusoidal Reference Position

We change the desired position from a constant voltage to $\theta_{ref} = 5 + 2\sin(\omega t)$ and observe that, using the controller we designed, the position follows the desired output with a small error that depends on the frequency ω. In Figure 6, we see that for $\omega = \frac{\pi}{2}$, there is a significant deviation of the actual position from the desired one, while for $\omega = \frac{\pi}{8}$, the system's response speed is sufficient to "track" the input.

3 State Dynamic Feedback

We use state dynamic feedback to make the motor's position converge to a desired value. Specifically, the controller to be designed should be capable of attenuating the disturbance introduced when the magnetic brake is released and eliminate the error that was present with the state feedback controller.

3.1 Controller Design

Let the controller be:

$$\begin{aligned} u &= -k_1 x_1 - k_2 x_2 - k_i z \\ \dot{u} &= y - r \end{aligned} \tag{40}$$

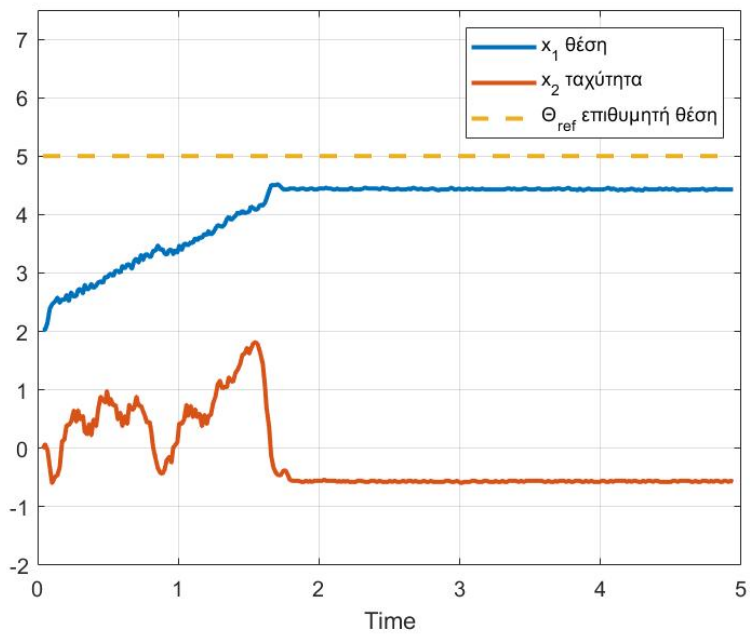

Figure 5: State linear feedback with $k_1 = 19.6$ and $k_2 = 8.5$. Use of magnetic brake.

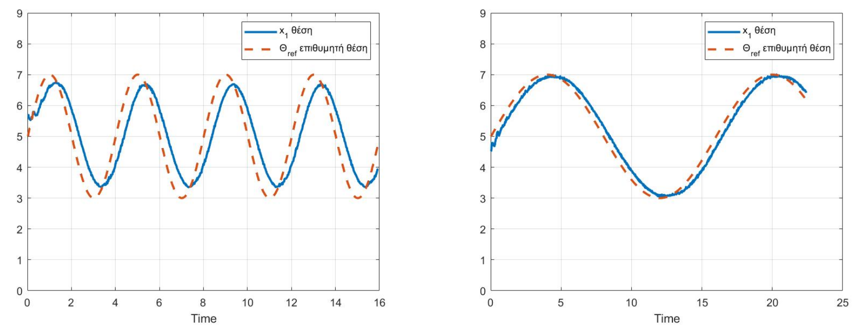

Figure 6: State linear feedback for sinusoidal input. Left $\omega = \frac{\pi}{2}$, Right $\omega = \frac{\pi}{8}$.

which is named as such (state dynamic feedback controller) because it processes the input and does not use it statically. By replacing it in equation 26 and adding the new state variable z, we obtain:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_\mu k_0}{k_T} & 0 \\ -\frac{k_T k_m}{T_m} k_1 & -\frac{1}{T_m} - \frac{k_T k_m}{T_m} k_2 & -\frac{k_T k_m}{T_m} k_i \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} r. \tag{41}$$

$$\begin{aligned} P_c(s) &= \begin{bmatrix} s & -\frac{k_\mu k_0}{k_T} & 0 \\ \frac{k_T k_m}{T_m} k_1 & s + \frac{1}{T_m} + \frac{k_T k_m}{T_m} k_2 & \frac{k_T k_m}{T_m} k_i \\ -1 & 0 & s \end{bmatrix} \\ &= s^3 + \frac{1}{Tm}(1 + k_T k_m k_2) s^2 + \frac{k_\mu k_0 k_m}{T_m} k_1 s + \frac{k_\mu k_0 k_m}{T_m} k_i \end{aligned} \tag{42}$$

The choice of k_1, k_2, and k_i must ensure that the roots of the characteristic polynomial are negative. For this reason, we apply the Routh-Hurwitz criterion.

$$\begin{array}{c|cc} s^3 & 1 & \frac{k_\mu k_0 k_m}{T_m} k_1 \\ s^2 & \frac{1}{Tm}(1 + k_T k_m k_2) & \frac{k_\mu k_0 k_m}{T_m} k_i \\ s^1 & \gamma_1 & 0 \\ s^0 & \delta_1 & \end{array}$$

where

$$\gamma_1 = \frac{\frac{k_\mu k_0 k_m}{T_m^2} k_1 (1 + k_T k_m k_2) - \frac{k_\mu k_0 k_m}{T_m} k_i}{\frac{1}{T_m}(1 + k_T k_m k_2)} \tag{43}$$

and

$$\delta_1 = \frac{k_\mu k_0 k_m}{T_m} k_i. \tag{44}$$

To avoid having roots in the right half-plane, there should be no sign changes in the elements of the first column of the Routh matrix. Since $1 > 0$, it must hold that $\frac{1}{T_m}(1 + k_T k_\mu k_2) > 0$, $\gamma_1 > 0$, and $\delta_1 > 0$. After some calculations, the following inequalities for the gain values are derived:

$$k_1 > \frac{T_m k_i}{1 + k_T k_m k_2}, \tag{45}$$

$$k_2 > -\frac{1}{k_T k_m} = -1.111... \tag{46}$$

and

$$k_i > 0. \tag{47}$$

3.2 Application

Using the above results, we implement the controller in MATLAB and apply it to the system. We choose values for k_1, k_2, and k_i that satisfy the equations 45, 46, and 47, respectively. After various trials, we arrive at the values of $k_1 = 5$, $k_2 = 3$, and $k_i = 5$. The system's response is shown in Figure 7 (left), and we can see that, in contrast to Figure 5, the state feedback controller successfully drives the motor to the desired position (exhibiting minimal error), achieving disturbance rejection.

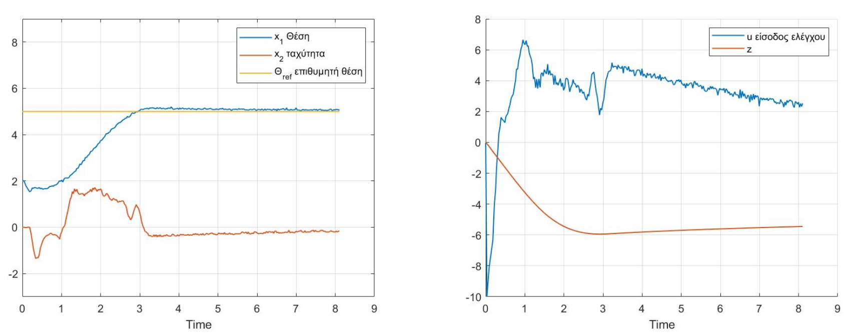

Figure 7: State Dynamic Feedback with $k_1 = 19.6$ and $k_2 = 8.5$. Use of magnetic brake. System response (left). New state variable z and control input u (right).

We can observe in Figure 7 (left) that the state feedback controller reaches the steady state in 3 seconds and is relatively slower compared to the state feedback controller. Additionally, we notice that in the initial moments, the motor develops negative velocity, causing the position to fall below the initial position before starting the rising trajectory towards the desired value of $\theta_{ref} = 5$.

In an attempt to increase the response speed, we examine the values of the new state variable z and the control input u (Figure 7, right).

We can see that the control input starts from an undesired value of -10 (sending the motor in the opposite direction of the desired one). For this reason, without changing the controller, we consider an initial condition of $z(0) = -2$ to increase the system's response speed. Indeed, we can observe in Figure 8 that now the position value does not fall below 2, and the time required to reach the steady state is reduced to 2.5 seconds.

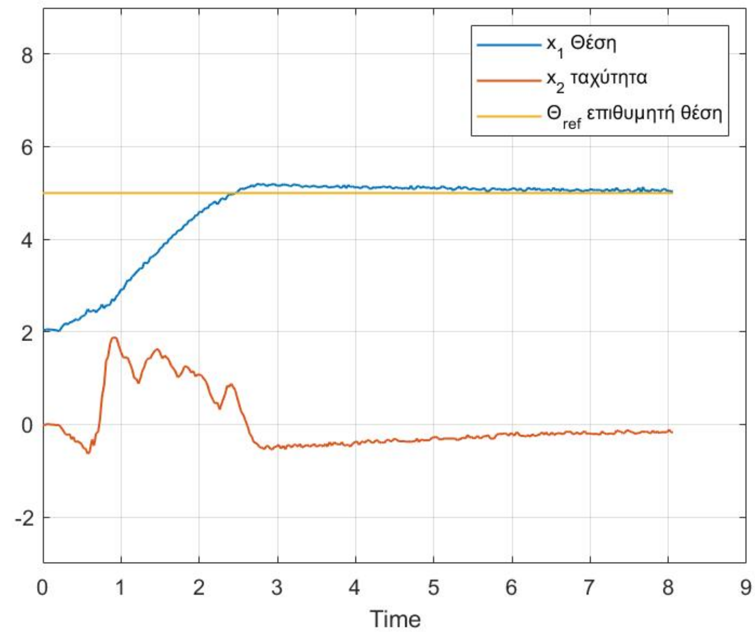

Figure 8: State Dynamic Feedback with $k_1 = 19.6$ and $k_2 = 8.5$. Initial condition $z(0) = -2$.

4 Linear output feedback controller

Finally the design of an output observer was required. Additionally, a linear output feedback controller was to be designed to utilize the observer and drive the system to a desired output, similar to the previous exercises.

4.1 Observer Design

Before designing an observer, it is essential to confirm the observability of the system. We compute the observability matrix W

$$W = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{k_\mu k_0}{k_T} \end{bmatrix} \tag{48}$$

We observe that the columns of W are linearly independent, which means that the system is observable since $rank(W) = 2$.

The observer has the form:

$$\dot{\hat{\boldsymbol{x}}} = \begin{bmatrix} \dot{\hat{x_1}} \\ \dot{\hat{x_2}} \end{bmatrix} = A\hat{\boldsymbol{x}} + B\boldsymbol{u} + L(y - C\hat{x}) \tag{49}$$

where

$$L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = W^{-1}\tilde{W} \begin{bmatrix} p_1 - a_1 \\ p_2 - a_2 \end{bmatrix}. \tag{50}$$

a_1 and a_2 are the coefficients of the characteristic polynomial of the system

$$\begin{aligned} p(s) &= det(sI - A) \\ &= \begin{vmatrix} s & -\frac{k_\mu k_0}{k_T} \\ 0 & s + \frac{1}{T_m} \end{vmatrix} \\ &= s^2 + \frac{1}{T_m}s = s^2 + a_1 s + a_2, \end{aligned} \tag{51}$$

so

$$a_1 = \frac{1}{T_m} a_2 = 0, \tag{52}$$

p_1 and p_2 the coefficients of the characteristic polynomial with the desired observer eigenvalues λ_1 and λ_2

$$\begin{aligned} p_d(s) &= (s - \lambda_1)(s - \lambda_2) \\ &= s^2 - (\lambda_1 + \lambda_2)s + \lambda_1\lambda_2 \\ &= s^2 + p_1 s + p_2 \end{aligned} \tag{53}$$

so

$$\begin{aligned} p_1 &= -(\lambda_1 + \lambda_2), \\ p_2 &= \lambda_1\lambda_2, \end{aligned} \tag{54}$$

W^{-1} is the inverse of the controllability matrix W

$$W^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{k_T}{k_\mu k_0} \end{bmatrix}, \tag{55}$$

and $\tilde{W}$ is the observability matrix of the observable canonical form

$$\tilde{W} = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{T_m} & 1 \end{bmatrix}. \tag{56}$$

Therefore, substituting into equation 50

$$L = \begin{bmatrix} p_1 - \frac{1}{T_m} \\ \frac{k_t}{k_\mu k_0} p_2 - \frac{k_T}{k_\mu k_0 T_m}(p_1 - \frac{1}{T_m}) \end{bmatrix} \tag{57}$$

and the observer, substituting into equation 49

$$\begin{bmatrix} \dot{\hat{x_1}} \\ \dot{\hat{x_2}} \end{bmatrix} = \begin{bmatrix} 0 & \frac{k_\mu k_0}{k_T} \\ 0 & -\frac{1}{T_m} \end{bmatrix} \begin{bmatrix} \hat{x_1} \\ \hat{x_2} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_T k_m}{T_m} \end{bmatrix} u + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} (x_1 - \hat{x_1}) \tag{58}$$

4.2 Application

Using the results above, we implement the controller in MATLAB and apply it to the system.

4.2.1 Observer Application

We consider eigenvalues λ_1 and λ_2. We excite the system with a step input $u = 7$ and obtain the results shown in Figures 9 and 10.

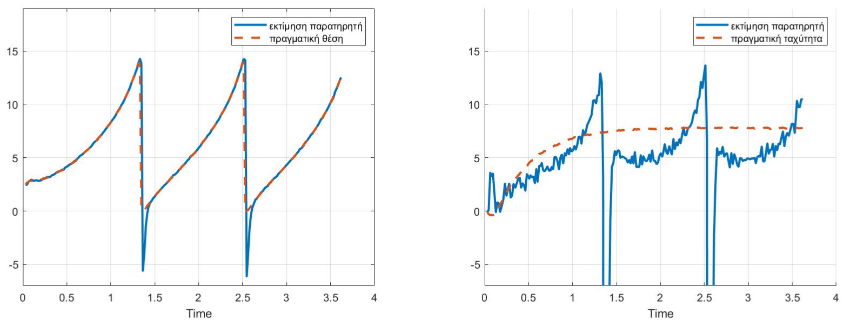

Figure 9: Observer with $\lambda_1 = -50$ and $\lambda_2 = -40$.

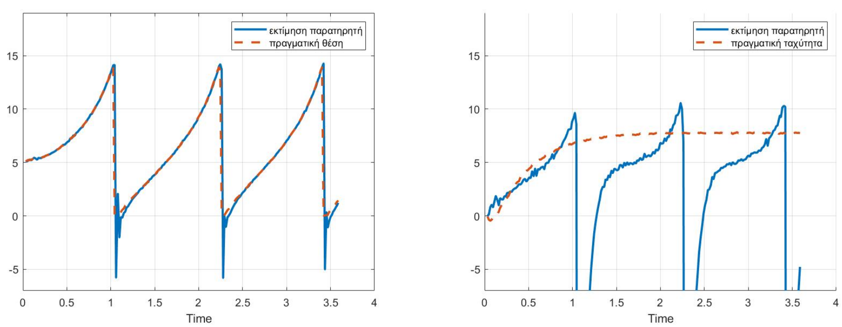

Figure 10: Observer with $\lambda_1 = -80$ and $\lambda_2 = -10$.

We observe that for both pairs of eigenvalues tested, the position estimation of the observer works exceptionally well. However, the velocity estimation shows a significant deviation. This might be due to possible modeling imperfections.

The fluctuations in the plots are caused by the sawtooth waveform of the position, as shown in Figure 2. According to Equation 58, the term $(x_1 - \hat{x_1})$ changes abruptly in the observer's estimation.

Figure 11 illustrates the velocity estimation of the observer when the motor's velocity is not constant.

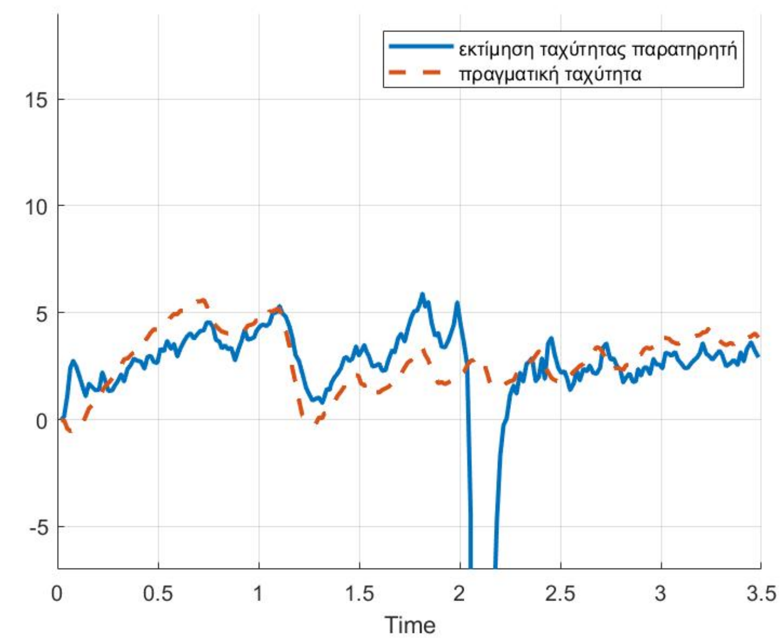

Figure 11: Observe random velocity.

4.2.2 Application of linear output feedback controller

We use a controller of the form $u = -k_1\hat{x_1} - k_2\hat{x_2} + k_r r$ with gains as calculated during the 2nd lab exercise, which are $k_1 = 19.6$, $k_2 = 8.5$, and $k_r = k_1$. We "drive" the motor from the position $\theta_0 = 2$ to the position $\theta_{ref} = 5$, and obtain the data shown in Figure 12 for the same pairs of eigenvalues used above.

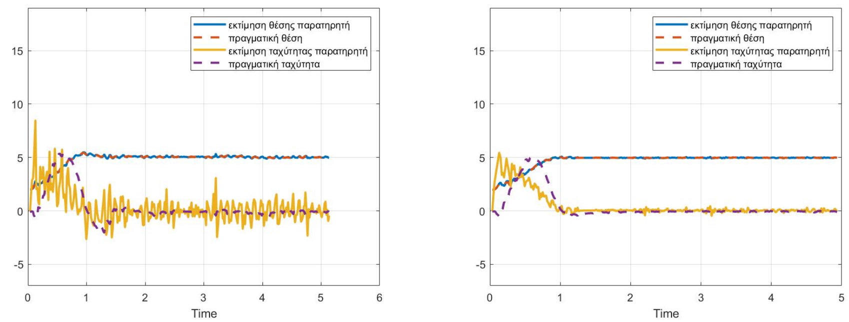

Figure 12: Linear output feedback control. Use of observer with: $\lambda_1 = -40$ and $\lambda_2 = -50$ (left) and $\lambda_1 = -80$ and $\lambda_2 = -10$ (right).

We observe that for both pairs of eigenvalues, the controller drives and stabilizes the system at the desired position $\theta_{ref} = 5$. However, it's worth noting that when using eigenvalues $\lambda_1 = -40$ and $\lambda_2 = -50$, the velocity estimation of the observer exhibits significant oscillations. This issue is mitigated to a satisfactory extent by using eigenvalues $\lambda_1 = -80$ and $\lambda_2 = -10$. Nevertheless, there still seems to be a deviation between the velocity estimation of the observer and the actual motor velocity.