MATH 495 - Capstone
Exit Exam by Tim Sizemore
Monday, February 3, 2014

[5 points] **Name:**____________________________

1. `[12 points]` Use the function $g(x)$ defined below to answer the following questions:

$$g(x) = \begin{cases} x+5 & \text{for } x < 0 \\ x^2+1 & \text{for } 0 \le x \le 2 \\ 3 & \text{for } x > 2 \end{cases}$$

   (a) $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x+5 = 5$

   (b) $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} x^2+1 = 1$

   (c) $\lim_{x\to 0} g(x)$does not exist

   (d) $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} 3 = 3$

2. `[16 points]` **Possums and Leslie Matrices**
The following table lists reproduction and survival rates for the female population of possum in the United States. Suppose possums give birth only once a year, which dictates a natural time step of one year. Due to natural life span and trac possums seldom if ever live longer than 5 years, which gives a natural stopping point for the age classes.

Birth and Survival Rates for Female Possums

Age (years)	Birth Rate	Survival Rate
0 - 1	0.0	0.6
1 - 2	1.3	0.8
2 - 3	1.8	0.8
3 - 4	0.9	0.4
4 - 5	0.2	0.0

Let x_1 represent the number of possums in the 0-1 year age group; x_2 the number of possums in the 1-2 year age group; x_3 the number of possums in the 2 - 3 year age group; x_4 the number of possums in the 3 - 4 year age group; and x_5 be the number of possums in the 4 - 5 year age group.
Calculate the rate at which the population is changing over each of those years by filling in the table below. Round your answers to THREE DECIMAL PLACES.

	Between Years 0 & 1	Between Years 1 & 2		Between Years 11 & 12	Between Years 12 & 13	Between Years 13 & 14
$k = \dfrac{\text{Pop. at Year} n+1}{\text{Pop. at Year} n}$						
Total Change: $k-1$						

3. `[16 points]` The temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2+4xy+y^2$. An ant on the plate walks around the circle of radius 5 centered at the origin. Where are the highest and lowest temperatures encountered by the ant? Use the Lagrange Multiplier Method to the maximum and minimum of $T(x, y)$ subject to the equation given by the ant's path.

$$\begin{aligned} \nabla T(x,y) &= \lambda \nabla g(x,y) \\ \langle 8x+4y, 4x+2y\rangle &= \lambda\langle 2x, 2y\rangle \end{aligned}$$

So our system of equations is

$$\begin{aligned} 8x+4y &= \lambda(2x) \quad (1) \\ 4x+2y &= \lambda(2y) \quad (2) \\ x^2+y^2 &= 25 \quad (3) \end{aligned}$$

Mulitplying 2 by -2 and adding to 1

$$\begin{aligned} -8x-4y &= -4\lambda y \\ 8x+4y &= 2\lambda x \\ &\Downarrow \\ 0 &= 2\lambda x - 4\lambda y \\ 0 &= 2\lambda(x-2y) \\ &\Downarrow \\ \lambda = 0 \quad &\text{or} \quad x = 2y \end{aligned}$$

If $\lambda = 0$, then

$$\begin{aligned} 4x+2y &= 0 \\ 4x &= -2y \\ x &= -\frac{1}{2}y \\ &\Downarrow \\ \left(-\frac{1}{2}\right)^2 + y^2 &= 25 \\ \frac{1}{4}y^2+y^2 &= 25 \\ \frac{5}{4}y^2 &= 25 \\ y^2 &= 20 \\ y &= \pm\sqrt{20} \\ y &= \pm 2\sqrt{5} \end{aligned}$$

As $x = -\frac{1}{2}y$ then $(-\sqrt{5}, 2\sqrt{5}), (\sqrt{5}, -2\sqrt{5})$

If $x = 2y$, then

$$\begin{aligned} (2y)^2+y^2 &= 25 \\ 5y^2 &= 25 \\ y^2 &= 25 \\ y &= \pm\sqrt{5} \end{aligned}$$

As $x = 2y$ then $(2\sqrt{5}, \sqrt{5}), (-2\sqrt{5}, -\sqrt{5})$.
Checking z-values:
$(2\sqrt{5}, \sqrt{5}) \to 80+40+5 = 125$ (max)
$(-2\sqrt{5}, -\sqrt{5}) \to 80+40+5 = 125$ (max)
$(-\sqrt{5}, 2\sqrt{5}) \to 20-40+20 = 0$ (min)
$(\sqrt{5}, -2\sqrt{5}) \to 20-40+20 = 0$ (min)

4. `[18 points]`Let A and B be subsets of a universal set U. Prove that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.
Proof:
(a) Prove $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

Let $x \in \overline{A \cup B}$.
Then $x \notin (A \cup B)$.
So, $\neg(x \in (A \cup B))$.
So, $\neg(x \in A or x \in B)$.
So, distributing the negation, $x \notin A$ and $x \notin B$.
Thus, $x \in \overline{A}$ and $x \in \overline{B}$.
So $x \in \overline{A} \cap \overline{B}$.
Since x was an arbitrary element, it is true for all elements x, if $x \in \overline{A \cup B}$, then $x \in \overline{A} \cap \overline{B}$. Hence, $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

(b) Prove $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Let $x \in \overline{A} \cap \overline{B}$.
So $x \in \overline{A}$ and $x \in \overline{B}$.
Thus, $x \notin A$ and $x \notin B$.
"Factoring out" a negation, we get $\neg(x \in A$ or $x \in B)$.
So, $\neg(x \in (A \cup B)$.
Thus, $x \notin (A \cup B)$.
So, $x \in \overline{A \cup B}$.
Since x was an arbitrary element, it is true for all elements x, if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A \cup B}$. Hence, $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

Since $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$, we know that $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

5. `[18 points]`**Verifications**

(a) Let W be the subset of $\mathbb{R}^{2\text{x}2}$ given by

$$W = \left\{ A \in \mathbb{R}^{2\text{x}2} \;\middle|\; A = \begin{bmatrix} a & a+2 \\ b & c \end{bmatrix} \right\}$$

Determine whether W forms a subspace of $V = \mathbb{R}^{2\text{x}2}$ over the field $F = \mathbb{R}$. Show all of your work.

(b) Let

$$\alpha_1 = \begin{bmatrix} 3 \\ -6 \end{bmatrix}, \; \alpha_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \text{ and } \alpha_3 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

Is $\alpha = \begin{bmatrix} 8 \\ -3 \end{bmatrix} \in \text{Span}(\alpha_1, \alpha_2, \alpha_3)$? Justify your answer.

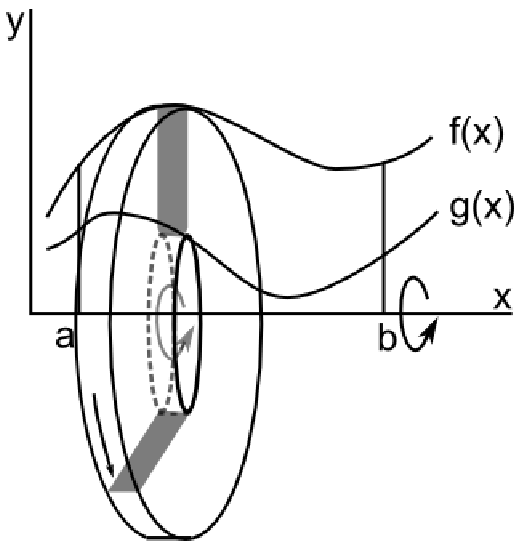
(c) Identify if the linear transformation, T, is invertible, showing your work. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x+y \\ -y \\ 2x \end{bmatrix}$$

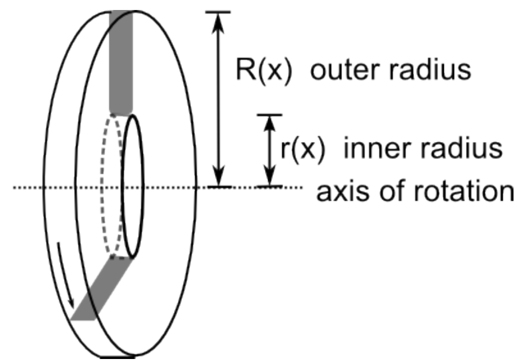
6. [15 points] **Washer Method**

When the area being rotated is **NOT** always flush to the axis of rotation, we can't use the disc method because our cross-section has a "hole" in it, giving us a **washer** rather than a disc. Once again, the representative rectangle is drawn **perpendicular** to the axis of rotation.

Vertical Rectangles (Horizontal axis of rotation)



a: Using Vertical Rectangles when Rotating



b: The Washer

Figure 1: Visualizing the Derivation of the Washer Method

Think of the rectangle *as pointing to the axis whose variable appears in the radius function.* Here we have an inner radius, $r(x)$, and an outer radius, $R(x)$, both functions of x. Thus, the volume of the washer is given by:

$$\begin{aligned} \text{Volume of washer} &= \text{(Volumer of outer disc) - (Volume of inner disc)} \\ &= \pi(R(x))^2\Delta x - \pi(r(x))^2\Delta x \end{aligned}$$

Thus, the volume of the region created by rotating this area about the axis of rotation is:

$$V = \int_a^b \pi(\underbrace{\quad R(x) \quad}_{\text{Outer Radius}})^2 - (\underbrace{\quad r(x) \quad}_{\text{Inner Radius}})^2 \, dx$$

Set up the equation for the volume generated by rotating the area between $y = \sqrt{x}$ and $y = 2$ and the y-axis about the x-axis.