Dynamical System

1 Introduction

A dynamical system, is a system in which a function describes the time dependence of a point in a geometrical space. The key concepts in dynamical system include:

1. **State Space:** This is the set of all possible states in which the system can be. For instance, in a simple mechanical system, the state space could be defined by the position and velocity of an object.
2. **State:** At any given time, the system is described by a specific point in its state space. This point is referred to as the state of the system.
3. **Dynamics:** The rules or laws that govern the evolution of the system over time. This is typically expressed as a set of differential equations or difference equations that determine how the state of the system changes.
4. **Deterministic, Stochastic:** A dynamical system is said to be **deterministic** if its future behavior is fully determined by its current state and the governing equations. In contrast, in **stochastic systems**, there's inherent randomness, so even knowing the current state doesn't allow for perfect prediction of its future.
5. **Continuous, Discrete:** Some dynamical systems change continuously over time (e.g., governed by differential equations), while others evolve in discrete time steps (e.g., governed by difference equations).
6. **Bifurcations:** These refer to qualitative changes in the long-term behavior of a system as a parameter is varied. For example, as one adjusts a certain parameter, a system might transition from stable behavior to oscillatory behavior.
7. **Chaos:** Some dynamical systems exhibit chaotic behavior, where long-term prediction becomes impossible even though the system is deterministic. This is because of sensitive dependence on initial conditions, meaning tiny differences in the starting state of the system can lead to vastly different outcomes.

2 Interested Questions

For the problem with the initial value as x_0:

$$\frac{dx}{dt} = f(x) \tag{1}$$

We are interested in:

1. **Asymptotic behavior:** As $t \to \infty$, $x(t) \to ?$
2. **Equilibria (fixed pts)**: Does it exist? How many?

$$\frac{dx}{dt}|_{x=x^*} = 0 \tag{2}$$

3. **Periodic Behavior:** for some P?

$$x(t + P) = x(t) \tag{3}$$

4. **Parametric Behavior:** Suppose the model has parameters $x = x(t, \theta)$, what happens to x as θ changes?
5. **Initial Value Sensitivity:** What happens for different starting values?

3 Stability Analysis

3.1 Normal Steps

First, we need to find the **fixed points (equilibrium points, singular points)**. Fixed point is a point in the state space where the system does not evolve further if it reaches that state, or where the derivative at that point is zero. Then, we choose the points close to the fixed points to see whether they are approaching or leaving the fixed points by checking the derivatives.

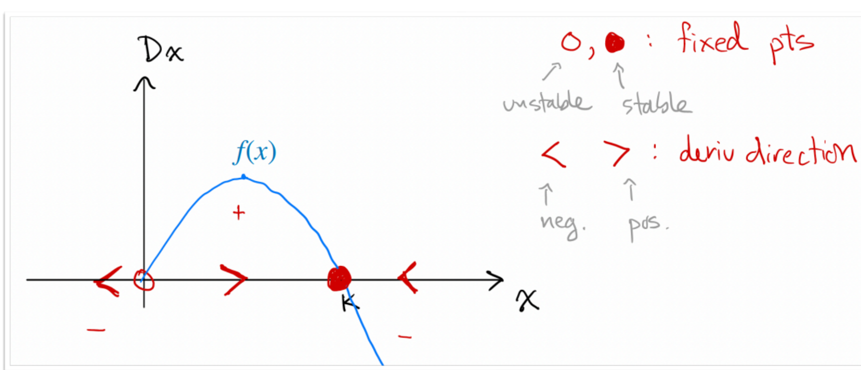

Figure 1: Stability Analysis

Sometimes, it may be stable on left but unstable on right, then we call this **semi-stable**.

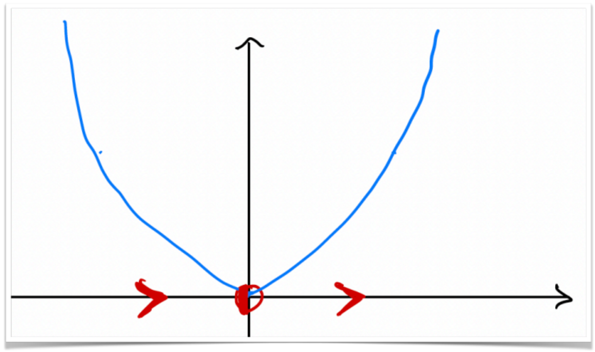

Figure 2: Semi Stable

3.2 Variable Rescaling

Sometimes, we can rescale the state variables and time to simplify a model. Recall the **logistic model** with two parameters r, k:

$$\frac{dx}{dt} = r(1 - \frac{x}{k})x \tag{4}$$

Assume we have:

$$x = \alpha\hat{x} \tag{5}$$

$$t = \beta\hat{t} \tag{6}$$

Then we can rearrange the equation:

$$\frac{dx}{dt} = \frac{d(\alpha\hat{x})}{d(\beta\hat{t})} = \frac{\alpha d\hat{x}}{\beta d\hat{t}} \tag{7}$$

Then we have:

$$\frac{\alpha d\hat{x}}{\beta d\hat{t}} = r(1 - \frac{\alpha\hat{x}}{k})\alpha\hat{x} \tag{8}$$

$$\frac{d\hat{x}}{d\hat{t}} = \beta r(1 - \frac{\alpha\hat{x}}{k})\alpha\hat{x} \tag{9}$$

Now we let:

$$\beta = \frac{1}{r}, \;\; \alpha = k \tag{10}$$

Finally we have:

$$\frac{d\hat{x}}{d\hat{t}} = (1 - \hat{x})\hat{x} \tag{11}$$

3.3 Algebraic Method

Assume:

$$x(t) = x^* + \eta(t) \tag{12}$$

Where $\eta(t)$ is a trajectory near the fixed point x^*. Based on the definition of the fixed point:

$$\frac{dx^*}{dt} = f(x^*) = 0 \tag{13}$$

$$\frac{d(x^* + \eta)}{dt} = f(x^* + \eta) \tag{14}$$

The Taylor Expansion will be:

$$f(x^* + \eta) = f(x^*) + \eta \cdot \frac{df}{dx}|_{x=x^*} + \frac{1}{2}\eta^2 \cdot \frac{d^2 f}{dx^2}|_{x=x^*} \tag{15}$$

Assume η is small, then we ignore the higher order terms. Therefore:

$$f(x^* + \eta) \approx \eta \cdot \frac{df}{dx}|_{x=x^*} \tag{16}$$

Recall that:

$$\frac{d(x^* + \eta)}{dt} = \frac{dx^*}{dt} + \frac{d\eta}{dt} = \frac{d\eta}{dt} \tag{17}$$

Then we have:

$$\frac{d\eta}{dt} \approx \eta \cdot \frac{df}{dx}|_{x=x^*} \tag{18}$$

If we define:

$$\lambda^* = \frac{df}{dx}|_{x=x^*} \tag{19}$$

Then we have:

$$\frac{d\eta}{dt} \approx \eta \cdot \lambda^* \tag{20}$$

Separate the variables and integrate both sides:

$$\frac{1}{\eta} d\eta = \lambda^* dt \tag{21}$$

$$\ln \eta = \lambda^* t + C \tag{22}$$

Take the exponential on both sides:

$$\eta = e^{\lambda^* t + C} = e^C \cdot e^{\lambda^* t} \tag{23}$$

Therefore, we can have the analytical solution:

$$\eta = \eta(0) exp(\lambda^* t) \tag{24}$$

The sign of λ^* will determines stability (here, we assume time is a positive number):

1. $\lambda^* < 0$: the exponential is close to 0, stable
2. $\lambda^* > 0$: the exponential is increasing rapidly, unstable
3. $\lambda^* = 0$: inconclusive, need to check second derivative

Suppose we have:

$$\frac{df}{dx} = 0, \ \frac{d^2 f}{dx^2} \neq 0 \tag{25}$$

Then, recall the Taylor Expansion:

$$\frac{d\eta}{dt} \approx \frac{1}{2}\eta^2 \cdot \frac{d^2 f}{dx^2}|_{x=x^*} \tag{26}$$

If we define:

$$\omega^* = -\frac{1}{2}\eta^2 \cdot \frac{d^2 f}{dx^2}|_{x=x^*} \tag{27}$$

Then we will have:

$$\frac{d\eta}{dt} \approx -\omega^* \eta^2 \tag{28}$$

$$-\int \frac{d\eta}{\eta^2} \approx \omega^* \int dt \tag{29}$$

$$\frac{1}{\eta} \approx \omega^* t + \frac{1}{\eta_0} \tag{30}$$

$$\eta(t) \approx \frac{\eta_0}{1 + \eta_0 \omega^* t} \tag{31}$$

If we assume $\omega^* < 0$, then we have the visualization as:

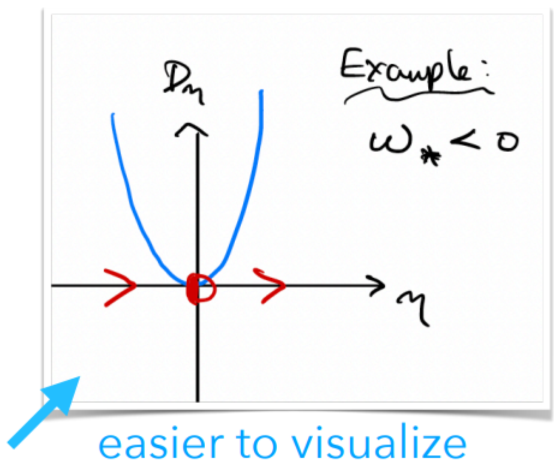

Figure 3: Second Derivative

Suppose $\omega_* < 0$. Then

$$\eta_0 < 0 \quad \rightarrow \quad \lim_{t \to \infty} \eta(t) = 0 \quad \text{stable from left}$$

$$\eta_0 > 0 \quad \rightarrow \quad \lim_{t \to 1/\eta_0 |\omega_*|} \eta(t) = \infty \quad \text{unstable from right}$$

Figure 4: Conclusion

4 Bifurcation

The bifurcation shows how **fixed points** change as parameters change.

4.1 Saddle Node Bifurcation (SNB)

Suppose we have:

$$\frac{dx}{dt} = f(x, r) \tag{32}$$

As r changes, the number of fixed point changes between:

$$2 \text{ pts} \leftrightarrow 1 \text{ pt} \leftrightarrow 0 \text{ pt} \tag{33}$$

Then we call this bifurcation as **Saddle Node Bifurcation (SNB)**. In details, as the r changes, first two fixed points (**one stable and one unstable**) will start to emerge (or collide) into one point (**semi-stable**). If r keeps changing, then there will be no fixed point. The direction could be reverse.

4.1.1 Case Study: Normal Form

The normal form representative of all SNBs could be expressed as:

$$\frac{dx}{dt} = r \pm x^2 \tag{34}$$

Take $r - x^2$ as an example:

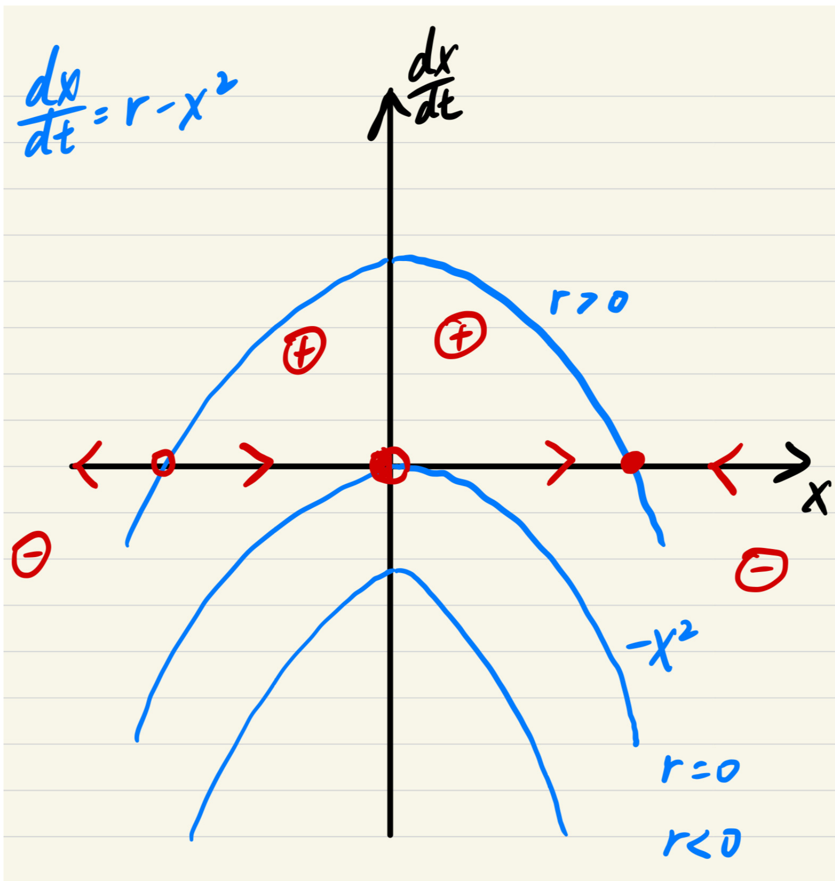

Figure 5: SNB Case

Some remarks:

1. When $r > 0$: there are two fixed points, one stable and the other is unstable
2. When $r = 0$: two fixed points emerge into one point, which is semi-stable
3. When $r < 0$: there will be no fixed point
4. **Bifuration occurs at** $x^* = 0$
5. **The critical bifurcation points is** $r_c = 0$

4.1.2 Case Study: Exponential Form

Another classical form of SNB can be expressed as:

$$\frac{dx}{dt} = r - x - exp(-x) \tag{35}$$

It is hard to find the fixed point directly, so we can rearrange the equation as:

$$r - x^* = exp(-x^*) \tag{36}$$

Then, to find the bifurcation point:

$$\frac{d}{dx}(r_c - x)|_{x^*} = \frac{d}{dx}exp(-x)|_{x^*} \tag{37}$$

$$-1 = -exp(-x^*) \tag{38}$$

Therefore we have:

$$x^* = 0, \ r_c = 1 \tag{39}$$

As r changes, the transitions are shown below:

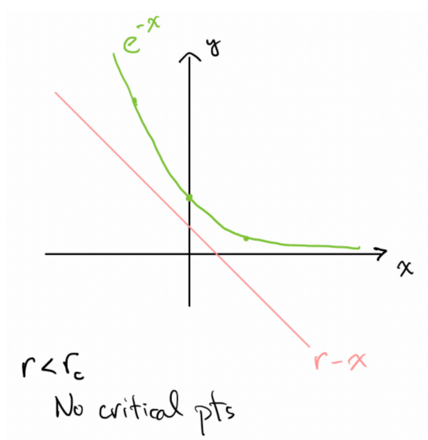

Figure 6: No Critical Point

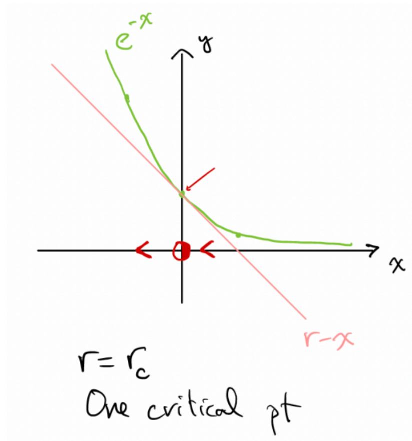

Figure 7: One Critical Point

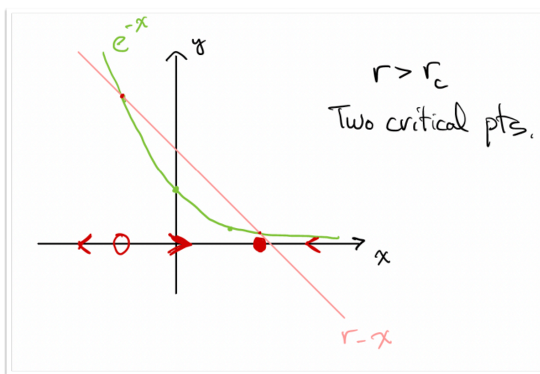

Figure 8: Two Critical Points

4.2 Transcritical Bifurcation (TCB)

Similarly, as r changes, the number of fixed point changes between:

$$2 \text{ pts} \leftrightarrow 1 \text{ pt} \tag{40}$$

Then we call this bifurcation as **Transcritical Bifurcation (TCB)**. In details, as the r changes, first two fixed points (**one stable and one unstable**) will start to emerge (or collide) into one point (**semi-stable**). If r keeps changing, they will split again into two points (**one stable and one unstable, but the stability exchanged**). The direction could be reversed.

4.2.1 Case Study: Normal Form

The normal form representative of all TCBs could be expressed as:

$$\frac{dx}{dt} = rx - x^2 = x(r - x) \tag{41}$$

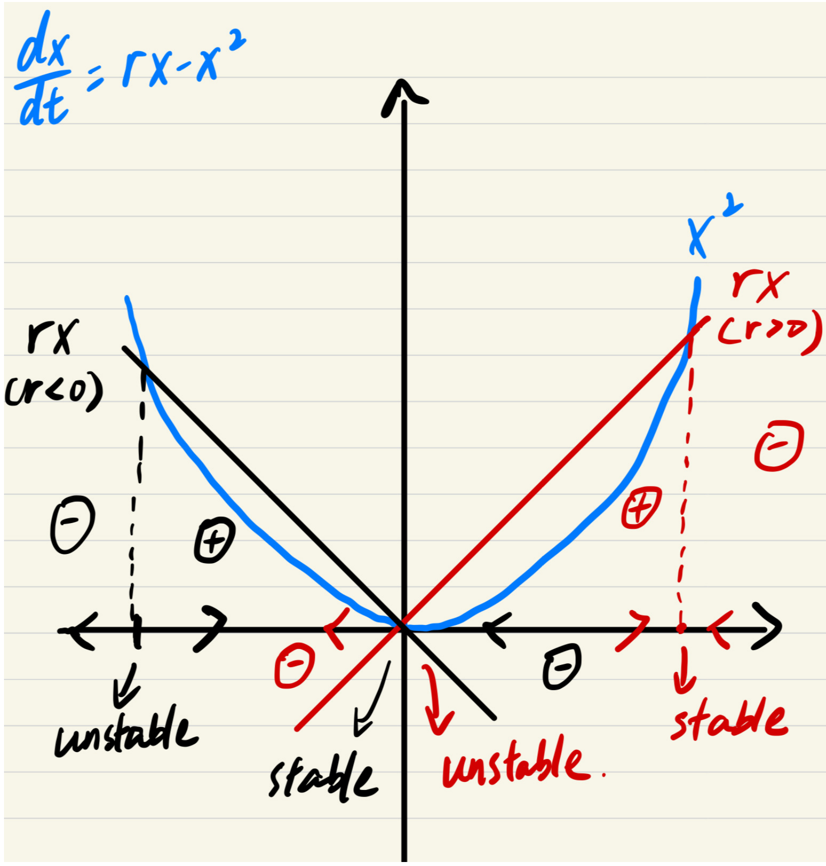

Figure 9: Transcritical Bifurcation

Some remarks:

1. When $r > 0$: two fixed points, one stable and one unstable
2. When $r < 0$: two fixed points, one stable and one unstable
3. When $r = 0$: one fixed point, semi-stable
4. **The critical bifurcation points is** $r_c = 0$

4.2.2 Case Study: Exponential Form

Assume we have the exponential form:

$$\frac{dx}{dt} = x(1 - x^2) - a(1 - exp(-bx)) \tag{42}$$

This undergoes TCB at $x^* = 0$ when $ab = 1$. The proof is shown below. For small x, the Taylor Expansion shows that:

$$1 - exp(-bx) = bx - b^2x^2/2 + O(x^3) \tag{43}$$

Therefore we have:

$$\frac{dx}{dt} = x - x^3 - abx + ab^2x^2/2 + O(x^3) \tag{44}$$

$$\frac{dx}{dt} = (1 - ab)x + ab^2x^2/2 + O(x^3) \tag{45}$$

If we let $x = \alpha\hat{x}$, then we have:

$$\alpha\frac{d\hat{x}}{dt} = (1 - ab)\hat{x} + ab^2\alpha^2\hat{x}^2/2 + O(x^3) \tag{46}$$

If we define:

$$\alpha = -\frac{2}{ab^2} \tag{47}$$

$$r = 1 - ab \tag{48}$$

The finally we have the form:

$$\frac{d\hat{x}}{dt} = r\hat{x} - \hat{x}^2 + O(\hat{x}^3) \tag{49}$$

4.3 Pitchfork Bifurcation

PFB refers to a sudden change in the qualitative or topological structure of a solution as a parameter is varied.

4.3.1 Supercritical (Stable) PFB

The general form is expressed as:

$$\frac{dx}{dt} = rx - x^3 \tag{50}$$

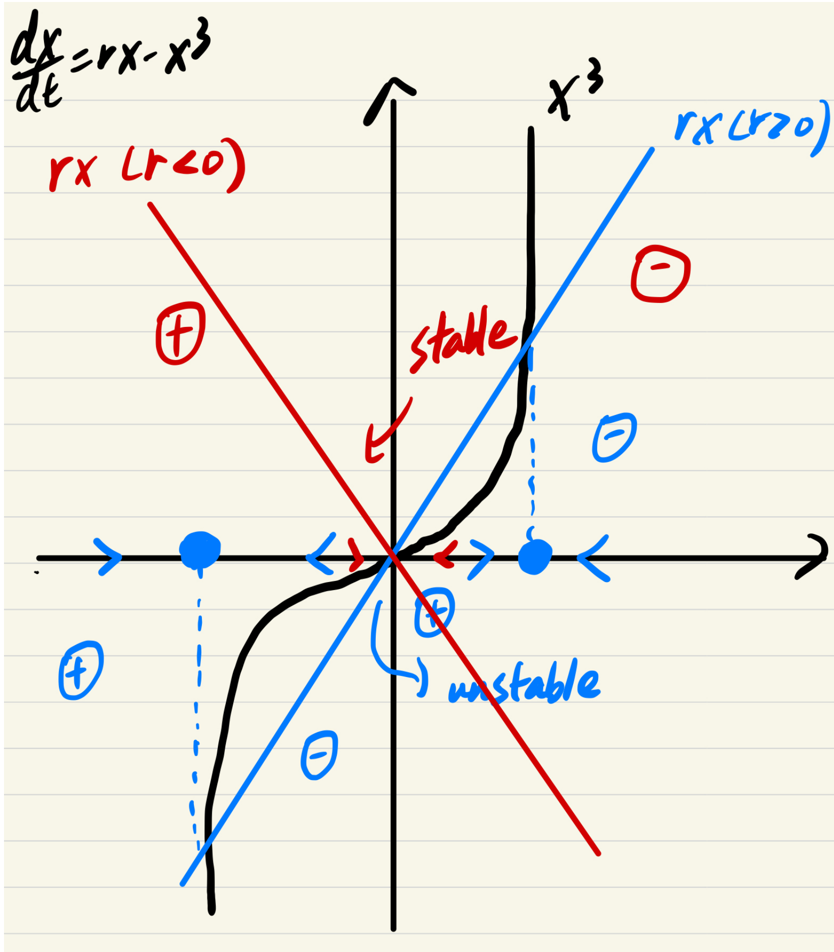

Figure 10: Supercritical PFB

Some remarks:

1. Before the bifurcation, there is **single stable fixed point**
2. After bifurcation, the previous fixed point becomes **unstable**, and two new **stable fixed points** appear.
3. The bifurcation direction could be reversed.

4.3.2 Subcritical (unstable) PFB

The general form is expressed as:

$$\frac{dx}{dt} = rx + x^3 \tag{51}$$

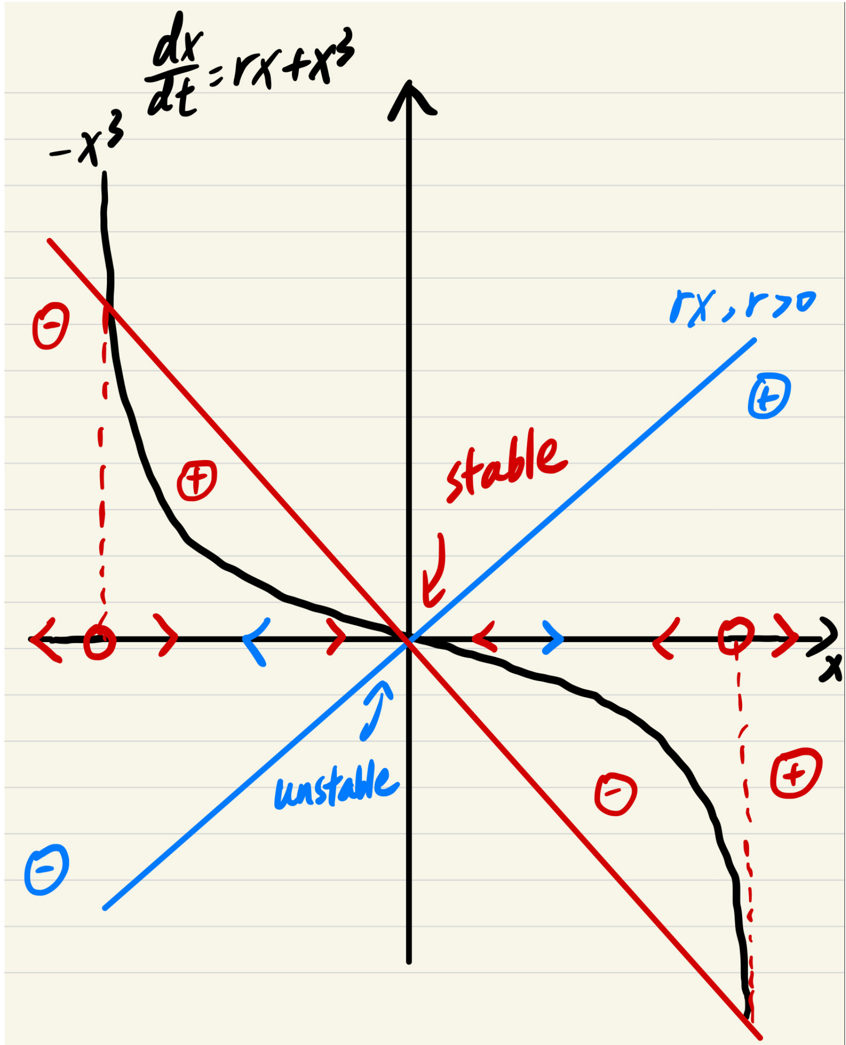

Figure 11: Subcritical PFB

1. Before the bifurcation, there is a **single stable fixed point** and two **unstable** fixed points.
2. After bifurcation, only unstable fixed point appears.
3. The direction of bifurcation could reverse.