Parallel Flow

1 Definition

The flow with parallel streamlines.

Figure 1: Parallel Flow.

2 Assumptions

1. $v = w = 0$
2. Newtonian Fluid
3. Incompressible (Constant Density), no body force
4. 2D (Infinite Wide), $\dfrac{\partial}{\partial z} = 0$
5. Steady Flow

3 Governing Equations

3.1 Continuity

Steady flow, constant density:

$$\nabla \cdot \underline{\boldsymbol{u}} = 0 \tag{1}$$

In 2D:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2}$$

Based on the assumptions, $v = 0$:

$$\frac{\partial u}{\partial x} = 0 \tag{3}$$

3.2 Momentum

Ignore the body force and assume constant density:

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{4}$$

In x direction:

$$\rho(\frac{\partial u}{\partial t} + (\underline{\boldsymbol{u}} \cdot \nabla)u) = -\frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x \partial x} + \frac{\partial^2 u}{\partial y \partial y}] \tag{5}$$

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x \partial x} + \frac{\partial^2 u}{\partial y \partial y}] \tag{6}$$

Because of continuity ($\frac{\partial u}{\partial x} = 0$), steady flow, and $v = 0$:

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y \partial y} \tag{7}$$

In y direction:

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \mu[\frac{\partial^2 v}{\partial x \partial x} + \frac{\partial^2 v}{\partial y \partial y}] \tag{8}$$

$$-\frac{\partial p}{\partial y} = 0 \tag{9}$$

Therefore, we know that $\mathbf{P} = \mathbf{P}(\mathbf{x})$ **Only!**

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y \partial y} \tag{10}$$

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + Ay + B \tag{11}$$

4 Couette Flow

4.1 Assumptions

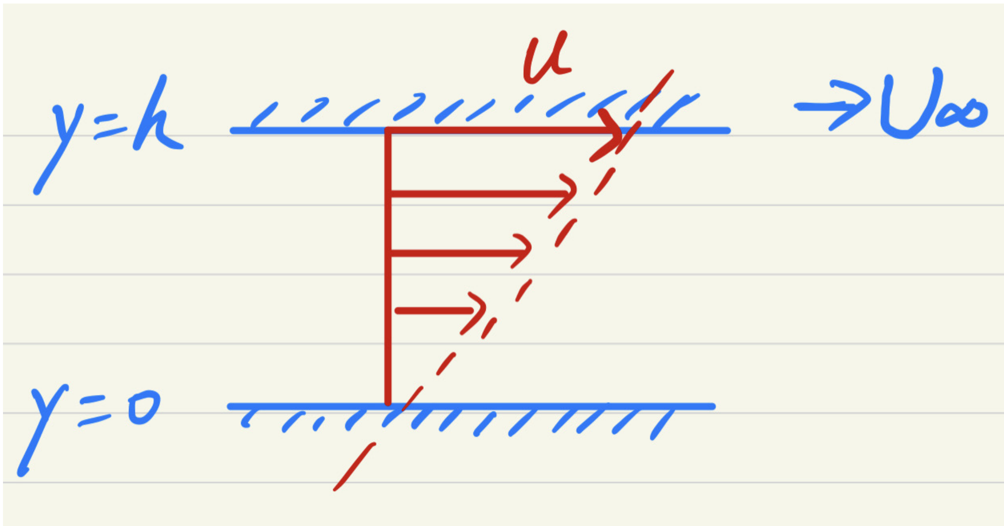

Figure 2: Couette Flow.

1. Parallel Flow Assumptions
2. No Pressure Gradient
3. One wall is moving at U_∞, the other wall is stationary

4.2 Boundary Conditions

1. $\dfrac{dp}{dx} = 0$
2. $u = U_\infty$ at $y = h$
3. $u = 0$ at $y = 0$

4.3 Governing Equations

$$u = \frac{1}{2\mu}\frac{dp}{dx}y^2 + Ay + B \tag{12}$$

At $y = 0$:

$$B = 0 \tag{13}$$

At $y = h$:

$$Ah + B = U_\infty \tag{14}$$

$$A = \frac{U_\infty}{h} \tag{15}$$

Therefore:

$$u = \frac{U_\infty}{h} y \tag{16}$$

5 Poiseuille Flow

5.1 Assumptions

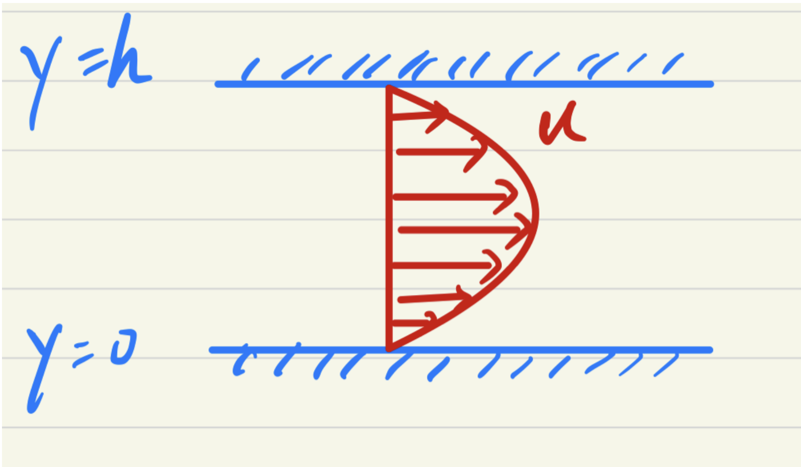

Figure 3: Poiseuille Flow.

1. Parallel Flow Assumptions
2. Non-zero Pressure Gradient
3. Both Walls Fixed

5.2 Boundary Condition

1. $u = 0$ at $y = 0$
2. $u = 0$ at $y = h$

5.3 Governing Equations

$$u = \frac{1}{2\mu}\frac{dp}{dx} y^2 + Ay + B \tag{17}$$

At $y = 0$:

$$B = 0 \tag{18}$$

At $y = h$:

$$0 = \frac{1}{2\mu}\frac{dp}{dx}h^2 + Ah + B \tag{19}$$

Therefore:

$$A = -\frac{1}{2\mu}\frac{dp}{dx}hy \tag{20}$$

And:

$$u = \frac{1}{2\mu}\frac{dp}{dx}y(y-h) \tag{21}$$

6 Superposition

6.1 Assumption

Couette Flow + Poiseuille Flow

6.2 Governing Equations

$$u = \frac{U_\infty}{h}y + \frac{1}{2\mu}\frac{dp}{dx}y(y-h) \tag{22}$$

Non-dimensional form:

$$\frac{u}{U} = \frac{y}{h} - \frac{h^2}{2\mu U}\frac{dp}{dx}\frac{y}{h}(1-\frac{y}{h}) \tag{23}$$

Define a non-dimensional parameter:

$$\Lambda = \frac{h^2}{2\mu U}\frac{dp}{dx} \tag{24}$$

Λ represents the pressure gradient. If $\Lambda < 0$, it is **favourable pressure gradient**. If $\Lambda > 0$, it is **adverse pressure gradient**, which may push fluid near the wall backwards and cause the flow separation ($\frac{\partial u}{\partial y} < 0$ at $y = 0$)

7 Unsteady Parallel Flow

7.1 Assumptions

1. Parallel Flow Assumptions

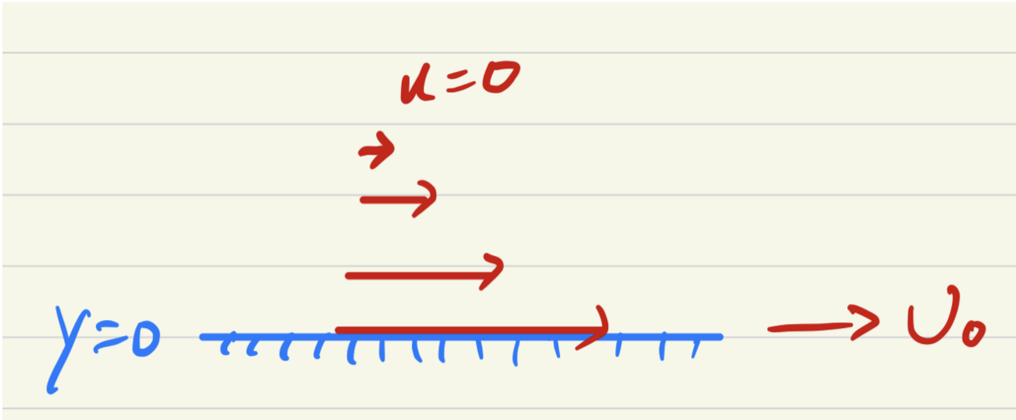

Figure 4: Unsteady Parallel Flow.

2. Fluid in unbounded space above a flat surface, initially at rest but suddenly moving at speed U_0
3. No Pressure Gradient

7.2 Initial and Boundary Condition

1. Initial Condition: at $t = 0, u = 0$ for all $y > 0$
2. Boundary Condition: at $t > 0, u = U_0$ at $y = 0$; $u = 0$ at $y \to \infty$

7.3 Governing Equations

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu[\frac{\partial^2 u}{\partial x \partial x} + \frac{\partial^2 u}{\partial y \partial y}] \tag{25}$$

Because of continuity ($\frac{\partial u}{\partial x} = 0$), no pressure gradient, and $v = 0$:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{26}$$

7.3.1 ODE and PDE

Unlike the **Ordinary Differential Equation (ODE)** in the steady flow problem, here we cannot avoid a **Partial Differential Equation (PDE)**. The differences between ODE and PDE are:

1. **ODEs**: These equations involve functions of only one independent variable and its derivatives. For example, the first-order ODE $\frac{dy}{dx} = f(x, y)$ involves one independent variable x and its derivative $\frac{dy}{dx}$. ODEs can be further classified as linear or nonlinear, homogeneous or nonhomogeneous, and so on, based on their specific features.

2. **PDEs**: These equations involve functions of more than one independent variable and their partial derivatives. For example, the two-dimensional heat equation:

$$\frac{\partial u}{\partial t} = \alpha(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{27}$$

involves two independent variables x and y, and their partial derivatives. PDEs describe a wide range of physical and mathematical phenomena, including waves, diffusion, quantum mechanics, and fluid dynamics.

7.3.2 Similarity Transformation

One of the best way to solve PDE is similarity transformation, so that we can change the PDE to ODE with a similarity variable as the single independent variable. But how to find this variable? Normally it has to be connected with the physics.

Thinking intuitively, viscosity quantifies the momentum diffusion from moving surface into fluid body. **For a more viscous fluid, motion will start earlier**. Or at a given time, effect reaches greater distance, $\propto \sqrt{\nu t}$. This could be obtained from dimensional analysis:

$$\sqrt{\nu t} = [\sqrt{m^2 \cdot s^{-1} \cdot s}] = [m] \tag{28}$$

Therefore, we choose the similarity parameter as:

$$\eta = \frac{y}{\sqrt{\nu t}} \tag{29}$$

which is a non-dimensional measure of "how far from the wall". Then this question becomes an ODE with η as a single independent variable. Derivatives in y and t can be transformed to derivatives in η via Chain Rule:

$$\frac{\partial ()}{\partial y} = \frac{\partial ()}{\partial \eta}\frac{\partial \eta}{\partial y} \tag{30}$$

$$\frac{\partial ()}{\partial t} = \frac{\partial ()}{\partial \eta}\frac{\partial \eta}{\partial t} \tag{31}$$

7.3.3 Final Solution

First, we choose the similarity variable as $\eta = \frac{y}{2\sqrt{\nu t}}$. Then, using the chain rule we get:

$$\frac{\partial \eta}{\partial y} = \frac{1}{2\sqrt{\nu t}}; \frac{\partial \eta}{\partial t} = -\frac{\eta}{2t} \tag{32}$$

Then, we transfer the equation:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{33}$$

into:

$$-\frac{\eta}{2t}\frac{\partial u}{\partial \eta} = \frac{\nu}{4\nu t}\frac{\partial^2 u}{\partial \eta^2} \tag{34}$$

Rearrange:

$$\frac{\partial^2 u}{\partial \eta^2} + 2\eta\frac{du}{d\eta} = 0 \tag{35}$$

Rewrite the boundary conditions as:

1. At $\eta = 0, u = U_0$
2. At $\eta \to \infty, u = 0$

Assume $f = \frac{du}{d\eta}$, then:

$$\frac{df}{d\eta} + 2\eta f = 0 \tag{36}$$

$$\frac{df}{f} = -2\eta d\eta \tag{37}$$

$$f = C_1 e^{-\eta^2} \tag{38}$$

Integrate again, we can get:

$$u = C_1 \int_0^{\eta} e^{-\eta^2}\, d\eta + C_2 \tag{39}$$

From the first boundary condition, we can get $C_2 = U_0$. Now, we introduce the error function, which is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-s^2}\, ds \tag{40}$$

With the properties of (easily proved):

$$erf(0) = 0, erf(\infty) = 1, erf(-\eta) = -erf(\eta) \tag{41}$$

Therefore, from the second boundary condition:

$$0 = C_1 \cdot \frac{\sqrt{\pi}}{2} + U_0 \tag{42}$$

$$C_1 = -U_0\frac{2}{\sqrt{\pi}} \tag{43}$$

Therefore,

$$u = U_0(1 - \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-s^2}\, ds) \tag{44}$$

$$u = U_0(1 - erf(\eta)) = U_0\boldsymbol{erfc}(\eta) \tag{45}$$