Basic Math

1 Gradient

1.1 Definition

The gradient of a scalar-valued differentiable function f of several variables is the vector field whose value at a point is the 'direction and rate of fastest increase' [3].

1.2 Gradient of Scalar is Vector

$$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} \tag{1}$$

1.3 Gradient of Vector is Tensor

$$\underline{\boldsymbol{f}} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k} \tag{2}$$

$$\begin{aligned} \nabla\underline{\boldsymbol{f}} &= \nabla(f_1, f_2, f_3) \\ &= \begin{bmatrix} \frac{\partial f}{\partial x}\hat{i} & \frac{\partial f}{\partial y}\hat{j} & \frac{\partial f}{\partial z}\hat{k} \\ \frac{\partial f}{\partial x}\hat{i} & \frac{\partial f}{\partial y}\hat{j} & \frac{\partial f}{\partial z}\hat{k} \\ \frac{\partial f}{\partial x}\hat{i} & \frac{\partial f}{\partial y}\hat{j} & \frac{\partial f}{\partial z}\hat{k} \end{bmatrix} \end{aligned} \tag{3}$$

2 Divergence

2.1 Definition

Divergence is a vector operator that operates on a vector field, producing a scalar field giving the quantity of the vector field's source at each point [2].

2.2 Divergence of Vector

$$\nabla \cdot \underline{\boldsymbol{f}} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \tag{4}$$

3 Laplacian

3.1 Definition

Divergence of the gradient.

3.2 Laplacian of Scalar

$$\begin{aligned}
\nabla \cdot (\nabla f) &= (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \cdot ((\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z})f) \\
&= ((\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}) \cdot (\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}))f \\
&= (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})f \\
&= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned} \tag{5}$$

3.3 Laplacian of Vector

$$\nabla^2 \underline{\boldsymbol{f}} = \nabla(\nabla \cdot \underline{\boldsymbol{f}}) - \nabla \times (\nabla \times \underline{\boldsymbol{f}}) \tag{6}$$

In Cartesian coordinates:

$$
\begin{aligned}
\nabla^2 \underline{\boldsymbol{f}} &= \begin{bmatrix} \nabla \cdot \nabla f_1 \\ \nabla \cdot \nabla f_2 \\ \nabla \cdot \nabla f_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} \\ \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2} \\ \frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2} \end{bmatrix}
\end{aligned} \tag{7}
$$

4 Curl

4.1 Definition

The curl is a vector operator that describes the infinitesimal circulation of a vector field in 3-D space [1]. A vector field whose curl is zero is called irrotational.

4.2 Curl of Vector

$$
\begin{aligned}
\nabla \times \underline{\boldsymbol{F}} &= \begin{Vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{Vmatrix} \\
&= (\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z})\hat{i} + (\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x})\hat{j} + (\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y})\hat{k} \\
&= \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{bmatrix}
\end{aligned} \tag{8}
$$

References

[1] URL https://en.wikipedia.org/wiki/Curl_(mathematics).

[2] URL https://en.wikipedia.org/wiki/Divergence.

[3] URL https://en.wikipedia.org/wiki/Gradient.