Continuum Model

1 Introduction

Continuum model is a model in which variables are considered to vary **continuously over space and/or time**, as opposed to being discretized or represented only at specific points. Continuum models are typically used when dealing with systems that are well-described by continuous variables, such as temperature in a fluid or stress in a solid, rather than by discrete events or entities.

2 Derivatives

Recall the definition of the derivative:

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \tag{1}$$

It represents the instantaneous rate of change.

3 Case Study

3.1 Population Prediction

Assume the **population** is expressed as f, the **birth rate** is expressed as r, and the **death rate** is expressed as s. Then we have:

$$\frac{df}{dt} = rf - sf = (r - s)f \tag{2}$$

If the initial value is:

$$f(0) = c \tag{3}$$

Then we have the analytical solution:

$$f(t) = c \cdot exp((r - s)t) \tag{4}$$

3.2 Population Cap

This time, suppose the environment only has resources to sustainably support a **maximum population** of size k, with only **birth rate** as f.

Logistically, we know that:

1. As the population grows toward the capacity, the rate should decline to 0.

2. At the capacity, the rate should be 0.

3. Beyond k, the rate should be negative to decrease the population.

Assume $R(f)$ be the population change rate, then we have:

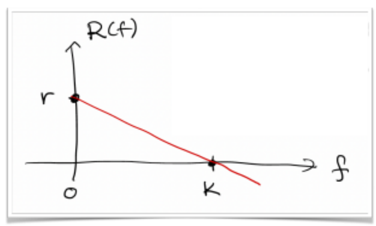

Figure 1: Rate of Change

Which can be expressed as:

$$R(f) = r - \frac{r}{k}f = r(1 - \frac{f}{k}) \tag{5}$$

So we have a new model, which is also called **logistic model of pop. growth**:

$$\frac{df}{dt} = Rf = r(1 - \frac{f}{k})f \tag{6}$$

This model is simple enough to have an analytic solution:

$$f(t) = \frac{c}{c/k + (1 - c/k)exp(-rt)} \tag{7}$$

3.3 Infection Spread

Now we come back to simulate the spread of infections. We have the following parameters:

1. k: Length of time of infection

2. τ: Probability of infection

3. $I(t)$: Infected, $I(t) \geq 0$

4. $S(t)$: Susceptible, $S(t) \leq 1$

5. $R(t)$: Recovered, $R(t) \leq 1$

Now we assume this is a **well-mixed** model, which means we ignore **spatial dimensions** and assume everyone is connected to everyone else.

Logistically, we know:

$$R(t) = 1 - I(t) - S(t) \tag{8}$$

$$\text{Recovery Rate} = \frac{I(t)}{k} \tag{9}$$

$$\text{Rate of New Infections} = \tau \cdot I(t) \cdot S(t) \tag{10}$$

Therefore, we have the equations:

$$\frac{dI(t)}{dt} = \tau I(t)S(t) - \frac{I(t)}{k} \tag{11}$$

$$\frac{dS(t)}{dt} = -\tau I(t)S(t) \tag{12}$$

$$\frac{dR(t)}{dt} = \frac{I(t)}{k} \tag{13}$$

If we want to reflect the idea that the infection lasts for k days, we need to introduce **delay** into the model:

$$\frac{dI(t)}{dt} = \tau I(t)S(t) - \tau I(t-k)S(t-k) \tag{14}$$

$$\frac{dS(t)}{dt} = -\tau I(t)S(t) \tag{15}$$

$$\frac{dR(t)}{dt} = \tau I(t-k)S(t-k) \tag{16}$$