Stagnation Thermal Boundary Layer

1 Overview

In fluid dynamics, stagnation properties are associated with the stagnation point of a fluid flow. The stagnation point is a location in a flow field where the velocity of the fluid is zero. The stagnation properties, often referred to as total or stagnation quantities, represent the conditions a fluid element would experience if it were brought to rest (stagnated) isentropically, i.e., without friction and heat transfer, from the free stream conditions.

2 Governing Equation

2.1 Derivation

Recall some stagnation properties:

$$h_0 = h + \frac{1}{2}u_i u_i \tag{1}$$

$$h_0 = C_p T_0 \tag{2}$$

Simplified Enthalpy Equation:

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = u\frac{\partial P}{\partial x} + k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{3}$$

Ignoring the forces, the kinetic energy equation is:

$$\begin{aligned}
\rho u_i \frac{Du_i}{Dt} &= u_i \frac{\partial \tau_{ji}}{\partial x_j} \\
&= u_i \frac{\partial}{\partial x_j}(-P\delta_{ji} + 2\mu S_{ji} - \frac{2}{3}\Delta S_{ji}) \\
&= -u_i \frac{\partial P}{\partial x_i} + 2\mu u_i \frac{\partial}{\partial x_j} S_{ji}
\end{aligned} \tag{4}$$

Expand:

$$2\mu u_i \frac{\partial}{\partial x_j} S_{ji} = \mu u_i \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{5}$$

Based on the boundary layer approximation, only u and $\partial u/\partial y$ matter:

$$2\mu u_i \frac{\partial}{\partial x_j} S_{ji} = \mu u \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \tag{6}$$

Add the enthalpy equation and the kinetic equation together to get the stagnation enthalpy equation:

$$\rho \frac{Dh}{Dt} + \rho \frac{D(\frac{1}{2}u_i u_i)}{Dt} = u\frac{\partial P}{\partial x} + k\frac{\partial^2 T}{\partial y^2} + \mu(\frac{\partial u}{\partial y})^2 - u\frac{\partial P}{\partial x} + \mu u \frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) \tag{7}$$

$$\rho C_p[u\frac{\partial T_0}{\partial x} + v\frac{\partial T_0}{\partial y}] = k\frac{\partial^2 T}{\partial y^2} + \mu\frac{\partial}{\partial y}(u\frac{\partial u}{\partial y}) \tag{8}$$

2.2 Unit Prandtl Number Assumption

If $Pr = 1$, then:

$$Pr = \frac{\mu C_p}{k} = 1 \tag{9}$$

$$\mu = \frac{k}{C_p} \tag{10}$$

Divide the previous equation by C_p:

$$\rho[u\frac{\partial T_0}{\partial x} + v\frac{\partial T_0}{\partial y}] = \mu\frac{\partial^2 T}{\partial y^2} + \frac{\mu}{C_p}\frac{\partial}{\partial y}(u\frac{\partial u}{\partial y}) = \mu\frac{\partial^2}{\partial y^2}[T + \frac{\frac{1}{2}u^2}{C_p}] \tag{11}$$

Based on the definition of stagnation temperature:

$$T_0 = T[1 + \frac{1}{2C_p}(u^2 + v^2)] \tag{12}$$

From boundary layer assumption, $u >> v$, then:

$$T_0 \approx T[1 + \frac{u^2}{2C_p}] \tag{13}$$

Then with the **unit Prandtl number assumption:**

$$\rho[u\frac{\partial T_0}{\partial x} + v\frac{\partial T_0}{\partial y}] = \mu\frac{\partial^2 T_0}{\partial y^2} \tag{14}$$

In this case, the wall can be isothermal (specified temperature) or adiabatic (no heat transferred). From observation, $T_0 = const$ is a solution, then:

$$T + \frac{u^2}{2C_p} = const \tag{15}$$

Differentiating in y:

$$\frac{\partial T}{\partial y} + \frac{1}{C_p}(u\frac{\partial u}{\partial y}) = 0 \tag{16}$$

$$C_p\frac{\partial T}{\partial y} = -u\frac{\partial u}{\partial y} \tag{17}$$

At the wall, due to no-slip condition, $u = 0$, so $\partial T/\partial y = 0$. This solution corresponds to an adiabatic wall.

3 Recovery Factor

If T_0 constant and zero heat flux, there is an adiabatic compression at the wall, all kinetic energy is converted into heat. However, the energy conversion is not 100% in real life. Therefore, we define the recovery factor as:

$$r = \frac{T_{ad,w} - T_\infty}{T_{0,\infty} - T_\infty} = \frac{T_{ad,w} - T_\infty}{U_\infty^2/(2C_p)} \tag{18}$$

Some remarks:

1. $T_{ad,w}$ is the adiabatic wall temperature, or temperature of the adiabatic wall
2. $T_{0,\infty}$ is the free-stream stagnation temperature
3. If $Pr = 1$, $r = 1$, $T_{ad,w} = T_{0,\infty}$