2D Linearization

1 Procedures

Recall the 2D dynamic system:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \tag{1}$$

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) \tag{2}$$

First, we need to find the fixed points so that:

$$\vec{f}(\vec{x}^*) = 0 \tag{3}$$

Then, we assume a small vector:

$$\vec{s} = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} \tag{4}$$

So for the fixed point, we have:

$$\vec{x}(t) = \vec{x}^* + \vec{s} \tag{5}$$

Then using the **multivariate Taylor Expansion**, we can linearize the equation at $\vec{x}^*$ (dropping the higher term):

$$\vec{f}(\vec{x}^* + \vec{s}) \approx \vec{f}(\vec{x}^*) + J_f(\vec{x}^*) \cdot \vec{s} \tag{6}$$

$$\vec{f}(\vec{s}) \approx J_f(\vec{x}^*) \cdot \vec{s} \tag{7}$$

Where $J_f(\vec{x}^*)$ is the **Jacobian of f at** $\vec{x}^*$.

2 Jacobian Matrix

Now we take a closer look at the Jacobian. Recall that:

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \tag{8}$$

Then the Jacobian matrix is defined as:

$$J_f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \tag{9}$$

2.1 Eigenvalues Calculation

Now we are interested in calculating the eigenvalues. The procedure is simple:

$$\det(A - \lambda I) = 0 \tag{10}$$

Assume:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{11}$$

Then:

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \tag{12}$$

$$det() = (a - \lambda)(d - \lambda) - bc = 0 \tag{13}$$

Then we can get the eigenvectors:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{14}$$

Then use the equation to find the eigen pair:

$$Av = \lambda v \tag{15}$$

2.2 Diagonal Jacobian

Recall the predator-prey model:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_1 x_2 \\ -r x_2 + x_1 x_2 \end{bmatrix} \tag{16}$$

Therefore, we can get the Jacobian matrix as:

$$J_f(\vec{x}) = \begin{bmatrix} 1-x_2 & -x_1 \\ x_2 & x_1 - r \end{bmatrix} \tag{17}$$

First we try the fixed point $[0, 0]^T$ (notice that the Jacobian matrix only has meaning at fixed point, only this way could get the eigenvalue for the system):

$$\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}$$

Then we have:

$$J^* = J_f(\vec{x}^*) = \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix} \tag{19}$$

Which is a **diagonal Jacobian**, the terms in diagonal direction are just **eigenvalues at this fixed point**. Plug in back to previous equation:

$$\frac{d}{dt}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \tag{20}$$

Which means:

$$\frac{ds_1}{dt} = \lambda_1 s_1, \ \frac{ds_2}{dt} = \lambda_2 s_2 \tag{21}$$

Or in other words:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \vec{s}(0)\begin{bmatrix} exp(\lambda_1 t) \\ exp(\lambda_2 t) \end{bmatrix} = \begin{bmatrix} exp(t) \\ exp(-rt) \end{bmatrix} \tag{22}$$

Which means, when close to $[0,0]^T$, x_1 is **exponentially blowing-up**, and x_2 is **exponentially decaying**:

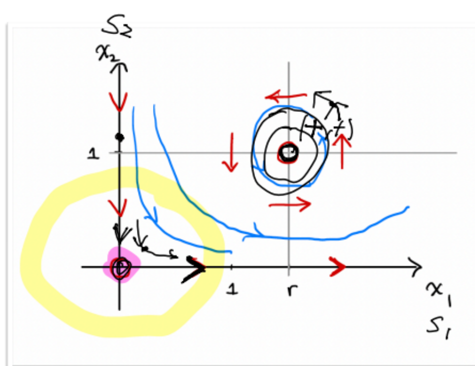

Figure 1: Diagonal Jacobian

2.3 Antidiagonal Jacobian

Similarly, when $\vec{x}^* = [r,1]^T$, we have:

$$J^* = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix} \tag{23}$$

Which is an **antidiagonal Jacobian**. Therefore we have:

$$\frac{d}{dt}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s_2 \\ s_1 \end{bmatrix} \tag{24}$$

Now we can find the eigenvalues at this fixed point:

$$J^* - \lambda I = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -r \\ 1 & -\lambda \end{bmatrix} \tag{25}$$

Then, using the determinant formula:

$$det \begin{bmatrix} -\lambda & -r \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + r = 0 \tag{26}$$

Solving this equation:

$$\lambda_1 = i\sqrt{r}, \ \ \lambda_2 = -i\sqrt{r} \tag{27}$$

3 Eigenvalues and Phase Diagram

Eigenvalues' real and imaginary parts will represent the system's stability, and also will affect the final phase diagram.

3.1 1D System, 1D Phase Diagram

3.1.1 Stability

Recall the previous content, in 1D the solution has the general form:

$$x(t) = \exp\left(\lambda^* t\right) \tag{28}$$

The sign of λ^* will determines stability (here, we assume time is a positive number):

1. $\lambda^* < 0$: the exponential is close to 0, stable
2. $\lambda^* > 0$: the exponential is increasing rapidly, unstable
3. $\lambda^* = 0$: inconclusive, need to check second derivative

Notice that 1D system **has only one eigenvalue, and this value must be real value.** The complex eigenvalues indicate **oscillatory behavior due to the interaction between multiple dimensions.** This does not exist in 1D, so the eigenvalue must be real.

3.1.2 Phase Diagram

Figure 2: 1D Phase Diagram

In 1D, the phase diagram is just a line, but it may have multiple fixed points. The stability of each fixed point will be different.

3.2 2D System, 2D Phase Diagram

3.2.1 Complex Value Form

For the general 2D case, we need to assume:

$$\vec{s}(t) = \exp(\lambda t)\vec{v} \tag{29}$$

Recall the equation:

$$\frac{d\vec{s}}{dt} = J^* \vec{s} \tag{30}$$

Therefore we have:

$$\lambda \exp(\lambda t)\vec{v} = J^* \exp(\lambda t)\vec{v} \tag{31}$$

$$\lambda \vec{v} = J^* \vec{v} \tag{32}$$

Then, solution $(\lambda, \vec{v})$ is an **eigenpair** of J^*. For an $n \times n$ matrix, there are up tp n **distinct eigenpairs**. Eigenpairs could be **complex-valued**! They will occur in **complex-conjugate pairs**:

$$\lambda_1 = \alpha + i\beta,\ \vec{v}_1 = a + ib \tag{33}$$

- If λ_1, λ_2 are distinct, the solution could be written as a linear combination:

$$\vec{s}(t) = a_1 \exp(\lambda_1 t)\vec{v}_1 + a_2 \exp(\lambda_2 t)\vec{v}_2 \tag{34}$$

- If λ_1 and λ_2 are **real** and $\lambda_1 > \lambda_2$, then we will have **dominance**, which means the solution will tend toward $\vec{v}_1$:

$$\vec{s}(t) = \exp(\lambda_1 t)\vec{v}_1[a_1\vec{v}_1 + a_2 \exp((\lambda_2 - \lambda_1)t)\vec{v}_2] \tag{35}$$

  When $t \to \infty$,

$$\vec{s}(t) = \exp(\lambda_1 t) \cdot a_1 \vec{v}_1 \tag{36}$$

- If λ_1 and λ_2 are **complex-valued**, then we have:

$$\exp(\lambda t) = \exp(\alpha t)\exp(i\beta t) = \underbrace{\exp(\alpha t)}_{\text{Stability}}[\underbrace{\cos(\beta t) + i\sin(\beta t)}_{\text{Oscillations}}] \tag{37}$$

3.2.2 Stability and Oscillations

The imaginary part β of the complex eigenvalues determines the **frequency and direction of the oscillations** in the phase space. It is also the reason why **spiral will show in the phase diagram**.

The sign of β influences the direction of rotation in the phase plane:

$$\begin{cases} \beta < 0 : & \textbf{Counterclockwise Spiral} \\ \beta > 0 : & \textbf{Clockwise Spiral} \end{cases} \tag{38}$$

Notice for the center node (circle), the two eigenvalues will both be pure imaginary and same magnitude, at this time direction of rotation is determined by the Jacobian matrix:

$$\begin{cases} bc < 0 : & \textbf{Clockwise Circle} \\ bc > 0 : & \textbf{Counterclockwise Circle} \end{cases} \tag{39}$$

The stability is mainly controlled by the real part of the eigenvalues (α). The dependence is shown below (with β):

$$\begin{cases} \alpha < 0 : & \textbf{Stable Spiral} \\ \alpha > 0 : & \textbf{Unstable Spiral} \\ \alpha = 0 : & \textbf{Circle (neutrally stable)} \end{cases} \tag{40}$$

3.2.3 Phase Diagram

We can find some special nodes in phase diagram using eigenvalues:

1. **Stable Node (Sink):** both eigenvalues are real and negative

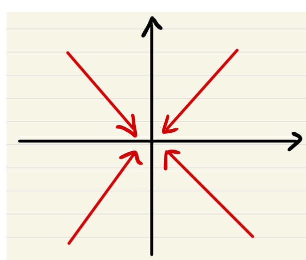

Figure 3: Stable Node (Sink)

2. **Unstable Node(Source):** both eigenvalues are real and positive

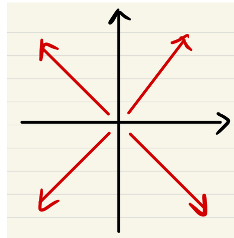

Figure 4: Unstable Node (Source)

3. **Saddle Node:** Both eigenvalues are real. One eigenvalue is positive, and the other is negative, and it is unstable.

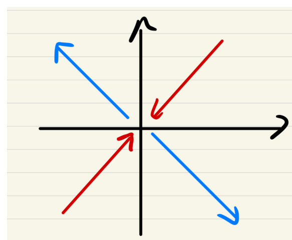

Figure 5: Saddle Node

4. **Stable Spiral Node:** Eigenvalues are complex conjugates with **negative real parts** (including both β cases)

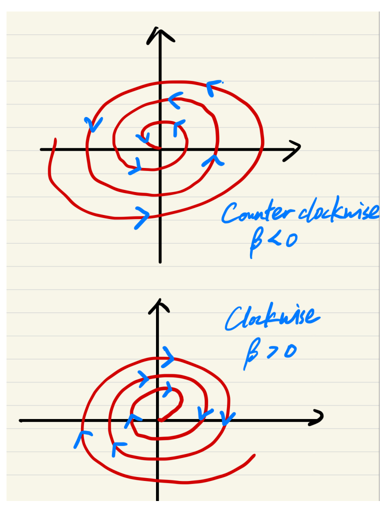

Figure 6: Stable Spiral Node

5. **Unstable Spiral Node:** Eigenvalues are complex conjugates with **positive real parts** (include both bc cases), and it is neutrally stable.

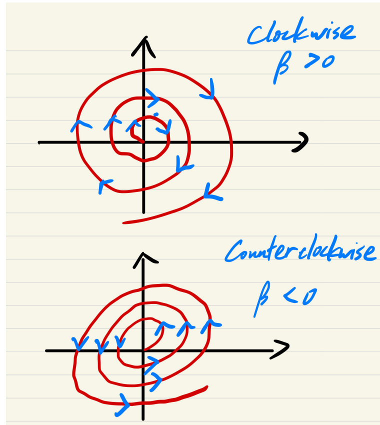

Figure 7: Unstable Spiral Node

6. **Center Node:** Eigenvalues are purely imaginary, complex conjugates with **zero real parts** (include both β cases)

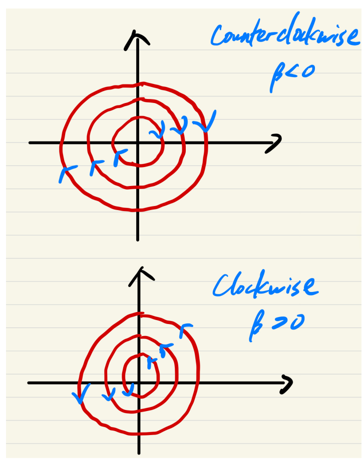

Figure 8: Center Node