Master Energy Equation

1 Thrust Loading vs Wing Loading

The **thrust loading** or **thrust-to-weight ratio** (T/W) is calculated by dividing the total thrust generated by the aircraft's engines by the total weight of the aircraft. This ratio is a key indicator of the aircraft's ability to accelerate and climb. A higher thrust-to-weight ratio means the aircraft can produce more thrust relative to its weight, leading to better performance in terms of acceleration and climb rate. Conversely, a lower thrust-to-weight ratio indicates less thrust for a given weight, resulting in reduced performance.

Wing loading in aircraft design is a measure of the distribution of weight (or load) over the wing area. It's a crucial factor in determining an aircraft's flight characteristics, including its lift, speed, and maneuverability. The wing loading is calculated by dividing the total weight of the aircraft by the wing area. This ratio is usually expressed in pounds per square foot (psf) or kilograms per square meter (kg/m^2).

A few key aspects of how wing loading affects aircraft performance include:

1. **Lift and Stall Speed**: Higher wing loading means the aircraft must travel faster to generate enough lift to take off or stay airborne. This results in a higher stall speed (the minimum speed at which the aircraft can fly without losing lift). Lower wing loading allows the aircraft to take off and land at lower speeds, making it more suitable for short runways.

2. **Maneuverability**: Aircraft with lower wing loading tend to be more maneuverable and have better low-speed handling characteristics. This is crucial for aircraft that need to perform tight turns or fly at low speeds, such as fighter jets or stunt planes.

3. **Ride Comfort:** Higher wing loading generally provides a smoother ride in turbulent conditions, as the aircraft is less affected by air pockets and gusts. This is often a consideration in the design of commercial airliners.

4. **Efficiency:** Wing loading also impacts the aerodynamic efficiency of an aircraft. A higher wing loading can lead to reduced drag and improved fuel efficiency during cruise, which is a critical factor for long-range commercial aircraft.

Thrust loading vs wing loading for cargo and passenger aircraft:

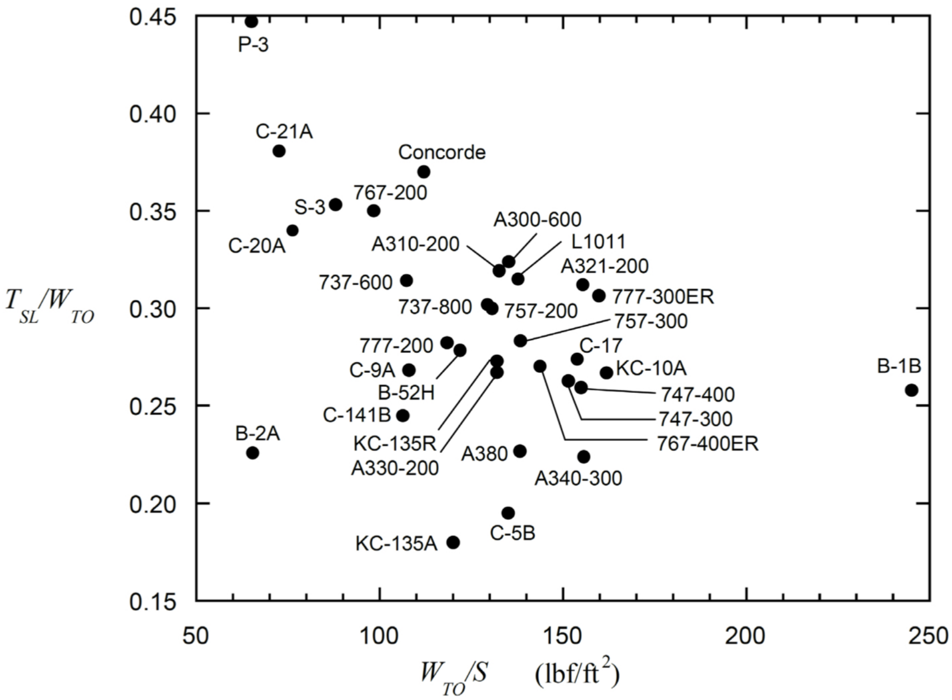

Figure 1: Cargo and passenger aircraft

This plot is also called **constraint analysis plot**, as shown below:

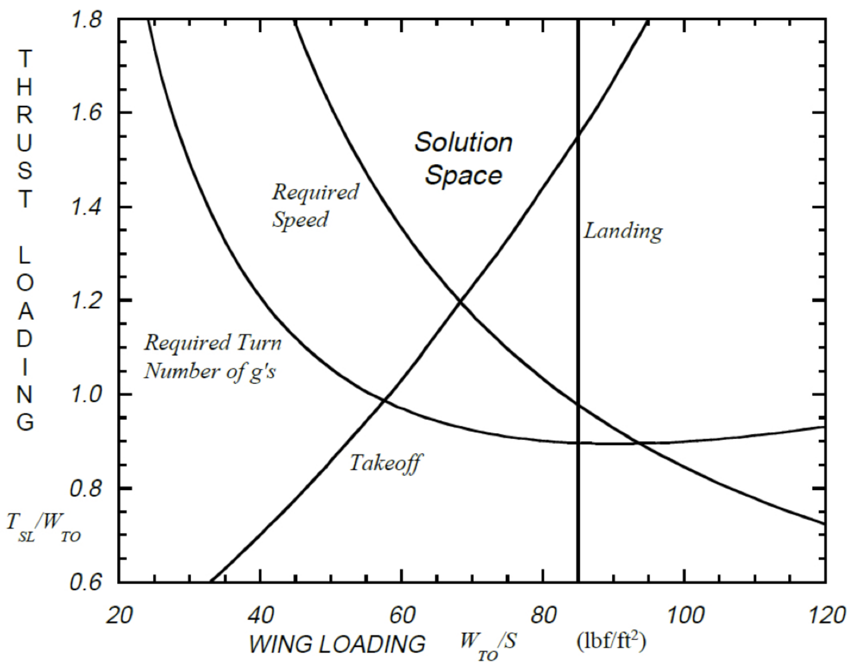

Figure 2: Constraint Plot

In this plot, the constraint analysis can be visualized to identify the solution space, and the performance constraints and requirements are set as functions of Thrust Loading and Wing Loading.

2 Aircraft Sizing

The traditional aircraft sizing process is shown below:

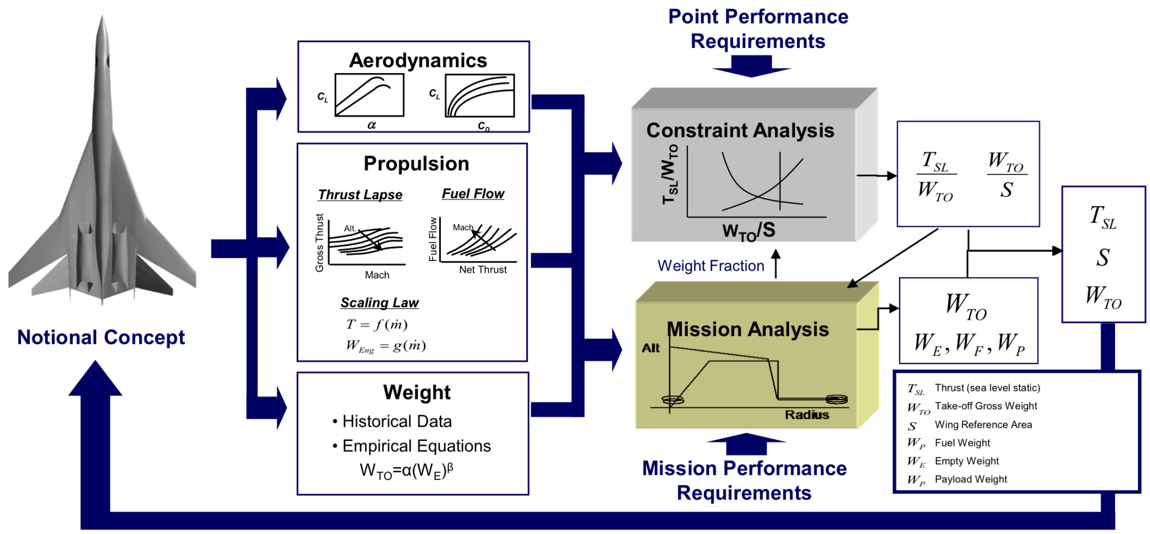

Figure 3: Constraint Plot

3 Energy Balance of the System

3.1 Overview

Before all the calculations, the assumptions are needed to specify:

1. Aircraft is represented as a moving point mass
2. Installed thrust and aerodynamic drag act in the same direction as the velocity

The energy balance could be expressed as:

$$(T - (D + R))V = W\frac{dh}{dt} + \frac{W}{g_0}\frac{d}{dt}(\frac{V^2}{2}) \tag{1}$$

Here:

1. T: Thrust
2. D: Induced drag and parasite drag
3. R: Drag due to non-clean configuration
4. V: Velocity
5. W: Weight
6. h: Height

3.2 Induced Drag

Induced drag is a type of aerodynamic drag that occurs as a result of the creation of lift on a wing or airfoil. It arises due to the pressure differences between the top and bottom surfaces of a wing as it generates lift. High pressure on the bottom and low pressure on the top create a pattern of airflow that results in vortices at the wingtips. These vortices cause the air to circulate around the wingtip from bottom to top, reducing the effective angle of attack of the wing and tilting the resultant lift vector rearwards. This rearward component of the lift vector is what is experienced as induced drag.

At low airspeeds or high angles of attack (like during takeoff and landing), induced drag is higher due to the increased strength of the wingtip vortices. Here are the explanation:

1. **Lift**: At lower speeds, an aircraft must generate the same amount of lift to counteract its weight and stay airborne. Because lift is a product of the airspeed, air density, wing area, and the lift coefficient, when the airspeed decreases, the aircraft must increase the lift coefficient to maintain the same lift. This is typically achieved by increasing the angle of attack.

2. **Angle of Attack**: As the angle of attack increases to compensate for lower airspeeds, the pressure differential between the upper and lower wing surfaces becomes greater. This difference is what produces lift, but it also intensifies the wingtip vortices, which are a primary cause of induced drag. These vortices represent a circulation of air that effectively 'spills' from the high-pressure area below the wing to the low-pressure area above at the wingtips.

3. **Downwash**: The increased wingtip vortices at lower speeds create more downwash behind the wings, which tilts the total aerodynamic force backwards, thus increasing the backward (drag) component of this force, known as induced drag.

3.3 Parasite Drag

Parasite drag is a type of aerodynamic drag that is not associated with the production of lift and it increases with the square of the aircraft's velocity. Usually it contains:

1. **Form Drag:** This results from the shape of the aircraft and its components. Air flowing over and around the body of the aircraft creates pressure differences and turbulence that contribute to resistance. Streamlined shapes are used to reduce form drag.

2. **Skin Friction Drag:** As air flows over the surface of the aircraft, it creates a layer of air that sticks to the surface, known as the boundary layer. The viscosity of the air within this layer causes resistance against the movement of the aircraft, which is skin friction drag.

3. **Interference Drag**: Occurs when varying currents of air over the aircraft meet and interact, typically at the junction between different parts of the aircraft (like the wing and the fuselage). These interactions can cause additional turbulence and drag.

3.4 Total Drag

Based on the previous analysis, when the speed is increasing, parasite drag will increase but induced drag will decrease. The total drag change is shown below:

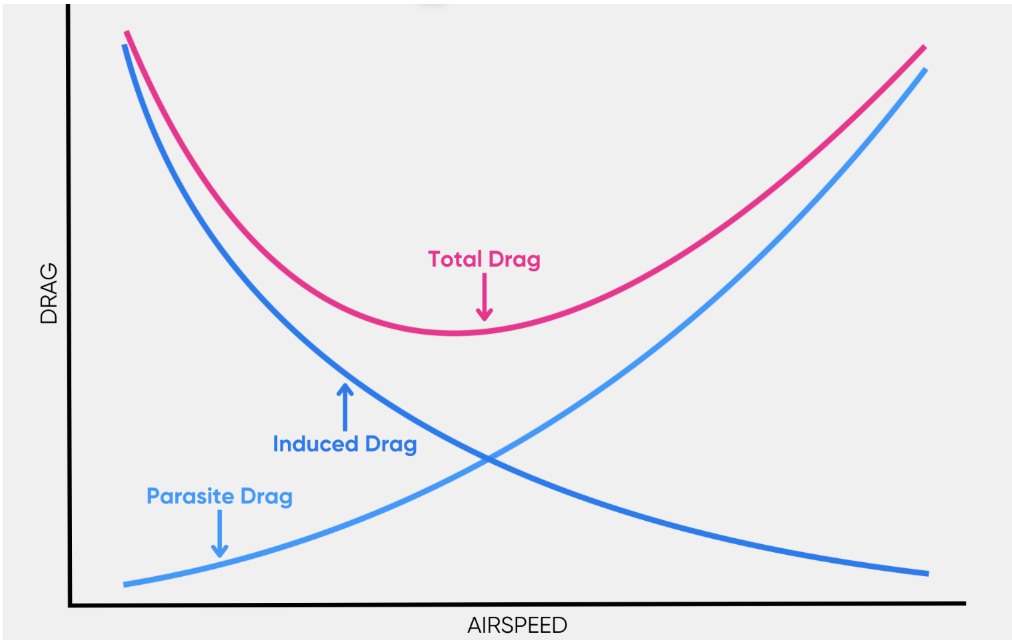

Figure 4: Total Drag

4 Derivation

Dimensionless form of the energy equation:

$$\frac{T-(D+R)}{W}=\frac{1}{V}\frac{d}{dt}(h+\frac{V^2}{2g_0}) \tag{2}$$

If the left hand side is multiplied by velocity, we get **weight specific excess power**, as shown below:

$$V\cdot\frac{T-(D+R)}{W}=P_s=\frac{d}{dt}(h+\frac{V^2}{2g_0}) \tag{3}$$

Aircraft engines are typically rated for their performance at sea level conditions because it provides a standard, consistent baseline. However, the actual performance of an engine in flight varies significantly with altitude due to changes in air density and temperature. By correcting sea level thrust for lapse rate, engineers and pilots can more accurately predict how an engine will perform in the less dense and colder air at cruising altitudes. Therefore, we need to correct the sea level thrust using lapse rate:

$$T=\alpha T_{SL} \tag{4}$$

The weight of an aircraft varies significantly during a flight due to fuel burn. As fuel is consumed, the aircraft becomes lighter, which affects its performance characteristics, such as takeoff distance, climb rate, cruising altitude, and fuel efficiency. Therefore, we need to correct weight using fuel/payload correction:

$$W = \beta W_{TO} \tag{5}$$

Therefore:

$$\frac{\alpha T_{SL} - (D + R)}{\beta W_{TO}} = \frac{1}{V}\frac{d}{dt}(h + \frac{V^2}{2g_0}) \tag{6}$$

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}((\frac{D + R}{\beta W_{TO}}) + \frac{1}{V}\frac{d}{dt}(h + \frac{V^2}{2g_0})) \tag{7}$$

5 Master Equation

5.1 Lift

Some expressions for lift:

$$L = nW = qC_L S \tag{8}$$

Here:

1. n: load factor, $n = 1$ for straight and level flight
2. q: dynamic pressure, $\frac{1}{2}\rho V^2$
3. S: wing area

Therefore we can get:

$$C_L = \frac{nW}{qS} = \frac{n\beta}{q}(\frac{W_{TO}}{S}) \tag{9}$$

5.2 Drag

The expression for drag is:

$$D = qC_D S \tag{10}$$

The **parabolic lift-drag polar** equation is shown below:

$$C_D = K_1 C_L^2 + K_2 C_L + C_{D_0} \tag{11}$$

Substitute C_D and C_L:

$$D = qS(K_1 C_L^2 + K_2 C_L + C_{D_0}) \tag{12}$$

$$D = qS(K_1(\frac{n\beta}{q}\frac{W_{TO}}{S})^2 + K_2(\frac{n\beta}{q}\frac{W_{TO}}{S}) + C_{D_0}) \tag{13}$$

5.3 Master Equation

Combine all of this, we get the master equation:

$$\frac{T_{SL}}{W_{to}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{to}}\left[K_1\left(\frac{n\beta W_{to}}{qS}\right)^2 + K_2\left(\frac{n\beta W_{to}}{qS}\right) + C_{D_o} + \frac{R}{qS}\right] + \frac{1}{V}\frac{d}{dt}\left(h + \frac{V^2}{2g_o}\right)\right\} \tag{14}$$

6 Case Study

6.1 Case 1: Constant Altitude/Speed Cruise

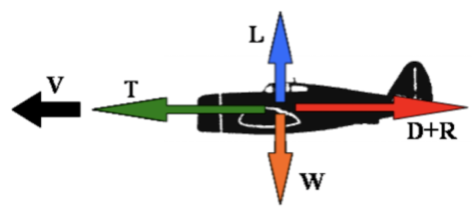

Figure 5: Constant Altitude/Speed Cruise

The assumptions include:

1. $\frac{dh}{dt} = 0$: Constant altitude
2. $\frac{dV}{dt} = 0$: Constant Speed
3. $n = 1$: Lift equals weight
4. $R = 0$: Clean configuration
5. $h\&V$: constant values

Now the master equation could be simplified to:

$$\frac{T_{SL}}{W_{to}} = \frac{\beta}{\alpha}\left\{\frac{qS}{\beta W_{to}}\left[K_1\left(\frac{\beta W_{to}}{qS}\right)^2 + K_2\left(\frac{\beta W_{to}}{qS}\right) + C_{D_o}\right]\right\} \tag{15}$$

$$\frac{T_{SL}}{W_{to}} = \frac{\beta}{\alpha}\left\{\underbrace{K_1\left(\frac{\beta}{q}\frac{W_{to}}{S}\right)}_{\text{Linear Term}} + \underbrace{K_2}_{\text{Constant Term}} + \underbrace{\frac{C_{D_0}}{\frac{\beta}{q}\frac{W_{to}}{S}}}_{\text{Inverse Term}}\right\} \tag{16}$$

Now, we go back to the constraint analysis plot:

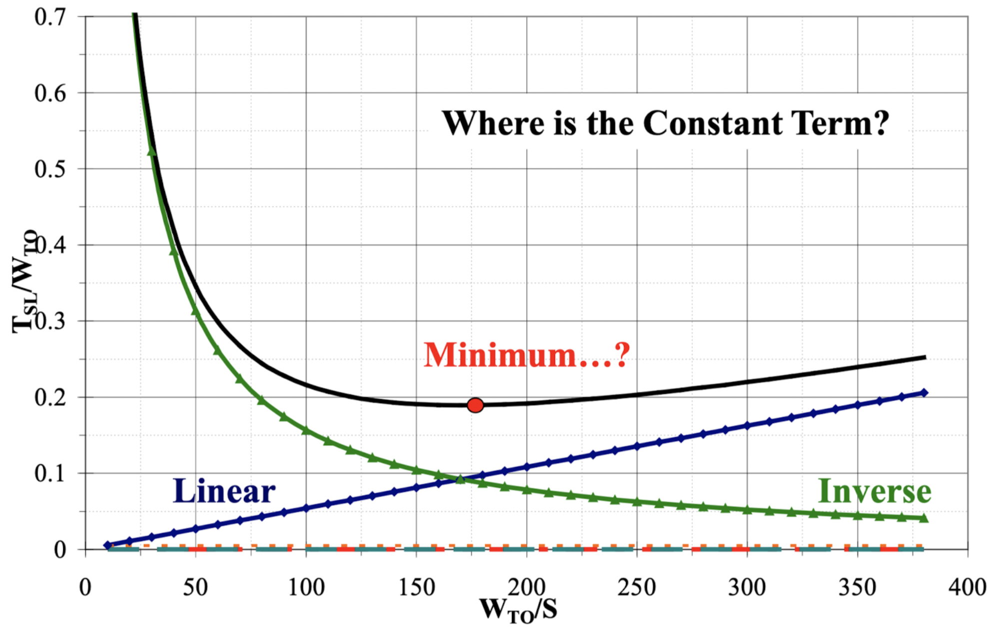

Figure 6: Case 1 Constraint Plot

To find the **minimum thrust to weight ratio**, we take the partial derivative with respect to wing loading and set it equal to zero:

$$0 = \frac{d}{d\left(\frac{W_{to}}{S}\right)}\left[\frac{\beta}{\alpha}\left(K_1\frac{\beta}{q}\left(\frac{W_{to}}{S}\right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{to}}{S}\right)}\right)\right] \tag{17}$$

Finally we can get:

$$\frac{W_{TO}}{S} = \frac{q}{\beta}\sqrt{\frac{C_{D_o}}{K_1}} \tag{18}$$

When the dynamic pressure q is very large ($C_D \approx C_{D_0}$), we have:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left(\frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)}\right) \tag{19}$$

And therefore:

$$\left(\frac{T_{SL}}{W_{TO}}\right)\left(\frac{W_{TO}}{S}\right) = \frac{qC_{D_o}}{\alpha} \tag{20}$$

6.2 Case 2: Constant Speed Climb

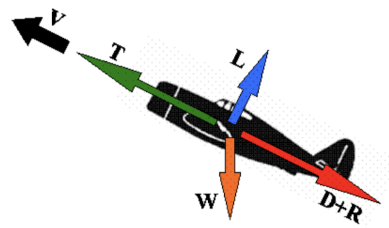

Figure 7: Constant Speed Climb

The assumptions include:

1. $\frac{dV}{dt} = 0$: Constant Speed
2. $n \approx 1$: Lift approximately equals weight
3. $R = 0$: Not on the ground
4. $h, \frac{dh}{dt}, V$: Values are given

After simplification, we get:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha} \left\{ K_1 \frac{\beta}{q} \left(\frac{W_{TO}}{S} \right) + K_2 + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} + \frac{1}{V}\frac{dh}{dt} \right\} \tag{21}$$

Comparing with case 1, case 2 only adds the final term climb rate, which is a constant. The constraint plot is shown below:

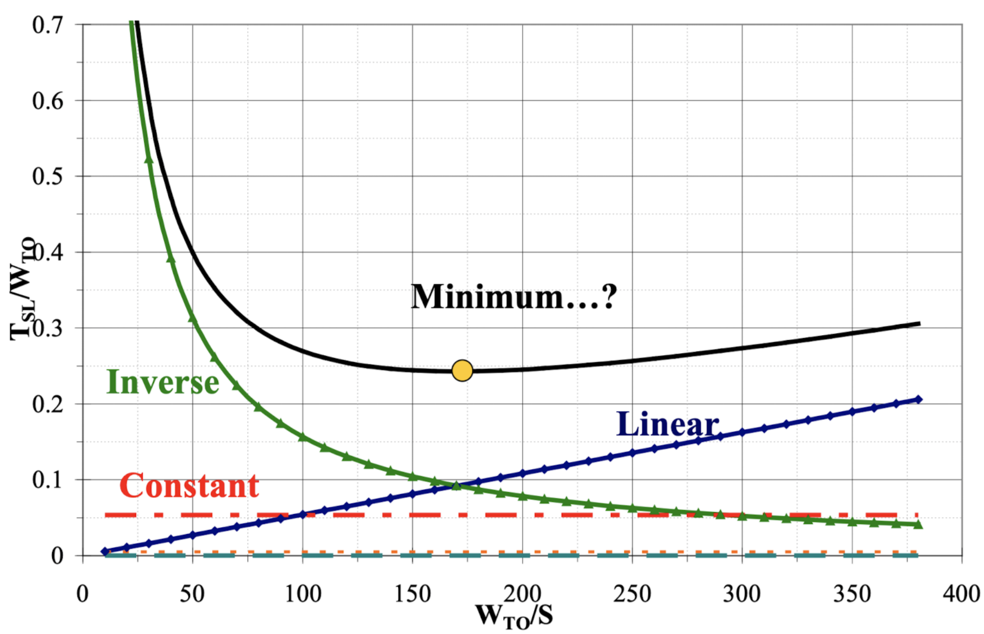

Figure 8: Case 2 Constraint Plot

Because the added term is a constant, the minimum thrust weight ratio appears at the same value of wing loading as case 1, but the thrust weight ratio will be different.

6.3 Case 3: Constant Altitude/Speed Turn

6.3.1 Load Factor

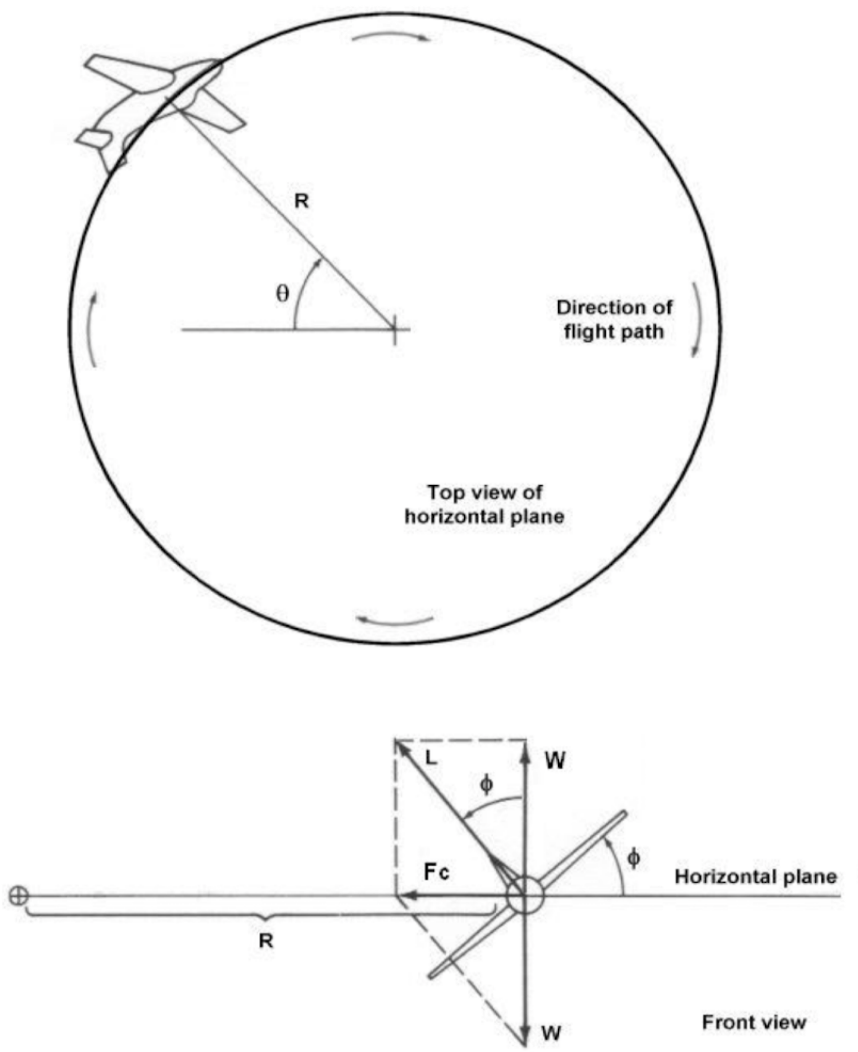

Figure 9: Speed Turn

Here:

1. R: Turn radius
2. L: Lift
3. W: Weight
4. F_c: Centripetal force

5. ϕ: Bank angle

Then, for a turn at constant altitude:

$$W = L\cos\phi \tag{22}$$

Then we define the load factor as:

$$n = \frac{L}{W} \tag{23}$$

Based on the geometry rules, we have:

$$L^2 = F_c^2 + W^2 \tag{24}$$

$$\frac{L^2}{W^2} = \frac{F_c^2}{W^2} + 1 \tag{25}$$

Now we get the expression for load factor:

$$\frac{L}{W} = \sqrt{\frac{F_c^2}{W^2} + 1} \tag{26}$$

Recall that the centripetal force could be expressed as:

$$F_c = ma_c \tag{27}$$

$$n = ((\frac{ma_c}{mg_0})^2 + 1)^{\frac{1}{2}} = ((\frac{a_c}{g_0})^2 + 1)^{\frac{1}{2}} \tag{28}$$

Recall that the definition of a_c:

$$a_c = \omega^2 R = \frac{V^2}{R} = V\omega \tag{29}$$

Therefore we have 2 versions of expressions:

$$n = ((\frac{\omega V}{g_0})^2 + 1)^{\frac{1}{2}} \tag{30}$$

$$n = ((\frac{V^2}{g_0 R})^2 + 1)^{\frac{1}{2}} \tag{31}$$

6.3.2 Assumptions

The assumptions include:

1. $\frac{dh}{dt} = 0$: Constant altitude
2. $\frac{dV}{dt} = 0$: Constant speed
3. $R = 0$: Not on the ground
4. h, n, V: Values are given

6.3.3 Derivation

Now the master equation could be simplified as:

$$\frac{T_{SL}}{W_{to}} = \frac{\beta}{\alpha}\left\{K_1 n^2 \frac{\beta}{q}\left(\frac{W_{to}}{S}\right) + K_2 n + \frac{C_{D_o}}{\frac{\beta}{q}\left(\frac{W_{to}}{S}\right)}\right\} \tag{32}$$

Also take the derivative of the wing loading, we know at the minimum thrust weight ratio, the wing loading expression is a function of load factor:

$$\frac{W_{to}}{S} = \frac{q}{n\beta}\sqrt{\frac{C_{D_o}}{K_1}} \tag{33}$$

The constraint plot is shown below:

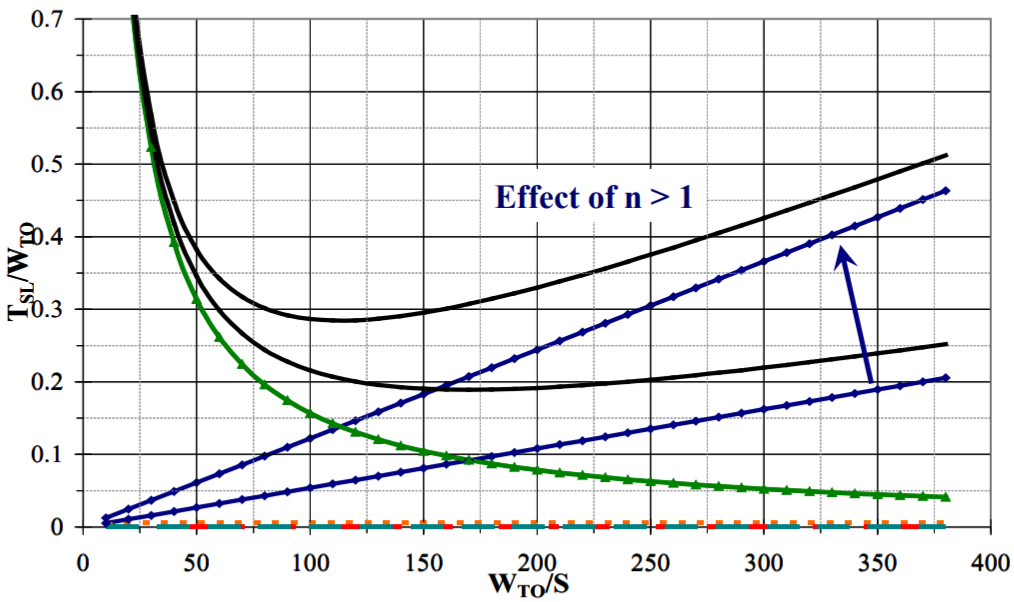

Figure 10: Case 3 Constraint Plot

6.4 Case 4: Horizontal Acceleration

The assumptions of this case include:

1. $n \approx 1$: Lift approximately equals weight
2. $R = 0$: Clean Configuration
3. $dh/dt, h, t$: Values are given

Now the master equation is simplified as:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left[K_1 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D0}}{\frac{\beta}{q}\left(\frac{W_{TO}}{S}\right)} + \frac{1}{g_0}\frac{dV}{dt}\right] \tag{34}$$

The additional constant term is to define an acceleration constraint:

$$\frac{1}{g_0}\frac{dV}{dt} = \frac{1}{g_0}\left(\frac{V_{\text{Final}} - V_{\text{Initial}}}{\Delta t_{\text{Allowable}}}\right) \tag{35}$$

The constraint plot is shown below:

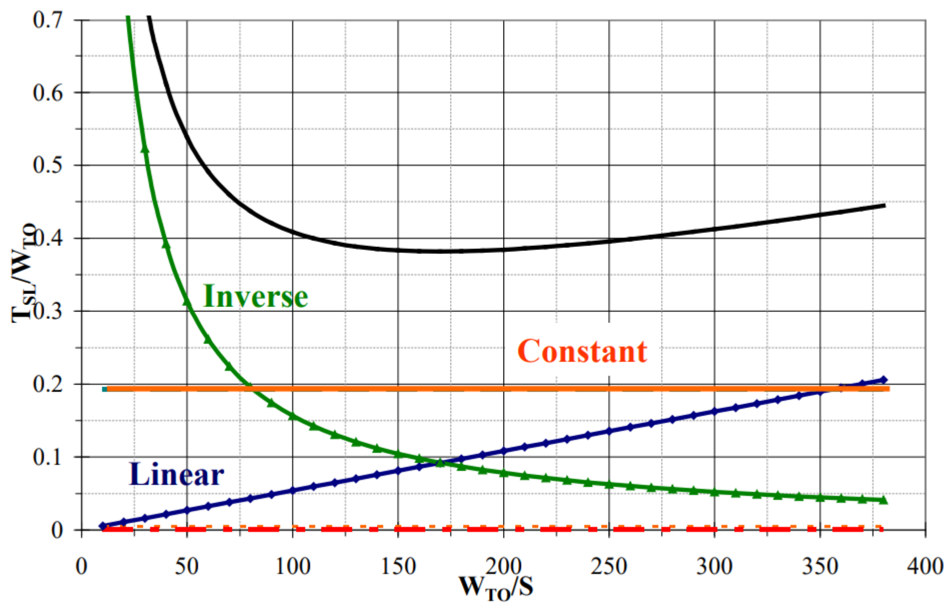

Figure 11: Case 4 Constraint Plot

6.5 Case 5: Takeoff Ground Roll (lots of thrust)

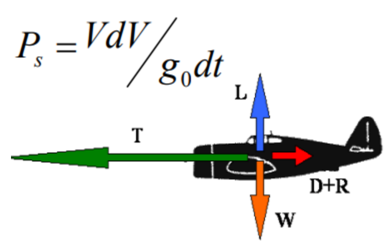

Figure 12: Case 5

The assumptions of this case include:

1. $\frac{dh}{dt} = 0$: Constant altitude
2. $R \neq 0$: Not clean configuration + Ground
3. Thrust is very large

At this case, we assume that thrust at take off is much larger than the drag, so the drag-related terms can be eliminated. Therefore, the master equation is simplified to:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha g_0}\frac{dV}{dt} = \frac{\beta}{\alpha g_0}\frac{dV}{ds/V} \tag{36}$$

Solving for ds yields:

$$ds = \frac{\beta}{\alpha g_o}\left(\frac{W_{TO}}{T_{SL}}\right) V\,dV \tag{37}$$

If the **takeoff ground roll distance** s_G and the **takeoff velocity** V_{TO} are given, then start from $s = 0$, $V = 0$ to do the integration we get:

$$s_G = \frac{\beta}{\alpha}\left(\frac{W_{TO}}{T_{SL}}\right)\frac{V_{TO}^2}{2g_o} \tag{38}$$

Usually take off velocity is defined through the stall velocity, with a safety constant $k_{TO} > 1$ (usually 1.2 - 1.3):

$$V_{TO} = k_{TO} V_{STALL} \tag{39}$$

Using the maximum lift conditions, we get:

$$qC_{L_{max}}S = \frac{1}{2}\rho V_{STALL}^2 C_{L_{max}} S = \beta W_{TO} \tag{40}$$

Combine all the equations together, we get:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta^2}{\alpha}\frac{k_{TO}^2}{s_G \rho g_o C_{L_{max}}}\left(\frac{W_{TO}}{S}\right) \tag{41}$$

6.6 Case 6: Takeoff Ground Roll (not so much thrust)

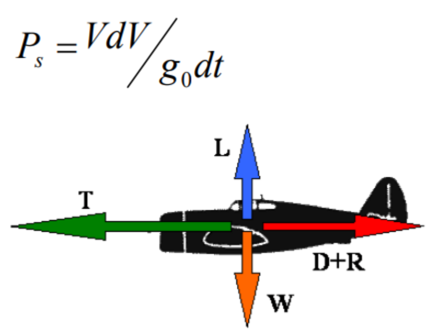

Figure 13: Case 6

The assumptions of this case include:

1. $\frac{dh}{dt} = 0$ Constant altitude
2. $n \approx 1$: Lift approximately equals weight

3. $D = qC_D S$: Drag calculation

Based on this assumption, the master equation could be simplified to:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left(\frac{D+R}{\beta W_{TO}} + \frac{1}{V}\frac{d}{dt}\left(\frac{V^2}{2g_o}\right)\right) \tag{42}$$

In this case, because thrust is in the order of magnitude of the drag and resistance forces, so we express the resistance due to ground friction and non-clean configuration:

$$R = qC_{DR}S + \mu_{TO}(\beta W_{TO} - qC_L S) \tag{43}$$

Combine with the drag term:

$$\frac{D+R}{\beta W_{TO}} = \left(C_D + C_{DR} - \mu_{TO}C_L\right) qS + \mu_{TO}\frac{\beta W_{TO}}{\beta W_{TO}} \tag{44}$$

Rearrange:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{\xi_{TO}\frac{q}{\beta}\left(\frac{S}{W_{TO}}\right) + \mu_{TO} + \frac{1}{g_o}\frac{dV}{dt}\right\} \tag{45}$$

Where:

$$\xi_{TO} = (C_D + C_{DR} - \mu_{TO}C_L) \tag{46}$$

Same integration as Case 5, we get:

$$s_G = -\frac{\beta\left(\frac{W_{TO}}{S}\right)}{\rho g_o \xi_{TO}} \ln\left[1 - \frac{\xi_{TO}}{\left[\frac{\alpha}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right) - \mu_{TO}\right]\frac{C_{L_{max}}}{k_{TO}^2}}\right] \tag{47}$$

In the limit as all terms in ξ_{TO} go to zero, and $ln(1-\epsilon)$ goes to $-\epsilon$, then we have the expression:

$$s_G \to \frac{\beta\left(\frac{W_{TO}}{S}\right)}{\rho g_o \xi_{TO}}\left[\frac{\xi_{TO}}{\frac{\alpha}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right)\frac{C_{L_{max}}}{k_{TO}^2}}\right] \tag{48}$$

Then we can also solve:

$$\frac{T_{SL}}{W_{TO}} \to \frac{\beta^2}{\alpha}\frac{k_{TO}^2}{s_G \rho g_o C_{L_{max}}}\left(\frac{W_{TO}}{S}\right) \tag{49}$$

6.6.1 Obstacle not cleared during the transition

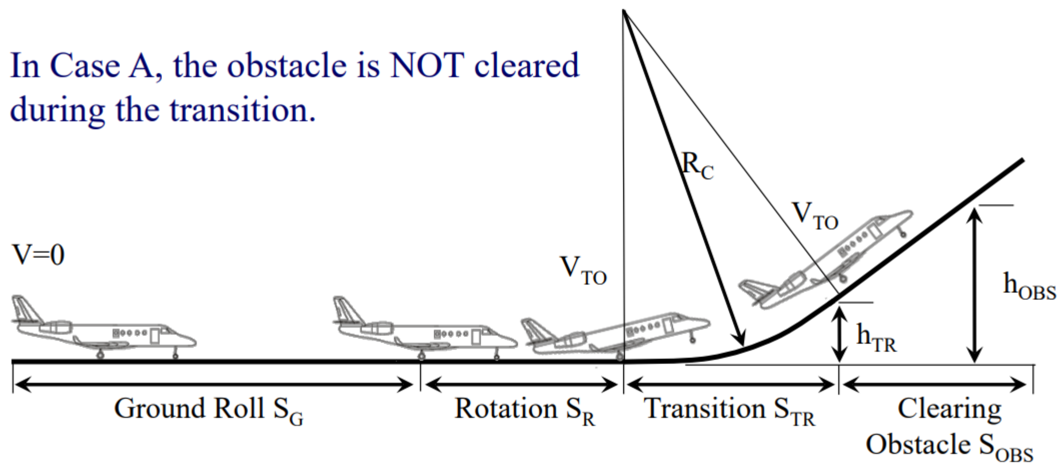

Figure 14: Case A

Now we take a look at each distance.

1. **Rotation Distance:** Distance to rotate is calculated as the product of rotation time and velocity during rotation:

$$s_R = t_R V_{TO} \tag{50}$$

Normally the time is 3 seconds for modern aircraft. The take off velocity could also be expressed in terms of the definition of lift. For takeoff lift:

$$L_{TO} = \frac{1}{2}\rho V_{TO}^2 S C_{L-TO} = W \tag{51}$$

For maximum lift:

$$V_{TO} = k_{TO}\sqrt{\left(\frac{W_{TO}}{S}\right)\frac{2\beta}{\rho C_{L_{max}}}} \tag{52}$$

Then the rotation distance could be expressed as:

$$s_R = t_R V_{TO} = t_R k_{TO}\sqrt{\left(\frac{2\beta}{\rho C_{L_{max}}}\right)\left(\frac{W_{TO}}{S}\right)} \tag{53}$$

2. **Transition Distance**: is defined as the distance that necessary to bring the aircraft to its climb angle:

$$s_{TR} = R_C \sin\theta_{CL} \tag{54}$$

The radius of the transition arch could be expressed in terms of takeoff velocity:

$$R_C = \frac{V_{TO}^2}{g_0(0.8k_{TO}^2 - 1)} \tag{55}$$

Therefore, the transition distance could be expressed as:

$$s_{TR} = \frac{k_{TO}^2 \sin\theta_{CL}}{g_o(0.8k_{TO}^2 - 1)} \frac{2\beta}{\rho C_{L_{max}}} \frac{W_{TO}}{S} \tag{56}$$

3. **Distance to clear obstacle:** Outside the turn to climb, the flight path is a **straight line** at an angle to the ground defined by the vehicle climb angle. Therefore:

$$S_{obs} = \frac{h_{obs} - h_{TR}}{\tan\theta_{CL}} \tag{57}$$

The climb angle is defined by climb rate and velocity:

$$\frac{1}{V}\frac{dh}{dt} = \sin\theta_{CL} = \frac{T - D}{W} \tag{58}$$

Altitude at the end of transition could be expressed as:

$$h_{TR} = \frac{V_{TO}^2(1 - \cos\theta_{CL})}{g_o(0.8k_{TO}^2 - 1)} = \frac{k_{TO}^2(1 - \cos\theta_{CL})}{g_o(0.8k_{TO}^2 - 1)} \frac{2\beta}{\rho C_{L_{max}}} \frac{W_{TO}}{S} \tag{59}$$

6.6.2 Obstacle cleared during the transition

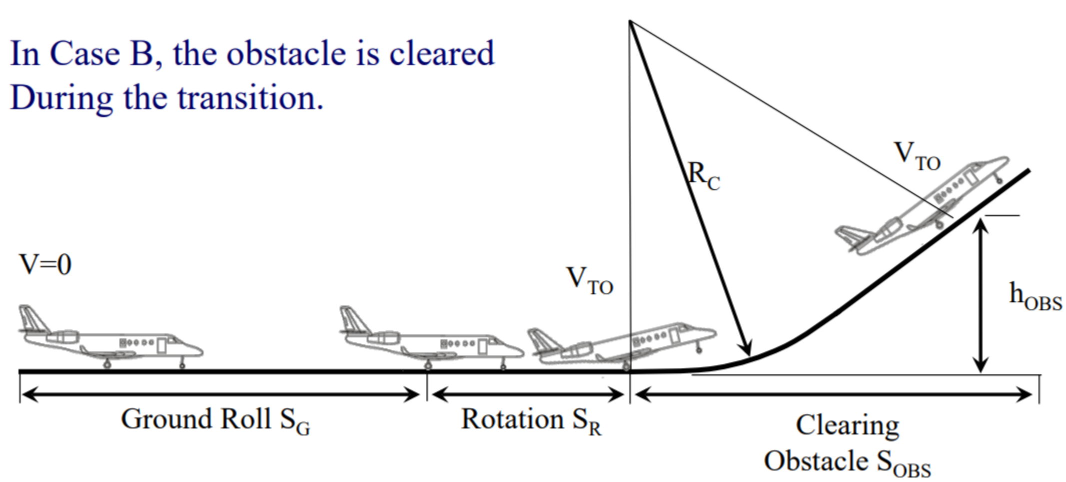

Figure 15: Case B

The total takeoff distance expression is the same:

$$S_{TO} = S_G + S_R + S_{obs} \tag{60}$$

The distance to required height expression is nearly the same:

$$S_{obs} = R_C \sin\theta_{obs} = \frac{V_{TO}^2 \sin\theta_{obs}}{g_o(0.8k_{TO}^2 - 1)} \tag{61}$$

Where:

$$\theta_{obs} = \cos^{-1}\left(1 - \frac{h_{obs}}{R_C}\right) \tag{62}$$

6.7 Case 7: Braking Roll

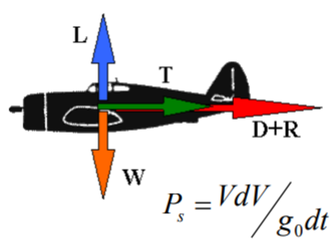

Figure 16: Case 7

The assumptions for this case:

1. $\frac{dh}{dt} = 0$: Constant altitude
2. $n \approx 1$: Lift approximately equals weight
3. $D = qC_DS$

Then, the master equation is simplified as:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left(\frac{D+R}{\beta W_{TO}} + \frac{1}{V}\frac{d}{dt}\left(\frac{V^2}{2g_o}\right)\right) \tag{63}$$

In this case, the resistance expression due to ground friction/braking and aerodynamic braking is:

$$R = qC_{DR}S + \mu_B(BW_{TO} - qC_LS) \tag{64}$$

Then the complete drag term could be re-written as:

$$\frac{D+R}{\beta W_{TO}} = \frac{(C_D + C_{DR} + \mu_B C_L)qS + \mu_B \beta W_{TO}}{\beta W_{TO}} \tag{65}$$

Then the master equation could be rearranged as:

$$T_{SL} = \frac{\beta}{\alpha}\left(\xi_L \frac{q}{\beta}\left(\frac{S}{W_{TO}}\right) + \mu_B + \frac{1}{g_o}\frac{dV}{dt}\right) \tag{66}$$

Here, we have:

$$\xi_L = (C_D + C_{DR} - \mu_B C_L) \tag{67}$$

Similar with case 6, now we integrate over $s = 0, V = V_{TD}$ (touch down velocity) to full stop conditions $s = s_B, V = 0$. The final expression is:

$$S_B = \frac{\beta}{\rho g (C_D + C_{DR} - \mu_B C_L)}\left(\frac{W_{TO}}{S}\right) \ln\left[1 + \left(\frac{(C_D + C_{DR} - \mu_B C_L)}{[\frac{(-\alpha)}{\beta}\left(\frac{T_{SL}}{W_{TO}}\right) + \mu_B]\frac{C_{L_{max}}}{k_{TD}^2}}\right)\right] \tag{68}$$

In this case, β will be the value at the end of the mission. $-\alpha$ is used to model thrust reversers, this value of α should be representative of the fraction of thrust that is effective reversed, usually 0.65-0.85 for commercial airliners.

If the thrust reverser $(-\alpha)$ is large enough, the expression will be simplified as:

$$S_B \to \left(\frac{\beta(W_{TO}/S)}{\rho g_0 \xi_L}\right)\left(\frac{\xi_L}{\frac{(-\alpha)T_{SL}}{\beta W_{TO}}\frac{C_{L_{max}}}{k_{TD}^2}}\right) \tag{69}$$

In this case, we can solve the constraint ratio as:

$$\frac{T_{SL}}{W_{TO}} \to \left(\frac{\beta^2}{-\alpha}\right)\left(\frac{k_{TD}^2}{S_B \rho g_0 C_{L_{max}}}\right)\left(\frac{W_{TO}}{S}\right) \tag{70}$$

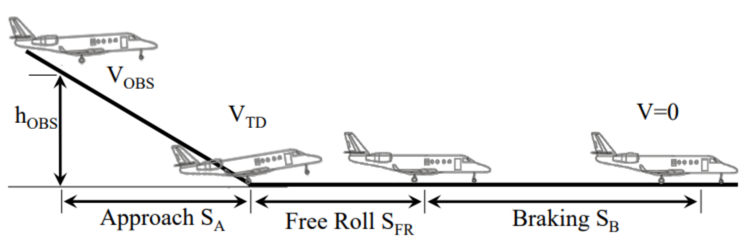

Figure 17: Total Landing Distance

The total landing distance is calculated as the sum of the approach distance, the free roll distance and the braking roll distance (discussed). k_{TD} is defined from the preceding segment.

1. **Approach Distance**: Given the **landing clearance obstacle** height h_{obs}, **safety factor for approach** k_{obs}, and the **safety factor for touch down** k_{TD}, we have the expression as:

$$S_A = \frac{2\beta}{\rho g_0 (C_D + C_{DR})} \left(\frac{W_{TO}}{S}\right) \left(\frac{k_{obs}^2 - k_{TD}^2}{k_{obs}^2 + k_{TD}^2}\right) + \frac{C_{L_{max}}}{C_D + C_{DR}} \frac{2h_{obs}}{k_{obs}^2 + k_{TD}^2} \tag{71}$$

Notice here k_{obs} defines the velocity when clearing the landing obstacle height V_{obs} in terms of the stall velocity V_{STALL}. V_{obs} and V_{app} are used interchangeably:

$$V_{obs} = k_{obs} V_{STALL} = V_{app} = k_{app} V_{STALL} \tag{72}$$

Recall the expression of V_{STALL}:

$$V_{app} = k_{app} V_{stall} = k_{app} \sqrt{\frac{2\beta}{\rho C_{L_{max}}} \left(\frac{W_{TO}}{S}\right)} \tag{73}$$

2. **Free Roll Distance**

   Usually the difference between approach and touchdown velocity is very small, but touchdown velocity could be expressed using stall velocity and k_{TD}:

$$V_{TD} = k_{TD} V_{stall} = k_{TD} \sqrt{\frac{2\beta}{\rho C_{L_{max}}} \left(\frac{W_{TO}}{S}\right)} \tag{74}$$

   If we assume the velocity at touchdown does not diminish during free roll, the free roll distance is calculated as a product of the free roll time (usually 3s) and the touch down velocity:

$$S_{FR} = t_{FR} V_{TD} = t_{FR} k_{TD} \sqrt{\frac{2\beta}{\rho C_{L_{max}}} \left(\frac{W_{TO}}{S}\right)} \tag{75}$$

6.8 Case 8: Service Ceiling

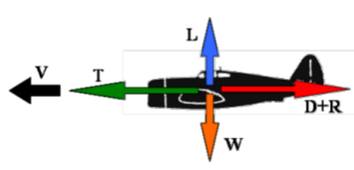

Figure 18: Case 8

Service ceiling is the altitude at which an aircraft's maximum climb rate has a specific value. The assumptions of this case include:

1. $\frac{dV}{dt} = 0$: No acceleration

2. $n = 1$: Straight and level
3. $R = 0$: Clean configuration
4. $P_s = \frac{dh}{dt} > 0$: value is given
5. h, C_L are given

The master equation is transferred to:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left[K_1 \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) + K_2 + \frac{C_{D0}}{\frac{\beta}{q}\left(\frac{W_{TO}}{qS}\right)} + \frac{1}{V}\frac{dh}{dt}\right] \tag{76}$$

The operating conditions for service ceiling may be determined by lift coefficient or by velocity for a **given altitude** and **climb rate**. If V is known:

$$C_L = \frac{\beta}{q}\left(\frac{W_{TO}}{S}\right) = \frac{\beta}{\frac{1}{2}\rho V^2}\left(\frac{W_{TO}}{S}\right) \tag{77}$$

If C_L is known:

$$V = \sqrt{\frac{2\beta}{\sigma \rho_{SL} C_L}\left(\frac{W_{TO}}{S}\right)} \tag{78}$$

When V and C_L are known, the equation could be reduced as:

$$\frac{T_{SL}}{W_{TO}} = \frac{\beta}{\alpha}\left\{K_1 C_L + K_2 + \frac{C_{D0}}{C_L} + \left(\frac{1}{V}\frac{dh}{dt}\right)\right\} \tag{79}$$