Normal Shock

1 Shock

1.1 Overview

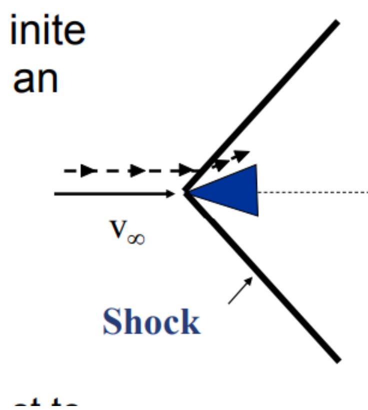

Figure 1: Shock

Recall the situation that **finite size body is moving faster than speed of sound**, then the fluid upstream **can not know body is coming before it gets there,** because the pressure information travels at speed of sound. So flow must suddenly adjust to presence of body, this adjustment occurs through the presence of a **shock**.

1.2 Sound Waves vs Shock Waves

Sound waves are weak, with minor density change ($d\rho/\rho \approx 0$). Also, they are **reversible and isentropic.**

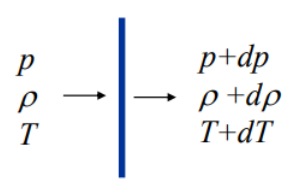

Figure 2: Sound Wave

Shock waves have strong compression ($\rho_2 > \rho_1$). The region for shock is very thin, so the changes in fluid properties are **nearly discontinuous.** The rapid change in properties is due to **internal viscous stresses**, so shock wave is **irreversible**. If we exclude the radiation, we can assume the shock wave is **adiabatic**. **Adiabatic + Irreversible = Nonisentropic.**

1.3 Formation

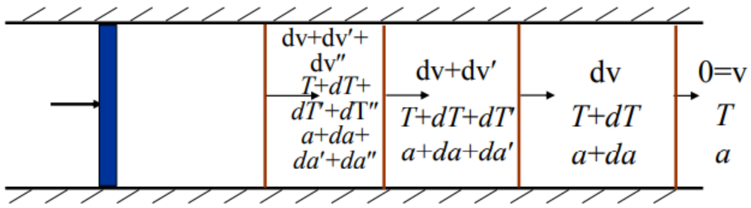

Figure 3: Shock Formation

Imagine we suddenly move the piston in tube in a very high speed, then it will produce **a series of discrete 1D compression waves**. Each pulse of piston produces **weak compression** wave travelling at **speed of sound** in moving gas in front of it. Each wave travels in **wake** of previous waves, even though they are all in speed of sound, **each wave travels slightly faster** because after the wave passing, the temperature will increase, so the velocity and speed of sound will increase too.

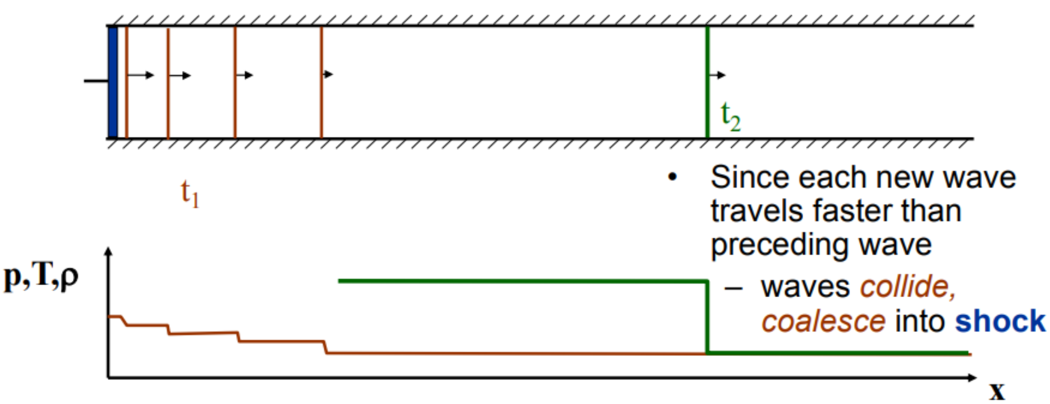

Figure 4: Coalescence of Compression Wave

For the **expansion waves**, for example successive waves see colder gas with lower a, **each new wave is slower than last.** This time, they **can not create shock** because they could not coalesce.

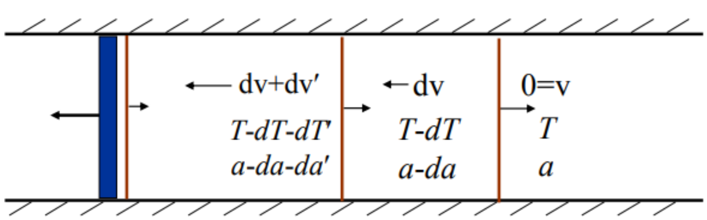

Figure 5: Expansion Wave

2 Normal Shock (Static Properties)

2.1 Definition

The shock with wave is **perpendicular to flow direction**.

2.2 Governing Equations

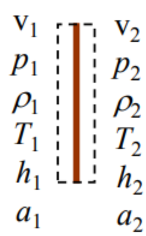

Figure 6: Problem Setup

Assumptions:

1. 1D
2. Stationary Shock (Steady)
3. Inviscid except inside shock
4. Adiabatic
5. Only flow work
6. Shock is non-equilibrium process internally, but we assume **flow before (1) and after (2) shock are in equilibrium.**

Then we have **mass conservation:**

$$\boxed{\frac{\dot{m}}{A} = \rho_1 u_1 = \rho_2 u_2} \tag{1}$$

Momentum Conservation:

$$p_1 A - p_2 A = \dot{m}_2 u_2 - \dot{m}_1 u_1 \tag{2}$$

So:

$$\boxed{p_1 - p_2 = \frac{\dot{m}}{A}(u_2 - u_1)} \tag{3}$$

Also:

$$p_1 A - p_2 A = \rho_2 u_2^2 A - \rho_1 u_1^2 A \tag{4}$$

So:

$$\boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2} \tag{5}$$

Energy Conservation:

$$\boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} = h_o} \tag{6}$$

Perfect Gas State Equations:

$$\boxed{p = \rho R T} \tag{7}$$

$$\boxed{dh = c_p dT} \tag{8}$$

$$\boxed{a^2 = \gamma R T} \tag{9}$$

So we have 6 equations, 6 unknowns (all the properties after the shock, properties before the shock are all known.)

2.3 Shock Hugoniot Equation

From energy equation:

$$h_2 - h_1 = \frac{1}{2}(u_1^2 - u_2^2) = \frac{1}{2}(u_1 - u_2)(u_1 + u_2) \tag{10}$$

Recall the momentum equation:

$$(u_1 - u_2) = \frac{p_2 - p_1}{\frac{\dot{m}}{A}} \tag{11}$$

And mass equation:

$$(u_1 + u_2) = (\frac{1}{\rho_1} + \frac{1}{\rho_2})\frac{\dot{m}}{A} \tag{12}$$

Therefore we can get the **Shock Hugoniot Equation:**

$$\boxed{h_2 - h_1 = \frac{1}{2}(p_2 - p_1)(\frac{1}{\rho_1} + \frac{1}{\rho_2})} \tag{13}$$

2.4 Entropy Change

For TPG/CPG, entropy state equation is:

$$s_2 - s_1 = c_v \ln \frac{T_2}{T_1} - R \ln \frac{\rho_2}{\rho_1} \tag{14}$$

Recall that:

$$c_v = \frac{R}{\gamma - 1} \tag{15}$$

Therefore:

$$\frac{s_2 - s_1}{c_v} = \ln [\frac{T_2}{T_1}(\frac{\rho_2}{\rho_1})^{-(\gamma-1)}] = \ln [\frac{p_2/\rho_2}{p_1/\rho_1}(\frac{\rho_2}{\rho_1})^{1-\gamma}] \tag{16}$$

Finally we have:

$$\boxed{\frac{s_2 - s_1}{c_v} = \ln [\frac{p_2}{p_1}(\frac{\rho_2}{\rho_1})^{-\gamma}]} \tag{17}$$

Entropy change as a function of pressure and density ratios across shock.

2.5 Velocity Ratio

$$\frac{u_2}{u_1} = \frac{M_2 a_2}{M_1 a_1} \tag{18}$$

$$\boxed{\frac{u_2}{u_1} = \frac{M_2}{M_1}\sqrt{\frac{T_2}{T_1}}} \tag{19}$$

2.6 Density Ratio

From mass conservation:

$$\rho_1 u_1 = \rho_2 u_2 \tag{20}$$

Therefore:

$$\boxed{\frac{\rho_2}{\rho_1} = \frac{u_1}{u_2} = \frac{M_1}{M_2}\sqrt{\frac{T_1}{T_2}}} \tag{21}$$

2.7 Temperature Ratio

$$h_{o1} = h_{o2} \rightarrow T_{o1} = T_{o2} \tag{22}$$

Therefore:

$$\frac{T_2}{T_1} = \frac{T_2/T_{o2}}{T_1/T_{o1}} \frac{T_{o2}}{T_{o1}} \tag{23}$$

$$\boxed{\frac{T_2}{T_1} = \frac{(1 + \frac{\gamma-1}{2} M_1^2)}{(1 + \frac{\gamma-1}{2} M_2^2)}} \tag{24}$$

2.8 Pressure Ratio

Recall the momentum equation:

$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{25}$$

Notice that:

$$\rho u^2 = \rho(M^2 a^2) = \rho(M^2 \gamma RT) = \rho(M^2 \gamma \frac{p}{\rho}) = p\gamma M^2 \tag{26}$$

Therefore:

$$p_1 + p_1 \gamma M_1^2 = p_2 + p_2 \gamma M_2^2 \tag{27}$$

$$\boxed{\frac{p_2}{p_1} = \frac{(1 + \gamma M_1^2)}{(1 + \gamma M_2^2)}} \tag{28}$$

2.9 Mach Number

Now the most tricky but most important relation appears. From mass conservation:

$$\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1} \tag{29}$$

For the LHS:

$$\frac{u_1}{u_2} = \frac{M_1 \sqrt{\gamma R T_1}}{M_2 \sqrt{\gamma R T_2}} \tag{30}$$

$$\frac{T_2}{T_1} = \frac{(1 + \frac{\gamma-1}{2} M_1^2)}{(1 + \frac{\gamma-1}{2} M_2^2)} \tag{31}$$

For the RHS:

$$\frac{\rho_2}{\rho_1} = \frac{p_2/RT_2}{p_1/RT_1} \tag{32}$$

$$\frac{p_2}{p_1} = \frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)} \tag{33}$$

Combining all these equation, after some magics, we get:

$$\boxed{\frac{M_2}{1+\gamma M_2^2}\sqrt{1+\frac{\gamma-1}{2}M_2^2} = \frac{M_1}{1+\gamma M_1^2}\sqrt{1+\frac{\gamma-1}{2}M_1^2}} \tag{34}$$

Solving this equation, we get:

$$\boxed{M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1}} \tag{35}$$

In shock's reference frame, flow before normal shock (M_1) is always supersonic, the flow after normal shock (M_2) is always subsonic.

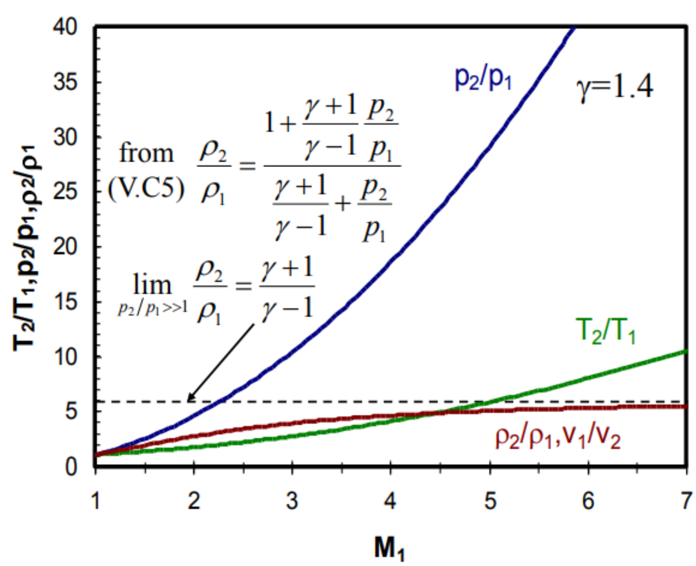

Figure 7: Property Ratios

Property ratios summary:

1. T, p, ρ **increase** across shock and ratios increase with M_1
2. u **decreases**
3. p ratio increase across normal shock is greatest static property change

3 Normal Shock (Stagnation Properties)

Now we want to know the stagnation properties relations.

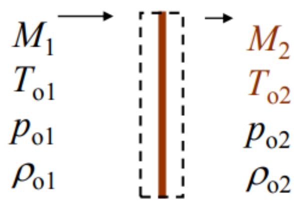

Figure 8: Stagnation Properties

3.1 Stagnation Temperature

Due to energy conservation, we know:

$$\boxed{T_{o2} = T_{o1}} \tag{36}$$

3.2 Stagnation Pressure

$$\frac{p_{o2}}{p_{o1}} = \frac{p_{o2}/p_2}{p_{o1}/p_1}\frac{p_2}{p_1} = \left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right)^{\frac{\gamma}{\gamma-1}}\frac{(1+\gamma M_1^2)}{(1+\gamma M_2^2)} \tag{37}$$

Recall that:

$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1}M_1^2 - 1} \tag{38}$$

After some magic math, we get:

$$\boxed{\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}} \tag{39}$$

$$\boxed{\frac{p_{o2}}{p_{o1}} = \left[\frac{\frac{\gamma+1}{2}M_1^2}{1+\frac{\gamma-1}{2}M_1^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{2\gamma}{\gamma+1}M_1^2 - \frac{\gamma-1}{\gamma+1}\right]^{\frac{1}{1-\gamma}}} \tag{40}$$

There are also some other useful expressions. Recall that:

$$\frac{T_2}{T_1} = \frac{(1+\frac{\gamma-1}{2}M_1^2)}{(1+\frac{\gamma-1}{2}M_2^2)} \tag{41}$$

Therefore we have:

$$\boxed{\frac{p_{o2}}{p_{o1}} = \left(\frac{T_1}{T_2}\right)^{\frac{\gamma}{\gamma-1}}\frac{p_2}{p_1}} \tag{42}$$

Also we know:

$$\frac{p_{o2}}{p1} = \frac{p_{o2}}{p2}\frac{p_2}{p_1} \tag{43}$$

So we have:

$$\boxed{\frac{p_{o2}}{p_1} = (1 + \frac{\gamma - 1}{2}M_2^2)^{\frac{\gamma}{\gamma-1}}\frac{p_2}{p_1}} \tag{44}$$

3.3 Stagnation Density

$$\frac{\rho_{o2}}{\rho_{o1}} = \frac{p_{o2}/RT_{o2}}{p_{o1}/RT_{o1}} \tag{45}$$

We know the stagnation temperature is constant, so:

$$\boxed{\frac{\rho_{o2}}{\rho_{o1}} = \frac{p_{o2}}{p_{o1}}} \tag{46}$$

3.4 Sonic Area Ratio

Recall the mass flow rate expression:

$$\cancelto{1}{\frac{\dot{m}_2}{\dot{m}_1}} = \frac{A_2^*}{A_1^*}\frac{p_{o2}/\sqrt{RT_{o2}}}{p_{o1}/\sqrt{RT_{o1}}}\cancelto{1}{\frac{f(\gamma)}{f(\gamma)}} \tag{47}$$

Therefore:

$$\boxed{\frac{A_2^*}{A_1^*} = \frac{p_{o1}}{p_{o2}}} \tag{48}$$

3.5 Summary

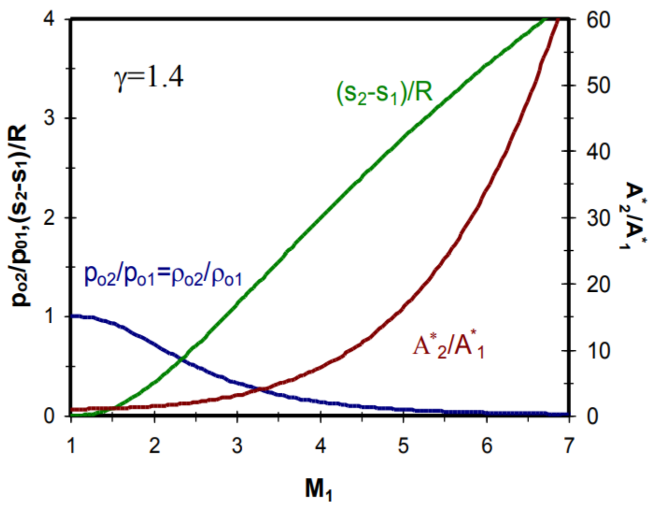

Figure 9: Stagnation Properties Ratio

Some remarks:

1. Stagnation temperature constant
2. Stagnation pressure and stagnation density **drop**
3. Entropy **increases**
4. **Sonic area increases**, so larger throat required after shock to reach sonic flow. (They have same mass flow rate, but after the shock the stagnation pressure drops, $\dot{m} \propto A^* p_o$)

4 Moving Normal Shock

So far, we focus on the changes across shock wave for the case of the shock **not moving**, which means observer 'sitting' on the shock, moving with shock:

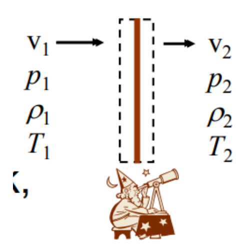

Figure 10: Stationary Shock

Now we want to consider the shock to be moving, so the observer not moving at same speed as shock:

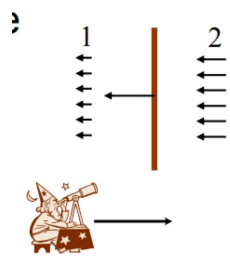

Figure 11: Moving Shock

To achieve this, we need to perform **Galilean Transform** to convert moving shock to stationary shock:

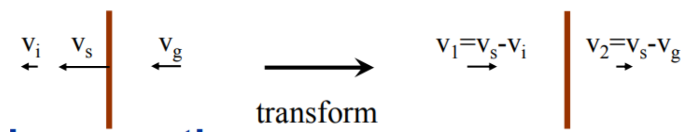

Figure 12: Galilean Transform

After the transformation, **static properties** are not affected, we can still use all the equations, but with:

$$M_1 = \frac{v_s - v_i}{a_i} \tag{49}$$

$$M_2 = \frac{v_s - v_g}{a_g} \tag{50}$$

However, the **stagnation properties** depend on velocity, not the same after transform.

5 Reflected Normal Shocks

When moving shock runs into a boundary