Stagnation

1 Bernoulli Equation

Recall the derivation of Naiver Stokes Equation (Here), if we assume the only body force is gravitational force, and assume the flow is inviscid, then we get the 3D **Euler Equation**:

$$\rho \frac{D\underline{\boldsymbol{u}}}{Dt} = \rho \underline{\boldsymbol{g}} - \nabla p \tag{1}$$

Under the Euler's Equation condition, if the flow is steady, **isodensity**, frictionless, one dimensional and ignore the gravitational force, the integration along a streamline between any points 1 and 2 will give the **Bernoulli Equation**:

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} \tag{2}$$

Integration:

$$\frac{1}{2}\rho u^2 + p = const \tag{3}$$

$$(p + \frac{1}{2}\rho V^2)_1 = (p + \frac{1}{2}\rho V^2)_2 \tag{4}$$

2 Isodensity Stagnation Properties

Based on Bernoulli Equation, we define the **stagnation pressure** as:

$$p_o = p + \frac{1}{2}\rho u^2 \tag{5}$$

Similarly, we define **stagnation enthalpy** as:

$$h_o = h + \frac{1}{2}u^2 \tag{6}$$

Notice that stagnation enthalpy is constant for steady, adiabatic flow without work. From this, we can derive the **stagnation temperature** formula:

$$c_p T_o = c_p T + \frac{1}{2}u^2 \tag{7}$$

$$T_o = T + \frac{u^2}{2c_p} \tag{8}$$

If we require constant enthalpy for Euler Equation, then:

$$dh = d(h_o - \frac{1}{2}u^2) = -d(\frac{1}{2}u^2) \tag{9}$$

Also we have:

$$\rho d(\frac{1}{2}u^2) + dp = 0 \tag{10}$$

$$-\rho dh + dp = 0 \tag{11}$$

$$dh - \frac{dp}{\rho} = 0 \tag{12}$$

Recall the Gibbs Equation:

$$dh - \frac{d\rho}{p} = Tds \tag{13}$$

Then the conditions for **Euler Equation (inviscid, no external work and adiabatic)** actually require the flow to be **isentropic**! Because one way to achieve isentropic is **reversible (no entropy production)** and **adiabatic (no entropy transfer)**. Inviscid and no friction assumptions can make the process reversible.

3 Non-Isodensity Stagnation Properties

Notice that the previous stagnation pressure is defined **under the Bernoulli equation assumptios, which requires isodensity.** Now we want to find the general expression for the stagnation pressure. Here we define p_o as **pressure if we slow a flow to zero velocity in a process without external work, heat transfer or entropy production, in other word, isentropic.**

Recall the isentropic relation for TPG:

$$\frac{p_2}{p_1} = e^{\frac{1}{R}\int_{T_1}^{T_2}\frac{c_p dT}{T}} \tag{14}$$

Replace the temperature with stagnation temperature:

$$\frac{p_o}{p} = exp(\int_{T}^{T_0}\frac{(c_p/R)dT}{T}) \tag{15}$$

If we assume CPG:

$$\frac{p_o}{p} = (\frac{T_o}{T})^{\frac{c_p}{R}} = (\frac{T_o}{T})^{\frac{\gamma}{\gamma-1}} \tag{16}$$

Recall the definition of stagnation temperature (this does not need to assume isodensity)

$$\frac{T_o}{T} = 1 + \frac{u^2}{2c_pT} \tag{17}$$

Then we get:

$$\frac{p_o}{p} = (1 + \frac{u^2}{2c_pT})^{\gamma/(\gamma-1)} \tag{18}$$

Recall that:

$$c_p = R\frac{\gamma}{\gamma - 1} \tag{19}$$

$$\frac{T_o}{T} = 1 + \frac{\gamma - 1}{2}\frac{u^2}{\gamma RT} \tag{20}$$

Finally we have:

$$\frac{p_o}{p} = (1 + \frac{\gamma - 1}{2}\frac{u^2}{\gamma RT})^{\gamma/(\gamma-1)} \tag{21}$$

4 Isodensity vs Incompressible

In summary, for CPG:

$$\frac{p_o}{p} = (1 + \frac{\gamma - 1}{\gamma}\frac{1}{2}\frac{\rho}{p}u^2)^{\gamma/(\gamma-1)} \tag{22}$$

and the assumptions include:

1. No external work
2. Adiabatic
3. Reversible

And for Bernoulli Equation:

$$\frac{p_o}{p} = 1 + \frac{1}{2}\frac{\rho}{p}u^2 \tag{23}$$

and the assumptions include:

1. No body forces
2. Inviscid
3. Isodensity

Comparing these two expression, we can conclude that **isodensity requires:**

1. Small p and T changes
2. Adiabatic
3. Negligible KE changes, so low speed

However, **is isodenstiy the same as incompressible? No!** Recall the continuity equation:

$$\frac{\partial \rho}{\partial t} + \underline{\boldsymbol{u}} \cdot \nabla \rho + \rho(\nabla \cdot \underline{\boldsymbol{u}}) = 0 \tag{24}$$

If a flow is incompressible, **density is constant within a small element volume, dV, which moves at the flow velocity u.** In other words, this constraint implies that the material derivative of the density must vanish to ensure incompressible flow.

Therefore we have:

$$\nabla \cdot \underline{\boldsymbol{u}} = 0 \tag{25}$$

We can also define the incompressible flow using compressibility:

$$\beta = \frac{1}{\rho}\frac{d\rho}{dp} \approx 0 \tag{26}$$

Later, we will prove the speed of sound is defined as:

$$a^2 = \frac{\partial p}{\partial \rho}|_s \tag{27}$$

Notice that incompressible is only an ideal assumption, which is not real. Therefore, if we assume incompressible, we are assuming **speed of sound approaching infinity**, which means the ratio of velocity to the speed of sound approaching zero, in other words, **low Mach number.** Therefore, **we can say that low-Mach number flows can be approximated as "incompressible".**

However, if we assume the flow is incompressible, we could not say the **density is constant with time or space.** Recall the material derivative of density:

$$\frac{\partial \rho}{\partial t} + \underline{\boldsymbol{u}} \cdot \nabla \rho = 0 \tag{28}$$

As long as the total derivative is zero, the flow is incompressible. We do not need to let every term be zero, as long as they cancel out.Therefore, an incompressible flow **could be unsteady.** In a steady burning candle flame for instance, the flow is incompressible but the density is varying spatially. In conclusion: **assuming constant density implies incompressibility but assuming incompressibility does not imply constant density.**

References

1. https://www.physicsforums.com/threads/constant-density-vs-incompressible.902510/