2D Linearization

1 Procedures

Recall the 2D dynamic system:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \tag{1}$$

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) \tag{2}$$

First, we need to find the fixed points so that:

$$\vec{f}(\vec{x}^*) = 0 \tag{3}$$

Then, we assume a small vector:

$$\vec{s} = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} \tag{4}$$

So for the fixed point, we have:

$$\vec{x}(t) = \vec{x}^* + \vec{s} \tag{5}$$

Then using the **multivariate Taylor Expansion**, we can linearize the equation at $\vec{x}^*$ (dropping the higher term):

$$\vec{f}(\vec{x}^* + \vec{s}) \approx \vec{f}(\vec{x}^*) + J_f(\vec{x}^*) \cdot \vec{s} \tag{6}$$

$$\vec{f}(\vec{s}) \approx J_f(\vec{x}^*) \cdot \vec{s} \tag{7}$$

Where $J_f(\vec{x}^*)$ is the **Jacobian of f at** $\vec{x}^*$.

2 Jacobian Matrix

Now we take a closer look at the Jacobian. Recall that:

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \tag{8}$$

Then the Jacobian matrix is defined as:

$$J_f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \tag{9}$$

2.1 Eigenvalues Calculation

Now we are interested in calculating the eigenvalues. The procedure is simple:

$$\det(A - \lambda I) = 0 \tag{10}$$

Assume:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{11}$$

Then:

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \tag{12}$$

$$det() = (a - \lambda)(d - \lambda) - bc = 0 \tag{13}$$

Then we can get the eigenvectors:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{14}$$

Then use the equation to find the eigen pair:

$$Av = \lambda v \tag{15}$$

2.2 Diagonal Jacobian

Recall the predator-prey model:

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_1 x_2 \\ -r x_2 + x_1 x_2 \end{bmatrix} \tag{16}$$

Therefore, we can get the Jacobian matrix as:

$$J_f(\vec{x}) = \begin{bmatrix} 1 - x_2 & -x_1 \\ x_2 & x_1 - r \end{bmatrix} \tag{17}$$

First we try the fixed point $[0, 0]^T$ (notice that the Jacobian matrix only has meaning at fixed point, only this way could get the eigenvalue for the system):

$$\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}$$

Then we have:

$$J^* = J_f(\vec{x}^*) = \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix} \tag{19}$$

Which is a **diagonal Jacobian**, the terms in diagonal direction are just **eigenvalues at this fixed point**. Plug in back to previous equation:

$$\frac{d}{dt}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \tag{20}$$

Which means:

$$\frac{ds_1}{dt} = \lambda_1 s_1, \; \frac{ds_2}{dt} = \lambda_2 s_2 \tag{21}$$

Or in other words:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \vec{s}(0)\begin{bmatrix} exp(\lambda_1 t) \\ exp(\lambda_2 t) \end{bmatrix} = \begin{bmatrix} exp(t) \\ exp(-rt) \end{bmatrix} \tag{22}$$

Which means, when close to $[0,0]^T$, x_1 is **exponentially blowing-up**, and x_2 is **exponentially decaying**:

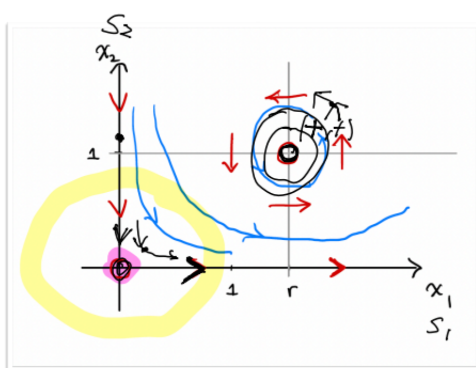

Figure 1: Diagonal Jacobian

2.3 Antidiagonal Jacobian

Similarly, when $\vec{x}^* = [r,1]^T$, we have:

$$J^* = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix} \tag{23}$$

Which is an **antidiagonal Jacobian**. Therefore we have:

$$\frac{d}{dt}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix}\begin{bmatrix} s_2 \\ s_1 \end{bmatrix} \tag{24}$$

Now we can find the eigenvalues at this fixed point:

$$J^* - \lambda I = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & -r \\ 1 & -\lambda \end{bmatrix} \tag{25}$$

Then, using the determinant formula:

$$det \begin{bmatrix} -\lambda & -r \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + r = 0 \tag{26}$$

Solving this equation:

$$\lambda_1 = i\sqrt{r}, \;\; \lambda_2 = -i\sqrt{r} \tag{27}$$

3 Eigenvalues and Phase Diagram

Eigenvalues' real and imaginary parts will represent the system's stability, and also will affect the final phase diagram.

3.1 1D System, 1D Phase Diagram

3.1.1 Stability

Recall the previous content, in 1D the solution has the general form:

$$x(t) = \exp(\lambda^* t) \tag{28}$$

The sign of λ^* will determines stability (here, we assume time is a positive number):

1. $\lambda^* < 0$: the exponential is close to 0, stable
2. $\lambda^* > 0$: the exponential is increasing rapidly, unstable
3. $\lambda^* = 0$: inconclusive, need to check second derivative

Notice that 1D system **has only one eigenvalue, and this value must be real value.** The complex eigenvalues indicate **oscillatory behavior due to the interaction between multiple dimensions.** This does not exist in 1D, so the eigenvalue must be real.

But in general, we could calculate the derivative to determine the stability. For a system:

$$\frac{dx}{dt} = f(x) \tag{29}$$

If $f'(x) > 0$, then the system is unstable; if $f'(x) < 0$, then the system is stable; if $f'(x) = 0$, then the system is semistable.

3.1.2 Phase Diagram

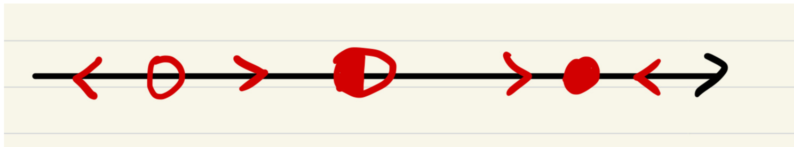

Figure 2: 1D Phase Diagram

In 1D, the phase diagram is just a line, but it may have multiple fixed points. The stability of each fixed point will be different.

3.2 2D System, 2D Phase Diagram

3.2.1 Complex Value Form

For the general 2D case, we need to assume:

$$\vec{s}(t) = \exp{(\lambda t)}\vec{v} \tag{30}$$

Recall the equation:

$$\frac{d\vec{s}}{dt} = J^*\vec{s} \tag{31}$$

Therefore we have:

$$\lambda \exp{(\lambda t)}\vec{v} = J^* \exp{(\lambda t)}\vec{v} \tag{32}$$

$$\lambda\vec{v} = J^*\vec{v} \tag{33}$$

Then, solution $(\lambda, \vec{v})$ is an **eigenpair** of J^*. For an $n \times n$ matrix, there are up tp n **distinct eigenpairs**. Eigenpairs could be **complex-valued**! They will occur in **complex-conjugate pairs**:

$$\lambda_1 = \alpha + i\beta,\ \ \vec{v}_1 = a + ib \tag{34}$$

- If λ_1, λ_2 are distinct, the solution could be written as a linear combination:

$$\vec{s}(t) = a_1 \exp{(\lambda_1 t)}\vec{v}_1 + a_2 \exp{(\lambda_2 t)}\vec{v}_2 \tag{35}$$

- If λ_1 and λ_2 are **real** and $\lambda_1 > \lambda_2$, then we will have **dominance**, which means the solution will tend toward $\vec{v}_1$:

$$\vec{s}(t) = \exp{(\lambda_1 t)}\vec{v}_1[a_1\vec{v}_1 + a_2 \exp{((\lambda_2 - \lambda_1)t)}\vec{v}_2] \tag{36}$$

When $t \to \infty$,

$$\vec{s}(t) = \exp{(\lambda_1 t)} \cdot a_1\vec{v}_1 \tag{37}$$

- If λ_1 and λ_2 are **complex-valued**, then we have:

$$\exp(\lambda t) = \exp(\alpha t)\exp(i\beta t) = \underbrace{\exp(\alpha t)}_{\text{Stability}}\underbrace{[\cos(\beta t) + i\sin(\beta t)]}_{\text{Oscillations}} \tag{38}$$

3.2.2 Stability and Oscillations

The imaginary part β of the complex eigenvalues determines the **frequency and direction of the oscillations** in the phase space. It is also the reason why **spiral will show in the phase diagram**.

The sign of β influences the direction of rotation in the phase plane:

$$\begin{cases} \beta < 0: & \textbf{Clockwise Spiral} \\ \beta > 0: & \textbf{Counterclockwise Spiral} \end{cases} \tag{39}$$

If the two eigenvalues have same magnitudes but different sign in imaginary part, at this time direction of rotation is determined by the Jacobian matrix (same trend as β):

$$\begin{cases} bc < 0: & \textbf{Clockwise} \\ bc > 0: & \textbf{Counterclockwise} \end{cases} \tag{40}$$

The stability is mainly controlled by the real part of the eigenvalues (α). The dependence is shown below (with β):

$$\begin{cases} \alpha < 0: & \textbf{Stable Spiral} \\ \alpha > 0: & \textbf{Unstable Spiral} \\ \alpha = 0: & \textbf{Circle or Elliptical} \end{cases} \tag{41}$$

3.2.3 Impacts of Eigenvectors

Eigenvectors are also very important for the 2D phase diagram. For Saddle node, each eigenvalue has a corresponding eigenvector that **defines the direction of the stable and unstable manifolds (axis)**. The graph is shown in next section.

- Eigenvector of positive eigenvalue: direction of the unstable axis.
- Eigenvector of negative eigenvalue: direction of the stable axis.

Here is a detailed instruction to find the eigenvectors. Suppose we have a Jacobian matrix at the fixed point:

$$\boldsymbol{J}^* = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{42}$$

Assume the eigenvectors are λ_1 and λ_2, then for λ_1, we need to solve:

$$(\boldsymbol{J} - \lambda_1 \boldsymbol{I})\boldsymbol{v_1} = 0 \tag{43}$$

$$\begin{bmatrix} a-\lambda_1 & b \\ c & d-\lambda_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \tag{44}$$

Now we can find the relation between x_1 and x_2:

$$(a-\lambda_1)x_1 + bx_2 = 0 \tag{45}$$

After the normalization, we will get the eigenvector. For example, if:

$$x_2 = -2x_1 \tag{46}$$

Then the eigenvector is (could be scaled):

$$\boldsymbol{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \tag{47}$$

3.2.4 Phase Diagram

We can find some special nodes in phase diagram using eigenvalues and eigenvectors. In this section, all the discussion is about a single fixed point. Assume for this single point, there are eigenvalues λ_1 and λ_2, and eigenvectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$.

1. $\alpha_1, \alpha_2 < 0, \beta_1 = \beta_2 = 0$: Both eigenvalues are real and negative, and this node will be a **stable node (Sink)**. Notice that if both eigenvalues are real, the curves could still be distorted, except the case $\alpha_1 = \alpha_2$:

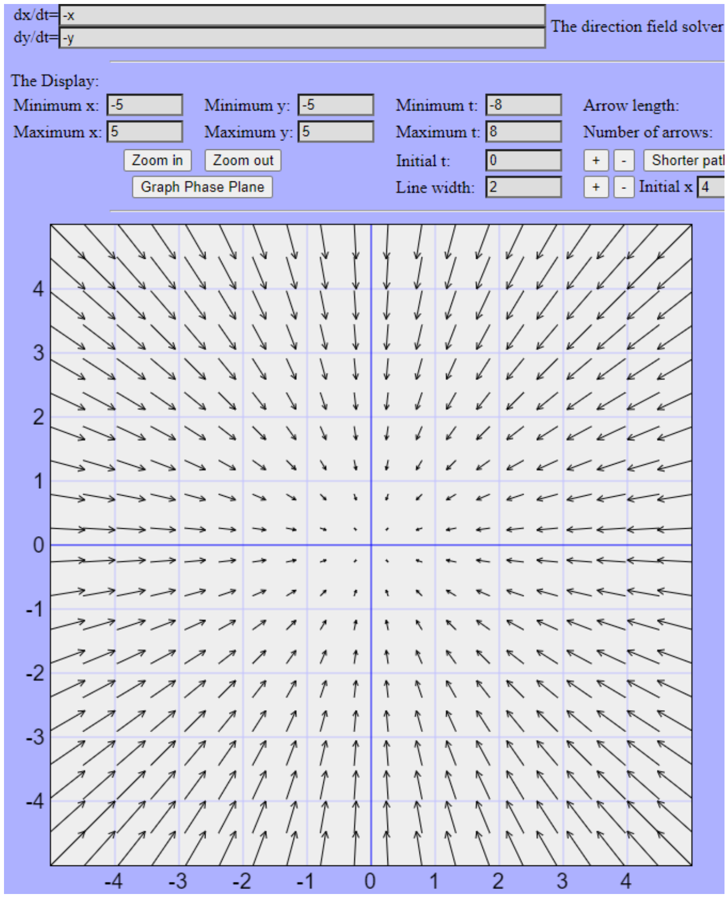

Figure 3: Sink node with same magnitude eigenvalues

When $\alpha_1 \neq \alpha_2$, then two eigenvalues will have interference. For example (fixed point as origin):

$$\frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} \boldsymbol{x} \tag{48}$$

The eigenvalues and eigenvectors are:

$$\lambda_1 = -1, \quad \boldsymbol{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{49}$$

$$\lambda_2 = -4, \quad \boldsymbol{v_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{50}$$

Hence the general solution is given by:

$$\boldsymbol{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{51}$$

Notice here there are 3 variables: x_1, x_2, t, that's why we could get 2D phase diagram. The x-axis represents x_1 and y-axis represents x_2. In the phase diagram, each curve is a trajectory line, and **the initial point ($t = 0$) could be any point in the phase diagram, the trajectory line will determine where this point will go in $t > 0$.**

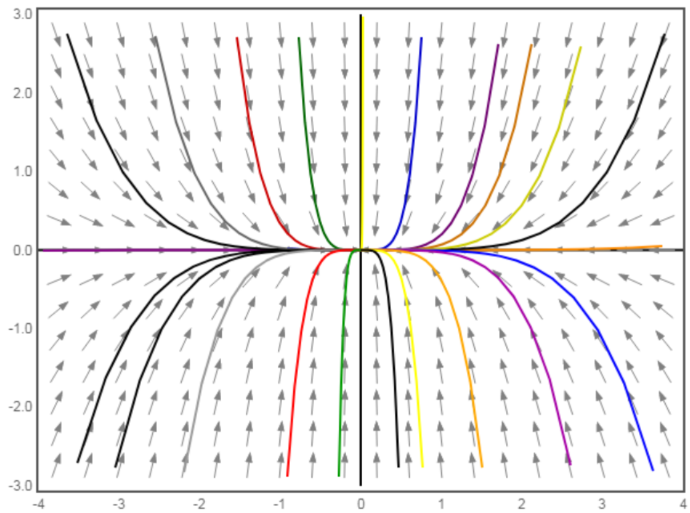

Figure 4: Sink node with different magnitude eigenvalues

In this example, we observe that:

$$|\lambda_2| = 4 > 1 = |\lambda_1| \tag{52}$$

Therefore, the straight line solution $c_2 e^{-4t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ will have more strength, and all the curves are more distorted to that direction (more like that direction).

2. $\alpha_1, \alpha_2 > 0, \beta_1 = \beta_2 = 0$: Both eigenvalues are real and positive, and this node will be a **unstable node (Source)**. This discussion is similar with sink, so when $\alpha_1 = \alpha_2$:

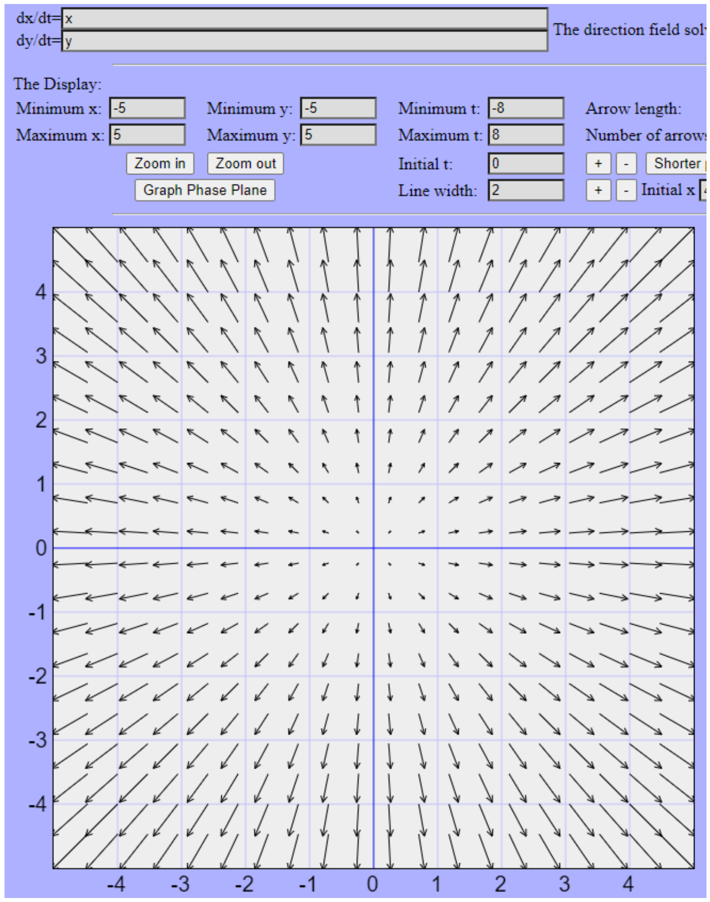

Figure 5: Source node with same magnitude eigenvalues

When $\alpha_1 \neq \alpha_2$, assume we have (fixed point as origin):

$$\frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \boldsymbol{x} \tag{53}$$

The eigenvalues and eigenvectors are:

$$\lambda_1 = 4, \quad \boldsymbol{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{54}$$

$$\lambda_2 = 1, \quad \boldsymbol{v_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{55}$$

Hence the general solution is given by:

$$\boldsymbol{x}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \tag{56}$$

Based on the observation:

$$|\lambda_1| = 4 > 1 = |\lambda_2| \tag{57}$$

Therefore, the straight line solution $c_1 e^{4t}\begin{bmatrix}1\\1\end{bmatrix}$ will have more strength, and all the curves are more distorted to that direction (more like that direction).

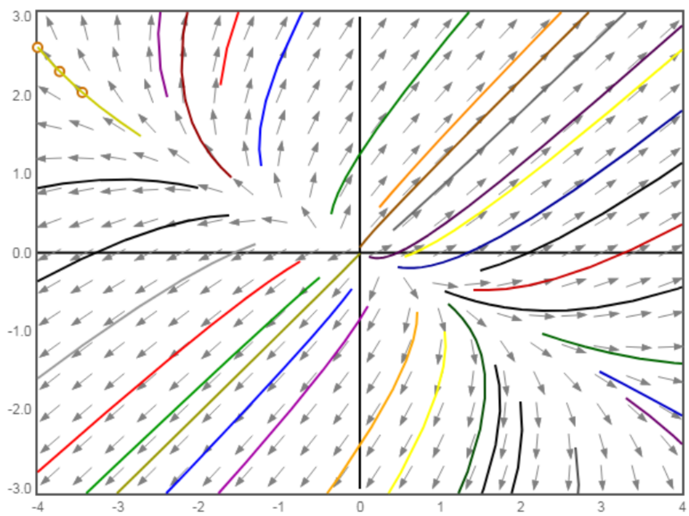

Figure 6: Source node with different magnitude eigenvalues

3. $\alpha_1 > 0, \alpha_2 < 0, \beta_1 = \beta_2 = 0$ Both eigenvalues are real, one eigenvalue is positive, and the other is negative, this node is called **Saddle Node**, and it is unstable. The unstable and stable axis are determined by the eigenvectors.

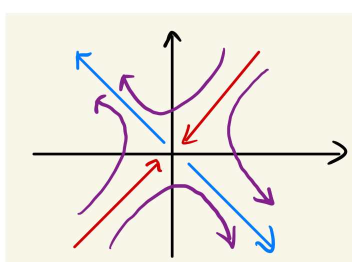

Figure 7: Saddle Node

4. $\beta_1 = \beta_2 = 0, \alpha_1 = \alpha_2$: Eigenvalues are purely real and equal, this node is called **Degenerate Node (Improper Node)**, stable if negative, unstable if positive. Notice there are other constraints of this point.

   - $\Delta = (\text{Tr}[A])^2 - 4\det[A] = 0$
   - The A matrix is not **diagonalizable**. This is the actual reason for **degeneration**.

For example, if we have:

$$\frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} \tag{58}$$

Then:

$$\Delta = (1+1)^2 - 4 \cdot 1 = 0 \tag{59}$$

However, because this matrix is diagonalizable, **this node is not a degenerate point, it is a source node (mentioned before)**. Instead, if we have:

$$\frac{d\boldsymbol{x}}{dt} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \boldsymbol{x} \tag{60}$$

Then this system satisfies two requirements, there is a degenerate point:

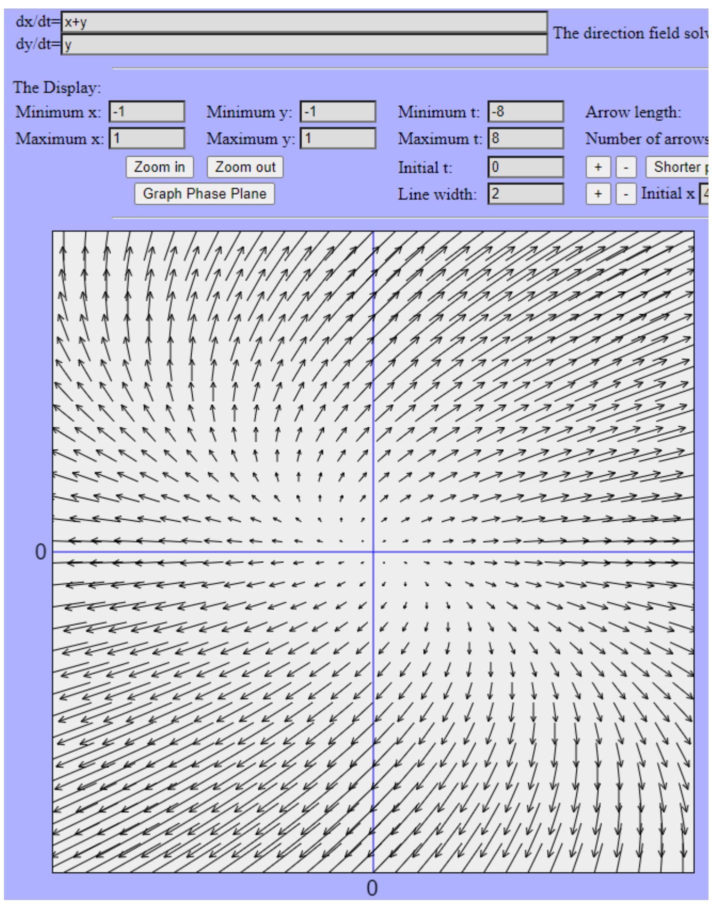

Figure 8: Degenerate Node

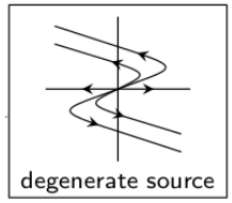

Figure 9: Unstable Degenerate Node

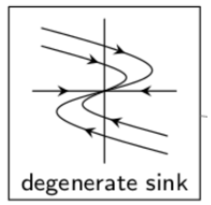

Figure 10: Stable Degenerate Node

5. $\beta_1, \beta_2 \neq 0, \alpha_1, \alpha_2 < 0$: Eigenvalues are complex conjugates with **negative real parts**, then this node is called **Stable Spiral Node** (including both β cases).

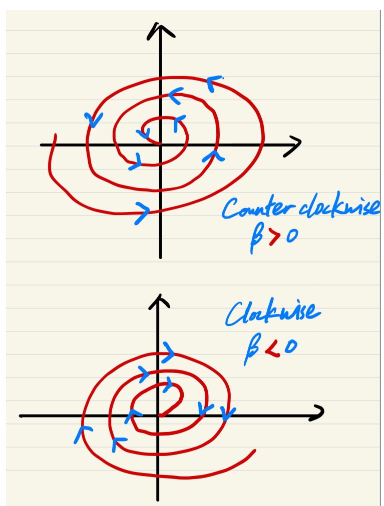

Figure 11: Stable Spiral Node

6. $\beta_1, \beta_2 \neq 0, \alpha_1, \alpha_2 > 0$: Eigenvalues are complex conjugates with **positive real parts**, then this node is called **Unstable Spiral Node** (include both β cases).

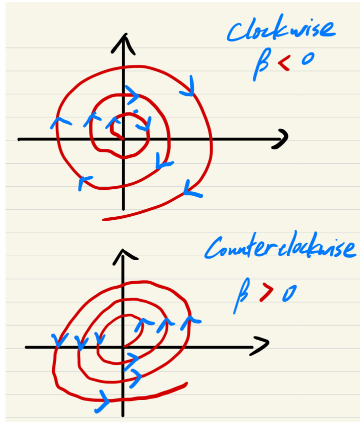

Figure 12: Unstable Spiral Node

7. $\beta_1, \beta_2 \neq 0, \alpha_1 = \alpha_2 = 0$: Eigenvalues are purely imaginary, complex conjugates with **zero real parts**, then this node is called **Center Node** (include both bc cases). Normally at this time, we have $\beta_1 = -\beta_2$. The center node could be circle or elliptical, dependent on the state variables.

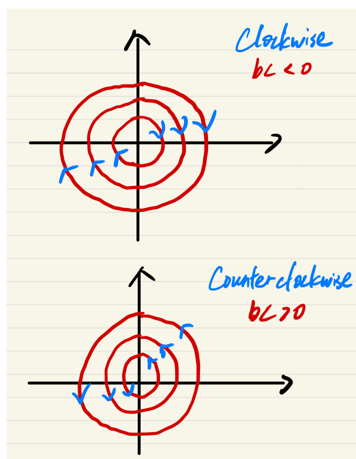

Figure 13: Center Node