2D Linearization

1 Procedures

Recall the 2D dynamic system:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \tag{1}$$

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) \tag{2}$$

First, we need to find the fixed points so that:

$$\vec{f}(\vec{x}^*) = 0 \tag{3}$$

Then, we assume a small vector:

$$\vec{s} = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} \tag{4}$$

So for the fixed point, we have:

$$\vec{x}(t) = \vec{x}^* + \vec{s} \tag{5}$$

Then using the **multivariate Taylor Expansion**, we can linearize the equation at $\vec{x}^*$ (dropping the higher term):

$$\vec{f}(\vec{x}^* + \vec{s}) \approx \vec{f}(\vec{x}^*) + J_f(\vec{x}^*) \cdot \vec{s} \tag{6}$$

$$\vec{f}(\vec{s}) \approx J_f(\vec{x}^*) \cdot \vec{s} \tag{7}$$

Where $J_f(\vec{x}^*)$ is the **Jacobian of f at** $\vec{x}^*$.

2 Jacobian Matrix

Now we take a closer look at the Jacobian. Recall that:

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \tag{8}$$

Then the Jacobian matrix is defined as:

$$J_f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \tag{9}$$

2.1 Diagonal Jacobian

Recall the predator-prey model:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_1 x_2 \\ -r x_2 + x_1 x_2 \end{bmatrix} \tag{10}$$

Therefore, we can get the Jacobian matrix as:

$$J_f(\vec{x}) = \begin{bmatrix} 1{-}x_2 & -x_1 \\ x_2 & x_1 - r \end{bmatrix} \tag{11}$$

First we try:

$$\vec{x^*} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12}$$

Then we have:

$$J^* = J_f(\vec{x^*}) = \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \tag{13}$$

Which is a **diagonal Jacobian**. Plug in back to previous equation:

$$\frac{d}{dt}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \tag{14}$$

Which means:

$$\frac{ds_1}{dt} = \lambda_1 s_1, \ \frac{ds_2}{dt} = \lambda_2 s_2 \tag{15}$$

Or in other words:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \vec{s}(0)\begin{bmatrix} exp(\lambda_1 t) \\ exp(\lambda_2 t) \end{bmatrix} = \begin{bmatrix} exp(t) \\ exp(-rt) \end{bmatrix} \tag{16}$$

Which means, when close to $[0,0]^T$, x_1 is **exponentially blowing-up**, and x_2 is **exponentially decaying**:

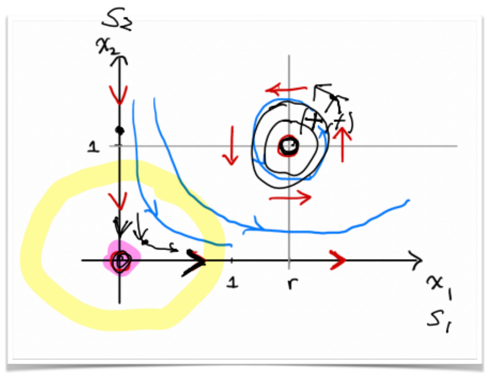

Figure 1: Diagonal Jacobian

2.2 Antidiagonal Jacobian

Similarly, when $\vec{x}^* = [r, 1]^T$, we have:

$$J^* = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix} \tag{17}$$

Which is an **antidiagonal Jacobian**. Therefore we have:

$$\frac{d}{dt}\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} -rs_2 \\ s_1 \end{bmatrix} \tag{18}$$

Which is like a swaping or rotation.

3 Eigenvalues

3.1 Connection with Jacobian

Diagonal and antidiagonal Jacobians are just two special cases. What about the general case? Now we need to assume:

$$\vec{s}(t) = \exp{(\lambda t)}\vec{v} \tag{19}$$

Recall the equation:

$$\frac{d\vec{s}}{dt} = J^*\vec{s} \tag{20}$$

Therefore we have:

$$\lambda \exp{(\lambda t)}\vec{v} = J^* \exp{(\lambda t)}\vec{v} \tag{21}$$

$$\lambda\vec{v} = J^*\vec{v} \tag{22}$$

Then, solution $(\lambda, \vec{v})$ is an **eigenpair** of J^*. In 1D ($x(t) = \exp{(\lambda t)}$), $\lambda < 0$ means we are going toward the fixed point, and $\lambda > 0$ means we are going away from the fixed point. In 2D, it is the same but along an **eigenvector**, which is just $\vec{v}$.

3.2 Eigenfacts

Now review some basic properties of eigenvalues:

1. For an $n \times n$ matrix, there are up tp n **distinct eigenpairs**

2. Eigenpairs could be **complex-valued**! They will occur in **complex-conjugate pairs**:

$$\lambda_1 = \alpha + i\beta,\ \ \vec{v}_1 = a + ib \tag{23}$$

3. If λ_1, λ_2 are distinct, the solution could be written as a linear combination:

$$\vec{s}(t) = a_1 \exp{(\lambda_1 t)} \vec{v}_1 + a_2 \exp{(\lambda_2 t)} \vec{v}_2 \tag{24}$$

If λ_1 and λ_2 are **real** and $\lambda_1 > \lambda_2$, then we will have **dominance**, which means the solution will tend toward $\vec{v}_1$:

$$\vec{s}(t) = \exp{(\lambda_1 t)} \vec{v}_1 [a_1 \vec{v}_1 + a_2 \exp{((\lambda_2 - \lambda_1)t)} \vec{v}_2] \tag{25}$$

When $t \to \infty$,

$$\vec{s}(t) = \exp{(\lambda_1 t)} \cdot a_1 \vec{v}_1 \tag{26}$$

4. If λ_1 and λ_2 are **complex-valued**, then we have:

$$\exp{(\lambda t)} = \exp{(\alpha t)} \exp{(i\beta t)} = \underbrace{\exp{(\alpha t)}}_{\text{Stability}} \underbrace{[\cos{(\beta t)} + i \sin{(\beta t)}]}_{\text{Oscillations}} \tag{27}$$

The stability dependence is shown below:

$$\begin{cases} \alpha < 0: & \text{stable spiral} \\ \alpha > 0: & \text{unstable spiral} \\ \alpha = 0: & \text{circle} \end{cases} \tag{28}$$

3.3 Eigenvalues Calculation

Now we are interested in calculating the eigenvalues. The procedure is simple:

$$\det(A - \lambda I) = 0 \tag{29}$$

Assume:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{30}$$

Then:

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \tag{31}$$

$$det() = (a - \lambda)(d - \lambda) - bc = 0 \tag{32}$$

Then we can get the eigenvectors:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{33}$$