Continuity Equation

1 Control Analysis

1.1 Control Volume

A control volume is a specified region in space through which mass, momentum, and energy can be exchanged with the surroundings. This volume is defined by an imaginary boundary, which is usually a closed surface. The control volume approach is typically used in the analysis of open systems, where mass and energy can cross the boundaries of the volume. This approach is particularly useful for studying fluid flow and heat transfer problems, as it allows the analysis of the conservation laws (mass, momentum, and energy) within the defined region.

1.2 Control Mass

A control mass, also known as a closed system, is a fixed quantity of matter that is separated from its surroundings by an imaginary boundary. The mass within this boundary remains constant, and no mass crosses the boundary. However, energy can be exchanged with the surroundings in the form of heat and work. Control mass analysis is useful for studying processes where the mass remains constant, such as the behavior of gases in a piston-cylinder assembly or the response of a solid object to temperature changes.

1.3 Control Surface

A control surface is an imaginary two-dimensional boundary that encloses a control volume. It is used to define the region of interest in control volume analysis. The control surface separates the control volume from its surroundings and is used to analyze the flow of mass, momentum, and energy across the boundary. The behavior of the system within the control volume is determined by applying the conservation laws to the control surface.

2 Conservation of Mass

2.1 Definition

$$Rate\, of\, change\, of\, mass\, in\, CV = Net\, mass\, flow\, rate\, goes\, across\, CS\, \boldsymbol{into}\, CV \tag{1}$$

2.2 Continuity Equation Derivation

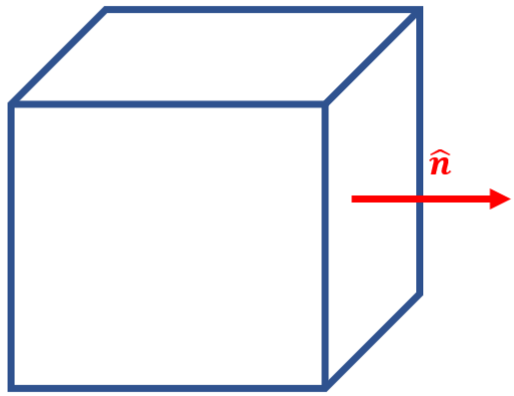

Figure 1: Normal Vector of Control Surface

As shown in the picture, the normal vector of the control surface is always pointing out of the control volume. Therefore, when we try to calculate the mass flow rate goes into the CV, we need to add a negative sign:

$$\frac{\partial}{\partial t}\int\limits_{CV} \rho \, dV = -\int\limits_{CS} \rho \underline{\boldsymbol{u}} \hat{\boldsymbol{n}} dS \tag{2}$$

Using divergence theorem:

$$\frac{\partial}{\partial t}\int\limits_{CV} \rho \, dV = -\int\limits_{CV} \nabla \cdot (\rho \underline{\boldsymbol{u}}) \, dV \tag{3}$$

$$\frac{\partial}{\partial t}\int\limits_{CV} \rho \, dV + \int\limits_{CV} \nabla \cdot (\rho \underline{\boldsymbol{u}}) \, dV = 0 \tag{4}$$

$$\int\limits_{CV} [\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{\boldsymbol{u}})] \, dV = 0 \tag{5}$$

Assume dV is infinitesimal, then:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{\boldsymbol{u}}) = 0 \tag{6}$$

$$\frac{\partial \rho}{\partial t} + \underline{\boldsymbol{u}} \cdot \nabla \rho + \rho (\nabla \cdot \underline{\boldsymbol{u}}) = 0 \tag{7}$$

$$\frac{D\rho}{Dt} + \rho (\nabla \cdot \underline{\boldsymbol{u}}) = 0 \tag{8}$$

If steady flow:

$$\rho(\nabla \cdot \underline{\boldsymbol{u}}) = 0 \tag{9}$$

If we assume the density is constant:

$$\nabla \cdot \underline{\boldsymbol{u}} = 0 \tag{10}$$

Written in Einstein's summation convention:

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{11}$$

References