

2D Linearization

1 Procedures

Recall the 2D dynamic system:

$$\frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \quad (1)$$

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) \quad (2)$$

First, we need to find the fixed points so that:

$$\vec{f}(\vec{x}^*) = 0 \quad (3)$$

Then, we assume a small vector:

$$\vec{s} = \begin{bmatrix} s_1(t) \\ s_2(t) \end{bmatrix} \quad (4)$$

So for the fixed point, we have:

$$\vec{x}(t) = \vec{x}^* + \vec{s} \quad (5)$$

Then using the **multivariate Taylor Expansion**, we can linearize the equation at \vec{x}^* (dropping the higher term):

$$\vec{f}(\vec{x}^* + \vec{s}) \approx \vec{f}(\vec{x}^*) + J_f(\vec{x}^*) \cdot \vec{s} \quad (6)$$

$$\vec{f}(\vec{s}) \approx J_f(\vec{x}^*) \cdot \vec{s} \quad (7)$$

Where $J_f(\vec{x}^*)$ is the **Jacobian of f at \vec{x}^*** .

2 Jacobian Matrix

Now we take a closer look at the Jacobian. Recall that:

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \quad (8)$$

Then the Jacobian matrix is defined as:

$$J_f(\vec{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \quad (9)$$

2.1 Diagonal Jacobian

Recall the [predator-prey model](#):

$$\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_1 x_2 \\ -r x_2 + x_1 x_2 \end{bmatrix} \quad (10)$$

Therefore, we can get the Jacobian matrix as:

$$J_f(\vec{x}) = \begin{bmatrix} 1-x_2 & -x_1 \\ x_2 & x_1 - r \end{bmatrix} \quad (11)$$

First we try:

$$\vec{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (12)$$

Then we have:

$$J^* = J_f(\vec{x}^*) = \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad (13)$$

Which is a **diagonal Jacobian**. Plug in back to previous equation:

$$\frac{d}{dt} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} \quad (14)$$

Which means:

$$\frac{ds_1}{dt} = \lambda_1 s_1, \quad \frac{ds_2}{dt} = \lambda_2 s_2 \quad (15)$$

Or in other words:

$$\begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \vec{s}(0) \begin{bmatrix} \exp(\lambda_1 t) \\ \exp(\lambda_2 t) \end{bmatrix} = \begin{bmatrix} \exp(t) \\ \exp(-rt) \end{bmatrix} \quad (16)$$

Which means, when close to $[0, 0]^T$, x_1 is **exponentially blowing-up**, and x_2 is **exponentially decaying**:

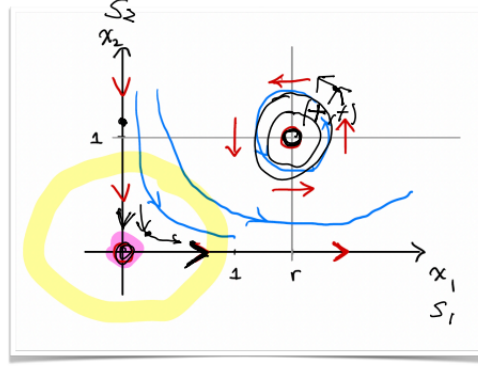


Figure 1: Diagonal Jacobian

2.2 Antidiagonal Jacobian

Similarly, when $\vec{x}^* = [r, 1]^T$, we have:

$$J^* = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix} \quad (17)$$

Which is an **antidiagonal Jacobian**. Therefore we have:

$$\frac{d}{dt} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} -rs_2 \\ s_1 \end{bmatrix} \quad (18)$$

Which is like a swapping or rotation.

3 Eigenvalues

3.1 Connection with Jacobian

Diagonal and antidiagonal Jacobians are just two special cases. What about the general case? Now we need to assume:

$$\vec{s}(t) = \exp(\lambda t) \vec{v} \quad (19)$$

Recall the equation:

$$\frac{d\vec{s}}{dt} = J^* \vec{s} \quad (20)$$

Therefore we have:

$$\lambda \exp(\lambda t) \vec{v} = J^* \exp(\lambda t) \vec{v} \quad (21)$$

$$\lambda \vec{v} = J^* \vec{v} \quad (22)$$

Then, solution (λ, \vec{v}) is an **eigenpair** of J^* . In 1D ($x(t) = \exp(\lambda t)$), $\lambda < 0$ means we are going toward the fixed point, and $\lambda > 0$ means we are going away from the fixed point. In 2D, it is the same but along an **eigenvector**, which is just \vec{v} .

3.2 Eigenfacts

Now review some basic properties of eigenvalues:

1. For an $n \times n$ matrix, there are up to n **distinct eigenpairs**
2. Eigenpairs could be **complex-valued**! They will occur in **complex-conjugate pairs**:

$$\lambda_1 = \alpha + i\beta, \quad \vec{v}_1 = a + ib \quad (23)$$

3. If λ_1, λ_2 are distinct, the solution could be written as a linear combination:

$$\vec{s}(t) = a_1 \exp(\lambda_1 t) \vec{v}_1 + a_2 \exp(\lambda_2 t) \vec{v}_2 \quad (24)$$

If λ_1 and λ_2 are **real** and $\lambda_1 > \lambda_2$, then we will have **dominance**, which means the solution will tend toward \vec{v}_1 :

$$\vec{s}(t) = \exp(\lambda_1 t) \vec{v}_1 [a_1 \vec{v}_1 + a_2 \exp((\lambda_2 - \lambda_1)t) \vec{v}_2] \quad (25)$$

When $t \rightarrow \infty$,

$$\vec{s}(t) = \exp(\lambda_1 t) \cdot a_1 \vec{v}_1 \quad (26)$$

4. If λ_1 and λ_2 are **complex-valued**, then we have:

$$\exp(\lambda t) = \exp(\alpha t) \exp(i\beta t) = \underbrace{\exp(\alpha t)}_{\text{Stability}} \underbrace{[\cos(\beta t) + i \sin(\beta t)]}_{\text{Oscillations}} \quad (27)$$

The stability dependence is shown below:

$$\begin{cases} \alpha < 0 : & \text{stable spiral} \\ \alpha > 0 : & \text{unstable spiral} \\ \alpha = 0 : & \text{circle} \end{cases} \quad (28)$$

3.3 Eigenvalues Calculation

Now we are interested in calculating the eigenvalues. The procedure is simple:

$$\det(A - \lambda I) = 0 \quad (29)$$

Assume:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (30)$$

Then:

$$A - \lambda I = \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} \quad (31)$$

$$\det() = (a - \lambda)(d - \lambda) - bc = 0 \quad (32)$$

Then we can get the eigenvectors:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (33)$$

Then use the equation to find the eigen pair:

$$Av = \lambda v \quad (34)$$

Recall the predator-prey model, when the fixed point is $[0, 0]^T$, the Jacobian matrix is:

$$J^* = \begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix} \quad (35)$$

Using the determinant method to calculate the eigenvalues, we have:

$$\lambda_1 = 1, \quad \lambda_2 = -r \quad (36)$$

Therefore for λ_1 , we have the following equation:

$$\begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} \rightarrow \begin{bmatrix} v_{11} \\ v_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (37)$$

Then for λ_2 , we have the following equation:

$$\begin{bmatrix} 1 & 0 \\ 0 & -r \end{bmatrix} \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = -r \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} \rightarrow \begin{bmatrix} v_{12} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (38)$$

Similarly, when the fixed point is $[r, 1]^T$, the Jacobian matrix is:

$$J^* = \begin{bmatrix} 0 & -r \\ 1 & 0 \end{bmatrix} \quad (39)$$

Then using the determinant method to calculate the eigenvalues, we have:

$$\lambda_1 = i\sqrt{\gamma}, \quad \lambda_2 = -i\sqrt{\gamma} \quad (40)$$

So we could find the eigenpairs:

$$v_1 = \begin{bmatrix} i\sqrt{\gamma} \\ 1 \end{bmatrix} \quad (41)$$

$$v_2 = \begin{bmatrix} -i\sqrt{\gamma} \\ 1 \end{bmatrix} \quad (42)$$

3.4 Nodes in Phase Diagram

We can find some special nodes in phase diagram using eigenvalues:

1. **Stable Node:** both eigenvalues are real and negative
2. **Unstable Node:** both eigenvalues are real and positive
3. **Stable Spiral Node:** Eigenvalues are complex conjugates with **negative real parts**
4. **Unstable Spiral Node:** Eigenvalues are complex conjugates with **positive real parts**
5. **Center Node:** Eigenvalues are purely imaginary (complex conjugates with **zero real parts**)
6. **Saddle Node:** One eigenvalue is positive, and the other is negative
7. **Stable Star Node:** Both eigenvalues are real, negative and have the same magnitude
8. **Unstable Star Node:** Both eigenvalues are real, positive and have the same magnitude