Stagnation Flow

1 Definition

Flow approaching a 90-degree barrier, with non-parallel streamlines.

2 Solution

2.1 Stream Function

To solve the stagnation flow problem, we define a stream function as the inviscid flow solution:

$$\psi(x, y) = Bxy \tag{1}$$

and we also define:

$$u = \frac{\partial \psi}{\partial y} \tag{2}$$

$$v = -\frac{\partial \psi}{\partial x} \tag{3}$$

Therefore, we can easily conclude that the unit of stream function is $[m \cdot s^{-1} \cdot m] = [m^2 \cdot s^{-1}]$. Therefore the unit of B can be calculated as $[s^{-1}]$.

Notice that this stream function automatically satisfy the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0 \tag{4}$$

2.2 Similarity Solution

The viscous effects are especially strong near the wall, so we change the stream function to:

$$\psi(x, y) = Bxf(y) \tag{5}$$

Any choice of f could satisfy continuity, but the no-slip condition still requires:

$$\frac{df}{dy} = 0, y = 0 \tag{6}$$

So the velocity components will be:

$$u = \frac{\partial \psi}{\partial y} = Bx\frac{df}{dy} = Bxf' \tag{7}$$

$$v = -\frac{\partial \psi}{\partial x} = -Bf \tag{8}$$

Apply these into the momentum equations:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{9}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{10}$$

After some magic math :), we get an ODE:

$$f''' + \frac{B}{\nu}(ff'' - f'^2) = -\frac{B}{\nu} \tag{11}$$

Now is the time to choose the similarity variable. Notice we have both $y[m]$ and $B[s^{-1}]$ and we want to use a single non-dimensional variable to combine them. With the help of $\nu[m^2s^{-1}]$, we get:

$$\eta = \frac{y}{\sqrt{\frac{\nu}{B}}} = y\sqrt{\frac{B}{\nu}} = [m]\cdot\sqrt{\frac{[s^{-1}]}{[m^2s^{-1}]}} = [1] \tag{12}$$

We also define a non-dimensional stream function

$$\psi/(x\sqrt{B\nu}) = F(\eta)$$

Express velocity components in terms of F:

$$u = \frac{\partial \psi}{\partial y} = x\sqrt{B\nu}F'\sqrt{\frac{B}{\nu}} = BxF'$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{B\nu}F$$

Compare two expressions we got for u so far:

$$u = Bxf' = BxF' \Rightarrow f' = F'$$

Beware; this is only saying $df/dy = dF/\eta$, so this does *not* mean f and F are the same. Comparing expressions for v as well:

$$v = -Bf' = -\sqrt{B\nu}F \Rightarrow f = \sqrt{\nu/B}F$$

Noting that $f = f(y)$ whereas $F = F(\eta)$ we can start to relate their derivatives with respect to y and η (related by $\eta = y\sqrt{B/\nu}$)

$$f'(y) = \frac{df}{dy} = \sqrt{\frac{\nu}{B}}\frac{dF}{d\eta}\frac{d\eta}{dy} = F'(\eta)$$

Differentiating again, with primes continuing to denote derivatives with respect to the appropriate variable, we get

$$f'' = \sqrt{B/\nu}F'' \quad ; \quad f''' = (B/\nu)F'''$$

Substituting these into the ODE for f (on preceding page) gives

$$\frac{B}{\nu}F''' + \frac{B}{\nu}\left(\sqrt{\frac{\nu}{B}}\sqrt{\frac{B}{\nu}}FF'' - F'^2\right) = -\frac{B}{\nu}$$

which is an "universal" ODE (good for any B, ν)

$$F''' + FF'' + 1 - F'^2 = 0$$

The boundary conditions for $F(\eta)$ are:

1. No slip at the wall: $F'(0) = 0$
2. Impermeability at the wall: $F(0) = 0$
3. $u = Bx$ at freestream (inviscid there): $F'(\infty) = 1$.