2D Dynamical System

1 Introduction

Sometimes, 1D dynamic system could not satisfy the requirements of the problem. Instead, we can use 2D dynamic system to express the problem. For example, the state vector will be:

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{1}$$

The vector function could be:

$$\vec{f}(\vec{x}) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{bmatrix} \tag{2}$$

Then the dynamics system could be expressed as:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) \tag{3}$$

2 Examples

2.1 Pendulum

One of the most classical problem is the pendulum problem. Assume the mass on a rigid bar is attached to a pivot, and it swings back and forth.

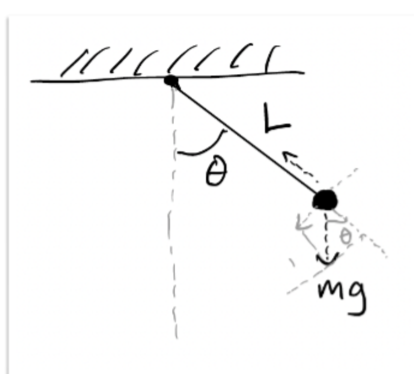


Figure 1: Pendulum

Assume the angular velocity as ω, then apply the Newton's Law:

$$\frac{d}{dt}\begin{bmatrix}\theta \\ \omega\end{bmatrix} = \begin{bmatrix}\omega \\ -\frac{g}{L}\sin\theta\end{bmatrix} \tag{4}$$

Which is in the form of:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}) \tag{5}$$

2.2 Two Interacting Drivers

Assume two drivers on the road trying to match the speed, with the speed limit as ω:

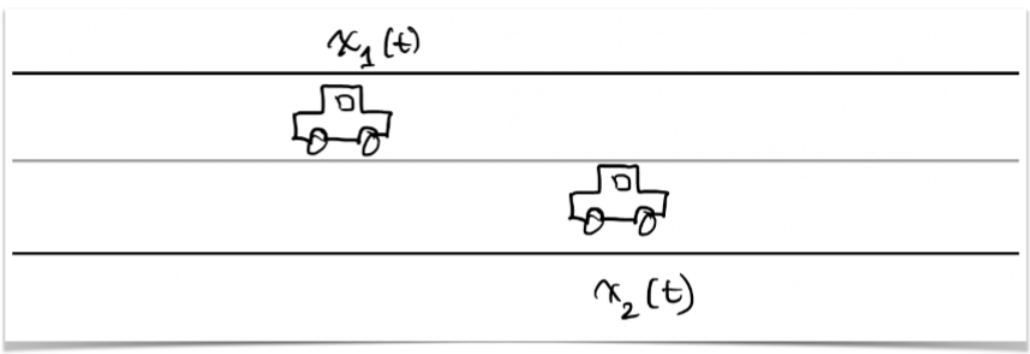


Figure 2: Two Interacting Drivers

Here, we define the distance between 2 cars as:

$$\phi = x_2 - x_1 \tag{6}$$

And we define the **interaction strength** as:

$$g(\phi)\begin{cases} > 0 & \text{if } \phi > 0 \\ = 0 & \text{if } \phi = 0 \\ < 0 & \text{if } \phi < 0 \end{cases} \tag{7}$$

And the relation functions are:

$$\frac{dx_1}{dt} = \omega + \varepsilon g(x_2 - x_1) \tag{8}$$

$$\frac{dx_2}{dt} = \omega - \varepsilon g(x_2 - x_1) \tag{9}$$

Therefore, we can rewrite the system as:

$$\frac{d\phi}{dt} = -2\varepsilon g(\phi) \tag{10}$$

Now if we try to find the fixed point, then:

$$-2\varepsilon g(\phi) = 0 \tag{11}$$

From the previous relation we know $\phi = 0$, which means the drivers line up. Also:

$$-2\varepsilon g(\phi) \begin{cases} < 0 & \text{if } \phi > 0 \\ = 0 & \text{if } \phi = 0 \\ > 0 & \text{if } \phi < 0 \end{cases} \tag{12}$$

Therefore we know this fixed point is **stable**. We call this system exhibits **synchronization.** Synchronization refers to the coordination of different processes, models, or simulations to ensure that they operate together in a consistent and timely manner. In this case, no matter how ϕ changes, the system is stable.

2.3 Racetrack

Suppose two cars running on a circle track:

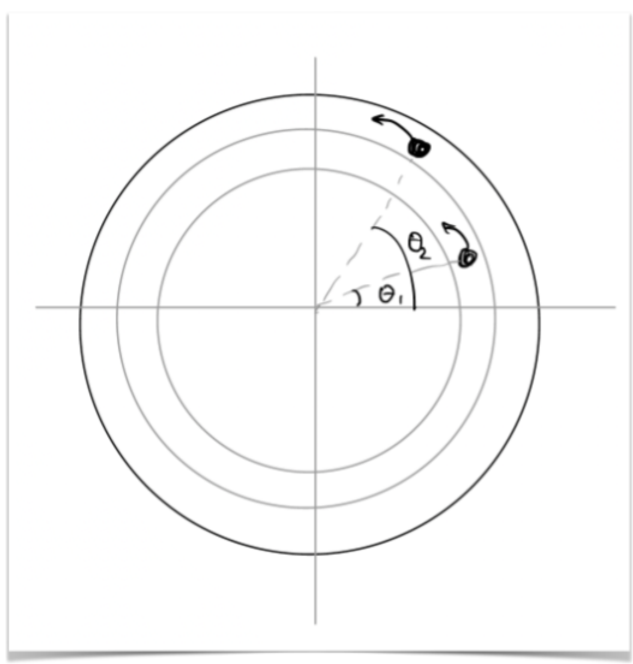


Figure 3: Race Track

Here, we define ω as the angular speed, and:

$$\phi = \theta_2 - \theta_1 \tag{13}$$

Similar with the previous case:

$$\frac{d\phi}{dt} = -2\varepsilon g(\phi) \tag{14}$$

But now we have constraint on $g(\phi)$. It needs to be angular, and also:

$$g(\phi) = g(\phi + 2\pi) \tag{15}$$

2.4 Predator-Prey

Another classic model is the predator-prey model. We have the following parameters:

1. The population of sheep: $x_1(t)$
2. The population of wolves: $x_2(t)$
3. Sheep population increases in absence of wolves, rate $r_1 > 0$
4. Wolves die in absence of sheep, rate $r_2 > 0$
5. $a_1, a_2 > 0$ are the interaction coefficients

Therefore, for the sheep:

$$\frac{dx_1}{dt} = r_1 x_1 - a_1 x_1 x_2 \tag{16}$$

Then for the wolves:

$$\frac{dx_2}{dt} = -r_2 x_2 + a_2 x_1 x_2 \tag{17}$$

Using the rescaling method:

$$x_1 = \alpha \hat{x}_1 \tag{18}$$

$$x_2 = \beta \hat{x}_2 \tag{19}$$

$$t = \tau \hat{t} \tag{20}$$

Therefore we have:

$$\frac{\alpha}{\tau} \frac{d\hat{x}_1}{d\hat{t}} = r_1 \alpha \hat{x}_1 - a_1 \alpha \beta \hat{x}_1 \hat{x}_2 \tag{21}$$

$$\frac{\beta}{\tau} \frac{d\hat{x}_2}{d\hat{t}} = -r_2 \beta \hat{x}_2 + a_2 \alpha \beta \hat{x}_1 \hat{x}_2 \tag{22}$$

Then we choose the parameters in a smart way:

$$\tau = \frac{1}{r_1} \tag{23}$$

$$\beta = \frac{r_1}{a_1} \tag{24}$$

$$\alpha = \frac{r_1}{a_2} \tag{25}$$

And also define a new parameter as:

$$r = \frac{r_2}{r_1} \tag{26}$$

Finally we have:

$$\frac{d\hat{x_1}}{d\hat{t}} = \hat{x_1} - \hat{x_1}\hat{x_2} \tag{27}$$

$$\frac{d\hat{x_2}}{d\hat{t}} = -\gamma\hat{x_2} + \hat{x_1}\hat{x_2} \tag{28}$$

Covert back into 1-parameter system:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_1x_2 \\ -rx_2 + x_1x_2 \end{bmatrix} \tag{29}$$

Now we try to find the fixed points. First we want to find the x_1 **Nullcline**, which is just:

$$\frac{dx_1}{dt} = 0 \tag{30}$$

So, we have:

$$x_1^* = x_1^* x_2^* \tag{31}$$

$$x_1^* = 0 \ \text{ or } \ x_2^* = 1 \tag{32}$$

Notice that they **will not happen at the same time.** Then we find the x_2 **Nullcline** (the intersection points between two Nullclines are the fix points):

$$\frac{dx_2}{dt} = 0 \tag{33}$$

$$rx_2^* = x_1^* x_2^* \tag{34}$$

When $x_1^* = 0$, we have:

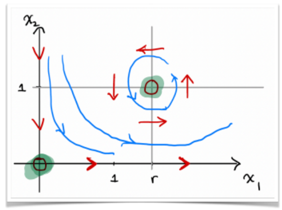
$$rx_2^* = 0, \ \ x_2^* = 0 \tag{35}$$

When $x_2^* = 1$, we have:

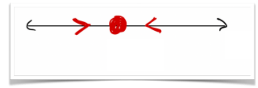
$$x_1^* = r \tag{36}$$

Therefore the fixed points will be $[0, 0]^T$ and $[r, 1]^T$. The phase diagram is shown below:

Phase diagram (2D)



Recall 1D idea:



Consider the nullclines:

$Dx_1 = 0$ or $Dx_2 = 0$

$(Dx_2 = ?)$ $(Dx_1 = ?)$

Nullclines $Dx_1 = 0$: (1) $x_1 = 0 \to Dx_2 = -rx_2$

(2) $x_2 = 1 \to Dx_2 = -r + x_1 = \begin{cases} > 0 & x_1 > r \\ < 0 & x_1 < r \end{cases}$

Nullclines $Dx_2 = 0$: (1) $x_2 = 0 \to Dx_1 = x_1$

44 (2) $x_1 = r \to Dx_1 = r(1 - x_2) = \begin{cases} > 0 & x_2 < 1 \\ < 0 & x_2 > 1 \end{cases}$

Figure 4: Phase Diagram