Jet

1 Free Shear Flow

1.1 Overview

Free shear flows are types of fluid flows far away from solid boundaries in which the velocity varies across the flow, causing shearing action. This variation in velocity causes momentum to be transferred from one layer to another, leading to shear stress.

1.2 Types

1. **Jets:** These occur when fluid is ejected from a nozzle into a still fluid.
2. **Wakes:** These occur when fluid flows past a bluff body, causing a region of low pressure and turbulent flow behind the body.
3. **Mixing layers:** These occur when two parallel streams of fluid at different velocities come into contact.
4. **Shear layers:** These occur when there is a significant velocity difference across a relatively thin layer of fluid.

2 Jets

2.1 Assumptions

1. Incompressible, 2D
2. Steady flow
3. Momentum diffuses from a central source, spreads over increasingly wider region downstream, but momentum is conserved overall.

2.2 Governing Equations

Start from incompressible, 2D, steady flow:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

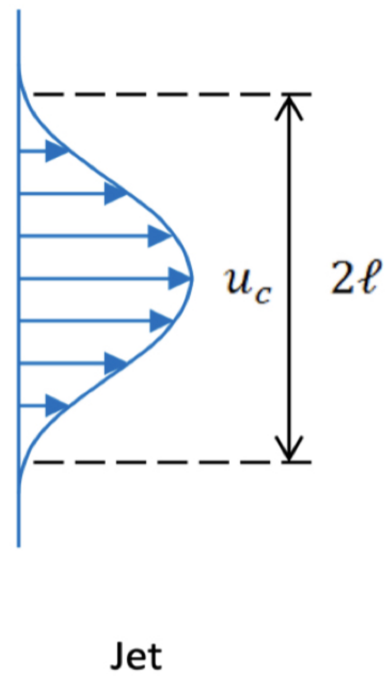

Figure 1: Jet.

$$\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})=-\frac{\partial p}{\partial x}+\mu[\frac{\partial^2 u}{\partial x\partial x}+\frac{\partial^2 u}{\partial y\partial y}] \tag{2}$$

$$\rho(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})=-\frac{\partial p}{\partial y}+\mu[\frac{\partial^2 v}{\partial x\partial x}+\frac{\partial^2 v}{\partial y\partial y}] \tag{3}$$

Because no solid boundaries to impose any pressure gradients, and x momentum is much larger than y momentum:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{4}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y\partial y} \tag{5}$$

LHS:

$$\begin{aligned}&u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+u[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}]\\&=2u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+u\frac{\partial v}{\partial y}\\&=2u\frac{\partial u}{\partial x}+\frac{\partial(uv)}{\partial y}\end{aligned} \tag{6}$$

Put it back:

$$\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y} = \nu \frac{\partial^2 u}{\partial y \partial y} \tag{7}$$

Integrate over y:

$$\int_{-\infty}^{\infty} [\frac{\partial u^2}{\partial x} + \frac{\partial (uv)}{\partial y}]\, dy = \int_{-\infty}^{\infty} \nu \frac{\partial^2 u}{\partial y \partial y}\, dy \tag{8}$$

$$\int_{-\infty}^{\infty} \frac{\partial u^2}{\partial x}\, dy + uv|_{-\infty}^{\infty} = \nu \frac{\partial u}{\partial y}|_{-\infty}^{\infty} \tag{9}$$

Boundary Conditions:

1. $u(x, \infty) = 0$
2. $u(x, 0) = U_0(x)$
3. $u(x, -\infty) = 0$

Therefore,

$$\int_{-\infty}^{\infty} \frac{\partial u^2}{\partial x}\, dy = 0 \tag{10}$$

$$\frac{\partial}{\partial x} \int_{-\infty}^{\infty} u^2\, dy = 0 \tag{11}$$

$$\int_{-\infty}^{\infty} u^2\, dy = const \tag{12}$$

Here, we define M as the momentum flux,

$$M = \rho \int_{-\infty}^{\infty} u^2\, dy \tag{13}$$

Now we assume $u(x, y) = U_0(x) g(\eta)$:

$$\frac{M}{\rho} = \int_{\infty}^{\infty} U_0^2(x) g^2(\eta) \delta(x)\, d\eta \tag{14}$$

Based on momentum conservation, we get:

$$U_0^2(x)\delta(x) = const \tag{15}$$

O.M. estimates terms in momentum equation

$$u\frac{\partial u}{\partial x} \sim U_0\frac{dU_0}{dx} \quad \text{whereas} \quad \nu\frac{\partial^2 u}{\partial y^2} \sim \nu\frac{U_0}{\delta^2}$$

If convective terms and diffusion terms are in balance

$$U_0\frac{dU_0}{dx} \sim \nu\frac{U_0}{\delta^2} \Rightarrow \delta^2\frac{dU_0}{dx} = const \tag{2}$$

Relations (1) and (2) will allow us to deduce the forms of $U_0(x)$ and $\delta(x)$, if we suppose they are power laws in x. (More later...)

Evolution of centerline velocity and thickness

We deduced (1) $U_0^2\delta = const$ and (2) $\delta^2 dU_0/dx = const$.

Also expect that U_0 decreases with x, while δ increases with x, in both cases monotonically. If series expansions are used, at sufficiently large x rates of increase or decrease will be driven by term with largest positive/negative exponent. Power-law dependences thus reasonable.

$$\text{Let} \;\; \delta(x) \propto x^m \;\; ; \;\; U_0(x) \propto x^n$$

Then, (1) $\Rightarrow m + 2n = 0$; (2) $\Rightarrow 2m + n - 1 = 0$. We get

$$\delta(x) \propto x^{2/3} \;\; ; \;\; U_0(x) \propto x^{-1/3}$$

(Note: this means $Re = U_0\delta/\nu$ increases with x: this flow has a tendency to become turbulent downstream.)

Seeking a similarity solution

Define $\eta = y/\delta(x)$ and use stream function (for both u and v)

$$\eta = (y/B)x^{-2/3} \quad ; \quad \psi \sim U_0\delta = Ax^{1/3}f(\eta)$$

where A and B are dimensional constants, to be determined later.

As done in other contexts earlier, use chain rule to transform each term in streamwise momentum equation (recall, $dp/dx = 0$)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{B}x^{-2/3} \quad ; \quad \frac{\partial \eta}{\partial x} = -\frac{2}{3}\frac{\eta}{x}$$

$$u = \frac{\partial \psi}{\partial y} = \frac{A}{B}f' x^{-1/3} \; ; \;\; v = -\frac{\partial \psi}{\partial x} = -\frac{A}{3}x^{-2/3}[f - 2\eta f'], \;\; \text{etc.}$$

After some straightforward manipulation, transformed equation is

$$f''' + (AB/3\nu)(f'^2 + ff'') = 0$$

Subject to the boundary conditions:

(a) $u = 0$ at $y = \pm\infty$ i.e. zero freestream velocity if discharging into stagnation surroundings. That is, $f'(\infty) = 0$.

(b) Symmetric velocity profile: $f'(\eta) = f(-\eta) \Rightarrow f''(0) = 0$

(c) $y = 0$ being a streamline: $v = 0$ at $y = 0 \Rightarrow f(0) = 0$

Math leading to solution of the transformed ODE

Although nonlinear, product rule allows integrating once to

$$f'' + (AB/3\nu)(ff') = C_1$$

Boundary conditions (b) and (c) imply entire LHS is zero at $\eta = 0$. This means C_1 must be 0. Then, a second integration gives

$$f' + (1/2)(AB/3\nu)(f^2) = C_2$$

On the centerline, $f(0) = 0$ while $f'(0) = 1$ (since $u = U_0$ there). This requires $C_2 = 1$. Further, since we still have freedom with the constants A and B, let's set the coefficient on the LHS to unity, by stipulating $AB/(3\nu) = 2$.

With these steps, we get

$$f' = 1 - f^2 \; .$$

This belongs to a class of ODEs called the Riccati Equation, where the RHS is a second-order polynomial of the unknown function, with coefficients that may be fixed or are functions of the independent variable alone). (See books on Differential Equations, if interested) The solution is

$$f = \tanh(\eta)$$

For the hyperbolic trigonometric functions: useful facts include

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}) \;\; ; \;\; \cosh(x) = \tfrac{1}{2}(e^x + e^{-x})$$
$$\cosh^2(x) - \sinh^2(x) \equiv 1$$
$$\tanh(x) = (e^x - e^{-x})/(e^x + e^{-x}) \; .$$

The streamwise velocity profile is given by

$$u = \frac{A}{B} x^{-1/3} \operatorname{sech}^2 \left[\frac{y}{B} x^{-2/3} \right]$$

The usual choices of A and B are such that

$$A = \left(\frac{9\nu M}{2\rho} \right)^{1/3} \;\; ; \;\; B = \left(\frac{48\nu^2\rho}{M} \right)^{1/3}$$

Using these results, it can be verified that, due to entrainment and mass conservation respectively:

$$\int_{-\infty}^{\infty} u \, dy \;\; \uparrow \text{with } x \;\; ; \;\; \int_{-\infty}^{\infty} u^2 \, dy = M/\rho \;\; \text{(constant)}$$

For reference, in the case of a round jet, $U_0 \propto x^{-1}$ while $\delta \propto x$. (Of course will decrease and increase with x respectively, but at a rate that is geometry-dependent.)