Navier-Stokes Equations

1 Overview

1.1 History

The Navier-Stokes equations describe the motion of fluid substances, including liquids, gases, and plasmas, and are named after the French mathematician and physicist Claude-Louis Navier and the Irish mathematician and physicist George Gabriel Stokes.

The history of the Navier-Stokes equations can be traced back to the early 19th century. It begins with the work of French mathematician and engineer Claude-Louis Navier, who in 1822 formulated the equations governing the motion of viscous fluids based on Isaac Newton's second law of motion and the concept of viscous stress.

In 1845, George Gabriel Stokes, an Irish mathematician and physicist, independently derived similar equations for the motion of fluids. Stokes' equations were more general than Navier's, as they also applied to inviscid (non-viscous) fluids. Stokes combined his work with Navier's, and the resulting equations became known as the Navier-Stokes equations.

The Navier-Stokes equations are partial differential equations that describe the conservation of momentum and mass in fluid flow. They have been widely used in engineering, physics, and meteorology to model and analyze fluid dynamics problems. Despite their importance and widespread applications, there is still no general solution to the equations, and the existence and smoothness of solutions remain open questions in mathematics. This has led to the development of various numerical methods, such as finite element methods and finite volume methods, to approximate solutions for practical problems.

1.2 Definition

Based on the definition from NASA page [1], the NS Equations include conservation of mass, momentum and energy equations (Fig. 1):

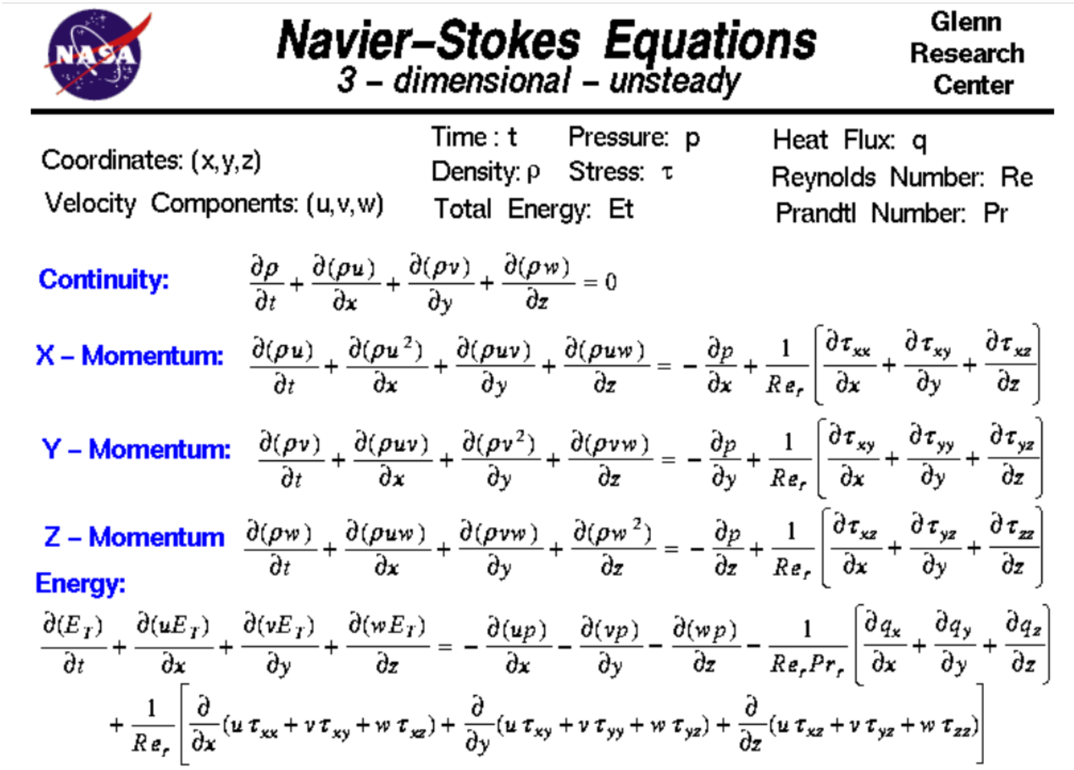

Figure 1: NS Equations Definitions from NASA

2 NS Equations Derivation (Momentum Part)

2.1 Conservation of Momentum

$$Rate\,of\,change\,of\,mass\,in\,CV = Net\,mass\,flow\,rate\,goes\,across\,CS\,\boldsymbol{into}\,CV \tag{1}$$

2.2 Derivation (without Force)

$$\frac{\partial}{\partial t}\int_{CV} \rho \underline{\boldsymbol{u}}\, dV = -\int_{CS} \rho \underline{\boldsymbol{u}}(\underline{\boldsymbol{u}}\hat{\boldsymbol{n}})\, dS + \underline{\boldsymbol{F}} \tag{2}$$

Ignore the force term first, and use the divergence theorem:

$$\frac{\partial}{\partial t}\int_{CV} \rho \underline{\boldsymbol{u}}\, dV = -\int_{CV} \nabla \cdot (\rho \underline{\boldsymbol{u}}\underline{\boldsymbol{u}})\, dV \tag{3}$$

$$LHS = \int_{CV} [\underline{\boldsymbol{u}}\frac{\partial \rho}{\partial t} + \rho\frac{\partial \underline{\boldsymbol{u}}}{\partial t}]\, dV \tag{4}$$

$$RHS = -\int_{CV} \rho[(\underline{\boldsymbol{u}} \cdot \nabla)\underline{\boldsymbol{u}}] + \underline{\boldsymbol{u}}[\rho(\nabla \cdot \underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} \cdot \nabla\rho] \, dV \tag{5}$$

Combine them, we can get:

$$\underline{\boldsymbol{u}}\frac{\partial \rho}{\partial t} + \rho\frac{\partial \underline{\boldsymbol{u}}}{\partial t} = -[\rho[(\underline{\boldsymbol{u}} \cdot \nabla)\underline{\boldsymbol{u}}] + \underline{\boldsymbol{u}}[\rho(\nabla \cdot \underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} \cdot \nabla\rho]] \tag{6}$$

$$\underline{\boldsymbol{u}}(\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} \cdot \nabla\rho) + \rho(\frac{\partial \underline{\boldsymbol{u}}}{\partial t} + (\underline{\boldsymbol{u}} \cdot \nabla)\underline{\boldsymbol{u}}) = 0 \tag{7}$$

From continuity equation, we know:

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot \underline{\boldsymbol{u}}) + \underline{\boldsymbol{u}} \cdot \nabla\rho = 0 \tag{8}$$

Therefore:

$$\rho\frac{D\underline{\boldsymbol{u}}}{Dt} = 0 \tag{9}$$

or:

$$\int_{CV} \rho\frac{D\underline{\boldsymbol{u}}}{Dt} \, dV = 0 \tag{10}$$

2.3 Force Acting on Fluid

The force acting on the fluid could be divided into two parts, which is body force and surface force.

2.3.1 Body Force

Body force refers to a force that acts throughout the volume of a body. Forces due to gravity, electric fields and magnetic fields are examples of body forces. In tensor form, body force can be expressed as:

$$\int_{CV} \rho f_i \, dV \tag{11}$$

2.3.2 Surface Force

Surface force refers to a force which is exerted to the surface of an object. This force is usually expressed by the integration of stress over the surface:

$$\int_{CS} \tau_{ji} n_j \, dS = \int_{CV} \frac{\partial \tau_{ji}}{\partial x_j} \, dV \tag{12}$$

The stress can be divided into normal stress and shear stress, the basic definition could be find in Viscosity section. In tensor form, τ_{ij} represents the force per unit area exerted along the j^{th} unit vector, on a surface whose normal is in the i^{th} direction. Notice that the stress tensor is symmetric (Proof), τ_{ji} is used in previous equation, to be consistent with the body force so that the force direction is in i^{th} direction.

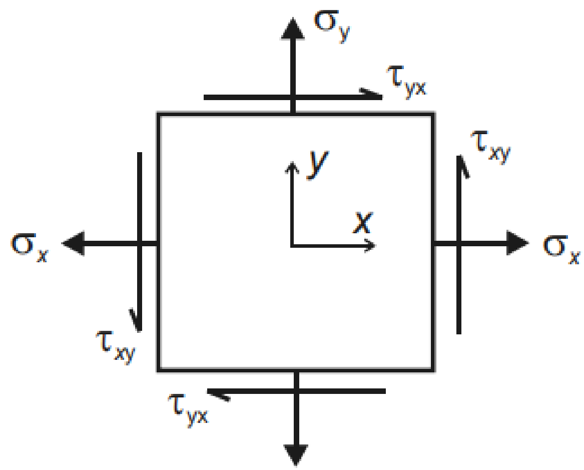

Figure 2: Normal Stress and Shear Stress[2].

Simple Proof: Consider a square fluid element, dx on each side. Shear stress on the right and top faces may induce a moment about the lower-left corner. If the moment is sustained, fluid element will rotate at rate depending on its moment of inertia:

$$I\omega = (\tau_{xy} - \tau_{yx})dx \tag{13}$$

Based on the definition of moment of inertia, $I \propto (dx)^2$, if one shrinks the element to a very small volume ($dx \to 0$), the rotational acceleration of the element ω will diverge to infinity unless the shear stress difference $\tau_{xy} - \tau_{yx}$ also tends to zero at least as fast as $(dx)^2 \to 0$. Since infinite rotational acceleration is not physically possible the stress tensor should be symmetric, so $\tau_{xy} = \tau_{yx}$.

2.3.3 Local Relative Motion

Using Taylor Series, we get:

$$du_i = \frac{\partial u_i}{\partial x_j} dx_j \tag{14}$$

Therefore, the Velocity Gradient Tensor is:

$$A = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} \tag{15}$$

Or in tensor form:

$$A_{ij} = \frac{\partial u_i}{\partial x_j} \tag{16}$$

Any second order square matrix could be split into the sum of a symmetric part and an anti-symmetric part:

$$\underline{\underline{\boldsymbol{A}}} = \frac{1}{2}(\underline{\underline{\boldsymbol{A}}} + \underline{\underline{\boldsymbol{A}}}^T) + \frac{1}{2}(\underline{\underline{\boldsymbol{A}}} - \underline{\underline{\boldsymbol{A}}}^T) \tag{17}$$

Where the transpose of A_{ij} is A_{ji}. Then we define the **strain rate** and **rotation rate** as:

$$S_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = S_{ji} \tag{18}$$

$$R_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) = R_{ji} \tag{19}$$

2.3.4 Related to Changes in Shape, Orientation and Size

Overview:

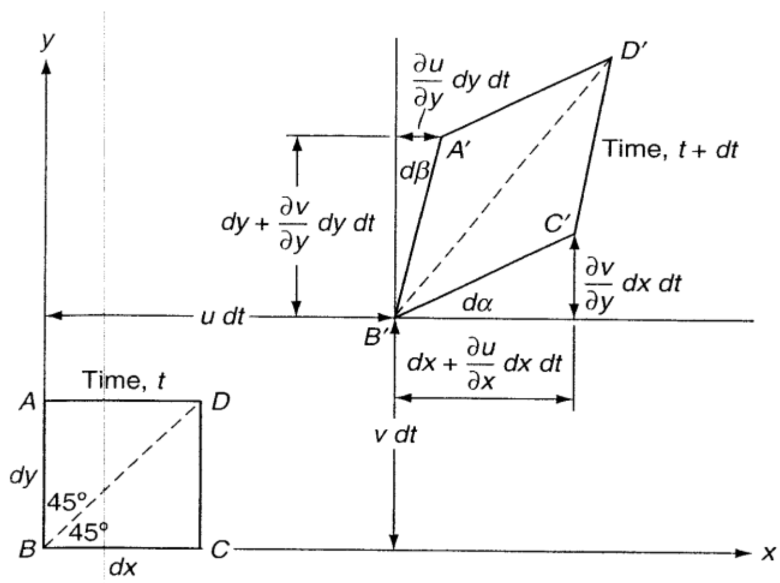

Figure 3: Distortion of Fluid Element[3].

1. $\dfrac{\partial u}{\partial y}$ being the only nonzero V.G. and positive: The fluid element is distorted in x direciton, but the volume stays the same.

2. $\dfrac{\partial v}{\partial x}$ being the only nonzero V.G. and positive: The fluid element is distorted in y direction, but the volume stays the same.

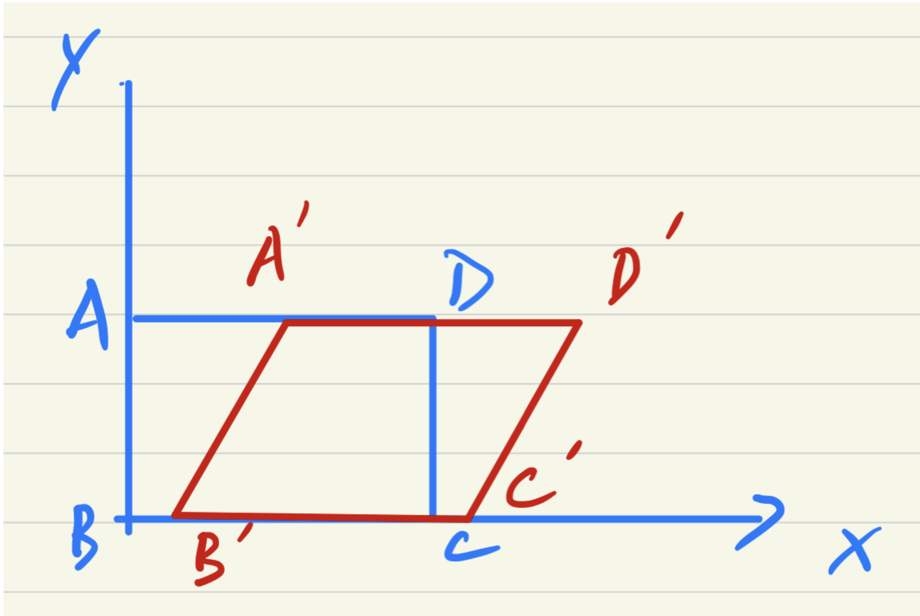

Figure 4: Case 1; Only positive $\dfrac{\partial u}{\partial y}$

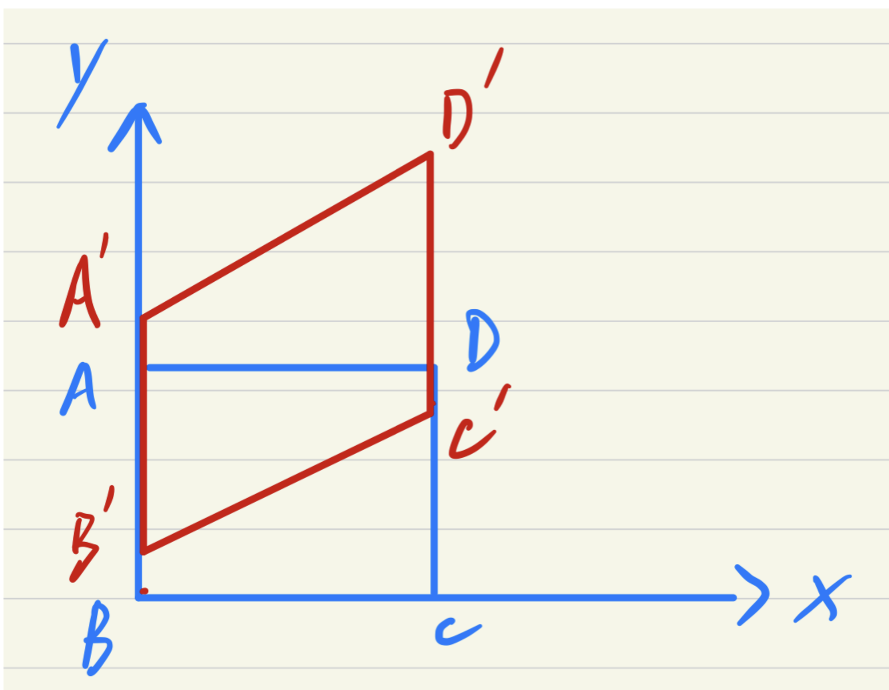

Figure 5: Case 2; Only positive $\dfrac{\partial v}{\partial x}$

3. $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial x}$ being the only nonzero V.G.s, both positive and equal to each other: The fluid element is distorted in both directions, but the volume stays the same. Notice that there is no rotation here, the change of position is due to the time.

4. $\dfrac{\partial u}{\partial x}$ being the only nonzero V.G. and positive: The fluid element is expanding in

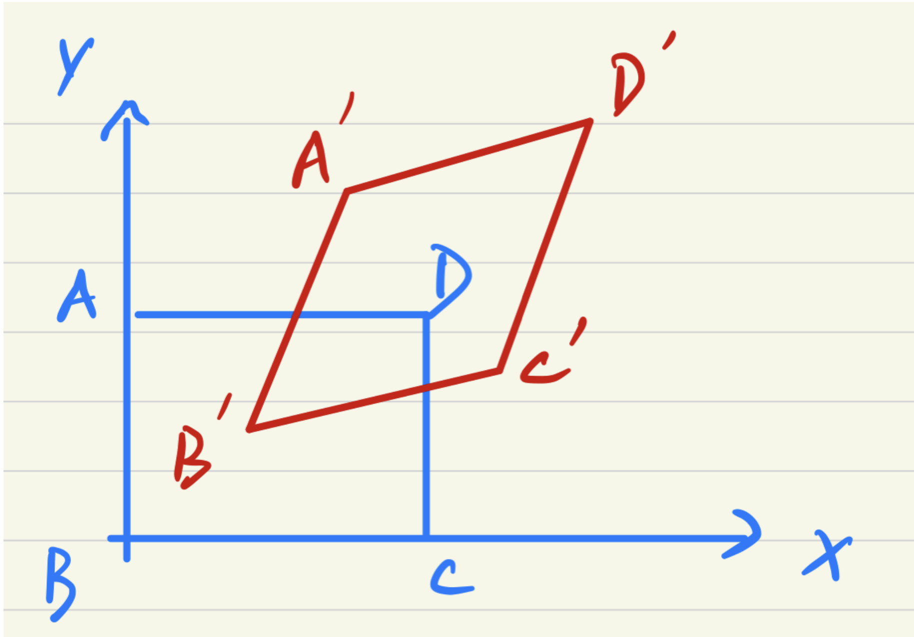

Figure 6: Case 3; Only positive $\dfrac{\partial v}{\partial x}$ and $\dfrac{\partial u}{\partial y}$, equal to each other

x direction, volume increases.

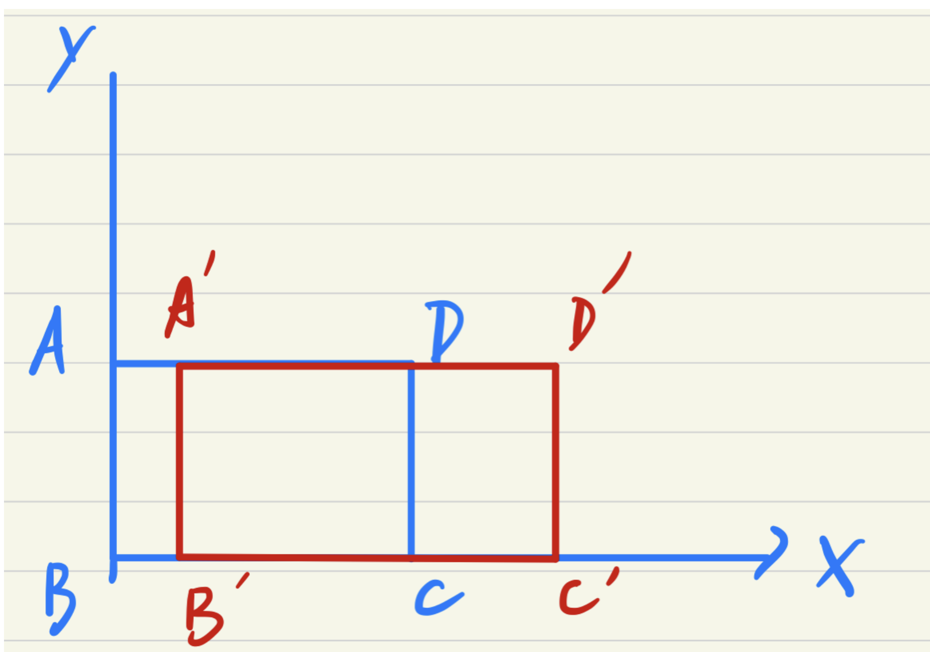

Figure 7: Case 4; Only positive $\dfrac{\partial u}{\partial x}$

5. $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial y}$ being the only nonzero V.G.s, equal in magnitude but opposite in sign: The fluid element shape is changing, but volume stays the same.

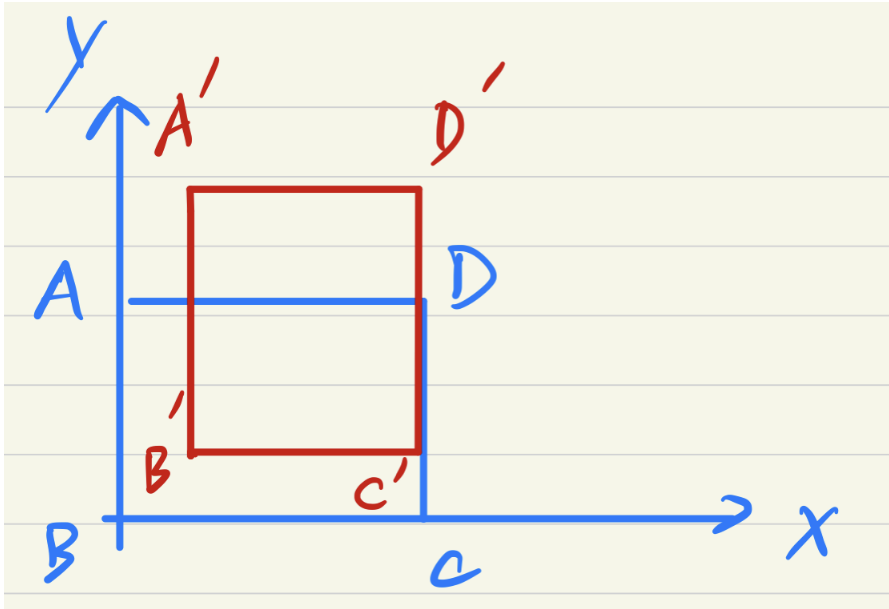

Figure 8: Case 5; Only positive $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$, equal but opposite sign.

2.4 Derivation (with Force)

Combine previous equations in tensor form:

$$\rho\frac{Du_i}{Dt} = \rho f_i + \frac{\partial \tau_{ji}}{\partial x_j} \tag{20}$$

2.4.1 Pressure

Pressure contributes to normal stresses only, is compressive and same in all directions:

$$\tau_{ij}^p = -p\delta_{ij} \tag{21}$$

If the fluid is not in motion, there is no viscous effect so this would be the entire τ_{ij}. Therefore:

$$\tau_{ii} = -p\delta_{ii} \tag{22}$$

$$p = -\frac{1}{3}\tau_{ii} \tag{23}$$

Note that this is the mechanical pressure. If fluid is in motion, there will be viscous effects and the fluid is not in equilibrium, which means the pressure is not the same as the thermodynamic pressure (equilibrium required), which appears in $P = \rho RT$ for perfect gas.

2.4.2 Viscous Stress

To separate the pressure effect from the stress tensor, we define σ_{ij} as the viscous stress tensor:

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij} \tag{24}$$

Recall that the viscous effects are tied to relative motion of adjacent fluid elements. Also we already know τ_{ij} is symmetric and pressure only affects the diagonal terms in stress tensor, so σ_{ij} is also symmetric.

Based on the definition of Newtonian fluids:

$$\sigma_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \lambda(\nabla \cdot \underline{\boldsymbol{u}})\delta_{ij} \tag{25}$$

Where λ is the bulk viscosity, related to volumetric changes of fluid.

2.4.3 Stokes Hypothesis

In previous section, the mechanical pressure if no motion is defined as:

$$p = -\frac{1}{3}\tau_{ii} \tag{26}$$

If the fluid is in motion, the mechanical pressure should still stay the same, which means:

$$\sigma_{ii} = 0 \tag{27}$$

$$\nabla \cdot \underline{\boldsymbol{u}} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \Delta \tag{28}$$

$$\begin{aligned} 0 &= 2\mu S_{ij} + \lambda\Delta\delta_{ii} \\ &= (2\mu + 3\lambda)\Delta \end{aligned} \tag{29}$$

Therefore, $\lambda = -\frac{2\mu}{3}$.

2.4.4 Final Derivation

$$\tau_{ji} = -p\delta_{ji} + 2\mu S_{ji} - \frac{2}{3}\mu\Delta\delta_{ji} \tag{30}$$

$$\begin{aligned} \frac{\partial \tau_{ji}}{\partial x_j} &= \frac{\partial}{\partial x_j}(-p\delta_{ji} + 2\mu S_{ji} - \frac{2}{3}\mu\Delta\delta_{ji}) \\ &= -\frac{\partial p}{\partial x_i} + 2\mu\frac{\partial S_{ji}}{\partial x_j} - \frac{2}{3}\mu\frac{\partial \Delta}{\partial x_i} \end{aligned} \tag{31}$$

$$\begin{aligned}
\frac{\partial S_{ji}}{\partial x_j} &= \frac{\partial}{\partial x_j}[\frac{1}{2}(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j})] \\
&= \frac{1}{2}(\frac{\partial^2 u_j}{\partial x_i \partial x_j}+\frac{\partial^2 u_i}{\partial x_j \partial x_j}) \\
&= \frac{1}{2}[\frac{\partial}{\partial x_i}(\frac{\partial u_j}{\partial x_j})+\frac{\partial^2 u_i}{\partial x_j \partial x_j}] \\
&= \frac{1}{2}(\frac{\partial \Delta}{\partial x_i}+\frac{\partial^2 u_i}{\partial x_j \partial x_j})
\end{aligned} \tag{32}$$

Plug back in:

$$\frac{\partial \tau_{ji}}{\partial x_j} = -\frac{\partial p}{\partial x_i}+\frac{1}{3}\mu\frac{\partial \Delta}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{33}$$

$$\rho\frac{Du_i}{Dt} = \rho f_i-\frac{\partial p}{\partial x_i}+\frac{1}{3}\mu\frac{\partial \Delta}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{34}$$

Or in vector form:

$$\rho\frac{D\underline{\boldsymbol{u}}}{Dt} = \rho\underline{\boldsymbol{f}}-\nabla p+\frac{1}{3}\mu\nabla\Delta+\mu\nabla^2\underline{\boldsymbol{u}} \tag{35}$$

If we ignore the body force and assume the density is constant, from continuity:

$$\nabla\cdot\underline{\boldsymbol{u}} = \Delta = 0 \tag{36}$$

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i}+\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{37}$$

3 Relations with Other Equations

3.1 Euler Equation

If we assume the only body force is gravitational force, and assume the flow is inviscid, then we get the Euler Equation from NS Equation:

$$\rho\frac{D\underline{\boldsymbol{u}}}{Dt} = \rho\underline{\boldsymbol{g}}-\nabla p \tag{38}$$

3.2 Bernoulli Equation

Under the Euler's Equation condition, if the flow is steady, incompressible, frictionless, one dimensional, the integration along a streamline between any points 1 and 2 will give the Bernoulli Equation:

$$(p+\frac{1}{2}\rho u^2+\rho gz)_1 = (p+\frac{1}{2}\rho u^2+\rho gz)_2 \tag{39}$$

References

[1] URL https://www.grc.nasa.gov/www/k-12/airplane/nseqs.htmll.

[2] URL https://civilengineeronline.com/mech/stress-trans.php.

[3] F. M. White. Viscous Fluid Flow. McGraw-Hill, 2006.