The Kummer Construction of Calabi-Yau and Hyper-Kähler Metrics on the $K3$ Surface, and Large Families of Volume Non-collapsed Limiting Compact Hyper-Kähler Orbifolds

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November 2024

Abstract

Right after Yau's resolution of the Calabi conjecture in the late 1970s, physicists Page [32] and Gibbons-Pope [16] conjectured that one may approximate Ricci-flat Kähler metrics on the $K3$ surface with metrics having "almost special holonomy" constructed via "resolving" the 16 orbifold singularities of a flat $\mathbb{T}^4/\mathbb{Z}_2$ with Eguchi-Hanson metrics. Constructions of such metrics with special holonomy from such a "gluing" construction of approximate special holonomy metrics have since been called "Kummer constructions" of special holonomy metrics, and their proposal was rigorously carried out in the 1990s by Kobayashi [23] and LeBrun-Singer [27], and in the 2010s by Donaldson [13].

In this paper, we provide two new rigorous proofs of Page-Gibbons-Pope's proposal based on singular perturbation and weighted function space analysis. Each proof is done from a different perspective:

- solving the complex Monge-Ampere equation (Calabi-Yau)
- perturbing closed definite triples (hyper-Kähler)

Both proofs yield Eguchi-Hanson metrics as ALE bubbles/rescaled limits. Moreover, our analysis in the former perspective yields estimates which *improve* Kobayashi's estimates, and our analysis in the latter perspective results in the construction of *large* families of Ricci-flat Kähler metrics on the $K3$ surface, yielding the full 58 dimensional moduli space of such.

Finally, as a byproduct of our analysis, we produce a plethora of large families of compact hyper-Kähler *orbifolds* which all arise as volume non-collapsed Gromov-Hausdorff limit spaces of the aforementioned constructed large families of Ricci-flat Kähler metrics on the $K3$ surface. Moreover, these compact hyper-Kähler orbifolds are explicitly exhibited as points in the "holes" of the moduli space of Ricci-flat Kähler-Einstein metrics on $K3$ under the period map.

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1   § Introduction & Survey §

A $K3$ **surface** is the underlying smooth oriented 4-manifold of a connected compact simply connected complex 2-manifold with trivial canonical bundle.

By a theorem of Kodaira, any pair of $K3$ surfaces are diffeomorphic to one another, whence we may speak of *the* $K3$ surface (up to diffeomorphism).[1] Hence any connected compact simply connected complex 2-fold with trivial canonical bundle is realized by a pair $(K3, J)$ with J an *integrable* complex structure, whence are called **complex $K3$ surfaces** with specified complex structure.

By a theorem of Siu, all complex $K3$ surfaces $(K3, J)$ are **Kähler**. That is, they admit Kähler metrics, which are Riemannian metrics with holonomy contained in $\mathrm{U}\left(\frac{n}{2}\right)$ (n being the real dimension of the underlying manifold). Yau's resolution of the Calabi conjecture therefore implies that any complex $K3$ surface $(K3, J)$ admits a unique **Ricci-flat Kähler metric** in *any* given Kähler class. Moreover, all Ricci-flat Kähler metrics have holonomy[2] contained in $\mathrm{SU}\left(\frac{n}{2}\right)$, and Riemannian metrics with such holonomy are called **Calabi-Yau** and are automatically Ricci-flat Kähler. Furthermore, Riemannian metrics with holonomy further contained in $\mathrm{Sp}\left(\frac{n}{4}\right)$ (which is itself contained in $\mathrm{SU}\left(\frac{n}{2}\right)$) are called **hyper-Kähler**, and are automatically Ricci-flat.

A crucial difference between Calabi-Yau metrics and hyper-Kähler metrics are that the former comes with a *fixed* (metric compatible) complex structure, while the latter comes with a *an entire 2-sphere/S^2s worth* of (metric compatible) complex structures. Hence a Calabi-Yau metric is Ricci-flat *Kähler* WRT a *single* complex structure and is a *complex geometric* object, whereas a hyper-Kähler metric is Ricci-flat *Kähler* WRT a *family* of complex structures, hence by virtue of this 2-sphere ambiguity, is a *real geometric* object.

However, in real dimension 4, we have the special isomorphism of Lie groups $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$, whence being Calabi-Yau and hyper-Kähler are *equivalent*. Therefore, all Ricci-flat Kähler metrics on the $K3$ surface are automatically both Calabi-Yau and hyper-Kähler.

Moreover, the **local Torelli Theorem** [22] tells us that there is a local diffeomorphism

$$\mathcal{P} : \mathcal{M}_{\mathrm{Vol}=1}^{K3} \to \mathrm{Gr}^+(3, 19)/\Gamma := \Gamma\backslash\mathrm{SO}(3, 19)/\mathrm{SO}(3) \times \mathrm{SO}(19)$$
$$g \mapsto \mathcal{H}_g^+$$

called the **period map** between the **moduli space** of all *unit volume* Ricci-flat Kähler metrics on $K3$ (hence the full moduli space without any volume normalization is denoted as $\mathcal{M}^{K3} = \mathbb{R}_+ \times \mathcal{M}_{\mathrm{Vol}=1}^{K3}$), and the Grassmannian of all positive 3-planes in $\mathbb{R}^{3,19} \cong H^2(K3; \mathbb{R})$ mod $\Gamma \leq \mathrm{GL}(22, \mathbb{Z})$ the automorphisms of the lattice $H^2(K3; \mathbb{Z})$ equipped with the intersection form $([\alpha], [\beta]) \mapsto \int_{K3} \alpha \wedge \beta$. This local diffeomorphism sends each g to $\mathcal{H}_g^+$, the 3-dimensional subspace of **self-dual harmonic 2-forms** WRT g.

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[1] Though, there do exist *topological $K3$ surfaces*, which are real 4-manifolds homeomorphic to a $K3$ but *not* diffeomorphic to it.
[2] Technically speaking, this is supposed to be the *restricted holonomy*, which is the identity component of the holonomy group.

However, this local diffeomorphism $\mathcal{P} : \mathscr{M}_{\text{Vol}=1}^{K3} \to \text{Gr}^+(3, 19)/\Gamma$ is *not* surjective: it is only surjective onto

$$\text{Gr}^+(3, 19)/\Gamma - \bigcup_{\substack{\Sigma \in H_2(K3;\mathbb{Z}) \\ \Sigma \cdot \Sigma = -2}} \left\{ H \in \text{Gr}^+(3, 19)/\Gamma : [\alpha](\Sigma) = 0, \forall [\alpha] \in H \right\}$$

which is the complement of a *codimension 3* subset.

In 1987, Kobayashi and Todorov [24] proved that points in these codimension 3 "holes" should correspond to Ricci-flat metrics with **orbifold singularities**. In fact, upon compactifying $\mathscr{M}^{K3}$ WRT the **Gromov-Hausdorff topology** to get $\overline{\mathscr{M}^{K3}}$, the results of general Riemannian convergence theory (which appeared shortly after Kobayashi-Todorov), and more specifically the results of Anderson [1] Bando-Kasue-Nakajima [3] and Nakajima [30] may be immediately applied to tell us that *any* Gromov-Hausdorff limit of any sequence $\{g_i\} \subset \mathscr{M}^{K3}$ which is **volume non-collapsed** must be a compact **hyper-Kähler orbifold** with finitely many isolated singular points. Moreover, such a sequence admits noncompact hyper-Kähler **ALE spaces** as **bubbles/rescaled limits**, which model the convergence near the (orbifold) singularities. In fact, upon denoting the **partial Gromov-Hausdorff compactification** of $\mathscr{M}^{K3}$ ($\mathscr{M}_{\text{Vol}=1}^{K3}$) by adding in all compact (unit volume) hyper-Kähler orbifolds as $\overline{\mathscr{M}^{K3}}^{\partial}$ ($\overline{\mathscr{M}_{\text{Vol}=1}^{K3}}^{\partial}$), it follows from Kobayashi-Todorov [24] and Anderson [2] that the period map extends to a *bijection*

$$\mathcal{P} : \overline{\mathscr{M}_{\text{Vol}=1}^{K3}}^{\partial} \to \text{Gr}^+(3, 19)/\Gamma$$

Hence when adding in all volume non-collapsed limits, these volume non-collapsed limits should correspond to points in the codimension 3 subset $\bigcup_{\substack{\Sigma \in H_2(K3;\mathbb{Z}) \\ \Sigma \cdot \Sigma = -2}} \{H \in \text{Gr}^+(3, 19)/\Gamma : [\alpha](\Sigma) = 0, \forall [\alpha] \in H\}$ in which the original period map fails to be surjective. Moreover, **continuity** of the period map WRT the Gromov-Hausdorff topology across these codimension 3 orbifold metrics was proved by Anderson in [2] (with a recent shorter proof by Odaka-Oshima [31]), who in fact showed that $\overline{\mathscr{M}_{\text{Vol}=1}^{K3}}^{\partial}$ and $\text{Gr}^+(3, 19)/\Gamma$ are **isometric** as Riemannian manifolds where the former's metric is the L^2 **metric** (which induces a stronger topology than the Gromov-Hausdorff topology) and we view $\overline{\mathscr{M}_{\text{Vol}=1}^{K3}}^{\partial}$ as the L^2 **completion** of $\mathscr{M}_{\text{Vol}=1}^{K3}$, and the latter with the (locally) symmetric metric coming from the Killing form of SO(3, 19).

Since both Yau's theorem and the general convergence theorems above respectively only give us abstract existence and abstract convergence behavior of these metrics on $K3$, it is therefore instructive to find more explicit/"hands-on" examples of these Ricci-flat metrics & their moduli behavior.

After Yau's proof, physicists Page [32] and Gibbons-Pope [16] suggested that a more concrete description of these Ricci-flat Kähler metrics on $K3$ may be obtained via first viewing $K3$ as a **Kummer surface** Km, and then approximating those Ricci-flat metrics by gluing in 16 copies of a particular ALE hyper-Kähler metric called the **Eguchi-Hanson metric** around each of the 16 singular points of the flat orbifold $\mathbb{T}^4/\mathbb{Z}_2$. Such a proposal has become known in the literature as the **Kummer construction** of (special, usually Ricci-flat) metrics, and will be referred to as such. This proposal was first done rigorously by Kobayashi [23] and LeBrun-Singer [27] (following an earlier attempt by Topiwala [36]) in the early to mid 1990s, and later in the 2010s by Donaldson [13].

Hence the original Kummer construction of Ricci-flat Kähler metrics on $K3$ is by now a classical topic in geometric analysis, having become the blueprint for and spawned multiple different variations and generalizations throughout the decades.

LeBrun-Singer's proof of the original Kummer construction uses **twistor theory**, while Kobayashi and Donaldson's proofs are more modern and uses PDE, specifically the **complex Monge-Ampere equation**. The former's proof is via the **continuity method**, and (with recent corrections by Lye [28], who went on to further study **stable geodesics** on the resulting metric on $K3$) consists primarily of deriving **a priori estimates** for the unknown scalar function appearing in the complex Monge-Ampere equation. Donaldson's proof, on the other hand, is based off **singular perturbation** and primarily uses the **implicit function theorem**. Proving the existence of Ricci-flat Kähler metrics using complex Monge-Ampere requires a *fixed* complex structure, whence without further work, can only be used to produce Calabi-Yau metrics. While being Calabi-Yau and hyper-Kähler in real dimension 4 are equivalent, fixing a complex structure in the Kummer construction does *not* yield an open subset/full parameter

family of the $58 = \dim_{\mathbb{R}} \mathscr{M}^{K3} = \dim_{\mathbb{R}} \mathscr{M}^{K3}_{\text{Vol}=1} + 1$ dimensional[3] moduli space of all Ricci-flat Kähler metrics on $K3$. This is because the complex structure on $K3$ in the Kummer construction is constructed via patching *the same choice* of complex structure on the 16 copies of the hyper-Kähler Eguchi-Hanson spaces to the background complex structure on $\mathbb{T}^4/\mathbb{Z}_2$. And since such a choice of complex structure is a choice of direction in S^2, fixing the same direction in all 16 S^2s of complex structures on the Eguchi-Hansons does *not* yield enough parameters to add up to 58.

Moreover, neither Kobayashi/Lye nor Donaldson use **weighted function spaces** for their estimates when dealing with the 16 conical singularities. Indeed, Kobayashi and Lye's a priori estimates are all unweighted since their strategy is to reprove Yau's a priori estimates in his solution to the Calabi conjecture to have constants which are *uniform* in the **gluing parameters** involved, and Donaldson exploits the conformal equivalence between the cone metric over $\mathbb{R}P^3$ and the cylindrical metric on $\mathbb{R}P^3$ to bypass the usage of polynomial weights. To the author's best knowledge, this machinery has yet to be used to give an alternative rigorous proof of Page-Gibbons-Pope's proposal, in either the Calabi-Yau or the hyper-Kähler perspectives.

In this paper, we rigorously carry out two more proofs of Page-Gibbons-Pope's proposal of constructing Ricci-flat Kähler metrics on $K3$ using singular perturbation and weighted analysis, from *both* the Calabi-Yau and hyper-Kähler perspectives. For the former, we find a Ricci-flat Kähler metric in a **given Kähler class** by solving a complex Monge-Ampere equation which necessitates having a *fixed* background complex structure, and for the latter we produce a hyper-Kähler metric via **closed definite triple perturbation** which reduces to solving a **nonlinear elliptic PDE *system***. Both PDEs admit **singular solutions** as limiting cases whence necessitating care in the analysis, and where weighted analysis proves extremely fruitful.

More specifically, in both the Calabi-Yau and hyper-Kähler perspectives we

- produce Ricci-flat Kähler metrics on the $K3$ surface which Gromov-Hausdorff converge to the flat orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

- explicitly exhibit Eguchi-Hanson metrics as ALE bubbles/rescaled limits near each singular orbifold point.

Moreover, as a byproduct of the weighted analysis developed, in the hyper-Kähler perspective we

- explicitly construct a **plethora** of compact hyper-Kähler *orbifolds* as explicit volume non-collapsed limits of **58-parameter** families of hyper-Kähler metrics on $K3$, thus yielding a large subset of points in the codimension 3 "holes" of $\text{Gr}^+(3, 19)/\Gamma$.

- exhibit an **iterated volume non-collapsed** phenomena on which each of these constructed compact hyper-Kähler orbifolds degenerate to one another.

That is, we both exhibit a multitude of elements in $\overline{\mathscr{M}^{K3}}^{\partial} - \mathscr{M}^{K3}$ $(\overline{\mathscr{M}^{K3}_{\text{Vol}=1}}^{\partial} - \mathscr{M}^{K3}_{\text{Vol}=1})$ and construct a full open neighborhood in $\overline{\mathscr{M}^{K3}}^{\partial}$ $(\overline{\mathscr{M}^{K3}_{\text{Vol}=1}}^{\partial})$ around these limit compact hyper-Kähler orbifolds, and then show how they degenerate to one another starting from a full 58-parameter family of hyper-Kähler metrics on $K3$.

In more detail, we prove:

Theorem 1.1 (Main Theorem I, see Theorem 3.7.3). *Let $K3$ denote the $K3$ surface.*
Equip $K3$ with a fixed complex structure J making it into a Kummer surface (Km, J).
*Then $0 < \exists \epsilon_0 \ll 1$ such that **there exists a 1-parameter family of Calabi-Yau metrics***

$$\{(g_\epsilon, \omega_\epsilon)\}_{\epsilon \in (0, \epsilon_0)}$$

on $K3$ which are Ricci-flat Kähler WRT the fixed complex structure J.

The solved for functions $\{f_\epsilon\}_{\epsilon \in (0, \epsilon_0)}$ in the complex Monge-Ampere equation (thus $\omega_\epsilon = \omega_0 + i\partial\bar{\partial}f_\epsilon$ for some initial approximately Ricci-flat Kähler form ω_0) produced satisfy the uniform estimate

$$\|f_\epsilon\|_{C^k_{g_\epsilon}(\text{Km})} \le C\epsilon^{3 + \frac{\delta}{2} - k}$$

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[3] $\dim_{\mathbb{R}} \mathscr{M}^{K3}_{\text{Vol}=1} = \dim_{\mathbb{R}} \text{Gr}^+(3, 19) = \dim_{\mathbb{R}} \text{SO}(3, 19) - \dim_{\mathbb{R}} \text{SO}(3) - \dim_{\mathbb{R}} \text{SO}(19) = 231 - 3 - 171 = 57$ since $\Gamma \le \text{GL}(22, \mathbb{Z})$ is discrete.

for $\delta \in (-2, 0)$.

Moreover when $\epsilon \searrow 0$,

$$(\mathrm{Km}, J, g_\epsilon, \omega_\epsilon) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0\right)$$

converges in the Gromov-Hausdorff topology to the flat Kähler orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0\right)$ with the singular orbifold Kähler metric g_0, ω_0.

Moreover, we may decompose $K3$ into a union of sets $\mathrm{Km}^{reg} \cup \bigsqcup_{i=1}^{16} \mathrm{Km}^{b_i}$ such that

(1) $(\mathrm{Km}^{reg}, J, g_\epsilon, \omega_\epsilon)$ *collapses to the flat Kähler manifold $\left(\mathbb{T}^4/\mathbb{Z}_2 - \mathrm{Fix}(\mathbb{Z}_2), J_0, g_0, \omega_0\right)$ with bounded curvature away from $\mathrm{Fix}(\mathbb{Z}_2)$ (**regular region**).*

(2) *For each $i = 1, \ldots, 16$, $\left(\mathrm{Km}^{b_i}, \frac{1}{\epsilon^2}g_\epsilon\right)$ converges to $\left(T^*\mathbb{C}P^1, g_{EH,1}\right)$ in C^∞_{loc} (**ALE bubble region**).*

$\blacklozenge$

Theorem 1.2 (Main Theorem II, see Theorem 4.7.3). *Enumerate the 16 singular points $\mathrm{Fix}(\mathbb{Z}_2) = \{p_1, \ldots, p_{16}\}$ of $\mathbb{T}^4/\mathbb{Z}_2$.*

*Then $\forall I \subseteq \{1, \ldots, 16\}$, $0 < \exists \epsilon^0_I \ll 1$ such that **there exists a** $10 + 3|I|$-**parameter family of (orbifold) hyper-Kähler triples and (orbifold) hyper-Kähler metrics***

$$\left\{\left(\boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)\right\}_{\substack{f \in \text{flat tori moduli} \\ \epsilon_i \in \left(0, \epsilon^0_I\right) \\ e_i \in S^2 \\ i \in I}}$$

*on the I-**th partial smoothing** $\mathrm{Km}^{|I|} = \mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}$ **of** $\mathbb{T}^4/\mathbb{Z}_2$, constructed by only smoothing the $|I|$ singular points $p_i \in \mathrm{Fix}(\mathbb{Z}_2), i \in I$.*

Moreover as $\epsilon^0_I \searrow 0$,

$$\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}, \boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}\right)$$

converges in the Gromov-Hausdorff topology to the flat hyper-Kähler orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}\right)$ with the singular orbifold hyper-Kähler metric $\boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}$. Moreover, we may decompose each compact orbifold $\mathrm{Km}^{|I|}$ into a union of sets $\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{reg} \cup \bigsqcup_{i \in I}\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{b_i}$ such that

(1) $\left(\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{reg}, \boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)$ *collapses to the flat Kähler orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2 - \{p_i\}_{i\in I}, \omega_0, g_{\omega_0}\right)$ with bounded curvature away from $p_i \in \mathrm{Fix}(\mathbb{Z}_2)$ for $i \in I$ (**regular region**).*

(2) *For each $i \in I$, $\left(\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{b_i}, \frac{1}{\epsilon^2}\boldsymbol{\omega}_{f,\epsilon_I,e_I}, \frac{1}{\epsilon^2}g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)$ converges to $\left(T^*S^2, A^e\boldsymbol{\omega}_{EH,1}, g_{\boldsymbol{\omega}_{EH,1}}\right)$ in C^∞_{loc} (**ALE bubble region**).*

*In particular, when $I = \{1, \ldots, 16\}$, $0 < \exists \epsilon^0 \ll 1$ such that **there exists a** $10 + 3 \cdot 16 = 58$-**parameter family of hyper-Kähler triples and hyper-Kähler metrics***

$$\left\{\left(\boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)\right\}_{\substack{f \in \text{flat tori moduli} \\ \epsilon_i \in \left(0, \epsilon^0\right) \\ e_i \in S^2 \\ i = 1, \ldots, 16}}$$

*on the **$K3$ surface** $\mathrm{Km}^{16} = K3 = \mathrm{Km}_{f,\epsilon_I,e_I}$.*

Moreover as $\epsilon^0 \searrow 0$,

$$\left(\mathrm{Km}_{f,\epsilon_I,e_I}, \boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}\right)$$

converges in the Gromov-Hausdorff topology to the flat hyper-Kähler orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}\right)$ with the singular orbifold hyper-Kähler metric $\boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}$. Moreover, we may decompose $K3$ into a union of sets $\mathrm{Km}^{reg}_{f,\epsilon_I,e_I} \cup \bigsqcup_{i=1}^{16} \mathrm{Km}^{b_i}_{f,\epsilon_I,e_I}$ such that

(1) $\left(\mathrm{Km}^{reg}_{f,\epsilon_I,e_I}, \boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}}\right)$ collapses to the flat Kähler manifold $\left(\mathbb{T}^4/\mathbb{Z}_2 - \mathrm{Fix}(\mathbb{Z}_2), \boldsymbol{\omega}_0, g_{\omega_0}\right)$ with bounded curvature away from $\mathrm{Fix}(\mathbb{Z}_2)$ (**regular region**).

(2) For each $i = 1, \ldots, 16$, $\left(\mathrm{Km}^{b_i}_{f,\epsilon_I,e_I}, \frac{1}{\epsilon^2}\boldsymbol{\omega}_{f,\epsilon_I,e_I}, \frac{1}{\epsilon^2}g_{\omega_{f,\epsilon_I,e_I}}\right)$ converges to $\left(T^*S^2, A^e\boldsymbol{\omega}_{EH,1}, g_{\omega_{EH,1}}\right)$ in C^∞_{loc} (**ALE bubble region**).

Therefore, letting $k \in \{0, \ldots, 16\}$ be a chosen nonnegative integer (the **depth**), upon considering the following strictly nested sequence

$$\{1, \ldots, 16\} = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_k$$

with $I_{16} := \emptyset$, then we have the following sequence of **iterated volume non-collapsed degenerations** of our 58-parameter family of hyper-Kähler metrics $\boldsymbol{\omega}_{f,\epsilon_{I_0},e_{I_0}}, g_{\omega_{f,\epsilon_{I_0},e_{I_0}}}$ on the K3 surface $\mathrm{Km}^{|I_0|}_{f,\epsilon_{I_0},e_{I_0}} = \mathrm{Km}_{f,\epsilon_{I_0},e_{I_0}}$:

where we

- degenerate from our hyper-Kähler K3 surface $\left(\mathrm{Km}_{f,\epsilon_{I_0},e_{I_0}}, g_{\omega_{f,\epsilon_{I_0},e_{I_0}}}\right)$ to $\left(\mathrm{Km}^{|I_j|}_{f,\epsilon_{I_j},e_{I_j}}, \boldsymbol{\omega}_{f,\epsilon_{I_j},e_{I_j}}, g_{\omega_{f,\epsilon_{I_j},e_{I_j}}}\right)$ via only collapsing the gluing parameters $\epsilon_i > 0$ corresponding to $i \in I_0 - I_j = I^c_j$

- degenerate from $\left(\mathrm{Km}^{|I_{j-1}|}_{f,\epsilon_{I_{j-1}},e_{I_{j-1}}}, \boldsymbol{\omega}_{f,\epsilon_{I_{j-1}},e_{I_{j-1}}}, g_{\omega_{f,\epsilon_{I_{j-1}},e_{I_{j-1}}}}\right)$ to $\left(\mathrm{Km}^{|I_j|}_{f,\epsilon_{I_j},e_{I_j}}, \boldsymbol{\omega}_{f,\epsilon_{I_j},e_{I_j}}, g_{\omega_{f,\epsilon_{I_j},e_{I_j}}}\right)$ via only collapsing the gluing parameters $\epsilon_i > 0$ corresponding to $i \in I_{j-1} - I_j$

- and degenerate from $\left(\mathrm{Km}^{|I_j|}_{f,\epsilon_{I_j},e_{I_j}}, \boldsymbol{\omega}_{f,\epsilon_{I_j},e_{I_j}}, g_{\omega_{f,\epsilon_{I_j},e_{I_j}}}\right)$ to $\left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_f, g_{\omega_f}\right)$ via collapsing all gluing parameters $\epsilon_i > 0, i \in I_j$.

♦

Remark 1.3. As remarked already, the uniform a priori estimates that Kobayashi [23] and Lye [28] prove are for the complex Monge-Ampere equation, whence their estimates requires a fixed integrable background complex structure and hold only in the Calabi-Yau perspective. Using the notation of Theorem 1.1, the estimates they prove are

$$\|f_\epsilon\|_{C^k_{g_\epsilon}(\mathrm{Km})} \le C\epsilon^{2-\frac{k}{2}}$$

Their proof follows Yau's proof of the Calabi conjecture, whence first uses a **uniform Sobolev inequality**, a **uniform Poincare inequality**, and **Moser iteration** to get the C^0-**estimate**, then the **maximum principle** to get the C^2-**estimate**, then further local estimates to get Holder and higher order estimates. Kobayashi in fact also proves slightly weaker uniform a priori estimates more generally when resolving *any* compact Ricci-flat Kähler orbifold whose resolutions yields $K3$, namely $\|f_\epsilon\|_{C^k_{g_\epsilon}(\mathrm{Km})} \le C\epsilon^{\frac{13}{8}-\frac{k}{2}}$.

The estimates that we prove from the Calabi-Yau perspective in Theorem 1.1, namely

$$\|f_\epsilon\|_{C^k_{g_\epsilon}(\mathrm{Km})} \le C\epsilon^{3+\frac{\delta}{2}-k}$$

for $\delta \in (-2, 0)$ is thus an improvement over Kobayashi/Lye's estimates for the case of resolving $\mathbb{T}^4/\mathbb{Z}_2$ as in the original Kummer construction, and is achieved via deriving an estimate of f_ϵ in certain **weighted Holder spaces** (see Theorem 3.7.2).

Moreover, our weighted analysis has the advantage of carrying over to the hyper-Kähler case, whence in Theorem 4.7.1 we prove similar weighted Holder estimates from the hyper-Kähler perspective, which readily carry over to the large families of hyper-Kähler orbifolds we construct in Theorem 1.2.

In fact, in *Remark* 3.5.5 and *Remark* 4.3.3 we derive **uniform Ricci curvature estimates** for our constructed approximate metric, whence (since we clearly have uniform volume non-collapse and uniformly bounded diameter) we *also* have uniform Poincare and uniform Sobolev inequalities in our setting. Whence we may also carry out a Moser iteration argument and rederive the analogue of Yau/Kobayashi/Lye's C^0 and higher order estimates.   ♣

Remark 1.4. Similar analysis done here has been carried out in slightly different settings.

Biquard-Minerbe [4] did a non-compact Kummer construction with weighed spaces in order to construct noncompact hyper-Kähler **ALF, ALG, and ALH spaces** (which respectively have cubic, quadratic, and linear volume growth), and was done via solving the complex Monge-Ampere equation.

Brendle-Kapouleas [5] did the original Kummer construction considered here on $K3$, but they glued in the 16 Eguchi-Hanson spaces with half of them having *opposite orientation* (arranged in a checkers board pattern). As a result, Brendle-Kapouleas had to solve the full Einstein equation Ric = 0 because the swapped orientation forces one to leave even the Kähler realm (since it prevents there being *any* resulting integrable complex structure) whence forcing the (approximate) metric to have *generic holonomy*. Note that in our case, when we vary the parameters of the construction from the hyper-Kähler perspective (Theorem 1.2), while we no longer patch together complex structures, we *still* have all Eguchi-Hanson spaces glued in with the *same* orientation since definite triples require the data of an orietation (see Definition 4.1.2), thus we're still *special holonomy*. Brendle-Kapouleas end up showing that the relevant linearization has an obstructed kernel (even up to several higher orders), thus nullifying the construction and dashing any hopes of using the original Kummer construction to construct Ricci-flat metrics on *closed* manifolds with *generic holonomy* (as opposed to Calabi-Yau, hyper-Kähler, etc., with *special* holonomy $\mathrm{SU}\left(\frac{n}{2}\right), \mathrm{Sp}\left(\frac{n}{4}\right)$).   ♣

Remark 1.5. It is interesting to consider the *full* compactification $\overline{\mathscr{M}^{K3}_{\mathrm{Vol}=1}}$, whence to study the limit spaces of sequences in $\mathscr{M}^{K3}$ which are **volume collapsed**. Indeed, since the period map extends to a bijection $\mathcal{P} : \overline{\mathscr{M}^{K3}_{\mathrm{Vol}=1}}^{\partial} \to \mathrm{Gr}^+(3, 19)/\Gamma$ when adding in all volume non-collapsed limits, as $\mathrm{Gr}^+(3, 19)/\Gamma$ is still *non-compact* we say that a sequence $\{g_i\} \subset \mathscr{M}^{K3}$ is volume collapsed to mean that $\{\mathcal{P}(g_i)\}$ diverges to infinity in $\mathrm{Gr}^+(3, 19)/\Gamma$.

Much recent work has been done in these and related directions, namely in classifying elements in the "boundary at ∞"/the set of **volume collapsed limit spaces** $\overline{\mathscr{M}^{K3}_{\mathrm{Vol}=1}} - \overline{\mathscr{M}^{K3}_{\mathrm{Vol}=1}}^{\partial}$, exhibiting open subsets of $\overline{\mathscr{M}^{K3}_{\mathrm{Vol}=1}}$/explicit 58 parameter families of volume collapsed sequences convergent to those collapsed limit spaces, and exhibiting more general noncompact hyper-Kähler gravitational instantons as bubbles/models of the convergence near singularities and volume-collapsed regions.

We give an impressionistic survey of these new (and old) results:

Foscolo [14] exhibited a full open subset/58 parameter family of sequences $\{g_i\} \subset \mathscr{M}^{K3}$ which exhibit **codimension 1 collapse** down to $\mathbb{T}^3/\mathbb{Z}_2$ which have *both* **cyclic ALF** (i.e. **multi Taub-NUT**) and **dihedral ALF** hyper-Kähler spaces as bubbles near (no more than 16 arbitrary) punctures and the 8 punctures corresponding to the 8 fixed points of $\mathbb{Z}_2 \circlearrowright \mathbb{T}^3$, respectively. This is done via a similar Kummer-type/gluing construction and closed definite perturbation as considered here, where (roughly speaking) we are shrinking one of the S^1-factors of $\mathbb{T}^4/\mathbb{Z}_2$ down to zero (or rather, the S^1-fibers of a not necessarily trivial principal U(1)-bundle over an open subset of a punctured $\mathbb{T}^3$). Moreover, the **charges** of these cyclic & dihedral ALF spaces must satisfy a *charge conservation* equation, which, by the residue theorem, guarantees the existence of a suitable harmonic function on an open subset of a punctured $\mathbb{T}^3$ with *prescribed asymptotics* near the punctures, which is needed for the **Gibbons-Hawking ansatz** of U(1)-invariant 4-dimensional hyper-Kähler metrics that is involved in the construction of all cyclic ALF spaces and the asymptotics of dihedral ALF spaces. Foscolo's constructed family of metrics (in the case where one of the building blocks consists of the **Atiyah-Hitchin manifold**, or the D_1 **ALF space**) was also shown to admit a **strictly stable minimal sphere** which is *not* holomorphic WRT any complex structure compatible with the metric, thus providing alternative examples of this phenomena on $K3$ which was originally shown to exist by Micallef-Wolfson [29].

Gross-Wilson [18] constructed sequences $\{g_i\} \subset \mathcal{M}^{K3}$ on an **elliptic $K3$ surface** with 24 singular I_1 fibers[4] exhibiting **codimension 2 collapse** down to $\left(\mathbb{C}P^1, g_{ML}\right)$ with the (singular) **McLean metric**. This is done via a gluing construction similar to the one considered here, where the model away from singular fibers provided by a **semi-flat metric**, and near the singular fibers by the hyper-Kähler **Ooguri-Vafa** metric, hence is a bubble modeling the convergence near the 24 singular I_1 fibers. The proof is done via complex Monge-Ampere and the continuity method, hence all metrics are Kähler WRT a fixed elliptic complex structure. Moreover, this setting is that of a Kähler metric on $K3$ approaching the "**large complex structure limit**", which by a hyper-Kähler rotation of any complex structure on $K3$ giving the existence of special Lagrangian fibrations, is equivalent to having a *fixed* complex structure on an elliptic $K3$ and letting the Kähler metric degenerate **adiabatically** (i.e. the volume of the fibers shrinking to zero). The main guiding heuristic here, namely that Kähler metrics WRT a complex structure in the "large complex structure limit" should degenerate in a certain way and (roughly) produce special Lagrangian (torus) fibrations, is called the "**SYZ conjecture**".

Gross-Tosatti-Zhang [17] generalized Gross-Wilson's construction to general elliptic $K3$ surfaces, i.e. non-generic configurations of singular fibers. This was also done via complex Monge-Ampere, hence the resulting sequence of Ricci-flat metrics are Kähler WRT a fixed elliptic complex structure. However, no description of the metric degeneration near the singular fibers/bubbles were explored. Chen-Viaclovsky-Zhang [8], using closed definite triple perturbation (but still with a fixed elliptic complex structure) redid Gross-Tosatti-Zhang but with much more precise descriptions of metric degeneration near singular fibers, exhibiting hyper-Kähler ALE, cyclic ALF, and hyper-Kähler **ALG** spaces as bubbles modeling the convergence near the singular fibers.

Hein-Sun-Viaclovsky-Zhang [19] constructed sequences $\{g_i\} \subset \mathcal{M}^{K3}$ which exhibits both **multi-scale/iterative** and **codimension 3 collapse** down to the unit interval $\left([0,1], dt^2\right)$. Generically, the collapse is modeled on S^1-fiber bundles over $\mathbb{T}^2$ with the fiber shrinking at a *faster* rate than the base, and away from the generic region the collapse is modeled on Taub-NUT and **ALH Tian-Yau** spaces appearing as bubbles. This was done on the purely *metric* level (hence via closed definite triple perturbation), with no background complex structure.

Chen-Viaclovsky-Zhang [9] in a different work proved Torelli-type theorems on hyper-Kähler **ALG*** spaces, and also exhibits codimension 2 collapse of hyper-Kähler metrics on $K3$ down to $\left(\mathbb{C}P^1, g_{ML}\right)$ with ALG* and Taub-NUT bubbles modeling the collapse near the singular fibers.

Around the same time as these developments, Sun-Zhang [34] classified *all* possible collapsed limit spaces/elements of $\overline{\mathcal{M}^{K3}_{\mathrm{Vol}=1}} - \overline{\mathcal{M}^{K3}_{\mathrm{Vol}=1}}^{\partial}$ (hence completing the full picture of the Gromov-Hausdorff compactification), and showed that $\overline{\mathcal{M}^{K3}_{\mathrm{Vol}=1}} - \overline{\mathcal{M}^{K3}_{\mathrm{Vol}=1}}^{\partial} = \left\{\mathbb{T}^3/\mathbb{Z}_2, S^2, [0,1]\right\}$, hence the flat $\mathbb{T}^3/\mathbb{Z}_3$ (codimension 1 collapse), a singular special Kähler metric on S^2 (codimension 2 collapse), or the flat unit interval $[0,1]$ (codimension 3 collapse, in which further structure was examined by Honda-Sun-Zhang [20]). This was done via proving **uniqueness of tangent cones** near singularities of any collapsed limit.

Finally, we mention recent work of Sun-Zhang [35] who proved new collapsing behavior of *arbitrary dimension* Calabi-Yau metrics in the "**small complex structure limit**" (hence the other extreme of Gross-Wilson/SYZ), specifically in the case of a family of smooth projective Calabi-Yau hypersurfaces viewed as a complex 1-parameter family $\{X_t\}_{t \in \Delta}$ of complex deformations of a simple normal crossing variety X_0. That is, they prove that one can carry over this algebraic degeneration as $|t| \searrow 0$ to the *metric* level and obtain precise geometric information on this metric degeneration, namely that the family collapses all the way down to the unit interval $\left([0,1], dt^2\right)$ in such a way as to exhibit multiscale/iterated collapse, with generic fibers an S^1-bundle over the smooth intersection divisor, and singular S^1-bundles with Taub-NUT $\times$ flat and Tian-Yau bubbles near the singular S^1 fibers and the endpoints of $[0,1]$, respectively, and with all S^1 fibers shrinking faster than their bases. ♣

This paper is organized as follows:

In Section 2, we list some common notations used throughout.

In Section 3.1, we list some preliminary definitions used and give a derivation of the Monge-Ampere equation for (more general) Kähler-Einstein metrics.

---

[4] Elliptic $K3$ surfaces, which (usually) have a *fixed* complex structure and always have singular fibers, have their singular fibers classified: the list of types of singular fibers are $I_0^*, II, III, IV, II^*, III^*, IV^*, I_\nu, I_\nu^*, \forall \nu \in \mathbb{N}$.

In Section 3.2, we define the Eguchi-Hanson metric on the blowup of $\mathbb{C}^2/\mathbb{Z}_2$ and prepare it for the gluing process, as well as recall some facts about flat tori.

In Section 3.3, we construct the "Kummer surface" Km by blowing up the orbifold points of $\mathbb{T}^4/\mathbb{Z}_2$ in a manner suitable for gluing, and we graft in the prepped Eguchi-Hanson metrics to get an approximately Ricci-flat Kähler metric ω_ϵ on Km.

In Section 3.4, we define the relevant weighted Holder spaces that will be used in the construction.

In Section 3.5, we setup the nonlinear PDE problem of perturbing the approximately Ricci-flat ω_ϵ to a genuinely Ricci-flat Kähler metric within its Kähler class as a fully nonlinear elliptic PDE of Monge-Ampere type, in a manner suitable for the implicit function theorem.

In Section 3.6, we do a blow-up analysis argument to invert the linearization of the nonlinear map defined in the previous section in the relevant weighted Holder spaces. This is the heart of the argument, as we need the weighted Holder spaces to get *uniform* bounded operator norm on the inverse of the linearization.

In Section 3.7, we finish the proof of Theorem 1.1 by applying the implicit function theorem to get our solution to the Monge-Ampere equation.

In Sections 4.2, 4.3, 4.4, and 4.6 we upgrade our analysis to established in the previous sections to deal with closed definite triple perturbation from the hyper-Kähler perspective. As we are no longer solving for a scalar function but for a 2-*form* in our PDE, the analysis substantially differs.

Finally, in Section 4.7 we finish the proof of Theorem 1.2 by applying the implicit function theorem to get our solution to the nonlinear elliptic PDE system involved, as well as discuss some consequences of what has been proved from two perspectives:

- the perspective of "further probing" the codimension 3 subset of all (unit volume) volume non-collapsed limits of hyper-Kähler metrics on $K3$ inside $\mathrm{Gr}^+(3,19)/\Gamma$ (*Remark* 4.7.4 and *Remark* 4.7.5)

- the broader perspective of general Riemannian convergence theory, specifically the work of Cheeger and Naber [7] (*Remark* 4.7.6).

2   § (Global) Notation §

Notation 2.1.   • Everything is assumed $C^{\infty,\omega}$ when appropriate/unless explicitly mentioned otherwise, and all manifolds are connected (unless otherwise stated).

- For g a Riemannian metric and $E \subseteq (M, g)$ some subset, we will let $|\cdot|_E^g := \mathrm{dist}_g(E, \cdot)$ be the *metric* distance function WRT g away from E.

  Clearly $|\cdot|_E^g \in \mathrm{Lip}^1(M, [0, \infty))$ is Lipschitz continuous with Lipschitz constant 1, and is C^∞/smooth away from E and its cut locus.

- $B_R^g(E)$ denotes the open ball $\{|\cdot|_E^g < R\}$. Hence this is a *metric* open tubular neighborhood of E.

- $D_R^g(E)$ denotes the disk $\{|\cdot|_E^g \leq R\}$. Hence this is a *metric* closed tubular neighborhood of E.

- We sometimes use $f|_p$ to denote f a function (tensor, section, etc.) evaluated at p, i.e. $f(p)$, to simplify notation.

- $|\cdot|_g$ will denote a (usually (p, q)-tensor) norm that's induced from a given specified Riemannian metric g.

- ∇_g will denote the Levi-Civita connection induced from a given Riemannian metric g.

- O_g will denote the usual Landau symbol but with all tensor norms measured with respect to the specified Riemannian metric g.

<div align="right">○</div>

3 § Calabi-Yau Metrics on $K3$ and the Classical Kummer Construction §

3.1 § Kähler-Einstein Metrics, the Complex Monge-Ampere Equation, and Calabi-Yau Metrics §

We collect some preliminary definitions and derivations pertaining to Kähler-Einstein metrics, mostly following Walpuski's excellent Riemannian geometry lecture notes [37].

Definition 3.1.1. A **Kähler manifold** is a *real smooth* $2n$ dimensional manifold M^{2n} with the following data:

(1) an almost complex structure $J \in \Gamma(\mathrm{Hom}(TM, TM))$, hence pointwise on each tangent space $J^2 = -\mathrm{id}$

(2) a Riemannian metric $g \in \Gamma(\mathrm{Sym}^2 T^*M)$

(3) a non-degenerate 2-form $\omega \in \Omega^2(M) := \Gamma(\Lambda^2 T^*M)$ (nondegeneracy here means that, pointwise/on the vector space level, the map $V \to V^*$ via $v \mapsto \omega(v, \cdot)$ is an isomorphism, which is equivalent to $\frac{\omega^n}{n!}$ being a volume form).

such that the 3 pieces of data above are compatible in the following way:

$$\omega(\cdot, \cdot) = g(J\cdot, \cdot)$$

and one of the following equivalent conditions are satisfied:

- J is integrable (i.e. M^{2n} is actually a complex n-manifold (M^n, J) *WRT* J, aka $N_J = 0$ the Nijenhuis tensor of J vanishes) and $d\omega = 0$ (i.e. ω is symplectic)

- $\nabla_g J = 0$

- $\nabla_g \omega = 0$

- The **holonomy group** $\mathrm{Hol}(g)$ of the Levi-Civita connection of g is contained in $\mathrm{U}(n)$.

Hence the data of (M^{2n}, J, g, ω) satisfying the compatibility condition and one of the 3 equivalent conditions a **Kähler manifold**, and we call $[\omega] \in H^2(M)$ the **Kähler class** and ω the **Kähler form**, and we also have $\frac{\omega^n}{n!} = \mathrm{d}V_g$. ●

Remark 3.1.2. Note that if we just have the 3 pieces of data of (J, g, ω) satisfying the compatibility condition, but not one of the following equivalent conditions, we call this an **almost Hermitian manifold**. ♣

Remark 3.1.3. More importantly, note that *any cardinality 2 subset of $\{J, g, \omega\}$ automatically determines/**defines** the leftover element in its complement* via the compatibility condition. Roughly speaking, this follows because $\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n)$ (this may be made rigorous using the language of G-structures on the frame bundle $\mathrm{Fr}(TM)$, which we unfortunately don't have the space to digress into). In other words, Kähler geometry is the *intersection* of complex and Riemannian geometry, and complex and symplectic geometry, and symplectic and Riemannian geometry. This is sometimes called the **"2-out-of-3"** property and is one of the reasons why Kähler geometry is very rich. ♣

Remark 3.1.4. Henceforth on a *complex* manifold (M^n, J) we will sometimes abuse notation and call ω the Kähler metric instead of g, precisely because 2-out-of-3 precisely means that on (M^n, J), the existence of a Kähler form ω (compatible with J) *automatically* gives us a Riemannian metric g (compatible with J) by the compatibility condition $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$. ♣

Now on any Kähler manifold (M^{2n}, J, g, ω), we have that the **canonical bundle** $K_M := \bigwedge_{\mathbb{C}}^n T^*M^{(1,0)}$ associated to the *underlying complex manifold* (M^n, J) has an induced Hermitian metric coming from the Hermitian metric $h := g - i\omega$ on TM (which as *complex* vector bundles is isomorphic to $TM^{(1,0)}$, a *holomorphic* vector bundle), and the induced Levi-Civita connection on K_M coming from the Levi-Civita connection of g is precisely the Chern connection of the induced Hermitian metric on K_M, i.e. it is unitary and has $(0,1)$-part the $\bar{\partial}$ operator inducing the holomorphic structure on K_M. Not only that, but the curvature 2-form F of this connection on K_M is precisely $i\mathrm{Ric}_\omega$, where $\mathrm{Ric}_\omega(\cdot, \cdot) := \mathrm{Ric}_g(J\cdot, \cdot) \in \Omega^2(M)$. Hence since $c_1(M^n, J) = -\frac{i}{2\pi}[F]$ by Chern-Weil theory[5], this gives us that $c_1(M^n, J) = \frac{1}{2\pi}[\mathrm{Ric}_\omega]$.

We summarize this and other properties needed in the sequel in the below proposition (proofs found in [37]):

Proposition 3.1.5. Let (M^{2n}, J, g, ω) be a Kähler manifold. Then the following properties hold:

- The first Chern class $c_1(M^n, J) \in H^2(M)$ of M is $c_1(M^n, J) = \frac{1}{2\pi}[\mathrm{Ric}_\omega]$.

- For any two Kähler forms ω_1, ω_0, the difference between their Ricci forms is $\mathrm{Ric}_{\omega_1} - \mathrm{Ric}_{\omega_0} = -i\partial\bar{\partial}\log\left(\frac{\omega_1^n}{\omega_0^n}\right)$, with the division of two top degree forms having the obvious meaning.

Remark 3.1.6. Note that $c_1(M^n, J) = \frac{1}{2\pi}[\mathrm{Ric}_\omega]$ holds for *any* Kähler form ω on (M^n, J), and hence $c_1(M^n, J)$ **does not depend** on the Kähler metric ω. In fact, as suggested by the notation, $c_1(M^n, J)$ only depends on the (integrable) complex structure J (besides obviously depending on the underlying smooth manifold M^{2n}), whence is an underlying *topological* invariant of the *complex* manifold (M^n, J). ♣

From now on, *fix the (integrable) complex structure J* on our smooth manifold M^{2n}, aka we're now working on a fixed *complex* manifold (M^n, J).

We now want to examine the necessary and sufficient conditions for a given closed (i.e. boundaryless and compact) Kähler manifold (M^n, J, g_0, ω_0) to admit a **Kähler-Einstein metric** in the same Kähler class $[\omega_0]$, namely a Kähler 2-form $\omega \in [\omega_0]$ (hence real $(1,1)$ and positive) such that $\mathrm{Ric}_\omega = \lambda\omega$ for $\lambda \in \mathbb{R}$, whence the resulting Riemannian metric g *compatible with J* is also Einstein with constant λ. Clearly from Proposition 3.1.5 we have that a necessary condition is that $c_1(M^n, J) = \frac{\lambda}{2\pi}[\omega] = \frac{\lambda}{2\pi}[\omega_0]$ since $[\omega] = [\omega_0]$.

It is therefore natural to ask the converse:

Question 3.7. Suppose $c_1(M^n, J) = \frac{\lambda}{2\pi}[\omega_0]$ for a fixed closed Kähler manifold (M^n, J, g_0, ω_0). Does there exist a Kähler form $\omega \in [\omega_0]$ such that $\mathrm{Ric}_\omega = \lambda\omega$?

Now by Proposition 3.1.5, we have that $c_1(M^n, J) = \frac{\lambda}{2\pi}[\omega_0] \Rightarrow [\mathrm{Ric}_{\omega_0}] = \lambda[\omega_0] \Rightarrow \mathrm{Ric}_{\omega_0} = \lambda\omega_0 + i\partial\bar{\partial}\rho$ by the $\partial\bar{\partial}$ lemma with $\rho \in C^\infty(M)$ unique up to a constant. We call ρ the **Ricci potential** of ω_0, and from now on we impose the following "volume normalization condition" to pin down the potential:

$$\int_M e^\rho \omega_0^n = \int_M \omega_0^n = n! \, \mathrm{Vol}_{g_0}(M)$$

Now again by the $\partial\bar{\partial}$ lemma, any Kähler form ω which is in the same cohomology class as ω_0 must be of the form $\omega = \omega_0 + i\partial\bar{\partial}f$ for some $f \in C^\infty(M)$. Hence if $\exists\omega$ a Kähler form in $[\omega_0]$ such that $\mathrm{Ric}_\omega = \lambda\omega$, then by part 2 of Proposition 3.1.5, $\omega = \omega_0 + i\partial\bar{\partial}f$ must satisfy

---

[5] We have that on any complex manifold (M^n, J) that $c_1(M^n, J) := c_1(\det_\mathbb{C} TM) = c_1\left(\det_\mathbb{C} TM^{(1,0)}\right) = -c_1\left(\det_\mathbb{C} T^*M^{(1,0)}\right)$, and Chern-Weil theory tells us that the first Chern class of a line bundle is $\frac{i}{2\pi}$ times the de Rham cohomology class of the curvature 2-form of *any arbitrary* connection on that line bundle, thus giving us $c_1(M^n, J) = -\frac{i}{2\pi}[F]$ for F the curvature 2-form of *any* connection on the line bundle $\det_\mathbb{C} T^*M^{(1,0)} =: \bigwedge_\mathbb{C}^n T^*M^{(1,0)} =: K_M$.

$$\text{Ric}_\omega - \text{Ric}_{\omega_0} = \lambda\big(\omega_0 + i\partial\bar\partial f\big) - \lambda\omega_0 - i\partial\bar\partial\rho$$
$$= i\partial\bar\partial(\lambda f - \rho)$$
$$= -i\partial\bar\partial\log\left(\frac{\omega^n}{\omega_0^n}\right)$$

giving us

$$i\partial\bar\partial\left(\log\left(\frac{\omega^n}{\omega_0^n}\right) + \lambda f - \rho\right) = 0$$

or equivalently

$$\log\left(\frac{\omega^n}{\omega_0^n}\right) + \lambda f - \rho = c \implies \frac{\omega^n}{\omega_0^n} = e^{-\lambda f} e^\rho e^c$$

with the constant $c := -\log\left(\dfrac{\int_M e^{-\lambda f} e^\rho \omega_0^n}{\int_M \omega_0^n}\right)$.

Now if $\lambda = 0$, then by the ρ normalization, $c = 0$. If $\lambda \ne 0$, then f solves $\frac{\omega^n}{\omega_0^n} = e^{-\lambda f} e^\rho e^c$ if and only if $f + \frac{c}{\lambda}$ solves $\frac{\omega^n}{\omega_0^n} = e^{-\lambda f} e^\rho$. Hence $\forall \lambda \in \mathbb{R}$, we end up with the following equation for $f \in C^\infty(M)$:

$$\frac{\big(\omega_0 + i\partial\bar\partial f\big)^n}{\omega_0^n} = e^{-\lambda f} e^\rho$$

Therefore we have proved the following

Proposition 3.1.8. Let (M^n, J, g_0, ω_0) be a closed *complex* manifold with *fixed* background integrable complex structure J and *arbitrary* Kähler form ω_0 compatible with J (whence yielding a Riemannian metric g_0).

If $c_1(M^n, J) = \lambda[\omega_0]$ for $\lambda \in \mathbb{R}$ with the Ricci potential normalization $\int_M e^\rho \omega_0^n = \int_M \omega_0^n$ (since $c_1(M^n, J) = \frac{\lambda}{2\pi}[\omega_0] \implies \text{Ric}_{\omega_0} = \lambda\omega_0 + i\partial\bar\partial\rho$), then if $\exists f \in C^\infty(M)$ such that

- (*Positivity*): $\omega := \omega_0 + i\partial\bar\partial f$ is a positive $(1,1)$-form (that is, $g(\cdot, \cdot) := \omega(\cdot, J\cdot)$ defines a real Riemannian metric)

- (*Complex Monge-Ampere*): f solves
$$\frac{\big(\omega_0 + i\partial\bar\partial f\big)^n}{\omega_0^n} = e^{-\lambda f} e^\rho$$

then $\omega_0 + i\partial\bar\partial f =: \omega \in [\omega_0]$ (which is automatically closed) is a Kähler form in the same cohomology class as the original Kähler metric ω_0 yielding a Kähler-Einstein metric $g := \omega(\cdot, J\cdot)$ with Einstein constant λ that's compatible with the *fixed* original integrable complex structure J.

In fact, further imposing the additional condition

- (*Integral Zero*):
$$\int_M f\omega_0^n = 0$$

makes f *unique*, whence the resulting $\omega \in [\omega_0]$ is the *unique* Kähler form in the starting Kähler class $[\omega_0]$ yielding a Kähler-Einstein metric with Einstein constant λ compatible with J.

Remark 3.1.9. The 2nd condition (complex Monge-Ampere) *automatically implies* the 1st condition (positivity). Indeed, being a positive $(1,1)$-form is an *open condition* (a continuous Hermitian matrix-valued function having positive eigenvalues at a point implies having positive eigenvalues on an open neighborhood at that point), whence the 2nd condition implies the 1st via covering M with local coordinate charts, continuity of the expression for ω whence continuity of all local coordinate functions of ω, and positivity of the RHS of the 2nd condition implying that the *determinant* of the local coordinate component Hermitian matrix of ω being *positive*. ♣

The equation $\frac{(\omega_0 + i\partial\bar{\partial}f)^n}{\omega_0^n} = e^{-\lambda f}e^\rho$ is a fully nonlinear scalar elliptic equation of complex *Monge-Ampere* type, i.e. in local coordinates the LHS looks like $\det(\text{Hess}_{\mathbb{C}}\, f)$. Therefore proving the *existence* of a smooth solution f to this equation (which by this proposition implies the existence of Kähler-Einstein metrics) is quite difficult, and becomes harder as the sign of λ changes from negative to 0 to positive. Aubin proved that a solution f *always* exists for $\lambda < 0$, Yau proved that a solution f *always* exists for the $\lambda = 0$ (i.e. Ricci-flat) case (called the **Calabi conjecture**), and very recently Chen-Donaldson-Sun [10] [11] [12] solved the $\lambda > 0$ case via proving that existence of KE metrics with $\lambda > 0$ is equivalent to an algebraic-geometric "K-stability" condition (since here there are *obstructions* to existence).

Next, we note the following special case of the above (whose proof is elementary, see Huybrechts' book [21]):

Proposition 3.1.10. Let (M^n, J, g_0, ω_0) be a fixed closed Kähler manifold. Suppose K_M is trivial, hence there exists a nowhere vanishing **holomorphic volume form**/nowhere vanishing holomorphic $(n, 0)$-form Ω (an example being the $K3$ surface) and $c_1(M^n, J) = 0$.

Then if $\exists f \in C^\infty(M)$ such that

- (*Positivity*): $\omega := \omega_0 + i\partial\bar{\partial}f$ is a positive $(1, 1)$-form

- (*Complex Monge-Ampere*): f solves

$$\frac{\left(\omega_0 + i\partial\bar{\partial}f\right)^n}{\Omega \wedge \bar{\Omega}} = C$$

  for some constant $C > 0$.[6]

then $\omega_0 + i\partial\bar{\partial}f =: \omega \in [\omega_0]$ is a Kähler form in the same cohomology class as the original Kähler metric ω_0 yielding a **Ricci-flat Kähler** metric $g := \omega(\cdot, J\cdot)$ compatible with the *fixed* original integrable complex structure J.

Further imposing the additional condition

- (*Integral Zero*):

$$\int_M f\omega_0^n = 0$$

makes f and whence $\omega \in [\omega_0]$ the *unique* Kähler form yielding a Ricci-flat metric g compatible with the original J.

Such Ricci-flat metrics which are Kähler WRT a *fixed* integrable complex structure form a special class of metrics, namely:

Definition 3.1.11. A **Calabi-Yau manifold** is a *real smooth* $2n$-dimensional manifold M^{2n} with the following data:

(1) An integrable complex structure J, whence making M^{2n} into a *complex* manifold (M^n, J).

(2) g is a Riemannian metric which is Kähler WRT J with Kähler form ω, whence making (M^n, J) a Kähler manifold (M^n, J, g, ω).

(3) Ω a nowhere vanishing holomorphic $(n, 0)$-form WRT J.

such that one of the following equivalent conditions are satisfied

- $\nabla_g\Omega = 0$ and Ω has the (pointwise) normalization $|\Omega|_g = 2^{\frac{n}{2}}$.

- The holonomy group $\text{Hol}(g)$ of the Levi-Civita connection of g is contained in $\text{SU}(n)$, and the associated parallel $(n, 0)$-form is Ω and has normalization $|\Omega|_g = 2^{\frac{n}{2}}$

- The pieces of data above are compatible in the following way:

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}}\left(\frac{i}{2}\right)^n \Omega \wedge \bar{\Omega}$$

---

[6] Most often people work with the convention that $C = n!(-1)^{\frac{n(n-1)}{2}}\left(\frac{i}{2}\right)^n$ upon (say) rescaling Ω, see Definition 3.1.11.

Thus since this compatibility condition is precisely the complex Monge-Ampere equation, by Proposition 3.1.10 we therefore have that **Calabi-Yau metrics are Ricci-flat**. •

Remark 3.1.12. The normalization $|\Omega|_g = 2^{\frac{n}{2}}$, which is equivalent to the specifically chosen constant factor $(-1)^{\frac{n(n-1)}{2}}\left(\frac{i}{2}\right)^n$ in the complex Monge-Ampere equation in Definition 3.1.11, is chosen specifically to make $\operatorname{Re}\Omega$ a **calibration**. ♣

Notation 3.1.13. Therefore, by *Remark* 3.1.12, we will oftentimes interchange the terms "Ricci-flat Kähler" and "Calabi-Yau".

Indeed, the latter is just the former but with a suitable normalization of the nowhere vanishing holomorphic volume form. Moreover, the former may be converted into the latter because Kähler Ricci-flat implies holonomy contained in $\mathrm{SU}(n)$, whence by the **holonomy principle** we get a nowhere vanishing parallel $(n, 0)$-form, which is closed by antisymmetrization of the Levi-Civita connection. ∘

Remark 3.1.14. Just like *Remark* 3.1.2, we have that if we weaken Definition 3.1.11 for our manifold $\left(M^{2n}, J, g, \omega\right)$ to instead be almost Hermitian, but equipped with a *complex* nowhere vanishing $(n, 0)$-form Ω with normalization $|\Omega|_g = 2^{\frac{n}{2}}$ and compatibility condition $\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}}\left(\frac{i}{2}\right)^n \Omega \wedge \overline{\Omega}$, then the tuple $\left(M^{2n}, J, g, \omega, \Omega\right)$ is called an $\mathrm{SU}(n)$-**structure on** M^{2n}. ♣

Lastly, we note a linear algebraic fact that will be conceptually useful later on (whose proof follows from the Euclidean case in $\mathbb{R}^{2n} = \mathbb{C}^n$):

Proposition 3.1.15. Let M^{2n} be a smooth manifold. Suppose that M^{2n} admits a nowhere vanishing $(n, 0)$-form Ω[7]. Then

- Ω *uniquely* determines the almost complex structure J_Ω over M via determining the $(1, 0)$-forms

$$\Lambda_{\mathbb{C}}^{1,0} T_p^* M := \ker\left(\alpha\big|_p \mapsto \Omega\big|_p \wedge \alpha\big|_p\right), \qquad \forall p \in M$$

- $d\Omega = 0$ implies that J_Ω is *integrable*, whence $\left(M^{2n}, \Omega\right)$ is in fact a *complex* manifold (M^n, J_Ω) WRT J_Ω and Ω becomes a *holomorphic* nowhere vanishing $(n, 0)$-form WRT J_Ω.

Hence a complex manifold (M^n, J) admitting a nowhere vanishing *holomorphic* volume form Ω automatically has trivial canonical bundle K_M (as the global frame of the complex line bundle K_M is literally given by Ω).

3.2 § Eguchi-Hanson Metrics & Flat Tori as "Building Blocks" §

In this section, we first define & state several important properties of the Eguchi-Hanson metric, and then recall some facts about flat tori.

First, we make the following observation which will be crucially used numerous times in the sequel.

Proposition 3.2.1. Let $M \circlearrowleft G$ be a free and proper action of a *finite* group G on a smooth manifold M, whence $\pi : M \twoheadrightarrow M/G$ is a principal G-bundle with the base space M/G a smooth manifold and hence is a (regular) $|G|$-fold covering. Then any smooth function (tensor, section, etc.) f on M/G may be lifted to a smooth G-invariant function (tensor, section, etc.) $\pi^* f$ on M.

Conversely, any smooth G-invariant function (tensor, section, etc.) on M descends down to a smooth function (tensor, section, etc.) on M/G.

In particular, when $G = \mathbb{Z}_2$ and $M = \mathbb{R}^n$ or $M = \mathbb{T}^n$ and the action sends $1 \mapsto \mathrm{id}_M$ and $-1 \mapsto -\mathrm{id}_M$ (aka via the standard coordinate involution $x \mapsto -x$), then any smooth function (tensor, section, etc.) on the *free* part $\mathbb{R}^n/\mathbb{Z}_2 - \{0\}$ or $\mathbb{T}^n/\mathbb{Z}_2 - \mathrm{Fix}(\mathbb{Z}_2)$ may be lifted to a smooth *even* function (tensor, section, etc.) on $\mathbb{R}^n$ or $\mathbb{T}^n$. That is, the lifted $\widetilde{f}$ satisfies $\widetilde{f}(x) = \widetilde{f}(-x)$.

Conversely, any smooth even function (tensor, section, etc.) on $\mathbb{R}^n$ or $\mathbb{T}^n$ descends down to a smooth function (tensor, section, etc.) on the free part $\mathbb{R}^n/\mathbb{Z}_2 - \{0\}$ or $\mathbb{T}^n/\mathbb{Z}_2 - \mathrm{Fix}(\mathbb{Z}_2)$.

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[7] This is called an $\mathrm{SL}(n, \mathbb{C})$-**structure** in the language of G-structures.

Start with Euclidean $\mathbb{R}^4$. We may equip $\mathbb{R}^4$ with the Euclidean integrable complex structure J_0 coming from the tautological identification $\mathbb{R}^4 = \mathbb{C}^2$.

Let's now look at the complex orbifold $(\mathbb{C}^2/\mathbb{Z}_2, J_0)$, or $(\mathbb{C}^2, J_0)$ mod the action $\mathbb{Z}_2 \subset \mathbb{C}^2$ via $1 \mapsto \mathrm{id}_{\mathbb{C}^2}, -1 \mapsto -\mathrm{id}_{\mathbb{C}^2}$, which is clearly holomorphic WRT J_0. Whence $\mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_2$ is a double cover outside of the orbifold singularity at the origin, i.e. $\mathbb{C}^2 - \{0\} \overset{2:1}{\twoheadrightarrow} \mathbb{C}^2/\mathbb{Z}_2 - \{0\}$. Blow up the singularity at the origin with a **crepant resolution** (i.e. preserving the canonical bundle, i.e. pullback of canonical bundle downstairs is canonical bundle upstairs) to get $\left(\widetilde{\mathbb{C}^2/\mathbb{Z}_2}, J_{\widetilde{\mathbb{C}^2/\mathbb{Z}_2}}\right) \overset{\mathrm{Biholo}}{\cong} \left(T^*\mathbb{C}P^1, J_{O(-2)}\right) = O(-2) = \left(T^*S^2, J_{O(-2)}\right)$ [8] and a map $\pi : \left(T^*\mathbb{C}P^1, J_{O(-2)}\right) \twoheadrightarrow \left(\mathbb{C}^2/\mathbb{Z}_2, J_0\right)$ which is a biholomorphism outside the origin, i.e. $\pi : T^*\mathbb{C}P^1 - \mathbb{C}P^1 \overset{\mathrm{Biholo}}{\cong} \mathbb{C}^2/\mathbb{Z}_2 - \{0\}$ since $\pi^{-1}(0) \cong \mathbb{C}P^1$ is the exceptional divisor/zero section.

Therefore we have that $dz^1 \wedge dz^2$, the standard nowhere vanishing $(2,0)$-form/nowhere vanishing *holomorphic* volume form on $(\mathbb{C}^2, J_0)$, gets preserved under $\mathbb{Z}_2$ (as it's clearly even) and descends down to a nowhere vanishing holomorphic $(2,0)$-form on $(\mathbb{C}^2/\mathbb{Z}_2, J_0)$ (it extends over the origin by Hartog), and gets pulled back to a nowhere vanishing holomorphic $(2,0)$-form $\Omega_{T^*\mathbb{C}P^1}$ on $\left(T^*\mathbb{C}P^1, J_{O(-2)}\right)$ since the resolution $\pi : \left(T^*\mathbb{C}P^1, J_{O(-2)}\right) \twoheadrightarrow \left(\mathbb{C}^2/\mathbb{Z}_2, J_0\right)$ is crepant. Thus $\Omega_{T^*\mathbb{C}P^1}$ by Proposition 3.1.15 uniquely determines the (integrable) complex structure $J_{O(-2)}$ on $T^*\mathbb{C}P^1$ (since $d\Omega_{T^*\mathbb{C}P^1} = 0$).

Now define the smooth function $\phi_0 : (0, \infty) \to \mathbb{R}$ via $\phi_0(r) := \frac{r^2}{2}$. Then setting $r \mapsto |\cdot|_0^{g_0} := \left(|z^1|^2 + |z^2|^2\right)^{\frac{1}{2}}$ (the standard absolute value on $\mathbb{C}^2$), we have that $i\partial\bar{\partial}\phi_0 =: \omega_0$ gives us the standard *Euclidean* flat Kähler metric, where of course g_0 is the *Euclidean* flat metric on $\mathbb{R}^4 = \mathbb{C}^2$. Repeating the same arguments as above, we have this time around that g_0, ω_0 gets pulled back up and smoothly defined on $T^*\mathbb{C}P^1 - \mathbb{C}P^1$. Whence we may speak of $|\cdot|_{\mathbb{C}P^1}^{g_0} := \mathrm{dist}_{g_0}(\mathbb{C}P^1, \cdot)$, the distance function away from the exceptional divisor/zero section *measured WRT g_0*.

Now define the smooth function $\phi_{FS} : (0, \infty) \to \mathbb{R}$ via $\phi_{FS}(r) := \log r = \frac{1}{2}\log r^2$. Setting $\phi_{FS} = \phi_{FS}\left(|\cdot|_0^{g_0}\right)$, this function clearly descends down the quotient $\mathbb{C}^2 - \{0\} \twoheadrightarrow \mathbb{C}P^1$, and $i\partial\bar{\partial}\phi_{FS} =: \omega_{FS}$ yields us the usual **Fubini-Study** Kähler metric g_{FS}, ω_{FS} on $(\mathbb{C}P^1, J_{\mathbb{C}P^1})$.

Let $s > 0$. Now define the smooth function $\phi_{EH,s} : (0, \infty) \to \mathbb{R}$ via

$$\phi_{EH,s}(r) := \frac{1}{2}\left(s \log r^2 + \left(s^2 + r^4\right)^{\frac{1}{2}} - s \log\left(s + \left(s^2 + r^4\right)^{\frac{1}{2}}\right)\right)$$

Setting $r \mapsto |\cdot|_0^{g_0}$ but this time noting that $\left(|\cdot|_0^{g_0}\right)^4$ is *holomorphic* (since $\left(\left|(z^1, z^2)\right|_0^{g_0}\right)^4 := \left|(z^1, z^2)\right|^4 = |z^1|^4 + 2|z^1 z^2|^2 + |z^2|^4$ clearly satisfies $\bar{\partial}|z|^4 = 0$ by the chain rule), we have that upon repeating the same arguments above (namely $\mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_2 - \{0\}$, then extending to $\mathbb{C}^2/\mathbb{Z}_2$ by Hartog, then lifting through the resolution $T^*\mathbb{C}P^1$) that

$$\omega_{EH,s} := i\partial\bar{\partial}\phi_{EH,s} = s\pi^*(\omega_{FS}) + i\partial\bar{\partial}\left(\frac{1}{2}\left(\left(s^2 + r^4\right)^{\frac{1}{2}} - s \log\left(s + \left(s^2 + r^4\right)^{\frac{1}{2}}\right)\right)\right)$$

is a Kähler form defined *everywhere* on the total space of $\pi : \left(T^*\mathbb{C}P^1, J_{O(-2)}\right) \twoheadrightarrow \left(\mathbb{C}P^1, J_{\mathbb{C}P^1}\right)$ with associated Riemannian metric $g_{EH,s}$, and where $r := |\cdot|_{\mathbb{C}P^1}^{g_0}$. For $\lambda \in \mathbb{C}^\times$, upon denoting by $R_\lambda \in \mathrm{SU}(2)$ the scaling action given by $R_\lambda := \lambda\mathrm{id}_{\mathbb{C}^2}$ sending $(z^1, z^2) \mapsto (\lambda z^2, \lambda z^2)$, which clearly descends down $\mathbb{C}^2 - \{0\} \twoheadrightarrow \mathbb{C}P^1$ and lifts to a global $\mathbb{C}^\times$ action on $T^*\mathbb{C}P^1$ (hence $R_\lambda \in \mathrm{Diff}\left(T^*\mathbb{C}P^1\right)$), upon observing that $\phi_{EH,s} = sR_{\frac{1}{s^{\frac{1}{2}}}}^*\left(\phi_{EH,1}\right)$, we have $\forall s > 0$ that

$$R_{s^{\frac{1}{2}}}^*\left(\frac{1}{s}g_{EH,s}\right) = g_{EH,1}$$

Hence, *up to scaling* there is one Kähler metric $g_{EH,1}, \omega_{EH,1}$ defined on $\left(T^*\mathbb{C}P^1, J_{O(-2)}\right)$, which we call the **Eguchi-Hanson metric**. We now summarize & state the important properties of this metric in the following

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[8] Since the smooth manifold $\left(T^*S^2, J_{O(-2)}\right)$ with that given integrable complex structure is precisely the complex manifold $T^*\mathbb{C}P^1$ with that same complex structure.

Proposition 3.2.2. Let $s > 0$ be a positive real number, and let $g_{EH,s}, \omega_{EH,s}$ be as above.

- $\left(T^*\mathbb{C}P^1, J_{O(-2)}, g_{EH,s}, \omega_{EH,s}\right)$ is a complete Ricci-flat Riemannian 4-manifold which is an **ALE**[9] **Calabi-Yau**[10] **gravitational instanton**[11] asymptotic at ∞ to $\mathbb{R}^4/\mathbb{Z}_2$ with rate $-4 < 0$. Hence $\exists K \subset T^*\mathbb{C}P^1$ a compact subset, a constant $R > 0$ *sufficiently large*, and a diffeomorphism $F : \mathbb{R}^4/\mathbb{Z}_2 - D_R^{g_0}(0) \to T^*\mathbb{C}P^1 - K$ such that

$$\left|\nabla_{g_0}^k\left(F^* g_{EH,s} - g_0\right)\right|_{g_0} = O\left(\left(|\cdot|_0^{g_0}\right)^{-4-k}\right) \qquad \text{and} \qquad \left|\nabla_{g_0}^k\left(F^* \omega_{EH,s} - \omega_0\right)\right|_{g_0} = O\left(\left(|\cdot|_0^{g_0}\right)^{-4-k}\right)$$

on $\mathbb{R}^4/\mathbb{Z}_2 - D_R^{g_0}(0) = \mathbb{C}^2/\mathbb{Z}_2 - D_R^{g_0}(0)$. In fact, this follows from the Taylor expansion $\left|\nabla_{g_0}^k\left(\phi_{EH,s} - \phi_0\right)\right|_{g_0} \leq c_0(k) r^{-2-k}$ for $r > 0$ sufficiently large and $c_0(k) > 0, k \in \mathbb{N}_0$ positive constants.

- In fact, this diffeomorphism outside a compact subset comes from $\pi : T^*\mathbb{C}P^1 - \mathbb{C}P^1 \stackrel{\text{Biholo}}{\cong} \frac{\mathbb{C}^2 - \{0\}}{\mathbb{Z}_2}$, i.e.

$$\pi : T^*\mathbb{C}P^1 - D_R^{g_0}(\mathbb{C}P^1) \stackrel{\text{Biholo}}{\cong} \mathbb{C}^2/\mathbb{Z}_2 - D_R^{g_0}(0).$$

- From the $s\pi^*(\omega_{FS})$ term in $\omega_{EH,s}$ and the fact that the Fubini-Study metric on $\mathbb{C}P^1$ is isometric to the round metric of $S^2\left(\frac{1}{2}\right)$, we have that the exceptional divisor/zero section $\mathbb{C}P^1 \subset \left(T^*\mathbb{C}P^1, g_{EH,s}\right)$[12] has sectional curvature $\frac{4}{s}$, volume $s\pi$, diameter $\frac{s^{\frac{1}{2}}}{2}$, and self intersection number $\mathbb{C}P^1 \cdot \mathbb{C}P^1 = -2$.[13]

  In particular, as $s \to 0$, the curvature of the exceptional $\mathbb{C}P^1$ blows up, but its diameter/volume shrinks.

- The Eguchi-Hanson metric $\left(T^*\mathbb{C}P^1, J_{O(-2)}, g_{EH,s}, \omega_{EH,s}\right)$ is U(2)-**invariant**. In particular, we immediately have that the Eguchi-Hanson metric is U(1)-invariant[14], i.e.

$$R_\lambda^*\left(g_{EH,s}\right) = g_{EH,s}, \qquad \forall \lambda \in U(1) \subset \mathbb{C}^\times$$

- $g_{EH,s}$ and $g_{EH,1}$ (hence similarly for $\omega_{EH,s}, \omega_{EH,1}$) are isometric up to scaling, i.e.

$$R_{s^{\frac{1}{2}}}^*\left(\frac{1}{s} g_{EH,s}\right) = g_{EH,1}$$

We now want to "prepare" the Eguchi-Hanson metric by interpolating it with the Euclidean metric on an "annulus" via a suitable cutoff function on $T^*\mathbb{C}P^1$.

Data 3.2.3. Fix a constant $C_1 > 1$. Define a smooth monotone cutoff function $\chi : \mathbb{R}_{\geq 0} \to [0,1]$ satisfying
$$\chi(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & 2 \leq x \end{cases}, |\nabla \chi| \leq C_1 \text{ on } \{1 \leq x \leq 2\}, \text{ hence } \chi \text{ satisfies supp } \nabla^k \chi \subseteq \{1 \leq x \leq 2\} \text{ and } \left|\nabla^k \chi\right| \leq c_\chi(k)$$
for some positive constants $c_\chi(k) > 0$ where $c_\chi(0) = 1$ and $c_\chi(1) = C_1$. WLOG let us also denote by χ the smooth monotone function on $T^*\mathbb{C}P^1$ by $\chi := \chi \circ |\cdot|_{\mathbb{C}P^1}^{g_0}$. Hence $\chi = \begin{cases} 1 & \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 1\right\} \\ 0 & \left\{2 \leq |\cdot|_{\mathbb{C}P^1}^{g_0}\right\} \end{cases}$, supp $\nabla^k \chi_\epsilon \subseteq \left\{1 \leq |\cdot|_{\mathbb{C}P^1}^{g_0} \leq 2\right\}$, and

$\left|\nabla^k \chi_\epsilon\right|_{g_0} \leq c_\chi(k)$.

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[9] Asymptotically Locally Euclidean.

[10] say, after appropriate normalization of $\Omega_{T^*\mathbb{C}P^1}$

[11] A complete non-compact Riemannian manifold with $\left\|\text{Rm}_g\right\|_{L^2} < \infty$.

[12] Also called the **bolt** of $T^*\mathbb{C}P^1$.

[13] This is because the Euler characteristic of S^2 is 2, $T^*\mathbb{C}P^1$ has the *opposite* orientation of $T\mathbb{C}P^1$, and the Euler class of a (real) vector bundle is the self-intersection number of the zero section. Alternatively, this follows because of the self-intersection and $c_1(L^*) = -c_1(L)$ and $c_1(L) = e(L_\mathbb{R})$ for complex line bundles L.

[14] This is easily seen because of how $\mathbb{C}^\times \subset T^*\mathbb{C}P^1$. Namely, it starts via R_λ on $\mathbb{C}^2$, which clearly descends down $\mathbb{C}^2 - \{0\} \twoheadrightarrow \mathbb{C}P^1$ and lifts to $T^*\mathbb{C}P^1$. However, if $\lambda \in U(1)$, then $|\lambda| = 1$, whence by definition of $\phi_{EH,s}$ being inputted with $|\cdot|_{\mathbb{C}P^1}^{g_0}$, $\phi_{EH,s}$ is clearly U(1)-invariant and hence so are $\omega_{EH,s}$ and $g_{EH,s}$.

Remark 3.2.4. The resulting Ricci-flat metric which will be constructed in fact does *not* depend on this cutoff function χ. Indeed, though the *initial* Kähler form and metric that we will perturb from does depend on χ, its *cohomology class* is independent of χ, and the resulting Ricci-flat Kähler form is the *unique* such form in the cohomology class of the initial Kähler form. ♣

Data 3.2.5. Throughout this thesis, let $\epsilon > 0$ be a positive number, the **gluing parameter**[15].

Let us first define $\chi_\sigma := \chi\left(\sigma |\cdot|_{\mathbb{C}P^1}^{g_0}\right)$, aka $\chi_\sigma := R_\sigma^* \chi$, given any $\sigma > 0$. Hence $\chi_\sigma = \begin{cases} 1 & \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq \frac{1}{\sigma}\right\} \\ 0 & \left\{\frac{2}{\sigma} \leq |\cdot|_{\mathbb{C}P^1}^{g_0}\right\} \end{cases}$,

$\operatorname{supp}\nabla^k \chi_\epsilon \subseteq \left\{\frac{1}{\sigma} \leq |\cdot|_{\mathbb{C}P^1}^{g_0} \leq \frac{2}{\sigma}\right\}$, and $\left|\nabla^k \chi_\epsilon\right|_{g_0} \leq c_\chi(k)\sigma^k$.

Define $\widehat{\phi_{EH-0,\epsilon}} := \chi_{\epsilon^{\frac{1}{2}}} \phi_{EH,1} + \left(1 - \chi_{\epsilon^{\frac{1}{2}}}\right)\phi_0$, and define $\widehat{\omega_{EH-0,\epsilon}} := i\partial\bar{\partial}\left(\widehat{\phi_{EH-0,\epsilon}}\right)$. Immediate from the definitions and the fact that being a positive $(1,1)$-form is open (*Remark* 3.1.9) is

Proposition 3.2.6. For $\epsilon > 0$ from Data 3.2.5 **sufficiently small** so that $\widehat{\omega_{EH-0,\epsilon}}$ is a positive $(1,1)$-form, we have that $\widehat{\omega_{EH-0,\epsilon}}$ is a Kähler form on $\left(T^*\mathbb{C}P^1, J_{O(-2)}\right)$ compatible with $J_{O(-2)}$. Hence $\left(T^*\mathbb{C}P^1, J_{O(-2)}, \widehat{\omega_{EH-0,\epsilon}}, \widehat{g_{EH-0,\epsilon}}\right)$ is a Kähler manifold such that

$$\widehat{\omega_{EH-0,\epsilon}} = \begin{cases} \omega_{EH,1} & \text{on } \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq \frac{1}{\epsilon^{\frac{1}{2}}}\right\} \\ \omega_0 & \text{on } \left\{\frac{2}{\epsilon^{\frac{1}{2}}} \leq |\cdot|_{\mathbb{C}P^1}^{g_0}\right\} \end{cases}$$

Moreover, by 2-out-of-3, **all of the above hold verbatim for the associated Riemannian metric $\widehat{g_{EH-0,\epsilon}}$.**

Define $\chi_\epsilon := \chi\left(\frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon^{\frac{1}{2}}}\right) = R^*_{\frac{1}{\epsilon^{\frac{1}{2}}}} \chi$. Hence $\chi_\epsilon = \begin{cases} 1 & \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq \epsilon^{\frac{1}{2}}\right\} \\ 0 & \left\{2\epsilon^{\frac{1}{2}} \leq |\cdot|_{\mathbb{C}P^1}^{g_0}\right\} \end{cases}$, $\operatorname{supp}\nabla^k \chi_\epsilon \subseteq \left\{\epsilon^{\frac{1}{2}} \leq |\cdot|_{\mathbb{C}P^1}^{g_0} \leq 2\epsilon^{\frac{1}{2}}\right\}$, and

$\left|\nabla^k \chi_\epsilon\right|_{g_0} \leq c_\chi(k)\frac{1}{\epsilon^{\frac{k}{2}}}$. Now define $\widehat{\phi_{EH,\epsilon}} := \chi_\epsilon \phi_{EH,\epsilon^2} + (1-\chi_\epsilon)\phi_0$. Hence $\widehat{\phi_{EH,\epsilon}} = \epsilon^2 \chi\left(\frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon^{\frac{1}{2}}}\right)\phi_{EH,1}\left(\frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon}\right) +$

$\epsilon^2\left(1-\chi\left(\frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon^{\frac{1}{2}}}\right)\right)\phi_0\left(\frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon}\right) = \epsilon^2\chi\left(\frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon^{\frac{1}{2}}}\right)\phi_{EH,1}\left(\frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon}\right) + \left(1 - \chi\left(\frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon^{\frac{1}{2}}}\right)\right)\phi_0\left(|\cdot|_{\mathbb{C}P^1}^{g_0}\right)$ by homogeneity of ϕ_0.

Now define $\widetilde{\omega_{EH,\epsilon}} := i\partial\bar{\partial}\left(\widehat{\phi_{EH,\epsilon}}\right)$. Immediate from the definitions, openness of positive $(1,1)$-forms, Leibniz rule, the identity $\phi_{EH,\epsilon^2} = \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(\phi_{EH,1}\right)$, as well as a local ON frame computation is

Proposition 3.2.7. For $\epsilon > 0$ from Data 3.2.5 **sufficiently small** so that $\widetilde{\omega_{EH,\epsilon}}$ is a positive $(1,1)$-form, we have that $\widetilde{\omega_{EH,\epsilon}}$ is a Kähler form on $\left(T^*\mathbb{C}P^1, J_{O(-2)}\right)$ compatible with $J_{O(-2)}$. Hence $\left(T^*\mathbb{C}P^1, J_{O(-2)}, \widetilde{\omega_{EH,\epsilon}}, \widetilde{g_{EH,\epsilon}}\right)$ is a Kähler manifold such that

-

$$\widetilde{\omega_{EH,\epsilon}} = \begin{cases} \omega_{EH,\epsilon^2} & \text{on } \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq \epsilon^{\frac{1}{2}}\right\} \\ \omega_0 & \text{on } \left\{2\epsilon^{\frac{1}{2}} \leq |\cdot|_{\mathbb{C}P^1}^{g_0}\right\} \end{cases} \quad \text{and} \quad \widetilde{g_{EH,\epsilon}} = \begin{cases} g_{EH,\epsilon^2} & \text{on } \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq \epsilon^{\frac{1}{2}}\right\} \\ g_0 & \text{on } \left\{2\epsilon^{\frac{1}{2}} \leq |\cdot|_{\mathbb{C}P^1}^{g_0}\right\} \end{cases}$$

-

$$\left|\nabla^k_{g_0}\left(\widetilde{g_{EH,\epsilon}} - g_0\right)\right|_{g_0} \leq c_2(k)\epsilon^{2-\frac{k}{2}} \quad \text{and} \quad \left|\nabla^k_{g_0}\left(\widetilde{\omega_{EH,\epsilon}} - \omega_0\right)\right|_{g_0} \leq c_1(k+2)\epsilon^{2-\frac{k}{2}}$$

on the annuli $\left\{\epsilon^{\frac{1}{2}} \leq |\cdot|_{\mathbb{C}P^1}^{g_0} \leq 2\epsilon^{\frac{1}{2}}\right\} = \left\{\frac{1}{\epsilon^{\frac{1}{2}}} \leq \frac{|\cdot|_{\mathbb{C}P^1}^{g_0}}{\epsilon} \leq \frac{2}{\epsilon^{\frac{1}{2}}}\right\}$ when $\epsilon > 0$ is **sufficiently small** (specifically $\epsilon < \frac{1}{R^2}$), for some positive constants $c_1(k) > 0, k \in \mathbb{N}_0$ in which[16] $\left|\nabla^k_{g_0}\left(\widehat{\phi_{EH,\epsilon}} - \phi_0\right)\right|_{g_0} \leq c_1(k)\epsilon^{3-\frac{k}{2}}$, and for some positive constants $c_2(k) > 0, k \in \mathbb{N}_0$.

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[15] Parameters feature quite prominently in gluing constructions, and this $\epsilon > 0$ will be referred back to throughout the rest of this paper!

[16] In fact, the proof yields that $c_1(k) = \sum_{i=0}^k c_0(i)c_\chi(k-i)$, where we recall that $c_0(k)$ are the constants in the decay of the Kähler potential $\phi_{EH,s}$ back in Proposition 3.2.2.

-
$$\widetilde{\omega_{EH,\epsilon}} = \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(\widetilde{\omega_{EH-0,\epsilon}}\right) \qquad \text{and} \qquad \widetilde{g_{EH,\epsilon}} = \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(\widetilde{g_{EH-0,\epsilon}}\right)$$

hence

$$\widetilde{\omega_{EH,\epsilon}} = \begin{cases} \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(\omega_{EH,1}\right) & \text{on } \left\{|\cdot|^{g_0}_{\mathbb{C}P^1} \le \epsilon^{\frac{1}{2}}\right\} \\ \omega_0 = \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(\omega_0\right) & \text{on } \left\{2\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\mathbb{C}P^1}\right\} \end{cases} \quad \text{and} \quad \widetilde{g_{EH,\epsilon}} = \begin{cases} \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(g_{EH,1}\right) & \text{on } \left\{|\cdot|^{g_0}_{\mathbb{C}P^1} \le \epsilon^{\frac{1}{2}}\right\} \\ g_0 = \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(g_0\right) & \text{on } \left\{2\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\mathbb{C}P^1}\right\} \end{cases}$$

Thus we also have, **for $\epsilon < 1$,**

$$\mathrm{d}V_{\widetilde{g_{EH,\epsilon}}} = \begin{cases} \mathrm{d}V_{g_{EH,\epsilon^2}} & \text{on } \left\{|\cdot|^{g_0}_{\mathbb{C}P^1} \le \epsilon^{\frac{1}{2}}\right\} \\ \left(1 + O\left(\epsilon^2\right)\right)\mathrm{d}V_{g_0} & \text{on } \left\{\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\mathbb{C}P^1}\right\} \end{cases}$$

Lastly, we recall a few very standard facts about flat tori.

Proposition 3.2.8. Let $\Lambda \subset \mathbb{R}^n$ be a full rank lattice, hence is equipped with a *choice* of isomorphism $\Lambda \cong \mathbb{Z}^n$. Consider the n-torus $\mathbb{T}^n := \mathbb{R}^n/\Lambda$.

- $(\mathbb{T}^n, g_0)$ is a flat Riemannian manifold (since the *Euclidean* flat metric g_0 is clearly $\mathbb{Z}^n$-invariant)
- The **moduli space** of all flat metrics on $\mathbb{T}^n$ is $\mathrm{GL}(n, \mathbb{Z})\backslash \mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$, which has dimension $\frac{n(n+1)}{2}$.
- The same holds for orbifolds $\mathbb{T}^4/G$ for G a finite group acting isometrically on $\mathbb{T}^4$.

Indeed, $\mathrm{GL}(n, \mathbb{Z})\backslash \mathrm{GL}(n, \mathbb{R})$ parametrizes the choice of isomorphism $\Lambda \cong \mathbb{Z}^n$, and $\mathrm{O}(n)$ is precisely the stablizer in $\mathrm{GL}(n, \mathbb{R})$ of the Euclidean metric g_0.

3.3 § The "Kummer Construction" §

Recall that a $K3$ surface is the underlying smooth oriented 4-manifold of a connected compact simply connected complex 2-manifold with trivial canonical bundle. Since any two $K3$ surfaces are automatically diffeomorphic, differing constructions of any complex 2-fold that are connected, compact, simply connected, and have trivial canonical bundle all yield the same $K3$ surface, up to diffeomorphism. We rigorously construct here one such definition of a $K3$ surface, namely as a **Kummer surface** Km, which is the resolution of an abelian variety quotiented out by an involution (which is closed, connected, simply connected and has trivial canonical bundle), and then graft in our Eguchi-Hanson metric $\widetilde{\omega_{EH,\epsilon}}$ from the previous section to get a family of Kähler metrics ω_ϵ on $\mathscr{K}$.

First consider the 4-torus $\mathbb{T}^4 = \mathbb{R}^4/\Lambda$ where $\Lambda \cong \mathbb{Z}^4$ is a full rank lattice in $\mathbb{R}^4$. Equip $\mathbb{R}^4$ with the standard Euclidean Kähler structure J_0, g_0, ω_0, getting $\left(\mathbb{C}^2, J_0, g_0, \omega_0\right)$. Each of the 3 tensors J_0, g_0, ω_0 are $\mathbb{Z}^n$-invariant, hence by Proposition 3.2.1 descends down to $\mathbb{T}^4$ yielding a **flat Kähler** 4-**torus** $\left(\mathbb{T}^4, J_0, g_0, \omega_0\right)$. Furthermore, the flat Kähler datum J_0, g_0, ω_0 are all $\mathbb{Z}_2$-invariant under the standard involution/$\mathbb{Z}_2$-action $z \mapsto -z$ on $\mathbb{T}^4 = \mathbb{C}^2/\Lambda$. In fact, the same argument above works verbatim on $dz^1 \wedge dz^2$. Hence, upon passing to the quotient orbifold $\mathbb{T}^4/\mathbb{Z}_2$, we in fact get a **flat Kähler orbifold** $\left(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0\right)$ with 16 isolated orbifold point-singularities, each modeled on $\mathbb{R}^4/\mathbb{Z}_2 = \mathbb{C}^2/\mathbb{Z}_2$ [17], and that the underlying **complex orbifold** $\left(\mathbb{T}^4/\mathbb{Z}_2, J_0\right)$ admits a nowhere vanishing *holomorphic* volume form $dz^1 \wedge dz^2$ which uniquely determines J_0 by Proposition 3.1.15. We perform a crepant resolution $\left(\widetilde{\mathbb{T}^4/\mathbb{Z}_2}, J\right)$ by blowing up each of the 16 orbifold singularities *and where we denote the resulting integrable complex structure induced by the pullback of $dz^1 \wedge dz^2$ by J*, and the result is a connected compact simply connected complex 2-fold with trivial canonical bundle, whence yielding a $K3$ surface we denote by Km. Specifically, since each

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[17] Note that topologically we have that $\mathbb{R}^4/\mathbb{Z}_2 \cong \mathrm{Cone}\left(\mathbb{R}P^3\right)$, and it is precisely the fact that the metric on $\mathrm{Cone}\left(\mathbb{R}P^3\right)$ is conformally equivalent to the metric on a cylinder over $\mathbb{R}P^3$ that Donaldson exploits in [13] to bypass the usage of weighted Holder spaces. Part of the reason why this works is because the cylindrical metric of $\mathbb{R}P^3$ has *bounded geometry* as opposed to the conical metric with its cone singularity.

orbifold singularity is modeled on $\left(\mathbb{C}^2/\mathbb{Z}_2, J_0\right)$, we have that the resolution at that singular point is *precisely the same resolution* $\pi : \left(T^*\mathbb{C}P^1, J_{O(-2)}\right) \twoheadrightarrow \left(\mathbb{C}^2/\mathbb{Z}_2, J_0\right)$ for the Eguchi-Hanson space. In fact, it is this fact which led Page, Gibbons, and Pope [32] [16] to conjecture that one may "graft in" 16 "small Eguchi-Hanson metrics" around the 16 exceptional divisors of $\widetilde{\mathbb{T}^4/\mathbb{Z}_2}$ onto the flat metric on its complement. Thus we also have a projection map $\pi : \left(\widetilde{\mathbb{T}^4/\mathbb{Z}_2}, J\right) \twoheadrightarrow \left(\mathbb{T}^4/\mathbb{Z}_2, J_0\right)$ which is a biholomorphism on the complement of the 16 exceptional $\mathbb{C}P^1$s.

In more detail, let $S := \text{sing}\left(\mathbb{T}^4/\mathbb{Z}_2\right)$ be the set of 16 singular points, aka $\text{Fix}(\mathbb{Z}_2)$.

Thus, let $|\cdot|_S^{g_0} := \text{dist}_{g_0}(S, \cdot)$ be the metric distance away from the singular points WRT g_0.

Data 3.3.1. Since this entire construction will work with any full rank lattice $\Lambda \cong \mathbb{Z}^4$, WLOG let $\Lambda = \mathbb{Z}^4$, hence the singular points are the points whose real and imaginary parts are $0 \mod \frac{1}{2}$ the distance between each of the 16 singular points WRT g_0 is $\frac{1}{2}$.

Let our sufficiently small $\epsilon > 0$ from Data 3.2.5 and Proposition 3.2.7 be even smaller so that $3\epsilon^{\frac{1}{2}} < \frac{1}{2}$ (if it wasn't already small enough to also ensure this).

Now consider 16 copies of the closed tubular neighborhood of radius $3\epsilon^{\frac{1}{2}}$ around the exceptional divisor on $\left(T^*\mathbb{C}P^1, J_{O(-2)}\right)$, i.e. $\bigsqcup_{i \in \{1,\dots,16\}} T^*\mathbb{C}P^1 - \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} > 3\epsilon^{\frac{1}{2}}\right\} = \bigsqcup_{i \in \{1,\dots,16\}} \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\} = \bigsqcup_{i \in \{1,\dots,16\}} D_{3\epsilon^{\frac{1}{2}}}^{g_0}(\mathbb{C}P^1)$.

Now define

$$(\text{Km}_\epsilon, J, g_\epsilon, \omega_\epsilon) := \left(\widetilde{\mathbb{T}^4/\mathbb{Z}_2}, J, g_\epsilon, \omega_\epsilon\right)$$

$$:= \frac{\left(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0\right) \bigsqcup_{i \in \{1,\dots,16\}} \left(T^*\mathbb{C}P^1 - \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} > 3\epsilon^{\frac{1}{2}}\right\}, J_{O(-2)}, \widetilde{g_{EH,\epsilon}}, \widetilde{\omega_{EH,\epsilon}}\right)}{\sim}$$

with the equivalence relation being the identification $\left\{|\cdot|_S^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\} \sim \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\}$. That is, each of the 16 connected components of $\left\{|\cdot|_S^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\}$, which is $\left\{|\cdot|_p^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\}$ for each $p \in S$, gets identified with a single copy of $\left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\} \subset T^*\mathbb{C}P^1$, and this identification is done via first lifting/pulling back $\left\{|\cdot|_p^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\} \subset \mathbb{T}^4/\mathbb{Z}_2$ to $\mathbb{T}^4$ via the (branched) double cover/orbifold projection, then to (a proper subset of the fundamental domain in) $\mathbb{C}^2$ via the quotient $\mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\Lambda = \mathbb{T}^4$, translating so that $p \mapsto 0$, then pushing down via $\mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_2$ onto $\mathbb{C}^2/\mathbb{Z}_2$, then lifting/pulling back via the crepant resolution map $T^*\mathbb{C}P^1 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_2$ to get $\left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\} \subset T^*\mathbb{C}P^1$. Thus the *underlying* projection map $\pi : \text{Km}_\epsilon \twoheadrightarrow \mathbb{T}^4/\mathbb{Z}_2$ is given by the identity on $\left\{3\epsilon^{\frac{1}{2}} < |\cdot|_S^{g_0}\right\}$, and on $\left\{|\cdot|_S^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\} \sim \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\}$ via the crepant resolution map $\pi : \left(T^*\mathbb{C}P^1, J_{O(-2)}\right) \twoheadrightarrow \left(\mathbb{C}^2/\mathbb{Z}_2, J_0\right)$.

This gives us our underlying Kummer $K3$ surface Km, *or* Km_ϵ *whenever we want to track the dependence on* ϵ, *as well as the background integrable complex structure* J. Indeed, since $(\text{Km}_\epsilon, J) = \left(\widetilde{\mathbb{T}^4/\mathbb{Z}_2}, J\right)$ is a crepant resolution of $\left(\mathbb{T}^4/\mathbb{Z}_2, J_0\right)$, we have that there exists a nowhere vanishing holomorphic $(2,0)$-form/holomorphic volume form Ω on (Km_ϵ, J) (given by $dz^1 \wedge dz^2$ on $\left\{|\cdot|_S^{g_0} > 3\epsilon^{\frac{1}{2}}\right\}$ and identifying $dz^1 \wedge dz^2$ on $\left\{|\cdot|_S^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\}$ with $\Omega_{T^*\mathbb{C}P^1}$ on $\left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\}$, since both holomorphic volume forms descend from the *same* Euclidean holomorphic volume form on $\mathbb{C}^2$), and Ω uniquely determines J and trivializes the canonical bundle from Proposition 3.1.15.

It now remains to define ω_ϵ.

Letting $E_i \cong \mathbb{C}P^1$ denote one of the 16 exceptional divisors and $\pi^{-1}(S) = \bigsqcup_{i \in \{1,\dots,16\}} E_i$, we set

$$\omega_\epsilon := \begin{cases} \omega_0 & \text{on } \text{Km}_\epsilon - \bigsqcup_{i \in \{1,\dots,16\}} D_{3\epsilon^{\frac{1}{2}}}^{g_0}(E_i) = \text{Km}_\epsilon - \left\{|\cdot|_{\pi^{-1}(S)}^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\} = \left\{3\epsilon^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_0}\right\} \\ \widetilde{\omega_{EH,\epsilon}} & \text{on (each 16 connected components of)} \left\{|\cdot|_{\pi^{-1}(S)}^{g_0} \leq 3\epsilon^{\frac{1}{2}}\right\} \end{cases}$$

which is clearly well defined and smooth on all of Km_ϵ by Proposition 3.2.7, and we now have our $K3$ surface $(\text{Km}_\epsilon, J, g_\epsilon, \omega_\epsilon)$ with our Kähler form ω_ϵ and associated Riemannian metric g_ϵ from 2-out-of-3, **depending on the small** $\epsilon > 0$.

Remark 3.3.2. Here we have that $|\cdot|^{g_0}_{\pi^{-1}(S)} := \mathrm{dist}_{g_0}\left(\pi^{-1}(S), \cdot\right) = \mathrm{dist}_{g_0}(S, \pi(\cdot)) =: |\pi(\cdot)|^{g_0}_S$ because $\pi : (\mathrm{Km}_\epsilon, J) \twoheadrightarrow$ $\left(\mathbb{T}^4/\mathbb{Z}_2, J_0\right)$ is a *biholomorphism* on the complement of the exceptional divisors $\pi^{-1}(S) = \sqcup_{i \in \{1,\dots,16\}} E_i$ and the projection $\mathbb{T}^4 \twoheadrightarrow \mathbb{T}^4/\mathbb{Z}_2$ is a double cover on the complement of the orbifold singularities S along which g_0 descends, whence we have the flat metric g_0 on $\mathbb{T}^4 - S \overset{2:1}{\twoheadrightarrow} \mathbb{T}^4/\mathbb{Z}_2 - S \overset{\text{Biholo}}{\cong} \mathrm{Km}_\epsilon - \pi^{-1}(S)$.

In particular, $\forall r > 0$ we may identify $\left\{ r < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \subset \mathrm{Km}_\epsilon$ with $\left\{ r < |\cdot|^{g_0}_S \right\} \subset \mathbb{T}^4/\mathbb{Z}_2 - S$ via $\pi : (\mathrm{Km}_\epsilon, J) \twoheadrightarrow$ $\left(\mathbb{T}^4/\mathbb{Z}_2, J_0\right)$ and its lift in $\left\{ r < |\cdot|^{g_0}_S \right\} \subset \mathbb{T}^4 - S$ via the double cover $\mathbb{T}^4 - S \overset{2:1}{\twoheadrightarrow} \mathbb{T}^4/\mathbb{Z}_2 - S$, where it is of course understood that $|\cdot|^{g_0}_S := \mathrm{dist}_{g_0}(S, \cdot)$.

On a similar vein, we only did the identification $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le 3\epsilon^{\frac{1}{2}} \right\} \sim \left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \le 3\epsilon^{\frac{1}{2}} \right\}$ in the underlying construction of Km_ϵ to guarantee that our ω_ϵ is well defined and smooth on all of Km_ϵ. Thus $\forall r \in \left(0, \frac{1}{2}\right)$ (since WLOG $\Lambda = \mathbb{Z}^4$) we may identify *each of the 16 connected components of* $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le r \right\} \subset \mathrm{Km}_\epsilon$ with $\left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \le r \right\} \subset T^*\mathbb{C}P^1$ via the same exact process as the original identification above (connected component starts out in $\mathbb{T}^4/\mathbb{Z}_2$, lift to $\mathbb{T}^4$, lift to $\mathbb{C}^2$, translate, project down to $\mathbb{C}^2/\mathbb{Z}_2$, lift to $T^*\mathbb{C}P^1$ via crepant resolution). ♣

Observe also the following properties which immediately follows from construction of ω_ϵ and Proposition 3.2.7:

Proposition 3.3.3. For $\epsilon > 0$ satisfying Data 3.3.1, we have that ω_ϵ satisfies

$$\omega_\epsilon = \begin{cases} \omega_{EH,\epsilon^2} & \text{on } \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le \epsilon^{\frac{1}{2}} \right\} \\ \omega_0 & \text{on } \left\{ 2\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \end{cases} \quad \text{and} \quad g_\epsilon = \begin{cases} g_{EH,\epsilon^2} & \text{on } \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le \epsilon^{\frac{1}{2}} \right\} \\ g_0 & \text{on } \left\{ 2\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \end{cases}$$

and, **for sufficiently small** $\epsilon > 0$ to satisfy Proposition 3.2.7,

$$\left| \nabla^k_{g_0}(\omega_\epsilon - \omega_0) \right|_{g_0} \le c_1(k+2)\epsilon^{2-\frac{k}{2}} \qquad \text{and} \qquad \left| \nabla^k_{g_0}(g_\epsilon - g_0) \right|_{g_0} \le c_2(k)\epsilon^{2-\frac{k}{2}}$$

on $\left\{ \epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \le 2\epsilon^{\frac{1}{2}} \right\} = \left\{ \frac{1}{\epsilon^{\frac{1}{2}}} \le \frac{|\cdot|^{g_0}_{\pi^{-1}(S)}}{\epsilon} \le \frac{2}{\epsilon^{\frac{1}{2}}} \right\}$.

Thus we also have, **for** $\epsilon < 1$ (which Data 3.3.1 guarantees),

$$\mathrm{d}V_{g_\epsilon} = \begin{cases} \mathrm{d}V_{g_{EH,\epsilon^2}} & \text{on } \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le \epsilon^{\frac{1}{2}} \right\} \\ \left(1 + O\left(\epsilon^2\right)\right)\mathrm{d}V_{g_0} & \text{on } \left\{ \epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \end{cases}$$

and that the cohomology class $[\omega_\epsilon] \in H^2(K3)$ is independent of the cutoff function χ from Data 3.2.3 (*Remark* 3.2.4).

Remark 3.3.4. Note that, as previously mentioned, a theorem of Kodaira implies that for any pair $\epsilon_1, \epsilon_2 > 0$ of *positive* real numbers, the resulting $\mathrm{Km}_{\epsilon_1}, \mathrm{Km}_{\epsilon_2}$ are diffeomorphic, and since we've fixed the underlying integrable complex structure J, they are in fact biholomorphic. ♣

Remark 3.3.5. From the construction of Km_ϵ and ω_ϵ itself, we have that if $\epsilon \searrow 0$, that as Riemannian manifolds $(\mathrm{Km}_\epsilon, J, g_\epsilon, \omega_\epsilon) \overset{GH}{\longrightarrow} \left(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0\right)$ converges in the **Gromov-Hausdorff topology** (see [6]) to the orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0\right)$ with the singular orbifold Kähler metric ω_0 with conical $\mathbb{C}^2/\mathbb{Z}_2$ singularities at each of the 16 singular points. Moreover, this convergence is satisfies **volume non-collapse**. Note that outside of the set S of 16 singular points, we have that ω_0 is the flat Kähler metric on $\mathbb{T}^4/\mathbb{Z}_2 - S$ from before.

Unfortunately, as with the language of G-structures and many other fascinating notions, we do not have the space to delve into this absolutely fantastic topic (see [6], [33]). ♣

3.4 § Weighted (Holder) Spaces §

We now define the weighted Holder spaces that we will be using for our setting, first for $\left(T^*\mathbb{C}P^1, J_{O(-2)}, g_{EH}, \omega_{EH}\right)$, then for $\left(\mathbb{T}^4, J_0, g_0, \omega_0\right)$ and by extension $\left(\mathbb{T}^4 - S, J_0, g_0, \omega_0\right) = \left(\mathbb{T}^4/\mathbb{Z}_2 - S, J_0, g_0, \omega_0\right)$, and lastly for $\left(\mathrm{Km}_\epsilon, J, g_\epsilon, \omega_\epsilon\right)$. As Holder spaces spaces *only depend on the Riemannian metric*, they do *not* require an integrable complex structure nor a Kähler form (though by 2-out-of-3 both of these uniquely determine the metric), *and hence are defined on the underlying smooth manifolds* $\left(T^*S^2, g_{EH,s}\right)$, $\left(\mathbb{T}^4, g_0\right)$, $\left(\mathbb{T}^4 - S, g_0\right)$, *and* $(\mathrm{Km}_\epsilon, g_\epsilon)$.

Notation 3.4.1. *Let $\alpha \in (0,1)$, $\delta \in \mathbb{R}$, $k \in \mathbb{N}_0$, and $\mathfrak{s}$ be a (p,q)-tensor. For any $j \geq 0$ let* $\left\|\nabla_g^j \mathfrak{s}\right\|_{C_{\delta,g}^0} := \left\|\rho^{-\delta+j} \nabla_g^j \mathfrak{s}\right\|_{C_g^0}$

denote the weighted C^0-norm corresponding to a Riemannian metric g for ρ a weight function.

We oftentimes call δ the **weight** or **rate** parameter. ○

Recalling $C_1 > 0$ from Data 3.2.3, on $T^*\mathbb{C}P^1$ we let $\widetilde{\rho}_0 : T^*\mathbb{C}P^1 \to [1, \infty)$ be a smooth monotone function satisfying

$$\widetilde{\rho}_0 = \begin{cases} |\cdot|_{\mathbb{C}P^1}^{g_0} & \text{on } \left\{2 \leq |\cdot|_{\mathbb{C}P^1}^{g_0}\right\} \\ 1 & \text{on } \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 1\right\} \end{cases}, \qquad |\nabla\widetilde{\rho}_0|_{g_0} \leq C_1 \text{ on } \left\{1 \leq |\cdot|_{\mathbb{C}P^1}^{g_0} \leq 2\right\}$$

On $\left(T^*\mathbb{C}P^1, J_{O(-2)}, g_{EH,s}, \omega_{EH,s}\right)$, we now define the $\|\cdot\|_{C_{\delta,g_{EH,s}}^{k,\alpha}(T^*\mathbb{C}P^1)}$ norm as:

$$\|\mathfrak{s}\|_{C_{\delta,g_{EH,s}}^{k,\alpha}(T^*\mathbb{C}P^1)} := \sum_{j \leq k} \left\|\nabla_{g_{EH}}^j \mathfrak{s}\right\|_{C_{\delta,g_{EH,s}}^0(T^*\mathbb{C}P^1)} + \sup_{\substack{x \neq y \\ d_{g_{EH,s}}(x,y) \leq \mathrm{inj}_{g_{EH,s}}}} \min\{\widetilde{\rho}_0(x), \widetilde{\rho}_0(y)\}^{-\delta+k+\alpha} \frac{\left|\nabla_{g_{EH,s}}^k \mathfrak{s}\big|_x - \nabla_{g_{EH,s}}^k \mathfrak{s}\big|_y\right|_{g_{EH,s}}}{d_{g_{EH,s}}(x,y)^\alpha}$$

$$:= \sum_{j \leq k} \left\|\widetilde{\rho}_0^{-\delta+j} \nabla_{g_{EH,s}}^j \mathfrak{s}\right\|_{C_{g_{EH,s}}^0(T^*\mathbb{C}P^1)} + \sup_{\substack{x \neq y \\ d_{g_{EH,s}}(x,y) \leq \mathrm{inj}_{g_{EH,s}}}} \min\{\widetilde{\rho}_0(x), \widetilde{\rho}_0(y)\}^{-\delta+k+\alpha} \frac{\left|\nabla_{g_{EH,s}}^k \mathfrak{s}\big|_x - \nabla_{g_{EH,s}}^k \mathfrak{s}\big|_y\right|_{g_{EH,s}}}{d_{g_{EH,s}}(x,y)^\alpha}$$

where we interpret the difference in the numerator of the (weighted) Holder seminorm via parallel transport along the unique minimal geodesic connecting x and y.

Note that all the covariant derivatives and parallel transport and higher order tensor (i.e. not functions) norms are all with respect to $g_{EH,s}, \omega_{EH,s}$ the Eguchi-Hanson metric on $\left(T^*\mathbb{C}P^1, J_{O(-2)}\right)$. We now define the space of weighted Holder (p,q)-tensors $C_{\delta,g_{EH,s}}^{k,\alpha}\left(T^*\mathbb{C}P^1\right)$ as $C_{\delta,g_{EH,s}}^{k,\alpha}\left(T^*\mathbb{C}P^1\right) := \left\{\mathfrak{s} \in C_{g_{EH,s}}^{k,\alpha}\left(T^*\mathbb{C}P^1\right) : \|\mathfrak{s}\|_{C_{\delta,g_{EH,s}}^{k,\alpha}(T^*\mathbb{C}P^1)} < \infty\right\}$ but with the $\|\cdot\|_{C_{\delta,g_{EH,s}}^{k,\alpha}(T^*\mathbb{C}P^1)}$ norm instead.[18] It is a Banach space whose elements, i.e. tensors $\mathfrak{s}$ with finite $\|\cdot\|_{C_{\delta,g_{EH,s}}^{k,\alpha}(T^*\mathbb{C}P^1)}$ norm, say $\|\mathfrak{s}\|_{C_{\delta,g_{EH,s}}^{k,\alpha}(T^*\mathbb{C}P^1)} \leq C$, decay roughly as $\nabla_{g_{EH,s}}^j \mathfrak{s} = O_{g_{EH,s}}\left(\widetilde{\rho}_0^{\delta-j}\right), j \leq k$.

Remark 3.4.2. Note that because our weight function $\widetilde{\rho}_0 : T^*\mathbb{C}P^1 \to [1, \infty)$ is smooth nondecreasing and satisfying $\widetilde{\rho}_0 = \begin{cases} |\cdot|_{\mathbb{C}P^1}^{g_0} & \text{on } \left\{2 \leq |\cdot|_{\mathbb{C}P^1}^{g_0}\right\} \\ 1 & \text{on } \left\{|\cdot|_{\mathbb{C}P^1}^{g_0} \leq 1\right\} \end{cases}$, it is in particular *unbounded from above*, and therefore we have that the space $C_{\delta,g_{EH,s}}^{k,\alpha}\left(T^*\mathbb{C}P^1\right)$ ***does not contain the constant tensors ! ! !*** This is a crucial property for weighted Holder spaces which allows for us to completely kill the kernel of the weighted Laplacian, since as we'll see, a suitable choice of weight parameter $\delta \in \mathbb{R}$ (specifically $\delta \in (-2, 0)$) kills the harmonic functions *but leaves only the constants left to deal with.* ♣

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[18] Note that the standard Holder space $C_{g_{EH,s}}^{k,\alpha}\left(T^*\mathbb{C}P^1\right)$ has the covariant derivatives, parallel transport, higher order tensor norms with respect to $g_{EH,s}, \omega_{EH,s}$.

On $\left(\mathbb{T}^4, J_0, g_0, \omega_0\right)$ and $\left(\mathbb{T}^4 - \mathcal{S}, J_0, g_0, \omega_0\right)$, we let $\rho_0 : \mathbb{T}^4 \to [0,1]$ be smooth, monotone, and satisfy

$$\rho_0 = \begin{cases} 1 & \text{on } \left\{\frac{1}{5} \leq |\cdot|_{\mathcal{S}}^{g_0}\right\} \\ |\cdot|_{\mathcal{S}}^{g_0} & \text{on } \left\{|\cdot|_{\mathcal{S}}^{g_0} \leq \frac{1}{8}\right\} \end{cases}, \qquad |\nabla \rho_0| \leq \frac{35 C_1}{3} \text{ on } \left\{\frac{1}{8} \leq |\cdot|_{\mathcal{S}}^{g_0} \leq \frac{1}{5}\right\}$$

For $\mathbb{T}^4 - \mathcal{S}$, we of course just restrict the ρ_0 above to that subset, whence is equal to $|\cdot|_{\mathcal{S}}^{g_0}$ on $\left\{0 < |\cdot|_{\mathcal{S}}^{g_0} \leq \frac{1}{8}\right\}$ instead.

Define

$$\|\mathfrak{s}\|_{C^{k,\alpha}_{\delta,g_0}(\mathbb{T}^4)} := \sum_{j \leq k} \left\|\nabla^j_{g_0} \mathfrak{s}\right\|_{C^0_{\delta,g_0}(\mathbb{T}^4)} + \sup_{\substack{x \neq y \\ d_{g_0}(x,y) \leq \text{inj}_{g_0}}} \min\{\rho_0(x), \rho_0(y)\}^{-\delta + k + \alpha} \frac{\left|\nabla^k_{g_0}\mathfrak{s}\big|_x - \nabla^k_{g_0}\mathfrak{s}\big|_y\right|_{g_0}}{d_{g_0}(x,y)^\alpha}$$

$$:= \sum_{j \leq k} \left\|\rho_0^{-\delta+j} \nabla^j_{g_0} \mathfrak{s}\right\|_{C^0_{g_0}(\mathbb{T}^4)} + \sup_{\substack{x \neq y \\ d_{g_0}(x,y) \leq \text{inj}_{g_0}}} \min\{\rho_0(x), \rho_0(y)\}^{-\delta + k + \alpha} \frac{\left|\nabla^k_{g_0}\mathfrak{s}\big|_x - \nabla^k_{g_0}\mathfrak{s}\big|_y\right|_{g_0}}{d_{g_0}(x,y)^\alpha}$$

where we interpret the difference in the numerator of the (weighted) Holder seminorm via parallel transport along the unique minimal geodesic connecting x and y, and all covariant derivatives (and parallel transport from) the Levi-Civita connection of g_0, ω_0 and the higher order tensor (i.e. not functions) norms with respect to g_0, ω_0. We now define the space of weighted Holder (p,q)-tensors $C^{k,\alpha}_{\delta,g_0}\left(\mathbb{T}^4\right)$ as $C^{k,\alpha}_{\delta,g_0}\left(\mathbb{T}^4\right) := \left\{\mathfrak{s} \in C^{k,\alpha}_{g_0}\left(\mathbb{T}^4\right) : \|s\|_{C^{k,\alpha}_{\delta,g_0}(\mathbb{T}^4)} < \infty\right\}$ but with the $\|\cdot\|_{C^{k,\alpha}_{\delta,g_0}(\mathbb{T}^4)}$ norm instead.[19] Verbatim as before, tensors with finite $\|\cdot\|_{C^{k,\alpha}_{\delta,g_0}(\mathbb{T}^4)}$ norm decay roughly as $\nabla^j_{g_0}\mathfrak{s} = O_{g_0}\left(\rho_0^{\delta - j}\right), j \leq k$.

Remark 3.4.3. In contrast to $T^*\mathbb{C}P^1$, note that because the weight function ρ_0 has *bounded* sup norm (due *in part* because it is smooth and the underlying space is (a subset of a space which is) *compact*), we have that $C^{k,\alpha}_{\delta,g_0}\left(\mathbb{T}^4\right)$ (specialized to the case of functions/$(0,0)$-tensors) *contains the constant functions ! ! !* This is important because this produces a 1 dimensional subspace in the kernel of any differential operator on $\mathbb{T}^4$ with domain any $C^{k,\alpha}_{\delta,g_0}\left(\mathbb{T}^4\right)$, in particular the weighted Laplacian. Hence, upon choosing a suitable weight $\delta \in \mathbb{R}$ (again, specifically $\delta \in (2 - \dim_{\mathbb{R}} M, 0)$) to kill the harmonic functions, we **still have to deal with the constant functions** in order to completely kill the kernel ! ! !

Note also that this entire discussion & definitions above hold verbatim for $\left(\mathbb{T}^4 - \mathcal{S}, J_0, g_0, \omega_0\right)$ as well. ♣

Now we define the weighted Holder spaces for $(\text{Km}_\epsilon, J, g_\epsilon, \omega_\epsilon)$. For that, we need $\epsilon > 0$ to now satisfy the following

Data 3.4.4. Let our sufficiently small $\epsilon > 0$ be from before (specifically Propositions 3.2.7 and 3.3.3 and Data 3.3.1), but now even smaller to also ensure $2\epsilon^{\frac{1}{2}} < \frac{3}{25} < \frac{1}{8}$ and hence automatically (from a simple inequality bash) $\epsilon^2 < \epsilon^{\frac{3}{2}} < \epsilon < 2\epsilon < \epsilon^{\frac{1}{2}} < \frac{3}{50} < \frac{1}{16}$ (if it wasn't already small enough to also ensure this).

Let $G_\epsilon : [0, \infty) \to [\epsilon, 1]$ be a smooth monotone function satisfying

$$G_\epsilon(x) = \begin{cases} 1 & \frac{1}{5} \leq x \\ x & 2\epsilon \leq x \leq \frac{1}{8} \\ \epsilon & x \leq \epsilon \end{cases}, \qquad |\nabla G_\epsilon| \leq C_1 \text{ on } \{\epsilon \leq x \leq 2\epsilon\}, \qquad |\nabla G_\epsilon| \leq \frac{35 C_1}{3} \text{ on } \left\{\frac{1}{8} \leq x \leq \frac{1}{5}\right\}$$

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[19] Note that the standard Holder space $C^{k,\alpha}_{g_0}\left(\mathbb{T}^4\right)$ again has the covariant derivatives, parallel transport, higher order tensor norms with respect to g_0, ω_0.

Note that $\left\{2\epsilon \le x \le \frac{1}{8}\right\} \ne \emptyset$ because our $\epsilon > 0$ has been chosen up to this point to be small enough so that $\epsilon < \frac{1}{16}$ from Data 3.4.4! Now define $\rho_\epsilon := G_\epsilon \circ |\cdot|^{g_0}_{\pi^{-1}(S)}$. Hence we have that $\rho_\epsilon : \mathrm{Km}_\epsilon \to [\epsilon, 1]$ is smooth, monotone, and satisfies

$$\rho_\epsilon = \begin{cases} 1 & \text{on } \left\{\frac{1}{5} \le |\cdot|^{g_0}_{\pi^{-1}(S)}\right\} \\ |\cdot|^{g_0}_{\pi^{-1}(S)} & \text{on } \left\{2\epsilon \le |\cdot|^{g_0}_{\pi^{-1}(S)} \le \frac{1}{8}\right\} \\ \epsilon & \text{on } \left\{|\cdot|^{g_0}_{\pi^{-1}(S)} \le \epsilon\right\} \end{cases}, \quad |\nabla\rho_\epsilon|_{g_0} \le C_1 \text{ on } \left\{\epsilon \le |\cdot|^{g_0}_{\pi^{-1}(S)} \le 2\epsilon\right\}, \quad |\nabla\rho_\epsilon|_{g_0} \le \frac{35 C_1}{3} \text{ on } \left\{\frac{1}{8} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \le \frac{1}{5}\right\}$$

Define

$$\|\mathfrak{s}\|_{C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} := \sum_{j \le k} \left\|\nabla^j_{g_\epsilon}\mathfrak{s}\right\|_{C^0_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} + \sup_{\substack{x \ne y \\ d_{g_\epsilon}(x,y) \le \mathrm{inj}_{g_\epsilon}}} \min\{\rho_\epsilon(x), \rho_\epsilon(y)\}^{-\delta+k+\alpha} \frac{\left|\nabla^k_{g_\epsilon}\mathfrak{s}\big|_x - \nabla^k_{g_\epsilon}\mathfrak{s}\big|_y\right|_{g_\epsilon}}{d_{g_\epsilon}(x,y)^\alpha}$$

$$:= \sum_{j \le k} \left\|\rho_\epsilon^{-\delta+j}\nabla^j_{g_\epsilon}\mathfrak{s}\right\|_{C^0_{g_\epsilon}(\mathrm{Km}_\epsilon)} + \sup_{\substack{x \ne y \\ d_{g_\epsilon}(x,y) \le \mathrm{inj}_{g_\epsilon}}} \min\{\rho_\epsilon(x), \rho_\epsilon(y)\}^{-\delta+k+\alpha} \frac{\left|\nabla^k_{g_\epsilon}\mathfrak{s}\big|_x - \nabla^k_{g_\epsilon}\mathfrak{s}\big|_y\right|_{g_\epsilon}}{d_{g_\epsilon}(x,y)^\alpha}$$

where we interpret the difference in the numerator of the (weighted) Holder seminorm via parallel transport along the unique minimal geodesic connecting x and y, and all covariant derivatives (and parallel transport from) the Levi-Civita connection of $g_\epsilon, \omega_\epsilon$ and the higher order tensor (i.e. not functions) norms with respect to $g_\epsilon, \omega_\epsilon$. We now define the space of weighted Holder (p,q)-tensors $C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)$ as $C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon) := \left\{\mathfrak{s} \in C^{k,\alpha}_{g_\epsilon}(\mathrm{Km}_\epsilon) : \|s\|_{C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} < \infty\right\}$ but with the $\|\cdot\|_{C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)}$ norm instead.[20]

With this characterization, we have that a tensor $\mathfrak{s} \in C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)$ with $\|\mathfrak{s}\|_{C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} \le C$ decays roughly as $\nabla^j_{g_\epsilon}\mathfrak{s} = O_{g_\epsilon}\left(\rho_\epsilon^{\delta-j}\right)$, $j \le k$ around each exceptional divisor.

Moreover, we have the following proposition for (tensor) products:

Proposition 3.4.5. Let $\delta < l$ for some $l > 0$. Then $\exists C_2 > 0$, *independent of* $\epsilon > 0$, such that

$$\|f \otimes g\|_{C^{0,\alpha}_{\delta-l,g_\epsilon}(\mathrm{Km}_\epsilon)} \le C_2 \epsilon^{\delta-l} \|f\|_{C^{0,\alpha}_{\delta-l,g_\epsilon}(\mathrm{Km}_\epsilon)} \|g\|_{C^{0,\alpha}_{\delta-l,g_\epsilon}(\mathrm{Km}_\epsilon)}$$

Proof. First we have that

$$\|f \otimes g\|_{C^{0,\alpha}_{\delta-l,g_\epsilon}(\mathrm{Km}_\epsilon)} \le C_2 \left\|\rho_\epsilon^{\delta-l}\right\|_{C^0} \|f\|_{C^{0,\alpha}_{\delta-l,g_\epsilon}(\mathrm{Km}_\epsilon)} \|g\|_{C^{0,\alpha}_{\delta-l,g_\epsilon}(\mathrm{Km}_\epsilon)}$$

by definition of the weighted Holder norms on $(\mathrm{Km}_\epsilon, g_\epsilon)$, the fact that $\rho_\epsilon^{-\delta+l} = \rho_\epsilon^{\delta-l}\left(\rho_\epsilon^{-\delta+l}\right)^2$, the definition of the C^0-norm, and the definition of the induced metric on tensors. Whence, because by definition of the distance function ρ_ϵ, we have that $\rho_\epsilon \ge \epsilon$ and $\delta - l < 0$, whence the result follows immediately. ∎

Remark 3.4.6. Lastly, just as for $\mathbb{T}^4 - S$, we have that due to ρ_0 having *bounded* sup norm (due *in part* because it is smooth and Km_ϵ is *compact*), we have that $C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)$ (specialized to the case of functions/$(0,0)$-tensors) **contains the constant functions**, whence produces a 1 dimensional subspace in the kernel of any differential operator on Km_ϵ with domain any $C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)$, in particular the weighted Laplacian. Hence, upon choosing a suitable weight $\delta \in \mathbb{R}$ (again, specifically $\delta \in (2 - \dim_\mathbb{R} M, 0)$) to kill the harmonic functions, we **still have to deal with the constant functions** in order to completely kill the kernel ! ! ! ♣

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[20] Note that the standard Holder space $C^{k,\alpha}_{g_\epsilon}(\mathrm{Km}_\epsilon)$ again has the covariant derivatives, parallel transport, higher order tensor norms with respect to $g_\epsilon, \omega_\epsilon$.

3.5 § Nonlinear Setup §

Let's now bring everything together and find our $\widetilde{\omega_\epsilon} \in [\omega_\epsilon]$ Kähler on $(\mathrm{Km}_\epsilon, J, g_\epsilon, \omega_\epsilon)$ whose associated $\widetilde{g_\epsilon}$ is Ricci-flat.

Notation 3.5.1. As we will be working with a *fixed* $\epsilon > 0$, denote the sought after Ricci-flat Kähler form $\widetilde{\omega_\epsilon} \in [\omega_\epsilon]$ as ω, *keeping in mind that as this will end up being the unique Ricci-flat Kähler form in $[\omega_\epsilon]$, that **it depends on the gluing parameter** $\epsilon > 0$ as the starting Kähler class $[\omega_\epsilon]$ depends on $\epsilon > 0$.* ○

By Proposition 3.1.8 from Section 3.1, we now want to solve for $f \in C^\infty(\mathrm{Km}_\epsilon)$ a smooth function/$(0,0)$-tensor defining a Kähler form $\omega := \omega_\epsilon + i\partial\bar\partial f \in [\omega_\epsilon]$ in the following **complex Monge-Ampere equation with singular limit**[21]:

$$\frac{\omega^2}{\omega_\epsilon^2} = \frac{\left(\omega_\epsilon + i\partial\bar\partial f\right)^2}{\omega_\epsilon^2} = e^{\phi_\epsilon}$$

where ϕ_ϵ is the Ricci potential $\mathrm{Ric}_{\omega_\epsilon} = i\partial\bar\partial\phi_\epsilon$ satisfying the volume normalization $\int_{\mathrm{Km}_\epsilon} e^{\phi_\epsilon}\omega_\epsilon^2 = \int_{\mathrm{Km}_\epsilon} \omega_\epsilon^2$. Because we have our nowhere vanishing holomorphic $(2,0)$-form Ω on $(\mathrm{Km}_\epsilon, J)$ (which determines J), by Proposition 3.1.10 this is equivalent to solving

$$\frac{\omega^2}{\Omega \wedge \overline{\Omega}} = \frac{\left(\omega_\epsilon + i\partial\bar\partial f\right)^2}{\Omega \wedge \overline{\Omega}} = C$$

for some constant $C > 0$. In fact, we therefore have that $\phi_\epsilon = \log\left(C\frac{\Omega\wedge\overline{\Omega}}{\omega_\epsilon^2}\right)$, and since $C > 0$ is a positive real constant, WLOG we may set $C = 1$ via multiplying the nowhere vanishing holomorphic $(2,0)$-form by $\frac{1}{C^{\frac{1}{2}}}$, i.e. replacing $\Omega \mapsto \frac{1}{C^{\frac{1}{2}}}\Omega$ (i.e. changing the trivialization of K_{Km_ϵ}), and thus $\phi_\epsilon = \log\left(\frac{\Omega\wedge\overline{\Omega}}{\omega_\epsilon^2}\right)$.

Now let Δ_g be the standard Hodge Laplacian $dd^* + d^*d$ acting on 0-forms (hence d^*d) with Hodge star coming from the Riemannian metric g which itself is from a given Kähler form and integrable complex structure ω, J via 2-out-of-3. Note that this has **nonnegative** spectrum. Now since $\Delta_g f = -2n\frac{i\partial\bar\partial f\wedge\omega^{n-1}}{\omega^n}$ on any $\dim_\mathbb{C} = n$ dimensional Kähler manifold $\left(M^{2n}, J, g, \omega\right)$, we now setup our nonlinear problem as follows:

Specializing to functions/$(0,0)$-tensors, define $C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 := \left\{f \in C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon) : \int_{\mathrm{Km}_\epsilon} f\omega_\epsilon^2 = 0\right\} \subset C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)$. Because ρ_ϵ has *bounded* sup norm, we have that $C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 \subset C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)$ is a *closed* Banach subspace, inheriting the same Banach norm $\|\cdot\|_{C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)}$.

Our nonlinear problem is therefore:

$$\mathcal{F}_\epsilon : C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)^0$$

$$f \mapsto \frac{\left(\omega_\epsilon + i\partial\bar\partial f\right)^2}{\omega_\epsilon^2} - e^{\phi_\epsilon}$$

where we solve

$$\mathcal{F}_\epsilon(f) := \frac{\left(\omega_\epsilon + i\partial\bar\partial f\right)^2}{\omega_\epsilon^2} - e^{\phi_\epsilon}$$

$$= \underbrace{\left(1 - e^{\phi_\epsilon}\right)}_{=\mathcal{F}_\epsilon(0)} \underbrace{-\frac{1}{2}\Delta_{g_\epsilon} f}_{=D_0\mathcal{F}_\epsilon(f)} + \underbrace{\frac{i\partial\bar\partial f \wedge i\partial\bar\partial f}{\omega_\epsilon^2}}_{\text{nonlinearity}}$$

$$= 0$$

---

[21] The singular limit comes from when $\epsilon \searrow 0$, which produces the flat Kähler metric on the singular orbifold $\mathbb{R}^4/\mathbb{Z}_2$, see *Remark* 3.3.5.

Remark 3.5.2. Note that since we are on a *compact* manifold, for the domain and codomain weighted Holder spaces $C^{2,\alpha}_{\delta,g_\epsilon}(\text{Km}_\epsilon), C^{0,\alpha}_{\delta-2,g_\epsilon}(\text{Km}_\epsilon)$ in the nonlinear problem above, by the Fredholm alternative for the Laplacian, **we have to restrict to the closed subspace of functions with integral zero**, i.e. $\int_{\text{Km}_\epsilon} f\omega^2_\epsilon = 0$, in order to guarantee the solvability of Laplace's equation $\Delta_{g_\epsilon} f = 0$ (or else, as remarked before, we have a nontrivial 1 dimensional subspace of the kernel consisting of the *constant* functions, hence guaranteeing a nontrivial kernel). ♣

Remark 3.5.3. Likewise, notice also that by restricting to functions with integral zero, we **automatically get uniqueness** of the resulting Kähler form $\omega \in [\omega_\epsilon]$ yielding a Ricci-flat Kähler metric (as explained by Propositions 3.1.8 and 3.1.10)! ♣

Our method of attack to prove the existence of f solving $\mathcal{F}_\epsilon(f) = 0$ above is via the **implicit function theorem**, itself derived from the **Banach fixed point theorem**. For this to work, we need to invert the linearization $D_0\mathcal{F}_\epsilon$ *uniformly in* $\epsilon > 0$, have a Lipschitz bound on the nonlinearity, and have some good bounds on $\mathcal{F}_\epsilon(0)$. We tackle the linearization in the next sections, and we start the next section with a preliminary step for the smallness of $\mathcal{F}_\epsilon(0)$.

Proposition 3.5.4. Recall the Ricci potential $\phi_\epsilon = \log\left(\frac{\Omega \wedge \overline{\Omega}}{\omega^2_\epsilon}\right)$ which also satisfies the volume normalization $\int_{\text{Km}_\epsilon} e^{\phi_\epsilon}\omega^2_\epsilon = \int_{\text{Km}_\epsilon} \omega^2_\epsilon$. Then we have that **for $\epsilon > 0$ from Data** 3.4.4 **sufficiently small as in Proposition** 3.3.3 and $c_1(k)$ from Propositions 3.2.7 and 3.3.3:

- $\text{supp } \phi_\epsilon = \left\{\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon^{\frac{1}{2}}\right\}$

- $\left|\nabla^k_{g_0}\phi_\epsilon\right|_{g_0} \leq c_1(k+2)\epsilon^{2-\frac{k}{2}}$ on the annulus $\left\{\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon^{\frac{1}{2}}\right\}$

Proof. The support follows immediately because ω_ϵ is the (scaled) Eguchi-Hanson on $\left\{|\cdot|^{g_0}_{\pi^{-1}(S)} \leq \epsilon^{\frac{1}{2}}\right\}$ and is flat on $\left\{2\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_0}_{\pi^{-1}(S)}\right\}$, hence the Ricci potential ϕ_ϵ of ω_ϵ must vanish on those regions, and the support is the *closure* of the complement of the vanishing set.

Secondly, since ω_0 is defined on the complement of the 16 exceptional divisors, it clearly is Ricci-flat and satisfies $\omega_0^2 = \Omega \wedge \overline{\Omega}$.

Hence

$$\left|\phi_\epsilon\right| = \left|\log\left(\frac{\Omega \wedge \overline{\Omega}}{\omega^2_\epsilon}\right)\right|$$

$$= \left|\log\left(\frac{\omega_0^2}{\omega^2_\epsilon}\right)\right| = \left|2\log\left(\frac{1}{1 + \frac{\mathfrak{D}}{\omega_0}}\right)\right|$$

$$= 2\left|\log\left(1 + \frac{\mathfrak{D}}{\omega_0}\right)\right|$$

$$\leq c_1(2)\epsilon^2$$

since $\mathfrak{D} := \omega_\epsilon - \omega_0$ and on the annulus $\left\{\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon^{\frac{1}{2}}\right\}$ we have that $|\mathfrak{D}| \leq c_1(2)\epsilon^2$ when $\epsilon > 0$ is sufficiently small (Proposition 3.2.7), and the Taylor expansion of $\log(1+x)$ finishes it off. The estimate for higher derivatives follows immediately from $\left|\nabla^k_{g_0}\mathfrak{D}\right|_{g_0} \leq c_1(k+2)\epsilon^{2-\frac{k}{2}}$ and the exact same argument. ∎

Remark 3.5.5. Therefore since the Ricci potential ϕ_ϵ satisfies $\text{Ric}_{\omega_\epsilon} = i\partial\bar{\partial}\phi_\epsilon$, we have that

$$\left|\nabla^k_{g_0}\text{Ric}_{\omega_\epsilon}\right|_{g_0} \leq c_1(k+4)\epsilon^{1-\frac{k}{2}} \qquad \text{and} \qquad \left|\nabla^k_{g_0}\text{Ric}_{g_\epsilon}\right|_{g_0} \leq c_3(k)\epsilon^{1-\frac{k}{2}}$$

holds on the annulus $\left\{\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon^{\frac{1}{2}}\right\}$ for some positive constants $c_3(k) > 0, k \in \mathbb{N}_0$, and $\text{Ric}_{\omega_\epsilon}, \text{Ric}_{g_\epsilon}$ vanishes outside $\left\{\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon^{\frac{1}{2}}\right\}$.

In fact, using local ON frames WRT g_0 we see that

$$\left|\text{Ric}_{g_\epsilon}\right|_{g_\epsilon} \leq c_3(0)\epsilon$$

holds on the annulus $\left\{\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon^{\frac{1}{2}}\right\}$, whence we easily get that

$$-3\Lambda g_\epsilon \leq \text{Ric}_{g_\epsilon} \leq 3\Lambda g_\epsilon$$

holds *everywhere* on Km_ϵ for $\Lambda := \frac{c_3(0)\epsilon}{3}$, aka $(\text{Km}_\epsilon, J, g_\epsilon, \omega_\epsilon)$ has **uniformly bounded Ricci curvature**.   ♣

3.6 § Blow-up analysis §

We now want to invert, for suitable values of $\delta \in \mathbb{R}$ **and for small enough** $\epsilon > 0$ **as from Data** 3.4.4, the linearization $D_0\mathcal{F}_\epsilon : C^{2,\alpha}_{\delta,g_\epsilon}(\text{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\text{Km}_\epsilon)^0$ of $\mathcal{F}_\epsilon : C^{2,\alpha}_{\delta,g_\epsilon}(\text{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\text{Km}_\epsilon)^0$ at the origin.

Since $D_0\mathcal{F}_\epsilon = -\frac{1}{2}\Delta_{g_\epsilon}f$, it suffices to invert $\Delta_{g_\epsilon} : C^{2,\alpha}_{\delta,g_\epsilon}(\text{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\text{Km}_\epsilon)^0$, which is a bounded linear operator. The key is to have the operator norm of the inverse be bounded **uniformly in** $\epsilon > 0$.

Thus we want to prove that for $\delta \in (-2, 0)$ **and** $\epsilon > 0$ **sufficiently small**, $\Delta_{g_\epsilon} : C^{2,\alpha}_{\delta,g_\epsilon}(\text{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\text{Km}_\epsilon)^0$ is a bounded linear isomorphism whose operator norm of the inverse is bounded by a constant *independent of the parameter* $\epsilon > 0$.

First, we record the result of a computation that will be extensively used in the sequel. It follows from the induced metric on (p, q)-tensors, the equation for the Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2}g^{pk}\left(\frac{\partial g_{pj}}{\partial x^i} + \frac{\partial g_{pi}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p}\right)$, the local coordinate expression $\Delta_g f = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^a}\left(\sqrt{\det g} \cdot g^{ab}\frac{\partial f}{\partial x^b}\right)$, and the fact that $\lambda \, \text{dist}_g(p, q) = \text{dist}_{\lambda^2 g}(p, q)$ for any $\lambda > 0$:

Proposition 3.6.1. Let (M^n, g) be an arbitrary Riemannian manifold. Let $C > 0$ be a constant. Let ρ be a smooth nonnegative function, let $\delta \in \mathbb{R}$, let $\mathfrak{s}$ be a (p, q)-tensor, let $|\cdot|_g$ denote all higher order tensor norms WRT the Riemannian metric g, and recall $\left\|\nabla^j_g\mathfrak{s}\right\|_{C^0_{\delta,g}} := \left\|\rho^{-\delta+j}\nabla^j_g\mathfrak{s}\right\|_{C^0_g}$.

(1) Under a conformal rescaling $g \mapsto C^2 g$, we have that

$$\nabla_{C^2 g} = \nabla_g$$

(2) Under a conformal rescaling $g \mapsto C^2 g$, we have that

$$\left|\nabla^j\mathfrak{s}\right|_{C^2 g} = \left(\frac{1}{C}\right)^{j+q-p}\left|\nabla^j\mathfrak{s}\right|_g$$

(3) Under a conformal rescaling $g \mapsto C^2 g$, we have that

$$\Delta_{C^2 g} = \frac{1}{C^2}\Delta_g$$

(4) Denote by $[\mathfrak{s}]_{C^{k,\alpha}_{\delta,g}} := \sup_{\substack{x \neq y \\ d_g(x,y) \leq \text{inj}_g}} \min\{\rho(x), \rho(y)\}^{-\delta+k+\alpha}\frac{\left|\left.\nabla^k_g\mathfrak{s}\right|_x - \left.\nabla^k_g\mathfrak{s}\right|_y\right|_g}{d_g(x,y)^\alpha}$ the weighted Holder seminorm on (p, q)-tensors (with all higher order tensor norms for $k \geq 1$ defined with respect to g). Immediately from (1) and (2), under a conformal rescaling $g \mapsto C^2 g$:

$$[\mathfrak{s}]_{C^{k,\alpha}_{\delta,C^2 g}} = \left(\frac{1}{C}\right)^{\delta+q-p}[\mathfrak{s}]_{C^{k,\alpha}_{\delta,g}}$$

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with the weight of the RHS $C^{k,\alpha}_{\delta,g}$ being $\frac{\rho}{C}$, i.e.

$$\sup_{\substack{x \neq y \\ d_{C^2g}(x,y) \leq \mathrm{inj}_{C^2g}}} \min\{\rho(x), \rho(y)\}^{-\delta+k+\alpha} \frac{\left|\left|\nabla^k_g \mathfrak{s}\right|_x - \nabla^k_g \mathfrak{s}\right|_y\right|_{C^2g}}{d_{C^2g}(x,y)^\alpha}$$

$$= \left(\frac{1}{C}\right)^{\delta+q-p} \sup_{\substack{x \neq y \\ d_g(x,y) \leq \mathrm{inj}_g}} \min\left\{\frac{\rho}{C}\Big|_x, \frac{\rho}{C}\Big|_y\right\}^{-\delta+k+\alpha} \frac{\left|\left|\nabla^k_g \mathfrak{s}\right|_x - \nabla^k_g \mathfrak{s}\right|_y\right|_g}{d_g(x,y)^\alpha}$$

(5) Under a conformal rescaling $g \mapsto C^2g$, we have from (4) that

$$\|\mathfrak{s}\|_{C^{k,\alpha}_{\delta,C^2g}} = \left(\frac{1}{C}\right)^{\delta+q-p} \|\mathfrak{s}\|_{C^{k,\alpha}_{\delta,g}}$$

where again the weight of the RHS is $\frac{\rho}{C}$.

(6) Under a conformal rescaling $g \mapsto C^2g$, we have from (1) and (2) that for the *unweighted/standard* Holder norm that

$$\|\mathfrak{s}\|_{C^{k,\alpha}_g} \xrightarrow{\text{conformal rescaling } g \mapsto C^2g} \|\mathfrak{s}\|_{C^{k,\alpha}_{C^2g}} = \sum_{j \leq k} \left(\frac{1}{C}\right)^{j+q-p} \left\|\nabla^j_g \mathfrak{s}\right\|_{C^0_g}$$

$$+ \left(\frac{1}{C}\right)^{k+q-p+\alpha} \sup_{\substack{x \neq y \\ d_g(u_1,u_2) \leq \mathrm{inj}_g}} \frac{\left|\left|\nabla^k_g \mathfrak{s}\right|_x - \nabla^k_g \mathfrak{s}\right|_y\right|_g}{d_g(x,y)^\alpha}$$

(7) Transferring the standard Schauder estimate $\|f\|_{C^{2,\alpha}_g} \leq C\left(\|f\|_{C^0_g} + \|\Delta_g f\|_{C^{0,\alpha}_g}\right)$ for functions/$(0,0)$-tensors on metric balls $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right)$ of radii (r_1, r_2)[22] (clearly with $r_1 < r_2$) under the conformal rescaling $g \mapsto C^2g$, we end up getting via (3) and (6):

$$\|f\|_{C^{2,\alpha}_{C^2g}} \leq C\left(\|f\|_{C^0_{C^2g}} + \|\Delta_{C^2g} f\|_{C^{0,\alpha}_{C^2g}}\right)$$

which is equal to

$$\|f\|_{C^0_g} + \frac{1}{C}\|\nabla_g f\|_{C^0_g} + \left(\frac{1}{C}\right)^2 \left\|\nabla^2_g f\right\|_{C^0_g} + \left(\frac{1}{C}\right)^{2+\alpha} \left[\nabla^2_g f\right]_{C^{0,\alpha}_g}$$

$$\leq C\left(\|f\|_{C^0_g} + \left(\frac{1}{C}\right)^2 \|\Delta_g f\|_{C^0_g} + \left(\frac{1}{C}\right)^{2+\alpha} \left[\Delta_g f\right]_{C^{0,\alpha}_g}\right)$$

on the *same* metric balls $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right) = \left(B^{C^2g}_{Cr_1}(p), B^{C^2g}_{Cr_2}(p)\right)$, but with different radii WRT the *new* metric C^2g (recall $\lambda \, \mathrm{dist}_g(p, q) = \mathrm{dist}_{\lambda^2g}(p, q)$ for any $\lambda > 0$ and points p, q!).

Conversely, if we already had the standard Schauder estimate for C^2g, namely $\|f\|_{C^{2,\alpha}_{C^2g}} \leq C\left(\|f\|_{C^0_{C^2g}} + \|\Delta_{C^2g} f\|_{C^{0,\alpha}_{C^2g}}\right)$ on $\left(B^{C^2g}_{Cr_1}(p), B^{C^2g}_{Cr_2}(p)\right)$, then unraveling it would yield the same estimate above on $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right) = \left(B^{C^2g}_{Cr_1}(p), B^{C^2g}_{Cr_2}(p)\right)$.

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[22] That is, the LHS of the inequality is on $B^g_{r_1}(p)$, and the RHS of the inequality is on $B^g_{r_2}(p)$, as these are *interior* Schauder estimates.

Next, we record some useful *uniform* estimates on our weight function $\rho_\epsilon := G_\epsilon \circ |\cdot|^{g_0}_{\pi^{-1}(S)} : \mathrm{Km}_\epsilon \to [\epsilon, 1]$ as well as on the weight function $\widetilde{\rho}_0$ used on $T^*\mathbb{C}P^1$, all of which follow immediately by construction:

Proposition 3.6.2. The following hold **uniformly in** $\epsilon > 0$, *i.e.* $\forall \epsilon > 0$:

(1) Restricted to $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le 2\epsilon \right\}$, we have the *pointwise everywhere/uniform* estimate

$$\epsilon \le \rho_\epsilon \le 2\epsilon$$

(2) Restricted to $\left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$, we have that ρ_ϵ *pointwise everywhere/uniformly* satisfies

$$\frac{3}{25} \le \rho_\epsilon \le 1$$

Proposition 3.6.3. Recall $\widetilde{\rho}_0 : T^*\mathbb{C}P^1 \to [1, \infty)$ the smooth monotone function satisfying $\widetilde{\rho}_0 = \begin{cases} |\cdot|^{g_0}_{\mathbb{C}P^1} & \text{on } \left\{ 2 \le |\cdot|^{g_0}_{\mathbb{C}P^1} \right\} \\ 1 & \text{on } \left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \le 1 \right\} \end{cases}$.

Let $R_0 > 1$ be some fixed constant, and let $p \in \left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < R_0 \right\}$ be an *arbitrary* point.

Then $\exists \delta_{2,p} > 0$ depending on p such that, restricted to $\left\{ |p|^{g_0}_{\mathbb{C}P^1} - \delta_{2,p} \le |\cdot|^{g_0}_{\mathbb{C}P^1} \le |p|^{g_0}_{\mathbb{C}P^1} + \delta_{2,p} \right\}$, we have the *pointwise everywhere/uniform* estimate

$$\widetilde{\rho}_0(p) - \frac{1}{1000}\widetilde{\rho}_0(p) \le \widetilde{\rho}_0 \le \widetilde{\rho}_0(p) + \frac{1}{1000}\widetilde{\rho}_0(p)$$

That is, we have that

$$\frac{999}{1000} \le \frac{\widetilde{\rho}_0(q)}{\widetilde{\rho}_0(p)} \le \frac{1001}{1000}, \qquad \forall q \in \left\{ |p|^{g_0}_{\mathbb{C}P^1} - \delta_{2,p} \le |\cdot|^{g_0}_{\mathbb{C}P^1} \le |p|^{g_0}_{\mathbb{C}P^1} + \delta_{2,p} \right\}$$

Now we arrive at the main isomorphism result in 3 parts: first a weighted Schauder estimate (which is first done locally), then an improved weighted Schauder estimate (which is the heart of the proof), and then the isomorphism result itself.

Proposition 3.6.4 (Local Weighted Schauder Estimate). **Let** $\epsilon > 0$ **be from Data** 3.4.4. Then $\forall f \in C^{2,\alpha}_{\delta, g_\epsilon}(\mathrm{Km}_\epsilon)$ and $\forall p \in \mathrm{Km}_\epsilon$, there exists an $r = r_p > 0$ (hence may depend on p) such that:

$$\|f\|_{C^{2,\alpha}_{\delta, g_\epsilon}\left(B^{g_\epsilon}_{r_p}(p) \right)} \le C_1 \left(\|f\|_{C^0_{\delta, g_\epsilon}\left(B^{g_\epsilon}_{2r_p}(p) \right)} + \left\| \Delta_{g_\epsilon} f \right\|_{C^{0,\alpha}_{\delta-2, g_\epsilon}\left(B^{g_\epsilon}_{2r_p}(p) \right)} \right)$$

$$:= C_1 \left(\left\| \rho_\epsilon^{-\delta} f \right\|_{C^0_{g_\epsilon}\left(B^{g_\epsilon}_{2r_p}(p) \right)} + \left\| \Delta_{g_\epsilon} f \right\|_{C^{0,\alpha}_{\delta-2, g_\epsilon}\left(B^{g_\epsilon}_{2r_p}(p) \right)} \right)$$

for $C_1 > 0$ **independent of** ϵ and $p \in \mathrm{Km}_\epsilon$.

Proof. The idea here is that, *depending on which region our point p is in*, we do a specific conformal rescaling on the metric (as well as rescale the function $f \mapsto kf$ by a positive scalar), use the fact that each (rescaled) region has **bounded geometry** to get *standard* Schauder estimates with *uniform* constants, then finish off with our precise uniform estimates on our weight functions (Proposition 3.6.2 and Proposition 3.6.3) via undoing the conformal rescaling & examining how the resulting standard Schauder estimates transform via Proposition 3.6.1. Since we do this on *finitely* many regions, we take the max of all the uniform constants $C > 0$ involved to get a global constant C_1.

Thus let $p \in \mathrm{Km}_\epsilon$.

We consider 3 cases: $p \in \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 2\epsilon \right\}$, $p \in \left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\}$, and $p \in \left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$. Clearly these 3 regions cover all of Km_ϵ.

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(1) **Case 1;** $p \in \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 2\epsilon \right\}$: WLOG let us focus on only 1 of the 16 connected components of $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 2\epsilon \right\}$, hence suppose that $\pi^{-1}(S) = E$ consists of only a single exceptional $E \cong \mathbb{C}P^1$. By construction (Proposition 3.3.3 as well as Data 3.4.4 telling us that $2\epsilon < \epsilon^{\frac{1}{2}}$), on this region $g_\epsilon = g_{EH,\epsilon^2}$. Recalling that $R_\epsilon^*\left(\frac{1}{\epsilon^2} g_{EH,\epsilon^2} \right) = g_{EH,1}$ from Proposition 3.2.2 as well as the identification $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le 3\epsilon^{\frac{1}{2}} \right\} \sim \left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \le 3\epsilon^{\frac{1}{2}} \right\}$ from the underlying construction of Km_ϵ, we have that upon doing a conformal rescaling $g_\epsilon \mapsto \frac{1}{\epsilon^2} g_\epsilon$ that $\left(\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 2\epsilon \right\}, \frac{1}{\epsilon^2} g_\epsilon \right)$ is isometric under R_ϵ to $\left(\left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} < 2 \right\}, g_{EH,1} \right) \subset \left(T^*\mathbb{C}P^1, g_{EH,1} \right)$, which is a precompact region of the Eguchi-Hanson space $\left(T^*\mathbb{C}P^1, g_{EH,1} \right)$. Since $\left(T^*\mathbb{C}P^1, g_{EH,1} \right)$ has **bounded geometry**[23], we thus have, for $r_2 > r_1 > 0$ small enough so that $B_{r_2}^{g_{EH,1}}(p) \subset \left(\left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} < 2 \right\}, g_{EH,1} \right)$ (hence both r_1, r_2 may depend on p) where we abuse notation and denote the point where p lands in $\left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} < 2 \right\}$ as p, the standard Schauder estimate $\|kf\|_{C^{2,\alpha}_{g_{EH,1}}} \le C_0 \left(\|kf\|_{C^0_{g_{EH,1}}} + \|k\Delta_{g_{EH,1}} f\|_{C^{0,\alpha}_{g_{EH,1}}} \right)$ on the metric balls $\left(B_{r_1}^{g_{EH,1}}(p), B_{r_2}^{g_{EH,1}}(p) \right)$ (where we have multiplied our function by a positive scalar $k > 0$, to be determined), *where $C_0 > 0$ is independent of p.*

Transferring everything back to $\left(\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 2\epsilon \right\}, \frac{1}{\epsilon^2} g_\epsilon \right)$ via applying R_ϵ to $\left(\left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} < 2 \right\}, g_{EH,1} \right)$ and using $\frac{1}{\epsilon^2} g_{EH,\epsilon^2} = R_{\frac{1}{\epsilon}}^*\left(g_{EH,1} \right)$ and the identification $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le 3\epsilon^{\frac{1}{2}} \right\} \sim \left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \le 3\epsilon^{\frac{1}{2}} \right\}$, we have that our standard Schauder estimate is now $\|kf\|_{C^{2,\alpha}_{\frac{1}{\epsilon^2} g_\epsilon}} \le C_0 \left(\|kf\|_{C^0_{\frac{1}{\epsilon^2} g_\epsilon}} + \left\| k\Delta_{\frac{1}{\epsilon^2} g_\epsilon} f \right\|_{C^{0,\alpha}_{\frac{1}{\epsilon^2} g_\epsilon}} \right)$ on the metric balls $\left(B_{R_1}^{\frac{1}{\epsilon^2} g_\epsilon}(p), B_{R_2}^{\frac{1}{\epsilon^2} g_\epsilon}(p) \right)$ for some $0 < R_1 < R_2$ (which may depend on p) such that $B_{R_2}^{\frac{1}{\epsilon^2} g_\epsilon}(p) \subset \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 2\epsilon \right\}$ and $B_{R_2}^{\frac{1}{\epsilon^2} g_\epsilon}(p) \subset R_\epsilon\left(B_{r_2}^{g_{EH,1}}(p) \right)$ (and vice versa for R_1). Now apply Proposition 3.6.1(7) with $C = \frac{1}{\epsilon}$ and $g = g_\epsilon$, set $k = \epsilon^{-\delta}$, and upon noting that on $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 2\epsilon \right\}$ we have the *uniform* estimate $\epsilon \le \rho_\epsilon \le 2\epsilon$ (Proposition 3.6.2(1)), we get upon recalling the definition of the weighted Holder norms our sought after *weighted* Schauder estimate $\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}} \le \widetilde{C_0} \left(\|f\|_{C^0_{\delta,g_\epsilon}} + \|\Delta_{g_\epsilon} f\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}} \right)$ on the metric balls $\left(B_{R_1}^{\frac{1}{\epsilon^2} g_\epsilon}(p), B_{R_2}^{\frac{1}{\epsilon^2} g_\epsilon}(p) \right) = \left(B_{\epsilon R_1}^{g_\epsilon}(p), B_{\epsilon R_2}^{g_\epsilon}(p) \right)$ upon setting $r_p := \epsilon R_1$ and $R_2 := 2R_1$. Note that $\widetilde{C_0} > 0$ is still independent of p as well as $\epsilon > 0$ since $\epsilon \le \rho_\epsilon \le 2\epsilon$ tells us that we may replace ρ_ϵ with ϵ, up to introducing harmless factors of 2 and $\frac{1}{2}$ into C_0.

(2) **Case 2;** $p \in \left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\}$: Again WLOG let us focus only on 1 of the 16 connected components of $\left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\}$. Hence g_ϵ is an interpolation of g_{EH,ϵ^2} and g_0 on $\left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\}$, namely $g_\epsilon = \begin{cases} \widetilde{g_{EH,\epsilon}} & \text{on } \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le 3\epsilon^{\frac{1}{2}} \right\} \\ g_0 & \text{on } \left\{ 3\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \end{cases}$ and where $\widetilde{g_{EH,\epsilon}} = \begin{cases} g_{EH,\epsilon^2} & \text{on } \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \le \epsilon^{\frac{1}{2}} \right\} \\ g_0 & \text{on } \left\{ 2\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \end{cases}$.

Now by *Remark* 3.3.2, we may identify $\left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\}$ with $\left\{ \epsilon < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8} \right\} \subset T^*\mathbb{C}P^1$, and upon recalling $\widetilde{g_{EH,\epsilon}} = \epsilon^2 R_{\frac{1}{\epsilon}}^*\left(\widetilde{g_{EH-0,\epsilon}} \right) \Leftrightarrow R_\epsilon^*\left(\frac{1}{\epsilon^2} \widetilde{g_{EH,\epsilon}} \right) = \widetilde{g_{EH-0,\epsilon}}$ and $g_0 = \epsilon^2 R_{\frac{1}{\epsilon}}^*(g_0) \Leftrightarrow R_\epsilon^*\left(\frac{1}{\epsilon^2} g_0 \right) = g_0$ from Proposition 3.2.7 directly implying by construction of $\widetilde{g_{EH-0,\epsilon}}$ from Proposition 3.2.6 (as well as by construction of g_ϵ) that $R_\epsilon^*\left(\frac{1}{\epsilon^2} g_\epsilon \right) = \widetilde{g_{EH-0,\epsilon}}$, we have that upon doing a conformal rescaling $g_\epsilon \mapsto \frac{1}{\epsilon^2} g_\epsilon$ that $\left(\left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\}, \frac{1}{\epsilon^2} g_\epsilon \right)$ is isometric under R_ϵ to $\left(\left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\}, \widetilde{g_{EH-0,\epsilon}} \right) \subset \left(T^*\mathbb{C}P^1, \widetilde{g_{EH-0,\epsilon}} \right)$, which

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[23] Indeed, the exceptional $\mathbb{C}P^1$ is no longer collapsing since we're in $g_{EH,1}$. Hence, roughly speaking, the conformal rescaling $g_\epsilon \mapsto \frac{1}{\epsilon^2} g_\epsilon$ "rescaled the shrinking exceptional divisor to have unit volume", whence dispensing with the only source of unbounded curvature (namely the shrinking exceptional divisor).

is a precompact region of the Riemannian manifold $(T^*\mathbb{C}P^1, \widehat{g_{EH-0,\epsilon}})$. Now by Proposition 3.2.6, since

$$\widehat{g_{EH-0,\epsilon}} = \begin{cases} g_{EH,1} & \text{on } \left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \leq \frac{1}{\epsilon^{\frac{1}{2}}} \right\} \\ g_0 & \text{on } \left\{ \frac{2}{\epsilon^{\frac{1}{2}}} \leq |\cdot|^{g_0}_{\mathbb{C}P^1} \right\} \end{cases}$$ aka $\widehat{g_{EH-0,\epsilon}}$ is an interpolation of $g_{EH,1}$ on a large compact region and

with the flat Euclidean metric g_0 on the complement of a (slightly) larger (pre)compact region, it is immediate that the Riemannian manifold $(T^*\mathbb{C}P^1, \widehat{g_{EH-0,\epsilon}})$ is complete with **bounded geometry**[24].

Next, note that upon transferring over the weight function ρ_ϵ via the above process, we end up getting $R_\epsilon^*(\rho_\epsilon)$,

and since this satisfies $R_\epsilon^*(\rho_\epsilon) = \begin{cases} \epsilon|\cdot|^{g_0}_{\mathbb{C}P^1} & \text{on } \left\{ 2 \leq |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\} \\ \epsilon & \text{on } \left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \leq 1 \right\} \end{cases}$, we immediately see that $R_\epsilon^*(\frac{1}{\epsilon}\rho_\epsilon) = \widetilde{\rho_0}$ on

$\left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\}$, the weight function used to defined the weighted Holder spaces on $T^*\mathbb{C}P^1$.

Now recall that our point p is now $p \in \left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\}$, where we abuse notation and denote the point where p lands in $\left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\}$ as p. Let $\delta_{2,p} > 0$ be the same constant as in Proposition 3.6.3 but shrunken so that we have that $\left\{ |p|^{g_0}_{\mathbb{C}P^1} - \delta_{2,p} \leq |\cdot|^{g_0}_{\mathbb{C}P^1} \leq |p|^{g_0}_{\mathbb{C}P^1} + \delta_{2,p} \right\} \subset \left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\}$ by openness. Now let $\sigma \in (0,1)$ be chosen such that, upon letting $R := \widetilde{\rho_0}(p)$ for notation's sake, we have that $B^{\widehat{g_{EH-0,\epsilon}}}_{2\sigma R}(p) \subset \left\{ |p|^{g_0}_{\mathbb{C}P^1} - \delta_{2,p} \leq |\cdot|^{g_0}_{\mathbb{C}P^1} \leq |p|^{g_0}_{\mathbb{C}P^1} + \delta_{2,p} \right\} \subset \left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\}$. Therefore, by Proposition 3.6.3, we have that $\frac{999}{1000} \leq \frac{\widetilde{\rho_0}(q)}{\widetilde{\rho_0}(p)} = \frac{\widetilde{\rho_0}(q)}{R} \leq \frac{1001}{1000}$ holds $\forall q \in B^{\widehat{g_{EH-0,\epsilon}}}_{2\sigma R}(p)$.

Now because $(T^*\mathbb{C}P^1, \widehat{g_{EH-0,\epsilon}})$ has bounded geometry and we're on a precompact region $\left(\left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\}, \widehat{g_{EH-0,\epsilon}} \right)$, we have the standard Schauder estimate $\|kf\|_{C^{2,\alpha}_{\widehat{g_{EH-0,\epsilon}}}} \leq \mathcal{K} \left(\|kf\|_{C^0_{\widehat{g_{EH-0,\epsilon}}}} + \|k\Delta_{\widehat{g_{EH-0,\epsilon}}}f\|_{C^{0,\alpha}_{\widehat{g_{EH-0,\epsilon}}}} \right)$ on the metric balls $\left(B^{\widehat{g_{EH-0,\epsilon}}}_{\sigma R}(p), B^{\widehat{g_{EH-0,\epsilon}}}_{2\sigma R}(p) \right)$ (where we have multiplied our function by a positive scalar $k > 0$, to be determined), *where $\mathcal{K} > 0$ is independent of p.*

Performing a conformal rescaling $\widehat{g_{EH-0,\epsilon}} \mapsto \frac{1}{R^2}\widehat{g_{EH-0,\epsilon}}$ and then applying Proposition 3.6.1(7) with $C = \frac{1}{R}$ and $g = \widehat{g_{EH-0,\epsilon}}$ gives us

$$\|kf\|_{C^0_{\widehat{g_{EH-0,\epsilon}}}} + R\|\nabla_{\widehat{g_{EH-0,\epsilon}}}kf\|_{C^0_{\widehat{g_{EH-0,\epsilon}}}} + R^2\|\nabla^2_{\widehat{g_{EH-0,\epsilon}}}kf\|_{C^0_{\widehat{g_{EH-0,\epsilon}}}} + R^{2+\alpha}\left[\nabla^2_{\widehat{g_{EH-0,\epsilon}}}kf \right]_{C^{0,\alpha}_{\widehat{g_{EH-0,\epsilon}}}}$$

$$\leq \mathcal{K}\left(\|kf\|_{C^0_{\widehat{g_{EH-0,\epsilon}}}} + R^2\|\Delta_{\widehat{g_{EH-0,\epsilon}}}kf\|_{C^0_{\widehat{g_{EH-0,\epsilon}}}} + R^{2+\alpha}\left[\Delta_{\widehat{g_{EH-0,\epsilon}}}kf \right]_{C^{0,\alpha}_{\widehat{g_{EH-0,\epsilon}}}} \right)$$

on the metric balls $\left(B^{\widehat{g_{EH-0,\epsilon}}}_{\sigma R}(p), B^{\widehat{g_{EH-0,\epsilon}}}_{2\sigma R}(p) \right) = \left(B^{\frac{1}{R^2}\widehat{g_{EH-0,\epsilon}}}_{\sigma}(p), B^{\frac{1}{R^2}\widehat{g_{EH-0,\epsilon}}}_{2\sigma}(p) \right)$.

Now recalling that $\frac{999}{1000} \leq \frac{\widetilde{\rho_0}(q)}{R} \leq \frac{1001}{1000}$ holds $\forall q \in B^{\widehat{g_{EH-0,\epsilon}}}_{2\sigma R}(p)$ and hence $\forall q \in B^{\widehat{g_{EH-0,\epsilon}}}_{\sigma R}(p)$ as well, we get upon recalling the definition of the weighted Holder norms the following weighted Schauder estimate $\|R^\delta kf\|_{C^{2,\alpha}_{\delta,\widehat{g_{EH-0,\epsilon}}}} \leq \widetilde{C_1}\left(\|R^\delta kf\|_{C^0_{\delta,\widehat{g_{EH-0,\epsilon}}}} + \|R^\delta k\Delta_{\widehat{g_{EH-0,\epsilon}}}f\|_{C^{0,\alpha}_{\delta-2,\widehat{g_{EH-0,\epsilon}}}} \right)$ on the metric balls $\left(B^{\widehat{g_{EH-0,\epsilon}}}_{\sigma R}(p), B^{\widehat{g_{EH-0,\epsilon}}}_{2\sigma R}(p) \right) = \left(B^{\frac{1}{R^2}\widehat{g_{EH-0,\epsilon}}}_{\sigma}(p), B^{\frac{1}{R^2}\widehat{g_{EH-0,\epsilon}}}_{2\sigma}(p) \right)$, where the weight function here is $\widetilde{\rho_0}$. Similarly as in the first case, $\widetilde{C_1} > 0$ is

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[24] Indeed, similar to the previous case for the Eguchi-Hanson space $(T^*\mathbb{C}P^1, g_{EH,1})$ having bounded geometry, because we're flat outside $\left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \leq \frac{2}{\epsilon^{\frac{1}{2}}} \right\}$ and Eguchi-Hanson $g_{EH,1}$ with parameter 1 on $\left\{ |\cdot|^{g_0}_{\mathbb{C}P^1} \leq \frac{1}{\epsilon^{\frac{1}{2}}} \right\}$ and interpolating in between, we may argue as before (namely the exceptional $\mathbb{C}P^1$ no longer shrinking) since there are no sources of unbounded curvature for either the flat Euclidean metric region (since its the Euclidean metric) or the interpolation annulus (since we're interpolating the Euclidean metric with $g_{EH,1}$ and $g_{EH,1}$ has bounded curvature everywhere).

still independent of all the parameters involved because we've only modified $\mathcal{K}$ by factors of $\frac{999}{1000}$ and $\frac{1001}{1000}$ since we've applied the uniform estimate $\frac{999}{1000} \leq \frac{\widetilde{\rho}_0(q)}{R} \leq \frac{1001}{1000}$ on those metric balls.

Now let us transfer everything back to $\left(\left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\}, \frac{1}{\epsilon^2} g_\epsilon \right)$ via applying R_ϵ to $\left(\left\{ 1 < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8\epsilon} \right\}, \widehat{\overline{g}_{EH-0,\epsilon}} \right)$ and using the identification $\left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\} \sim \left\{ \epsilon < |\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{1}{8} \right\} \subset T^*\mathbb{C}P^1$, as well as using the relation $\frac{1}{\epsilon^2} g_\epsilon = R^*_{\frac{1}{\epsilon}} \left(\widehat{\overline{g}_{EH-0,\epsilon}} \right)$ and the relation $\frac{1}{\epsilon} \rho_\epsilon = R^*_{\frac{1}{\epsilon}} \left(\widetilde{\rho}_0 \right)$. We get that our weighted Schauder estimate has now be-

come $\left\| R^\delta k f \right\|_{C^{2,\alpha}_{\delta, \frac{1}{\epsilon^2} g_\epsilon}} \leq \widetilde{C_1} \left(\left\| R^\delta k f \right\|_{C^0_{\delta, \frac{1}{\epsilon^2} g_\epsilon}} + \left\| R^\delta k \Delta_{\frac{1}{\epsilon^2} g_\epsilon} f \right\|_{C^{0,\alpha}_{\delta-2, \frac{1}{\epsilon^2} g_\epsilon}} \right)$ on the metric balls $\left(B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_1}(p), B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_2}(p) \right)$

for some $0 < R_1 < R_2$ (which may depend on p) such that $B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_2}(p) \subset \left\{ \epsilon < |\cdot|^{g_0}_{\pi^{-1}(S)} < \frac{1}{8} \right\}$ and $B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_2}(p) \subset R_\epsilon \left(B^{\widehat{g_{EH-0,\epsilon}}}_{2\sigma R}(p) \right)$ (and vice versa for R_1), where the weight function used here is $\frac{1}{\epsilon} \rho_\epsilon$. Now we undo the original conformal rescaling of $\frac{1}{\epsilon^2} g_\epsilon$ via performing a conformal rescaling $\frac{1}{\epsilon^2} g_\epsilon \mapsto \epsilon^2 \frac{1}{\epsilon^2} g_\epsilon = g_\epsilon$ on the metric balls $\left(B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_1}(p), B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_2}(p) \right)$ hence yielding $\left(B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_1}(p), B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_2}(p) \right) = \left(B^{g_\epsilon}_{\epsilon R_1}(p), B^{g_\epsilon}_{\epsilon R_2}(p) \right)$, applying Proposition 3.6.1(5) with $C = \epsilon$ and $g = \frac{1}{\epsilon^2} g_\epsilon$ on each of the 3 weighted Holder norm terms *where g denotes the metric used in the weighted Holder norm*, and then setting $k = \epsilon^\delta R^{-\delta}$, yielding the weighted Schauder estimate

$\|f\|_{C^{2,\alpha}_{\delta, g_\epsilon}} \leq \widetilde{C_1} \left(\|f\|_{C^0_{\delta, g_\epsilon}} + \left\| \Delta_{\frac{1}{\epsilon^2} g_\epsilon} f \right\|_{C^{0,\alpha}_{\delta-2, g_\epsilon}} \right)$ on the metric balls $\left(B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_1}(p), B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_2}(p) \right) = \left(B^{g_\epsilon}_{\epsilon R_1}(p), B^{g_\epsilon}_{\epsilon R_2}(p) \right)$.

Now Proposition 3.6.1(3) tells us that $\Delta_{\frac{1}{\epsilon^2} g_\epsilon} = \epsilon^2 \Delta_{g_\epsilon}$, and since Data 3.4.4 tells us that $\epsilon^2 < \frac{1}{16} < 1$, we finally get our sought after weighted Schauder estimate $\|f\|_{C^{2,\alpha}_{\delta, g_\epsilon}} \leq \widetilde{C_1} \left(\|f\|_{C^0_{\delta, g_\epsilon}} + \left\| \Delta_{g_\epsilon} f \right\|_{C^{0,\alpha}_{\delta-2, g_\epsilon}} \right)$ on the metric balls $\left(B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_1}(p), B^{\frac{1}{\epsilon^2} g_\epsilon}_{R_2}(p) \right) = \left(B^{g_\epsilon}_{\epsilon R_1}(p), B^{g_\epsilon}_{\epsilon R_2}(p) \right)$ upon setting $r_p := \epsilon R_1$ and $R_2 := 2R_1$.

(3) **Case 3; $p \in \left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$:** By Data 3.4.4, since $2\epsilon^{\frac{1}{2}} < \frac{3}{25}$, by construction of g_ϵ (Proposition 3.3.3) we have that on this region $g_\epsilon = g_0$ the flat metric. This, combined with Proposition 3.6.2(2) which tells us that our weight function satisfies on this region the *uniform* estimate $\frac{3}{25} \leq \rho_\epsilon \leq 1$, the proof of the weighted Schauder estimate is exactly as in the first case, but this time using the identification $\left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \sim \left\{ \frac{3}{25} < |\cdot|^{g_0}_{S} \right\} \subset \mathbb{T}^4 - S \subset \mathbb{T}^4$ from *Remark* 3.3.2 and without having to do any conformal rescaling.

In more detail, upon transferring everything over from $\left(\left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}, g_\epsilon \right)$ to (the obviously precompact region) $\left(\left\{ \frac{3}{25} < |\cdot|^{g_0}_{S} \right\}, g_0 \right) \subset (\mathbb{T}^4, g_0)$, and since flat tori clearly have bounded geometry, we thus automatically have the standard Schauder estimate $\|f\|_{C^{2,\alpha}_{g_0}} \leq C_{1.5} \left(\|f\|_{C^0_{g_0}} + \|\Delta_{g_0} f\|_{C^{0,\alpha}_{g_0}} \right)$ on the metric balls $\left(B^{g_0}_{R_1}(p), B^{g_0}_{R_2}(p) \right)$ where $0 < R_1 < R_2$ is chosen such that $B^{g_0}_{R_2}(p) \subset \left\{ \frac{3}{25} < |\cdot|^{g_0}_{S} \right\}$, and *where $C_{1.5} > 0$ is independent of p*. Undoing the identification $\left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \sim \left\{ \frac{3}{25} < |\cdot|^{g_0}_{S} \right\}$ gives us that same estimate over $\left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$ on the metric balls $\left(B^{g_0}_{R_1}(p), B^{g_0}_{R_2}(p) \right)$ (which are clearly preserved under the identification). Since we have our uniform estimate $\frac{3}{25} \leq \rho_\epsilon \leq 1$ on $\left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$, we have on any subset of $\left\{ \frac{3}{25} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$ that the weighted Holder norms WRT ρ_ϵ and the standard Holder norms are *equivalent norms*, whence (up to changing the constant $C_{1.5}$ by harmless factors) we immediately get our sought after weighted Schauder estimate $\|f\|_{C^{2,\alpha}_{\delta, g_\epsilon}} \leq \widetilde{C_{1.5}} \left(\|f\|_{C^0_{\delta, g_\epsilon}} + \|\Delta_{g_\epsilon} f\|_{C^{0,\alpha}_{\delta-2, g_\epsilon}} \right)$ on the metric balls $\left(B^{g_\epsilon}_{R_1}(p), B^{g_\epsilon}_{R_2}(p) \right)$ upon setting $r_p := R_1$ and $R_2 := 2R_1$, with $\widetilde{C_{1.5}} > 0$ still independent of all parameters involved.

Setting $C_1 := \max\left\{\widetilde{C_0}, \widetilde{C_1}, \widetilde{C_{1.5}}\right\}$ and noting that these 3 constants are both *independent of p* as well as *independent of $\epsilon > 0$* yields us our local weighted Schauder estimate $\forall p \in \mathrm{Km}_\epsilon$, as was to be shown. $\blacksquare$

Corollary 3.6.5 (Weighted Schauder Estimate). **Let $\epsilon > 0$ be from Data 3.4.4 and sufficiently small**. We have the following estimate $\forall f \in C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)$:

$$\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} \leq C_2 \left(\|f\|_{C^0_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} + \|\Delta_{g_\epsilon} f\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)} \right)$$

$$:= C_2 \left(\left\|\rho_\epsilon^{-\delta} f\right\|_{C^0_{g_\epsilon}(\mathrm{Km}_\epsilon)} + \|\Delta_{g_\epsilon} f\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)} \right)$$

for $C_2 > 0$ **independent of ϵ**. $\blacksquare$

Proof. Suppose for contradiction that the estimate fails. This implies by countable choice that $\forall i \in \mathbb{N}, \exists \epsilon_i \in \left(0, \frac{1}{i}\right)$ whence $\exists \epsilon_i \searrow 0$ and $\exists f_i \in C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})$ a sequence such that (after normalizing)

$$\|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} = 1$$

$$\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} + \|\Delta_{g_{\epsilon_i}} f_i\|_{C^{0,\alpha}_{\delta-2,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} < \frac{1}{i}$$

Now since $\|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} := \|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} + \|\nabla_{g_{\epsilon_i}} f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} + \|\nabla^2_{g_{\epsilon_i}} f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} + [f_i]_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} = 1$ and $\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} \leq \frac{1}{i}$, we have by a simple inequality bash that either $\frac{1-\frac{1}{i}}{3} \leq \|\nabla_{g_{\epsilon_i}} f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})}$ or $\frac{1-\frac{1}{i}}{3} \leq \|\nabla^2_{g_{\epsilon_i}} f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})}$ or $\frac{1-\frac{1}{i}}{3} \leq [f_i]_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})}$ holds $\forall i \in \mathbb{N}$, aka since $1 - \frac{1}{i}$ is strictly monotone increasing in $i \in \mathbb{N}$ we have either $\frac{1}{6} \leq \|\nabla_{g_{\epsilon_i}} f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})}$ or $\frac{1}{6} \leq \|\nabla^2_{g_{\epsilon_i}} f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})}$ or $\frac{1}{6} \leq [f_i]_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})}$ holds $\forall i \in \mathbb{N} - \{1\}$. Hence $\exists p_i \in \mathrm{Km}_{\epsilon_i}$ and an $r_{p_i} > 0$ such that either $\frac{1}{6} \leq \left|\rho_{\epsilon_i}(p_i)^{-\delta+1} \nabla_{g_{\epsilon_i}} f_i(p_i)\right|_{g_{\epsilon_i}}$ or $\frac{1}{6} \leq \left|\rho_{\epsilon_i}(p_i)^{-\delta+2} \nabla^2_{g_{\epsilon_i}} f_i(p_i)\right|_{g_{\epsilon_i}}$ or $\frac{1}{6} \leq [f_i]_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}\left(B^{g_\epsilon}_{r_{p_i}}(p_i)\right)}$ holds $\forall i \in \mathbb{N} - \{1\}$. But since $A \subseteq B \implies \|\cdot\|_{C^{k,\alpha}(A)} \leq \|\cdot\|_{C^{k,\alpha}(B)}$ (aka locality of Holder norms), we have by our local estimate Proposition 3.6.4 that $\|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}\left(B^{g_{\epsilon_i}}_{r_{p_i}}(p_i)\right)} \leq$

$C_1\left(\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}\left(B^{g_{\epsilon_i}}_{2r_i}(p_i)\right)} + \|\Delta_{g_{\epsilon_i}} f_i\|_{C^{0,\alpha}_{\delta-2,g_{\epsilon_i}}\left(B^{g_{\epsilon_i}}_{2r_{p_i}}(p_i)\right)}\right) \leq C_1\left(\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} + \|\Delta_{g_{\epsilon_i}} f_i\|_{C^{0,\alpha}_{\delta-2,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})}\right) \leq \frac{C_1}{i} \searrow 0$ as $i \nearrow \infty$. But then we would have either $\frac{1}{6} \leq \left|\rho_{\epsilon_i}(p_i)^{-\delta+1} \nabla_{g_{\epsilon_i}} f_i(p_i)\right|_{g_{\epsilon_i}} \leq \|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}\left(B^{g_{\epsilon_i}}_{r_{p_i}}(p_i)\right)} \searrow 0$ or $\frac{1}{6} \leq \left|\rho_{\epsilon_i}(p_i)^{-\delta+2} \nabla^2_{g_{\epsilon_i}} f_i(p_i)\right|_{g_{\epsilon_i}} \leq \|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}\left(B^{g_{\epsilon_i}}_{r_{p_i}}(p_i)\right)} \searrow 0$ or $\frac{1}{6} \leq [f_i]_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}\left(B^{g_\epsilon}_{r_{p_i}}(p_i)\right)} \leq \|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}\left(B^{g_{\epsilon_i}}_{r_{p_i}}(p_i)\right)} \searrow 0$ as $i \nearrow \infty$, aka a contradiction in all 3 cases. $\blacksquare$

We now want to remove the C^0 term on the RHS of our above weighted Schauder estimate. This allows us to get a **uniform, i.e. independent of $\epsilon > 0$, bound on the inverse** of Δ_{g_ϵ}:

Proposition 3.6.6 (Improved Weighted Schauder Estimate). **Let the rate/weight parameter satisfy $\delta \in (-2, 0)$. Let $\epsilon > 0$ be as from Data 3.4.4.**

Then for $\epsilon > 0$ ***sufficiently small*** there exists a constant $C_3 > 0$ **independent of $\epsilon > 0$** such that we have the following estimate for $\Delta_{g_\epsilon} : C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)^0$:

$$\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} \leq C_3 \|\Delta_{g_\epsilon} f\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)}, \qquad \forall f \in C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0$$

Note that $f \in C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0$ i.e. the estimate is restricted to the closed subspace of functions with integral zero, i.e. $\int_{\mathrm{Km}_\epsilon} f \omega_\epsilon^2 = 0$.

Remark 3.6.7. This inequality precisely means that when $\delta \in (-2, 0)$ and the gluing parameter $\epsilon > 0$ is sufficiently small, $\Delta_{g_\epsilon} : C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)^0$ is injective, *uniformly* bounded below, and has *closed range*. ♣

Proof. We prove this by contradiction via a "blow-up" analysis on each region $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 3\epsilon^{\frac{1}{2}} \right\}$ and $\left\{ 2\epsilon^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$ of the cover $\mathrm{Km}_\epsilon \subset \left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 3\epsilon^{\frac{1}{2}} \right\} \cup \left\{ 2\epsilon^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$.

By our weighted Schauder estimate proved above (Corollary 3.6.5), we may instead prove

$$\|f\|_{C^0_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} \leq C_4 \|\Delta_{g_\epsilon} f\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)}$$

with the same assumptions ($\delta \in (-2, 0)$, etc.), whence $C_3 := C_2(C_4 + 1)$. Now assume for contradiction that the statement for this latter inequality fails. Then this implies by countable choice that $\exists \epsilon_i \in \left(0, \frac{1}{i}\right)$ whence $\exists \epsilon_i \searrow 0$ and $\exists f_i \in C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})^0$ a sequence such that (after normalizing) we get a tuple of equations

$$\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} = 1$$

$$\|\Delta_{g_{\epsilon_i}} f_i\|_{C^{0,\alpha}_{\delta-2,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} \leq \frac{1}{i}$$

Note that our weighted Schauder estimate Corollary 3.6.5 immediately gives us

$$\|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} \leq 2C_2$$

Let us now examine the following two regions: $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \subset \mathrm{Km}_{\epsilon_i}$ and $\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}} \right\} \subset \mathrm{Km}_{\epsilon_i}$.

(1) **Case 1;** $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$: Firstly, from *Remark* 3.3.2 let us identify each region $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\} \subset \mathrm{Km}_{\epsilon_i}$ with $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\} \subset \mathbb{T}^4/\mathbb{Z}_2 - S$ (hence we're in a *fixed* ambient manifold). Similarly from that remark, let us denote the lift of $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\} \subset \mathbb{T}^4/\mathbb{Z}_2 - S$ to $\mathbb{T}^4 - S$ via the double cover as $\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}} \subset \mathbb{T}^4 - S$, and denote the resulting lift of the metric g_{ϵ_i} as $\widehat{g_{\epsilon_i}}$.

Next, we note that by construction (Proposition 3.3.3) that each $g_{\epsilon_i} = g_0$ on each $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$ hence on $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}$ and hence $\widehat{g_{\epsilon_i}} = g_0$ on $\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}}$. Now since both $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}$, $\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}}$ are in *fixed* ambient manifolds $\mathbb{T}^4/\mathbb{Z}_2 - S$, $\mathbb{T}^4 - S$ respectively, and since $\mathbb{Z}_2$ acts on $\mathbb{T}^4 - S$ and on each region $\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}} \subset \mathbb{T}^4 - S$ in the same way, combined with the fact that our metric is precisely g_0 we may easily construct **equivariant Gromov-Hausdorff ϵ-approximations** for arbitrarily small ϵ [25] and see that we have the following [26] **equivariant Gromov-Hausdorff convergence** [27] (see [33]) as $i \nearrow \infty$:

$$\left(\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}}, \widehat{g_{\epsilon_i}} \right) \xrightarrow{eqGH} \left(\mathbb{T}^4 - S, g_0 \right)$$

$$\downarrow_{2:1} \qquad\qquad\qquad\qquad \downarrow_{2:1}$$

$$\left(\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}, g_{\epsilon_i} \right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2 - S, g_0 \right)$$

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[25] In fact, just take $\epsilon = 3\epsilon_i^{\frac{1}{2}}$.

[26] Note that our spaces here are *non-compact*. Hence the (equivariant) Gromov-Hausdorff topology is induced by the (equivariant) Gromov-Hausdorff *pseudo*-metric. The only time compactness of the metric spaces is ever involved is to show that $d_{GH}(X, Y) = 0 \implies X \overset{\text{isometric}}{\cong} Y$.

[27] In fact, since our manifolds are all open subsets of a *fixed* ambient manifold, and our metric tensors are all the same and are all induced from the ambient metric (which is flat) via restriction, since the flat metric clearly is Einstein and since each manifold is non-collapsed we may upgrade this convergence to C^∞_{loc}-**convergence** via harmonic coordinates and elliptic regularity. Moreover, we may choose to thicken our region to instead be $\left\{ \frac{1}{2}\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}$ and still get the Gromov-Hausdorff convergence to $(\mathbb{T}^4/\mathbb{Z}_2 - S, g_0)$.

Abuse notation and denote by f_i to mean *both* the restriction of each $f_i \in C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})^0$ to $\overline{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)}\right\}}$

as well as the transferal of f_i to $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S\right\} \subset \mathbb{T}^4/\mathbb{Z}_2 - S$, and denote the lift of each f_i to $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S\right\}$ as $\widehat{f_i}$. Hence $\widehat{f_i}$ is even/$\mathbb{Z}_2$-invariant by Proposition 3.2.1.

Now since we have $\|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} \leq 2C_2$, we thus get $\|f_i\|_{C^{2,\alpha}_{\delta,g_{\epsilon_i}}\left(\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S\right\}\right)} \leq 2C_2$ and hence the same

bound for each $\widehat{f_i}$, namely

$$\left\|\widehat{f_i}\right\|_{C^{2,\alpha}_{\delta,\widehat{g_{\epsilon_i}}}\left(\overline{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S\right\}}\right)} \leq 2C_2$$

Therefore, by Arzela-Ascoli, upon relabeling the original sequence of f_i we have that

$$\widehat{f_i} \xrightarrow{C^{2,\alpha'}_{\delta,\widehat{g_{\epsilon_i}}},loc} \widehat{f_\infty}$$

$$\left\|\widehat{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,g_0}(\mathbb{T}^4-S)} \leq 2C_2$$

with $0 < \alpha' < \alpha < 1$, for some function $\widehat{f_\infty}$ on $\mathbb{T}^4 - S$. Here, the *loc* convergence means on compact

subsets, i.e. the convergence $\xrightarrow{C^{2,\alpha'}_{\delta,\widehat{g_{\epsilon_i}}},loc}$ denotes that (recall $\widehat{g_{\epsilon_i}} = g_0$) $\left\|\widehat{f_i} - \widehat{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,\widehat{g_{\epsilon_i}}}(K)} \searrow 0$ hence implies

$\left\|\widehat{f_i}\right\|_{C^{2,\alpha'}_{\delta,\widehat{g_{\epsilon_i}}}(K)} \to \left\|\widehat{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,g_0}(K)}$, holding for all compact subsets $K \Subset \overline{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S\right\}} \subset \mathbb{T}^4 - S$. The second bound

holds everywhere on $\mathbb{T}^4 - S$ (i.e. $\left\|\widehat{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,g_0}(\mathbb{T}^4-S)} \leq 2C_2$ as opposed to $\left\|\widehat{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,g_0,loc}(\mathbb{T}^4-S)} \leq 2C_2$) because

we may let our compact subsets have unit diameter (since $\|f\|_{C^{k,\alpha'}(K)} \leq \max\left\{1, (\operatorname{diam} K)^{\alpha-\alpha'}\right\} \|f\|_{C^{k,\alpha}(K)}$).

Moreover, by the equivariant Gromov-Hausdorff convergence, we have that $\widehat{f_\infty}$ is even/$\mathbb{Z}_2$-invariant and
hence descends down to a function f_∞ on $\mathbb{T}^4/\mathbb{Z}_2 - S$ by Proposition 3.2.1. Moreover, the weight function
used in $C^{2,\alpha'}_{\delta,g_0}(\mathbb{T}^4 - S)$ is precisely the limiting weight function which satisfies

$$\rho_0 = \begin{cases} 1 & \text{on } \left\{\frac{1}{5} \leq |\cdot|^{g_0}_S\right\} \\ |\cdot|^{g_0}_S & \text{on } \left\{0 < |\cdot|^{g_0}_S \leq \frac{1}{8}\right\} \end{cases}, \qquad |\nabla\rho_0| \leq \frac{35C_1}{3} \text{ on } \left\{\frac{1}{8} \leq |\cdot|^{g_0}_S \leq \frac{1}{5}\right\}$$

aka in the limit we get that $\lim_{i \nearrow \infty} \rho_{\epsilon_i} = \rho_0$ is precisely the weight function used to define $C^{k,\alpha}_{\delta,g_0}(\mathbb{T}^4 - S)$
back in Section 3.4.

Playing the same game with $\left\|\Delta_{g_{\epsilon_i}} f_i\right\|_{C^{0,\alpha}_{\delta-2,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} \leq \frac{1}{i}$, namely that this bound directly implies the bounds

$\left\|\Delta_{g_{\epsilon_i}} f_i\right\|_{C^{0,\alpha}_{\delta-2,g_{\epsilon_i}}\left(\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S\right\}\right)} \leq \frac{1}{i}$ and $\left\|\Delta_{\widehat{g_{\epsilon_i}}} \widehat{f_i}\right\|_{C^{0,\alpha}_{\delta-2,\widehat{g_{\epsilon_i}}}\left(\overline{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S\right\}}\right)} \leq \frac{1}{i}$, that we may take the limit as $i \nearrow \infty$

and get

$$\Delta_{g_0} \widehat{f_\infty} = 0$$

on $(\mathbb{T}^4 - S, g_0)$ (since $\Delta_{g_0} \widehat{f_\infty} = 0$ on all compact subsets of $\mathbb{T}^4 - S$, hence on all of $\mathbb{T}^4 - S$). Now as
explained back in Section 3.4, $\left\|\widehat{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,g_0}(\mathbb{T}^4-S)} := \left\|\widehat{f_\infty}\right\|_{C^0_{\delta,g_0}(\mathbb{T}^4-S)} + \left\|\nabla_{g_0}\widehat{f_\infty}\right\|_{C^0_{\delta,g_0}(\mathbb{T}^4-S)} + \left\|\nabla^2_{g_0}\widehat{f_\infty}\right\|_{C^0_{\delta,g_0}(\mathbb{T}^4-S)} +$

$\left[\widehat{f_\infty}\right]_{C^{2,\alpha'}_{\delta,g_0}(\mathbb{T}^4-S)} \leq 2C_2 < \infty$ for $0 < \alpha' < \alpha < 1$ means that $\widehat{f_\infty}$ satisfies the decay

$$\nabla^j_{g_0} \widehat{f_\infty} = O_{g_0}\left(\rho_0^{\delta-j}\right), \qquad \forall j \leq 2$$

That is, by the above properties of ρ_0, $\widehat{f_\infty}$ is bounded on $\left\{\frac{1}{8} \leq |\cdot|_S^{g_0}\right\}$ and satisfies

$$\left|\nabla_{g_0}^j \widehat{f_\infty}\right|_{g_0} \leq C_{5,j}\left(|\cdot|_S^{g_0}\right)^{\delta-j}, \qquad \forall j \leq 2$$

on $\left\{0 < |\cdot|_S^{g_0} \leq \frac{1}{8}\right\} \subset \mathbb{T}^4 - S$ for some $C_{5,j} > 0$.

Now $\Delta_{g_0} \widehat{f_\infty} = 0$ means that $\widehat{f_\infty}$ is harmonic on $\mathbb{T}^4 - S$ WRT the flat metric g_0 descending from the Euclidean space $\left(\mathbb{R}^4, g_0\right)$, hence by elliptic regularity $\widehat{f_\infty} \in C^\infty\left(\mathbb{T}^4 - S\right)$. Moreover, via lifting $\mathbb{T}^4 - S$ via $\mathbb{R}^4 \twoheadrightarrow \mathbb{R}^4/\Lambda =: \mathbb{T}^4$ to its fundamental domain in $\mathbb{R}^4$ (which ends up being a *punctured* open set), we have that near each puncture/point of S, say on $\left\{0 < |\cdot|_S^{g_0} < \frac{63}{500}\right\}$, that by virtue of being harmonic WRT g_0 the flat Euclidean metric, our $\mathbb{Z}_2$-invariant function $\widehat{f_\infty}$ admits a Laurent series expansion (no need for higher tensor norms since we're a $(0,0)$-tensor)

$$\widehat{f_\infty} = A_0 + O\left(\text{dist}_{g_0}(S, \cdot)\right) + \frac{D_0}{\left(\text{dist}_{g_0}(S, \cdot)\right)^2} + \sum_{m=1}^\infty D_m \frac{p_m\left(\text{dist}_{g_0}(S, \cdot)\right)}{\left(\text{dist}_{g_0}(S, \cdot)\right)^{2m+2}}$$

which converges uniformly and absolutely on compact subsets of $\left\{0 < \text{dist}_{g_0}(S, \cdot) < \frac{63}{500}\right\}$, where $A_0, D_0, D_m \in \mathbb{R}$ are constants and p_m is a homogeneous harmonic polynomial of degree m, and we recall from *Remark 3.3.2* that by definition $|\cdot|_S^{g_0} := \text{dist}_{g_0}(S, \cdot)$.

However, $-2 < \delta < 0$ whence $\text{dist}_{g_0}(S, \cdot)^\delta < \text{dist}_{g_0}(S, \cdot)^{-2}$ holds *on* $\left\{0 < \text{dist}_{g_0}(S, \cdot) < \frac{63}{500}\right\}$ *since* $\text{dist}_{g_0}(S, \cdot) < \frac{63}{500}$. Whence, since we have that $\left|\widehat{f_\infty}\right| \leq C_{5,0}\left(\text{dist}_{g_0}(S, \cdot)\right)^\delta$ on $\left\{0 < \text{dist}_{g_0}(S, \cdot) \leq \frac{1}{8}\right\}$ (the $|\cdot|_{g_0}$-norm on 0-forms is just absolute value), since the above Laurent series expansion holds on $\left\{0 < \text{dist}_{g_0}(S, \cdot) < \frac{63}{500}\right\}$ hence holds on $\left\{0 < \text{dist}_{g_0}(S, \cdot) \leq \frac{1}{8}\right\} \subset \left\{0 < \text{dist}_{g_0}(S, \cdot) < \frac{63}{500}\right\}$, that $\left|\widehat{f_\infty}\right| \leq C_{5,0}\left(\text{dist}_{g_0}(S, \cdot)\right)^\delta$ forces $D_0 = 0$ and each $D_m = 0$, whence

$$\widehat{f_\infty} = A_0 + O\left(\text{dist}_{g_0}(S, \cdot)\right)$$

on $\left\{0 < \text{dist}_{g_0}(S, \cdot) < \frac{63}{500}\right\}$.

Whence $\widehat{f_\infty}$ has *removable singularities* at each point of S, whence extends to a smooth harmonic function $\widehat{f_\infty}$ on the flat 4-torus $\left(\mathbb{T}^4, g_0\right)$. Moreover, we have by our new decay $\widehat{f_\infty} = A_0 + O\left(\text{dist}_{g_0}(S, \cdot)\right)$ for $\widehat{f_\infty}$ on $\left\{\text{dist}_{g_0}(S, \cdot) < \frac{63}{500}\right\}$ that $\widehat{f_\infty}$ is bounded on $\left\{\text{dist}_{g_0}(S, \cdot) < \frac{63}{500}\right\}$ since $\text{dist}_{g_0}(S, \cdot) < \frac{63}{500}$. Whence since $\widehat{f_\infty}$ is bounded on $\left\{\frac{1}{8} \leq \text{dist}_{g_0}(S, \cdot)\right\}$, we thus have that $\widehat{f_\infty}$ is a *bounded* harmonic function on the compact 4-manifold without boundary $\left(\mathbb{T}^4, g_0\right)$, whence $\widehat{f_\infty}$ is a **constant** by the maximum principle.

Now recall that each $f_i \in C_{\delta, g_{\epsilon_i}}^{2,\alpha}\left(\text{Km}_{\epsilon_i}\right)^0$, whence the sequence of real numbers $\left\{\int_{\text{Km}_{\epsilon_i}} f_i \omega_{\epsilon_i}^2\right\} \subset \mathbb{R}$ is just the constant sequence $\{0\} \subset \mathbb{R}$. Write each $\int_{\text{Km}_{\epsilon_i}} f_i \omega_{\epsilon_i}^2 = \int_{\left\{|\cdot|_{\pi^{-1}(S)}^{g_0} \leq 2\epsilon_i^{\frac{1}{2}}\right\}} f_i \omega_{\epsilon_i}^2 + \int_{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|_{\pi^{-1}(S)}^{g_0}\right\}} f_i \omega_{\epsilon_i}^2$. Hence each

$$\int_{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|_{\pi^{-1}(S)}^{g_0}\right\}} f_i \omega_{\epsilon_i}^2 = -\int_{\left\{|\cdot|_{\pi^{-1}(S)}^{g_0} \leq 2\epsilon_i^{\frac{1}{2}}\right\}} f_i \omega_{\epsilon_i}^2$$ from the integral zero assumption.

Since each $\|f_i\|_{C_{\delta, g_{\epsilon_i}}^0\left(\text{Km}_{\epsilon_i}\right)} = 1$, by definition of the weighted Holder norms we have that each $f_i = O\left(\rho_{\epsilon_i}^\delta\right)$. Moreover, since we have that $\epsilon_i \leq \rho_{\epsilon_i}$, $\delta < 0$ gives us $\rho_{\epsilon_i}^\delta \leq \epsilon_i^\delta$ and hence each $f_i = O\left(\epsilon_i^\delta\right)$. Therefore upon

using the identity $\frac{\omega^2}{2!} = dV_g$ we have that

$$\left| \int_{\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon_i^{\frac{1}{2}} \right\}} f_i \omega_{\epsilon_i}^2 \right| = 2 \left| \int_{\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon_i^{\frac{1}{2}} \right\}} f_i dV_{g_{\epsilon_i}} \right|$$

$$= O\left(\epsilon_i^\delta\right) \int_{\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon_i^{\frac{1}{2}} \right\}} dV_{g_{\epsilon_i}}$$

$$= O\left(\epsilon_i^\delta\right) \text{Vol}_{g_0}\left(\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon_i^{\frac{1}{2}} \right\} \right)$$

$$= O\left(\epsilon_i^\delta\right) \epsilon_i^{\frac{4}{2}}$$

$$= O\left(\epsilon_i^{\delta+2}\right)$$

Thus since $-2 < \delta \iff 0 < \delta + 2$, we therefore have that $\epsilon_i^{\delta+2} \searrow 0$ as $i \to \infty$, whence we have that $\int_{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}} f_i \omega_{\epsilon_i}^2 = - \int_{\left\{ |\cdot|^{g_0}_{\pi^{-1}(S)} \leq 2\epsilon_i^{\frac{1}{2}} \right\}} f_i \omega_{\epsilon_i}^2 \to 0$.

Next, lift to the 2-fold cover and use the identity $\frac{\omega^2}{2!} = dV_g$, whence each

$$\int_{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}} f_i \omega_{\epsilon_i}^2 = \int_{\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}}} \widehat{f_i} dV_{\widehat{g_{\epsilon_i}}}$$

Now I claim that $\int_{\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}}} \widehat{f_i} dV_{\widehat{g_{\epsilon_i}}} \to \int_{\mathbb{T}^4 - S} \widehat{f_\infty} dV_{g_0}$. Indeed, this follows[28] immediately, as *each* of our manifolds in question $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}$ are open subsets of $(\mathbb{T}^4, g_0)$ and are all equipped with the *same* metric as the target, namely the induced metric $g_0 = \widehat{g_{\epsilon_i}}$ from restriction of g_0 to $\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}} \subset \mathbb{T}^4$ (see Footnote 27), and Proposition 3.3.3 gives us that each $dV_{g_{\epsilon_i}} = \left(1 + O(\epsilon_i^2)\right) dV_{g_0}$ on $\left\{ \epsilon_i^{\frac{1}{2}} \leq |\cdot|^{g_0}_{\pi^{-1}(S)} \right\}$.

Whence since S (a finite set of points) has $\mathcal{H}^4$-measure zero, we thus have

$$\int_{\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}}} \widehat{f_i} dV_{\widehat{g_{\epsilon_i}}} \to \int_{\mathbb{T}^4} \widehat{f_\infty} dV_{g_0}$$

$$\int_{\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}}} \widehat{f_i} dV_{\widehat{g_{\epsilon_i}}} \to 0$$

whence

$$\int_{\mathbb{T}^4} \widehat{f_\infty} dV_{g_0} = 0$$

As $\widehat{f_\infty}$ is a constant, we therefore conclude that $\widehat{f_\infty} = 0$.

Therefore, we have that $\left\| \widehat{f_i} \right\|_{C^{2,\alpha'}_{\delta, \widehat{g_{\epsilon_i}}}(K)} \searrow 0$ for all compact subsets $K \Subset \widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}} \subset \mathbb{T}^4 - S$, whence transferring everything back to Km_{ϵ_i} and using $\|\cdot\|_{C^0_{\delta, g_{\epsilon_i}}} \leq \|\cdot\|_{C^{2,\alpha'}_{\delta, g_{\epsilon_i}}}$, we thus have that

$$\|f_i\|_{C^0_{\delta, g_{\epsilon_i}}(K)} \searrow 0$$

---

[28] Alternatively, one may appeal to **Colding's Volume Convergence Theorem** [6], which tells us that lower Ricci-bound is sufficient to upgrade the Gromov-Hausdorff convergence of a volume non-collapsed sequence of Riemannian manifolds to convergence in the **measured Gromov-Hausdorff topology** where the measures involved are the canonical Radon measure induced from the Riemannian volume densities. Thus we may appeal to such (see *Remark* 3.5.5) since each $\widehat{g_{\epsilon_i}} = g_0$ and g_0 is manifestly Ricci bounded below, and because $\left(\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_S \right\}}, \widehat{g_{\epsilon_i}} \right)$ satisfies volume non-collapse.

holds for all compact subsets $K \Subset \left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|_{\pi^{-1}(S)}^{g_0} \right\} \subset \mathrm{Km}_{\epsilon_i}$.

(2) **Case 2**; $\left\{ |\cdot|_{\pi^{-1}(S)}^{g_0} < 3\epsilon_i^{\frac{1}{2}} \right\}$: By construction, we have that on $\left\{ |\cdot|_{\pi^{-1}(S)}^{g_0} < 3\epsilon_i^{\frac{1}{2}} \right\}$ that $g_{\epsilon_i} = \widetilde{g_{EH,\epsilon_i}}$. Moreover, by Proposition 3.2.7 we have that $\widetilde{g_{EH,\epsilon_i}} = \epsilon_i^2 R_{\frac{1}{\epsilon_i}}^* \left(\widetilde{g_{EH-0,\epsilon_i}} \right)$.

Therefore, just as in proving **Case 2** of Proposition 3.6.4, upon performing a conformal rescaling $g_{\epsilon_i} \mapsto \frac{1}{\epsilon_i^2} g_{\epsilon_i}$, we have that each Riemannian manifold $\left(\left\{ |\cdot|_{\pi^{-1}(S)}^{g_0} < 3\epsilon_i^{\frac{1}{2}} \right\}, \frac{1}{\epsilon_i^2} g_{\epsilon_i} \right)$ is isometric under R_{ϵ_i} to the Riemannian submanifold $\left(\left\{ |\cdot|_{\mathbb{C}P^1}^{g_0} < \frac{3}{\epsilon_i^{\frac{1}{2}}} \right\}, \widetilde{g_{EH-0,\epsilon_i}} \right) \subset \left(T^*\mathbb{C}P^1, \widetilde{g_{EH-0,\epsilon_i}} \right)$.

Hence upon taking the limit as $i \nearrow \infty$, we have by Proposition 3.2.6 that $\left(\left\{ |\cdot|_{\mathbb{C}P^1}^{g_0} < \frac{3}{\epsilon_i^{\frac{1}{2}}} \right\}, \widetilde{g_{EH-0,\epsilon_i}} \right)$ exhausts to $\left(T^*\mathbb{C}P^1, g_{EH,1} \right)$ in the limit. More precisely, upon picking a point $p_i \in \left\{ |\cdot|_{\pi^{-1}(S)}^{g_0} < 3\epsilon_i^{\frac{1}{2}} \right\}$ for each i, we have the following convergence in the **pointed Gromov-Hausdorff topology**[29]:

$$\left(\left\{ |\cdot|_{\pi^{-1}(S)}^{g_0} < 3\epsilon_i^{\frac{1}{2}} \right\}, \frac{1}{\epsilon_i^2} g_{\epsilon_i}, p_i \right) \xrightarrow{pGH} \left(T^*\mathbb{C}P^1, g_{EH,1}, p_\infty \right)$$

Transferring over the weight function ρ_{ϵ_i} on $\left\{ |\cdot|_{\pi^{-1}(S)}^{g_0} < 3\epsilon_i^{\frac{1}{2}} \right\}$ via the above isometry gives us $R_{\epsilon_i}^* \left(\rho_{\epsilon_i} \right)$ which satisfies $R_{\epsilon_i}^* \left(\frac{1}{\epsilon_i} \rho_{\epsilon_i} \right) = \begin{cases} \frac{1}{\epsilon_i} & \text{on } \left\{ \frac{1}{5\epsilon_i} \leq |\cdot|_{\mathbb{C}P^1}^{g_0} \right\} \\ |\cdot|_{\mathbb{C}P^1}^{g_0} & \text{on } \left\{ 2 \leq |\cdot|_{\mathbb{C}P^1}^{g_0} \leq \frac{1}{8\epsilon_i^{\frac{1}{2}}} \right\} \\ 1 & \text{on } \left\{ |\cdot|_{\mathbb{C}P^1}^{g_0} \leq 1 \right\} \end{cases}$. Call $\widetilde{\rho_{\epsilon_i}} := R_{\epsilon_i}^* \left(\frac{1}{\epsilon_i} \rho_{\epsilon_i} \right)$ for notation.

Therefore, as $\|f_i\|_{C^0_{\delta, g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} := \left\| \rho_{\epsilon_i}^{-\delta} f_i \right\|_{C^0_{g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} = \left\| \rho_{\epsilon_i}^{-\delta} \epsilon_i^\delta \epsilon_i^{-\delta} f_i \right\|_{C^0_{g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})}$, we must also scale our functions $f_i \mapsto \epsilon_i^{-\delta} f_i$ to preserve the weighted Holder norm upon transferring to $\left(\left\{ |\cdot|_{\mathbb{C}P^1}^{g_0} < \frac{3}{\epsilon_i^{\frac{1}{2}}} \right\}, \widetilde{g_{EH-0,\epsilon_i}} \right)$ (after first restricting to $\left\{ |\cdot|_{\pi^{-1}(S)}^{g_0} < 3\epsilon_i^{\frac{1}{2}} \right\}$). Thus call $\widetilde{f_i} := \epsilon_i^{-\delta} f_i$ for notation.

Therefore $\|f_i\|_{C^{2,\alpha}_{\delta, g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} \leq 2C_2$ implies that we have the uniform bound $\left\| \widetilde{f_i} \right\|_{C^{2,\alpha}_{\delta, \widetilde{g_{EH-0,\epsilon_i}}} \left(\left\{ |\cdot|_{\mathbb{C}P^1}^{g_0} < \frac{3}{\epsilon_i^{\frac{1}{2}}} \right\} \right)} \leq 2C_2$ where the weight function used is precisely $\widetilde{\rho_{\epsilon_i}}$. Whence since $T^*\mathbb{C}P^1 = \bigcup_{i\in\mathbb{N}} \left\{ |\cdot|_{\mathbb{C}P^1}^{g_0} < \frac{3}{\epsilon_i^{\frac{1}{2}}} \right\}$ is a nested exhaustion of precompact subsets, we therefore have by Arzela-Ascoli that (upon relabeling the original sequence of f_i once more)

$$\widetilde{f_i} \xrightarrow{C^{2,\alpha'}_{\delta, \widetilde{g_{EH-0,\epsilon_i}}, loc}} \widetilde{f_\infty}$$

$$\left\| \widetilde{f_\infty} \right\|_{C^{2,\alpha'}_{\delta, g_{EH,1}} \left(T^*\mathbb{C}P^1 \right)} \leq 2C_2$$

---

[29] In fact, upon shrinking each region to $\left(\left\{ |\cdot|_{\mathbb{C}P^1}^{g_0} < \frac{1}{\epsilon_i^{\frac{1}{2}}} \right\}, \widetilde{g_{EH-0,\epsilon_i}} \right)$ where we have $\widetilde{g_{EH-0,\epsilon_i}} = g_{EH,1}$, since our manifolds are all open subsets of the fixed ambient manifold $(T^*\mathbb{C}P^1, g_{EH,1})$, and our metric tensors are all the same and are all induced from the ambient metric (which is $\widetilde{g_{EH-0,\epsilon_i}} = g_{EH,1}$) via restriction, since the ambient metric $g_{EH,1}$ is Ricci-flat hence Einstein and since each manifold is non-collapsed we may upgrade this convergence to C^∞_{loc}-**convergence** via harmonic coordinates and elliptic regularity.

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with $0 < \alpha' < \alpha < 1$, for some function $\widetilde{f_\infty}$ on $T^*\mathbb{C}P^1$ and where the convergence $\xrightarrow{C^{2,\alpha'}_{\delta,\widetilde{g_{EH-0,\epsilon_i}},loc}}$ means that $\left\|\widetilde{f_i} - \widetilde{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,\widetilde{g_{EH-0,\epsilon_i}}}(K)} \searrow 0$ hence $\left\|\widetilde{f_i}\right\|_{C^{2,\alpha'}_{\delta,\widetilde{g_{EH-0,\epsilon_i}}}(K)} \rightarrow \left\|\widetilde{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,g_{EH,1}}(K)}$, for all compact subsets $K \Subset$

$\left\{|\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\}$, and with $\left\|\widetilde{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,g_{EH,1}}(T^*\mathbb{C}P^1)} \leq 2C_2$ following upon restricting to unit diameter compact subsets. Moreover, the weight function used in $C^{2,\alpha'}_{\delta,g_{EH,1}}(T^*\mathbb{C}P^1)$ is precisely the limiting weight function

which satisfies $\widetilde{\rho_0} = \begin{cases} |\cdot|^{g_0}_{\mathbb{C}P^1} & \text{on } \left\{2 \leq |\cdot|^{g_0}_{\mathbb{C}P^1}\right\} \\ 1 & \text{on } \left\{|\cdot|^{g_0}_{\mathbb{C}P^1} \leq 1\right\} \end{cases}$, aka in the limit we get that $\lim_{i \nearrow \infty} \widetilde{\rho_{\epsilon_i}} = \widetilde{\rho_0}$ is precisely the weight function used to define the weighted Holder spaces for $T^*\mathbb{C}P^1$.

Playing the same game with $\left\|\Delta_{g_{\epsilon_i}} f_i\right\|_{C^{0,\alpha}_{\delta-2,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} \leq \frac{1}{i}$, namely that this bound directly implies the bounds $\left\|\Delta_{g_{\epsilon_i}} f_i\right\|_{C^{0,\alpha}_{\delta-2,\widetilde{g_{EH-0,\epsilon_i}}}\left(\left\{|\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\}\right)} \leq \frac{1}{i}$, that we may take the limit as $i \nearrow \infty$ and get

$$\Delta_{g_{EH,1}} \widetilde{f_\infty} = 0$$

on $(T^*\mathbb{C}P^1, g_{EH,1})$ (since $\Delta_{g_{EH,1}} \widetilde{f_\infty} = 0$ on all compact subsets of $T^*\mathbb{C}P^1$, hence on all of $T^*\mathbb{C}P^1$ by a compact exhaustion), hence by elliptic regularity $\widetilde{f_\infty} \in C^\infty(T^*\mathbb{C}P^1)$.

Now $\left\|\widetilde{f_\infty}\right\|_{C^{2,\alpha'}_{\delta,g_{EH,1}}(T^*\mathbb{C}P^1)} \leq 2C_2$ means that $\widetilde{f_\infty}$ satisfies the decay

$$\nabla^j_{g_{EH,1}} \widetilde{f_\infty} = O_{g_{EH,1}}\left(\widetilde{\rho_0}^{\,\delta-j}\right), \qquad \forall j \leq 2$$

hence by the properties of $\widetilde{\rho_0}$, we have that $\widetilde{f_\infty}$ is bounded on $\left\{|\cdot|^{g_0}_{\mathbb{C}P^1} \leq 2\right\}$ and satisfies

$$\left|\nabla^j_{g_{EH,1}} \widetilde{f_\infty}\right|_{g_0} \leq C_{6,j}\left(|\cdot|^{g_0}_{\mathbb{C}P^1}\right)^{\delta-j}, \qquad \forall j \leq 2$$

on $\left\{2 \leq |\cdot|^{g_0}_{\mathbb{C}P^1}\right\}$ for some $C_{6,j} > 0$.

But since $\delta < 0$, we have that

$$\left|\widetilde{f_\infty}\right| \leq C_{6,0}\left(|\cdot|^{g_0}_{\mathbb{C}P^1}\right)^{\delta} \leq C_{6,0} 2^{\delta} < \infty$$

on $\left\{2 \leq |\cdot|^{g_0}_{\mathbb{C}P^1}\right\}$, and hence $\widetilde{f_\infty}$ is bounded on all of $T^*\mathbb{C}P^1$. Moreover, $\left|\widetilde{f_\infty}\right| \leq C_{6,0}\left(|\cdot|^{g_0}_{\mathbb{C}P^1}\right)^{\delta}$ and $\delta < 0$ means that $\widetilde{f_\infty} \rightarrow 0$ as $|\cdot|^{g_0}_{\mathbb{C}P^1} \nearrow \infty$.

Thus we have that $\widetilde{f_\infty}$ is harmonic WRT $g_{EH,1}$, bounded on all of $T^*\mathbb{C}P^1$, and decays to 0 at infinity, whence via maximum principle and connectedness of $T^*\mathbb{C}P^1$, we get $\widetilde{f_\infty} = 0$.

Therefore, we have that $\left\|\widetilde{f_i}\right\|_{C^{2,\alpha'}_{\delta,\widetilde{g_{EH-0,\epsilon_i}}}(K)} \searrow 0$, for all compact subsets $K \Subset \left\{|\cdot|^{g_0}_{\mathbb{C}P^1} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\} \subset T^*\mathbb{C}P^1$. Whence transferring everything back to Km_{ϵ_i} and using $\|\cdot\|_{C^0_{\delta,g_{\epsilon_i}}} \leq \|\cdot\|_{C^{2,\alpha'}_{\delta,g_{\epsilon_i}}}$, we thus have that

$$\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}} \searrow 0$$

holds for all compact subsets $K \Subset \left\{|\cdot|^{g_0}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\} \subset \mathrm{Km}_{\epsilon_i}$.

Hence we have shown that

$$\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(K_1)} \searrow 0$$

$$\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(K_2)} \searrow 0$$

$\forall K_1 \Subset \left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)}\right\}$ and $\forall K_2 \Subset \left\{|\cdot|^{g_0}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\}$ compact. Whence since $\mathrm{Km}_{\epsilon_i} \subset \left\{|\cdot|^{g_0}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\} \cup \left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_0}_{\pi^{-1}(S)}\right\}$, we therefore have that

$$\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(K)} \searrow 0$$

on all compact subsets $K \Subset \mathrm{Km}_{\epsilon_i}$, particularly $K = \mathrm{Km}_{\epsilon_i}$. But this contradicts the fact that each $\|f_i\|_{C^0_{\delta,g_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} = 1$. Whence we have reached a contradiction to which the proof follows, as was to be shown. ∎

With these in hand, we may finally prove

Theorem 3.6.8. *For $\delta \in (-2,0)$ **and** $\epsilon > 0$ **sufficiently small**, the following is a **bounded linear isomorphism** for both the domain and codomain $C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon), C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)$ restricted to the closed subspace of functions with integral zero, i.e. $\int_{\mathrm{Km}_\epsilon} f\omega_\epsilon^2 = 0$:*

$$\Delta_{g_\epsilon} : C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)^0$$

*More crucially, we have that the operator norm of the inverse is bounded by a constant **independent of** $\epsilon > 0$:*

$$\left\|\Delta_{g_\epsilon}^{-1}\right\| \leq K$$

♦

Proof. The Laplacian in the unweighted Holder spaces is Fredholm of index zero. Now for *fixed* δ, ϵ, we have that $C^{k,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon), C^{2,\alpha}(\mathrm{Km}_\epsilon)$ have equivalent norms (they're already the same underlying vector space and ρ_ϵ is by construction uniformly bounded below by ϵ and above by 1). Hence $\Delta_{g_\epsilon} : C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)^0$ is also Fredholm of index zero. As remarked in *Remark* 3.6.7, the improved weighted Schauder estimate $\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} \leq C_3\|\Delta_{g_\epsilon}f\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)}$ as well as the integral zero condition gives us that $\ker \Delta_{g_\epsilon} = \{0\}$ when $\epsilon > 0$ **is sufficiently small** and $\delta \in (-2,0)$, hence $\mathrm{ind}(\Delta_{g_\epsilon}) = 0$ implies surjective and hence bounded linear isomorphism, and $\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} \leq C_3\|\Delta_{g_\epsilon}f\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)}$ also gives us our $\epsilon > 0$ **independent** bound on the operator norm of the inverse, as was to be shown. ∎

3.7 § Finishing the Proof §

Recall from Section 3.5 our nonlinear problem:

$$\mathcal{F}_\epsilon : C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)^0$$

$$f \mapsto \frac{\left(\omega_\epsilon + i\partial\bar{\partial}f\right)^2}{\omega_\epsilon^2} - e^{\phi_\epsilon}$$

where we solve

$$\mathcal{F}_\epsilon(f) := \frac{\left(\omega_\epsilon + i\partial\bar{\partial}f\right)^2}{\omega_\epsilon^2} - e^{\phi_\epsilon}$$

$$= \underbrace{\left(1 - e^{\phi_\epsilon}\right)}_{=\mathcal{F}_\epsilon(0)} \underbrace{-\frac{1}{2}\Delta_{g_\epsilon}f}_{=D_0\mathcal{F}_\epsilon(f)} + \underbrace{\frac{i\partial\bar{\partial}f \wedge i\partial\bar{\partial}f}{\omega_\epsilon^2}}_{\text{nonlinearity}}$$

$$= 0$$

We want to apply the following

Theorem 3.7.1 (Implicit Function Theorem). *Let X, Y be two Banach spaces. Let $\Phi : X \to Y$ be a C^1 map. Write $\Phi(x) = \Phi(0) + D_0\Phi(x) + N(x)$ with $N(x) := \Phi(x) - \Phi(0) - D_0\Phi(x)$ being the nonlinear term. Suppose we have 3 positive constants $L, r_0, N > 0$ such that*

(1) $D_0\Phi : X \to Y$ *admits a bounded linear right inverse $R : Y \to X$ such that $\|R\|_{op} \leq L$ ($L > 0$ controls right inverse)*

(2) *A bound $\|N(x) - N(y)\|_Y \leq N\|x - y\|_X(\|x\|_X + \|y\|_X)$,* $\qquad \forall x, y \in D_{r_0}^X(0)$ *("Lipschitz" bounds on nonlinearity)*

(3) $\|\Phi(0)\|_Y \leq \frac{r}{2L}$ *with $r < \min\{r_0, \frac{1}{2NL}\}$ ("smallness" of $\Phi(0)$)*

Then $\Phi(x) = 0$ has a unique solution $x \in D_{2L\|\Phi(0)\|_Y}^X(0)$. ♦

Proof. This is essentially the Banach fixed point theorem. We have that $\Phi(x) = 0 \iff \mathcal{A}(x) = x$ with $\mathcal{A}(x) := -R(\Phi(0) + N(x))$. We want to show that $\mathcal{A} : D_r^X(0) \to D_r^X(0)$ is a contraction mapping. First we have that $\operatorname{im}\mathcal{A} \subset D_r^X(0)$ because for $x \in D_r^X(0)$,

$$
\begin{aligned}
\|\mathcal{A}(x)\|_X &= \|R(\Phi(0) + N(x))\|_X \\
&\leq L(\|\Phi(0)\|_Y + \|N(x)\|_Y) \\
&\leq L(\|\Phi(0)\|_Y + 4L^2 N\|\Phi(0)\|_Y^2) \\
&< L\|\Phi(0)\|_Y(1 + 1) \\
&= 2L\|\Phi(0)\|_Y
\end{aligned}
$$

Next we have that

$$
\begin{aligned}
\|\mathcal{A}(x) - \mathcal{A}(y)\|_X &= \|R(N(x) - N(y))\|_X \\
&\leq L\|N(x) - N(y)\|_Y \\
&\leq LN\|x - y\|_X(\|x\|_X + \|y\|_X) \\
&\leq 4NL^2\|\Phi(0)\|_Y\|x - y\|_X \\
&< \frac{4NL^2}{4NL^2}\|x - y\|_X \\
&= \|x - y\|_Y
\end{aligned}
$$

Hence we're done, as was to be shown. ∎

We've now arrived at:

Theorem 3.7.2 (Big Theorem I). *Let $\delta \in (-2, 0)$. Let $\epsilon > 0$ from Data 3.2.5 be*

(1) *sufficiently small so that each pre-glued Eguchi-Hanson metric from Propositions 3.2.6 and 3.2.7 are Kähler and satisfy their annular estimates, and $\epsilon < 1$ to satisfy the volume form estimates.*

(2) $3\epsilon^{\frac{1}{2}} < \frac{1}{2}$ *from Data 3.3.1 for the construction of Km_ϵ.*

(3) $2\epsilon^{\frac{1}{2}} < \frac{3}{25}$ *from Data 3.4.4 for the construction of ρ_ϵ.*

(4) *sufficiently small for the smallness of $\mathcal{F}_\epsilon(0)$ from Proposition 3.5.4.*

(5) *sufficiently small for the weighted Schauder estimate from Corollary 3.6.5.*

(6) *sufficiently small for the improved Schauder estimate from Proposition 3.6.6.*

(7) *sufficiently small for the the uniform bounded linear isomorphism $\Delta_{g_\epsilon} : C_{\delta, g_\epsilon}^{2,\alpha}(\mathrm{Km}_\epsilon)^0 \to C_{\delta-2, g_\epsilon}^{0,\alpha}(\mathrm{Km}_\epsilon)^0$ from Theorem 3.6.8 (again for both the domain and codomain restricted to the closed subspace of functions with integral zero, i.e. $\int_{\mathrm{Km}_\epsilon} f\omega_\epsilon^2 = 0$).*

(8) $\epsilon^{1+\frac{\delta}{2}} < \frac{1}{4L^2 C_7 C_8}$ *for $L, C_7, C_8 > 0$ constants which will be defined in the proof.*

*Then for each such $\epsilon > 0$, the equation $\mathcal{F}_\epsilon(f) = 0$ has a **unique** solution f satisfying:*

- $\int_{\mathrm{Km}_\epsilon} f \omega_\epsilon^2 = 0$ *(**integral zero**)*

- $f \in C^\infty(\mathrm{Km}_\epsilon)$ *(**smoothness**)*

- $\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} \le 2L \|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)} < C_9 \epsilon^{3-\frac{\delta}{2}}$ *for $C_9 > 0$ chosen so that $C_9 > 2LC_8$ (**smallness of norm**)*

*Hence by Proposition 3.1.8, Remark 3.1.9, and Proposition 3.1.10, for each such $\epsilon > 0$ we have that $\omega_\epsilon + i\partial\bar\partial f \in [\omega_\epsilon]$ is the **unique** Kähler form in the Kähler class of ω_ϵ on $(\mathrm{Km}_\epsilon, J)$ which gives a Ricci-flat Kähler metric compatible with J.* ♦

Proof. We setup the input needed for the implicit function theorem (Theorem 3.7.1).

(1) We have from Theorem 3.6.8 that $-\frac{1}{2}\Delta_{g_\epsilon} : C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0 \to C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)^0$ is a bounded linear isomorphism with $\left\|2\Delta_{g_\epsilon}^{-1}\right\| \le 2K =: L$ and $L > 0$ *uniform* in $\epsilon > 0$.

(2) Firstly, we have that for any 4-form Θ, that since $\Theta = \frac{\Theta}{\omega\wedge\omega}\omega\wedge\omega$ where $\frac{\Theta}{\omega\wedge\omega}$ is the coefficient *function*, that combined with the identities $dV_g = \frac{\omega^2}{2}$ and $\left|dV_g\right|_g = 1$, we have that (where the tensor norms here are on 4-forms) $|\Theta|_g = \left|\frac{\Theta}{\omega\wedge\omega}\omega\wedge\omega\right|_g = 2\left|\frac{\Theta}{\omega\wedge\omega}dV_g\right|_g = 2\left|\frac{\Theta}{\omega\wedge\omega}\right|$. Whence it suffices to bound the 4-form norm of $i\partial\bar\partial f \wedge i\partial\bar\partial f - i\partial\bar\partial g \wedge i\partial\bar\partial g$.

Therefore, using the fact that $a \wedge a - b \wedge b = (a - b) \wedge (a + b)$ for 2-forms a, b, Proposition 3.4.5 with $l = 2$ on expressions with products, and the definition of the weighted Holder norms, for the nonlinearity we have that there exists a *uniform* constant $C_7 > 0$ such that

$$\|N(f) - N(g)\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)} \le C_7 \epsilon^{\delta-2}\|f - g\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)}\left(\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} + \|g\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)}\right), \qquad \forall f, g \in D_{r_0}^{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)}(0)$$

with $N := C_7 \epsilon^{\delta-2}$ and $r_0 > 0$ to be determined later.

(3) Proposition 3.5.4 and the Taylor expansion of the exponential function gives us supp $\mathcal{F}_\epsilon(0) \subseteq \left\{\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \le 2\epsilon^{\frac{1}{2}}\right\}$ and $|\mathcal{F}_\epsilon(0)| \le c_1(2)\epsilon^2$ on its support. By definition of the weighted Holder norm and the definition of ρ_ϵ restricted to $\left\{\epsilon^{\frac{1}{2}} \le |\cdot|^{g_0}_{\pi^{-1}(S)} \le 2\epsilon^{\frac{1}{2}}\right\}$ combined with Data 3.4.4 (namely $2\epsilon < \epsilon^{\frac{1}{2}}$) giving the bound $\epsilon^{\frac{1}{2}} \le \rho_\epsilon \le 2\epsilon^{\frac{1}{2}}$, we thus have that $\|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)} \le C_8 \epsilon^{2-\frac{\delta-2}{2}} = C_8 \epsilon^{3-\frac{\delta}{2}}$ for some *uniform* constant $C_8 > 0$.

Set $r_0 := C_9 \epsilon^{3-\frac{\delta}{2}}$ with $C_9 > 0$ chosen so that $C_9 > 2LC_8$. We now need to show that

$$\|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)} \le \frac{r}{2L}$$

with $r < \min\left\{r_0, \frac{1}{2NL}\right\} = \min\left\{C_9 \epsilon^{3-\frac{\delta}{2}}, \frac{\epsilon^{2-\delta}}{2LC_7}\right\}$. We have that

$$\|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)} \le C_8 \epsilon^{3-\frac{\delta}{2}} = \frac{2LC_8 \epsilon^{3-\frac{\delta}{2}}}{2L}$$

and so we choose our $r := 2LC_8 \epsilon^{3-\frac{\delta}{2}}$. Clearly $r < r_0$, and it remains to show that $r < \frac{\epsilon^{2-\delta}}{2LC_7}$. But $\epsilon^{3-\frac{\delta}{2}} = \epsilon^{2-\delta+P}$ where $P := 1 + \frac{\delta}{2}$, which is *positive* because $\delta \in (-2, 0)$. Hence since $x > y \Rightarrow c^x < c^y$ for $c \in (0, 1)$ small enough, **making $\epsilon > 0$ small enough so that** $\epsilon^{1+\frac{\delta}{2}} < \frac{1}{4L^2 C_7 C_8}$ forces $r < \frac{\epsilon^{2-\delta}}{2LC_7}$. Therefore, *for each such $\epsilon > 0$*

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Theorem 3.7.1 gives us the existence of a unique solution $f \in D^{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)^0}_{2L\|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)}}(0)$ to $\mathcal{F}_\epsilon(f) = 0$, giving us

existence, uniqueness, integral zero, and the bound $\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} \leq 2L\|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,g_\epsilon}(\mathrm{Km}_\epsilon)} < C_9\epsilon^{3-\frac{\delta}{2}}$ since $r < r_0$. Smoothness follows from elliptic regularity, as our nonlinear equation has smooth coefficients & 2nd order elliptic linearization and the produced function f is in $C^{2,\alpha} \subset C^2$. Lastly, Propositions 3.1.8 and 3.1.10, as well as *Remark 3.1.9* regarding positivity of the resulting Kähler form being automatic, finish off the existence of our unique Ricci-flat Kähler metric for each sufficiently small $\epsilon > 0$, as was to be shown. ∎

Now we have from the smallness of the norm $\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} < C_9\epsilon^{3-\frac{\delta}{2}}$ of the solution to the nonlinear problem above, the definition of the weighted Holder spaces, the global bound $\epsilon \leq \rho_\epsilon$ and the fact that $\delta \in (-2, 0)$ gives $\|f\|_{C^{2,\alpha}_{\delta,g_\epsilon}(\mathrm{Km}_\epsilon)} < C_9\epsilon^{3-\frac{\delta}{2}} \implies \|f\|_{C^0(\mathrm{Km}_\epsilon)} < C_9\epsilon^{3+\frac{\delta}{2}}, \|\nabla_{g_\epsilon}f\|_{C^0_{g_\epsilon}(\mathrm{Km}_\epsilon)} < C_9\epsilon^{2+\frac{\delta}{2}}, \|\nabla^2_{g_\epsilon}f\|_{C^0_{g_\epsilon}(\mathrm{Km}_\epsilon)} < C_9\epsilon^{1+\frac{\delta}{2}}$ with all higher order tensor norms defined with respect to $g_\epsilon, \omega_\epsilon$. In particular, this implies that $\|i\partial\bar\partial f\|_{C^0_{g_\epsilon}(\mathrm{Km}_\epsilon)} < C_{10}\epsilon^{1+\frac{\delta}{2}}$ with the higher order tensor norm defined with respect to $g_\epsilon, \omega_\epsilon$ for some *uniform* $C_{10} > 0$, whence $\|i\partial\bar\partial f\|_{C^0_{g_\epsilon}(\mathrm{Km}_\epsilon)}$ tends to zero as $\epsilon \searrow 0$ due to $\delta \in (-2, 0)$.

Combining these with the fact that $\|(\omega_\epsilon + i\partial\bar\partial f) - \omega_\epsilon\|_{C^0_{g_\epsilon}(\mathrm{Km}_\epsilon)} = \|i\partial\bar\partial f\|_{C^0_{g_\epsilon}(\mathrm{Km}_\epsilon)} < C_{10}\epsilon^{1+\frac{\delta}{2}}$ implies the identity map $\mathrm{id}_{\mathrm{Km}_\epsilon}$ is an $O(1)\epsilon^{1+\frac{\delta}{2}}$-Gromov-Hausdorff approximation via 2-out-of-3, *Remark 3.3.5*, the two Gromov-Hausdorff convergences in the proof of Proposition 3.6.6 (Footnotes 27 and 29), Proposition 3.2.2 regarding the curvature blow-up of the bolt of g_{EH,ϵ^2} and Propositions 3.2.7 and 3.3.3 on the construction of g_ϵ, and the triangle inequality for the Gromov-Hausdorff metric, we thus have proven Theorem 1.1, i.e.:

Theorem 3.7.3 (Main Theorem I). *Let K3 denote the K3 surface.*

Equip K3 with the fixed complex structure J and compatible family of Kähler metrics $g_\epsilon, \omega_\epsilon$ constructed in Section 3.3, hence getting the family of Kähler metrics $(\mathrm{Km}_\epsilon, J, g_\epsilon, \omega_\epsilon)$ on a family of Kummer surfaces $(\mathrm{Km}_\epsilon, J)$.

*Then denoting by $\widetilde{\omega_\epsilon} := \omega_\epsilon + i\partial\bar\partial f_\epsilon \in [\omega_\epsilon]$ and $\widetilde{g_\epsilon}$ the unique Ricci-flat Kähler metric in the Kähler class $[\omega_\epsilon]$ produced from Theorem 3.7.2, we have that $0 < \exists\epsilon_0 \ll 1$ such that **there exists a 1-parameter family of Calabi-Yau[30] metrics***

$$\{(\widetilde{g_\epsilon}, \widetilde{\omega_\epsilon})\}_{\epsilon \in (0,\epsilon_0)}$$

on K3 which are Ricci-flat Kähler WRT the fixed complex structure J.

The solved for functions $\{f_\epsilon\}_{\epsilon \in (0,\epsilon_0)}$ produced from Theorem 3.7.2 satisfy the uniform estimate

$$\|f_\epsilon\|_{C^k_{g_\epsilon}(\mathrm{Km}_\epsilon)} \leq C_9\epsilon^{3+\frac{\delta}{2}-k}$$

for $\delta \in (-2, 0)$.

Moreover when $\epsilon \searrow 0$,

$$(\mathrm{Km}_\epsilon, J, \widetilde{g_\epsilon}, \widetilde{\omega_\epsilon}) \xrightarrow{GH} (\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0)$$

converges in the Gromov-Hausdorff topology to the flat Kähler orbifold $(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0)$ with the singular orbifold Kähler metric g_0, ω_0 (which is equal to the flat Kähler metric on $\mathbb{T}^4/\mathbb{Z}_2 - S$).

Moreover, we may decompose K3 into a union of sets[31] $\mathrm{Km}_\epsilon^{reg} \cup \bigsqcup_{p \in S} \mathrm{Km}_\epsilon^{b_p}$ such that

(1) $\left(\mathrm{Km}_\epsilon^{reg}, J, \widetilde{g_\epsilon}, \widetilde{\omega_\epsilon}\right)$ *collapses to the flat Kähler manifold $(\mathbb{T}^4/\mathbb{Z}_2 - S, J_0, g_0, \omega_0)$ with bounded curvature away from S (**regular region**).*

(2) *For each $p \in S$, $\left(\mathrm{Km}_\epsilon^{b_p}, \frac{1}{\epsilon^2}\widetilde{g_\epsilon}\right)$ converges to $(T^*\mathbb{C}P^1, g_{EH,1})$ in C^∞_{loc} (**ALE bubble region**).*

◆

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[30] upon suitably normalizing the nowhere vanishing holomorphic $(2, 0)$-form on each Km_ϵ

[31] say, $\mathrm{Km}_\epsilon^{reg} := \left\{\frac{\epsilon^{\frac{1}{2}}}{2} < |\cdot|^{g_0}_{\pi^{-1}(S)}\right\}$ and each $\mathrm{Km}_\epsilon^{b_p}$ being the 16 connected components of $\left\{|\cdot|^{g_0}_{\pi^{-1}(S)} \leq \epsilon^{\frac{1}{2}}\right\}$.

4 § Hyper-Kähler Metrics on $K3$ and Volume Non-Collapsed Limit Spaces §

4.1 § Hyper-Kähler Metrics in 4-dimensions & Perturbing Closed Definite Triples §

Let us first recall the basics of hyper-Kähler metrics in dimension 4 which will be needed, mostly following Foscolo's paper [14] and survey [15].

First, recall that for (M^4, μ_0) an oriented smooth 4-manifold with a volume form μ_0, that each $\Lambda^2 T_x^* M$ has a non-degenerate quadratic form of signature $(3, 3)$ via $\frac{\cdot \wedge \cdot}{\mu_0}$.

Notation 4.1.1. Now let $\boldsymbol{\omega} := (\omega_1, \omega_2, \omega_3) \in \Omega^2(M) \otimes \mathbb{R}^3$ denote a triple of *real* 2-forms, *and from now on we consider these triples as* **column** *vectors*. This always yields us a (smooth) symmetric matrix-valued function $Q \in \Gamma(M, \operatorname{Sym}^2(\mathbb{R}^3))$ via **componentwise division**:

$$\frac{1}{2} \frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\mu_0} = Q$$

That is, $\frac{1}{2} \omega_i \wedge \omega_j = Q_{ij} \mu_0$. ○

Definition 4.1.2. On (M^4, μ_0) an oriented smooth 4-manifold with a volume form μ_0, for a triple of 2-forms $\boldsymbol{\omega}$ the following are equivalent:

- $\operatorname{span}(\boldsymbol{\omega}) := \operatorname{span}(\omega_1, \omega_2, \omega_3) \subseteq \Lambda^2 T_x^* M$ is a 3-dimensional positive definite subspace for every $x \in M$.
- The associated Q is a positive definitive matrix everywhere.

In such an event, we call $\boldsymbol{\omega}$ a **definite triple**. ●

Now for $\boldsymbol{\omega}$ a definite triple, we may associate an **associated volume form** $\mu_{\boldsymbol{\omega}}$ and a new matrix called the **associated intersection matrix** $Q_{\boldsymbol{\omega}}$ via $\mu_{\boldsymbol{\omega}} := (\det Q)^{\frac{1}{3}} \mu_0$ and $Q_{\boldsymbol{\omega}} := (\det Q)^{-\frac{1}{3}} Q$. Hence what we've done is normalized the determinant of Q to have unit determinant, and have appropriately normalized the given volume form so that we satisfy

$$\frac{1}{2} \frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\mu_{\boldsymbol{\omega}}} = Q_{\boldsymbol{\omega}}$$

as well as

$$\mu_{\boldsymbol{\omega}} = \frac{1}{2 \operatorname{Tr}(Q_{\boldsymbol{\omega}})} \operatorname{Tr}\left(\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T\right)$$

Thus both $\mu_{\boldsymbol{\omega}}, Q_{\boldsymbol{\omega}}$ are *independent* of the choice of volume form μ_0. Moreover, the following are equivalent:

- A choice of conformal class $[g]$ on M.
- The choice of a 3-dimensional positive definite subspace of $\Lambda^2 T_x^* M, \forall x \in M$.

Hence every definite triple $\boldsymbol{\omega}$ defines a Riemannian metric $g_{\boldsymbol{\omega}}$ via requiring $\operatorname{span}(\boldsymbol{\omega})|_x = \Lambda_{g_{\boldsymbol{\omega}}}^+ T_x^* M, \forall x \in M$ and $dV_{g_{\boldsymbol{\omega}}} = \mu_{\boldsymbol{\omega}}$, where $\Lambda_{g_{\boldsymbol{\omega}}}^+, \Omega_{g_{\boldsymbol{\omega}}}^+$ denote the self-dual *2-forms* WRT $*_{g_{\boldsymbol{\omega}}}$.

Definition 4.1.3. On (M^4, μ_0) with a definite triple $\boldsymbol{\omega}$, we say that

- $\boldsymbol{\omega}$ is **closed** or **hypersymplectic** when $d\boldsymbol{\omega} := (d\omega_1, d\omega_2, d\omega_3) = 0$.
- $\boldsymbol{\omega}$ is an $SU(2)$-**structure** when $Q_{\boldsymbol{\omega}} = \operatorname{id}$, i.e. $\frac{1}{2} \frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\mu_{\boldsymbol{\omega}}} = \operatorname{id}$.
- $\boldsymbol{\omega}$ is **hyper-Kähler** when it is both closed and an $SU(2)$-structure.

Note that the 3rd point is equivalent to $\operatorname{Hol}(g_{\boldsymbol{\omega}}) \subset \operatorname{Sp}(1)$, *whence* **hyper-Kähler metrics are Ricci-flat**. ●

Remark 4.1.4. Note that when $(M^4, \boldsymbol{\omega})$ is an SU(2)-structure, we have that

$$\mu_{\boldsymbol{\omega}} = \frac{1}{6}\boldsymbol{\omega}^T \wedge \boldsymbol{\omega} = \frac{1}{6}\operatorname{Tr}\!\left(\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T\right)$$

Note the order swap, i.e. $\mu_{\boldsymbol{\omega}} = \frac{1}{6}\sum_{i=1}^{3}\omega_i \wedge \omega_i$. Indeed, either notice that $\operatorname{Tr}(\mathrm{id}) = 3$ or write out $\frac{1}{2}\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T = \mathrm{id} \cdot \mu_{\boldsymbol{\omega}}$, giving us $\frac{1}{2}\omega_i \wedge \omega_j = \delta_{ij}\mu_{\boldsymbol{\omega}}$ whence $\mu_{\boldsymbol{\omega}} = \frac{1}{2}\omega_i \wedge \omega_i, \forall i \in \{1, 2, 3\}$. Whence summing over all 3 equivalent expressions for $\mu_{\boldsymbol{\omega}}$ and then dividing by 3 gives us our sought after expression. Therefore, being an SU(2)-structure may be equivalently stated as $\boldsymbol{\omega}$ satisfying one of the following *equivalent* equations:

$$Q_{\boldsymbol{\omega}} = \mathrm{id} \iff \frac{1}{2}\frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\mu_{\boldsymbol{\omega}}} = \mathrm{id} \iff \frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\boldsymbol{\omega}^T \wedge \boldsymbol{\omega}} = \frac{1}{3}\mathrm{id}$$

♣

Remark 4.1.5. Being hyper-Kähler is related to being Calabi-Yau (or Ricci-flat Kähler), as well as the usual definition of hyper-Kähler, as follows: First, *choose a direction in $S^2 = S^2(1) \subset \mathbb{R}^3$, say $e_1 \in S^2 \subset \mathbb{R}^3$.*

For $(M^4, \boldsymbol{\omega}, g_{\boldsymbol{\omega}})$ hyper-Kähler, we have that upon writing $\omega_c := \omega_2 + i\omega_3$, $\overline{\omega_c} := \omega_2 - i\omega_3$, that this makes ω_c a complex 2-form, defining an almost complex structure J_1 ala Proposition 3.1.15 via defining $\Lambda_{\mathbb{C}}^{1,0}T_x^*M := \ker(\alpha|_x \mapsto \alpha|_x \wedge \omega_c|_x), \forall x \in M$. In fact, as the notation suggests, the almost complex structure J_1 to which ω_c is complex with respect to is **precisely** the almost complex structure J_1 by which ω_1 is compatible with $g_{\boldsymbol{\omega}}$.

$d\omega_c = 0$ forces the differential ideal generated by the $(1, 0)$-forms be closed, whence J_1 is integrable by Newlander-Nirenberg. Moreover, ω_1 and ω_c are respectively real $(1, 1)$ and holomorphic $(2, 0)$ forms *with respect to* J_1, and since ω_1 is closed & nondegenerate, this makes (M^4, J_1, g, ω_1) *Kähler* with $g := g_{\boldsymbol{\omega}}$ the induced metric by 2-out-of-3. Moreover, $\frac{1}{2}\frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\mu_{\boldsymbol{\omega}}} = Q_{\boldsymbol{\omega}} = \mathrm{id}$ gives[32] us $\omega_1^2 = \frac{1}{2}\omega_c \wedge \overline{\omega_c}$, aka by Proposition 3.1.10, g is *Ricci-flat*.

Since the choice of direction $e_1 \in S^2 \subset \mathbb{R}^3$ was *arbitrary*, we get that $g_{\boldsymbol{\omega}}$ is Ricci-flat Kähler WRT to an S^2s worth of compatible integrable complex structures. In other words, $(J_e, \omega_e)_{e \in S^2}$ are all compatible WRT the *fixed* metric $g_{\boldsymbol{\omega}}$. Holonomially this is because $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ (and more generally $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m)$), and schematically

$$(M^4, \boldsymbol{\omega}, g_{\boldsymbol{\omega}}) \text{ hyper-Kähler} \implies (M^4, J_e, g_{\boldsymbol{\omega}}, \omega_e) \text{ Ricci-flat Kähler}, \forall e \in S^2$$

Conversely, only in $\dim_{\mathbb{R}} M = 4$ because $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$, given (M^4, J, g, ω) Ricci-flat Kähler (whence M^4 is clearly orientable since it admits complex structures), we have that the real and imaginary parts of the associated nowhere vanishing holomorphic $(2, 0)$-form/holomorphic volume form Ω (*Remark* 3.1.13) give us $\omega_2 := \operatorname{Re}\Omega, \omega_3 := \operatorname{Im}\Omega$ respectively which are closed, and we let $\omega_1 := \omega$, whence giving us our hyper-Kähler triple $\boldsymbol{\omega}$. Moreover, we still end up with $g_{\boldsymbol{\omega}} = g$ our original given Kähler *metric*. Schematically,

$$(M^4, J, g, \omega) \text{ Ricci-flat Kähler} \implies (M^4, \boldsymbol{\omega}, g_{\boldsymbol{\omega}}) \text{ hyper-Kähler}$$
$$\text{where } \boldsymbol{\omega} := (\omega, \operatorname{Re}\Omega, \operatorname{Im}\Omega)$$
$$\text{and } g_{\boldsymbol{\omega}} = g$$

Hence, when passing from Ricci-flat Kähler (M^4, J, g, ω) to hyper-Kähler $(M^4, \boldsymbol{\omega}, g_{\boldsymbol{\omega}})$, *we forget the given complex structure J and work with the underlying oriented smooth 4-manifold M^4.* ♣

Now suppose $(M^4, \boldsymbol{\omega})$ is hypersymplectic/closed, and (interpreted componentwise) $\|Q_{\boldsymbol{\omega}} - \mathrm{id}\|_{C^0(M^4)} < \sigma$ for $0 < \sigma \ll 1$. We want to perturb $\boldsymbol{\omega}$ into a hyper-Kähler triple, whence by *Remark* 4.1.4 we seek a triple of *closed* 2-forms $\boldsymbol{\eta}$ such that $\boldsymbol{\omega} + \boldsymbol{\eta}$ is hyper-Kähler, i.e.

$$\frac{(\boldsymbol{\omega} + \boldsymbol{\eta}) \wedge (\boldsymbol{\omega} + \boldsymbol{\eta})^T}{(\boldsymbol{\omega} + \boldsymbol{\eta})^T \wedge (\boldsymbol{\omega} + \boldsymbol{\eta})} = \frac{1}{3}\mathrm{id}$$

---

[32] Indeed, $\frac{1}{2}\frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\mu_{\boldsymbol{\omega}}} = \mathrm{id} \iff \frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\boldsymbol{\omega}^T \wedge \boldsymbol{\omega}} = \frac{1}{3}\mathrm{id}$ thus we have that $\omega_i \wedge \omega_j = \frac{1}{3}\delta_{ij}(\omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3)$, whence setting $i = j = 1$ and noting that $\omega_c \wedge \overline{\omega_c} = \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3$ finishes it off.

Decompose $\boldsymbol{\eta} = \boldsymbol{\eta}^+ + \boldsymbol{\eta}^-$ into self-dual and anti-self-dual parts WRT g_ω, respectively, whence $\boldsymbol{\eta}^\pm := \frac{\boldsymbol{\eta} \pm *_{g_\omega} \boldsymbol{\eta}}{2}$.
Write the self-dual part as $\boldsymbol{\eta}^+ = A\boldsymbol{\omega}$ for some matrix $A \in \Gamma(M, \mathrm{End}(\mathbb{R}^3))$.

Given any tuple of 2-forms $\boldsymbol{\gamma}$, denote by $\boldsymbol{\gamma} * \boldsymbol{\gamma} := \frac{1}{2} \frac{\boldsymbol{\gamma} \wedge \boldsymbol{\gamma}^T}{\mu_\omega} \in \Gamma(M, \mathrm{Sym}^2(\mathbb{R}^3))$ that particular symmetric 3×3 matrix valued function on M. Rewrite the perturbation above via factoring out the product *using the fact that the wedge product of a self-dual form and an anti-self-dual form vanishes*[33], then multiplying the LHS by $1 = \frac{2\mu_\omega}{2\mu_\omega}$, then using the relation $\frac{1}{2} \frac{\boldsymbol{\omega} \wedge \boldsymbol{\omega}^T}{\mu_\omega} = Q_\omega$, and finally passing the denominator of the LHS to the RHS. We thus get

$$Q_\omega + Q_\omega A^T + AQ_\omega + AQ_\omega A^T + \boldsymbol{\eta}^- * \boldsymbol{\eta}^- = \frac{1}{3}\left(\mathrm{Tr}(Q_\omega) + \mathrm{Tr}(AQ_\omega) + \mathrm{Tr}\left(Q_\omega A^T\right) + \mathrm{Tr}\left(AQ_\omega A^T\right) + \mathrm{Tr}(\boldsymbol{\eta}^- * \boldsymbol{\eta}^-)\right)\mathrm{id}$$

Therefore letting $M_0 := M - \frac{1}{3}\mathrm{Tr}(M) \cdot \mathrm{id}$ be the trace-free part of any 3×3 matrix M, our perturbation equation is thus

$$\left(Q_\omega A^T + AQ_\omega + AQ_\omega A^T\right)_0 = (-Q_\omega - \boldsymbol{\eta}^- * \boldsymbol{\eta}^-)_0$$

Now consider the map

$$\Gamma(M, \mathrm{End}(\mathbb{R}^3)) \to \Gamma(M, \mathrm{Sym}^2(\mathbb{R}^3))$$
$$A \mapsto Q_\omega A^T + AQ_\omega + AQ_\omega A^T$$

Note that the differential of this map at 0 is $A \mapsto Q_\omega A^T + AQ_\omega$, and since $\|Q_\omega - \mathrm{id}\|_{C^0} < \sigma$ for $0 < \sigma \ll 1$, *pointwise* this linear map induces an isomorphism $\mathrm{Sym}^2(\mathbb{R}^3) \to \mathrm{Sym}^2(\mathbb{R}^3)$ for $\sigma \ll 1$ sufficiently small. Whence upon restricting to trace-free symmetric matrices we invoke the **inverse function theorem** to define a smooth map $\mathcal{F} : \mathrm{Sym}^2_0(\mathbb{R}^3) \to \mathrm{Sym}^2_0(\mathbb{R}^3)$ such that $(Q_\omega A^T + AQ_\omega + AQ_\omega A^T)_0 = S \iff A = \mathcal{F}(S)$, whence we may rewrite the rewritten perturbation equation above as

$$\boldsymbol{\eta}^+ = \mathcal{F}\left((-Q_\omega - \boldsymbol{\eta}^- * \boldsymbol{\eta}^-)_0\right)\boldsymbol{\omega}$$

Remark 4.1.6. In the event that $\boldsymbol{\omega}$ is hyper-Kähler, aka $Q_\omega = \mathrm{id}$, we have that the 3 dimensional kernel of the linearization $A \mapsto Q_\omega A^T + AQ_\omega$ at the origin is in *bijection* with **infinitesimal hyper-Kähler rotations**. ♣

Now let $\mathcal{H}^+_{g_\omega}$ be the 3-dimensional space of *self-dual* harmonic *2-forms* WRT g_ω. We have that $\mathcal{H}^+_{g_\omega} = \mathrm{span}_{\mathbb{R}}(\boldsymbol{\omega})$, aka $\mathcal{H}^+_{g_\omega}$ consists of *constant* linear combinations of the 3 components of $\boldsymbol{\omega}$, because $\boldsymbol{\omega}$ is closed and self-dual hence harmonic, and linearly independent because $\boldsymbol{\omega}$ is definite.

In addition to $\boldsymbol{\eta}$ being closed, let us now impose the following

Ansatz 4.1.7. Let

$$\boldsymbol{\eta} = d\boldsymbol{a} + \boldsymbol{\zeta}$$

where $\boldsymbol{a} \in \Omega^1(M) \otimes \mathbb{R}^3$ and $\boldsymbol{\zeta} \in \mathcal{H}^+_{g_\omega} \otimes \mathbb{R}^3$.

With this ansatz, we may now finally rewrite the rewritten perturbation equation above as the following **nonlinear PDE system**:

$$d^+_{g_\omega}\boldsymbol{a} + \boldsymbol{\zeta} = \mathcal{F}\left(\left(-Q_\omega - d^-_{g_\omega}\boldsymbol{a} * d^-_{g_\omega}\boldsymbol{a}\right)_0\right)\boldsymbol{\omega}$$

where

$$(\boldsymbol{a}, \boldsymbol{\zeta}) \in \left(\Omega^1(M) \oplus \mathcal{H}^+_{g_\omega}\right) \otimes \mathbb{R}^3$$

Here of course $d^\pm_g := \frac{d \pm *_g d}{2}$ are the self-dual and anti-self-dual parts of the exterior derivative, and $d^*_g = -*_g d*_g$ is the adjoint of the exterior derivative acting on 1-forms on an oriented Riemannian 4-manifold.

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[33] Indeed, $\mathfrak{a}^+ \wedge \mathfrak{q}^- = (*_g \mathfrak{a}^+) \wedge (*_g \mathfrak{q}^-) = -\mathfrak{a}^+ \wedge \mathfrak{q}^-$.

However, the above nonlinear system of PDE is *not* elliptic: a simple dimension count yields that it inputs an element of $\left(\Omega^1(M) \oplus \mathcal{H}_g^+\right) \otimes \mathbb{R}^3$, which is $7 \cdot 3 = 21$-dimensional, and outputs an element of $\Omega_g^+(M) \otimes \mathbb{R}^3$, which is $3 \cdot 3 = 9$-dimensional. We therefore also impose the following **gauge-fixing condition**:

$$d_{g_\omega}^* \boldsymbol{a} = 0$$

Whence our nonlinear PDE system now becomes

$$d_{g_\omega}^+ \boldsymbol{a} + \zeta = \mathcal{F}\left(\left(-Q_\omega - d_{g_\omega}^- \boldsymbol{a} * d_{g_\omega}^- \boldsymbol{a}\right)_0\right)\omega, \qquad d_{g_\omega}^* \boldsymbol{a} = 0$$

This time around, we input an element of $\left(\Omega^1(M) \oplus \mathcal{H}_g^+\right) \otimes \mathbb{R}^3$, which is $7 \cdot 3 = 21$-dimensional, and output an element of $\left(\Omega^0(M) \oplus \Omega_g^+(M)\right) \otimes \mathbb{R}^3$, which is $(3 + 4) \cdot 3 = 21$-dimensional. Moreover, the linearization is

$$\left(D_{g_\omega} \circ \mathrm{pr}_1 + (0, \mathrm{pr}_2)\right) \otimes \mathbb{R}^3 = \left(D_{g_\omega} \circ \mathrm{pr}_1 + 0 \oplus \mathrm{id}_{\mathcal{H}_{g_\omega}^+}\right) \otimes \mathbb{R}^3$$

with $D_{g_\omega} := d_{g_\omega}^* \oplus d_{g_\omega}^+ : \Omega^1(M) \to \Omega^0(M) \oplus \Omega_{g_\omega}^+(M)$ the **leading order term** and $\mathrm{id}_{\mathcal{H}_{g_\omega}^+} : \mathcal{H}_{g_\omega}^+ \to \mathcal{H}_{g_\omega}^+ \subset \Omega_{g_\omega}^+(M)$ the **lower order term**. A straightforward computation gives us that the principal symbol of $D_g : \Omega^1(M) \to \Omega^0(M) \oplus \Omega_g^+(M)$ is

$$\sigma_1\left(D_g\right)(x, \xi) = \frac{1}{2}\left(-\xi^\sharp \lrcorner \cdot, *_g(\xi \wedge \cdot)\right) : T_x^* M \to \mathbb{R} \oplus \Lambda_g^+ T_x^* M, \qquad \forall (x, \xi) \in T^* M$$

which is clearly an isomorphism away from the zero section of T^*M since $\dim_{\mathbb{R}} M = 4$ and $\sigma_1\left(D_g\right)(x, \xi)$ is clearly injective for $\xi \neq 0$, whence D_g *is a first order elliptic PDO* and our nonlinear PDE system above is in fact a **nonlinear elliptic** system.

Moreover, we have the following standard facts from Hodge theory:

Proposition 4.1.8. Let $\left(M^4, g\right)$ be an oriented Riemannian 4-manifold. Let $\mathfrak{a} \in \Omega^1(M)$ be any 1-form. Then[34]

$$D_g \mathfrak{a} = 0 \implies \Delta_g \mathfrak{a} = 0$$

Proof. Indeed, $D_g \mathfrak{a} = 0$ means that $d_g^* \mathfrak{a} = 0$ and $d_g^+ \mathfrak{a} = \frac{d\mathfrak{a} + *_g d\mathfrak{a}}{2} = 0$ hence $*_g d\mathfrak{a} = -d\mathfrak{a}$. Now since $d_g^* \mathfrak{a} = 0$, we have that $\Delta_g \mathfrak{a} := \left(dd_g^* + d_g^* d\right)\mathfrak{a} = 0 + d_g^* d\mathfrak{a}$. But $*_g d\mathfrak{a} = -d\mathfrak{a}$ and $d_g^* = - *_g d *_g$ on 2-forms means that $d_g^* d\mathfrak{a} = - *_g d *_g d\mathfrak{a} = *_g d^2 \mathfrak{a} = 0$, as was to be shown. ∎

Proposition 4.1.9. Let $\left(M^4, g\right)$ be an oriented Riemannian 4-manifold. Suppose that $b_1(M) = 0$, and moreover suppose that $\left(M^4, g\right)$ admits a Fredholm theory for $\Delta_g := \left(d + d_g^*\right)^2$ (e.g. when M^4 is closed). Then we have the following:

- $\Omega^1(M) = d\left(\Omega^0(M)\right) \oplus \mathring{\Omega}_g^1(M)$

- $\Omega_g^+(M) = \mathcal{H}_g^+ \oplus d_g^+\left(\mathring{\Omega}_g^1(M)\right)$

Here $\mathring{\Omega}_g^1(M) := \Omega^1(M) \cap \ker d_g^* = \left\{\mathfrak{a} \in \Omega^1(M) : d_g^* \mathfrak{a} = 0\right\}$, and all decompositions are L_g^2-orthogonal.

Proof. We firstly have that $\Omega^1(M) = d\left(\Omega^0(M)\right) \oplus d_g^*\left(\Omega^2(M)\right) \oplus \mathcal{H}_g^1$ by definition. But since $b_1(M) = 0$, $d_g^*\left(\Omega^2(M)\right) \subset \mathring{\Omega}_g^1(M)$, and $d\left(\Omega^0(M)\right), \mathring{\Omega}_g^1(M)$ are L_g^2-orthogonal, $\Omega^1(M) = d\left(\Omega^0(M)\right) \oplus \mathring{\Omega}_g^1(M)$ follows. $\Omega_g^+(M) = \mathcal{H}_g^+ \oplus d_g^+\left(\mathring{\Omega}_g^1(M)\right)$ is thus immediate, as was to be shown. ∎

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[34] In fact, D_g is the **Dirac operator** associated to g when g is a 4-dimensional hyper-Kähler metric, so this makes sense.

Therefore, Proposition 4.1.9 immediately tells us that $d_g^+ \oplus \mathrm{id}_{\mathcal{H}_g^+} : \mathring{\Omega}_g^1(M) \oplus \mathcal{H}_g^+ \twoheadrightarrow \Omega_g^+(M)$ hence $\left(d_g^+ \oplus \mathrm{id}_{\mathcal{H}_g^+}\right) \otimes \mathbb{R}^3 : \left(\mathring{\Omega}_g^1(M) \oplus \mathcal{H}_g^+\right) \otimes \mathbb{R}^3 \to \Omega_g^+(M) \otimes \mathbb{R}^3$ is *surjective*, and

$$\ker\left(d_g^+ \oplus \mathrm{id}_{\mathcal{H}_g^+}\right) = \mathcal{H}_g^1$$

the space of harmonic 1-forms. But since Proposition 4.1.9 requires our (M^4, g) to satisfy $b_1(M) = 0$, we therefore have that $d_g^+ \oplus \mathrm{id}_{\mathcal{H}_g^+}$ hence $\left(d_g^+ \oplus \mathrm{id}_{\mathcal{H}_g^+}\right) \otimes \mathbb{R}^3$ is *always* an isomorphism.

Henceforth, when our $(M^4, \boldsymbol{\omega})$ is hypersymplectic/closed, M^4 is *closed* and satisfies $b_1(M^4) = 0$, and $\|Q_\omega - \mathrm{id}\|_{C^0(M^4)} < \sigma$ for $0 < \sigma \ll 1$, we have that we may perturb $\boldsymbol{\omega}$ to a genuinely hyper-Kähler triple provided we may be able to prove the existence of a smooth solution to the following **nonlinear *elliptic* system**:

$$\Phi(\boldsymbol{a}, \boldsymbol{\zeta}) = 0$$

where

$$\Phi : \left(\mathring{\Omega}_g^1(M) \oplus \mathcal{H}_{g_\omega}^+\right) \otimes \mathbb{R}^3 \to \Omega_{g_\omega}^+(M) \otimes \mathbb{R}^3$$
$$(\boldsymbol{a}, \boldsymbol{\zeta}) \mapsto d_{g_\omega}^+ \boldsymbol{a} + \boldsymbol{\zeta} - \mathcal{F}\left(\left(-Q_\omega - d_{g_\omega}^- \boldsymbol{a} * d_{g_\omega}^- \boldsymbol{a}\right)_0\right) \boldsymbol{\omega}$$

and with $D_0\Phi = \left(d_g^+ \oplus \mathrm{id}_{\mathcal{H}_g^+}\right) \otimes \mathbb{R}^3$ always an isomorphism (whence why we've already factored in the gauge-fixing term into the domain via restricting to $\mathring{\Omega}_g^1(M)$).

Remark 4.1.10. Note that throughout, it is *necessary* to deform the cohomology classes of $\omega_1, \omega_2, \omega_3$, because every hyper-Kähler triple must satisfy $\frac{1}{2}\langle [\boldsymbol{\omega}] \cup [\boldsymbol{\omega}]^T, [M] \rangle = \begin{pmatrix} \mathrm{Vol}_{g_\omega}(M) & 0 & 0 \\ 0 & \mathrm{Vol}_{g_\omega}(M) & 0 \\ 0 & 0 & \mathrm{Vol}_{g_\omega}(M) \end{pmatrix}$. ♣

Now let us gather together the same building blocks for the Kummer construction that has already been carried out, but this time from the hyper-Kähler perspective.

4.2 § Eguchi-Hanson Metrics & Flat Tori as "Building Blocks" §

Let $g_{\omega_0}, \boldsymbol{\omega}_0$ be the Euclidean hyper-Kähler metric on $\mathbb{R}^4$, whence $g_{\omega_0} = g_0$ and $\boldsymbol{\omega}_0 = (\omega_0, \mathrm{Re}\, dz^1 \wedge dz^2, \mathrm{Im}\, dz^1 \wedge dz^2)$ by *Remark* 4.1.5. Let $g_{EH,s}, \omega_{EH,s}$ be the Eguchi-Hanson metric and Kähler form constructed as from Section 3.2. The analogue of Proposition 3.2.2 is

Proposition 4.2.1. Let $s > 0$ be a positive real number. Let $\left(T^*\mathbb{CP}^1, J_{\mathcal{O}(-2)}, g_{EH,s}, \omega_{EH,s}\right)$ be our Eguchi-Hanson Kähler manifold.

- By *Remark* 4.1.5, we have that since $\left(T^*\mathbb{CP}^1, J_{\mathcal{O}(-2)}, g_{EH,s}, \omega_{EH,s}\right)$ is Ricci-flat Kähler, that the Eguchi-Hanson metric is also **hyper-Kähler** via

$$\left(T^*\mathbb{CP}^1, J_{\mathcal{O}(-2)}, g_{EH,s}, \omega_{EH,s}\right) \implies \left(T^*S^2, \boldsymbol{\omega}_{EH,s}, g_{\boldsymbol{\omega}_{EH,s}}\right)$$

  where we *forget the integrable complex structure* $J_{\mathcal{O}(-2)}$ (whence getting T^*S^2 instead of $T^*\mathbb{CP}^1$) and we have that

$$\boldsymbol{\omega}_{EH,s} := \left(\omega_{EH,s}, \mathrm{Re}\, \Omega_{T^*\mathbb{CP}^1}, \mathrm{Im}\, \Omega_{T^*\mathbb{CP}^1}\right)$$

  and

$$g_{\boldsymbol{\omega}_{EH,s}} = g_{EH,s}$$

- $\left(T^*S^2, \boldsymbol{\omega}_{EH,s}, g_{\boldsymbol{\omega}_{EH,s}}\right)$ is a complete Ricci-flat Riemannian 4-manifold which is an **ALE hyper-Kähler gravitational instanton** asymptotic at ∞ to $\mathbb{R}^4/\mathbb{Z}_2$ with rate $-4 < 0$. Hence $\exists K \subset T^*S^2$ a compact subset, a constant $R > 0$ *sufficiently large*, and a diffeomorphism $F : \mathbb{R}^4/\mathbb{Z}_2 - D_R^{g_{\boldsymbol{\omega}_0}}(0) \to T^*S^2 - K$ such that

$$\left.\left|\nabla_{g_{\boldsymbol{\omega}_0}}^k \left(F^* g_{\boldsymbol{\omega}_{EH,s}} - g_{\boldsymbol{\omega}_0}\right)\right|\right|_{g_{\boldsymbol{\omega}_0}} = O_{g_{\boldsymbol{\omega}_0}}\left(\left(|\cdot|_0^{g_{\boldsymbol{\omega}_0}}\right)^{-4-k}\right) \quad \text{and} \quad \left.\left|\nabla_{g_{\boldsymbol{\omega}_0}}^k \left(F^* \boldsymbol{\omega}_{EH,s} - \boldsymbol{\omega}_0\right)\right|\right|_{g_{\boldsymbol{\omega}_0}} = O_{g_{\boldsymbol{\omega}_0}}\left(\left(|\cdot|_0^{g_{\boldsymbol{\omega}_0}}\right)^{-4-k}\right)$$

  on $\mathbb{R}^4/\mathbb{Z}_2 - D_R^{g_{\boldsymbol{\omega}_0}}(0)$, where the latter inequality is interpreted componentwise.

- In fact, this diffeomorphism outside a compact subset comes from $\pi : T^*\mathbb{C}P^1 - \mathbb{C}P^1 \overset{\text{Biholo}}{\cong} \frac{\mathbb{C}^2-\{0\}}{\mathbb{Z}_2}$, i.e.
  $\pi : T^*\mathbb{C}P^1 - D_R^{g_{\boldsymbol{\omega}_0}}(\mathbb{C}P^1) \overset{\text{Biholo}}{\cong} \mathbb{C}^2/\mathbb{Z}_2 - D_R^{g_{\boldsymbol{\omega}_0}}(0)$, *since all biholomorphisms are diffeomorphisms upon forgetting the complex structures.*

- From the $s\pi^*(\omega_{FS})$ term in $\omega_{EH,s} = \left(\boldsymbol{\omega}_{EH,s}\right)_1$ and the fact that the Fubini-Study metric on $\mathbb{C}P^1$ is isometric to the round metric of $S^2\left(\frac{1}{2}\right)$, we have that the exceptional divisor/zero section $S^2 \subset \left(T^*S^2, g_{\boldsymbol{\omega}_{EH,s}}\right)$ has sectional curvature $\frac{4}{s}$, volume $s\pi$, diameter $\frac{s^{\frac{1}{2}}}{2}$, and self intersection number $S^2 \cdot S^2 = -2$.

  In particular, as $s \to 0$, the curvature of the exceptional S^2 blows up, but its diameter/volume shrinks.

- The Eguchi-Hanson metric $\left(T^*S^2, \boldsymbol{\omega}_{EH,s}, g_{\boldsymbol{\omega}_{EH,s}}\right)$ is **U(2)-invariant**. In particular, we immediately have that the Eguchi-Hanson metric is U(1)-invariant[35], i.e.

$$R_\lambda^*\left(g_{\boldsymbol{\omega}_{EH,s}}\right) = g_{\boldsymbol{\omega}_{EH,s}}, \qquad \forall \lambda \in \mathrm{U}(1) \subset \mathbb{C}^\times$$

- $g_{\boldsymbol{\omega}_{EH,s}}$ and $g_{\boldsymbol{\omega}_{EH,1}}$ (hence similarly for $\boldsymbol{\omega}_{EH,s}, \boldsymbol{\omega}_{EH,1}$) are isometric up to scaling, i.e.

$$R_{s^{\frac{1}{2}}}^*\left(\frac{1}{s}g_{\boldsymbol{\omega}_{EH,s}}\right) = g_{\boldsymbol{\omega}_{EH,1}}$$

  In fact, we may say more: from Kronheimer's classification of hyper-Kähler ALE gravitational instantons [26] [25] (see [22]), **all hyper-Kähler ALE spaces asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$ are isomorphic to $\left(T^*S^2, \boldsymbol{\omega}_{EH,1}, g_{\boldsymbol{\omega}_{EH,1}}\right)$ up to dilation/scaling & hyper-Kähler rotations**, namely

$$\left(T^*S^2, \boldsymbol{\omega}_{t,A}, g_{\boldsymbol{\omega}_{t,A}}\right)$$

  where the isomorphism is given by a diffeomorphism $P \in \mathrm{Diff}\left(T^*S^2\right)$ satisfying

$$P^*\left(g_{\boldsymbol{\omega}_{t,A}}\right) = t^2 g_{\boldsymbol{\omega}_{EH,1}} \quad \text{and} \quad P^*\left(\boldsymbol{\omega}_{t,A}\right) = t^2 A \boldsymbol{\omega}_{EH,1}$$

  for some $(t, A) \in \mathbb{R}_{>0} \times \mathrm{SO}(3, \mathbb{R})$, where (recalling our column convention from Notation 4.1.1) the RHS denotes matrix multiplication.

  Recalling that hyper-Kähler metrics have an $S^2(1)$s worth of Kähler forms which are compatible with it, that $\mathrm{SO}(3, \mathbb{R})$ acts transitively on $S^2(1) \subset \mathbb{R}^3$, and that WLOG the direction in S^2 corresponding to $\boldsymbol{\omega}_{EH,1}$ is e_1, we thus have that upon **counting parameters** that we have $t \in \mathbb{R}_{>0}$ and a choice of direction $e \in S^2(1) \subset \mathbb{R}^3$ to which $A \in \mathrm{SO}(3, \mathbb{R})$ sends $e_1 \mapsto e$.

  In other words, *all hyper-Kähler ALE spaces asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$ form a **3-parameter family**, as each is given by a choice of dilation/scaling constant $t \in \mathbb{R}_{>0}$ to scale $g_{\boldsymbol{\omega}_{EH,1}}$ and a choice of direction $e \in S^2(1)$ to*

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[35] In fact, this U(1)-invariance is what allows us to write the Eguchi-Hanson metric (from the hyper-Kähler perspective) via the **Gibbons-Hawking ansatz**. The harmonic function used to get Eguchi-Hanson therefore is just a Dirac monopole with 2 singularities and *zero mass* (or else we would get ALF).

send $\omega_{EH,1}$ to under "rotation". For future reference, denote the 3-parameter family of all hyper-Kähler ALE spaces asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$ as

$$\left\{\left(T^*S^2, \omega_{t,e}^{ALE}, g_{\omega_{t,e}^{ALE}}\right)\right\}_{(t,e)\in\mathbb{R}_{>0}\times S^2}$$

where ofc $\omega_{t,e}^{ALE}, g_{\omega_{t,e}^{ALE}} \overset{\text{isometric}}{\cong} A^e\omega_{EH,t^2}, g_{\omega_{EH,t^2}} \overset{\text{isometric}}{\cong} t^2A^e\omega_{EH,1}, t^2g_{\omega_{EH,1}}$ where $A^e \in SO(3,\mathbb{R})$ maps the $\omega_{EH,1}$ direction in S^2 to $e \in S^2$.

Let $\epsilon > 0$ be the gluing parameter from Data 3.2.5.
The analogue of Proposition 3.2.6 is

Proposition 4.2.2. For $\epsilon > 0$ from Data 3.2.5 **sufficiently small** so that $\widehat{\omega_{EH-0,\epsilon}}$ is a positive $(1,1)$-form hence a Kähler form on $\left(T^*\mathbb{C}P^1, J_{O(-2)}\right)$ compatible with $J_{O(-2)}$.

Then letting $\widehat{\boldsymbol{\omega}_{EH-0,\epsilon}} := \left(\widehat{\omega_{EH-0,\epsilon}}, \operatorname{Re}\Omega_{T^*\mathbb{C}P^1}, \operatorname{Im}\Omega_{T^*\mathbb{C}P^1}\right)$ and $g_{\widehat{\boldsymbol{\omega}_{EH-0,\epsilon}}} := \widehat{g_{EH-0,\epsilon}}$, we have that $\left(T^*S^2, \widehat{\boldsymbol{\omega}_{EH-0,\epsilon}}, \widehat{g_{EH-0,\epsilon}}\right)$ is a hyper-Kähler manifold such that

$$\widehat{\omega_{EH-0,\epsilon}} = \begin{cases} \omega_{EH,1} & \text{on } \left\{|\cdot|_{S^2}^{g_{\omega_0}} \le \frac{1}{\epsilon^{\frac{1}{2}}}\right\} \\ \omega_0 & \text{on } \left\{\frac{2}{\epsilon^{\frac{1}{2}}} \le |\cdot|_{S^2}^{g_{\omega_0}}\right\} \end{cases}$$

Moreover, by 2-out-of-3, **all of the above hold verbatim for the associated Riemannian metric** $g_{\widehat{\boldsymbol{\omega}_{EH-0,\epsilon}}}$.

The analogue of Proposition 3.2.7 is

Proposition 4.2.3. For $\epsilon > 0$ from Data 3.2.5 **sufficiently small** so that $\widehat{\omega_{EH,\epsilon}}$ is a positive $(1,1)$-form hence a Kähler form on $\left(T^*\mathbb{C}P^1, J_{O(-2)}\right)$ compatible with $J_{O(-2)}$.

Then letting $\widehat{\boldsymbol{\omega}_{EH,\epsilon}} := \left(\widehat{\omega_{EH,\epsilon}}, \operatorname{Re}\Omega_{T^*\mathbb{C}P^1}, \operatorname{Im}\Omega_{T^*\mathbb{C}P^1}\right)$ and $g_{\widehat{\boldsymbol{\omega}_{EH,\epsilon}}} := \widehat{g_{EH,\epsilon}}$, we have that $\left(T^*S^2, \widehat{\boldsymbol{\omega}_{EH,\epsilon}}, \widehat{g_{EH,\epsilon}}\right)$ is a hyper-Kähler manifold such that

- $$\widehat{\omega_{EH,\epsilon}} = \begin{cases} \omega_{EH,\epsilon^2} & \text{on } \left\{|\cdot|_{S^2}^{g_{\omega_0}} \le \epsilon^{\frac{1}{2}}\right\} \\ \omega_0 & \text{on } \left\{2\epsilon^{\frac{1}{2}} \le |\cdot|_{S^2}^{g_{\omega_0}}\right\} \end{cases} \quad \text{and} \quad g_{\widehat{\omega_{EH,\epsilon}}} = \begin{cases} g_{\omega_{EH,\epsilon^2}} & \text{on } \left\{|\cdot|_{S^2}^{g_{\omega_0}} \le \epsilon^{\frac{1}{2}}\right\} \\ g_{\omega_0} & \text{on } \left\{2\epsilon^{\frac{1}{2}} \le |\cdot|_{S^2}^{g_{\omega_0}}\right\} \end{cases}$$

- With the former interpreted componentwise,

$$\left|\nabla_{g_{\omega_0}}^k \left(\widehat{\omega_{EH,\epsilon}} - \omega_0\right)\right|_{g_{\omega_0}} \le c_1(k+2)\epsilon^{2-\frac{k}{2}} \quad \text{and} \quad \left|\nabla_{g_0}^k \left(g_{\widehat{\omega_{EH,\epsilon}}} - g_{\omega_0}\right)\right|_{g_{\omega_0}} \le c_2(k)\epsilon^{2-\frac{k}{2}}$$

on the annuli $\left\{\epsilon^{\frac{1}{2}} \le |\cdot|_{S^2}^{g_{\omega_0}} \le 2\epsilon^{\frac{1}{2}}\right\} = \left\{\frac{1}{\epsilon^{\frac{1}{2}}} \le \frac{|\cdot|_{S^2}^{g_{\omega_0}}}{\epsilon} \le \frac{2}{\epsilon^{\frac{1}{2}}}\right\}$ when $\epsilon > 0$ is **sufficiently small** (specifically $\epsilon < \frac{1}{R^2}$), for the same positive constants $c_1(k), c_2(k) > 0, k \in \mathbb{N}_0$ as in Proposition 3.2.7.

- With the former interpreted componentwise,

$$\widehat{\omega_{EH,\epsilon}} = \epsilon^2 R_{\frac{1}{\epsilon}}^* \left(\widehat{\omega_{EH-0,\epsilon}}\right) \quad \text{and} \quad g_{\widehat{\omega_{EH,\epsilon}}} = \epsilon^2 R_{\frac{1}{\epsilon}}^* \left(g_{\widehat{\omega_{EH-0,\epsilon}}}\right)$$

hence

$$\widehat{\omega_{EH,\epsilon}} = \begin{cases} \epsilon^2 R_{\frac{1}{\epsilon}}^* \left(\omega_{EH,1}\right) & \text{on } \left\{|\cdot|_{S^2}^{g_{\omega_0}} \le \epsilon^{\frac{1}{2}}\right\} \\ \omega_0 = \epsilon^2 R_{\frac{1}{\epsilon}}^* \left(\omega_0\right) & \text{on } \left\{2\epsilon^{\frac{1}{2}} \le |\cdot|_{S^2}^{g_{\omega_0}}\right\} \end{cases} \quad \text{and} \quad g_{\widehat{\omega_{EH,\epsilon}}} = \begin{cases} \epsilon^2 R_{\frac{1}{\epsilon}}^* \left(g_{\omega_{EH,1}}\right) & \text{on } \left\{|\cdot|_{S^2}^{g_{\omega_0}} \le \epsilon^{\frac{1}{2}}\right\} \\ g_{\omega_0} = \epsilon^2 R_{\frac{1}{\epsilon}}^* \left(g_{\omega_0}\right) & \text{on } \left\{2\epsilon^{\frac{1}{2}} \le |\cdot|_{S^2}^{g_{\omega_0}}\right\} \end{cases}$$

Thus we also have, **for** $\epsilon < 1$,

$$dV_{g_{\widetilde{\omega_{EH,\epsilon}}}} = \begin{cases} dV_{g_{\omega_{EH,\epsilon^2}}} & \text{on } \left\{ |\cdot|_{S^2}^{g_{\omega_0}} \leq \epsilon^{\frac{1}{2}} \right\} \\ \left(1 + O\big(\epsilon^2\big)\right) dV_{g_{\omega_0}} & \text{on } \left\{ \epsilon^{\frac{1}{2}} \leq |\cdot|_{S^2}^{g_{\omega_0}} \right\} \end{cases}$$

This directly implies that the associated intersection matrix satisfies (componentwise)

$$\left\| Q_{\widetilde{\omega_{EH,\epsilon}}} - \mathrm{id} \right\|_{C^0(T^*S^2)} \leq \mathcal{E}_1 \epsilon^2$$

for some uniform constant $\mathcal{E}_1 > 0$. More specifically,

$$\left| Q_{\widetilde{\omega_{EH,\epsilon}}} - \mathrm{id} \right| \begin{cases} \leq \mathcal{E}_1 \epsilon^2 & \text{on } \left\{ \epsilon^{\frac{1}{2}} \leq |\cdot|_{S^2}^{g_{\omega_0}} \leq 2\epsilon^{\frac{1}{2}} \right\} \\ = 0 & \text{on } \left\{ |\cdot|_{S^2}^{g_{\omega_0}} \leq \epsilon^{\frac{1}{2}} \right\} \sqcup \left\{ 2\epsilon^{\frac{1}{2}} \leq |\cdot|_{S^2}^{g_{\omega_0}} \right\} \end{cases}$$

4.3 § The "Kummer Construction" §

Let $\mathbb{T}^4 = \mathbb{R}^4/\Lambda$ for a full rank lattice $\Lambda \cong \mathbb{Z}^4$ in $\mathbb{R}^4$. Upon equipping $\mathbb{R}^4$ with the standard Euclidean hyper-Kähler metric ω_0, g_{ω_0}, as each of these tensors are $\mathbb{Z}^4$-invariant, hence by Proposition 3.2.1 descends down to $\mathbb{T}^4$ yielding a **flat hyper-Kähler 4-torus** $\big(\mathbb{T}^4, \omega_0, g_{\omega_0}\big)$. Moreover, Proposition 3.2.8 still holds for our flat hyper-Kähler 4-torus $\big(\mathbb{T}^4, \omega_0, g_{\omega_0}\big)$, whence the moduli of all flat metrics on $\mathbb{T}^4$ is $\mathrm{GL}(4, \mathbb{Z}) \backslash \mathrm{GL}(4, \mathbb{R}) / O(4)$, which has dimension $\frac{4(4+1)}{2} = 10$.

Just as in Section 3.3, upon passing the flat hyper-Kähler datum ω_0, g_{ω_0} on $\mathbb{T}^4$ to the quotient orbifold $\mathbb{T}^4/\mathbb{Z}_2$ with the standard involution action $p \mapsto -p$ on $\mathbb{T}^4$, we get a **flat hyper-Kähler orbifold** $\big(\mathbb{T}^4/\mathbb{Z}_2, \omega_0, g_{\omega_0}\big)$ with 16 isolated orbifold point singularities $S := \mathrm{Fix}(\mathbb{Z}_2)$, each modeled on $\mathbb{R}^4/\mathbb{Z}_2$.

Let $|\cdot|_S^{g_{\omega_0}} := \mathrm{dist}_{g_{\omega_0}}(S, \cdot)$ be the usual metric distance away from S. Just as in Data 3.3.1, WLOG let $\Lambda = \mathbb{Z}^4$, and choose our sufficiently small $\epsilon > 0$ from Data 3.2.5 and Proposition 4.2.3 to be even smaller so that $3\epsilon^{\frac{1}{2}} < \frac{1}{2}$.

Now consider 16 copies of the closed tubular neighborhood of radius $3\epsilon^{\frac{1}{2}}$ around the exceptional divisor on T^*S^2, i.e. $\bigsqcup_{i \in \{1,\dots,16\}} T^*S^2 - \left\{ |\cdot|_{S^2}^{g_{\omega_0}} > 3\epsilon^{\frac{1}{2}} \right\} = \bigsqcup_{i \in \{1,\dots,16\}} \left\{ |\cdot|_{S^2}^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\} = \bigsqcup_{i \in \{1,\dots,16\}} D_{3\epsilon^{\frac{1}{2}}}^{g_{\omega_0}}(S^2)$.

Now define

$$\big(\mathrm{Km}_\epsilon, \omega_\epsilon, g_{\omega_\epsilon}\big) := \frac{\big(\mathbb{T}^4/\mathbb{Z}_2, \omega_0, g_{\omega_0}\big) \bigsqcup_{i \in \{1,\dots,16\}} \left(T^*S^2 - \left\{ |\cdot|_{S^2}^{g_{\omega_0}} > 3\epsilon^{\frac{1}{2}} \right\}, \widetilde{\omega_{EH,\epsilon}}, g_{\widetilde{\omega_{EH,\epsilon}}} \right)}{\sim}$$

(where we have $g_{\omega_\epsilon} = g_\epsilon$) with the equivalence relation being the identification $\left\{ |\cdot|_S^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\} \sim \left\{ |\cdot|_{S^2}^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\}$. That is, each of the 16 connected components of $\left\{ |\cdot|_S^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\}$, which is $\left\{ |\cdot|_p^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\}$ for each $p \in S$, gets identified with a single copy of $\left\{ |\cdot|_{S^2}^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\} \subset T^*S^2$, and this identification is done via first lifting/pulling back $\left\{ |\cdot|_p^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\} \subset \mathbb{T}^4/\mathbb{Z}_2$ to $\mathbb{T}^4$ via the (branched) double cover/orbifold projection, then to (a proper subset of the fundamental domain in) $\mathbb{R}^4$ via the quotient $\mathbb{R}^4 \twoheadrightarrow \mathbb{R}^4/\Lambda = \mathbb{T}^4$, translating so that $p \mapsto 0$, then pushing down via $\mathbb{R}^4 \twoheadrightarrow \mathbb{R}^4/\mathbb{Z}_2$ onto $\mathbb{R}^4/\mathbb{Z}_2$, then lifting/pulling back via the *underlying smooth map of the* crepant resolution map $T^*\mathbb{C}P^1 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_2$ to get $\left\{ |\cdot|_{S^2}^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\} \subset T^*S^2$. Thus the *underlying* projection map $\pi : \mathrm{Km}_\epsilon \twoheadrightarrow \mathbb{T}^4/\mathbb{Z}_2$ is given by the identity on $\left\{ 3\epsilon^{\frac{1}{2}} < |\cdot|_S^{g_{\omega_0}} \right\}$, and on $\left\{ |\cdot|_S^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\} \sim \left\{ |\cdot|_{S^2}^{g_{\omega_0}} \leq 3\epsilon^{\frac{1}{2}} \right\}$ via *the underlying smooth map of the* crepant resolution map $\pi : \big(T^*\mathbb{C}P^1, J_{O(-2)}\big) \twoheadrightarrow \big(\mathbb{C}^2/\mathbb{Z}_2, J_0\big)$.

As this construction is just the same as back in Section 3.3 but done *on the underlying oriented smooth manifold level*, we thus still get a Kummer $K3$ surface Km_ϵ

Now to define our closed/hypersymplectic definite triple $\boldsymbol{\omega}_\epsilon$, letting $E_i \cong S^2$ denote one of the 16 exceptional divisors and $\pi^{-1}(S) = \sqcup_{i \in \{1,\dots,16\}} E_i$, we set

$$\boldsymbol{\omega}_\epsilon := \begin{cases} \boldsymbol{\omega}_0 & \text{on } \mathrm{Km}_\epsilon - \sqcup_{i \in \{1,\dots,16\}} D^{g_{\omega_0}}_{3\epsilon^{\frac{1}{2}}}(E_i) = \mathrm{Km}_\epsilon - \left\{ |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq 3\epsilon^{\frac{1}{2}} \right\} = \left\{ 3\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\} \\ \widetilde{\boldsymbol{\omega}_{EH,\epsilon}} & \text{on } \left\{ |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq 3\epsilon^{\frac{1}{2}} \right\} \end{cases}$$

which is clearly well defined and smooth on all of Km_ϵ by Proposition 4.2.3, and we now have our $K3$ surface $\left(\mathrm{Km}_\epsilon, \boldsymbol{\omega}_\epsilon, g_{\boldsymbol{\omega}_\epsilon} \right)$ with our closed definite triple $\boldsymbol{\omega}_\epsilon$ and associated Riemannian metric $g_{\boldsymbol{\omega}_\epsilon}$, **depending on the small $\epsilon > 0$.**

We still have the analogue of *Remark* 3.3.2 by forgetting the complex structure and working on the underlying smooth manifold level.

The analogue of Proposition 3.3.3 is

Proposition 4.3.1. For $\epsilon > 0$ satisfying Data 3.3.1, we have that $\boldsymbol{\omega}_\epsilon$ satisfies

$$\boldsymbol{\omega}_\epsilon = \begin{cases} \boldsymbol{\omega}_{EH,\epsilon^2} & \text{on } \left\{ |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq \epsilon^{\frac{1}{2}} \right\} \\ \boldsymbol{\omega}_0 & \text{on } \left\{ 2\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\} \end{cases} \quad \text{and} \quad g_{\boldsymbol{\omega}_\epsilon} = \begin{cases} g_{\omega_{EH,\epsilon^2}} & \text{on } \left\{ |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq \epsilon^{\frac{1}{2}} \right\} \\ g_{\omega_0} & \text{on } \left\{ 2\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\} \end{cases}$$

and, **for sufficiently small $\epsilon > 0$** to satisfy Proposition 4.2.3 (and with the former interpreted componentwise),

$$\left| \nabla^k_{g_{\omega_0}} (\boldsymbol{\omega}_\epsilon - \boldsymbol{\omega}_0) \right|_{g_{\omega_0}} \leq c_1(k+2)\epsilon^{2-\frac{k}{2}} \quad \text{and} \quad \left| \nabla^k_{g_0} (g_{\boldsymbol{\omega}_\epsilon} - g_{\omega_0}) \right|_{g_{\omega_0}} \leq c_2(k)\epsilon^{2-\frac{k}{2}}$$

on $\left\{ \epsilon^{\frac{1}{2}} \leq |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq 2\epsilon^{\frac{1}{2}} \right\} = \left\{ \frac{1}{\epsilon^{\frac{1}{2}}} \leq \frac{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}}{\epsilon} \leq \frac{2}{\epsilon^{\frac{1}{2}}} \right\}$.

Thus we also have, **for $\epsilon < 1$** (which Data 3.3.1 guarantees),

$$dV_{g_{\boldsymbol{\omega}_\epsilon}} = \begin{cases} dV_{g_{\omega_{EH,\epsilon^2}}} & \text{on } \left\{ |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq \epsilon^{\frac{1}{2}} \right\} \\ \left(1 + O(\epsilon^2) \right) dV_{g_{\omega_0}} & \text{on } \left\{ \epsilon^{\frac{1}{2}} \leq |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\} \end{cases}$$

and that the cohomology class $[\boldsymbol{\omega}_\epsilon] \in H^2(K3) \otimes \mathbb{R}^3$ is independent of the cutoff function χ in its construction (*Remark* 3.2.4).

Moreover, this volume form decay directly implies that the associated intersection matrix satisfies (componentwise)

$$\left\| Q_{\boldsymbol{\omega}_\epsilon} - \mathrm{id} \right\|_{C^0(\mathrm{Km}_\epsilon)} \leq \mathcal{E}_1 \epsilon^2$$

for some uniform constant $\mathcal{E}_1 > 0$. More specifically,

$$\left| Q_{\boldsymbol{\omega}_\epsilon} - \mathrm{id} \right| \begin{cases} \leq \mathcal{E}_1 \epsilon^2 & \text{on } \left\{ \epsilon^{\frac{1}{2}} \leq |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq 2\epsilon^{\frac{1}{2}} \right\} \\ = 0 & \text{on } \left\{ |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq \epsilon^{\frac{1}{2}} \right\} \sqcup \left\{ 2\epsilon^{\frac{1}{2}} \leq |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\} \end{cases}$$

We thus still have that each pair $\mathrm{Km}_{\epsilon_1}, \mathrm{Km}_{\epsilon_2}$ are diffeomorphic for $\epsilon_1, \epsilon_2 > 0$, and *Remark* 3.3.5 analogously gives us

Remark 4.3.2. From the construction of Km_ϵ and $\boldsymbol{\omega}_\epsilon$ itself, we have that if $\epsilon \searrow 0$, that as Riemannian manifolds $\left(\mathrm{Km}_\epsilon, \boldsymbol{\omega}_\epsilon, g_{\boldsymbol{\omega}_\epsilon} \right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_0, g_{\omega_0} \right)$ converges in the Gromov-Hausdorff topology to the orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_0, g_{\omega_0} \right)$ with the singular orbifold hyper-Kähler metric $\boldsymbol{\omega}_0$ with conical $\mathbb{R}^4/\mathbb{Z}_2$ singularities at each of the 16 singular points. Moreover, this convergence is satisfies volume non-collapse. Note that outside of the set S of 16 singular points, we have that $\boldsymbol{\omega}_0$ is the flat hyper-Kähler metric on $\mathbb{T}^4/\mathbb{Z}_2 - S$ from before. ♣

Moreover, the analogue of *Remark* 3.5.5 is

Remark 4.3.3. Recall from *Remark* 4.1.5 that for (M^4, ω, g_ω) which is *hypersymplectic*, that the resulting $(M^4, J_e, g_\omega, \omega_e)$ is Kähler *for each direction* $e \in S^2$ (since $Q_\omega = \mathrm{id}$ only comes in to show that g_ω is Ricci-flat), i.e. each $(J_e, \omega_e)_{e \in S^2}$ is Kähler WRT the *fixed* metric g_ω. Hence WLOG upon fixing a direction $e \in S^2$, say e_1, we have that we may do the same estimates as in Proposition 3.5.4 for the Ricci-potential of $\mathrm{Ric}_{g_{\omega_\epsilon}}$ WRT J_{e_1} and argue in the same exact same way as *Remark* 3.5.5 to get that

$$\left| \nabla_{g_{\omega_0}}^k \mathrm{Ric}_{g_{\omega_\epsilon}} \right|_{g_{\omega_0}} \leq c_4(k) \epsilon^{1 - \frac{k}{2}}$$

holds on the annulus $\left\{ \epsilon^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \leq 2\epsilon^{\frac{1}{2}} \right\}$ for some positive constants $c_4(k) > 0, k \in \mathbb{N}_0$, that $\mathrm{Ric}_{g_{\omega_\epsilon}}$ vanishes outside $\left\{ \epsilon^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \leq 2\epsilon^{\frac{1}{2}} \right\}$, and that

$$\left| \mathrm{Ric}_{g_{\omega_\epsilon}} \right|_{g_{\omega_\epsilon}} \leq c_4(0) \epsilon$$

holds on the annulus $\left\{ \epsilon^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \leq 2\epsilon^{\frac{1}{2}} \right\}$, whence we easily get that

$$-3\Lambda g_{\omega_\epsilon} \leq \mathrm{Ric}_{g_{\omega_\epsilon}} \leq 3\Lambda g_{\omega_\epsilon}$$

holds *everywhere* on Km_ϵ for $\Lambda := \frac{c_4(0)\epsilon}{3}$, aka $(\mathrm{Km}_\epsilon, \omega_\epsilon, g_{\omega_\epsilon})$ has **uniformly bounded Ricci curvature**. ♣

4.4 § Weighted (Holder) Spaces §

As remarked back in Section 3.4, since Holder spaces only depend on the Riemannian metric, the same weighted Holder spaces as defined back in Section 3.4 still hold in the hyper-Kähler category, whence we may speak of $C_{\delta,g}^{k,\alpha}$ for (p,q)-tensors on $(T^*S^2, \omega_{EH,s}, g_{\omega_{EH,s}})$, $(\mathbb{T}^4, \omega_0, g_{\omega_0})$, $(\mathbb{T}^4 - S, \omega_0, g_{\omega_0})$, and $(\mathrm{Km}_\epsilon, \omega_\epsilon, g_{\omega_\epsilon})$.

For reference, this is

$$\|\mathfrak{s}\|_{C_{\delta,g}^{k,\alpha}} := \sum_{j \leq k} \left\| \nabla_g^j \mathfrak{s} \right\|_{C_{\delta,g}^0} + \sup_{\substack{x \neq y \\ d_g(x,y) \leq \mathrm{inj}_g}} \min\{\rho(x), \rho(y)\}^{-\delta + k + \alpha} \frac{\left| \nabla_g^k \mathfrak{s}\big|_x - \nabla_g^k \mathfrak{s}\big|_y \right|_g}{d_g(x,y)^\alpha}$$

$$:= \sum_{j \leq k} \left\| \rho^{-\delta + j} \nabla_g^j \mathfrak{s} \right\|_{C_g^0} + \sup_{\substack{x \neq y \\ d_g(x,y) \leq \mathrm{inj}_g}} \min\{\rho(x), \rho(y)\}^{-\delta + k + \alpha} \frac{\left| \nabla_g^k \mathfrak{s}\big|_x - \nabla_g^k \mathfrak{s}\big|_y \right|_g}{d_g(x,y)^\alpha}$$

where we interpret the difference in the numerator of the (weighted) Holder seminorm via parallel transport along the unique minimal geodesic connecting x and y, and all covariant derivatives (and parallel transport from) the Levi-Civita connection of g and the higher order tensor (i.e. not functions) norms with respect to g. The space of weighted Holder (p,q)-tensors $C_{\delta,g}^{k,\alpha}$ is thus $C_{\delta,g}^{k,\alpha} := \left\{ \mathfrak{s} \in C_g^{k,\alpha} : \|s\|_{C_{\delta,g}^{k,\alpha}} < \infty \right\}$ but with the $\|\cdot\|_{C_{\delta,g}^{k,\alpha}}$ norm instead.

As the weight function ρ_ϵ on Km_ϵ needed for the $C_{\delta,g}^{k,\alpha}$-norm on $(\mathrm{Km}_\epsilon, g_{\omega_\epsilon})$ requires $2\epsilon^{\frac{1}{2}} < \frac{3}{25}$, we thus still have Data 3.4.4 in effect.

Notation 4.4.1. For notational clarity (as we're no longer only dealing with scalar-valued functions), we let $C_{\delta,g}^{k,\alpha}(M)$ denote the weighted Holder space of $\mathbb{R}$-valued functions on M, and we let $C_{\delta,g}^{k,\alpha}(E)$ denote the weighted Holder space of sections of a metric $\mathbb{K}$-vector bundle $E \twoheadrightarrow M$.

Sometimes, especially when doing estimates, whenever the tensor type is obvious from the context we will resort back to the usual notation $C_{\delta,g}^{k,\alpha}(M)$ for norms of tensors. ○

Remark 4.4.2. In the sequel, we will be dealing with the following function spaces:

$$\mathcal{H}_g^+ \otimes \mathbb{R}^3 \cong \mathbb{R}^9$$

$$C_{\delta,g}^{1,\alpha}\left(\mathring{\Omega}_g^1(M^4)\right)$$

$$C_{\delta-1,g}^{0,\alpha}\left(\Lambda_g^+ T^* M^4\right) \otimes \mathbb{R}^3$$

$$\left(C_{\delta,g}^{1,\alpha}(T^* M^4) \oplus \mathcal{H}_g^+\right) \otimes \mathbb{R}^3 = \left(C_{\delta,g}^{1,\alpha}(T^* M^4) \otimes \mathbb{R}^3\right) \oplus \left(\mathcal{H}_g^+ \otimes \mathbb{R}^3\right)$$

$$C_{\delta-1,g}^{0,\alpha}\left(\mathbb{R} \oplus \Lambda_g^+ T^* M^4\right) \otimes \mathbb{R}^3 = \left(C_{\delta-1,g}^{0,\alpha}(M^4) \otimes \mathbb{R}^3\right) \oplus \left(C_{\delta-1,g}^{0,\alpha}\left(\Lambda_g^+ T^* M^4\right) \otimes \mathbb{R}^3\right)$$

$$\left(C_{\delta,g}^{1,\alpha}\left(T^* \mathring{M}^4\right) \oplus \mathcal{H}_g^+\right) \otimes \mathbb{R}^3 = \left(C_{\delta,g}^{1,\alpha}\left(T^* \mathring{M}^4\right) \otimes \mathbb{R}^3\right) \oplus \left(\mathcal{H}_g^+ \otimes \mathbb{R}^3\right)$$

where we recall that $\mathcal{H}_g^+$ denotes the space of self-dual harmonic forms WRT g and[36]

$$C_{\delta,g}^{1,\alpha}\left(\mathring{\Omega}_g^1(M^4)\right) := \left\{\mathfrak{a} \in C_{\delta,g}^{1,\alpha}(T^* M^4) : d_g^* \mathfrak{a} = 0\right\} = C_{\delta,g}^{1,\alpha}(T^* M^4) \cap \ker d_g^*$$

is in fact a *closed* Banach subspace of $C_{\delta,g}^{1,\alpha}(T^* M^4)$ WRT the $\|\cdot\|_{C_{\delta,g}^{1,\alpha}(T^* M^4)}$-norm since the (formal) adjoint of the exterior derivative $d_g^* : C_{\delta,g}^{1,\alpha}(T^* M^4) \to C_{\delta-1,g}^{0,\alpha}(M^4)$ is continuous.

That is, these function spaces consists of either a single or a pair of 3-*tuples of elements*. In addition to the $\|\cdot\|_{C_{\delta,g}^{k,\alpha}}$-norms which we've already defined, we therefore use the following norms when working with the above:

$$\|b\|_{\mathcal{H}_g^+} = \|b\|_{L_g^2} := \left(\int_{M^4} b \wedge *_g b\right)^{\frac{1}{2}}$$

$$\|a\|_{\mathcal{A}\otimes\mathbb{R}^3} = \left(\|a_1\|_{\mathcal{A}}^2 + \|a_2\|_{\mathcal{A}}^2 + \|a_3\|_{\mathcal{A}}^2\right)^{\frac{1}{2}}$$

$$\|(a,b)\|_{\mathcal{A}\oplus\mathcal{B}} = \|a\|_{\mathcal{A}} + \|b\|_{\mathcal{B}}$$

That is, we put the L_g^2-norm on $\mathcal{H}_g^+$, the product L^2-norm on 3-tuples, and the product L^1-norm on direct sums. ♣

4.5   § Nonlinear Setup §

Let us now bring everything together and find our hyper-Kähler triple $\widetilde{\omega}_\epsilon$ by perturbing our closed definite triple ω_ϵ using the setup in Section 4.1.

Notation 4.5.1. As we will be working with a *fixed* $\epsilon > 0$, denote the sought after hyper-Kähler triple $\widetilde{\omega}_\epsilon = \omega_\epsilon + \eta$ as ω, *keeping in mind that as this will end up being the unique hyper-Kähler triple that's perturbed from ω, that it* **depends on the gluing parameter** $\epsilon > 0$ *as the starting closed definite triple ω_ϵ depends on $\epsilon > 0$.* ○

From Section 4.1, since Km_ϵ is a closed manifold with $b_1(\mathrm{Km}_\epsilon) = 0$ we want to solve the following nonlinear *elliptic* system with singular limit (see *Remark* 4.3.2):

$$\Phi_\epsilon : \left(C_{\delta,g_{\omega_\epsilon}}^{1,\alpha}\left(\mathring{\Omega}_{g_{\omega_\epsilon}}^1(\mathrm{Km}_\epsilon)\right) \oplus \mathcal{H}_{g_{\omega_\epsilon}}^+\right) \otimes \mathbb{R}^3 \to C_{\delta-1,g_{\omega_\epsilon}}^{0,\alpha}\left(\Lambda_{g_{\omega_\epsilon}}^+ T^* \mathrm{Km}_\epsilon\right) \otimes \mathbb{R}^3$$

$$(a,\zeta) \mapsto d_{g_{\omega_\epsilon}}^+ a + \zeta - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d_{g_{\omega_\epsilon}}^- a * d_{g_{\omega_\epsilon}}^- a\right)_0\right)\omega_\epsilon$$

---

[36] Note the abuse of notation here, as Ω^1, $\mathring{\Omega}_g^1$ usually denotes *smooth* regularity.

where we solve

$$\Phi_\epsilon(a, \zeta) := d^+_{g_{\omega_\epsilon}} a + \zeta - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}} a * d^-_{g_{\omega_\epsilon}} a\right)_0\right)\omega_\epsilon$$

$$= \underbrace{-\mathcal{F}\left((-Q_{\omega_\epsilon})_0\right)\omega_\epsilon}_{=\Phi_\epsilon(0,0)} + \underbrace{d^+_{g_{\omega_\epsilon}} a + \zeta}_{=D_0\Phi_\epsilon(a,\zeta)} + \underbrace{\mathcal{F}\left((-Q_{\omega_\epsilon})_0\right)\omega_\epsilon - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}} a * d^-_{g_{\omega_\epsilon}} a\right)_0\right)\omega_\epsilon}_{\text{nonlinearity}}$$

$$= 0$$

Recall that $\mathcal{H}^+_{g_{\omega_\epsilon}}$ denotes the space of self-dual harmonic 2-forms WRT g_{ω_ϵ}, whence consists of constant linear combinations of the 3 components of ω_ϵ.

Similar to the Calabi-Yau case, our strategy to prove the existence of (a, ζ) solving $\Phi_\epsilon(a, \zeta) = 0$ above is via the implicit function theorem (Theorem 3.7.1).

4.6 § Blow-up analysis §

First, we recall that the linearization at the origin of Φ_ϵ, $D_0\Phi_\epsilon = \left(d^+_{g_{\omega_\epsilon}} \oplus \mathrm{id}_{\mathcal{H}^+_{g_{\omega_\epsilon}}}\right) \otimes \mathbb{R}^3$ always an isomorphism since Km_ϵ is closed and satisfies $b_1(\mathrm{Km}_\epsilon) = 0$. However, while $D_0\Phi_\epsilon$ is always an isomorphism, the key is to have the operator norm of its inverse be **uniformly bounded in $\epsilon > 0$**.

Our strategy is to first prove uniform estimates for our Dirac operator

$$D_{g_{\omega_\epsilon}} := d^*_{g_{\omega_\epsilon}} \oplus d^+_{g_{\omega_\epsilon}} : C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}(T^*\mathrm{Km}_\epsilon) \to C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}\left(\mathbb{R} \oplus \Lambda^+_{g_{\omega_\epsilon}} T^*\mathrm{Km}_\epsilon\right)$$

for certain weights δ, then show that these imply the sought after uniform estimates for our linearization

$$D_0\Phi_\epsilon = \left(d^+_{g_{\omega_\epsilon}} \oplus \mathrm{id}_{\mathcal{H}^+_{g_{\omega_\epsilon}}}\right) \otimes \mathbb{R}^3 : \left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right) \oplus \mathcal{H}^+_{g_{\omega_\epsilon}}\right) \otimes \mathbb{R}^3 \to C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}\left(\Lambda^+_{g_{\omega_\epsilon}} T^*\mathrm{Km}_\epsilon\right) \otimes \mathbb{R}^3$$

The analogue of Proposition 3.6.1 is the following, which follows from $d^*_{C^2 g} = \frac{1}{C^2} d^*_g$, $*_{C^2 g} = *_g$ on 2-forms, and the conformal invariance $\Lambda^+_{C^2 g} T^*M = \Lambda^+_g T^*M$ of self-dual 2-forms:

Proposition 4.6.1. Let (M^n, g) be an arbitrary Riemannian manifold. Let $C > 0$ be a constant. Let ρ be a smooth nonnegative function, let $\delta \in \mathbb{R}$, let $\mathfrak{w}$ be a $(0, m)$-tensor, let $|\cdot|_g$ denote all higher order tensor norms WRT g, and recall $\left\|\nabla^j_g \mathfrak{w}\right\|_{C^0_{\delta,g}} := \left\|\rho^{-\delta+j} \nabla^j_g \mathfrak{w}\right\|_{C^0_g}$. Recall from Proposition 3.6.1(1) that $\nabla_{C^2 g} = \nabla_g$.

(1) Under a conformal rescaling $g \mapsto C^2 g$, we have that

$$\left|\nabla^j \mathfrak{w}\right|_{C^2 g} = \left(\frac{1}{C}\right)^{j+m} \left|\nabla^j \mathfrak{w}\right|_g$$

(2) Under a conformal rescaling $g \mapsto C^2 g$, we have for the Hodge Laplacian $\Delta_g := dd^*_g + d^*_g d$ that

$$\Delta_{C^2 g} = \frac{1}{C^2} \Delta_g$$

(3) When $n = 4$ and (M^4, g) is oriented, under a conformal rescaling $g \mapsto C^2 g$ we have that

$$D_{C^2 g} = \left(\frac{1}{C^2} d^*_g, d^+_g\right) = \left(\frac{1}{C^2} d^*_g\right) \oplus d^+_g$$

(4) Immediately from Proposition 3.6.1(1) and (1), under a conformal rescaling $g \mapsto C^2 g$ we have that

$$[\mathfrak{w}]_{C^{k,\alpha}_{\delta,C^2 g}} = \left(\frac{1}{C}\right)^{\delta+m} [\mathfrak{w}]_{C^{k,\alpha}_{\delta,g}}$$

with the weight of the RHS $C^{k,\alpha}_{\delta,g}$ being $\frac{\rho}{C}$, i.e.

$$\sup_{\substack{x \neq y \\ d_{C^2 g}(x,y) \leq \mathrm{inj}_{C^2 g}}} \min\{\rho(x),\rho(y)\}^{-\delta+k+\alpha} \frac{\left|\nabla^k_g \mathfrak{w}\big|_x - \nabla^k_g \mathfrak{w}\big|_y\right|_{C^2 g}}{d_{C^2 g}(x,y)^\alpha}$$

$$= \left(\frac{1}{C}\right)^{\delta+m} \sup_{\substack{x \neq y \\ d_g(x,y) \leq \mathrm{inj}_g}} \min\left\{\frac{\rho}{C}\Big|_x, \frac{\rho}{C}\Big|_y\right\}^{-\delta+k+\alpha} \frac{\left|\nabla^k_g \mathfrak{w}\big|_x - \nabla^k_g \mathfrak{w}\big|_y\right|_g}{d_g(x,y)^\alpha}$$

(5) Under a conformal rescaling $g \mapsto C^2 g$, we have from (4) that

$$\|\mathfrak{w}\|_{C^{k,\alpha}_{\delta,C^2 g}} = \left(\frac{1}{C}\right)^{\delta+m} \|\mathfrak{w}\|_{C^{k,\alpha}_{\delta,g}}$$

where again the weight of the RHS is $\frac{\rho}{C}$.

(6) Under a conformal rescaling $g \mapsto C^2 g$, we have from Proposition 3.6.1(1) and (1) that for the *unweighted/standard* Holder norm that

$$\|\mathfrak{w}\|_{C^{k,\alpha}_g} \xrightarrow{\text{conformal rescaling } g \mapsto C^2 g} \|\mathfrak{w}\|_{C^{k,\alpha}_{C^2 g}} = \sum_{j \leq k} \left(\frac{1}{C}\right)^{j+m} \left\|\nabla^j_g \mathfrak{w}\right\|_{C^0_g}$$

$$+ \left(\frac{1}{C}\right)^{k+m+\alpha} \sup_{\substack{x \neq y \\ d_g(u_1,u_2) \leq \mathrm{inj}_g}} \frac{\left|\nabla^k_g \mathfrak{w}\big|_x - \nabla^k_g \mathfrak{w}\big|_y\right|_g}{d_g(x,y)^\alpha}$$

(7) Transferring the standard Schauder estimate $\|\mathfrak{a}\|_{C^{2,\alpha}_g} \leq C\left(\|\mathfrak{a}\|_{C^0_g} + \|\Delta_g \mathfrak{a}\|_{C^{0,\alpha}_g}\right)$ for 1-forms $\mathfrak{a}$ on metric balls $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right)$ of radii (r_1, r_2) (clearly with $r_1 < r_2$) under the conformal rescaling $g \mapsto C^2 g$, we end up getting via (2) and (6):

$$\|\mathfrak{a}\|_{C^{2,\alpha}_{C^2 g}} \leq C\left(\|\mathfrak{a}\|_{C^0_{C^2 g}} + \|\Delta_{C^2 g} \mathfrak{a}\|_{C^{0,\alpha}_{C^2 g}}\right)$$

which is equal to

$$\frac{1}{C}\|\mathfrak{a}\|_{C^0_g} + \left(\frac{1}{C}\right)^2 \|\nabla_g \mathfrak{a}\|_{C^0_g} + \left(\frac{1}{C}\right)^3 \|\nabla^2_g \mathfrak{a}\|_{C^0_g} + \left(\frac{1}{C}\right)^{3+\alpha} [\nabla^2_g \mathfrak{a}]_{C^{0,\alpha}_g}$$

$$\leq C\left(\frac{1}{C}\|\mathfrak{a}\|_{C^0_g} + \left(\frac{1}{C}\right)^3 \|\Delta_g \mathfrak{a}\|_{C^0_g} + \left(\frac{1}{C}\right)^{3+\alpha} [\Delta_g \mathfrak{a}]_{C^{0,\alpha}_g}\right)$$

on the *same* metric balls $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right) = \left(B^{C^2 g}_{Cr_1}(p), B^{C^2 g}_{Cr_2}(p)\right)$, but with different radii WRT the *new* metric $C^2 g$.

Conversely, if we already had the standard Schauder estimate for C^2g, namely $\|\mathfrak{a}\|_{C^{2,\alpha}_{C^2g}} \le C\left(\|\mathfrak{a}\|_{C^0_{C^2g}} + \|\Delta_{C^2g}\mathfrak{a}\|_{C^{0,\alpha}_{C^2g}}\right)$ on $\left(B^{C^2g}_{Cr_1}(p), B^{C^2g}_{Cr_2}(p)\right)$, then unraveling it would yield the same estimate above on $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right) = \left(B^{C^2g}_{Cr_1}(p), B^{C^2g}_{Cr_2}(p)\right)$.

(8) When $n = 4$ and $\left(M^4, g\right)$ is oriented, under a conformal rescaling $g \mapsto C^2g$, upon recalling the L^2-norm on direct sums convention from *Remark* 4.4.2 (since D_g lands in $\Omega^0 \oplus \Omega^+_g \subset \Omega^0 \oplus \Omega^2$), on 1-forms $\mathfrak{a}$ we have from (3) and upon setting $m = 0$ and $m = 2$ in (6) that

$$\left\|D_{C^2g}\mathfrak{a}\right\|_{C^{0,\alpha}_{C^2g}} = \left(\frac{1}{C}\right)^2\left\|d_g^*\mathfrak{a}\right\|_{C^0_g} + \left(\frac{1}{C}\right)^{2+\alpha}\left[d_g^*\mathfrak{a}\right]_{C^{0,\alpha}_g} + \left(\frac{1}{C}\right)^2\left\|d_g^+\mathfrak{a}\right\|_{C^0_g} + \left(\frac{1}{C}\right)^{2+\alpha}\left[d_g^+\mathfrak{a}\right]_{C^{0,\alpha}_g}$$

[37]Therefore, since we have from (6) that $\left\|\nabla_g\mathfrak{a}\right\|_{C^{0,\alpha}_{C^2g}} = \left(\frac{1}{C}\right)^2\left\|\nabla_g\mathfrak{a}\right\|_{C^0_g} + \left(\frac{1}{C}\right)^{2+\alpha}\left[\nabla_g\mathfrak{a}\right]_{C^{0,\alpha}_g}$ for $\mathfrak{a}$ a 1-form, the heuristic here is that *the $C^{0,\alpha}_g$-norm of $D_g\mathfrak{a}$ and $\nabla_g\mathfrak{a}$ have the same scaling behavior under the conformal rescaling $g \mapsto C^2g$, even though D_g takes values in sections of the bundle $\mathbb{R} \oplus \Lambda^+_g T^*M^4 \twoheadrightarrow M^4$.*

(9) When $n = 4$ and $\left(M^4, g\right)$ is oriented, transferring the standard Schauder estimate $\|\mathfrak{a}\|_{C^{1,\alpha}_g} \le C\left(\|\mathfrak{a}\|_{C^0_g} + \|D_g\mathfrak{a}\|_{C^{0,\alpha}_g}\right)$ for 1-forms $\mathfrak{a}$ on metric balls $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right)$ of radii (r_1, r_2) (clearly with $r_1 < r_2$) under the conformal rescaling $g \mapsto C^2g$, we end up getting via (3), (6) and (8):

$$\|\mathfrak{a}\|_{C^{1,\alpha}_{C^2g}} \le C\left(\|\mathfrak{a}\|_{C^0_{C^2g}} + \|D_{C^2g}\mathfrak{a}\|_{C^{0,\alpha}_{C^2g}}\right)$$

which is equal to

$$\frac{1}{C}\|\mathfrak{a}\|_{C^0_g} + \left(\frac{1}{C}\right)^2\left\|\nabla_g\mathfrak{a}\right\|_{C^0_g} + \left(\frac{1}{C}\right)^{2+\alpha}\left[\nabla_g\mathfrak{a}\right]_{C^{0,\alpha}_g}$$

$$\le C\left(\frac{1}{C}\|\mathfrak{a}\|_{C^0_g} + \left(\frac{1}{C}\right)^2\left\|d_g^*\mathfrak{a}\right\|_{C^0_g} + \left(\frac{1}{C}\right)^{2+\alpha}\left[d_g^*\mathfrak{a}\right]_{C^{0,\alpha}_g} + \left(\frac{1}{C}\right)^2\left\|d_g^+\mathfrak{a}\right\|_{C^0_g} + \left(\frac{1}{C}\right)^{2+\alpha}\left[d_g^+\mathfrak{a}\right]_{C^{0,\alpha}_g}\right)$$

on the *same* metric balls $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right) = \left(B^{C^2g}_{Cr_1}(p), B^{C^2g}_{Cr_2}(p)\right)$, but with different radii WRT the *new* metric C^2g.

Conversely, if we already had the standard Schauder estimate for C^2g, namely $\|\mathfrak{a}\|_{C^{1,\alpha}_{C^2g}} \le C\left(\|\mathfrak{a}\|_{C^0_{C^2g}} + \|D_{C^2g}\mathfrak{a}\|_{C^{0,\alpha}_{C^2g}}\right)$ on $\left(B^{C^2g}_{Cr_1}(p), B^{C^2g}_{Cr_2}(p)\right)$, then unraveling it would yield the same estimate above on $\left(B^g_{r_1}(p), B^g_{r_2}(p)\right) = \left(B^{C^2g}_{Cr_1}(p), B^{C^2g}_{Cr_2}(p)\right)$.

Propositions 3.6.2 and 3.6.3 still hold as these are just estimates on the weight functions ρ_ϵ and $\widetilde{\rho}_0$ on Km_ϵ and T^*S^2, respectively.

We therefore arrive at the analogue of the first key proposition, namely the local weighted Schauder estimate (Proposition 3.6.4), but for $D_{g_{\omega_\epsilon}}$:

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[37] Indeed, since $\left\|D_{C^2g}\mathfrak{a}\right\|_{C^{0,\alpha}_{C^2g}} = \frac{1}{C^2}\left\|d_g^*\mathfrak{a}\right\|_{C^{0,\alpha}_{C^2g}} + \left\|d_g^+\mathfrak{a}\right\|_{C^{0,\alpha}_{C^2g}}$, $\left\|d_g^*\mathfrak{a}\right\|_{C^{0,\alpha}_{C^2g}} = \left\|d_g^*\mathfrak{a}\right\|_{C^0_g} + \left(\frac{1}{C}\right)^\alpha\left[d_g^*\mathfrak{a}\right]_{C^{0,\alpha}_g}$, and $\left\|d_g^+\mathfrak{a}\right\|_{C^{0,\alpha}_{C^2g}} = \left(\frac{1}{C}\right)^2\left\|d_g^+\mathfrak{a}\right\|_{C^0_g} +$ $\left(\frac{1}{C}\right)^{2+\alpha}\left[d_g^+\mathfrak{a}\right]_{C^{0,\alpha}_g}$.

Proposition 4.6.2 (Local Weighted Schauder Estimate). **Let** $\epsilon > 0$ **be from Data** 3.4.4. Then $\forall \mathfrak{a} \in C^{1,\alpha}_{\delta, g_{\omega_\epsilon}}(T^* \mathrm{Km}_\epsilon)$ and $\forall p \in \mathrm{Km}_\epsilon$, there exists an $r = r_p > 0$ (hence may depend on p) such that:

$$\|\mathfrak{a}\|_{C^{1,\alpha}_{\delta, g_{\omega_\epsilon}}\left(T^* B^{g_{\omega_\epsilon}}_{r_p}(p)\right)} \leq C_{11}\left(\|\mathfrak{a}\|_{C^0_{\delta, g_{\omega_\epsilon}}\left(T^* B^{g_{\omega_\epsilon}}_{2r_p}(p)\right)} + \left\|D_{g_{\omega_\epsilon}}\mathfrak{a}\right\|_{C^{0,\alpha}_{\delta-1, g_{\omega_\epsilon}}\left(\mathbb{R}\oplus\Lambda^+_{g_{\omega_\epsilon}} T^* B^{g_{\omega_\epsilon}}_{2r_p}(p)\right)}\right)$$

$$:= C_{11}\left(\left\|\rho_\epsilon^{-\delta}\mathfrak{a}\right\|_{C^0_{g_{\omega_\epsilon}}\left(T^* B^{g_{\omega_\epsilon}}_{2r_p}(p)\right)} + \left\|D_{g_{\omega_\epsilon}}\mathfrak{a}\right\|_{C^{0,\alpha}_{\delta-1, g_{\omega_\epsilon}}\left(\mathbb{R}\oplus\Lambda^+_{g_{\omega_\epsilon}} T^* B^{g_{\omega_\epsilon}}_{2r_p}(p)\right)}\right)$$

for $C_{11} > 0$ **independent of** ϵ **and** $p \in \mathrm{Km}_\epsilon$.

Proof. The proof strategy is the exact same as the proof of Proposition 3.6.4, but with some key differences due to working with 1-forms and with D_g instead of the scalar Laplacian Δ_g.

In more detail, let $p \in \mathrm{Km}_\epsilon$.

We consider 3 cases: $p \in \left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 2\epsilon\right\}$, $p \in \left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\}$, and $p \in \left\{\frac{3}{25} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\}$. Clearly these 3 regions cover all of Km_ϵ.

(1) **Case 1;** $p \in \left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 2\epsilon\right\}$: WLOG let us focus on only 1 of the 16 connected components of $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 2\epsilon\right\}$, hence suppose that $\pi^{-1}(S) = E$ consists of only a single exceptional $E \cong S^2$. By construction (Proposition 4.3.1 as well as Data 3.4.4 telling us that $2\epsilon < \epsilon^{\frac{1}{2}}$), on this region $g_{\omega_\epsilon} = g_{\omega_{EH,\epsilon^2}}$. Recalling that $R^*_\epsilon\left(\frac{1}{\epsilon^2}g_{\omega_{EH,\epsilon^2}}\right) = g_{\omega_{EH,1}}$ from Proposition 4.2.1 as well as the identification $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq 3\epsilon^{\frac{1}{2}}\right\} \sim \left\{|\cdot|^{g_{\omega_0}}_{S^2} \leq 3\epsilon^{\frac{1}{2}}\right\}$ from the underlying construction of Km_ϵ, we have that upon doing a conformal rescaling $g_{\omega_\epsilon} \mapsto \frac{1}{\epsilon^2}g_{\omega_\epsilon}$ that $\left(\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 2\epsilon\right\}, \frac{1}{\epsilon^2}g_{\omega_\epsilon}\right)$ is isometric under R_ϵ to $\left(\left\{|\cdot|^{g_{\omega_0}}_{S^2} < 2\right\}, g_{\omega_{EH,1}}\right) \subset (T^*S^2, g_{\omega_{EH,1}})$, which is a precompact region of the Eguchi-Hanson space $(T^*S^2, g_{\omega_{EH,1}})$. Since $(T^*S^2, g_{\omega_{EH,1}})$ has bounded geometry, we thus have, for $r_2 > r_1 > 0$ small enough so that $B^{g_{\omega_{EH,1}}}_{r_2}(p) \subset \left(\left\{|\cdot|^{g_{\omega_0}}_{S^2} < 2\right\}, g_{\omega_{EH,1}}\right)$ (hence both r_1, r_2 may depend on p) where we abuse notation and denote the point where p lands in $\left\{|\cdot|^{g_{\omega_0}}_{S^2} < 2\right\}$ as p, the standard Schauder estimate $\|k\mathfrak{a}\|_{C^{1,\alpha}_{g_{\omega_{EH,1}}}} \leq C_{11.1}\left(\|k\mathfrak{a}\|_{C^0_{g_{\omega_{EH,1}}}} + \left\|kD_{g_{\omega_{EH,1}}}\mathfrak{a}\right\|_{C^{0,\alpha}_{g_{\omega_{EH,1}}}}\right)$ on the metric balls $\left(B^{g_{\omega_{EH,1}}}_{r_1}(p), B^{g_{\omega_{EH,1}}}_{r_2}(p)\right)$ (where we have multiplied our 1-form by a positive scalar $k > 0$, to be determined), *where* $C_{11.1} > 0$ *is independent of* p.

Transferring everything back to $\left(\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 2\epsilon\right\}, \frac{1}{\epsilon^2}g_{\omega_\epsilon}\right)$ via applying R_ϵ to $\left(\left\{|\cdot|^{g_{\omega_0}}_{S^2} < 2\right\}, g_{\omega_{EH,1}}\right)$ and using $\frac{1}{\epsilon^2}g_{\omega_{EH,\epsilon^2}} = R^*_{\frac{1}{\epsilon}}\left(g_{\omega_{EH,1}}\right)$ and the identification $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq 3\epsilon^{\frac{1}{2}}\right\} \sim \left\{|\cdot|^{g_{\omega_0}}_{S^2} \leq 3\epsilon^{\frac{1}{2}}\right\}$, we have that our standard Schauder estimate is now $\|k\mathfrak{a}\|_{C^{1,\alpha}_{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}} \leq C_{11.1}\left(\|k\mathfrak{a}\|_{C^0_{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}} + \left\|kD_{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}\mathfrak{a}\right\|_{C^{0,\alpha}_{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}}\right)$ on the metric balls $\left(B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_1}(p), B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p)\right)$ for some $0 < R_1 < R_2$ (which may depend on p) such that $B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p) \subset \left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 2\epsilon\right\}$ and $B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p) \subset R_\epsilon\left(B^{g_{\omega_{EH,1}}}_{r_2}(p)\right)$ (and vice versa for R_1). Now apply Proposition 4.6.1(9) with $C = \frac{1}{\epsilon}$ and $g = g_{\omega_\epsilon}$, set $k = \epsilon^{-\delta-1}$, and upon noting that on $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 2\epsilon\right\}$ we have the *uniform* estimate $\epsilon \leq \rho_\epsilon \leq 2\epsilon$ (Proposition 3.6.2(1)), we get upon recalling the definition of the weighted Holder norms our sought after *weighted* Schauder estimate $\|\mathfrak{a}\|_{C^{1,\alpha}_{\delta, g_{\omega_\epsilon}}} \leq \widetilde{C_{11.1}}\left(\|\mathfrak{a}\|_{C^0_{\delta, g_{\omega_\epsilon}}} + \left\|D_{g_{\omega_\epsilon}}\mathfrak{a}\right\|_{C^{0,\alpha}_{\delta-1, g_{\omega_\epsilon}}}\right)$ on the metric balls $\left(B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_1}(p), B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p)\right) = \left(B^{g_{\omega_\epsilon}}_{\epsilon R_1}(p), B^{g_{\omega_\epsilon}}_{\epsilon R_2}(p)\right)$ upon setting $r_p := \epsilon R_1$ and $R_2 := 2R_1$. Note that $\widetilde{C_{11.1}} > 0$ is still independent of p *as well as* $\epsilon > 0$ since $\epsilon \leq \rho_\epsilon \leq 2\epsilon$ tells us that we may replace ρ_ϵ with ϵ, up to introducing harmless factors of 2 and $\frac{1}{2}$ into $C_{11.1}$.

(2) **Case 2;** $p \in \left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\}$: Again WLOG let us focus only on 1 of the 16 connected components of $\left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\}$. Hence g_{ω_ϵ} is an interpolation of $g_{\omega_{EH,\epsilon^2}}$ and g_{ω_0} on $\left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\}$, namely

$$g_{\omega_\epsilon} = \begin{cases} g_{\widetilde{\omega_{EH,\epsilon}}} & \text{on } \left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \le 3\epsilon^{\frac{1}{2}}\right\} \\ g_{\omega_0} & \text{on } \left\{3\epsilon^{\frac{1}{2}} \le |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\} \end{cases} \text{ and where } g_{\widetilde{\omega_{EH,\epsilon}}} = \begin{cases} g_{\omega_{EH,\epsilon^2}} & \text{on } \left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \le \epsilon^{\frac{1}{2}}\right\} \\ g_{\omega_0} & \text{on } \left\{2\epsilon^{\frac{1}{2}} \le |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\} \end{cases}.$$

Now by *Remark* 3.3.2, we may identify $\left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\}$ with $\left\{\epsilon < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8}\right\} \subset T^*S^2$, and upon recalling $g_{\widetilde{\omega_{EH,\epsilon}}} = \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(g_{\widetilde{\omega_{EH-0,\epsilon}}}\right) \Leftrightarrow R^*_\epsilon\left(\frac{1}{\epsilon^2}g_{\widetilde{\omega_{EH,\epsilon}}}\right) = g_{\widetilde{\omega_{EH-0,\epsilon}}}$ and $g_{\omega_0} = \epsilon^2 R^*_{\frac{1}{\epsilon}}\left(g_{\omega_0}\right) \Leftrightarrow R^*_\epsilon\left(\frac{1}{\epsilon^2}g_{\omega_0}\right) = g_{\omega_0}$ from Proposition 4.2.3 directly implying by construction of $g_{\widetilde{\omega_{EH-0,\epsilon}}}$ from Proposition 4.2.2 (as well as by construction of g_{ω_ϵ}) that $R^*_\epsilon\left(\frac{1}{\epsilon^2}g_{\omega_\epsilon}\right) = g_{\widetilde{\omega_{EH-0,\epsilon}}}$, we have that upon doing a conformal rescaling $g_{\omega_\epsilon} \mapsto \frac{1}{\epsilon^2}g_{\omega_\epsilon}$ that $\left(\left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\}, \frac{1}{\epsilon^2}g_{\omega_\epsilon}\right)$ is isometric under R_ϵ to $\left(\left\{1 < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\}, g_{\widetilde{\omega_{EH-0,\epsilon}}}\right) \subset \left(T^*S^2, g_{\widetilde{\omega_{EH-0,\epsilon}}}\right)$, which is a precompact region of the Riemannian manifold $\left(T^*S^2, g_{\widetilde{\omega_{EH-0,\epsilon}}}\right)$. Now by Proposition 4.2.2, since

$$g_{\widetilde{\omega_{EH-0,\epsilon}}} = \begin{cases} g_{\omega_{EH,1}} & \text{on } \left\{|\cdot|^{g_{\omega_0}}_{S^2} \le \frac{1}{\epsilon^{\frac{1}{2}}}\right\} \\ g_{\omega_0} & \text{on } \left\{\frac{2}{\epsilon^{\frac{1}{2}}} \le |\cdot|^{g_{\omega_0}}_{S^2}\right\} \end{cases} \text{ aka } g_{\widetilde{\omega_{EH-0,\epsilon}}} \text{ is an interpolation of } g_{\omega_{EH,1}} \text{ on a large compact region}$$

and with the flat Euclidean metric g_{ω_0} on the complement of a (slightly) larger (pre)compact region, it is immediate that the Riemannian manifold $\left(T^*S^2, g_{\widetilde{\omega_{EH-0,\epsilon}}}\right)$ is complete with bounded geometry.

Next, note that upon transferring over the weight function ρ_ϵ via the above process, we end up getting $R^*_\epsilon(\rho_\epsilon)$, and since this satisfies $R^*_\epsilon(\rho_\epsilon) = \begin{cases} \epsilon|\cdot|^{g_{\omega_0}}_{S^2} & \text{on } \left\{2 \le |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\} \\ \epsilon & \text{on } \left\{|\cdot|^{g_{\omega_0}}_{S^2} \le 1\right\} \end{cases}$, we immediately see that $R^*_\epsilon\left(\frac{1}{\epsilon}\rho_\epsilon\right) = \widetilde{\rho_0}$ on $\left\{1 < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\}$, the weight function used to defined the weighted Holder spaces on T^*S^2.

Now recall that our point p is now $p \in \left\{1 < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\}$, where we abuse notation and denote the point where p lands in $\left\{1 < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\}$ as p. Let $\delta_{2,p} > 0$ be the same constant as in Proposition 3.6.3 but shrunken so that we have that $\left\{|p|^{g_{\omega_0}}_{S^2} - \delta_{2,p} \le |\cdot|^{g_{\omega_0}}_{S^2} \le |p|^{g_{\omega_0}}_{S^2} + \delta_{2,p}\right\} \subset \left\{1 < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\}$ by openness. Now let $\sigma \in (0, 1)$ be chosen such that, upon letting $R := \widetilde{\rho_0}(p)$ for notation's sake, we have that $B^{g_{\widetilde{\omega_{EH,\epsilon}}}}_{2\sigma R}(p) \subset \left\{|p|^{g_{\omega_0}}_{S^2} - \delta_{2,p} \le |\cdot|^{g_{\omega_0}}_{S^2} \le |p|^{g_{\omega_0}}_{S^2} + \delta_{2,p}\right\} \subset \left\{1 < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\}$. Therefore, by Proposition 3.6.3, we have that $\frac{999}{1000} \le \frac{\widetilde{\rho_0}(q)}{\widetilde{\rho_0}(p)} = \frac{\widetilde{\rho_0}(q)}{R} \le \frac{1001}{1000}$ holds $\forall q \in B^{g_{\widetilde{\omega_{EH-0,\epsilon}}}}_{2\sigma R}(p)$.

Now because $\left(T^*S^2, g_{\widetilde{\omega_{EH-0,\epsilon}}}\right)$ has bounded geometry and we're on a precompact region $\left(\left\{1 < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\}, g_{\widetilde{\omega_{EH-0,\epsilon}}}\right)$, we have the standard Schauder estimate $\|k\mathfrak{a}\|_{C^{1,\alpha}_{g_{\widetilde{\omega_{EH-0,\epsilon}}}}} \le \mathcal{K}\left(\|k\mathfrak{a}\|_{C^0_{g_{\widetilde{\omega_{EH-0,\epsilon}}}}} + \left\|kD_{g_{\widetilde{\omega_{EH-0,\epsilon}}}}\mathfrak{a}\right\|_{C^{0,\alpha}_{g_{\widetilde{\omega_{EH-0,\epsilon}}}}}\right)$ on the metric balls $\left(B^{g_{\widetilde{\omega_{EH-0,\epsilon}}}}_{\sigma R}(p), B^{g_{\widetilde{\omega_{EH-0,\epsilon}}}}_{2\sigma R}(p)\right)$ (where we have multiplied our function by a positive scalar $k > 0$, to be determined), *where $\mathcal{K} > 0$ is independent of p.*

Performing a conformal rescaling $g_{\widetilde{\omega_{EH-0,\epsilon}}} \mapsto \frac{1}{R^2}g_{\widetilde{\omega_{EH-0,\epsilon}}}$ and then applying Proposition 4.6.1(9) with $C = \frac{1}{R}$

and $g = g_{\widehat{\omega EH-0,\epsilon}}$ gives us

$$R\|k\mathfrak{a}\|_{C^0_{g_{\widehat{\omega EH-0,\epsilon}}}} + R^2\left\|\nabla_{g_{\widehat{\omega EH-0,\epsilon}}}k\mathfrak{a}\right\|_{C^0_{g_{\widehat{\omega EH-0,\epsilon}}}} + R^{2+\alpha}\left[\nabla_{g_{\widehat{\omega EH-0,\epsilon}}}k\mathfrak{a}\right]_{C^{0,\alpha}_{g_{\widehat{\omega EH-0,\epsilon}}}}$$

$$\leq \mathcal{K}\left(R\|k\mathfrak{a}\|_{C^0_{g_{\widehat{\omega EH-0,\epsilon}}}} + R^2\left\|d^*_{g_{\widehat{\omega EH-0,\epsilon}}}k\mathfrak{a}\right\|_{C^0_{g_{\widehat{\omega EH-0,\epsilon}}}} + R^{2+\alpha}\left[d^*_{g_{\widehat{\omega EH-0,\epsilon}}}k\mathfrak{a}\right]_{C^{0,\alpha}_{g_{\widehat{\omega EH-0,\epsilon}}}}\right.$$

$$\left. + R^2\left\|d^+_{g_{\widehat{\omega EH-0,\epsilon}}}k\mathfrak{a}\right\|_{C^0_{g_{\widehat{\omega EH-0,\epsilon}}}} + R^{2+\alpha}\left[d^+_{g_{\widehat{\omega EH-0,\epsilon}}}k\mathfrak{a}\right]_{C^{0,\alpha}_{g_{\widehat{\omega EH-0,\epsilon}}}}\right)$$

on the metric balls $\left(B^{g_{\widehat{\omega EH-0,\epsilon}}}_{\sigma R}(p), B^{g_{\widehat{\omega EH-0,\epsilon}}}_{2\sigma R}(p)\right) = \left(B^{\frac{1}{R^2}g_{\widehat{\omega EH-0,\epsilon}}}_{\sigma}(p), B^{\frac{1}{R^2}g_{\widehat{\omega EH-0,\epsilon}}}_{2\sigma}(p)\right)$.

Now recalling that $\frac{999}{1000} \leq \frac{\widetilde{\rho_0}(q)}{R} \leq \frac{1001}{1000}$ holds $\forall q \in B^{g_{\widehat{\omega EH-0,\epsilon}}}_{2\sigma R}(p)$ and hence $\forall q \in B^{g_{\widehat{\omega EH-0,\epsilon}}}_{\sigma R}(p)$ as well, we get upon recalling the definition of the weighted Holder norms the following weighted Schauder estimate $\left\|R^{\delta+1}k\mathfrak{a}\right\|_{C^{1,\alpha}_{\delta,g_{\widehat{\omega EH-0,\epsilon}}}} \leq \widetilde{C_{11.2}}\left(\left\|R^{\delta+1}k\mathfrak{a}\right\|_{C^0_{\delta,g_{\widehat{\omega EH-0,\epsilon}}}} + \left\|R^{\delta+1}kD_{g_{\widehat{\omega EH-0,\epsilon}}}\mathfrak{a}\right\|_{C^{0,\alpha}_{\delta-1,g_{\widehat{\omega EH-0,\epsilon}}}}\right)$ on the metric balls

$\left(B^{g_{\widehat{\omega EH-0,\epsilon}}}_{\sigma R}(p), B^{g_{\widehat{\omega EH-0,\epsilon}}}_{2\sigma R}(p)\right) = \left(B^{\frac{1}{R^2}g_{\widehat{\omega EH-0,\epsilon}}}_{\sigma}(p), B^{\frac{1}{R^2}g_{\widehat{\omega EH-0,\epsilon}}}_{2\sigma}(p)\right)$, where the weight function here is $\widetilde{\rho_0}$. Similarly as in the first case, $\widetilde{C_{11.2}} > 0$ is still independent of all the parameters involved because we've only modified $\mathcal{K}$ by factors of $\frac{999}{1000}$ and $\frac{1001}{1000}$ since we've applied the uniform estimate $\frac{999}{1000} \leq \frac{\widetilde{\rho_0}(q)}{R} \leq \frac{1001}{1000}$ on those metric balls.

Now let us transfer everything back to $\left(\left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\}, \frac{1}{\epsilon^2}g_{\omega_\epsilon}\right)$ via applying R_ϵ to $\left(\left\{1 < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8\epsilon}\right\}, g_{\widehat{\omega EH-0,\epsilon}}\right)$ and using the identification $\left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\} \sim \left\{\epsilon < |\cdot|^{g_{\omega_0}}_{S^2} < \frac{1}{8}\right\} \subset T^*S^2$, as well as using the relation $\frac{1}{\epsilon^2}g_{\omega_\epsilon} = R^*_{\frac{1}{\epsilon}}\left(g_{\widehat{\omega EH-0,\epsilon}}\right)$ and the relation $\frac{1}{\epsilon}\rho_\epsilon = R^*_{\frac{1}{\epsilon}}(\widetilde{\rho_0})$. We get that our weighted Schauder estimate

has now become $\left\|R^{\delta+1}k\mathfrak{a}\right\|_{C^{1,\alpha}_{\delta,\frac{1}{\epsilon^2}g_{\omega_\epsilon}}} \leq \widetilde{C_{11.2}}\left(\left\|R^{\delta+1}k\mathfrak{a}\right\|_{C^0_{\delta,\frac{1}{\epsilon^2}g_{\omega_\epsilon}}} + \left\|R^{\delta+1}kD_{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}\mathfrak{a}\right\|_{C^{0,\alpha}_{\delta-1,\frac{1}{\epsilon^2}g_{\omega_\epsilon}}}\right)$ on the metric

balls $\left(B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_1}(p), B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p)\right)$ for some $0 < R_1 < R_2$ (which may depend on p) such that $B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p) \subset \left\{\epsilon < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < \frac{1}{8}\right\}$ and $B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p) \subset R_\epsilon\left(B^{g_{\widehat{\omega EH-0,\epsilon}}}_{2\sigma R}(p)\right)$ (and vice versa for R_1), where the weight function used here is $\frac{1}{\epsilon}\rho_\epsilon$. Now we undo the original conformal rescaling of $\frac{1}{\epsilon^2}g_{\omega_\epsilon}$ via performing a conformal rescaling $\frac{1}{\epsilon^2}g_{\omega_\epsilon} \mapsto \epsilon^2\frac{1}{\epsilon^2}g_{\omega_\epsilon} = g_{\omega_\epsilon}$ on the metric balls $\left(B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_1}(p), B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p)\right)$ hence yielding $\left(B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_1}(p), B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p)\right) = \left(B^{g_{\omega_\epsilon}}_{\epsilon R_1}(p), B^{g_{\omega_\epsilon}}_{\epsilon R_2}(p)\right)$, applying Proposition 4.6.1(5) with $C = \epsilon$ and $g = \frac{1}{\epsilon^2}g_{\omega_\epsilon}$ on each of the 3 weighted Holder norm terms where g denotes the metric used in the weighted Holder norm, and then setting $k = \epsilon^{\delta+1}R^{-\delta-1}$,

yielding the weighted Schauder estimate $\|\mathfrak{a}\|_{C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}} \leq \widetilde{C_{11.2}}\left(\|\mathfrak{a}\|_{C^0_{\delta,g_{\omega_\epsilon}}} + \left\|D_{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}\mathfrak{a}\right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}}\right)$ on the metric balls $\left(B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_1}(p), B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p)\right) = \left(B^{g_{\omega_\epsilon}}_{\epsilon R_1}(p), B^{g_{\omega_\epsilon}}_{\epsilon R_2}(p)\right)$. Now Proposition 4.6.1(3) tells us that $D_{\frac{1}{\epsilon^2}g_{\omega_\epsilon}} = \left(\epsilon^2 d^*_{g_{\omega_\epsilon}}, d^+_{g_{\omega_\epsilon}}\right) = \left(\epsilon^2 d^*_{g_{\omega_\epsilon}}\right) \oplus d^+_{g_{\omega_\epsilon}}$, and since Data 3.4.4 tells us that $\epsilon^2 < \frac{1}{16} < 1$, we finally get our

sought after weighted Schauder estimate $\|\mathfrak{a}\|_{C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}} \leq \widetilde{C_{11.2}}\left(\|\mathfrak{a}\|_{C^0_{\delta,g_{\omega_\epsilon}}} + \left\|D_{g_{\omega_\epsilon}}\mathfrak{a}\right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}}\right)$ on the metric balls $\left(B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_1}(p), B^{\frac{1}{\epsilon^2}g_{\omega_\epsilon}}_{R_2}(p)\right) = \left(B^{g_{\omega_\epsilon}}_{\epsilon R_1}(p), B^{g_{\omega_\epsilon}}_{\epsilon R_2}(p)\right)$ upon setting $r_p := \epsilon R_1$ and $R_2 := 2R_1$.

(3) **Case 3;** $p \in \left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\}$: By Data 3.4.4, since $2\epsilon^{\frac{1}{2}} < \frac{3}{25}$, by construction of g_{ω_ϵ} (Proposition 3.3.3) we have that on this region $g_{\omega_\epsilon} = g_{\omega_0}$ the flat metric. This, combined with Proposition 3.6.2(2) which tells us that our weight function satisfies on this region the *uniform* estimate $\frac{3}{25} \leq \rho_\epsilon \leq 1$, the proof of the weighted Schauder estimate is exactly as in the first case, but this time using the identification $\left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\} \sim \left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{S} \right\} \subset \mathbb{T}^4 - S \subset \mathbb{T}^4$ from *Remark* 3.3.2 and without having to do any conformal rescaling.

In more detail, upon transferring everything over from $\left(\left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\}, g_{\omega_\epsilon} \right)$ to (the obviously precompact region) $\left(\left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{S} \right\}, g_{\omega_0} \right) \subset (\mathbb{T}^4, g_{\omega_0})$, and since flat tori clearly have bounded geometry, we thus automatically have the standard Schauder estimate $\|\mathfrak{a}\|_{C^{1,\alpha}_{g_{\omega_0}}} \leq C_{11.3} \left(\|\mathfrak{a}\|_{C^0_{g_{\omega_0}}} + \left\| D_{g_{\omega_0}} \mathfrak{a} \right\|_{C^{0,\alpha}_{g_{\omega_0}}} \right)$ on the metric balls $\left(B^{g_{\omega_0}}_{R_1}(p), B^{g_{\omega_0}}_{R_2}(p) \right)$ where $0 < R_1 < R_2$ is chosen such that $B^{g_{\omega_0}}_{R_2}(p) \subset \left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{S} \right\}$, and *where* $C_{11.3} > 0$ *is independent of* p. Undoing the identification $\left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\} \sim \left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{S} \right\}$ gives us that same estimate over $\left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\}$ on the metric balls $\left(B^{g_{\omega_0}}_{R_1}(p), B^{g_{\omega_0}}_{R_2}(p) \right)$ (which are clearly preserved under the identification). Since we have our uniform estimate $\frac{3}{25} \leq \rho_\epsilon \leq 1$ on $\left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\}$, we have on any subset of $\left\{ \frac{3}{25} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\}$ that the weighted Holder norms WRT ρ_ϵ and the standard Holder norms are *equivalent norms*, whence (up to changing the constant $C_{11.3}$ by harmless factors) we immediately get our sought after weighted Schauder estimate $\|\mathfrak{a}\|_{C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}} \leq \widetilde{C_{11.3}} \left(\|\mathfrak{a}\|_{C^0_{\delta,g_{\omega_\epsilon}}} + \left\| D_{g_{\omega_\epsilon}} \mathfrak{a} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}} \right)$ on the metric balls $\left(B^{g_{\omega_\epsilon}}_{R_1}(p), B^{g_{\omega_\epsilon}}_{R_2}(p) \right)$ upon setting $r_p := R_1$ and $R_2 := 2R_1$, with $\widetilde{C_{11.3}} > 0$ still independent of all parameters involved.

Setting $C_{11} := \max \left\{ \widetilde{C_{11.1}}, \widetilde{C_{11.2}}, \widetilde{C_{11.3}} \right\}$ and noting that these 3 constants are both *independent of* p as well as *independent of* $\epsilon > 0$ yields us our local weighted Schauder estimate $\forall p \in Km_\epsilon$, as was to be shown. ∎

Thus we get the analogue of Corollary 3.6.5 for $D_{g_{\omega_\epsilon}}$ whose proof is verbatim from before:

Corollary 4.6.3 (Weighted Schauder Estimate). **Let** $\epsilon > 0$ **be from Data** 3.4.4 **and sufficiently small**. We have the following estimate $\forall \mathfrak{a} \in C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}(T^* Km_\epsilon)$:

$$\|\mathfrak{a}\|_{C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}(T^* Km_\epsilon)} \leq C_{12} \left(\|\mathfrak{a}\|_{C^0_{\delta,g_{\omega_\epsilon}}(T^* Km_\epsilon)} + \left\| D_{g_{\omega_\epsilon}} \mathfrak{a} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}} \left(\mathbb{R} \oplus \Lambda^+_{g_{\omega_\epsilon}} T^* Km_\epsilon \right)} \right)$$

$$:= C_{12} \left(\left\| \rho_\epsilon^{-\delta} \mathfrak{a} \right\|_{C^0_{g_{\omega_\epsilon}}(T^* Km_\epsilon)} + \left\| D_{g_{\omega_\epsilon}} \mathfrak{a} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}} \left(\mathbb{R} \oplus \Lambda^+_{g_{\omega_\epsilon}} T^* Km_\epsilon \right)} \right)$$

for $C_{12} > 0$ independent of ϵ. ∎

Thus since we want to improve the above weighted Holder estimate for $D_{g_{\omega_\epsilon}}$ to uniform invertibility of $D_{g_{\omega_\epsilon}}$, the analogue of Proposition 3.6.6 is

Proposition 4.6.4 (Improved Weighted Schauder Estimate). **Let the rate/weight parameter satisfy** $\delta \in (-2, 0)$. **Let** $\epsilon > 0$ **be as from Data** 3.4.4.

Then for $\epsilon > 0$ ***sufficiently small*** there exists a constant $C_{13} > 0$ **independent of** $\epsilon > 0$ such that we have the following estimate for $D_{g_{\omega_\epsilon}} : C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}(T^* Km_\epsilon) \to C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}} \left(\mathbb{R} \oplus \Lambda^+_{g_{\omega_\epsilon}} T^* Km_\epsilon \right)$:

$$\|\mathfrak{a}\|_{C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}(T^* Km_\epsilon)} \leq C_{13} \left\| D_{g_{\omega_\epsilon}} \mathfrak{a} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}} \left(\mathbb{R} \oplus \Lambda^+_{g_{\omega_\epsilon}} T^* Km_\epsilon \right)}, \qquad \forall \mathfrak{a} \in C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}(T^* Km_\epsilon)$$

Remark 4.6.5. This inequality precisely means that when $\delta \in (-2, 0)$ and the gluing parameter $\epsilon > 0$ is sufficiently small, $D_{g_{\omega_\epsilon}} : C^{1,\alpha}_{\delta, g_{\omega_\epsilon}}(T^* \mathrm{Km}_\epsilon) \to C^{0,\alpha}_{\delta-1, g_{\omega_\epsilon}}\left(\mathbb{R} \oplus \Lambda^+_{g_{\omega_\epsilon}} T^* \mathrm{Km}_\epsilon\right)$ is injective, *uniformly* bounded below, and has *closed range*. ♣

Proof. Just like for Proposition 3.6.6, we prove this by contradiction via a "blow-up" analysis on each region $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 3\epsilon^{\frac{1}{2}}\right\}$ and $\left\{2\epsilon^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\}$ of the cover $\mathrm{Km}_\epsilon \subset \left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 3\epsilon^{\frac{1}{2}}\right\} \cup \left\{2\epsilon^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\}$. However, as we are dealing with 1-forms and with D_g, there are several differences that manifest in the analysis.

By our weighted Schauder estimate proved above (Corollary 4.6.3), we may instead prove

$$\|\mathfrak{a}\|_{C^0_{\delta, g_{\omega_\epsilon}}(T^* \mathrm{Km}_\epsilon)} \leq C_{14}\left\|D_{g_{\omega_\epsilon}} \mathfrak{a}\right\|_{C^{0,\alpha}_{\delta-1, g_{\omega_\epsilon}}\left(\mathbb{R}\oplus\Lambda^+_{g_{\omega_\epsilon}} T^* \mathrm{Km}_\epsilon\right)}$$

with the same assumptions ($\delta \in (-2, 0)$, etc.), whence $C_{13} := C_{12}(C_{14} + 1)$. Now assume for contradiction that the statement for this latter inequality fails. Then this implies by countable choice that $\exists \epsilon_i \in \left(0, \frac{1}{i}\right)$ whence $\exists \epsilon_i \searrow 0$ and $\exists \mathfrak{a}_i \in C^{1,\alpha}_{\delta, g_{\omega_{\epsilon_i}}}(T^* \mathrm{Km}_{\epsilon_i})$ a sequence such that (after normalizing) we get a tuple of equations

$$\|\mathfrak{a}_i\|_{C^0_{\delta, g_{\omega_{\epsilon_i}}}\left(T^* \mathrm{Km}_{\epsilon_i}\right)} = 1$$

$$\left\|D_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i\right\|_{C^{0,\alpha}_{\delta-1, g_{\omega_{\epsilon_i}}}\left(\mathbb{R}\oplus\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i}\right)} \leq \frac{1}{i}$$

Note that our weighted Schauder estimate Corollary 4.6.3 immediately gives us

$$\|\mathfrak{a}_i\|_{C^{1,\alpha}_{\delta, g_{\omega_{\epsilon_i}}}\left(T^* \mathrm{Km}_{\epsilon_i}\right)} \leq 2C_{12}$$

Let us now examine the following two regions: $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\} \subset \mathrm{Km}_{\epsilon_i}$ and $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\} \subset \mathrm{Km}_{\epsilon_i}$.

(1) **Case 1**; $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\}$: Firstly, from *Remark* 3.3.2 let us identify each region $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\} \subset \mathrm{Km}_{\epsilon_i}$ with $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\} \subset \mathbb{T}^4/\mathbb{Z}_2 - S$ (hence we're in a *fixed* ambient manifold). Similarly from that remark, let us denote the lift of $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\} \subset \mathbb{T}^4/\mathbb{Z}_2 - S$ to $\mathbb{T}^4 - S$ via the double cover as $\widehat{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\}} \subset \mathbb{T}^4 - S$, and denote the resulting lift of the metric $g_{\omega_{\epsilon_i}}$ as $\widehat{g_{\omega_{\epsilon_i}}}$.

Next, we note that by construction (Proposition 4.3.1) that each $g_{\omega_{\epsilon_i}} = g_{\omega_0}$ on each $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\}$ hence on $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\}$ and hence $\widehat{g_{\omega_{\epsilon_i}}} = g_{\omega_0}$ on $\widehat{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\}}$. Arguing as before, since both $\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\}$, $\widehat{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\}}$ are in *fixed* ambient manifolds $\mathbb{T}^4/\mathbb{Z}_2 - S$, $\mathbb{T}^4 - S$ respectively, since $\mathbb{Z}_2$ acts on $\mathbb{T}^4 - S$ and on each region $\widehat{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\}} \subset \mathbb{T}^4 - S$ in the same way, combined with the fact that our metric is precisely g_{ω_0} we may easily construct equivariant Gromov-Hausdorff ϵ-approximations for arbitrarily small ϵ[38] and see that we have the following equivariant Gromov-Hausdorff convergence as $i \nearrow \infty$:

$$
\begin{array}{ccc}
\left(\widehat{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\}}, \widehat{g_{\omega_{\epsilon_i}}}\right) & \xrightarrow{\ eqGH\ } & (\mathbb{T}^4 - S, g_{\omega_0}) \\
{\scriptstyle 2:1}\downarrow & & \downarrow{\scriptstyle 2:1} \\
\left(\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S}\right\}, g_{\omega_{\epsilon_i}}\right) & \xrightarrow{\ GH\ } & (\mathbb{T}^4/\mathbb{Z}_2 - S, g_{\omega_0})
\end{array}
$$

---

[38] In fact, just take $\epsilon = 3\epsilon_i^{\frac{1}{2}}$.

Abuse notation and denote by $\mathfrak{a}_i$ to mean *both* the restriction of each $\mathfrak{a}_i \in C^{1,\alpha}_{\delta,g_{\omega\epsilon_i}} \left(T^* \mathrm{Km}_{\epsilon_i} \right)$ to $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\}$ as well as the transferal of $\mathfrak{a}_i$ to $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S} \right\} \subset \mathbb{T}^4/\mathbb{Z}_2 - S$, and denote the lift of each $\mathfrak{a}_i$ to $\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S} \right\}$ as $\widehat{\mathfrak{a}_i}$. Hence $\widehat{\mathfrak{a}_i}$ is even/$\mathbb{Z}_2$-invariant by Proposition 3.2.1.

Now since we have $\|\mathfrak{a}_i\|_{C^{1,\alpha}_{\delta,g_{\omega\epsilon_i}} \left(T^* \mathrm{Km}_{\epsilon_i} \right)} \leq 2C_{12}$, we thus get $\|\mathfrak{a}_i\|_{C^{1,\alpha}_{\delta,g_{\omega\epsilon_i}} \left(\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S} \right\} \right)} \leq 2C_{12}$ and hence the same bound for each $\widehat{\mathfrak{a}_i}$, namely

$$\|\widehat{\mathfrak{a}_i}\|_{C^{1,\alpha}_{\delta,\widehat{g_{\omega\epsilon_i}}} \left(\widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S} \right\}} \right)} \leq 2C_{12}$$

Therefore, by Arzela-Ascoli, upon relabeling the original sequence of $\mathfrak{a}_i$ we have that

$$\widehat{\mathfrak{a}_i} \xrightarrow{C^{1,\alpha'}_{\delta,\widehat{g_{\omega\epsilon_i}}},loc} \widehat{\mathfrak{a}_\infty}$$
$$\|\widehat{\mathfrak{a}_\infty}\|_{C^{1,\alpha'}_{\delta,g_{\omega_0}} (\mathbb{T}^4 - S)} \leq 2C_2$$

with $0 < \alpha' < \alpha < 1$, for some 1-form $\widehat{\mathfrak{a}_\infty}$ on $\mathbb{T}^4 - S$. Here, the *loc* convergence means on compact subsets. The second bound holds everywhere on $\mathbb{T}^4 - S$ via the usual unit diameter covering argument. Moreover, by the equivariant Gromov-Hausdorff convergence, we have that $\widehat{\mathfrak{a}_\infty}$ is even/$\mathbb{Z}_2$-invariant and hence descends down to a 1-form $\mathfrak{a}_\infty$ on $\mathbb{T}^4/\mathbb{Z}_2 - S$ by Proposition 3.2.1. Moreover, the weight function used in $C^{1,\alpha'}_{\delta,g_{\omega_0}} (\mathbb{T}^4 - S)$ is precisely the limiting weight function, which (as before) is ρ_0 the weight function used to define $C^{k,\alpha}_{\delta,g_{\omega_0}} (\mathbb{T}^4 - S)$ back in Sections 3.4 and 4.4.

Playing the same game with $\left\| D_{g_{\omega\epsilon_i}} \mathfrak{a}_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega\epsilon_i}} \left(\mathbb{R} \oplus \Lambda^+_{g_{\omega\epsilon_i}} T^* \mathrm{Km}_{\epsilon_i} \right)} \leq \frac{1}{i}$, namely that this bound directly implies the bounds $\left\| D_{g_{\omega\epsilon_i}} \mathfrak{a}_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega\epsilon_i}} \left(\mathbb{R} \oplus \Lambda^+_{g_{\omega\epsilon_i}} T^* \left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S} \right\} \right)} \leq \frac{1}{i}$ and $\left\| D_{\widehat{g_{\omega\epsilon_i}}} \widehat{\mathfrak{a}_i} \right\|_{C^{0,\alpha}_{\delta-1,\widehat{g_{\omega\epsilon_i}}} \left(\mathbb{R} \oplus \Lambda^+_{g_{\omega\epsilon_i}} T^* \widehat{\left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g_{\omega_0}}_{S} \right\}} \right)} \leq \frac{1}{i}$, that we may take the limit as $i \nearrow \infty$ and get

$$D_{g_{\omega_0}} \widehat{\mathfrak{a}_\infty} = 0$$

on $\left(\mathbb{T}^4 - S, g_{\omega_0} \right)$ (since $D_{g_{\omega_0}} \widehat{\mathfrak{a}_\infty} = 0$ on all compact subsets of $\mathbb{T}^4 - S$, hence on all of $\mathbb{T}^4 - S$). Now as explained back in Sections 3.4 and 4.4, $\|\widehat{\mathfrak{a}_\infty}\|_{C^{1,\alpha'}_{\delta,g_{\omega_0}} (\mathbb{T}^4 - S)} := \|\widehat{\mathfrak{a}_\infty}\|_{C^0_{\delta,g_{\omega_0}} (\mathbb{T}^4 - S)} + \left\| \nabla_{g_{\omega_0}} \widehat{\mathfrak{a}_\infty} \right\|_{C^0_{\delta,g_{\omega_0}} (\mathbb{T}^4 - S)} + [\widehat{\mathfrak{a}_\infty}]_{C^{1,\alpha'}_{\delta,g_{\omega_0}} (\mathbb{T}^4 - S)} \leq 2C_2 < \infty$ for $0 < \alpha' < \alpha < 1$ means that $\widehat{\mathfrak{a}_\infty}$ satisfies the decay

$$\nabla^j_{g_{\omega_0}} \widehat{\mathfrak{a}_\infty} = O_{g_{\omega_0}} \left(\rho_0^{\delta-j} \right), \qquad \forall j \leq 1$$

That is, by the above properties of ρ_0, $\widehat{\mathfrak{a}_\infty}$ is bounded on $\left\{ \frac{1}{8} \leq |\cdot|^{g_{\omega_0}}_{S} \right\}$ and satisfies

$$\left| \nabla^j_{g_{\omega_0}} \widehat{\mathfrak{a}_\infty} \right|_{g_{\omega_0}} \leq C_{15,j} \left(|\cdot|^{g_{\omega_0}}_{S} \right)^{\delta-j}, \qquad \forall j \leq 1$$

on $\left\{ 0 < |\cdot|^{g_{\omega_0}}_{S} \leq \frac{1}{8} \right\} \subset \mathbb{T}^4 - S$ for some $C_{15,j} > 0$.

Now since $\mathbb{T}^4 := \mathbb{R}^4/\Lambda$ where $\Lambda \cong \mathbb{Z}^4$, since $\mathbb{R}^4$ has 4 **nowhere vanishing parallel** 1-**forms** dx^1, dx^2, dx^3, dx^4, namely the **global coordinate coordinate coframe**, which are parallel WRT g_{ω_0} and since all flat metrics on $\mathbb{T}^4$ come from flat metrics descending upstairs from $\mathbb{R}^4$, we have that $\mathbb{T}^4$ is **parallelizable** and we may globally trivialize $T^*\mathbb{T}^4$ with 4 nowhere vanishing parallel 1-forms $d\theta^1, d\theta^2, d\theta^3, d\theta^4$, namely the descendants of dx^1, dx^2, dx^3, dx^4. Hence $\theta^1, \theta^2, \theta^3, \theta^4$ are the 4 angular coordinate functions of $\mathbb{T}^4 \cong S^1 \times S^1 \times S^1 \times S^1$. Hence

since the Levi-Civita connection and the Hodge star of a fixed metric commute, we have that $d\theta^1, d\theta^2, d\theta^3, d\theta^4$ give us a global trivialization of $T^*\mathbb{T}^4$ by *harmonic* 1-forms. Thus since Proposition 4.1.8 tells us that

$$D_{g_{\omega_0}}\widehat{\mathfrak{a}_\infty} = 0 \implies \Delta_{g_{\omega_0}}\widehat{\mathfrak{a}_\infty} = 0$$

upon writing

$$\widehat{\mathfrak{a}_\infty} = \widehat{f_1}d\theta^1 + \widehat{f_2}d\theta^2 + \widehat{f_3}d\theta^3 + \widehat{f_4}d\theta^4$$

for f_1, f_2, f_3, f_4 coefficient *functions* on $\mathbb{T}^4 - S$, that harmonicity of each $d\theta^i$ WRT g_{ω_0} and $\Delta_{g_{\omega_0}}\widehat{\mathfrak{a}_\infty} = 0$ forces

$$\Delta_{g_{\omega_0}}f_1 = 0$$
$$\Delta_{g_{\omega_0}}f_2 = 0$$
$$\Delta_{g_{\omega_0}}f_3 = 0$$
$$\Delta_{g_{\omega_0}}f_4 = 0$$

In other words, f_1, f_2, f_3, f_4 are four harmonic *functions* on $\mathbb{T}^4 - S$. Moreover, since $\widehat{\mathfrak{a}_\infty}$ is bounded on $\left\{\frac{1}{8} \le |\cdot|_S^{g_{\omega_0}}\right\}$ and satisfies $\left|\nabla^j_{g_{\omega_0}}\widehat{\mathfrak{a}_\infty}\right|_{g_{\omega_0}} \le C_{15,j}\left(|\cdot|_S^{g_{\omega_0}}\right)^{\delta-j}, \forall j \le 1$ on $\left\{0 < |\cdot|_S^{g_{\omega_0}} \le \frac{1}{8}\right\} \subset \mathbb{T}^4 - S$, this means that *the four coefficient functions satisfy these very same bounds as well.*

Therefore we may play the same game as in the proof of Proposition 3.6.6 (namely elliptic regularity yielding each $f_i \in C^\infty(\mathbb{T}^4 - S)$, then Laurent series expansion near S, $-2 < \delta$ with the 2nd bound near S implying removable singularities near S, then the above bounds away from & near S implying each f_i is a bounded), yielding that each of the four coefficient functions f_1, f_2, f_3, f_4 of $\widehat{\mathfrak{a}_\infty}$ are **constant** by maximum principle. Thus we have that $\widehat{\mathfrak{a}_\infty} \in \Omega^1(\mathbb{T}^4)$ is just a constant 1-form (hence smooth).

Now recall that $\widehat{\mathfrak{a}_\infty} = f_1 d\theta^1 + f_2 d\theta^2 + f_3 d\theta^3 + f_4 d\theta^4$ is **even/$\mathbb{Z}_2$-invariant** and (upon restricting to $\mathbb{T}^4 - S$) descends down to a 1-form $\mathfrak{a}_\infty$ on $\mathbb{T}^4/\mathbb{Z}_2 - S$. But our involution $\mathbb{Z}^2 \circlearrowright \mathbb{T}^4$ acts via $x \mapsto -x$, and **since each $d\theta^i$ is a coordinate 1-form for the angular/S^1 coordinate θ^i**, we have that *each $d\theta^i$ is odd WRT $\mathbb{Z}_2$,* aka satisfies $d\theta^i\big|_{-x} = -d\theta^i\big|_x$ since the involution acts via $\theta^i \mapsto -\theta^i$. Thus to preserve even-ness/$\mathbb{Z}_2$-invariance of $\widehat{\mathfrak{a}_\infty}$, we must conclude that each coefficient function f_i is **odd** WRT $\mathbb{Z}_2$, that is, $f_i(-x) = -f_i(x)$. But each $f_i \in \mathbb{R}$ is a *constant*, whence odd-ness forces $f_1 = f_2 = f_3 = f_4 = 0$.

Whence we have concluded that $\widehat{\mathfrak{a}_\infty} = 0$.

Therefore, we have that $\|\widehat{\mathfrak{a}_i}\|_{C^{1,\alpha'}_{\delta,\widehat{g_{\omega_{\epsilon_i}}}}(K)} \searrow 0$ for all compact subsets $K \Subset \overline{\left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|_S^{g_{\omega_0}}\right\}} \subset \mathbb{T}^4 - S$, whence transferring everything back to Km_{ϵ_i} and using $\|\cdot\|_{C^0_{\delta,g_{\omega_{\epsilon_i}}}} \le \|\cdot\|_{C^{1,\alpha'}_{\delta,g_{\omega_{\epsilon_i}}}}$, we thus have that

$$\|\mathfrak{a}_i\|_{C^0_{\delta,g_{\omega_{\epsilon_i}}}(K)} \searrow 0$$

holds for all compact subsets $K \Subset \left\{2\epsilon_i^{\frac{1}{2}} < |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}}\right\} \subset \mathrm{Km}_{\epsilon_i}$.

(2) **Case 2;** $\left\{|\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} < 3\epsilon_i^{\frac{1}{2}}\right\}$: By construction, we have that on $\left\{|\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} < 3\epsilon_i^{\frac{1}{2}}\right\}$ that $g_{\omega_{\epsilon_i}} = g_{\widehat{\omega_{EH,\epsilon_i}}}$. Moreover, by Proposition 4.2.3 we have that $g_{\widehat{\omega_{EH,\epsilon_i}}} = \epsilon_i^2 R^*_{\frac{1}{\epsilon_i}}\left(g_{\widehat{\omega_{EH-0,\epsilon_i}}}\right)$.

Therefore, just as in proving **Case 2** of Proposition 4.6.2, upon performing a conformal rescaling $g_{\omega_{\epsilon_i}} \mapsto \frac{1}{\epsilon_i^2}g_{\omega_{\epsilon_i}}$, we have that each Riemannian manifold $\left(\left\{|\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} < 3\epsilon_i^{\frac{1}{2}}\right\}, \frac{1}{\epsilon_i^2}g_{\omega_{\epsilon_i}}\right)$ is isometric under R_{ϵ_i} to the Riemannian submanifold $\left(\left\{|\cdot|_{S^2}^{g_{\omega_0}} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\}, g_{\widehat{\omega_{EH-0,\epsilon_i}}}\right) \subset \left(T^*S^2, g_{\widehat{\omega_{EH-0,\epsilon_i}}}\right)$.

Hence upon taking the limit as $i \nearrow \infty$, we have by Proposition 4.2.2 that $\left(\left\{|\cdot|^{g_{\omega_0}}_{S^2} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\}, g_{\widehat{\omega EH-0,\epsilon_i}}\right)$ exhausts

to $\left(T^*S^2, g_{EH,1}\right)$ in the limit. More precisely, upon picking a point $p_i \in \left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\}$ for each i, we have the following convergence in the pointed Gromov-Hausdorff topology:

$$\left(\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\}, \frac{1}{\epsilon_i^2}g_{\omega_{\epsilon_i}}, p_i\right) \xrightarrow{pGH} \left(T^*S^2, g_{\omega_{EH,1}}, p_\infty\right)$$

Transferring over the weight function ρ_{ϵ_i} on $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\}$ via the above isometry gives us $R^*_{\epsilon_i}\left(\rho_{\epsilon_i}\right)$

which satisfies $R^*_{\epsilon_i}\left(\frac{1}{\epsilon_i}\rho_{\epsilon_i}\right) = \begin{cases} \frac{1}{\epsilon_i} & \text{on } \left\{\frac{1}{5\epsilon_i} \leq |\cdot|^{g_{\omega_0}}_{S^2}\right\} \\ |\cdot|^{g_{\omega_0}}_{S^2} & \text{on } \left\{2 \leq |\cdot|^{g_{\omega_0}}_{S^2} \leq \frac{1}{8\epsilon_i^{\frac{1}{2}}}\right\} \\ 1 & \text{on } \left\{|\cdot|^{g_{\omega_0}}_{S^2} \leq 1\right\} \end{cases}$. Call $\widetilde{\rho_{\epsilon_i}} := R^*_{\epsilon_i}\left(\frac{1}{\epsilon_i}\rho_{\epsilon_i}\right)$ for notation.

Therefore, as $\|\mathfrak{a}_i\|_{C^0_{\delta,g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} := \left\|\rho_{\epsilon_i}^{-\delta}\mathfrak{a}_i\right\|_{C^0_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} = \left\|\rho_{\epsilon_i}^{-\delta}\epsilon_i^\delta\epsilon_i^{-\delta}\mathfrak{a}_i\right\|_{C^0_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})}$, we must also scale our

1-forms $\mathfrak{a}_i \mapsto \epsilon_i^{-\delta}\mathfrak{a}_i$ to preserve the weighted Holder norm upon transferring to $\left(\left\{|\cdot|^{g_{\omega_0}}_{S^2} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\}, g_{\widehat{\omega EH-0,\epsilon_i}}\right)$

(after first restricting to $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\}$). Thus call $\widetilde{\mathfrak{a}}_i := \epsilon_i^{-\delta}\mathfrak{a}_i$ for notation.

Therefore $\|\mathfrak{a}_i\|_{C^{1,\alpha}_{\delta,g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} \leq 2C_{12}$ implies that we have the uniform bound $\left\|\widetilde{\mathfrak{a}}_i\right\|_{C^{1,\alpha}_{\delta,g_{\widehat{\omega EH-0,\epsilon_i}}}\left(\left\{|\cdot|^{g_{\omega_0}}_{S^2} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\}\right)} \leq$

$2C_{12}$ where the weight function used is precisely $\widetilde{\rho_{\epsilon_i}}$. Whence since $T^*S^2 = \bigcup_{i\in\mathbb{N}}\left\{|\cdot|^{g_{\omega_0}}_{S^2} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\}$ is a nested

exhaustion of precompact subsets, we therefore have by Arzela-Ascoli that (upon relabeling the original sequence of $\mathfrak{a}_i$ once more)

$$\widetilde{\mathfrak{a}}_i \xrightarrow{C^{1,\alpha'}_{\delta,g_{\widehat{\omega EH-0,\epsilon_i}},loc}} \widetilde{\mathfrak{a}}_\infty$$
$$\left\|\widetilde{\mathfrak{a}}_\infty\right\|_{C^{1,\alpha'}_{\delta,g_{\omega_{EH,1}}}(T^*S^2)} \leq 2C_{12}$$

with $0 < \alpha' < \alpha < 1$, for some 1-form $\widetilde{\mathfrak{a}}_\infty$ on T^*S^2 and where the convergence $\xrightarrow{C^{1,\alpha'}_{\delta,g_{\widehat{\omega EH-0,\epsilon_i}},loc}}$ means convergence on compact subsets, and with $\left\|\widetilde{\mathfrak{a}}_\infty\right\|_{C^{1,\alpha'}_{\delta,g_{\omega_{EH,1}}}(T^*S^2)} \leq 2C_{12}$ following upon restricting to unit diameter compact subsets. Moreover, the weight function used in $C^{1,\alpha'}_{\delta,g_{\omega_{EH,1}}}\left(T^*S^2\right)$ is precisely the limiting weight function, which (as before) is $\widetilde{\rho_0}$ the weight function used to define the weighted Holder spaces for T^*S^2 back in Sections 3.4 and 4.4.

Playing the same game with $\left\|D_{g_{\omega_{\epsilon_i}}}\mathfrak{a}_i\right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} \leq \frac{1}{i}$, namely that this bound directly implies the bounds

$\left\|D_{g_{\omega_{\epsilon_i}}}\mathfrak{a}_i\right\|_{C^{0,\alpha}_{\delta-1,g_{\widehat{\omega EH-0,\epsilon_i}}}\left(\left\{|\cdot|^{g_{\omega_0}}_{S^2} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\}\right)} \leq \frac{1}{i}$, that we may take the limit as $i \nearrow \infty$ and get

$$D_{g_{\omega_{EH,1}}}\widetilde{\mathfrak{a}}_\infty = 0$$

on $\left(T^*S^2, g_{\omega_{EH,1}}\right)$ (since $D_{g_{\omega_{EH,1}}}\widetilde{\mathfrak{a}_\infty} = 0$ on all compact subsets of T^*S^2, hence on all of T^*S^2 by a compact exhaustion), hence by Proposition 4.1.8 we get

$$\Delta_{g_{\omega_{EH,1}}}\widetilde{\mathfrak{a}_\infty} = 0$$

and thus by elliptic regularity $\widetilde{\mathfrak{a}_\infty} \in \Omega^1\left(T^*S^2\right)$ is a smooth 1-form as it is harmonic.

Now $\left\|\widetilde{\mathfrak{a}_\infty}\right\|_{C^{1,\alpha'}_{\delta,g_{\omega_{EH,1}}}(T^*S^2)} \leq 2C_{12}$ means that $\widetilde{\mathfrak{a}_\infty}$ satisfies the decay

$$\nabla^j_{g_{\omega_{EH,1}}}\widetilde{\mathfrak{a}_\infty} = O_{g_{\omega_{EH,1}}}\left(\widetilde{\rho}_0^{\delta-j}\right), \qquad \forall j \leq 1$$

hence by the properties of $\widetilde{\rho}_0$, we have that $\left|\widetilde{\mathfrak{a}_\infty}\right|_{g_{\omega_{EH,1}}}$ hence $\widetilde{\mathfrak{a}_\infty}$ is bounded *WRT the norm induced from* $g_{\omega_{EH,1}}$ on $\left\{|\cdot|^{g_{\omega_0}}_{S^2} \leq 2\right\}$ and satisfies

$$\left|\nabla^j_{g_{\omega_{EH,1}}}\widetilde{\mathfrak{a}_\infty}\right|_{g_{\omega_{EH,1}}} \leq C_{16,j}\left(|\cdot|^{g_{\omega_0}}_{S^2}\right)^{\delta-j}, \qquad \forall j \leq 1$$

on $\left\{2 \leq |\cdot|^{g_{\omega_0}}_{S^2}\right\}$ for some $C_{16,j} > 0$.

But since $\delta < 0$, we have that

$$\left|\widetilde{\mathfrak{a}_\infty}\right|_{g_{\omega_{EH,1}}} \leq C_{6,0}\left(|\cdot|^{g_{\omega_0}}_{S^2}\right)^{\delta} \leq C_{16,0}2^{\delta} < \infty$$

on $\left\{2 \leq |\cdot|^{g_{\omega_0}}_{S^2}\right\}$, and hence $\left|\widetilde{\mathfrak{a}_\infty}\right|_{g_{\omega_{EH,1}}}$ is bounded on all of T^*S^2. Moreover, $\left|\widetilde{\mathfrak{a}_\infty}\right|_{g_{\omega_{EH,1}}} \leq C_{16,0}\left(|\cdot|^{g_{\omega_0}}_{S^2}\right)^{\delta}$ and $\delta < 0$ means that, *since $\widetilde{\mathfrak{a}_\infty}$ is a 1-form*, $\left|\widetilde{\mathfrak{a}_\infty}\right|_{g_{\omega_{EH,1}}} \searrow 0$ as $|\cdot|^{g_{\omega_0}}_{S^2} \nearrow \infty$.

Thus we have established that $\left|\widetilde{\mathfrak{a}_\infty}\right|_{g_{\omega_{EH,1}}}$, hence $\left|\widetilde{\mathfrak{a}_\infty}\right|^2_{g_{\omega_{EH,1}}}$, is bounded on all of T^*S^2 and $\searrow 0$ as $|\cdot|^{g_{\omega_0}}_{S^2} \nearrow \infty$.

Now from the **Bochner-Weitzenböck formula for 1 forms** $-\frac{1}{2}\Delta_g|\alpha|^2_g = \left|\nabla_g\alpha\right|^2_g - \left\langle\Delta_g\alpha, \alpha\right\rangle_g + \mathrm{Ric}_g(\alpha, \alpha)$, since the Eguchi-Hanson space $\left(T^*S^2, g_{\omega_{EH,1}}\right)$ is Ricci-flat, we get from $\Delta_{g_{\omega_{EH,1}}}\widetilde{\mathfrak{a}_\infty} = 0$/harmonicity of $\widetilde{\mathfrak{a}_\infty}$ that

$$-\frac{1}{2}\Delta_{g_{\omega_{EH,1}}}\left|\widetilde{\mathfrak{a}_\infty}\right|^2_{g_{\omega_{EH,1}}} = \left|\nabla_{g_{\omega_{EH,1}}}\widetilde{\mathfrak{a}_\infty}\right|^2_{g_{\omega_{EH,1}}} \geq 0$$

That is, $\left|\widetilde{\mathfrak{a}_\infty}\right|^2_{g_{\omega_{EH,1}}}$ is **subharmonic** on T^*S^2 (recall that Δ_g has *nonnegative spectrum*), and since $\left|\widetilde{\mathfrak{a}_\infty}\right|^2_{g_{\omega_{EH,1}}}$ is bounded on all of T^*S^2 it therefore has a global maximum whence by the maximum principle $\left|\widetilde{\mathfrak{a}_\infty}\right|^2_{g_{\omega_{EH,1}}}$ is **constant**, and since $\left|\widetilde{\mathfrak{a}_\infty}\right|^2_{g_{\omega_{EH,1}}} \searrow 0$ as $|\cdot|^{g_{\omega_0}}_{S^2} \nearrow \infty$, we therefore conclude

$$\left|\widetilde{\mathfrak{a}_\infty}\right|^2_{g_{\omega_{EH,1}}} = 0 \iff \widetilde{\mathfrak{a}_\infty} = 0$$

Therefore, we have that $\left\|\widetilde{\mathfrak{a}_i}\right\|_{C^{1,\alpha'}_{\delta,g_{\widetilde{\omega_{EH-0,\epsilon_i}}}}(K)} \searrow 0$, for all compact subsets $K \Subset \left\{|\cdot|^{g_{\omega_0}}_{S^2} < \frac{3}{\epsilon_i^{\frac{1}{2}}}\right\} \subset T^*S^2$. Whence transferring everything back to Km_{ϵ_i} and using $\|\cdot\|_{C^0_{\delta,g_{\omega_{\epsilon_i}}}} \leq \|\cdot\|_{C^{1,\alpha'}_{\delta,g_{\omega_{\epsilon_i}}}}$, we thus have that

$$\|\mathfrak{a}_i\|_{C^0_{\delta,g_{\omega_{\epsilon_i}}}} \searrow 0$$

holds for all compact subsets $K \Subset \left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}}\right\} \subset \mathrm{Km}_{\epsilon_i}$.

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Hence we have shown that

$$\|\mathfrak{a}_i\|_{C^0_{\delta,g\omega_{\epsilon_i}}(K_1)} \searrow 0$$

$$\|\mathfrak{a}_i\|_{C^0_{\delta,g\omega_{\epsilon_i}}(K_2)} \searrow 0$$

$\forall K_1 \Subset \left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g\omega_0}_{\pi^{-1}(S)} \right\}$ and $\forall K_2 \Subset \left\{ |\cdot|^{g\omega_0}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}} \right\}$ compact. Whence since $\mathrm{Km}_{\epsilon_i} \subset \left\{ |\cdot|^{g\omega_0}_{\pi^{-1}(S)} < 3\epsilon_i^{\frac{1}{2}} \right\} \cup \left\{ 2\epsilon_i^{\frac{1}{2}} < |\cdot|^{g\omega_0}_{\pi^{-1}(S)} \right\}$, we therefore have that

$$\|\mathfrak{a}_i\|_{C^0_{\delta,g\omega_{\epsilon_i}}(K)} \searrow 0$$

on all compact subsets $K \Subset \mathrm{Km}_{\epsilon_i}$, particularly $K = \mathrm{Km}_{\epsilon_i}$. But this contradicts the fact that each $\|\mathfrak{a}_i\|_{C^0_{\delta,g\omega_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i})} = 1$. Whence we have reached a contradiction to which the proof follows, as was to be shown. ∎

With these in hand, we may finally prove

Theorem 4.6.6. *For $\delta \in (-1,0)$ **and** $\epsilon > 0$ from Data 3.4.4 **sufficiently small**, the following is a **bounded linear isomorphism**:*

$$D_0\Phi_\epsilon : \left(C^{1,\alpha}_{\delta,g\omega_\epsilon}\left(\mathring{\Omega}^1_{g\omega_\epsilon}(\mathrm{Km}_\epsilon) \right) \oplus \mathcal{H}^+_{g\omega_\epsilon} \right) \otimes \mathbb{R}^3 \to C^{0,\alpha}_{\delta-1,g\omega_\epsilon}\left(\Lambda^+_{g\omega_\epsilon} T^* \mathrm{Km}_\epsilon \right) \otimes \mathbb{R}^3$$

where $D_0\Phi_\epsilon = \left(d^+_{g\omega_\epsilon} \oplus \mathrm{id}_{\mathcal{H}^+_{g\omega_\epsilon}} \right) \otimes \mathbb{R}^3$.

*More crucially, we have that the operator norm of the inverse is bounded by $\mathcal{L} > 0$ a constant **independent of** $\epsilon > 0$:*

$$\left\| (D_0\Phi_\epsilon)^{-1} \right\| \leq \mathcal{L}$$

That is, under these conditions,

$$\forall t \in C^{0,\alpha}_{\delta-1,g\omega_\epsilon}\left(\Lambda^+_{g\omega_\epsilon} T^* \mathrm{Km}_\epsilon \right) \otimes \mathbb{R}^3$$

there exists a unique pair

$$(\boldsymbol{a}, \boldsymbol{\zeta}) \in \left(C^{1,\alpha}_{\delta,g\omega_\epsilon}\left(\mathring{\Omega}^1_{g\omega_\epsilon}(\mathrm{Km}_\epsilon) \right) \oplus \mathcal{H}^+_{g\omega_\epsilon} \right) \otimes \mathbb{R}^3$$

such that

$$D_0\Phi_\epsilon(\boldsymbol{a}, \boldsymbol{\zeta}) = d^+_{g\omega_\epsilon}\boldsymbol{a} + \boldsymbol{\zeta} = \boldsymbol{t}$$

and

$$\|\boldsymbol{a}\|_{C^{1,\alpha}_{\delta,g\omega_\epsilon}\left(\mathring{\Omega}^1_{g\omega_\epsilon}(\mathrm{Km}_\epsilon) \right)\otimes\mathbb{R}^3} + \|\boldsymbol{\zeta}\|_{\mathcal{H}^+_{g\omega_\epsilon}\otimes\mathbb{R}^3} \leq \mathcal{L}\|\boldsymbol{t}\|_{C^{0,\alpha}_{\delta-1,g\omega_\epsilon}\left(\Lambda^+_{g\omega_\epsilon} T^* \mathrm{Km}_\epsilon \right)\otimes\mathbb{R}^3}$$

◆

Proof. Firstly, the existence of a unique $(\boldsymbol{a}, \boldsymbol{\zeta})$ for any $\boldsymbol{t}$ is clear since the linearization $D_0\Phi_\epsilon$ is always an isomorphism hence bijective. We now just need to show the uniform operator norm estimate for $(D_0\Phi_\epsilon)^{-1}$. WLOG we prove this on the level of forms, whence by *Remark 4.4.2* the estimate follows on 3-tuples (up to changing the constant).

So suppose we do not have a uniform constant $\mathcal{L}_1 > 0$ such that

$$\|\mathfrak{a}\|_{C^{1,\alpha}_{\delta,g\omega_\epsilon}\left(\mathring{\Omega}^1_{g\omega_\epsilon}(\mathrm{Km}_\epsilon) \right)} + \|\zeta\|_{L^2_{g\omega_\epsilon}} \leq \mathcal{L}_1\|\mathfrak{t}\|_{C^{0,\alpha}_{\delta-1,g\omega_\epsilon}\left(\Lambda^+_{g\omega_\epsilon} T^* \mathrm{Km}_\epsilon \right)}$$

for all $\mathfrak{t}$ such that $d^+_{g\omega_\epsilon}\mathfrak{a} + \zeta = \mathfrak{t}$. Then this implies by countable choice that $\exists \epsilon_i \in \left(0, \frac{1}{i}\right)$ whence $\exists \epsilon_i \searrow 0$ and $\exists (\mathfrak{a}_i, \zeta_i) \in C^{1,\alpha}_{\delta,g\omega_{\epsilon_i}}\left(T^* \mathring{\mathrm{Km}}_{\epsilon_i} \right) \oplus \mathcal{H}^+_{g\omega_{\epsilon_i}}$ a sequence such that (after normalizing) we get a tuple of equations

$$\|\mathfrak{a}_i\|_{C^{1,\alpha}_{\delta,g\omega_{\epsilon_i}}\left(\mathring{\Omega}^1_{g\omega_{\epsilon_i}}(\mathrm{Km}_{\epsilon_i}) \right)} + \|\zeta_i\|_{L^2_{g\omega_{\epsilon_i}}} = 1$$

$$\left\| d^+_{g\omega_{\epsilon_i}}\mathfrak{a}_i + \zeta_i \right\|_{C^{0,\alpha}_{\delta-1,g\omega_{\epsilon_i}}\left(\Lambda^+_{g\omega_{\epsilon_i}} T^* \mathrm{Km}_{\epsilon_i} \right)} \leq \frac{1}{i}$$

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Thus as $i \nearrow \infty$ we clearly have that

$$\left\| d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)} \searrow 0$$

Now since each Km_{ϵ_i} is *closed*, being harmonic is equivalent to being closed & co-closed by integration by parts. Whence using integration by parts and the fact that each ζ_i is both self-dual and harmonic, we have that
$\left\langle d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i, \zeta_i \right\rangle_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} := \int_{\mathrm{Km}_{\epsilon_i}} \left(d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i \right) \wedge *_{g_{\omega_{\epsilon_i}}} \zeta_i = \int_{\mathrm{Km}_{\epsilon_i}} \left(d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i \right) \wedge \zeta_i = \| \zeta_i \|^2_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})}$. Now since
$\left\| d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)} \leq \frac{1}{i}$, recalling the definition of the $\| \cdot \|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}}}$ -norm we have that

$$\left| d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i \right|_{g_{\omega_{\epsilon_i}}} \leq C_{17} \frac{1}{i} \rho_\epsilon^{\delta-1}$$

Therefore by Cauchy-Schwartz applied to the *pointwise* Hodge-inner product and Holder's inequality for L^2_g,

$$
\begin{aligned}
0 \leq \| \zeta_i \|^2_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} &= \left\langle d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i, \zeta_i \right\rangle_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} \\
&= \int_{\mathrm{Km}_{\epsilon_i}} \left(d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i \right) \wedge \zeta_i \\
&= \int_{\mathrm{Km}_{\epsilon_i}} \left(d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i, \zeta_i \right)_{g_{\omega_\epsilon}} dV_{g_{\omega_{\epsilon_i}}} \\
&\leq C_{17} \frac{1}{i} \int_{\mathrm{Km}_{\epsilon_i}} \rho_{\epsilon_i}^{\delta-1} |\zeta_i|_{g_{\omega_{\epsilon_i}}} dV_{g_{\omega_{\epsilon_i}}} \\
&\leq C_{17} \frac{1}{i} \| \zeta_i \|_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} \left(\int_{\mathrm{Km}_{\epsilon_i}} \rho_{\epsilon_i}^{2\delta-2} dV_{g_{\omega_{\epsilon_i}}} \right)^{\frac{1}{2}} \\
&= C_{17} \frac{1}{i} \| \zeta_i \|_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} \left\| \rho_{\epsilon_i}^{\delta-1} \right\|_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})}
\end{aligned}
$$

Now recall that $g_{\omega_{\epsilon_i}}$ is Ricci-flat on $\left\{ |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq \epsilon_i^{\frac{1}{2}} \right\}$ and that $\left(\mathrm{Km}_{\epsilon_i}, g_{\omega_{\epsilon_i}} \right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, g_{\omega_0} \right)$ is volume non-collapsed (*Remark* 4.3.2). By Colding's Volume Convergence Theorem (Footnote 28) on $\left\{ |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq \epsilon_i^{\frac{1}{2}} \right\}$ and directly inspecting $dV_{g_{\omega_{\epsilon_i}}} = dV_{g_{\omega_0}} + O_{g_{\omega_0}}(\epsilon_i^2)$ on $\left\{ \epsilon_i^{\frac{1}{2}} \leq |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \right\}$ (or just directly applying Colding's Volume Convergence on all of Km_{ϵ_i} since we have a global Ricci lower bound from *Remark* 4.3.3), we have that

$$\left\| \rho_{\epsilon_i}^{\delta-1} \right\|^2_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} = \int_{\mathrm{Km}_{\epsilon_i}} \rho_{\epsilon_i}^{2\delta-2} dV_{g_{\omega_{\epsilon_i}}} \to \int_{\mathbb{T}^4/\mathbb{Z}_2} \rho_0^{2\delta-2} dV_{g_{\omega_0}} = \int_{\mathbb{T}^4/\mathbb{Z}_2 - S} \rho_0^{2\delta-2} dV_{g_{\omega_0}} = \frac{1}{2} \int_{\mathbb{T}^4 - S} \rho_0^{2\delta-2} dV_{g_{\omega_0}}$$

since S has measure zero and $\lim_{i \nearrow \infty} \rho_{\epsilon_i} = \rho_0$ the limiting weight function is the weight function used to define the weighted Holder spaces on $\mathbb{T}^4$ and is manifestly even whence descends down $\mathbb{T}^4 - S \xrightarrow{2:1} \mathbb{T}^4/\mathbb{Z}_2 - S$.

Thus since $2\delta - 2 < 0$ because $\delta < 0$ and since $\rho_0 = \begin{cases} 1 & \text{on } \left\{ \frac{1}{5} \leq |\cdot|^{g_{\omega_0}}_S \right\} \\ |\cdot|^{g_{\omega_0}}_S & \text{on } \left\{ |\cdot|^{g_{\omega_0}}_S \leq \frac{1}{8} \right\} \end{cases}$, we need to worry about controlling $\rho_0^{2\delta-2}$ near S. Thus on $\left\{ |\cdot|^{g_{\omega_0}}_S \leq \frac{1}{8} \right\}$, that upon using exponential polar coordinates WRT g_{ω_0} centered around each point $p \in S$ we have that $\rho_0 = |\cdot|^{g_{\omega_0}}_p$ is the radial distance function of those polar coordinates, and

whence

$$\int_{\left\{|\cdot|_S^{g_{\omega_0}} \leq \frac{1}{8}\right\}} \rho_0^{2\delta-2} dV_{g_0} = 16 \int_{\left\{|\cdot|_p^{g_{\omega_0}} \leq \frac{1}{8}\right\}} \left(|\cdot|_p^{g_{\omega_0}}\right)^{2\delta-2} dV_{g_0}$$

$$= 16 \int_0^{\frac{1}{8}} r^{2\delta-2} r^{4-1} dr \int_{S^3} d\theta$$

$$= O(1) \int_0^{\frac{1}{8}} r^{2\delta+1} dr = O(1) \left.\frac{r^{2\delta+2}}{2\delta+2}\right|_0^{\frac{1}{8}} = O(1) < \infty$$

precisely because $0 < 2\delta + 2 \iff -1 < \delta$. Whence we have that $\int_{\mathbb{T}^4-S} \rho_0^{2\delta-2} dV_{g_{\omega_0}} = O(1)$, whence since we have thus shown that

$$\left\|\rho_{\epsilon_i}^{\delta-1}\right\|_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})}^2 \rightarrow \frac{1}{2}\left\|\rho_0^{\delta-1}\right\|_{L^2_{g_{\omega_0}}(\mathbb{T}^4-S)}^2 < \infty$$

that is, the sequence of real numbers $\left\{\left\|\rho_{\epsilon_i}^{\delta-1}\right\|_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})}^2\right\}_i \subset \mathbb{R}$ is convergent, we therefore have that

$$\left\|\rho_{\epsilon_i}^{\delta-1}\right\|_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})}^2 \leq M$$

is *uniformly* bounded above as convergent sequences are bounded. Whence

$$\|\zeta_i\|_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} \leq C_{17}\frac{1}{i}M^{\frac{1}{2}}$$

whence

$$\|\zeta_i\|_{L^2_{g_{\omega_{\epsilon_i}}}(\mathrm{Km}_{\epsilon_i})} \searrow 0$$

as $i \nearrow \infty$.

Next, recall that our 3-dimensional space of self-dual harmonic 2-forms is $\mathcal{H}^+_{g_{\omega_{\epsilon_i}}} = \mathrm{span}_{\mathbb{R}}(\boldsymbol{\omega}_{\epsilon_i}) = \mathrm{span}_{\mathbb{R}}\left\{\omega_1^{\epsilon_i}, \omega_2^{\epsilon_i}, \omega_3^{\epsilon_i}\right\}$. Now I claim that $\omega_1^{\epsilon_i}, \omega_2^{\epsilon_i}, \omega_3^{\epsilon_i}$ each have *uniformly* bounded $C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}}$-norm, where the bound is *uniform in i*. Indeed, since $Q_{\boldsymbol{\omega}_{\epsilon_i}} = \mathrm{id}$ on $\left\{|\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \leq \epsilon_i^{\frac{1}{2}}\right\} \sqcup \left\{2\epsilon_i^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}}\right\}$, we have that $\boldsymbol{\omega}_{\epsilon_i} = \left(\omega_1^{\epsilon_i}, \omega_2^{\epsilon_i}, \omega_3^{\epsilon_i}\right)$ is hyper-Kähler and thus each $\omega_1^{\epsilon_i}, \omega_2^{\epsilon_i}, \omega_3^{\epsilon_i}$ is parallel WRT $\nabla_{g_{\omega_{\epsilon_i}}}$ and bounded WRT $g_{\omega_{\epsilon_i}}$. Moreover we have from Proposition 4.3.1 that $\left|\boldsymbol{\omega}_{\epsilon_i} - \boldsymbol{\omega}_0\right|_{g_{\omega_0}} \leq c_1(2)\epsilon_i^2$ aka componentwise $\boldsymbol{\omega}_{\epsilon_i} = \boldsymbol{\omega}_0 + O_{g_{\omega_0}}\left(\epsilon_i^2\right)$ on $\left\{\epsilon_i^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \leq 2\epsilon_i^{\frac{1}{2}}\right\}$. Whence using a local ON frame calculation as well as the decay $g_{\omega_{\epsilon_i}} = g_{\omega_0} + O_{g_{\omega_0}}\left(\epsilon_i^2\right)$ on $\left\{\epsilon_i^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \leq 2\epsilon_i^{\frac{1}{2}}\right\}$, we compute that (the 3 components of) $\left|\boldsymbol{\omega}_{\epsilon_i}\right|_{g_{\omega_{\epsilon_i}}}$ is bounded uniformly in i on the annular region $\left\{\epsilon_i^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \leq 2\epsilon_i^{\frac{1}{2}}\right\}$. Combined with the fact that $0 < -\delta + 1 \iff \delta < 1$ which we have because $\delta < 0$, we get that from $\epsilon_i \leq \rho_{\epsilon_i} \leq 1$ that $\rho_{\epsilon_i}^{-\delta+1} \leq 1$ is uniformly bounded above, whence we conclude that each $\omega_1^{\epsilon_i}, \omega_2^{\epsilon_i}, \omega_3^{\epsilon_i}$ has *uniformly* bounded $C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}}$-norm, i.e. $\boldsymbol{\omega}_{\epsilon_i}$ has *uniformly* bounded $C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}}\left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i}\right) \otimes \mathbb{R}^3$-norm from *Remark* 4.4.2.

Next up, for $m, n \in \{1, 2, 3\}$ we have by definition of the associated intersection matrix $\left(Q_{\boldsymbol{\omega}_{\epsilon_i}}\right)_{mn} = \frac{1}{2}\omega_m^{\epsilon_i} \wedge \omega_n^{\epsilon_i}$ and from Proposition 4.3.1 the fact that $Q_{\boldsymbol{\omega}_{\epsilon_i}} = \mathrm{id} + O(\epsilon_i^2)$ on $\left\{\epsilon_i^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \leq 2\epsilon_i^{\frac{1}{2}}\right\}$, that $dV_{g_{\omega_{\epsilon_i}}} = \left(1 + O(\epsilon_i^2)\right)dV_{g_{\omega_0}}$

on $\left\{\epsilon_i^{\frac{1}{2}} \le |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}}\right\}$ (with straight up equality with $\mathrm{d}V_{g_{\omega_0}}$ on $\left\{2\epsilon_i^{\frac{1}{2}} \le |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}}\right\}$), that

$$
\begin{aligned}
\left\langle \omega_m^{\epsilon_i}, \omega_n^{\epsilon_i} \right\rangle_{L^2_{g_{\omega_{\epsilon_i}}}} &= \int_{\mathrm{Km}_{\epsilon_i}} \omega_m^{\epsilon_i} \wedge *_{g_{\omega_{\epsilon_i}}} \omega_n^{\epsilon_i} \\
&= \int_{\mathrm{Km}_{\epsilon_i}} \omega_m^{\epsilon_i} \wedge \omega_n^{\epsilon_i} \\
&= \int_{\left\{|\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \le \epsilon_i^{\frac{1}{2}}\right\}} \omega_m^{\epsilon_i} \wedge \omega_n^{\epsilon_i} + \int_{\left\{\epsilon_i^{\frac{1}{2}} \le |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \le 2\epsilon_i^{\frac{1}{2}}\right\}} \omega_m^{\epsilon_i} \wedge \omega_n^{\epsilon_i} + \int_{\left\{2\epsilon_i^{\frac{1}{2}} \le |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}}\right\}} \omega_m^{\epsilon_i} \wedge \omega_n^{\epsilon_i} \\
&= 2\int_{\left\{|\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \le \epsilon_i^{\frac{1}{2}}\right\}} \delta_{mn} \mathrm{d}V_{g_{\omega_{\epsilon_i}}} + 2\int_{\left\{\epsilon_i^{\frac{1}{2}} \le |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \le 2\epsilon_i^{\frac{1}{2}}\right\}} \left(\delta_{mn} + O\left(\epsilon_i^2\right)\right) \mathrm{d}V_{g_{\omega_{\epsilon_i}}} + 2\int_{\left\{2\epsilon_i^{\frac{1}{2}} \le |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}}\right\}} \delta_{mn} \mathrm{d}V_{g_{\omega_{\epsilon_i}}} \\
&= 2\delta_{mn} \mathrm{Vol}_{g_{\omega_{\epsilon_i}}}\left(\mathrm{Km}_{\epsilon_i}\right) + 2\int_{\left\{\epsilon_i^{\frac{1}{2}} \le |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \le 2\epsilon_i^{\frac{1}{2}}\right\}} O\left(\epsilon_i^2\right)\left(1 + O\left(\epsilon_i^2\right)\right) \mathrm{d}V_{g_{\omega_0}} \\
&= 2\delta_{mn} \mathrm{Vol}_{g_{\omega_{\epsilon_i}}}\left(\mathrm{Km}_{\epsilon_i}\right) + O\left(\epsilon_i^2\right) \mathrm{Vol}_{g_{\omega_0}}\left(\left\{\epsilon_i^{\frac{1}{2}} \le |\cdot|_{\pi^{-1}(S)}^{g_{\omega_0}} \le 2\epsilon_i^{\frac{1}{2}}\right\}\right) \\
&= 2\delta_{mn} \mathrm{Vol}_{g_{\omega_{\epsilon_i}}}\left(\mathrm{Km}_{\epsilon_i}\right) + O\left(\epsilon_i^4\right) < \infty
\end{aligned}
$$

and where this bound is *uniform in i* since Km_{ϵ_i} is volume non-collapsed and has uniformly bounded Ricci (*Remark 4.3.3*) whence Colding's Volume Convergence Theorem tells us that $\mathrm{Vol}_{g_{\omega_{\epsilon_i}}}\left(\mathrm{Km}_{\epsilon_i}\right) \to \mathrm{Vol}_{g_{\omega_0}}\left(\mathbb{T}^4/\mathbb{Z}_2\right) < \infty$.

Whence we have shown that

$$
\left\|\omega_l^{\epsilon_i}\right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}}\left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i}\right)}, \left\langle \omega_m^{\epsilon_i}, \omega_n^{\epsilon_i} \right\rangle_{L^2_{g_{\omega_{\epsilon_i}}}} \qquad \forall l, m, n \in \{1, 2, 3\}
$$

are all *uniformly* bounded in i,

Now let us produce via Gram-Schmidt an $L^2_{g_{\omega_{\epsilon_i}}}$-orthonormal basis $\overline{\omega_1^{\epsilon_i}}, \overline{\omega_2^{\epsilon_i}}, \overline{\omega_3^{\epsilon_i}}$ of $\mathcal{H}^+_{g_{\omega_{\epsilon_i}}}$ from $\omega_1^{\epsilon_i}, \omega_2^{\epsilon_i}, \omega_3^{\epsilon_i}$. Because each $\overline{\omega_1^{\epsilon_i}}, \overline{\omega_2^{\epsilon_i}}, \overline{\omega_3^{\epsilon_i}}$ consists of linear combinations of $\omega_1^{\epsilon_i}, \omega_2^{\epsilon_i}, \omega_3^{\epsilon_i}$ with coefficients involving $L^2_{g_{\omega_{\epsilon_i}}}$-inner products $\left\langle \omega_m^{\epsilon_i}, \omega_n^{\epsilon_i} \right\rangle_{L^2_{g_{\omega_{\epsilon_i}}}}, \forall m, n \in \{1, 2, 3\}$, we have that

$$
\left\|\overline{\omega_m^{\epsilon_i}}\right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}}\left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i}\right)} \qquad \forall m \in \{1, 2, 3\}
$$

are all *uniformly* bounded in i.

Therefore, upon writing out $\zeta_i = \lambda_1^i \overline{\omega_1^{\epsilon_i}} + \lambda_2^i \overline{\omega_2^{\epsilon_i}} + \lambda_3^i \overline{\omega_3^{\epsilon_i}}$ where each $\lambda_k^i = \int_{\mathrm{Km}_{\epsilon_i}} \zeta_i \wedge \overline{\omega_k^{\epsilon_i}} \in \mathbb{R}$, we have that

$$
\|\zeta_i\|^2_{L^2_{g_{\omega_{\epsilon_i}}}\left(\mathrm{Km}_{\epsilon_i}\right)} = \left(\lambda_1^i\right)^2 + \left(\lambda_2^i\right)^2 + \left(\lambda_3^i\right)^2
$$

whence

$$
\lambda_1^i, \lambda_2^i, \lambda_3^i \to 0
$$

as $i \nearrow \infty$.

Whence by the triangle inequality we get

$$\left\| d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)} = \left\| d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i - \zeta_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)}$$

$$\leq \left\| d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i + \zeta_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)} + \left\| \zeta_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)}$$

$$\leq \frac{1}{i} + \left\| \zeta_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)} = \frac{1}{i} + \left\| \lambda_1^i \overline{\omega_1^{\epsilon_i}} + \lambda_2^i \overline{\omega_2^{\epsilon_i}} + \lambda_3^i \overline{\omega_3^{\epsilon_i}} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)}$$

$$\leq \frac{1}{i} + \left| \lambda_1^i \right| \left\| \overline{\omega_1^{\epsilon_i}} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)} + \left| \lambda_2^i \right| \left\| \overline{\omega_2^{\epsilon_i}} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)}$$

$$+ \left| \lambda_3^i \right| \left\| \overline{\omega_3^{\epsilon_i}} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)}$$

$$\leq \frac{1}{i} + O\left(\left| \lambda_1^i \right| \right) + O\left(\left| \lambda_2^i \right| \right) + O\left(\left| \lambda_3^i \right| \right)$$

where we've used the fact that each $\left\| \overline{\omega_m^{\epsilon_i}} \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)}$ is uniformly bounded in i to absorb those terms into $O\left(\left| \lambda_m^i \right| \right)$ as harmless constants.

Therefore since $\left\| \mathfrak{a}_i \right\|_{C^{1,\alpha}_{\delta,g_{\omega_{\epsilon_i}}} \left(\mathring{\Omega}^1_{g_{\omega_{\epsilon_i}}} \left(\mathrm{Km}_{\epsilon_i} \right) \right)} = 1 - \left\| \zeta_i \right\|_{L^2_{g_{\omega_{\epsilon_i}}}}$, we have upon normalizing each $\mathfrak{a}_i$ to have unit $C^{1,\alpha}_{\delta,g_{\omega_{\epsilon_i}}} \left(\mathring{\Omega}^1_{g_{\omega_{\epsilon_i}}} \left(\mathrm{Km}_{\epsilon_i} \right) \right)$-norm since $\left\| \zeta_i \right\|_{L^2_{g_{\omega_{\epsilon_i}}}} \searrow 0$, we have finally shown that

$$\left\| \mathfrak{a}_i \right\|_{C^{1,\alpha}_{\delta,g_{\omega_{\epsilon_i}}} \left(\mathring{\Omega}^1_{g_{\omega_{\epsilon_i}}} \left(\mathrm{Km}_{\epsilon_i} \right) \right)} = 1$$

$$\left\| d^+_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i \right\|_{C^{0,\alpha}_{\delta-1,g_{\omega_{\epsilon_i}}} \left(\Lambda^+_{g_{\omega_{\epsilon_i}}} T^* \mathrm{Km}_{\epsilon_i} \right)} \searrow 0$$

as $i \nearrow \infty$. But this flat out contradicts Proposition 4.6.4 since $\left\| \mathfrak{a}_i \right\|_{C^{1,\alpha}_{\delta,g_{\omega_{\epsilon_i}}} \left(\mathring{\Omega}^1_{g_{\omega_{\epsilon_i}}} \left(\mathrm{Km}_{\epsilon_i} \right) \right)} = \left\| \mathfrak{a}_i \right\|_{C^{1,\alpha}_{\delta,g_{\omega_{\epsilon_i}}} \left(T^* \mathrm{Km}_{\epsilon_i} \right)}$ with $d^*_{g_{\omega_{\epsilon_i}}} \mathfrak{a}_i = 0$, whence we have reached a contradiction, as was to be shown. ∎

4.7 § Finishing the Proof & Discussion §

Recall from Section 4.5 our nonlinear problem:

$$\Phi_\epsilon : \left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}} \left(\mathring{\Omega}^1_{g_{\omega_\epsilon}} \left(\mathrm{Km}_\epsilon \right) \right) \oplus \mathcal{H}^+_{g_{\omega_\epsilon}} \right) \otimes \mathbb{R}^3 \to C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}} \left(\Lambda^+_{g_{\omega_\epsilon}} T^* \mathrm{Km}_\epsilon \right) \otimes \mathbb{R}^3$$

$$(a, \zeta) \mapsto d^+_{g_{\omega_\epsilon}} a + \zeta - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}} a * d^-_{g_{\omega_\epsilon}} a \right)_0 \right) \omega_\epsilon$$

where we solve

$$\Phi_\epsilon(a, \zeta) := d^+_{g_{\omega_\epsilon}} a + \zeta - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}} a * d^-_{g_{\omega_\epsilon}} a \right)_0 \right) \omega_\epsilon$$

$$= \underbrace{-\mathcal{F}\left(\left(-Q_{\omega_\epsilon} \right)_0 \right) \omega_\epsilon}_{=\Phi_\epsilon(0,0)} + \underbrace{d^+_{g_{\omega_\epsilon}} a + \zeta}_{=D_0\Phi_\epsilon(a,\zeta)} + \underbrace{\mathcal{F}\left(\left(-Q_{\omega_\epsilon} \right)_0 \right) \omega_\epsilon - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}} a * d^-_{g_{\omega_\epsilon}} a \right)_0 \right) \omega_\epsilon}_{\text{nonlinearity}}$$

$$= 0$$

Applying Theorem 3.7.1, we now arrive at:

Theorem 4.7.1 (Big Theorem II). *Let $\delta \in (-1, 0)$. Let $\epsilon > 0$ **from Data** 3.2.5 be*

(1) *sufficiently small so that each pre-glued Eguchi-Hanson metric from Propositions 4.2.2 and 4.2.3 are hyper-Kähler and satisfy their annular estimates, and $\epsilon < 1$ to satisfy the volume form estimates.*

(2) $3\epsilon^{\frac{1}{2}} < \frac{1}{2}$ *from Data 3.3.1 for the construction of* Km_ϵ.

(3) $2\epsilon^{\frac{1}{2}} < \frac{3}{25}$ *from Data 3.4.4 for the construction of* ρ_ϵ.

(4) *sufficiently small for the weighted Schauder estimate from Corollary 4.6.3.*

(5) *sufficiently small for the improved Schauder estimate from Proposition 4.6.4.*

(6) *sufficiently small for the the uniform bounded linear isomorphism* $D_0\Phi_\epsilon : \left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right) \oplus \mathcal{H}^+_{g_{\omega_\epsilon}}\right) \otimes$
$\mathbb{R}^3 \to C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}\left(\Lambda^+_{g_{\omega_\epsilon}} T^*\mathrm{Km}_\epsilon\right) \otimes \mathbb{R}^3$ *where* $D_0\Phi_\epsilon = \left(d^+_{g_{\omega_\epsilon}} \oplus \mathrm{id}_{\mathcal{H}^+_{g_{\omega_\epsilon}}}\right) \otimes \mathbb{R}^3$ *from Theorem 4.6.6.*

(7) $\epsilon^{\frac{3}{2}+\frac{\delta}{2}} < \frac{1}{4L^2 C_{18}C_{19}}$ *for* $L, C_{18}, C_{19} > 0$ *constants which will be defined in the proof.*

*Then for each such $\epsilon > 0$, the equation $\Phi_\epsilon(\boldsymbol{a}, \boldsymbol{\zeta}) = 0$ has a **unique** solution $(\boldsymbol{a}, \boldsymbol{\zeta})$ satisfying:*

- $\boldsymbol{a} \in \Omega^1(\mathrm{Km}_\epsilon) \otimes \mathbb{R}^3$ **(smoothness)**

- $\|(\boldsymbol{a}, \boldsymbol{\zeta})\|_{\left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}^+_{g_{\omega_\epsilon}}\right)\otimes\mathbb{R}^3} \leq 2L\|\Phi_\epsilon(0,0)\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}\left(\Lambda^+_{g_{\omega_\epsilon}}T^*\mathrm{Km}_\epsilon\right)\otimes\mathbb{R}^3} < C_{20}\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}}$ *for* $C_{20} > 0$ *chosen so that* $C_{20} > 2LC_{19}$ **(smallness of norm)**

*Hence by Section 4.1 and Ansatz 4.1.7, for each such $\epsilon > 0$ we have that $\omega_\epsilon + d\boldsymbol{a} + \boldsymbol{\zeta} \in [\omega_\epsilon]$ is the **unique** hyper-Kähler triple perturbed from the given closed and "almost hyper-Kähler triple" ω_ϵ on Km_ϵ, whence gives us a hyper-Kähler metric $g_{\omega_\epsilon + d\boldsymbol{a} + \boldsymbol{\zeta}}$.* ◆

Proof. We setup the input needed for the implicit function theorem (Theorem 3.7.1).

(1) We have from Theorem 4.6.6 that $D_0\Phi_\epsilon : \left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right) \oplus \mathcal{H}^+_{g_{\omega_\epsilon}}\right) \otimes \mathbb{R}^3 \to C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}\left(\Lambda^+_{g_{\omega_\epsilon}} T^*\mathrm{Km}_\epsilon\right) \otimes \mathbb{R}^3$ is a bounded linear isomorphism with $\left\|(D_0\Phi_\epsilon)^{-1}\right\| \leq \mathcal{L} =: L$ and $L > 0$ *uniform in* $\epsilon > 0$.

(2) Let $r_1 > 0$ be some fixed positive number which will be specified later. Equip $\mathrm{Sym}^2_0(\mathbb{R}^3)$ with the Frobenius norm $\|M\|_F := \left(\sum_{i,j}|M_{ij}|^2\right)^{\frac{1}{2}} = \left(\mathrm{Tr}(M^T M)\right)^{\frac{1}{2}}$. Recall that $\mathcal{F} : \mathrm{Sym}^2_0(\mathbb{R}^3) \to \mathrm{Sym}^2_0(\mathbb{R}^3)$ is a smooth map. Whence precomposing $\mathcal{F}$ by a translation, say $\mathcal{F}(M_1 + \cdot)$ for $M_1 \in \mathrm{Sym}^2_0(\mathbb{R}^3)$, remains smooth on $\mathrm{Sym}^2_0(\mathbb{R}^3)$. Therefore by the mean value theorem, we have that $\|\mathcal{F}(M_1 + A_1) - \mathcal{F}(M_1 + A_2)\|_F \leq \mathcal{G}\|A_1 - A_2\|_F$ for when $A_1, A_2 \in D_{r_1}(0) \subset \mathrm{Sym}^2_0(\mathbb{R}^3)$ the closed disk of radius $r_1 > 0$ around the origin WRT the Frobenius norm, which is *compact* since $\mathrm{Sym}^2_0(\mathbb{R}^3)$ is finite dimensional, whence the constant $\mathcal{G} > 0$ is *uniform* in $A_1, A_2 \in D_{r_1}(0)$.

In our case, since $\mathsf{N}(\boldsymbol{a}, \boldsymbol{\zeta}) = \mathsf{N}(\boldsymbol{a}) := \mathcal{F}\left((-Q_{\omega_\epsilon})_0\right)\omega_\epsilon - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a}\right)_0\right)\omega_\epsilon$ only depends on $\boldsymbol{a}$ and that $\mathsf{N}(\boldsymbol{a_1}) - \mathsf{N}(\boldsymbol{a_2}) = \left(\mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2}\right)_0\right) - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1}\right)_0\right)\right)\omega_\epsilon$, we thus have upon letting $M_1 := -(Q_{\omega_\epsilon})_0$, $A_1 := -\left(d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1}\right)_0$, and $A_2 := -\left(d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2}\right)_0$, that *pointwise* in $\mathrm{Sym}^2_0(\mathbb{R}^3)$ we have that

$$|\mathsf{N}(\boldsymbol{a_1}) - \mathsf{N}(\boldsymbol{a_2})|_{g_\epsilon \otimes \mathbb{R}^3} = \left|\left(\mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1}\right)_0\right) - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2}\right)_0\right)\right)\omega_\epsilon\right|_{g_\epsilon \otimes \mathbb{R}^3}$$

$$\leq \left\|\mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1}\right)_0\right) - \mathcal{F}\left(\left(-Q_{\omega_\epsilon} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2}\right)_0\right)\right\|_F |\omega_\epsilon|_{g_\epsilon \otimes \mathbb{R}^3}$$

$$\leq \mathcal{G}\left\|\left(d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2}\right)_0\right\|_F |\omega_\epsilon|_{g_\epsilon \otimes \mathbb{R}^3}$$

$$\leq \mathcal{G}\left\|d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_1} - d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2} * d^-_{g_{\omega_\epsilon}}\boldsymbol{a_2}\right\|_F |\omega_\epsilon|_{g_\epsilon \otimes \mathbb{R}^3}$$

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where $|\boldsymbol{v}|_{g_\epsilon\otimes\mathbb{R}^3}:=\left(|v_1|_{g_\epsilon}+|v_2|_{g_\epsilon}+|v_3|_{g_\epsilon}\right)^{\frac{1}{2}}$ (recall *Remark* 4.4.2), that $\|M\|_{\mathrm{F},\mathcal{A}}:=\left(\sum_{i,j}|M_{ij}|_{\mathcal{A}}^2\right)^{\frac{1}{2}}$ the Frobenius norm coupled with the norm $\|\cdot\|_{\mathcal{A}}$, that $\|M\|_{\mathrm{F},\mathcal{A}}=\|M\|_{\mathcal{A}\otimes\mathbb{R}^3}$ for a 3×1 or 1×3 form valued matrix M, $\|AB\|_{\mathrm{F}}\leq\|A\|_{\mathrm{F}}\|B\|_{\mathrm{F}}$, and $\|A_0\|_{\mathrm{F}}\leq\|A\|_{\mathrm{F}}$.

Now recall that $\left|\mathrm{d}V_g\right|_g=1$ directly implies that $|\Theta|_g=\left|\frac{\Theta}{\mathrm{d}V_g}\mathrm{d}V_g\right|_g=\left|\frac{\Theta}{\mathrm{d}V_g}\right|$ for any 4-form Θ and that $\boldsymbol{\gamma}*\boldsymbol{\gamma}:=\frac{1}{2}\frac{\boldsymbol{\gamma}\wedge\boldsymbol{\gamma}^T}{\mu_\omega}=\frac{1}{2}\frac{\boldsymbol{\gamma}\wedge\boldsymbol{\gamma}^T}{\mathrm{d}V_{g_\omega}}$ for any triple $\boldsymbol{\gamma}$ of 2-forms. Thus $\left|\boldsymbol{\gamma}*\boldsymbol{\gamma}\right|_{ij}=\frac{1}{2}\left|\frac{\gamma_i\wedge\gamma_j}{\mathrm{d}V_{g_\omega}}\right|=\frac{1}{2}\left|\gamma_i\wedge\gamma_j\right|_{g_\omega}=\frac{1}{2}\left|\left(\boldsymbol{\gamma}\wedge\boldsymbol{\gamma}^T\right)_{ij}\right|_{g_{\omega_\epsilon}}$, and since $AA^T-BB^T=\mathfrak{S}\left[(A-B)(A+B)^T\right]$ where $\mathfrak{S}[M]:=\frac{M+M^T}{2}$ is the symmetrization map, upon using the definition of the Frobenius norm and continuity of the symmetrization map we have that

$$\left\|d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1*d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1-d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2*d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right\|_{\mathrm{F}}=\left\|d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1\wedge\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1\right)^T-d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\wedge\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)^T\right\|_{\mathrm{F},g_{\omega_\epsilon}}$$

$$=\left\|\mathfrak{S}\left[\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1-d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)\wedge\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1+d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)^T\right]\right\|_{\mathrm{F},g_{\omega_\epsilon}}$$

$$\leq\|\mathfrak{S}\|_{\mathrm{op}}\left\|\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1-d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)\wedge\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1+d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)^T\right\|_{\mathrm{F},g_{\omega_\epsilon}}$$

Thus we have the *pointwise* inequality

$$|\mathsf{N}(\boldsymbol{a}_1)-\mathsf{N}(\boldsymbol{a}_2)|_{g_\epsilon\otimes\mathbb{R}^3}\leq\mathsf{G}\|\mathfrak{S}\|_{\mathrm{op}}|\boldsymbol{\omega}_\epsilon|_{g_\epsilon\otimes\mathbb{R}^3}\left\|\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1-d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)\wedge\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1+d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)^T\right\|_{\mathrm{F},g_{\omega_\epsilon}}$$

Thus we multiply each side by $\rho_\epsilon^{-\delta+1}$, and upon recalling that $|\boldsymbol{\omega}_\epsilon|_{C^0_{g_\epsilon\otimes\mathbb{R}^3}}$ is uniformly bounded in ϵ from the proof of Theorem 4.6.6 and the definition of the weighted Holder norm, we finally have upon taking $\sup_{\mathrm{Km}_\epsilon}$ on both sides and arguing all of the above in a similar fashion for the Holder seminorm that

$$\|\mathsf{N}(\boldsymbol{a}_1)-\mathsf{N}(\boldsymbol{a}_2)\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}\left(\Lambda^+_{g_{\omega_\epsilon}}T^*\mathrm{Km}_\epsilon\right)\otimes\mathbb{R}^3}\leq\mathsf{G}_1\left\|\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1-d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)\wedge\left(d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_1+d_{g_{\omega_\epsilon}}^-\boldsymbol{a}_2\right)^T\right\|_{\mathrm{F},C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}}$$

for some *uniform* constant $\mathsf{G}_1>0$ and with the $\|\cdot\|_{\mathrm{F},C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}}$-norm having the obvious meaning. Thus, using Proposition 3.4.5 with $l=1$ on the product in the RHS term, noting the definition of the weighted Holder norms (namely that the weight is $\rho_\epsilon^{-\delta+1}$ for the $\nabla_{g_{\omega_\epsilon}}$-term), applying the triangle inequality, and adding the remaining terms in the norm of $\left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}^+_{g_{\omega_\epsilon}}\right)\otimes\mathbb{R}^3$ onto the RHS (all of which are *nonnegative*), we finally get that

$$\|\mathsf{N}\left(\boldsymbol{a}_1,\boldsymbol{\zeta}_1\right)-\mathsf{N}\left(\boldsymbol{a}_2,\boldsymbol{\zeta}_2\right)\|_{C^{0,\alpha}_{\delta-1,g_{\omega_\epsilon}}\left(\Lambda^+_{g_{\omega_\epsilon}}T^*\mathrm{Km}_\epsilon\right)\otimes\mathbb{R}^3}$$

$$\leq C_{18}\epsilon^{\delta-1}\left\|\left(\boldsymbol{a}_1,\boldsymbol{\zeta}_1\right)-\left(\boldsymbol{a}_2,\boldsymbol{\zeta}_2\right)\right\|_{\left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}^+_{g_{\omega_\epsilon}}\right)\otimes\mathbb{R}^3}$$

$$\cdot\left(\left\|\left(\boldsymbol{a}_1,\boldsymbol{\zeta}_1\right)\right\|_{\left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}^+_{g_{\omega_\epsilon}}\right)\otimes\mathbb{R}^3}+\left\|\left(\boldsymbol{a}_2,\boldsymbol{\zeta}_2\right)\right\|_{\left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}^+_{g_{\omega_\epsilon}}\right)\otimes\mathbb{R}^3}\right)$$

for some *uniform* constant $C_{18}>0$.

Lastly, we in fact set $r_1:=r_0^2$ and have that this above inequality makes sense only when

$$\left(\boldsymbol{a}_1,\boldsymbol{\zeta}_1\right),\left(\boldsymbol{a}_2,\boldsymbol{\zeta}_2\right)\in D_{r_0}^{\left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}^+_{g_{\omega_\epsilon}}\right)\otimes\mathbb{R}^3}(0)$$

because by definition of the weighted Holder norm, $\rho_\epsilon\leq 1$, and $0<-\delta+1$, we have that $\left\|\rho_\epsilon^{-\delta+1}\nabla_{g_{\omega_\epsilon}}\boldsymbol{a}\right\|_{C^0_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\otimes\mathbb{R}^3}=\left\|\rho_\epsilon^{-\delta+1}\right\|_{C^0(\mathrm{Km}_\epsilon)}\left\|\nabla_{g_{\omega_\epsilon}}\boldsymbol{a}\right\|_{C^0_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\otimes\mathbb{R}^3}=\left\|\nabla_{g_{\omega_\epsilon}}\boldsymbol{a}\right\|_{C^0_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\otimes\mathbb{R}^3}$ and

$$\left\|\nabla_{g_{\omega_\epsilon}}\boldsymbol{a}\right\|_{C^0_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\otimes\mathbb{R}^3}\leq\|(\boldsymbol{a},\boldsymbol{\zeta})\|_{\left(C^{1,\alpha}_{\delta,g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}^+_{g_{\omega_\epsilon}}\right)\otimes\mathbb{R}^3}\leq r_0$$

and for γ a triple of 2-forms

$$\|\gamma * \gamma\|_{\mathrm{F}} = \|\gamma \wedge \gamma^T\|_{\mathrm{F},g} \leq \|\gamma\|_{\mathrm{F},g}^2 = \|\gamma\|_{g \otimes \mathbb{R}^3}^2$$

Thus we have that $N := C_{18} \epsilon^{\delta - 1}$ and $r_0 > 0$ to be determined later.

(3) Since $\Phi_\epsilon(0,0) = -\mathcal{F}\big((-Q_{\omega_\epsilon})_0\big)\omega_\epsilon$, we have by the same arguments as for establishing the nonlinearity estimate above coupled with the fact that $\mathcal{F}(0) = 0$ that

$$
\begin{aligned}
\|\Phi_\epsilon(0,0)\|_{C_{\delta-1,g\omega_\epsilon}^{0,\alpha}\left(\Lambda_{g\omega_\epsilon}^+ T^* \mathrm{Km}_\epsilon\right)\otimes\mathbb{R}^3} &\overset{\leq}{} \mathcal{G}_2 \big\|(Q_{\omega_\epsilon})_0\big\|_{\mathrm{F},C_{\delta-1,g\omega_\epsilon}^{0,\alpha}} \\
&= \mathcal{G}_2 \Big\|Q_{\omega_\epsilon} - \tfrac{1}{3}\mathrm{Tr}(Q_{\omega_\epsilon}) \cdot \mathrm{id}\Big\|_{\mathrm{F},C_{\delta-1,g\omega_\epsilon}^{0,\alpha}} \\
&= \mathcal{G}_2 \big\|\mathrm{id} + O(\epsilon^2) - (1 + O(\epsilon^2))\mathrm{id}\big\|_{\mathrm{F},C_{\delta-1,g\omega_\epsilon}^{0,\alpha}} \\
&= \mathcal{G}_2 \big\|O(\epsilon^2)\big\|_{\mathrm{F},C_{\delta-1,g\omega_\epsilon}^{0,\alpha}} \\
&\leq C_{19}\epsilon^{2-\frac{\delta-1}{2}} \\
&= C_{19}\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}}
\end{aligned}
$$

where we have used the fact that $Q_{\omega_\epsilon} = \mathrm{id} + O(\epsilon^2)$ hence $\mathrm{Tr}(Q_{\omega_\epsilon}) = \big(1 + O(\epsilon^2)\big)\mathrm{Tr}(\mathrm{id}) = \big(1 + O(\epsilon^2)\big) \cdot 3$ holding both componentwise and globally on Km_ϵ (Proposition 4.3.1), as well as the fact that $\mathrm{supp}\big((Q_{\omega_\epsilon})_0\big) \subseteq \left\{\epsilon^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g\omega_0} \leq 2\epsilon^{\frac{1}{2}}\right\}$ since outside that region $Q_{\omega_\epsilon} = \mathrm{id}$, whence the factor of δ comes in because on $\left\{\epsilon^{\frac{1}{2}} \leq |\cdot|_{\pi^{-1}(S)}^{g\omega_0} \leq 2\epsilon^{\frac{1}{2}}\right\}$ our weight function ρ_ϵ when combined with Data 3.4.4 (namely $2\epsilon < \epsilon^{\frac{1}{2}}$) satisfies $\epsilon^{\frac{1}{2}} \leq \rho_\epsilon \leq 2\epsilon^{\frac{1}{2}}$ whence since $0 < -\delta + 1$ we have that $\rho_\epsilon^{-\delta+1} = O\big(\epsilon^{\frac{-\delta+1}{2}}\big)$.

All of this holds for some *uniform* constant $C_{19} > 0$.

Set $r_0 := C_{20}\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}}$ with $C_{20} > 0$ chosen so that $C_{20} > 2LC_{19}$. We now need to show that

$$\|\Phi_\epsilon(0,0)\|_{C_{\delta-1,g\omega_\epsilon}^{0,\alpha}\left(\Lambda_{g\omega_\epsilon}^+ T^* \mathrm{Km}_\epsilon\right)\otimes\mathbb{R}^3} \leq \frac{r}{2L}$$

with $r < \min\left\{r_0, \frac{1}{2NL}\right\} = \min\left\{C_{20}\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}}, \frac{\epsilon^{1-\delta}}{2LC_{18}}\right\}$. We have that

$$\|\Phi_\epsilon(0,0)\|_{C_{\delta-1,g\omega_\epsilon}^{0,\alpha}\left(\Lambda_{g\omega_\epsilon}^+ T^* \mathrm{Km}_\epsilon\right)\otimes\mathbb{R}^3} \leq C_{19}\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}} = \frac{2LC_{19}\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}}}{2L}$$

and so we choose our $r := 2LC_{19}\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}}$. Clearly $r < r_0$, and it remains to show that $r < \frac{\epsilon^{1-\delta}}{2LC_{18}}$. But $\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}} = \epsilon^{1-\delta+P}$ where $P := \frac{3}{2} + \frac{\delta}{2}$, which is *positive* because $\delta \in (-1,0)$. Hence since $x > y \Rightarrow c^x < c^y$ for $c \in (0,1)$ small enough, **making $\epsilon > 0$ small enough so that $\epsilon^{\frac{3}{2}+\frac{\delta}{2}} < \frac{1}{4L^2 C_{18}C_{19}}$** forces $r < \frac{\epsilon^{1-\delta}}{2LC_{18}}$. Therefore, *for each such $\epsilon > 0$* Theorem 3.7.1 gives us the existence of a unique solution $(a, \zeta) \in D_{2L\|\Phi_\epsilon(0)\|_{C_{\delta-1,g\omega_\epsilon}^{0,\alpha}\left(\Lambda_{g\omega_\epsilon}^+ T^* \mathrm{Km}_\epsilon\right)\otimes\mathbb{R}^3}}^{\left(C_{\delta,g\omega_\epsilon}^{1,\alpha}\left(\mathring{\Omega}_{g\omega_\epsilon}^1(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}_{g\omega_\epsilon}^+\right)\otimes\mathbb{R}^3}(0)$ to $\Phi_\epsilon(a, \zeta) = 0$, giving us existence, uniqueness, and the bound $\|(a, \zeta)\|_{\left(C_{\delta,g\omega_\epsilon}^{1,\alpha}\left(\mathring{\Omega}_{g\omega_\epsilon}^1(\mathrm{Km}_\epsilon)\right)\oplus\mathcal{H}_{g\omega_\epsilon}^+\right)\otimes\mathbb{R}^3} \leq 2L\|\Phi_\epsilon(0)\|_{C_{\delta-1,g\omega_\epsilon}^{0,\alpha}\left(\Lambda_{g\omega_\epsilon}^+ T^* \mathrm{Km}_\epsilon\right)\otimes\mathbb{R}^3} < C_{20}\epsilon^{2+\frac{1}{2}-\frac{\delta}{2}}$ since $r < r_0$. Smoothness follows from elliptic regularity, as our nonlinear equation has smooth coefficients & 1st order elliptic linearization and the produced solution (a, ζ) is in $C^{1,\alpha} \subset C^1$. Hence the setup in Section 4.1 as well as Anzatz 4.1.7 finish off the existence of our unique Ricci-flat hyper-Kähler metric for each sufficiently small $\epsilon > 0$, as was to be shown. ∎

Arguing the same way as *Remark* 4.3.3, namely that since *any* direction $e \in S^2$ gives us a Ricci-flat Kähler structure J_e, ω_e which is *compatible* with the given hyper-Kähler metric $g_{\omega_\epsilon + da + \zeta}$, then WLOG letting $e = e_1$ then getting $J_{e_1}, \mathrm{pr}_1(\omega_\epsilon + da + \zeta)$ aka $J_{e_1}, (\omega_\epsilon)_1 + da_1 + \zeta_1$ our Ricci-flat Kähler structure compatible with $g_{\omega_\epsilon + da + \zeta}$ that by definition of the weighted Holder spaces, the global bound $\epsilon \leq \rho_\epsilon$ and the fact that $\delta \in (-1, 0)$ and our smallness of norm bound $\|(a, \zeta)\|_{\left(C^{1,\alpha}_{\delta, g_{\omega_\epsilon}}\left(\mathring{\Omega}^1_{g_{\omega_\epsilon}}(\mathrm{Km}_\epsilon)\right) \oplus \mathcal{H}^+_{g_{\omega_\epsilon}}\right) \otimes \mathbb{R}^3} < C_{20} \epsilon^{2 + \frac{1}{2} - \frac{\delta}{2}}$ that

$$\|(\omega_\epsilon)_1 + da_1 + \zeta_1 - (\omega_\epsilon)_1\|_{C^0(\mathrm{Km}_\epsilon)} = \|da_1 + \zeta_1\|_{C^0(\mathrm{Km}_\epsilon)} \leq C_{21} \epsilon^{2 + \frac{1}{2} - \frac{\delta}{2}} \epsilon^{\delta - 1} = C_{21} \epsilon^{\frac{3}{2} + \frac{\delta}{2}}$$

for some *uniform* constant $C_{21} > 0$, whence $\|(\omega_\epsilon)_1 + da_1 + \zeta_1 - (\omega_\epsilon)_1\|_{C^0(\mathrm{Km}_\epsilon)}$ tends to 0 as $\epsilon \searrow 0$ due to $\delta \in (-1, 0)$.

Thus since $(\omega_\epsilon)_1 + da_1 + \zeta_1$ is compatible with $g_{\omega_\epsilon + da + \zeta}$ via 2-out-of-3 directly implying that the above inequality implies that the identity map $\mathrm{id}_{\mathrm{Km}_\epsilon}$ is an $O(1)\epsilon^{\frac{3}{2} + \frac{\delta}{2}}$-Gromov-Hausdorff approximation, *Remark* 4.3.2, the two Gromov-Hausdorff convergences in the proof of Proposition 4.6.4, Proposition 4.2.1 regarding the curvature blow-up of the bolt of $g_{\omega_{EH, \epsilon^2}}$ and Propositions 4.2.3 and 4.3.1 on the construction of g_{ω_ϵ}, and the triangle inequality for the Gromov-Hausdorff metric, we thus have the analogue of Theorem 3.7.3 in the hyper-Kähler context:

Theorem 4.7.2 (Main Theorem IIa). *Let K3 denote the K3 surface.*

Equip K3 with the family of closed definite triples ω_ϵ constructed in Section 4.3, hence getting the family of almost hyper-Kähler metrics $\left(\mathrm{Km}_\epsilon, \omega_\epsilon, g_{\omega_\epsilon}\right)$ on a family of Kummer surfaces Km_ϵ.

*Then denoting by $\widetilde{\omega_\epsilon} := \omega_\epsilon + da + \zeta \in [\omega_\epsilon]$ and $g_{\widetilde{\omega_\epsilon}}$ the unique Ricci-flat hyper-Kähler metric produced from Theorem 4.7.1 for each $\epsilon > 0$, we have that $0 < \exists \epsilon_0 \ll 1$ such that **there exists a 1-parameter family of hyper-Kähler triples and hyper-Kähler metrics***

$$\left\{\left(\widetilde{\omega_\epsilon}, g_{\widetilde{\omega_\epsilon}}\right)\right\}_{\epsilon \in (0, \epsilon_0)}$$

on K3.

Moreover when $\epsilon \searrow 0$,

$$\left(\mathrm{Km}_\epsilon, \omega_\epsilon, g_{\omega_\epsilon}\right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, \omega_0, g_{\omega_0}\right)$$

converges in the Gromov-Hausdorff topology to the flat hyper-Kähler orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, \omega_0, g_{\omega_0}\right)$ with the singular orbifold hyper-Kähler metric ω_0, g_{ω_0} (which is equal to the flat hyper-Kähler metric on $\mathbb{T}^4/\mathbb{Z}_2 - S$).

Moreover, we may decompose K3 into a union of sets[39] $\mathrm{Km}^{reg}_\epsilon \cup \bigsqcup_{p \in S} \mathrm{Km}^{b_p}_\epsilon$ such that

(1) *$\left(\mathrm{Km}^{reg}_\epsilon, \omega_\epsilon, g_{\omega_\epsilon}\right)$ collapses to the flat Kähler manifold $\left(\mathbb{T}^4/\mathbb{Z}_2 - S, \omega_0, g_{\omega_0}\right)$ with bounded curvature away from S (**regular region**).*

(2) *For each $p \in S$, $\left(\mathrm{Km}^{b_p}_\epsilon, \frac{1}{\epsilon^2} g_{\omega_\epsilon}\right)$ converges to $\left(T^* S^2, g_{\omega_{EH, 1}}\right)$ in C^∞_{loc} (**ALE bubble region**).*

$\blacklozenge$

In fact, what we have proved yields more:

Enumerate the 16 orbifold singularities $S = \{p_1, \ldots, p_{16}\}$ of $\mathbb{T}^4/\mathbb{Z}_2$.

Pick a fixed flat hyper-Kähler metric ω_0, g_{ω_0} on $\mathbb{T}^4/\mathbb{Z}_2$, as from Proposition 3.2.8 there is a $\frac{4(4+1)}{2} = 10$-parameter family of flat metrics on $\mathbb{T}^4/\mathbb{Z}_2$.

For any subset $I \subseteq \{1, \ldots, 16\}$, denote its complement as $I^c := \{1, \ldots, 16\} - I$ whence $I \sqcup I^c = \{1, \ldots, 16\}$.

For a fixed $I \subseteq \{1, \ldots, 16\}$, repeat the gluing construction in Section 4.3 on the base orbifold $\mathbb{T}^4/\mathbb{Z}_2$, but instead we

- pick a flat hyper-Kähler metric ω_f, g_{ω_f} on $\mathbb{T}^4/\mathbb{Z}_2$ where $f \in \mathrm{GL}(4, \mathbb{Z}) \backslash \mathrm{GL}(4, \mathbb{R})/O(4)$ the moduli space of flat metrics on $\mathbb{T}^4/\mathbb{Z}_2$ which is a $\frac{4(4+1)}{2} = 10$-parameter/dimensional space (Proposition 3.2.8).

---

[39] say, $\mathrm{Km}^{reg}_\epsilon := \left\{\frac{\epsilon^{\frac{1}{2}}}{2} < |\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)}\right\}$ and each Km^p_ϵ being the 16 connected components of $\left\{|\cdot|^{g_{\omega_0}}_{\pi^{-1}(S)} \leq \epsilon^{\frac{1}{2}}\right\}$.

- *leave $p_i \in S \subset \mathbb{T}^4/\mathbb{Z}_2$ for $i \in I^c$ unresolved.*

- recalling from Proposition 4.2.1

$$\left\{ \left(T^*S^2, \omega_{t,e}^{ALE}, g_{\omega_{t,e}^{ALE}} \right) \right\}_{(t,e) \in \mathbb{R}_{>0} \times S^2}$$

the 3-parameter family of hyper-Kähler ALE spaces asymptotic to $\mathbb{R}^4/\mathbb{Z}_2$ and where

$$\omega_{t,e}^{ALE}, g_{\omega_{t,e}^{ALE}} \stackrel{\text{isometric}}{\cong} A^e \omega_{EH,t^2}, g_{\omega_{EH,t^2}} \stackrel{\text{isometric}}{\cong} t^2 A^e \omega_{EH,1}, t^2 g_{\omega_{EH,1}}$$

for each $i \in I$ we choose a pair $(\epsilon_i, e_i) \in \mathbb{R}_{>0} \times S^2$ and glue $\left(T^*S^2, \omega_{\epsilon_i,e_i}^{ALE}, g_{\omega_{\epsilon_i,e_i}^{ALE}} \right)$ around the i-th singular point $p_i \in S$ using the isometry above to equivalently glue $\left(T^*S^2, A^{e_i} \omega_{EH,\epsilon_i^2}, g_{\omega_{EH,\epsilon_i^2}} \right)$ via the same cutoff function interpolation of Propositions 4.2.2 and 4.2.3.

This yields a compact *orbifold* $\text{Km}_{f,\epsilon_I,e_I}^{|I|}$ with $16 - |I|$ orbifold points $p_i \in \text{Km}_{f,\epsilon_I,e_I}^{|I|}, i \in I^c$ each locally modeled on $\mathbb{R}^4/\mathbb{Z}_2$ which we call the *I-th partial smoothing* $\text{Km}^{|I|} = \text{Km}_{f,\epsilon_I,e_I}^{|I|}$ of $\mathbb{T}^4/\mathbb{Z}_2$, equipped with the constructed closed *orbifold* definite triple which is *almost* hyper-Kähler. Since the resulting closed *orbifold* definite triple on $\text{Km}_{f,\epsilon_I,e_I}^{|I|}$ that gets constructed is a genuine hyper-Kähler triple near both the $|I|$ exceptional S^2s and near each orbifold point $p_i \in \text{Km}_{f,\epsilon_I,e_I}^{|I|}, i \in I^c$, namely the *smooth* Eguchi-Hanson hyper-Kähler triple $A^{e_i} \omega_{EH,\epsilon_i^2}$ and the flat *orbifold* hyper-Kähler triple ω_0 near each $S^2 \cong E_i, i \in I$ and $p_j \in \text{Km}_{\epsilon_I}, j \in I^c$ respectively, and satisfies the *same* associated intersection matrix decay $Q = \text{id} + O(\epsilon^2)$ on each annulus $\left\{ \epsilon_i^{\frac{1}{2}} \leq |\cdot|_{E_i}^{g_{\omega_0}} \leq 2\epsilon_i^{\frac{1}{2}} \right\}$ around each exceptional $S^2 \cong E_i$ for $i \in I$, the analysis developed in Sections 4.4, 4.5, 4.6, and in the proof of Theorem 4.7.1 to perturb approximate hyper-Kähler triples to genuine hyper-Kähler triples via solving the nonlinear elliptic PDE system may be carried out with only cosmetic adjustments in the presence of flat hyper-Kähler orbifold singularities (via lifting to a $\mathbb{Z}_2$-invariant problem on a *smooth* manifold since all orbifold singularities in our context have isotropy group $\mathbb{Z}_2$) to yield an **orbifold hyper-Kähler triple** $\omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}}$ on the compact orbifold $\text{Km}_{f,\epsilon_I,e_I}^{|I|}$ for when each of the gluing parameters $\{ \epsilon_i : i \in I \}$ are sufficiently small.

Whence since the proofs of both the **regular region** and **ALE bubble region** structures in Theorem 4.7.1 above follow from the Gromov-Hausdorff convergences in the proof of Proposition 4.6.4, which follow via examining the behavior as $\epsilon \searrow 0$ of the regions of Km_ϵ in the bulk and *near* each exceptional S^2 *after rescaling the metric*, respectively, we immediately have that such Gromov-Hausdorff convergences in the bulk and near each exceptional S^2 follow verbatim for the compact hyper-Kähler orbifolds $\left(\text{Km}_{f,\epsilon_I,e_I}^{|I|}, \omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}} \right)$ because the region near each exceptional S^2 is *localized* near S^2, whence looks the same irrespective of whether or not there are flat orbifold singularities in its complement. Thus these cosmetic adjustments yield us Theorem 1.2, namely

Theorem 4.7.3 (Main Theorem II). *Enumerate the 16 singular points $S = \{p_1, \ldots, p_{16}\}$ of $\mathbb{T}^4/\mathbb{Z}_2$.*

*Then $\forall I \subseteq \{1, \ldots, 16\}, 0 < \exists \epsilon_I^0 \ll 1$ such that **there exists a $10 + 3|I|$-parameter family of (orbifold) hyper-Kähler triples and (orbifold) hyper-Kähler metrics***

$$\left\{ \left(\omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}} \right) \right\}_{\substack{f \in \text{GL}(4,\mathbb{Z}) \backslash \text{GL}(4,\mathbb{R})/O(4) \\ \epsilon_i \in (0,\epsilon_I^0) \\ e_i \in S^2 \\ i \in I}}$$

on the I-th partial smoothing $\text{Km}^{|I|} = \text{Km}_{f,\epsilon_I,e_I}^{|I|}$ of $\mathbb{T}^4/\mathbb{Z}_2$.

Moreover, as $\epsilon_I^0 \searrow 0$,

$$\left(\text{Km}_{f,\epsilon_I,e_I}^{|I|}, \omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}} \right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, \omega_f, g_{\omega_f} \right)$$
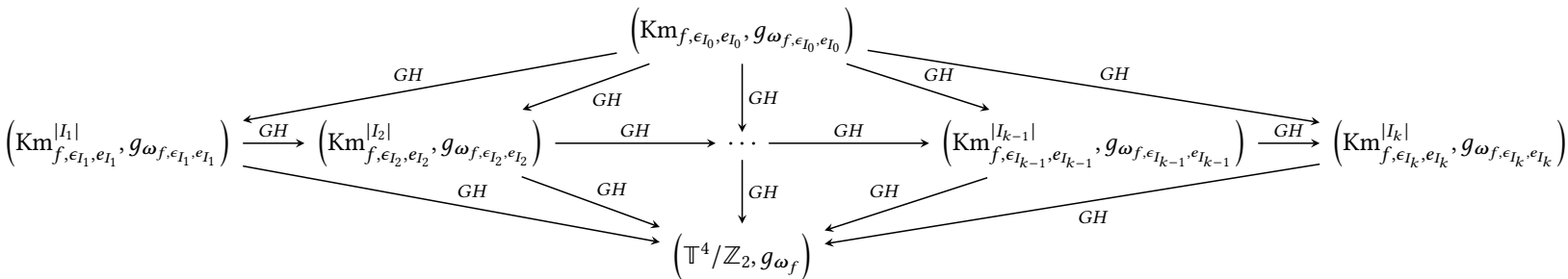
converges in the Gromov-Hausdorff topology to the flat hyper-Kähler orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}\right)$ *with the singular orbifold hyper-Kähler metric* $\boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}$. *Moreover, we may decompose each compact orbifold* $\mathrm{Km}^{|I|}$ *into a union of sets*[40] $\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{reg} \cup \bigsqcup_{i\in I}\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{b_i}$ *such that*

(1) $\left(\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{reg}, \boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)$ *collapses to the flat Kähler orbifold* $\left(\mathbb{T}^4/\mathbb{Z}_2 - \{p_i\}_{i\in I}, \boldsymbol{\omega}_0, g_{\boldsymbol{\omega}_0}\right)$ *with bounded curvature away from* $p_i \in S$ *for* $i \in I$ *(**regular region**).*

(2) *For each* $i \in I$, $\left(\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{b_i}, \frac{1}{\epsilon^2}\boldsymbol{\omega}_{f,\epsilon_I,e_I}, \frac{1}{\epsilon^2}g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)$ *converges to* $\left(T^*S^2, A^e\boldsymbol{\omega}_{EH,1}, g_{\boldsymbol{\omega}_{EH,1}}\right)$ *in* C^∞_{loc} *(**ALE bubble region**).*

In particular, when $I = \{1,\dots,16\}, 0 < \exists\epsilon^0 \ll 1$ *such that **there exists a** $10 + 3\cdot 16 = 58$-**parameter family of hyper-Kähler triples and hyper-Kähler metrics***

$$\left\{\left(\boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)\right\}_{\substack{f\in\mathrm{GL}(4,\mathbb{Z})\backslash\mathrm{GL}(4,\mathbb{R})/O(4) \\ \epsilon_i\in\left(0,\epsilon^0\right) \\ e_i\in S^2 \\ i=1,\dots,16}}$$

*on the **K3 surface*** $\mathrm{Km}^{16} = K3 = \mathrm{Km}_{f,\epsilon_I,e_I}$.

Moreover, as $\epsilon^0 \searrow 0$,

$$\left(\mathrm{Km}_{f,\epsilon_I,e_I}, \boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}\right)$$

converges in the Gromov-Hausdorff topology to the flat hyper-Kähler orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, \boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}\right)$ *with the singular orbifold hyper-Kähler metric* $\boldsymbol{\omega}_f, g_{\boldsymbol{\omega}_f}$. *Moreover, we may decompose $K3$ into a union of sets* $\mathrm{Km}^{reg}_{f,\epsilon_I,e_I} \cup \bigsqcup_{i=1}^{16}\mathrm{Km}^{b_i}_{f,\epsilon_I,e_I}$ *such that*

(1) $\left(\mathrm{Km}^{reg}_{f,\epsilon_I,e_I}, \boldsymbol{\omega}_{f,\epsilon_I,e_I}, g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)$ *collapses to the flat Kähler manifold* $\left(\mathbb{T}^4/\mathbb{Z}_2 - S, \boldsymbol{\omega}_0, g_{\boldsymbol{\omega}_0}\right)$ *with bounded curvature away from* S *(**regular region**).*

(2) *For each* $i = 1,\dots,16$, $\left(\mathrm{Km}^{b_i}_{f,\epsilon_I,e_I}, \frac{1}{\epsilon^2}\boldsymbol{\omega}_{f,\epsilon_I,e_I}, \frac{1}{\epsilon^2}g_{\boldsymbol{\omega}_{f,\epsilon_I,e_I}}\right)$ *converges to* $\left(T^*S^2, A^e\boldsymbol{\omega}_{EH,1}, g_{\boldsymbol{\omega}_{EH,1}}\right)$ *in* C^∞_{loc} *(**ALE bubble region**).*

Therefore, letting $k \in \{0,\dots,16\}$ *be a chosen nonnegative integer (the **depth**), upon considering the following strictly nested sequence*

$$\{1,\dots,16\} = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_k$$

with $I_{16} := \emptyset$, *then we have the following sequence of **iterated volume non-collapsed degenerations** of our 58-parameter family of hyper-Kähler metrics* $\boldsymbol{\omega}_{f,\epsilon_{I_0},e_{I_0}}, g_{\boldsymbol{\omega}_{f,\epsilon_{I_0},e_{I_0}}}$ *on the K3 surface* $\mathrm{Km}^{|I_0|}_{f,\epsilon_{I_0},e_{I_0}} = \mathrm{Km}_{f,\epsilon_{I_0},e_{I_0}}$:

where we

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say, $\left(\mathrm{Km}^{|I|}_{f,\epsilon_I,e_I}\right)^{reg} := \left\{\frac{(\max\epsilon_I)^{\frac{1}{2}}}{2} < |\cdot|^{g_{\boldsymbol{\omega}_f}}_{\bigsqcup_{i\in I}E_i}\right\}$ *and each* $\mathrm{Km}^{b_i}_\epsilon$ *being* $\left\{|\cdot|^{g_{\boldsymbol{\omega}_f}}_{E_i} \leq (\max\epsilon_I)^{\frac{1}{2}}\right\}$.

- *degenerate from our hyper-Kähler K3 surface* $\left(\mathrm{Km}_{f,\epsilon_{I_0},e_{I_0}}, g_{\omega_{f,\epsilon_{I_0},e_{I_0}}}\right)$ *to* $\left(\mathrm{Km}_{f,\epsilon_{I_j},e_{I_j}}^{|I_j|}, \omega_{f,\epsilon_{I_j},e_{I_j}}, g_{\omega_{f,\epsilon_{I_j},e_{I_j}}}\right)$ *via only collapsing the gluing parameters* $\epsilon_i > 0$ *corresponding to* $i \in I_0 - I_j = I_j^c$

- *degenerate from* $\left(\mathrm{Km}_{f,\epsilon_{I_{j-1}},e_{I_{j-1}}}^{|I_{j-1}|}, \omega_{f,\epsilon_{I_{j-1}},e_{I_{j-1}}}, g_{\omega_{f,\epsilon_{I_{j-1}},e_{I_{j-1}}}}\right)$ *to* $\left(\mathrm{Km}_{f,\epsilon_{I_j},e_{I_j}}^{|I_j|}, \omega_{f,\epsilon_{I_j},e_{I_j}}, g_{\omega_{f,\epsilon_{I_j},e_{I_j}}}\right)$ *via only collapsing the gluing parameters* $\epsilon_i > 0$ *corresponding to* $i \in I_{j-1} - I_j$

- *and degenerate from* $\left(\mathrm{Km}_{f,\epsilon_{I_j},e_{I_j}}^{|I_j|}, \omega_{f,\epsilon_{I_j},e_{I_j}}, g_{\omega_{f,\epsilon_{I_j},e_{I_j}}}\right)$ *to* $\left(\mathbb{T}^4/\mathbb{Z}_2, \omega_f, g_{\omega_f}\right)$ *via collapsing all gluing parameters* $\epsilon_i > 0, i \in I_j$.

♦

Remark 4.7.4. Consider the depth $k = 1$ case of the iterated volume non-collapsed degenerations of our 58-parameter family of hyper-Kähler metrics on $K3$ and where $|I_1| = 15$. That is, from $\left(\mathrm{Km}_{f,\epsilon_I,e_I}, \omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}}\right)$ with $I := \{1, \ldots, 16\}$ we only collapse a *single* gluing parameter ϵ_i for $i \in I - I_1 = I_1^c$, resulting in a compact hyper-Kähler orbifold $\left(\mathrm{Km}_{f,\epsilon_{I_1},e_{I_1}}^{|I_1|}, \omega_{f,\epsilon_{I_1},e_{I_1}}, g_{\omega_{f,\epsilon_{I_1},e_{I_1}}}\right)$ with a single orbifold singularity modeled on $\mathbb{R}^4/\mathbb{Z}_2$. This resulting compact orbifold $\mathrm{Km}^{|I_1|}$ thus has a $10 + 3|I_1| = 10 + 3 \cdot 15 = 55$-parameter family of orbifold hyper-Kähler metrics $\left(\omega_{f,\epsilon_{I_1},e_{I_1}}, g_{\omega_{f,\epsilon_{I_1},e_{I_1}}}\right)$. As remarked back in Section 1, the volume non-collapsed limits of (unit volume) hyper-Kähler metrics on $K3$ corresponds under the period map to points in the *codimension 3* subset

$$\bigcup_{\substack{\Sigma \in H_2(K3;\mathbb{Z}) \\ \Sigma \cdot \Sigma = -2}} \left\{ H \in \mathrm{Gr}^+(3, 19)/\Gamma : [\alpha](\Sigma) = 0, \forall [\alpha] \in H \right\}$$

of $\mathrm{Gr}^+(3, 19)/\Gamma$ in which the original period map $\mathcal{P} : \mathcal{M}_{\mathrm{Vol}=1}^{K3} \to \mathrm{Gr}^+(3, 19)/\Gamma$ fails to be surjective.

Therefore, upon rescaling the metrics involved to have unit volume and thus sacrificing a choice of parameter (namely where we fix the volume of the starting flat hyper-Kähler orbifold $\left(\mathbb{T}^4/\mathbb{Z}_2, \omega_f, g_{\omega_f}\right)$), Theorem 4.7.3 therefore produces a 54-parameter family of orbifold hyper-Kähler metrics $\left(\mathrm{Km}_{f,\epsilon_{I_1},e_{I_1}}^{|I_1|}, \omega_{f,\epsilon_{I_1},e_{I_1}}, g_{\omega_{f,\epsilon_{I_1},e_{I_1}}}\right)$ which arise as volume non-collapsed Gromov-Hausdorff limit spaces of a 57-parameter family of hyper-Kähler metrics $\left(\mathrm{Km}_{f,\epsilon_I,e_I}, \omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}}\right)$ on $K3$. Thus since

$$57 - 54 = 3$$

we have that this is an explicit example of the codimension 3 nature of the "holes" that the range of the original period map misses in its codomain $\mathrm{Gr}^+(3, 19)/\Gamma$. And indeed, from Proposition 4.2.1 and construction of $\left(\mathrm{Km}_{f,\epsilon_I,e_I}, \omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}}\right)$ we have that the diameter of the ith exceptional $E_i \cong S^2$ in the $K3$ surface $\mathrm{Km}_{f,\epsilon_I,e_I}$ has volume $\pi\epsilon_i^2$ and diameter $\frac{\epsilon_i}{2}$ and self-intersection number $E_i \cdot E_i = S^2 \cdot S^2 = -2$, so collapsing ϵ_i for $i \in I - I_1 = I_1^c$ to get to our compact hyper-Kähler orbifold $\left(\mathrm{Km}_{f,\epsilon_{I_1},e_{I_1}}^{|I_1|}, \omega_{f,\epsilon_{I_1},e_{I_1}}, g_{\omega_{f,\epsilon_{I_1},e_{I_1}}}\right)$ *precisely* gives us our $\Sigma \in H_2(K3;\mathbb{Z})$ with $\Sigma \cdot \Sigma = -2$ and a "large"/*top dimensional* subset of the points $H \in \mathrm{Gr}^+(3, 19)/\Gamma$ such that $[\alpha](\Sigma) = 0, \forall [\alpha] \in H$, namely:

$$\Sigma := E_i$$

where $i \in I - I_1 = I_1^c$ and

$$\left\{ \mathrm{span}_{\mathbb{R}}\left(\omega_{f,\epsilon_j,e_j}\right) : f \in \text{unit vol flat}, \epsilon_j \in \left(0, \epsilon_{I_1}^0\right), e_j \in S^2, j \in I_1 \right\} \subseteq \left\{ H \in \mathrm{Gr}^+(3, 19)/\Gamma : [\alpha](\Sigma) = 0, \forall [\alpha] \in H \right\}$$

and

$$\dim_{\mathbb{R}} \left\{ H \in \mathrm{Gr}^+(3, 19)/\Gamma : [\alpha](\Sigma) = 0, \forall [\alpha] \in H \right\}$$
$$= \dim_{\mathbb{R}} \left\{ \mathrm{span}_{\mathbb{R}}\left(\omega_{f,\epsilon_j,e_j}\right) : f \in \text{unit vol flat}, \epsilon_j \in \left(0, \epsilon_{I_1}^0\right), e_j \in S^2, j \in I_1 \right\}$$
$$= 54$$

Here we are implicitly viewing $\omega_{f,\epsilon_j,e_j}$ as a *singular* hyper-Kähler triple on $K3$ gotten by pullback along the smoothing map $\mathrm{Km}_{f,\epsilon_I,e_I} \twoheadrightarrow \mathrm{Km}^{|I_1|}_{f,\epsilon_{I_1},e_{I_1}}$. ♣

Remark 4.7.5. Therefore generalizing *Remark* 4.7.4 for arbitrary depth $k \in \{1, \ldots, 16\}$ (excluding the trivial $k = 0$ case) whence $I_k \subsetneq \{1, \ldots, 16\}$, we thus have that upon rescaling all metrics to have unit volume, that Theorem 4.7.3 produces a $9 + 3 \cdot |I_k|$-parameter family of orbifold hyper-Kähler metrics $\left(\mathrm{Km}^{|I_k|}_{f,\epsilon_{I_k},e_{I_k}}, \omega_{f,\epsilon_{I_k},e_{I_k}}, g_{\omega_{f,\epsilon_{I_k},e_{I_k}}}\right)$ which arise as volume non-collapsed Gromov-Hausdorff limit spaces of a 57-parameter family of hyper-Kähler metrics $\left(\mathrm{Km}_{f,\epsilon_I,e_I}, \omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}}\right)$ on $K3$. Thus since

$$57 - (9 + 3 \cdot |I_k|) = 48 - 3 \cdot |I_k|$$
$$= 3 \cdot (16 - |I_k|)$$
$$> 3$$

since $I_k \subsetneq \{1, \ldots, 16\}$, we have that these explicit examples elucidate/"further probe down" into the codimension 3 holes $\bigcup_{\substack{\Sigma \in H_2(K3;\mathbb{Z}) \\ \Sigma \cdot \Sigma = -2}} \{H \in \mathrm{Gr}^+(3, 19)/\Gamma : [\alpha](\Sigma) = 0, \forall [\alpha] \in H\}$ of $\mathrm{Gr}^+(3, 19)/\Gamma$ in which the original period map $\mathcal{P} : \mathscr{M}^{K3}_{\mathrm{Vol}=1} \to \mathrm{Gr}^+(3, 19)/\Gamma$ fails to be surjective *by exhibiting a* **stratification** *of that codimension 3 subset according to the size of* I_k^c. Thus our large family of orbifold hyper-Kähler metrics $\left(\mathrm{Km}^{|I_k|}_{f,\epsilon_{I_k},e_{I_k}}, \omega_{f,\epsilon_{I_k},e_{I_k}}, g_{\omega_{f,\epsilon_{I_k},e_{I_k}}}\right)$ corresponds inside the codimesion 3 subset $\bigcup_{\substack{\Sigma \in H_2(K3;\mathbb{Z}) \\ \Sigma \cdot \Sigma = -2}} \{H \in \mathrm{Gr}^+(3, 19)/\Gamma : [\alpha](\Sigma) = 0, \forall [\alpha] \in H\} \subset \mathrm{Gr}^+(3, 19)/\Gamma$ to

$$\bigcup_{i \in I - I_k = I_k^c} \left\{ \mathrm{span}_{\mathbb{R}}\left(\omega_{f,\epsilon_j,e_j}\right) : f \in \text{unit vol flat}, \epsilon_j \in \left(0, \epsilon_{I_k}^0\right), e_j \in S^2, j \in I_k \right\}$$

i.e. corresponds to the union of $16 - |I_k|$ subsets, each with dimension $9 + 3 \cdot |I_k|$ and each corresponding to the $16 - |I_k|$ exceptional $E_i \cong S^2$s for $i \in I_k^c$, which are clearly 2-cycles inside $K3$ with self-intersection -2, volume $\pi \epsilon_i^2$, and diameter $\frac{\epsilon_i}{2}$. ♣

Remark 4.7.6. Since Ricci-flat metrics are Einstein, and since each $(\mathrm{Km}_\epsilon, J, \widetilde{g}_\epsilon, \widetilde{\omega}_\epsilon)$ in Theorem 3.7.3 and each $\left(\mathrm{Km}_{f,\epsilon_I,e_I}, \omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}}\right)$ in Theorem 4.7.3 clearly have uniformly bounded diameter and satisfies (uniform) *volume non-collapsing*, the convergences

$$(\mathrm{Km}_\epsilon, J, \widetilde{g}_\epsilon, \widetilde{\omega}_\epsilon) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0\right)$$
$$\left(\mathrm{Km}_{f,\epsilon_I,e_I}, \omega_{f,\epsilon_I,e_I}, g_{\omega_{f,\epsilon_I,e_I}}\right) \xrightarrow{GH} \left(\mathbb{T}^4/\mathbb{Z}_2, \omega_f, g_{\omega_f}\right)$$
$$\left(\mathrm{Km}_{f,\epsilon_{I_0},e_{I_0}}, g_{\omega_{f,\epsilon_{I_0},e_{I_0}}}\right) \xrightarrow{GH} \left(\mathrm{Km}^{|I_j|}_{f,\epsilon_{I_j},e_{I_j}}, \omega_{f,\epsilon_{I_j},e_{I_j}}, g_{\omega_{f,\epsilon_{I_j},e_{I_j}}}\right)$$

proven in those two theorems exhibit examples in which the codimension 4 conjecture is *sharp*, since the orbifold singularities of $\left(\mathbb{T}^4/\mathbb{Z}_2, J_0, g_0, \omega_0\right)$, $\left(\mathbb{T}^4/\mathbb{Z}_2, \omega_f, g_{\omega_f}\right)$ and $\left(\mathrm{Km}^{|I_j|}_{f,\epsilon_{I_j},e_{I_j}}, \omega_{f,\epsilon_{I_j},e_{I_j}}, g_{\omega_{f,\epsilon_{I_j},e_{I_j}}}\right)$ clearly have (real) codimension 4. Indeed, the **codimension 4 conjecture**, now a theorem by Cheeger and Naber [7], states that the (pointed) Gromov-Hausdorff limit spaces of *non-collapsed* Riemannian manifolds with bounded Ricci curvature are smooth away from a closed set of (real) Hausdorff codimension 4. The compact Einstein case, proven by Anderson [1], Bando-Kasue-Nakajima [3], and Nakajima [30], in fact tells us that when the sequence of manifolds in question are Einstein 4-manifolds having *uniform* Ricci upper bound, uniform diameter upper bound, uniform Euler characteristic upper bound, and uniform volume *lower* bound, a subsequence converges to an Einstein *orbifold* with finitely many isolated singular points. Whence Theorem 3.7.3, Theorem 4.7.2, and each of the convergences in Theorem 4.7.3 gives us many explicit examples of this, as each of the underlying smooth manifolds in $(\mathrm{Km}_\epsilon, J, \widetilde{g}_\epsilon, \widetilde{\omega}_\epsilon)$ and $\left(\mathrm{Km}_{f,\epsilon_{I_0},e_{I_0}}, g_{\omega_{f,\epsilon_{I_0},e_{I_0}}}\right)$ are the $K3$ surface having Euler characteristic 24. ♣

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