The Kummer Construction of Ricci-Flat Kähler Metrics on $K3$ via Weighted Analysis

Thomas Jiang[†]

[†]email: thomas.jiang@duke.edu
Department of Mathematics, Duke University, Durham, NC, USA;

April 2023

Abstract

In this undergraduate thesis[1], we construct Ricci-flat Kähler metrics on $K3$ surfaces arising from the Kummer construction via gluing Eguchi-Hanson metrics around each exceptional divisor.

In the simplest setting of Calabi-Yau metrics (i.e. with fixed complex structure, as opposed to hyperKähler metrics, which are equivalent in complex dimension 2), this "Kummer construction" of Ricci-flat Kähler metrics on $K3$ has yet to be carried out with the machinery of weighted spaces.

We therefore carry out this construction using weighted Holder spaces as the main analytical setup.

§ Introduction §

A $K3$ surface is a compact simply connected complex 2-fold with trivial canonical bundle. By a theorem of Kodaira, any pair of $K3$ surfaces are diffeomorphic to one another.[2]

Since the $K3$ surface is Kähler, Yau's resolution of the Calabi conjecture shows that $K3$ surface admits a Ricci flat Kähler metric in any given Kähler class. However, Yau's theorem only gives us abstract existence of these Ricci-flat Kähler metrics. It

[1] This is the *original version* of the author's honors undergraduate thesis that was written during the month of April in 2023 at Stony Brook University. A revised version with numerous additions will appear soon.

[2] Though, there do exist *topological $K3$ surfaces*, which are real 4-manifolds homeomorphic to a $K3$ but *not* diffeomorphic to it.

is therefore instructive to find more explicit/"hands-on" examples of these Ricci-flat metrics.

After Yau's proof, physicists Gibbons and Pope [9] suggested that a more concrete description of some of these Ricci-flat Kähler metric on $K3$ may be obtained via first viewing $K3$ as a "Kummer surface", and then approximating those Ricci flat metrics by a glued-in Eguchi-Hanson metric. Such a proposal has become known in the literature as the "Kummer construction" of (special, usually Ricci flat) metrics, and will be referred to as such. This proposal was first done rigorously by Topiwala [14] and LeBrun-Singer [11] via twistor theory methods over 3-4 decades ago, and later last decade by Donaldson [7] using standard PDE methods.

Hence the original "Kummer construction" of Gibbons-Pope for these specific Ricci-flat Kähler metrics on $K3$ is by now a classical topic in geometric analysis, having become the blueprint for and spawned multiple different variations and generalizations throughout the decades.

While Donaldson's proof of the original Kummer construction used standard PDE methods, he exploits the conformal equivalence between the cone metric over $\mathbb{R}P^3$ and the cylindrical metric on $\mathbb{R}P^3$ to bypass the usage of weighted Holder spaces. Weighted Holder spaces have been the standard analytical setup used in more advanced and modern gluing constructions. To the author's best knowledge this machinery has yet to be used to give an alternative rigorous proof of Gibbons-Pope's proposal.

In this thesis, we carry out this construction using the full apparatus of weighted Holder spaces, thus giving an alternative rigorous proof of the original Kummer construction.

Remark 0.1. Note that this construction does *not* give the full moduli space of all Ricci-flat Kähler metrics on $K3$. Rather, the original Kummer construction gives an explicit description of a small region of the "edge/boundary" of such moduli space. This is because the metrics it produces degenerate[3] to the singular Kähler metric on the orbifold whose resolution gives the "Kummer surface" (this metric is in fact the flat orbifold metric on the compliment of the orbifold singularities). ♣

Remark 0.2. It is important to note here that we are dealing with *Calabi-Yau* metrics as opposed to *hyper-Kähler* metrics (which, in complex dimension two, are equivalent since $\mathrm{SU}(2) \cong \mathrm{USp}(1)$). The latter has a S^2-worth of complex structures, while the former has a *fixed* complex structure.

The Kummer construction for *hyper-Kähler* metrics on $K3$ has been rigorously carried out using weighted Holder spaces quite recently by Lorenzo Foscolo in [8], where he goes on to examine new collapsing behavior of these hyper-Kähler

[3] More specifically, Gromov-Hausdorff converges

metrics, as well as the full moduli space of Ricci-flat Kähler metrics on $K3$. Foscolo's paper is therefore quite a bit more technical than this thesis, though the overall strategy (as in every gluing construction) is exactly the same (i.e. glue, perturb, invert linearization in suitable weighted spaces, finish using implicit function theorem). ♣

Remark 0.3. Similar analysis has been carried out in slightly different settings. Biquard-Minerbe [2] did the Kummer construction with weighed Holder spaces for more general ALF, ALG, and ALH asymptotics. Brendle-Kapouleas [3] did the original Kummer construction considered here, but they glued in the 16 Eguchi-Hanson spaces with half of them having *opposite orientation* (arranged in a checkers board patter). As a result, Brendle-Kapouleas had to solve the full Einstein equation $\mathrm{Ric} = 0$ because the swapped orientation forces one to leave the Kähler realm (in fact, it breaks the complex structure preservation and forces the metric to have *generic holonomy*). They end up showing that the relevant linearization has an obstructed kernel, thus nullifying the construction and dashing any hopes of using the original Kummer construction to construct Ricci flat metrics on *closed* manifolds with *generic holonomy* (as opposed to Calabi-Yau, aka Ricci-flat Kähler, with *special* $\mathrm{SU}(n)$ holonomy). ♣

In the first section, we list some preliminary definitions used and give a derivation of the Monge-Ampere equation for (more general) Kähler-Einstein metrics, which we'll have to solve later on in our specific setting.

In the second section, we define the Eguchi-Hanson metric on the blowup of $\mathbb{C}^2/\mathbb{Z}_2$ and prepare it for the gluing process.

In the third section, we construct the "Kummer surface" $\mathscr{K}$ by blowing up the orbifold points of $\mathbb{T}^4/\{\pm 1\}$ in a manner suitable for gluing, and we graft in the prepped Eguchi-Hanson metrics to get an apprioximatly Ricci-flat Kähler metric ω_ϵ on $\mathscr{K}$.

In the fourth section, we define the relevant weighted Holder spaces, first on the flat Euclidean space $\mathbb{C}^n$, then on the Eguchi-Hanson space $\widetilde{\mathbb{C}^2/\mathbb{Z}_2}$, and then on $\mathscr{K}$.

In the fifth section, we setup the nonlinear PDE problem of perturbing the approximately Ricci flat ω_ϵ to a genuinely Ricci-flat Kähler metric within its Kähler class as a fully nonlinear elliptic PDE of Monge-Ampere type, in a manner suitable for the implicit function theorem.

In the sixth section, we invert the linearization of the nonlinear map defined in the previous section in the relevant weighted Holder spaces. This is the heart of the argument, as we need the weighted Holder spaces to get *uniform* bounded operator norm on the inverse of the linearization.

Finally, in the seventh and last section, we finish off the proof of the Kummer construction by applying the implicit function theorem to get our solution to the Monge-Ampere equation.

Acknowledgments. This is the *original version* of the author's honors undergraduate thesis that was written during the month of April in 2023 at Stony Brook University.

I am in tremendous debt to my advisor Professor Simon Donaldson for taking on an undergraduate student, bearing with my questions and slow understanding, teaching me a crazy amount of beautiful math (of which this thesis is only a portion of all the beautiful math that I've learned from him), and simply sacrificing time out of a super busy schedule to meet with me every 2-3 weeks. These last ~ 1.5 years under the tutilage of Professor Donaldson will go down as one of the best years of my mathematical life, and he has left an indelible impression on my mathematical trajectory. It has been a joyous and tremendous privilege to study and learn from one of my mathematical heros, and I hope to keep learning from him in the years and decades ahead as I enter graduate school and beyond.

Thanks also to Professor Marcus Khuri for referring me to Bartnick [1] for theorems regarding weighted Holder spaces, for bearing with my many questions regarding the analytical side of this thesis, and for also teaching me tons of beautiful math and research insider tricks.

Last but not least, thanks also to Jian Wang, Scott Sutherland, Mark McLean, Dashen Yan, Dylan Galt, and many others for teaching and helping me with so much, both mathematically and beyond.

Soli Deo Gloria! ✝

§ Section 0: Notation §

- Everything is assumed $C^{\infty,\omega}$ when appropriate/unless explicitly mentioned otherwise, and all manifolds are connected.
- B_R denotes the open ball $\{|z| < R\}$
- D_R denotes the disk $\{|z| \leq R\}$
- We sometimes denote by $B_R(E)$ or $\{|z| < R\}$ to be the *open* tubular neighborhood of an embedded submanifold E of some ambient manifold with $|\cdot|$ being the distance function from E.
- Likewise, we sometimes denote by $D_R(E)$ or $\{|z| \leq R\}$ to be the *closed* tubular neighborhood of an embedded submanifold E, etc..
- We sometimes use $f|_x$ to denote a function f evaluated at x, i.e. $f(x)$, to simplify notation.

§ Section 1: Preliminaries §

We collect some preliminary definitions and derivations, mostly following Walpuski's excellent Riemannian geometry lecture notes [15].

Definition 0.4. A *Kähler* manifold is a real $2n$ dimensional smooth manifold M with the following data:

(1) an almost complex structure $J \in \Gamma(\mathrm{Hom}(TM, TM))$, hence pointwise on each tangent space $J^2 = -\mathrm{id}$

(2) A Riemannian metric $g \in \Gamma(T^*M \odot T^*M)$

(3) a non-degenerate 2 form $\omega \in \Omega^2(M) := \Gamma\left(\bigwedge^2 T^*M\right)$ (nondegeneracy here means that, pointwise/on the vector space level, the map $V \longrightarrow V^*$ via $v \mapsto \omega(v, -)$ is an isomorphism, which is equivalent to $\frac{\omega^n}{n!}$ being a volume form).

such that the 3 pieces of data above are compatible in the following way:

$$\omega(-,-) = g(J-,-)$$

and one of the following equivalent conditions are satisfied:

- J is integrable (i.e. M is actually a complex n manifold) and $d\omega = 0$ (i.e. ω is symplectic)
- $\nabla J = 0$ where ∇ is the Levi-Civita connection of g
- The holonomy group of the Levi-Civita connection of g is contained in $\mathrm{U}(n)$.

Hence the data of (M, J, g, ω) satisfying the compatibility condition and one of the 3 equivalent conditions a *Kähler manifold*, and we call $[\omega] \in H^2(M)$ the *Kähler class* and ω the *Kähler form.* •

Remark 0.5. Note that if we just have the 3 pieces of data of (J, g, ω) satisfying the compatibility condition, but not one of the following equivalent conditions, we call this an *almost Hermitian manifold.* ♣

Remark 0.6. More importantly, note that *any cardinality 2 subset of* $\{J, g, \omega\}$ *automatically determines/**defines** the leftover element in its complement* via the compatability condition. Roughly speaking, this follows because $\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{Sp}(2n)$ (this may be made rigorous using the language of G-structures on the frame bundle $\mathrm{Fr}(TM)$, which we unfortunately don't have the space to digress into). In other words, Kähler geometry is the *intersection* of complex and Riemannian geometry, and complex and symplectic geometry, and symplectic and Riemannian geometry. This is sometimes called the **"2 out of 3"** property, and is one of the reasons why Kähler geometry is very rich. ♣

Now on any Kähler manifold (M, J, g, ω), we have that the canonical bundle $K_M := \bigwedge_{\mathbb{C}}^n T^*M^{(1,0)}$ has an induced Hermitian metric coming from the Hermitian metric $h := g - i\omega$ on TM (which as *complex* vector bundles is isomorphic to $TM^{(1,0)}$, a *holomorphic* vector bundle), and the induced Levi-Civita connection on K_M coming from the Levi-Civita connection of g is precisely the Chern connection of the induced Hermitian metric on K_M, i.e. it is unitary and has $(0,1)$ part the $\bar{\partial}$ operator inducing the holomorphic structure on K_M. Not only that, but the curvature 2 form F of this connection on K_M is precisely $i\mathrm{Ric}_\omega$, where $\mathrm{Ric}_\omega := \mathrm{Ric}_g(J-, -) \in \Omega^2(M)$. Hence since $c_1(M) = -\frac{i}{2\pi}[F]$ by Chern-Weil theory[4], this gives us that $c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}_\omega]$.

We summarize this and other properties needed in the sequel in the below proposition (proofs found in [15]):

Proposition 0.7. Let (M, J, g, ω) be a Kähler manifold.
Then the following properties hold:

- The first Chern class $c_1(M) \in H^2(M)$ of M is $c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}_\omega]$.
- For any two Kähler forms ω_1, ω_0, the difference between their Ricci forms is $\mathrm{Ric}_{\omega_1} - \mathrm{Ric}_{\omega_0} = -i\partial\bar{\partial}\log\left(\frac{\omega_1^n}{\omega_0^n}\right)$, with the division of two top degree forms having the obvious meaning.

Remark 0.8. Note that $c_1(M) = \frac{1}{2\pi}[\mathrm{Ric}_\omega]$ holds for **any** Kähler form ω on M, and hence $c_1(M)$ does **not** depend on the Kähler metric ω. In fact, $c_1(M)$ only depends on the (integrable) complex structure J. ♣

From now on, fix the complex structure J.

We now want to examine the necessary and sufficient conditions for a given closed (i.e. boundaryless and compact) Kähler manifold (M, ω_0)[5] to admit a Kähler-Einstein metric in the same Kähler class $[\omega_0]$, namely a Kähler 2-form $\omega \in [\omega_0]$

[4] We have that on any complex manifold $c_1(M) := c_1(\det_{\mathbb{C}} TM) = c_1\left(\det_{\mathbb{C}} TM^{(1,0)}\right) = -c_1\left(\det_{\mathbb{C}} T^*M^{(1,0)}\right)$, and Chern-Weil theory tells us that the first Chern class of a line bundle is $\frac{i}{2\pi}$ times the de Rham cohomology class of the curvature 2 form of *any arbitrary* connection on that line bundle, thus giving us $c_1(M) = -\frac{i}{2\pi}[F]$ for F the curvature 2 form of *any* connection on the line bundle $\det_{\mathbb{C}} T^*M^{(1,0)} =: \bigwedge_{\mathbb{C}}^n T^*M^{(1,0)} =: K_M$.

[5] We're now omitting J, g since we usually care about ω and we're fixing J.

such that $\mathrm{Ric}_\omega = \lambda\omega$ for $\lambda \in \mathbb{R}$. Clearly from Proposition 0.7 we have that a necessary condition is that $c_1(M) = \frac{\lambda}{2\pi}[\omega] = \frac{\lambda}{2\pi}[\omega_0]$ since $[\omega] = [\omega_0]$.

It is therefore natural to ask the converse:

Question 0.9. Suppose $c_1(M) = \frac{\lambda}{2\pi}[\omega_0]$ for a fixed closed Kähler manifold (M, ω_0). Does there exist a Kähler form $\omega \in [\omega_0]$ such that $\mathrm{Ric}_\omega = \lambda\omega$?

Now by Proposition 0.7, we have that $c_1(M) = \frac{\lambda}{2\pi}[\omega_0] \Rightarrow [\mathrm{Ric}_{\omega_0}] = \lambda[\omega_0] \Rightarrow \mathrm{Ric}_{\omega_0} = \lambda\omega_0 + i\partial\bar{\partial}\rho$ by the $\partial\bar{\partial}$ lemma with $\rho \in C^\infty(M)$ unique up to a constant. We call ρ the *Ricci potential* of ω_0, and from now on we impose the following "volume normalization condition" to pin down the potential:

$$\int_M e^\rho \omega_0^n = \int_M \omega_0^n$$

Now again by the $\partial\bar{\partial}$ lemma, any Kähler form ω which is in the same cohomology class as ω_0 must be of the form $\omega = \omega_0 + i\partial\bar{\partial}f$ for some $f \in C^\infty(M)$[6]. Hence if $\exists\omega$ a Kähler form in $[\omega_0]$ such that $\mathrm{Ric}_\omega = \lambda\omega$, then by part 2 of Proposition 0.7, $\omega = \omega_0 + i\partial\bar{\partial}f$ must satisfy

$$\begin{aligned}
\mathrm{Ric}_\omega - \mathrm{Ric}_{\omega_0} &= \lambda\left(\omega_0 + i\partial\bar{\partial}f\right) - \lambda\omega_0 - i\partial\bar{\partial}\rho \\
&= i\partial\bar{\partial}(\lambda f - \rho) \\
&= -i\partial\bar{\partial}\log\left(\frac{\omega^n}{\omega_0^n}\right)
\end{aligned}$$

giving us

$$i\partial\bar{\partial}\left(\log\left(\frac{\omega^n}{\omega_0^n}\right) + \lambda f - \rho\right) = 0$$

or equivalently

$$\log\left(\frac{\omega^n}{\omega_0^n}\right) + \lambda f - \rho = c \Rightarrow \frac{\omega^n}{\omega_0^n} = e^{-\lambda f} e^\rho e^c$$

with the constant $c := -\log\left(\frac{\int_M e^{-\lambda f} e^\rho \omega_0^n}{\int_M \omega_0^n}\right)$.

Now if $\lambda = 0$, then by the ρ normalization, $c = 0$. If $\lambda \neq 0$, then f solves $\frac{\omega^n}{\omega_0^n} = e^{-\lambda f} e^\rho e^c$ if and only if $f + \frac{c}{\lambda}$ solves $\frac{\omega^n}{\omega_0^n} = e^{-\lambda f} e^\rho$. Hence $\forall\lambda \in \mathbb{R}$, we end up with the following equation for $f \in C^\infty(M)$:

[6] Technically speaking, we also require $\omega = \omega_0 + i\partial\bar{\partial}f$ to be a positive $(1,1)$ form in order for it to actually be Kähler, but positivity is an open condition.

$$\frac{\left(\omega_0 + i\partial\bar{\partial} f\right)^n}{\omega_0^n} = e^{-\lambda f} e^{\rho} \tag{0.10}$$

Therefore we have proved the following

Proposition 0.11. Let (M, ω_0) be a fixed closed Kähler manifold.

If $c_1(M) = \lambda[\omega_0]$ with the Ricci potential normalization $\int_M e^{\rho}\omega_0^n = \int_M \omega_0^n$ (since $c_1(M) = \frac{\lambda}{2\pi}[\omega_0] \Rightarrow \mathrm{Ric}_{\omega_0} = \lambda\omega_0 + i\partial\bar{\partial}\rho$), then there exists a Kähler-Einstein metric $\omega \in [\omega_0]$ if $\omega = \omega_0 + i\partial\bar{\partial} f$ is a positive $(1,1)$ form and solves

$$\frac{\left(\omega_0 + i\partial\bar{\partial} f\right)^n}{\omega_0^n} = e^{-\lambda f} e^{\rho}$$

This equation is a fully nonlinear elliptic equation of *Monge-Ampere* type, i.e. in local coordinates the LHS looks like $\det(\mathrm{Hess}\, f)$. Therefore solving this PDE for f is quite difficult, and becomes harder as the sign of λ changes from negative to 0 to positive. Aubin solved the above for $\lambda < 0$, Yau solved the $\lambda = 0$ (i.e. Ricci flat) case (called the *Calabi conjecture*), and very recently Chen-Donaldson-Sun [4] [5] [6] solved the $\lambda > 0$ case via proving that existence of KE metrics with $\lambda > 0$ is equivalent to an algebraic-geometric "K-stability" condition.

Lastly, we note the following (whose proof is elementary, see Huybrechts' book [10])

Proposition 0.12. Let (M, ω_0) be a fixed closed Kähler manifold. Suppose K_M is trivial, hence there exists a nowhere vanishing **holomorphic** $(n, 0)$ form Ω (an example being the $K3$ surface).

Then there exists a Kähler form $\omega \in [\omega_0]$ with vanishing Ricci form if $\omega = \omega_0 + i\partial\bar{\partial} f$ is a positive $(1,1)$ form and solves

$$\frac{\left(\omega_0 + i\partial\bar{\partial} f\right)^n}{\Omega \wedge \bar{\Omega}} = C$$

for some constant $C > 0$.

§ Section 2: Eguchi-Hanson Spaces as the "Building Block" §

Following Biquard-Minerbe [2], we define the Eguchi-Hanson metric and prepare it for gluing.

Let's now look at $\mathbb{C}^2/\mathbb{Z}_2$, or $\mathbb{C}^2$ quotient the action $\mathbb{Z}_2 \subset \mathbb{C}^2$ via $1 \mapsto \mathrm{id}_{\mathbb{C}^2}$, $-1 \mapsto -\mathrm{id}_{\mathbb{C}^2}$. Blow up the singularity at the origin with a crepant resolution (i.e. preserving the canonical bundle, i.e. pullback of canonical bundle downstairs is canonical bundle upstairs) to get $\widetilde{\mathbb{C}^2/\mathbb{Z}_2} \overset{\text{Biholo}}{\cong} T^*\mathbb{C}P^1$ and a map $\pi : T^*\mathbb{C}P^1 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_2$ which is a biholomorphism outside the origin, i.e. $\pi : T^*\mathbb{C}P^1 - \mathbb{C}P^1 \overset{\text{Biholo}}{\cong} \frac{\mathbb{C}^2-\{0\}}{\mathbb{Z}_2}$ since $\pi^{-1}(0) \cong \mathbb{C}P^1$ is the exceptional divisor.

Hence we may put coordinates $\left\{u^i\right\}_{i\in\{1,2\}}$ on $T^*\mathbb{C}P^1 - \mathbb{C}P^1$ by pulling back the standard coordinates on $\mathbb{C}^2 - \{0\}$ which get pushed down to $\frac{\mathbb{C}^2-\{0\}}{\mathbb{Z}_2}$. From this, we have that $du^1 \wedge du^2$, a nowhere vanishing $(2,0)$ form on $\mathbb{C}^2$, gets preserved under $\mathbb{Z}_2$ and descends down to a nowhere vanishing $(2,0)$ form on $\mathbb{C}^2/\mathbb{Z}_2$, and gets pulled back to a nowhere vanishing $(2,0)$ form Ω on $T^*\mathbb{C}P^1$ since the resolution $\pi : T^*\mathbb{C}P^1 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_2$ is crepant. Not only that, but the flat Kähler metric $i\partial\bar{\partial}\left(\frac{|u|^2}{2}\right) =: \omega_0 =: i\partial\bar{\partial}\phi_0$ on $\mathbb{C}^2$ passes down to the quotient and gets pulled back up to $T^*\mathbb{C}P^1 - \mathbb{C}P^1$ and is defined exactly the same.

Now let $r : T^*\mathbb{C}P^1 - \mathbb{C}P^1 \longrightarrow \mathbb{R}_{\geq 0}$ by $r\left(u^1, u^2\right) := \left(\left|u^1\right|^2 + \left|u^2\right|^2\right)^{\frac{1}{2}}$, the radial distance of a point from the exceptional divisor with the pulled back coordinates above (which is also the distance from the exceptional divisor WRT the pullback of the flat metric on $\frac{\mathbb{C}^2-\{0\}}{\mathbb{Z}_2}$). We sometimes abuse notation and denote $r(u) = |u|$ or simply as r when the point u is clear from the context. Define $\phi_{EH} \in C^\infty\left(T^*\mathbb{C}P^1 - \mathbb{C}P^1\right)$ via $\phi_{EH}(u) := \frac{1}{2}\left(\left(1+r^4\right)^{\frac{1}{2}} + 2\log r - \log\left(1 + \left(1+r^4\right)\right)^{\frac{1}{2}}\right)$. Define $\omega_{EH} := i\partial\bar{\partial}\phi_{EH}$. This in fact gives us a Kähler form on all of $T^*\mathbb{C}P^1$ that is in fact Ricci flat. ω_{EH} is called the *Eguchi-Hanson* metric, and it is in fact asymptotic to the flat Kähler metric $i\partial\bar{\partial}\left(\frac{r^2}{2}\right) =: \omega_0$ as $|u| \nearrow \infty$. More specifically, we have that $\nabla^k_{\omega_0}(\phi_{EH} - \phi_0) = O\left(|u|^{-2-k}\right)$ and $\nabla^k_{\omega_0}(\omega_{EH} - \omega_0) = O\left(|u|^{-4-k}\right)$ as $|u| \nearrow \infty$ where the Levi-Civita connection is WRT the flat metric ω_0.

Remark 0.13. The pair $\left(T^*\mathbb{C}P^1, \omega_{EH}\right)$ is therefore a quintessential example of an *ALE space* asymptotic at ∞ to $\mathbb{C}^2/\mathbb{Z}_2$, with the diffeomorphism outside a compact subset coming from $\pi : T^*\mathbb{C}P^1 - \mathbb{C}P^1 \overset{\text{Biholo}}{\cong} \frac{\mathbb{C}^2-\{0\}}{\mathbb{Z}_2}$. **Note** that this biholomorphism also holds outside any *closed* tubular neighborhood of the exceptional divisor, i.e. $\pi : T^*\mathbb{C}P^1 - D_R\left(\mathbb{C}P^1\right) \overset{\text{Biholo}}{\cong} \frac{\mathbb{C}^2-D_R(0)}{\mathbb{Z}_2}$. ♣

Remark 0.14. For open or closed tubular neighborhoods of the exceptional divisor $\mathbb{C}P^1$ in the Eguchi-Hanson space, we will sometimes abuse notation and denote $\{|u| < R\}, \{|u| \leq R\}$ for $R > 0$ as being a subset of $T^*\mathbb{C}P^1$ *instead of* $T^*\mathbb{C}P^1 - \mathbb{C}P^1$ where the coordinates $\left\{u^i\right\}_{i\in\{1,2\}}$ are defined. **Henceforth**, $\{|u| < R\}, \{|u| \leq R\} \subset T^*\mathbb{C}P^1$ is therefore the subset of $T^*\mathbb{C}P^1$ which agrees with $\{|u| < R\}, \{|u| \leq R\}$ *on* $T^*\mathbb{C}P^1 - \mathbb{C}P^1$. ♣

We now want to "prepare" the EH metric by interpolating EH with the flat metric on an "annulus" via a suitable cutoff function on $T^*\mathbb{C}P^1$. Define a smooth cutoff function $\chi : \mathbb{R}_{\geq 0} \longrightarrow [0,1]$ satisfying $\chi(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & 2 \leq x \end{cases}$.

Let $\epsilon > 0$ be a small positive number (this epsilon parameter features quite prominently in all gluing constructions, and will be referred back to throughout the rest of this thesis!). Define $\chi_\epsilon(u) := \chi\left(\epsilon^{\frac{1}{2}}|u|\right)$. Hence $\chi_\epsilon(u) = \begin{cases} 1 & |u| \leq \frac{1}{\epsilon^{\frac{1}{2}}} \\ 0 & \frac{2}{\epsilon^{\frac{1}{2}}} \leq |u| \end{cases}$.

We now define $\phi_{EH,\epsilon} := \chi_\epsilon \phi_{EH} + (1-\chi_\epsilon)\phi_0$ and $\omega_{EH,\epsilon} := i\partial\bar{\partial}\phi_{EH,\epsilon}$. Immediate from the definitions is:

Proposition 0.15. $\omega_{EH,\epsilon}$ is a Kähler form on $T^*\mathbb{C}P^1$ **for small enough** $\epsilon > 0$ that satisfies: $\omega_{EH,\epsilon} = \begin{cases} \omega_{EH} & \text{on } \left\{|u| \leq \frac{1}{\epsilon^{\frac{1}{2}}}\right\} \\ \omega_0 & \text{on } \left\{\frac{2}{\epsilon^{\frac{1}{2}}} \leq |u|\right\} \end{cases}$ and satisfies $\left|\nabla^k_{\omega_0}(\omega_{EH,\epsilon} - \omega_0)\right| \leq c(k)\left(\frac{1}{\epsilon^{\frac{1}{2}}}\right)^{-4-k}$ on the annulus $\frac{1}{\epsilon^{\frac{1}{2}}} \leq |u| \leq \frac{2}{\epsilon^{\frac{1}{2}}}$ (with the higher order tensor norm for with respect to ω_0)

§ Section 3: The "Kummer Construction" §

Recall that a $K3$ surface is a compact simply connected complex 2-fold with trivial canonical bundle. Since any two $K3$ surfaces are diffeomorphic, differing constructions of any complex 2-fold that are compact, simply connected, and have trivial canonical bundle all yield the same $K3$ surface, up to diffeomorphism. We rigorously construct here one such definition of a $K3$ surface, namely as a Kummer surface $\mathscr{K}$, which is the resolution of an abelian variety quotiented out by an involution, and then graft in our Eguchi-Hanson metric $\omega_{EH,\epsilon}$ from the previous section to get a family of Kähler metrics ω_ϵ on $\mathscr{K}$.

Now let the small $\epsilon > 0$ from the previous section as before, but now even smaller to also ensure $2\epsilon^{\frac{1}{2}} < 1$ **(if it wasn't already small enough to also ensure this).**

View $\mathbb{T}^4 = \mathbb{C}^2/\Lambda$ as the quotient of $\mathbb{C}^2$ by any lattice Λ. WLOG[7] take $\Lambda = (8\mathbb{Z})^4 := \{(a+bi, c+di) : a,b,c,d \in 8\mathbb{Z}\}$. Quotient out $\mathbb{T}^4$ by the involution which sends $z \mapsto -z$, which clearly descends from $\mathbb{C}^2$ down to the quotient $\mathbb{T}^4$. We now get an orbifold $\mathbb{T}^4/\{\pm 1\}$ with 16 orbifold singularities at the points $\left(z^1, z^2\right)$ whose

[7] The choice of the factor of 8 in the lattice makes no difference at all, it merely allows me to have closed balls of radius 1 around each 16 orbifold points of $\mathbb{T}^4/\{\pm 1\}$ to be disjoint, and I prefer to have that radius be 1 instead of $\frac{1}{8}$. In fact, it turns out that the choice of lattice Λ is a *parameter* in the moduli space of these Ricci-flat Kähler metrics on $K3$.

real and imaginary parts are $0 \mod 4$, and with orbifold singularity modeled on $\mathbb{C}^2/\mathbb{Z}_2$.[8] We perform a crepant resolution $\widetilde{\mathbb{T}^4/\{\pm 1\}}$ by blowing up each of the 16 orbifold singularities, and the result is a compact simply connected complex 2-fold with trivial canonical bundle, i.e. a $K3$ surface. Specifically, since each orbifold singularity is modeled on $\mathbb{C}^2/\mathbb{Z}_2$, we have that the resolution at that singular point is precisely the same resolution $\pi : T^*\mathbb{C}P^1 \twoheadrightarrow \mathbb{C}^2/\mathbb{Z}_2$ for the Eguchi-Hanson space. In fact, it is this fact which led Gibbons and Pope [9] to conjecture that one may "graft in" 16 "small Eguchi-Hanson metrics" around the 16 exceptional divisors of $\widetilde{\mathbb{T}^4/\{\pm 1\}}$ onto the flat metric on its complement. Thus we also have a projection map $\pi : \widetilde{\mathbb{T}^4/\{\pm 1\}} \twoheadrightarrow \mathbb{T}^4/\{\pm 1\}$ which is a biholomorphism on the complement of the 16 exceptional $\mathbb{C}P^1$s.

In more detail, let $\mathcal{S} := \operatorname{sing}\left(\mathbb{T}^4/\{\pm 1\}\right)$ be the set of 16 singular points. For each $p \in \mathcal{S}$, pick normal coordinates $\left\{z^i\right\}_{i\in\{1,2\}}$ centered at p defining 16 disjoint (normal) coordinate open balls of radius 1 (hence are only defined for $|z| < 1$). In these coordinate open balls, remove the disk of radius $\epsilon^{\frac{1}{2}}$ **for the same small** $\epsilon > 0$ and get $\mathbb{T}^4/\{\pm 1\} - \sqcup_{i\in\{1,\dots,16\}} D_{\epsilon^{\frac{1}{2}}}(p_i)$. Now transfer the standard flat Kähler metric ω_0 on $\mathbb{C}^2 - \{0\}$ to $\mathbb{T}^4/\{\pm 1\} - \mathcal{S}$ and then onto $\mathbb{T}^4/\{\pm 1\} - \sqcup_{i\in\{1,\dots,16\}} D_{\epsilon^{\frac{1}{2}}}(p_i)$ since $\pi : \mathscr{K} \twoheadrightarrow \mathbb{T}^4/\{\pm 1\}$ is a biholomorphism on the complement of the exceptional divisors. We now have a pair $\left(\mathbb{T}^4/\{\pm 1\} - \sqcup_{i\in\{1,\dots,16\}} D_{\epsilon^{\frac{1}{2}}}(p_i), \omega_0\right)$.

With the same ϵ, on $T^*\mathbb{C}P^1$ with $\left\{u^i\right\}_{i\in\{1,2\}}$ coordinates, consider 16 copies of the open tubular neighborhood of radius $\frac{2}{\epsilon^{\frac{1}{2}}}$ around the exceptional divisor, i.e. $\bigsqcup_{i\in\{1,\dots,16\}} T^*\mathbb{C}P^1 - \left\{|u| \geq \frac{2}{\epsilon^{\frac{1}{2}}}\right\} = \bigsqcup_{i\in\{1,\dots,16\}} \left\{|u| < \frac{2}{\epsilon^{\frac{1}{2}}}\right\}$.

Now define

$$
\begin{aligned}
(\mathscr{K}, \omega_\epsilon) &:= \left(\widetilde{\mathbb{T}^4/\{\pm 1\}}, \omega_\epsilon\right) \\
&:= \frac{\left(\mathbb{T}^4/\{\pm 1\} - \sqcup_{i\in\{1,\dots,16\}} D_{\epsilon^{\frac{1}{2}}}(p_i), \omega_0\right) \bigsqcup_{i\in\{1,\dots,16\}} \left(T^*\mathbb{C}P^1 - \left\{|u| \geq \frac{2}{\epsilon^{\frac{1}{2}}}\right\}, \widetilde{\omega_{EH,\epsilon}}\right)}{\sim}
\end{aligned}
$$

with the equivalence relation being the identification $\left\{\epsilon^{\frac{1}{2}} < |z| < 2\epsilon^{\frac{1}{2}}\right\} \sim \left\{\frac{1}{\epsilon^{\frac{1}{2}}} < |u| < \frac{2}{\epsilon^{\frac{1}{2}}}\right\}$ via $z = \epsilon u$ (i.e. each 16 connected components of $\left\{\epsilon^{\frac{1}{2}} < |z| < 2\epsilon^{\frac{1}{2}}\right\}$ is identified with a single copy of $\left\{\frac{1}{\epsilon^{\frac{1}{2}}} < |u| < \frac{2}{\epsilon^{\frac{1}{2}}}\right\} \subset T^*\mathbb{C}P^1$). This gives us our underlying $K3$

[8] Note that topologically we have that $\mathbb{C}^2/\mathbb{Z}_2 \cong \operatorname{cone}\left(\mathbb{R}P^3\right)$, and it is precisely the fact that the metric on $\operatorname{cone}\left(\mathbb{R}P^3\right)$ is conformally equivalent to the metric on a cylinder over $\mathbb{R}P^3$ that Donaldson exploits in [7] to bypass the usage of weighted Holder spaces. Part of the reason why this works is because the cylindrical metric of $\mathbb{R}P^3$ has *bounded geometry* as opposed to the conical metric with its cone singularity.

surface $\mathscr{K}$.

It now remains to define $\widetilde{\omega_{EH,\epsilon}}$ and show how that is identified with ω_0 to get ω_ϵ on $\mathscr{K}$. Now on each of the 16 $T^*\mathbb{C}P^1$, recall $\omega_{EH,\epsilon} := i\partial\bar{\partial}\phi_{EH,\epsilon}$ with $\phi_{EH,\epsilon} := \chi_\epsilon\phi_{EH} + (1-\chi_\epsilon)\phi_0$ a function of $\left\{u^i\right\}_{i\in\{1,2\}}$ on $T^*\mathbb{C}P^1$ our interpolated Kähler form and potential on $T^*\mathbb{C}P^1$ with the **small** $\epsilon > 0$. Define $\widetilde{\phi_{EH,\epsilon}} := \epsilon^2\phi_{EH,\epsilon}\left(\frac{z}{\epsilon}\right)$, i.e. identifying $z = \epsilon u$ as in the underlying $K3$ surface gluing, and then scaling the result by ϵ^2. Define $\widetilde{\omega_{EH,\epsilon}} := i\partial\bar{\partial}\left(\widetilde{\phi_{EH,\epsilon}}\right)$. This satisfies (in the identified z coordinates) $\widetilde{\omega_{EH,\epsilon}} = \begin{cases} \epsilon^2\omega_{EH} & \text{on } \left\{|z| \le \epsilon^{\frac{1}{2}}\right\} \\ \omega_0 & \text{on } \left\{2\epsilon^{\frac{1}{2}} \le |z|\right\} \end{cases}$ and $\left|\nabla^k_{\omega_0}\left(\widetilde{\omega_{EH,\epsilon}} - \omega_0\right)\right| \le c(k)\epsilon^{2-\frac{k}{2}}$ on the annuli $\left\{\epsilon^{\frac{1}{2}} < |z| < 2\epsilon^{\frac{1}{2}}\right\}$ (since the flat metric is defined away from the exceptional divisor in $T^*\mathbb{C}P^1$). Hence we define

$$\omega_\epsilon := \begin{cases} \omega_0 & \text{on } \mathscr{K} - \sqcup_{i\in\{1,\ldots,16\}} B_{2\epsilon^{\frac{1}{2}}}(E_i) = \mathscr{K} - \left\{|z| < 2\epsilon^{\frac{1}{2}}\right\} \\ \widetilde{\omega_{EH,\epsilon}} & \text{on } \left\{|z| < 2\epsilon^{\frac{1}{2}}\right\} \end{cases}$$

which is clearly well defined on all of $\mathscr{K}$ and we now have our $K3$ surface $(\mathscr{K}, \omega_\epsilon)$ with our Kähler metric ω_ϵ **depending on the small $\epsilon > 0$.**

Since $\pi : \mathscr{K} \twoheadrightarrow \mathbb{T}^4/\{\pm 1\}$ is a biholomorphism on the complement of the exceptional divisors, we can pullback the normal coordinates $\left\{z^i\right\}_{i\in\{1,2\}}$ defined as open unit radius balls around each $p_i \in \mathbb{S}$ to get open unit radius tubular neighborhoods around each exceptional divisor on $\mathscr{K}$. Observe that there are the following 3 regions of $\mathscr{K}$ that are of importance:

(1) The region $\mathscr{K} - \sqcup_{i\in\{1,\ldots,16\}} B_1(E_i) = \mathscr{K} - \{|z| < 1\}$, i.e. the complement of the 16 unit radii open tubular neighborhoods around each exceptional divisor E_i. Again since $\pi : \mathscr{K} \twoheadrightarrow \mathbb{T}^4/\{\pm 1\}$ is a biholomorphism on the complement of the exceptional divisors, we may view this region as $\mathbb{T}^4/\{\pm 1\} - \{|z| < 1\}$, the complement of the 16 unit radius open normal coordinate balls around each singular point. Thus on this region the coordinates $\left\{z^i\right\}_{i\in\{1,2\}}$ aren't defined, but it is useful to view this region as $\{|z| \ge 1\}$. **We denote this region as $\mathscr{K}_{ext}$.**

(2) The annular region $\{|z| < 1\} - \{|z| < \epsilon\} = \{\epsilon \le |z| < 1\}$ **for the same small $\epsilon > 0$ as before**. Here we can either view this as a subset $\{\epsilon \le |z| < 1\} \subset \mathbb{T}^4/\{\pm 1\}$ by the biholomorphism above, or (each 16 connected components) as a subset $\left\{1 \le |u| < \frac{1}{\epsilon}\right\} \subset T^*\mathbb{C}P^1$. **We denote this region as $\mathscr{K}_{ann}$, and it has 16 connected components.**

Note that $\mathscr{K}_{ann}$, in the event $\epsilon \searrow 0$, has the interior radius collapsing down to zero.

(3) The region $\{|z| < \epsilon\}$, **again same small** $\epsilon > 0$. This region (or rather, each of the 16 connected components) is identified with $\{|u| < 1\} \subset T^*\mathbb{C}P^1$, i.e. $B_1(E) \subset T^*\mathbb{C}P^1$, the open neighborhood of the exceptional divisor of "radius" 1. **We denote this region as** $\mathscr{K}_{int}$.

Note that $\mathscr{K}_{int}$, in the event $\epsilon \searrow 0$, is collapsing down to a point.

Hence $\mathscr{K} = \mathscr{K}_{ext} \sqcup \mathscr{K}_{ann} \sqcup \mathscr{K}_{int}$.

Observe also that

$$\omega_\epsilon = \begin{cases} \epsilon^2 \omega_{EH} & \text{on each 16 connected components of } \left\{|z| \leq \epsilon^{\frac{1}{2}}\right\} \subset \mathscr{K}_{int} \sqcup \mathscr{K}_{ann} \\ \omega_0 & \text{on } \left\{2\epsilon^{\frac{1}{2}} \leq |z|\right\} \subset \mathscr{K}_{ann} \sqcup \mathscr{K}_{ext} \end{cases}$$

Remark 0.16. Note that we're omitting the ϵ in the notation for the underlying $K3$ surface $\mathscr{K}$. This is because, as previously mentioned, a theorem of Kodaira implies that for any pair $\epsilon_1, \epsilon_2 > 0$ of **positive** real numbers, the resulting $\mathscr{K}_{\epsilon_1}, \mathscr{K}_{\epsilon_2}$ are diffeomorphic, and since we've fixed the underlying integrable complex structure, they are in fact biholomorphic. ♣

Remark 0.17. From the above observations about the collapsing behavior of $\mathscr{K}_{ann}, \mathscr{K}_{int}$ and the construction of $\mathscr{K}$ and ω_ϵ itself, we have that if we were to take a sequence $\epsilon_i \searrow 0$, that the pair $\left(\mathscr{K}, \omega_{\epsilon_i}\right) \xrightarrow{GH} \left(\mathbb{T}^4/\{\pm 1\}, \omega_0\right)$ converges *in the Gromov-Hausdorff topology* to the orbifold $\left(\mathbb{T}^4/\{\pm 1\}, \omega_0\right)$ with the singular Kähler metric ω_0 with conical $\mathbb{C}^2/\{\pm 1\}$ singularities at each of the 16 singular points. Note that outside of the set $\mathbb{S}$ of 16 singular points, we have that ω_0 is the flat Kähler metric on $\mathbb{T}^4/\{\pm 1\} - \mathbb{S}$ from before, *hence the same notation ! ! !*

Unfortunately, as with the language of G structures and many other fascinating notions, we do not have the space to delve into this absolutely fantastic topic. ♣

§ Section 4: Weighted Holder Spaces §

Following chapter 8 of [13], we now define the weighted Holder spaces that we will be using for our setting, first for $(\mathbb{C}^n - \{0\}, \omega_0)$, then for $\left(T^*\mathbb{C}P^1, \omega_{EH}\right)$, and lastly for $(\mathscr{K}, \omega_\epsilon)$. **Unless otherwise stated**, let $\alpha \in (0,1), \delta \in \mathbb{R}, k \in \mathbb{N}_0$. To simplify notation in the sequel, for any $j \geq 0$ let $\left\|\nabla_g^j f\right\|_{L^\infty_\delta} := \left\|\rho^{-\delta+j} \nabla_g^j f\right\|_{L^\infty}$ denote the weighted L^∞ norm corresponding to a Riemannian metric g for ρ a weight function (with higher order tensor norms defined with respect to a metric depending on the circumstances).

On $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we let $\rho : \mathbb{R}^{2n} \longrightarrow [1, \infty)$ be a smooth nondecreasing function satisfying

$$\rho(x) = \begin{cases} 1 & |x| \leq 1 \\ |x| & 2 \leq |x| \end{cases}$$

We define the $\|f\|_{C^{k,\alpha}_\delta(\mathbb{R}^{2n})}$ norm as:

$$\|f\|_{C^{k,\alpha}_\delta(\mathbb{R}^{2n})} := \sum_{j \leq k} \left\| \nabla^j_{g_0} f \right\|_{L^\infty_\delta(\mathbb{R}^{2n})} + \sup_{x \neq y} \min\{\rho(x), \rho(y)\}^{-\delta+k+\alpha} \frac{\left| \nabla^k_{g_0} f \Big|_x - \nabla^k_{g_0} f \Big|_y \right|}{|x-y|^\alpha}$$

$$:= \sum_{j \leq k} \left\| \rho^{-\delta+j} \nabla^j_{g_0} f \right\|_{L^\infty(\mathbb{R}^{2n})} + \sup_{x \neq y} \min\{\rho(x), \rho(y)\}^{-\delta+k+\alpha} \frac{\left| \nabla^k_{g_0} f \Big|_x - \nabla^k_{g_0} f \Big|_y \right|}{|x-y|^\alpha}$$

with ∇_{g_0} the Levi-Civita connection of the flat metric g_0 on $\mathbb{R}^{2n}$. We then define the space $C^{k,\alpha}_\delta(\mathbb{R}^{2n})$ as $C^{k,\alpha}_\delta(\mathbb{R}^{2n}) := \left\{ f \in C^{k,\alpha}(\mathbb{R}^{2n}) : \|f\|_{C^{k,\alpha}_\delta(\mathbb{R}^{2n})} < \infty \right\}$ but with the $\|\cdot\|_{C^{k,\alpha}_\delta(\mathbb{R}^{2n})}$ norm instead. It is a Banach space whose elements, say f with $\|f\|_{C^{k,\alpha}_\delta(\mathbb{R}^{2n})} \leq C$, (roughly) decays like $\nabla^j_{g_0} f = O(\rho^{\delta-j})$, $j \leq k$. Note that this works verbatim for $C^{k,\alpha}_\delta(\mathbb{R}^{2n} - \{0\}) = C^{k,\alpha}_\delta(\mathbb{C}^n - \{0\})$.

Remark 0.18. Note that because our weight function $\rho : \mathbb{R}^{2n} \longrightarrow [1, \infty)$ is smooth nondecreasing and satisfying $\rho(x) = \begin{cases} 1 & |x| \leq 1 \\ |x| & 2 \leq |x| \end{cases}$, it is in particular *unbounded from above*, and therefore we have that the spaces $C^{k,\alpha}_\delta(\mathbb{R}^{2n}), C^{k,\alpha}_\delta(\mathbb{R}^{2n} - \{0\}) = C^{k,\alpha}_\delta(\mathbb{C}^n - \{0\})$ ***do not contain the constant functions ! ! !*** This is a crucial property which allows for us to completely kill the kernel of the weighted Laplacian, since as we'll see, a suitable choice of weight parameter $\delta \in \mathbb{R}$ (specifically $\delta \in (-2, 0)$) kills the harmonic functions, leaving only the constants left to deal with. ♣

On $(T^*\mathbb{C}P^1, \omega_{EH})$ with coordinates $\{u^i\}_{i \in \{1,2\}}$, we let $\gamma_2 : T^*\mathbb{C}P^1 \longrightarrow [0, 1]$ be a smooth cutoff function which satisfies $\gamma_2 = \begin{cases} 0 & \text{on } \{|u| \leq 1\} \\ 1 & \text{on } T^*\mathbb{C}P^1 - \{|u| < 2\} \end{cases}$. We thus have that for $f : T^*\mathbb{C}P^1 \longrightarrow \mathbb{R}$ any function that $\gamma_2 f$ itself may be viewed as a function on $\mathbb{C}^2 - \{0\}$ extended by zero since $\pi : T^*\mathbb{C}P^1 \longrightarrow \mathbb{C}^2/\mathbb{Z}_2$ is a biholomorphism away from the exceptional divisor, i.e. $T^*\mathbb{C}P^1 - D_1(\mathbb{C}P^1) \overset{\text{Biholo}}{\cong} \frac{\mathbb{C}^2 - D_1(0)}{\mathbb{Z}_2} \subset \mathbb{C}^2 - \{0\}$. We now define the $\|f\|_{C^{k,\alpha}_\delta(T^*\mathbb{C}P^1)}$ norm as:

$$
\begin{aligned}
\|f\|_{C^{k,\alpha}_{\delta}(T^*\mathbb{C}P^1)} &:= \|f\|_{C^{k,\alpha}(\{|u|\le 1\})} + \|\gamma_2 f\|_{C^{k,\alpha}_{\delta}(T^*\mathbb{C}P^1-\{|u|\le 1\})} \\
&= \|f\|_{C^{k,\alpha}(\{|u|\le 1\})} + \|\gamma_2 f\|_{C^{k,\alpha}_{\delta}(\mathbb{C}^2-\{0\})}
\end{aligned}
$$

Here on the closed tubular neighborhood $\{|u| \le 1\}$ of the exceptional divisor, we use the standard Holder norm

$$
\|f\|_{C^{k,\alpha}(\{|u|\le 1\})} := \sum_{j\le k} \left\| \nabla^j_{\omega_{EH}} f \right\|_{L^\infty(\{|u|\le 1\})} + \sup_{\substack{x,y\in\{|u|\le 1\} \\ x\neq y \\ d_{\omega_{EH}}(x,y)\le \mathrm{inj}_{\omega_0}}} \frac{\left| \nabla^k_{\omega_{EH}} f\big|_x - \nabla^k_{\omega_{EH}} f\big|_y \right|}{d_{\omega_{EH}}(x,y)^\alpha}
$$

Note that all the covariant derivatives and parallel transport and higher order tensor (i.e. not functions) norms are all with respect to ω_{EH} the Eguchi-Hanson metric on $T^*\mathbb{C}P^1$. We now define the space $C^{k,\alpha}_{\delta}\left(T^*\mathbb{C}P^1\right)$ as $C^{k,\alpha}_{\delta}\left(T^*\mathbb{C}P^1\right) := \left\{ f \in C^{k,\alpha}\left(T^*\mathbb{C}P^1\right) : \|f\|_{C^{k,\alpha}_{\delta}(T^*\mathbb{C}P^1)} < \infty \right\}$ but with the $\|\cdot\|_{C^{k,\alpha}_{\delta}(T^*\mathbb{C}P^1)}$ norm instead.[9] Verbatim from the remark above, $C^{k,\alpha}_{\delta}\left(T^*\mathbb{C}P^1\right)$ ***does not contain the constant functions ! ! !***

Now we define the weighted Holder spaces for $(\mathscr{K}, \omega_\epsilon)$ (**same** $\epsilon > 0$ **small**). Define a smooth nondecreasing function $\rho_\epsilon : \mathscr{K} \longrightarrow [\epsilon, 1]$ satisfying $\rho_\epsilon = \begin{cases} 1 & \text{on } \mathscr{K}_{ext} \\ |z| & \text{on } \left\{ \epsilon^{\frac{1}{2}} < |z| < 0.99 \right\} \subset \mathscr{K}_{ann} \\ \epsilon & \text{on } \mathscr{K}_{int} \end{cases}$.

Define

$$
\begin{aligned}
\|f\|_{C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})} &:= \sum_{j\le k} \left\| \nabla^j_{\omega_\epsilon} f \right\|_{L^\infty_\delta(\mathscr{K})} + \sup_{\substack{x\neq y \\ d_{\omega_\epsilon}(x,y)\le \mathrm{inj}_{\omega_\epsilon}}} \min\{\rho_\epsilon(x), \rho_\epsilon(y)\}^{-\delta+k+\alpha} \frac{\left| \nabla^k_{\omega_\epsilon} f\big|_x - \nabla^k_{\omega_\epsilon} f\big|_y \right|}{d_{\omega_\epsilon}(x,y)^\alpha} \\
&:= \sum_{j\le k} \left\| \rho_\epsilon^{-\delta+j} \nabla^j_{\omega_\epsilon} f \right\|_{L^\infty(\mathscr{K})} + \sup_{\substack{x\neq y \\ d_{\omega_\epsilon}(x,y)\le \mathrm{inj}_{\omega_\epsilon}}} \min\{\rho_\epsilon(x), \rho_\epsilon(y)\}^{-\delta+k+\alpha} \frac{\left| \nabla^k_{\omega_\epsilon} f\big|_x - \nabla^k_{\omega_\epsilon} f\big|_y \right|}{d_{\omega_\epsilon}(x,y)^\alpha}
\end{aligned}
$$

with all covariant derivatives (and parallel transport from) the Levi-Civita connection of ω_ϵ and the higher order tensor (i.e. not functions) norms with respect to ω_ϵ.

[9] Note that the standard Holder space $C^{k,\alpha}\left(T^*\mathbb{C}P^1\right)$ again has the covariant derivatives, parallel transport, higher order tensor norms with respect to ω_{EH}.

We now define the space $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})$ as $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K}) := \left\{f \in C^{k,\alpha}(\mathscr{K}) : \|f\|_{C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})} < \infty\right\}$ but with the $\|\cdot\|_{C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})}$ norm instead.[10]

With this characterization, we have that a function $f \in C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})$ with $\|f\|_{C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})} \leq C$ decays roughly as $\nabla^j f = O\left(\rho_\epsilon^{\delta-j}\right)$, $j \leq k$ around each exceptional divisor.

Remark 0.19. Note that because the weight function ρ_ϵ has *bounded* sup norm (due in part because it is smooth and $\mathscr{K}$ is *compact*), we have that $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})$ ***contains the constant functions ! ! !*** This is important because this produces a 1 dimensional subspace in the kernel of any differential operator on $\mathscr{K}$ with domain any $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})$, in particular the weighted Laplacian. Hence, upon choosing a suitable weight $\delta \in \mathbb{R}$ (again, specifically $\delta \in (-2, 0)$) to kill the harmonic functions, we **still have to deal with the constant functions** in order to completely kill the kernel ! ! ! ♣

§ Section 5: Nonlinear Setup §

Let's now bring everything together and find our $\omega \in [\omega_\epsilon]$ Kähler on $\mathscr{K}$ which is Ricci-flat (**here again $\epsilon > 0$ small from before**).

By Proposition 0.11 from section 1, we now want to solve for f defining a Kähler form $\omega := \omega_\epsilon + i\partial\bar{\partial} f \in [\omega_\epsilon]$ in the following equation:

$$\frac{\omega^2}{\omega_\epsilon^2} = \frac{\left(\omega_\epsilon + i\partial\bar{\partial} f\right)^2}{\omega_\epsilon^2} = e^{\phi_\epsilon}$$

where ϕ_ϵ is the Ricci potential $\underbrace{\mathrm{Ric}_{\omega_\epsilon} = i\partial\bar{\partial}\phi_\epsilon}$ satisfying the volume normalization $\int_{\mathscr{K}} e^{\phi_\epsilon}\omega_\epsilon^2 = \int_{\mathscr{K}} \omega_\epsilon^2$. Since $\mathscr{K} = \widetilde{\mathbb{T}^4/\{\pm 1\}}$ is a crepant resolution of $\mathbb{T}^4/\{\pm 1\}$, and since the standard nowhere vanishing holomorphic $(2, 0)$ form $dz^1 \wedge dz^2$ on $\mathbb{C}^2$ descends down to $\mathbb{T}^4$ and to $\mathbb{T}^4/\{\pm 1\}$, we have that there exists a nowhere vanishing holomorphic $(2, 0)$ form Ω on $\mathscr{K}$. Because we have our nowhere vanishing holomorphic $(2, 0)$ form Ω on $\mathscr{K}$, by Proposition 0.11 this is equivalent to solving

$$\frac{\omega^2}{\Omega \wedge \bar{\Omega}} = \frac{\left(\omega_\epsilon + i\partial\bar{\partial} f\right)^2}{\Omega \wedge \bar{\Omega}} = C$$

for some constant $C > 0$. In fact, we therefore have that $\phi_\epsilon = \log\left(C\frac{\Omega\wedge\bar{\Omega}}{\omega_\epsilon^2}\right)$, and since $C > 0$ is a positive real constant, WLOG we may set $C = 1$ via multiplying the holomorphic nowhere vanishing $(2, 0)$ form by $\frac{1}{C^{\frac{1}{2}}}$, i.e. replacing $\Omega \mapsto \frac{1}{C^{\frac{1}{2}}}\Omega$, and thus $\phi_\epsilon = \log\left(\frac{\Omega\wedge\bar{\Omega}}{\omega_\epsilon^2}\right)$.

[10] Note that the standard Holder space $C^{k,\alpha}(\mathscr{K})$ again has the covariant derivatives, parallel transport, higher order tensor norms with respect to ω_ϵ.

Now let Δ_ω be the standard Hodge Laplacian $d\delta + \delta d$ acting on 0-forms with Hodge star coming from the Riemannian metric g_ω which itself is from ω via "2 out of 3". Note that this has **nonnegative** spectrum. Now since $\Delta_\omega f = -2n\frac{i\partial\bar{\partial}f\wedge\omega^{n-1}}{\omega^n}$ on any $\dim_{\mathbb{C}} = n$ dimensional Kähler maniold (M,ω), we now setup our nonlinear problem as follows:

First define $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 := \left\{f \in C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K}) : \int_{\mathscr{K}} f\omega_\epsilon^2 = 0\right\} \subset C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})$. Because ρ_ϵ has *bounded* sup norm, we have that $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 \subset C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})$ is a **closed** Banach subspace, inheriting the same Banach norm $\|\cdot\|_{C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})}$.

Our nonlinear problem is therefore:

$$\begin{aligned} \mathcal{F}_\epsilon : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 &\to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0 \\ f &\mapsto \frac{\left(\omega_\epsilon + i\partial\bar{\partial}f\right)^2}{\omega_\epsilon^2} - e^{\phi_\epsilon} \end{aligned}$$

where we solve

$$\begin{aligned} \mathcal{F}_\epsilon(f) &:= \frac{\left(\omega_\epsilon + i\partial\bar{\partial}f\right)^2}{\omega_\epsilon^2} - e^{\phi_\epsilon} \\ &= \underbrace{\left(1 - e^{\phi_\epsilon}\right)}_{=\mathcal{F}_\epsilon(0)} \underbrace{-\frac{1}{2}\Delta_{\omega_\epsilon} f}_{=D_0\mathcal{F}_\epsilon(f)} + \underbrace{\frac{i\partial\bar{\partial}f\wedge i\partial\bar{\partial}f}{\omega_\epsilon^2}}_{\text{nonlinearity}} \\ &= 0 \end{aligned}$$

Remark 0.20. Note that since we are on a *compact* manifold, for the domain and codomain weighted Holder spaces $C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K}), C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})$ in the nonlinear problem above, by the Fredholm alternative for the Laplacian, **we have to restrict to the closed subspace of functions with integral zero**, i.e. $\int_{\mathscr{K}} f\omega_\epsilon^2 = 0$, in order to guarantee the solvability of Laplace's equation $\Delta_{\omega_\epsilon} f = 0$ (or else, as remarked before, we have a nontrivial 1 dimensional subspace of the kernel consisting of the *constant* functions, hence guaranteeing a nontrivial kernel). ♣

Our method of attack to prove the existence of f solving $\mathcal{F}_\epsilon(f) = 0$ above is via the implicit function theorem. For this to work, we need to invert the linearization $D_0\mathcal{F}_\epsilon$ uniformly in $\epsilon > 0$, have a Lipschitz bound on the nonlinearity, as well as some good bounds on $\mathcal{F}_\epsilon(0)$. We tackle the linearization in the next section and rest in the last section, and we end off this section with a preliminary step for the smallness of $\mathcal{F}_\epsilon(0)$.

Proposition 0.21. Recall the Ricci potential $\phi_\epsilon = \log\left(\frac{\Omega\wedge\bar{\Omega}}{\omega_\epsilon^2}\right)$ which also satisfies the volume normalization $\int_{\mathscr{K}} e^{\phi_\epsilon}\omega_\epsilon^2 = \int_{\mathscr{K}} \omega_\epsilon^2$. Then we have that

- $\operatorname{supp} \phi_\epsilon = \left\{\epsilon^{\frac{1}{2}} \le |z| \le 2\epsilon^{\frac{1}{2}}\right\}$
- $\left|\nabla^k_{\omega_0} \phi_\epsilon\right| \le c(k)\epsilon^{2-\frac{k}{2}}$ on the annulus $\left\{\epsilon^{\frac{1}{2}} \le |z| \le 2\epsilon^{\frac{1}{2}}\right\}$ (here the higher order tensor norm for $k \ge 1$ is with respect to ω_0, which is defined outside of the 16 exceptional divisors)

Proof. The support follows immediately because ω_ϵ is the (scaled) Eguchi-Hanson on $\left\{|z| \le \epsilon^{\frac{1}{2}}\right\}$ and is flat on $\left\{2\epsilon^{\frac{1}{2}} \le |z|\right\}$, hence the Ricci potential ϕ_ϵ of ω_ϵ must vanish on those regions, and the support is the *closure* of the compliment of the vanishing set.

Secondly, since ω_0 is defined on the compliment of the 16 exceptional divisors, it clearly is Ricci flat and satisfies $\omega_0^2 = \Omega \wedge \bar{\Omega}$.

Hence

$$\begin{aligned}
|\phi_\epsilon| &= \left|\log\left(\frac{\Omega \wedge \bar{\Omega}}{\omega_\epsilon^2}\right)\right| \\
&= \left|\log\left(\frac{\omega_0^2}{\omega_\epsilon^2}\right)\right| = \left|2\log\left(\frac{1}{1 + \frac{\mathfrak{D}}{\omega_0}}\right)\right| \\
&= \left|\log\left(1 + \frac{\mathfrak{D}}{\omega_0}\right)\right| \\
&\le c\epsilon^2
\end{aligned}$$

since $\mathfrak{D} \coloneqq \omega_\epsilon - \omega_0$ and on the annulus $\left\{\epsilon^{\frac{1}{2}} \le |z| \le 2\epsilon^{\frac{1}{2}}\right\}$ we have that $|\mathfrak{D}| \le c\epsilon^2$ from here, and the Taylor expansion of $\log(1+x)$ finishes it off. The estimate for higher derivatives follows immediately from $\left|\nabla^k_{\omega_0}\mathfrak{D}\right| \le c(k)\epsilon^{2-\frac{k}{2}}$ and the exact same argument. ∎

§ Section 6: Inverting the Linearization §

We now want to invert, for suitable values of $\delta \in \mathbb{R}$ **and for small enough** $\epsilon > 0$ **as before**, the linearization $D_0\mathcal{F}_\epsilon : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 \to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0$ of $\mathcal{F}_\epsilon : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 \to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0$ at the origin.

Since $D_0\mathcal{F}_\epsilon = -\frac{1}{2}\Delta_{\omega_\epsilon} f$, it suffices to invert $\Delta_{\omega_\epsilon} : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 \to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0$, which is a bounded linear operator. The key is to have the operator norm of the inverse be bounded **uniformly in** $\epsilon > 0$.

First we cite two results on the invertibility of the Laplacian on $\left(\mathbb{C}^2 - \{0\}, \omega_0\right)$ and $\left(T^*\mathbb{C}P^1, \omega_{EH}\right)$. An excellent reference for these results is the paper of Bartnick

[1] (done with weighted Sobolev spaces instead, but the same results hold for weighted Holder):

Proposition 0.22. **Let** $\delta \in (-2,0)$. Then the following are bounded linear isomorphisms:

- $\Delta_{\omega_0,\delta} : C^{k,\alpha}_{\delta}\left(\mathbb{C}^2 - \{0\}\right) \to C^{k-2,\alpha}_{\delta-2}\left(\mathbb{C}^2 - \{0\}\right)$
- $\Delta_{\omega_{EH},\delta} : C^{k,\alpha}_{\delta}\left(T^*\mathbb{C}P^1\right) \to C^{k-2,\alpha}_{\delta-2}\left(T^*\mathbb{C}P^1\right)$

Very roughly speaking, the proofs of the above follows from the first (i.e. flat Euclidean) case by the locally constant property of the Fredholm index and by the asymptotically flat property of ω_{EH}. The flat case proof roughly goes by studying the scaling properties of the singular integral operator representation of solutions to the Laplacian to show surjectivity, and then using polar coordinates and the spherical eigenfunction decomposition for the functions in the kernel plus the condition that δ is **not** an *indical root* (i.e. is **not** the growth rate of harmonic functions, of which $\delta \in (-2,0)$ implies for $\dim_{\mathbb{R}} = 4$) to force trivial kernel since as remarked before, the constant functions have already been excluded.[11].

Thus we want to prove that for $\delta \in (-2,0)$ and **and** $\epsilon > 0$ **sufficiently small**, $\Delta_{\omega_\epsilon} : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 \to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0$ is a bounded linear isomorphism whose operator norm of the inverse is bounded by a constant **independent of** $\epsilon > 0$.

First, we record a result of a computation that will be extensively used in the sequel. It follows from the chain rule, the equation for the Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2}g^{pk}\left(\frac{\partial g_{pj}}{\partial x^i} + \frac{\partial g_{pi}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^p}\right)$, and the local coordinate expression $\Delta_g f = -\frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^a}\left(\sqrt{\det g}\cdot g^{ab}\frac{\partial f}{\partial x^b}\right)$:

Proposition 0.23. Let $C > 0$ be a constant. Let $\rho|_z$ be any weight function originally defined on z coordinates, let $\delta \in \mathbb{R}$, and let $z = Cu$ be an identification between two sets of coordinates $\{z^i\}, \{u^i\}$. Define $\widetilde{f}(u) := C^{-\delta} f(Cu)$ for any function f originally defined on z coordinates. When obvious from the context that $\delta = 0$, abuse notation and denote $\widetilde{f}(u) := f(Cu)$ as well. To make notation cleaner, abuse notation and let the transferred-to-u-coordinates weight function be denoted as $\rho|_u$. Define $\widetilde{g}|_u := \frac{1}{C^2} g|_{Cu}$ for some metric tensor g originally defined on z coordinates. Then we have the following:

(1) Under a conformal rescaling $g \mapsto \frac{1}{C^2}g$ but with the underlying coordinates **fixed** (i.e. no transferring from $z \mapsto u$ or vice versa via $z = Cu$):

$$\nabla_{\frac{1}{C^2}g} = \nabla_g$$

[11] Unfortunately, we don't have time to go into the fascinating math that this all entails, and we must refer the reader to the excellent paper of Bartnick [1].

(2) Under **both** the conformal rescaling and the transferal from z to u coordinates (i.e. $\widetilde{g}\big|_u \coloneqq \frac{1}{C^2}g\big|_{Cu}$):

$$\left(\rho^{-\delta+j}\nabla^j_g f\right)\Big|_{Cu} = \left(\left(\frac{\rho}{C}\right)^{-\delta+j}\nabla^j_{\widetilde{g}}\widetilde{f}\right)\Bigg|_u$$

(3) Under a conformal rescaling $g \mapsto \frac{1}{C^2}g$ but with the underlying coordinates **fixed** (i.e. no transferring from $z \mapsto u$ or vice versa via $z = Cu$):

$$\Delta_{\frac{1}{C^2}g} = C^2\Delta_g$$

(4) Under transferal from z to u coordinates but **without** conformal rescaling $g \mapsto \frac{1}{C^2}g$:

$$\left(\Delta_g f\right)\big|_{Cu} = \frac{1}{C^2}\left(\Delta_g \widetilde{f}\right)\Big|_u$$

(5) Under **both** the conformal rescaling and the transferal from z to u coordinates (i.e. $\widetilde{g}\big|_u \coloneqq \frac{1}{C^2}g\big|_{Cu}$):

$$\left(\Delta_g f\right)\big|_{Cu} = \left(\Delta_{\widetilde{g}}\widetilde{f}\right)\Big|_u$$

(6) Denote by $[f]_{C^{k,\alpha}_{g,\delta}} \coloneqq \sup_{\substack{x\neq y\\ d_g(x,y)\leq \mathrm{inj}_g}} \min\{\rho(x),\rho(y)\}^{-\delta+k+\alpha}\frac{\left|\nabla^k_g f\big|_x - \nabla^k_g f\big|_y\right|}{d_g(x,y)^\alpha}$ the weighted Holder seminorm (with all higher order tensor norms for $k \geq 1$ defined with respect to g). Then under **both** the conformal rescaling and the transferal from z to u coordinates (i.e. $\widetilde{g}\big|_u \coloneqq \frac{1}{C^2}g\big|_{Cu}$):

$$[f]_{C^{k,\alpha}_{g,\delta}} = \left[\widetilde{f}\right]_{C^{k,\alpha}_{\widetilde{g},\delta}}$$

with the weight of $C^{k,\alpha}_{\widetilde{g},\delta}$ being $\frac{\rho}{C}$, i.e.

$$\begin{aligned}
&\sup_{\substack{Cu_1\neq Cu_2\\ d_g(Cu_1,Cu_2)\leq \mathrm{inj}_g}} \min\{\rho(Cu_1),\rho(Cu_2)\}^{-\delta+k+\alpha}\frac{\left|\nabla^k_g f\Big|_{Cu_1} - \nabla^k_g f\Big|_{Cu_2}\right|}{d_g(Cu_1,Cu_2)^\alpha}\\
&= \sup_{\substack{u_1\neq u_2\\ d_{\widetilde{g}}(u_1,u_2)\leq \mathrm{inj}_{\widetilde{g}}}} \min\left\{\frac{\rho}{C}\Big|_{u_1},\frac{\rho}{C}\Big|_{u_2}\right\}^{-\delta+k+\alpha}\frac{\left|\nabla^k_{\widetilde{g}}\widetilde{f}\Big|_{u_1} - \nabla^k_{\widetilde{g}}\widetilde{f}\Big|_{u_2}\right|}{d_{\widetilde{g}}(u_1,u_2)^\alpha}
\end{aligned}$$

and all higher order tensor norms for $k \geq 1$ defined with respect to $\widetilde{g}$

(7) Under **both** the conformal rescaling and the transferal from z to u coordinates (i.e. $\widetilde{g}|_u := \frac{1}{C^2} g|_{Cu}$), we have that

$$\|f\|_{C^{k,\alpha}_{g,\delta}} = \left\|\widetilde{f}\right\|_{C^{k,\alpha}_{\widetilde{g},\delta}}$$

with the weight of $C^{k,\alpha}_{\widetilde{g},\delta}$ being $\frac{\rho}{C}$ and all higher order tensor norms for $k \geq 1$ defined with respect to $\widetilde{g}$.

(8) Under **both** the conformal rescaling and the transferal from z to u coordinates (i.e. $\widetilde{g}|_u := \frac{1}{C^2} g|_{Cu}$), we have for the **unweighted** Holder space that

$$\|f\|_{C^{k,\alpha}_{g}} \xrightarrow{\text{transfer from z to u}} \sum_{j \leq k} \left(\frac{1}{C}\right)^j \left\|\nabla^j_{\widetilde{g}} \widetilde{f}\right\|_{L^\infty} + \left(\frac{1}{C}\right)^{k+\alpha} \sup_{\substack{u_1 \neq u_2 \\ d_{\widetilde{g}}(u_1,u_2) \leq \mathrm{inj}_{\widetilde{g}}}} \frac{\left|\nabla^k_{\widetilde{g}} \widetilde{f}\Big|_{u_1} - \nabla^k_{\widetilde{g}} \widetilde{f}\Big|_{u_2}\right|}{d_{\widetilde{g}}(u_1, u_2)^\alpha}$$

with all higher order tensor norms on the RHS for $k \geq 1$ defined with respect to $\widetilde{g}$

(9) Transferring the standard Schauder estimate $\|f\|_{C^{2,\alpha}} \leq \mathrm{C}\left(\|f\|_{L^\infty} + \left\|\Delta_g f\right\|_{C^{0,\alpha}}\right)$ on z coordinates under **both** the conformal rescaling and the transferal from z to u coordinates (i.e. $\widetilde{g}|_u := \frac{1}{C^2} g|_{Cu}$), we end up getting:

$$\begin{aligned} &\left\|\widetilde{f}\right\|_{L^\infty} + \frac{1}{C}\left\|\nabla_{\widetilde{g}} \widetilde{f}\right\|_{L^\infty} + \left(\frac{1}{C}\right)^2 \left\|\nabla^2_{\widetilde{g}} \widetilde{f}\right\|_{L^\infty} + \left(\frac{1}{C}\right)^{2+\alpha} \left[\nabla^2_{\widetilde{g}} \widetilde{f}\right]_{C^{0,\alpha}} \\ &\leq \mathrm{C}\left(\left\|\widetilde{f}\right\|_{L^\infty} + \left(\frac{1}{C}\right)^2 \left\|\Delta_{\widetilde{g}} \widetilde{f}\right\|_{L^\infty} + \left(\frac{1}{C}\right)^{2+\alpha} \left[\Delta_{\widetilde{g}} \widetilde{f}\right]_{C^{0,\alpha}}\right) \end{aligned}$$

with the left hand side on Ω and the right hand side on $\Theta \supsetneq \Omega$ and all higher order tensor norms defined with respect to $\widetilde{g}$.

Now we arrive at the main isomorphism result in 3 parts: first a weighted Schauder estimate, then an improved weighted Schauder estimate (which is the heart of the proof), and then the isomorphism result itself.

Proposition 0.24. (Weighted Schauder Estimate) **Let $\epsilon > 0$ from before be sufficiently small.** We have the following estimate:

$$\|f\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})} \le C\Big(\|f\|_{L^\infty_{\omega_\epsilon,\delta}(\mathscr{K})} + \big\|\Delta_{\omega_\epsilon} f\big\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})}\Big)$$
$$:= C\left(\left\|\rho_\epsilon^{-\delta} f\right\|_{L^\infty(\mathscr{K})} + \big\|\Delta_{\omega_\epsilon} f\big\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})}\right)$$

for $C > 0$ **independent of ϵ.**

Proof. The idea here is to use the standard interior Schauder estimates in Euclidean space on the ball of unit radius, then scale it to a smaller ball, then transfer everything over to $\mathscr{K}$ via a covering of normal coordinate balls.

Since Holder spaces are local, by partition of unity and the triangle inequality it suffices to prove the estimate on each of the 3 regions $\mathscr{K}_{ext}, \mathscr{K}_{ann}, \mathscr{K}_{int}$.

Recall our weight function $\rho_\epsilon : \mathscr{K} \longrightarrow [\epsilon, 1]$ satisfying $\rho_\epsilon = \begin{cases} 1 & \text{on } \mathscr{K}_{ext} \\ |z| & \text{on } \left\{\epsilon^{\frac{1}{2}} < |z| < 0.99\right\} \subset \mathscr{K}_{ann} \\ \epsilon & \text{on } \mathscr{K}_{int} \end{cases}$ for the weighted Holder spaces $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})$. Since $\rho_\epsilon = 1$ on $\mathscr{K}_{ext}$, on this region the weighted Holder spaces $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K})$ are the usual Holder spaces, and our weighted Schauder estimate on $\mathscr{K}_{ext}$ is just the usual Schauder estimate for the flat Laplacian Δ_{ω_0} since $\omega_\epsilon = \omega_0$ on $\mathscr{K}_{ext}$, and our estimate follows from covering $\mathscr{K}_{ext}$ with small enough normal coordinate balls.

It now remains therefore to prove our weighted Schauder estimates on $\mathscr{K}_{ann}, \mathscr{K}_{int}$. WLOG we focus on only 1 out of the 16 connected components of $\mathscr{K}_{ann}, \mathscr{K}_{int}$.

View $\mathscr{K}_{ann} \sqcup \mathscr{K}_{int}$ as $\left\{|u| < \frac{1}{\epsilon}\right\} \subset T^*\mathbb{C}P^1$ from the identification $z = \epsilon u$ in the definition of $\mathscr{K}$ from gluing back in section 3. Transfer ω_ϵ to the $\left\{u^i\right\}_{i\in\{1,2\}}$ coordinates on $T^*\mathbb{C}P^1$ and set $g_\epsilon|_u := \frac{1}{\epsilon^2}\omega_\epsilon|_{\epsilon u}$ (note the factor of $\frac{1}{\epsilon^2}$ to counteract the factor of ϵ^2 in the definition of ω_ϵ on $\mathscr{K}$). Our weight $\rho_\epsilon|_{\epsilon u}$ now is on $\left\{|u| < \frac{1}{\epsilon}\right\} \subset T^*\mathbb{C}P^1$ and satisfies $\rho_\epsilon|_{\epsilon u} = \begin{cases} 1 & \text{on } \left\{\frac{1}{\epsilon} \le |u|\right\} \\ \epsilon|u| & \text{on } \left\{\frac{1}{\epsilon^{\frac{1}{2}}} < |u| < \frac{0.99}{\epsilon}\right\} \\ \epsilon & \text{on } \{|u| < 1\} \end{cases}$. Abuse notation and denote $\rho_\epsilon|_{\epsilon u}$ by $\rho_\epsilon|_u$.

Now define $\widetilde{f}(u) := \epsilon^{-\delta} f(\epsilon u)$ for any function f originally defined on z coordinates.

Since we want to prove $\|f\|_{C^{2,\alpha}_{\delta,\epsilon}} \le C\Big(\|f\|_{L^\infty_{\omega_\epsilon,\delta}} + \big\|\Delta_{\omega_\epsilon} f\big\|_{C^{0,\alpha}_{\delta-2,\epsilon}}\Big)$, on the remaining region $\mathscr{K}_{ann} \sqcup \mathscr{K}_{int}$, upon transferring this estimate from z coordinates to u coordinates via $\mathscr{K}_{ann} \sqcup \mathscr{K}_{int} \sim \left\{|u| < \frac{1}{\epsilon}\right\}$ from above, we immediately have from Proposition 0.23 that this is equivalent to proving

$$\left\|\widetilde{f}\right\|_{C^{2,\alpha}_{g_\epsilon,\delta}} \leq C\left(\left\|\widetilde{f}\right\|_{L^\infty_{g_\epsilon,\delta}} + \left\|\Delta_{g_\epsilon}\widetilde{f}\right\|_{C^{2,\alpha}_{g_\epsilon,\delta}}\right)$$

on the region $\left\{|u| < \frac{1}{\epsilon}\right\}$.

As is in Proposition 0.23, we have that the weight that's in the weighted Holder spaces $C^{k,\alpha}_{g_\epsilon,\delta}$ above is $\frac{\rho_\epsilon}{\epsilon}$ satisfying $\frac{\rho_\epsilon}{\epsilon} = \begin{cases} \frac{1}{\epsilon} & \text{on } \left\{\frac{1}{\epsilon} \leq |u|\right\} \\ |u| & \text{on } \left\{\frac{1}{\epsilon^{\frac{1}{2}}} < |u| < \frac{0.99}{\epsilon}\right\} \\ 1 & \text{on } \{|u| < 1\} \end{cases}$.

Since $\frac{\rho_\epsilon}{\epsilon} = 1$ on $\{|u| < 1\} =: \mathscr{K}_{int}$, we have that on this region the above scaled Schauder estimates is just the usual Schauder estimates. This follows from covering this region with small enough normal coordinate balls such that Δ_{g_ϵ} is *uniformly* close enough in C^k to the euclidean flat Laplacian in that normal coordinate covering, and then appealing to the standard Schauder estimates for *uniformly* elliptic 2nd order linear PDOs.

It now remains to prove this estimate on the annular region $\left\{1 \leq |u| < \frac{1}{\epsilon}\right\}$. Assign to each point $j \in \left\{1 \leq |u| < \frac{1}{\epsilon}\right\}$ a real number r_j satisfying $1 \leq r_j < \frac{1}{\epsilon}$, and a large enough natural number $T \in \mathbb{N}$ such that each $\frac{r_j}{T} \leq \mathrm{inj}_{g_\epsilon}$ and Δ_{g_ϵ} is *uniformly* as close as we need in the C^k norm to the standard flat Laplacian on each normal coordinate ball $B_{g_\epsilon}\left(j, \frac{r_j}{T}\right)$. Hence each ball $B_{g_\epsilon}\left(j, \frac{r_j}{T}\right)$, since they are normal coordinate balls, are radially isometric (by Gauss' lemma) to balls of radius $\frac{r_j}{T}$ in Euclidean space. We then take the usual Schauder estimates for standard Holder spaces for the Euclidean flat Laplacian on the Euclidean ball of radius $\frac{1}{T}$, scale that Euclidean ball of radius $\frac{1}{T}$ to radius $\frac{r_j}{T}$, and transfer the resulting estimate to Δ_{g_ϵ} (which is uniformly elliptic in j because of how we set the above up so that Δ_{g_ϵ} is uniformly close enough in the C^k norm to the standard flat Laplacian) on $B_{g_\epsilon}\left(j, \frac{r_j}{T}\right)$. From Proposition 0.23 we thus get:

$$\begin{aligned}\left\|\widetilde{f}\right\|_{L^\infty\left(B_{g_\epsilon}\left(j,\frac{r_j}{T}\right)\right)} + r_j\left\|\nabla_{g_\epsilon}\widetilde{f}\right\|_{L^\infty\left(B_{g_\epsilon}\left(j,\frac{r_j}{T}\right)\right)} + r_j^2\left\|\nabla^2_{g_\epsilon}\widetilde{f}\right\|_{L^\infty\left(B_{g_\epsilon}\left(j,\frac{r_j}{T}\right)\right)} + r_j^{2+\alpha}\left[\nabla^2_{g_\epsilon}\widetilde{f}\right]_{C^{0,\alpha}\left(B_{g_\epsilon}\left(j,\frac{r_j}{T}\right)\right)} \\ \leq C\left(\left\|\widetilde{f}\right\|_{L^\infty\left(B_{g_\epsilon}\left(j,\frac{2r_j}{T}\right)\right)} + r_j^2\left\|\Delta_{g_\epsilon}\widetilde{f}\right\|_{L^\infty\left(B_{g_\epsilon}\left(j,\frac{2r_j}{T}\right)\right)} + r_j^{2+\alpha}\left[\Delta_{g_\epsilon}\widetilde{f}\right]_{C^{0,\alpha}\left(B_{g_\epsilon}\left(j,\frac{2r_j}{T}\right)\right)}\right)\end{aligned}$$

for $C > 0$ independent of $\epsilon > 0$. We can thus furthermore make the constant $C > 0$ independent of j because Δ_{g_ϵ} is uniformly elliptic in j.

Since $1 \leq r_j < \frac{1}{\epsilon}$, we have by the shape of $\frac{\rho_\epsilon}{\epsilon}$ that $\frac{\rho_\epsilon}{\epsilon}$ is comparable to r_j, i.e. $\frac{\rho_\epsilon}{\epsilon} \approx r_j$ aka $\frac{r_j}{2} \leq \frac{\rho_\epsilon}{\epsilon} \leq 2r_j$. Now multiply both sides of the above inequality by $r_j^{-\delta}$ and use $\frac{\rho_\epsilon}{\epsilon} \approx r_j$ to get that our weighted Schauder estimate $\left\|\widetilde{f}\right\|_{C^{2,\alpha}_{g_\epsilon,\delta}\left(B_{g_\epsilon}\left(j,\frac{r_j}{T}\right)\right)} \leq$

$C\left(\left\|\widetilde{f}\right\|_{L^\infty_{g_\epsilon,\delta}\left(B_{g_\epsilon}\left(j,\frac{2r_j}{T}\right)\right)}+\left\|\Delta_{g_\epsilon}\widetilde{f}\right\|_{C^{2,\alpha}_{g_\epsilon,\delta}\left(B_{g_\epsilon}\left(j,\frac{2r_j}{T}\right)\right)}\right)$ with weight $\frac{\rho_\epsilon}{\epsilon}$ holds on each ball $B_{g_\epsilon}\left(j,\frac{r_j}{T}\right)$. The fact that $C>0$ is independent of both $\epsilon>0$, j gives us the desired estimate on the annular region $\left\{1\le|u|<\frac{1}{\epsilon}\right\}$, and we're done. ■

We now want to remove the L^∞ term on the RHS of our above weighted Schauder estimate. This allows us to get a **uniform, i.e. independent of** $\epsilon>0$, **bound on the inverse of** Δ_{ω_ϵ}:

Proposition 0.25. (Improved Weighted Schauder Estimate) **Let** $\delta\in(-2,0)$. **Let** $\epsilon>0$ **from before be sufficiently small.**

For $\Delta_{\omega_\epsilon}:C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0\to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0$, we have the following estimate:

$$\|f\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})}\le C\left\|\Delta_{\omega_\epsilon}f\right\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})}$$

for $C>0$ **independent of** ϵ.

Note that $f\in C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0$ i.e. the estimate is restricted to the closed subspace of functions with integral zero, i.e. $\int_{\mathscr{K}}f\omega_\epsilon^2=0$.

Proof. We prove this by contradiction via a "blow-up" analysis[12] on (roughly speaking) each region $\mathscr{K}_{ext},\mathscr{K}_{int}$ (Eguchi-Hanson), $\mathscr{K}_{ann}$ (the neck) of $\mathscr{K}$.

By our weighted Schauder estimate just proved above, we may equivalently prove $\|f\|_{L^\infty_{\omega_\epsilon,\delta}(\mathscr{K})}\le C\left\|\Delta_{\omega_\epsilon}f\right\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})}$ with $\delta\in(-2,0)$, etc..

Now assume for contradiction that the inequality fails. Then $\exists\epsilon_i\searrow 0$ and $\{f_i\}_{i\in\mathbb{N}}$ sequences such that (after normalizing)

$$\|f_i\|_{L^\infty_{\omega_{\epsilon_i},\delta}(\mathscr{K})}=1$$

$$\left\|\Delta_{\omega_{\epsilon_i}}f_i\right\|_{C^{0,\alpha}_{\delta-2,\epsilon_i}(\mathscr{K})}\le\frac{1}{i}$$

Our weighted Schauder estimate proved previously gives us that

$$\|f_i\|_{C^{2,\alpha}_{\delta,\epsilon_i}(\mathscr{K})}\le 2C$$

Now choose a sequence of points $p_i\in\mathscr{K}$ such that $\rho_{\epsilon_i}^{-\delta}(p_i)|f_i(p_i)|=1$ (because $\|f_i\|_{L^\infty_{\omega_{\epsilon_i},\delta}(\mathscr{K})}=1$). Since $\mathscr{K}$ is compact, the sequence $\{p_i\}$ has a convergent subsequence. We now examine which region of $\mathscr{K}$ that subsequence converges to, and derive a contradiction in every case.

[12] I learned this through many of the sources already cited, but I am indebted in particular to chapter 3 (particularly Proposition 3.3.1) of Spotti's PhD thesis [12] which stands out in particular for explaining the contradiction/blow up strategy in crystal clear detail.

(1) **Case 1**: $\lim_{i\nearrow\infty}\rho_{\epsilon_i}(p_i) > 0$

This means that $p_i \longrightarrow p_\infty$ with $p_\infty \in \mathscr{K}$ outside the exceptional divisor set E of $\mathscr{K}$, and since $\mathscr{K}$ is the resolution of $\mathbb{T}^4/\{\pm 1\}$, it is biholomorphic outside of the exceptional divisors/singular points, we have that $p \in \mathbb{T}^4/\{\pm 1\} - \mathcal{S}$. Since $\|f_i\|_{C^{2,\alpha}_{\delta,\epsilon_i}(\mathscr{K}-E)} \leq 2C$, by Arzela-Ascoli we have that $f_i \longrightarrow f_\infty$ in $C^{2,\frac{\alpha}{2}}_{loc}(\mathscr{K}-E)$ (i.e. uniformly (sub)converges in $C^{2,\frac{\alpha}{2}}$ on compact subsets of $\mathscr{K}-E$). From $\left\|\Delta_{\omega_{\epsilon_i}} f_i\right\|_{C^{0,\alpha}_{\delta-2,\epsilon_i}(\mathscr{K})} \leq \frac{1}{i}$ gives us that $\Delta_{\omega_0} f_\infty = 0$. Transferring everything over to $\mathbb{T}^4/\{\pm 1\} - \mathcal{S}$ using the biholomorphism $\mathscr{K} - E \cong \mathbb{T}^4/\{\pm 1\} - \mathcal{S}$, we use local (normal) orbifold coordinates centered at each orbifold point $q \in \mathcal{S}$ to get that f_∞ is in fact a weak (i.e. distributional) solution to Δ_{ω_0} **when** $\delta \in (-2, 0)$. This is because, for $\phi \in C^\infty_0(B(0,R))$ smooth compactly supported test functions on $B(0,R) \subset \mathbb{C}^2$ with $0.99 > R > 0$ small enough (i.e. the normal orbifold coordinate ball of radius $0.99 > R > 0$), we have by integration by parts that (for $R > r > 0$)

$$\int_{B(0,R)-B(0,r)} f_\infty(\Delta_{\omega_0} u)dV_{\omega_0} = \underbrace{\int_{B(0,R)-B(0,r)} u(\Delta_{\omega_0} f_\infty)dV_{\omega_0}}_{=0 \text{ since } \Delta_{\omega_0} f_\infty = 0} + \int_{S^3(r)}\left(f_\infty\frac{\partial u}{\partial \nu} - u\frac{\partial f_\infty}{\partial \nu}\right)d\Sigma$$

with ν the unit inward pointing normal WRT the metric ω_0. Hence

$$\begin{aligned}\left|\int_{B(0,R)-B(0,r)} f_\infty(\Delta_{\omega_0} u)dV_{\omega_0}\right| &\leq C_1 r^3 \sup_{B(0,r)}|f_\infty| \sup_{S^3(r)}\left|\operatorname{grad}_{\omega_0} u\right| \\ &\quad + C_2 r^3 \sup_{B(0,r)}|u| \sup_{S^3(r)}\left|\operatorname{grad}_{\omega_0} f_\infty\right| \\ &\leq C r^3\left(r^\delta + r^{\delta-1}\right) \\ &\searrow 0 \text{ as } r \searrow 0 \text{ when } \delta \in (-2,0)\end{aligned}$$

since $\rho_0 = |z|$ when $|z| < 0.99$. Hence f_∞ is in fact a smooth solution of $\Delta_{\omega_0} f_\infty = 0$ by elliptic regularity. But by the maximum principle, we must have that $f_\infty = c$ is a constant. However, the integral zero condition $\int_{\mathscr{K}} f_i \omega_{\epsilon_i}^2 = 0$ as well as the fact that integral zero functions are closed subspaces in each weighted Holder space gives us $\int_{\mathbb{T}^4/\{\pm 1\}-\mathcal{S}} f_\infty \omega_0^2 = 0$ which implies that $f_\infty = 0$. But upon taking the limit as $i \nearrow \infty$ in $\rho_{\epsilon_i}^{-\delta}(p_i)|f_i(p_i)| = 1$, we get that $\rho_0^{-\delta}(p_\infty)|f_\infty(p_\infty)| = 1$ which implies $f_\infty \neq 0$, a contradiction.

Remark 0.26. By trichotomy, we have that the remaining two cases consists of:

- **Case 2:** $\lim_{i\nearrow\infty}\rho_{\epsilon_i}(p_i)=0$ **and** $\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}(p_i)}{\epsilon_i}<\infty$ **(say,** $<R<\infty$**)**
- **Case 3:** $\lim_{i\nearrow\infty}\rho_{\epsilon_i}(p_i)=0$ **and** $\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}(p_i)}{\epsilon_i}=\infty$

Case 2 has p_∞ inside the "Eguchi-Hanson" region, while Case 3 has p_∞ inside the "neck" region.

This is because, since $\epsilon_i\searrow 0$ faster than $\rho_{\epsilon_i}(p_i)\searrow 0$ due to $\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}(p_i)}{\epsilon_i}=\infty$ in Case 3, we have that the "Eguchi-Hanson" region collapses down faster than p_i could stay in it, hence p_∞ is left behind in the "neck" region.

Another way to see this is to examine Case 3 more closely (which we will do when we get to Case 3 below). In Case 3, by the shape of ρ_{ϵ_i} we have that $\rho_{\epsilon_i}(p_i)$ is comparable to $|p_i|$, i.e. $\rho_{\epsilon_i}(p_i)\approx|p_i|$. Now pick sequences $R_i\nearrow\infty, r_i\searrow 0$ such that

(1) $r_i|p_i|<|p_i|<R_i|p_i|$

(2) $R_i|p_i|\searrow 0$

(3) $\frac{r_i|p_i|}{\epsilon_i}\nearrow\infty$

Identify $\{r_i|p_i|<|z|<R_i|p_i|\}\sim\left\{\frac{r_i|p_i|}{\epsilon_i}<|u|<\frac{R_i|p_i|}{\epsilon_i}\right\}\subset T^*\mathbb{C}P^1$ via $z=\epsilon_i u$ as usual. Because Case 3 implies $\frac{r_i|p_i|}{\epsilon_i}\nearrow\infty$, our region $\left\{\frac{r_i|p_i|}{\epsilon_i}<|u|<\frac{R_i|p_i|}{\epsilon_i}\right\}\subset T^*\mathbb{C}P^1$ collapses at infinity, and we can't do anything on $T^*\mathbb{C}P^1$. Had $\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}(p_i)}{\epsilon_i}<R<\infty$ as in Case 2, we would get the entirety of $T^*\mathbb{C}P^1$ as $i\nearrow\infty$ because it would be impossible to find $R_i\nearrow\infty, r_i\searrow 0$ satisfying $r_i|p_i|<|p_i|<R_i|p_i|, R_i|p_i|\searrow 0$ (which is always possible) but which *also* satisfies $\frac{r_i|p_i|}{\epsilon_i}\nearrow\infty$. ♣

(2) **Case 2:** $\lim_{i\nearrow\infty}\rho_{\epsilon_i}(p_i)=0$ **and** $\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}(p_i)}{\epsilon_i}<\infty$ **(say,** $<R<\infty$**)**

In other words, $p_i\to p_\infty$ and p_∞ is in the $T^*\mathbb{C}P^1$ region of $\mathscr{K}$ because under $z=\epsilon_i u$, we have that $\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}(p_i)}{\epsilon_i}<R\Rightarrow\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}}{\epsilon_i}(p_i)<R$. More precisely, we have that $p_\infty\in\mathscr{K}_{ann}\sqcup\mathscr{K}_{int}$ (we assume here WLOG that we're on 1 of the 16 connected components of $\mathscr{K}_{ann}\sqcup\mathscr{K}_{int}$). As in the proof of the weighted Schauder estimates, view (that connected component of) $\mathscr{K}_{ann}\sqcup\mathscr{K}_{int}$ as $\left\{|u|<\frac{1}{\epsilon_i}\right\}\subset T^*\mathbb{C}P^1$ from the identification $z=\epsilon_i u$ in the definition of $\mathscr{K}$ from gluing back in section 3. Transfer ω_{ϵ_i} to the $\{u^v\}_{v\in\{1,2\}}$ coordinates on $T^*\mathbb{C}P^1$ and set $g_{\epsilon_i}:=\frac{1}{\epsilon_i^2}\omega_{\epsilon_i}$ (note the factor of $\frac{1}{\epsilon_i^2}$ to counteract the factor of ϵ_i^2 in the definition of ω_{ϵ_i} on $\mathscr{K}$). Our weight ρ_{ϵ_i} now is on $\left\{|u| < \frac{1}{\epsilon_i}\right\} \subset T^*\mathbb{C}P^1$ and satisfies $\rho_{\epsilon_i} = \begin{cases} 1 & \text{on } \left\{\frac{1}{\epsilon_i} \leq |u|\right\} \\ \epsilon_i|u| & \text{on } \left\{\frac{1}{\epsilon_i^{\frac{1}{2}}} < |u| < \frac{0.99}{\epsilon_i}\right\} \\ \epsilon_i & \text{on } \{|u| < 1\} \end{cases}$.

Define $\widetilde{f_i}(u)$ to be $\epsilon_i^{-\delta} f_i(\epsilon_i u)$. Then from Proposition 0.23 we have that $\widetilde{f_i}$ now satisfies:

$$\left\|\widetilde{f_i}\right\|_{L^\infty_{g_{\epsilon_i},\delta}\left(\left\{|u|<\frac{1}{\epsilon_i}\right\}\right)} = 1$$

$$\left(\frac{\rho_{\epsilon_i}}{\epsilon_i}\right)^{-\delta}(p_i)\left|\widetilde{f_i}(p_i)\right| = 1$$

$$\left\|\Delta_{g_{\epsilon_i}}\widetilde{f_i}\right\|_{C^{0,\alpha}_{\delta-2,\epsilon_i}\left(\left\{|u|<\frac{1}{\epsilon_i}\right\}\right)} \leq \frac{1}{i}$$

$$\left\|\widetilde{f_i}\right\|_{C^{2,\alpha}_{\delta,\epsilon_i}\left(\left\{|u|<\frac{1}{\epsilon_i}\right\}\right)} \leq 2C$$

where the weight for the weighted Holder spaces here being $\frac{\rho_{\epsilon_i}}{\epsilon_i}$ and the metric used being g_{ϵ_i}. Now we take the limit $i \nearrow \infty$, and we get

$$\widetilde{f_i} \xrightarrow{C^{2,\frac{\alpha}{2}}_{loc}} \widetilde{f_\infty} \text{ on } T^*\mathbb{C}P^1$$

$$p_i \to p_\infty \in \{|u| < R\} \subset T^*\mathbb{C}P^1$$

$$\left|\widetilde{f_\infty}(p_\infty)\right| > R^\delta > 0$$

$$\Delta_{g_0}\widetilde{f_\infty} = 0$$

Now note that $g_0 := \lim_{i\nearrow\infty} \frac{1}{\epsilon_i^2}\omega_{\epsilon_i}$ is in fact the standard Eguchi-Hanson metric ω_{EH} on $T^*\mathbb{C}P^1$. Now since the standard Holder compact embedding $k+\alpha > L+\gamma \Rightarrow C^{k,\alpha} \overset{cpt}{\subset} C^{L,\gamma}$ adapted to the weighted setting is $k+\alpha > L+\gamma, \delta < \beta \Rightarrow C^{k,\alpha}_\delta \overset{cpt}{\subset} C^{L,\gamma}_\beta$, we have that we may view $\widetilde{f_\infty} \in C^{2,\frac{\alpha}{2}}_\beta\left(T^*\mathbb{C}P^1\right)$ for $\beta \in (\delta, 0) \subset (-2, 0)$. Hence because part 2 of Proposition 0.22 gives us that $\Delta_{\omega_{EH},\beta} : C^{2,\frac{\alpha}{2}}_\beta\left(T^*\mathbb{C}P^1\right) \to C^{0,\frac{\alpha}{2}}_{\beta-2}\left(T^*\mathbb{C}P^1\right)$ is a bounded linear isomorphism for $\beta \in (-2, 0)$, we must have that $\widetilde{f_\infty} = 0$, contradicting $\left|\widetilde{f_\infty}(p_\infty)\right| > R^\delta > 0$.

(3) **Case 3:** $\lim_{i\nearrow\infty}\rho_{\epsilon_i}(p_i) = 0$ **and** $\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}(p_i)}{\epsilon_i} = \infty$

In other words, $p_i \to p_\infty$ and p_∞ is in the "neck" region of $\mathcal{K}$. More precisely, since $\lim_{i\nearrow\infty}\frac{\rho_{\epsilon_i}(p_i)}{\epsilon_i} = \infty$, we have that $\epsilon_i \searrow 0$ faster than $\rho_{\epsilon_i}(p_i) \searrow 0$, and so by the shape of ρ_{ϵ_i} we have that $\rho_{\epsilon_i}(p_i)$ is comparable to $|p_i|$, i.e. $\rho_{\epsilon_i}(p_i) \approx |p_i|$. Now pick sequences $R_i \nearrow \infty, r_i \searrow 0$ such that

(a) $r_i|p_i| < |p_i| < R_i|p_i|$

(b) $R_i|p_i| \searrow 0$

(c) $\frac{r_i|p_i|}{\epsilon_i} \nearrow \infty$

Then we clearly have that each $p_i \in \{r_i|p_i| < |z| < R_i|p_i|\} \subset \mathscr{K}_{ann} \sqcup \mathscr{K}_{int}$. Now since $\mathscr{K}_{ann} \sqcup \mathscr{K}_{int} \subset \{|z| < 1\}$ is contained in the normal coordinate open unit radius tubular neighborhood each exceptional divisor of $\mathscr{K}$, we may (WLOG after looking at only that connected component out of 16 which contains p_∞) scale by $z = |p_i|v$ to get onto $\mathbb{C}^2 - \{0\}$ (where we relabel the coordinates on $\mathbb{C}^2 - \{0\}$ to be $\{v^a\}_{a\in\{1,2\}}$). Hence, similar to the previous case, we identify $\{r_i|p_i| < |z| < R_i|p_i|\} \sim \{r_i < |v| < R_i\} \subset \mathbb{C}^2 - \{0\}$, scale the metric $g_i := \frac{1}{|p_i|^2}\omega_{\epsilon_i}$, transfer our weight ρ_{ϵ_i} to satisfy $\rho_{\epsilon_i} = \begin{cases} 1 & \text{on } \left\{\frac{1}{|p_i|} \leq |v|\right\} \\ \epsilon_i|v| & \text{on } \left\{\frac{\epsilon_i^{\frac{1}{2}}}{|p_i|} < |v| < \frac{0.99}{|p_i|}\right\} \\ \epsilon_i & \text{on } \left\{|v| < \frac{1}{|p_i|}\right\} \end{cases}$, define $\widetilde{f}_i(v)$ to be $|p_i|^{-\delta} f_i(|p_i|v)$, and get from Proposition 0.23 that $\widetilde{f}_i$ now satisfies:

$$\left\|\widetilde{f}_i\right\|_{L^\infty_{g_i,\delta}(\{r_i<|v|<R_i\})} = 1$$

$$\left(\frac{\rho_{\epsilon_i}}{|p_i|}\right)^{-\delta}(p_i)\left|\widetilde{f}_i(p_i)\right| = 1$$

$$\left\|\Delta_{g_i}\widetilde{f}_i\right\|_{C^{0,\alpha}_{\delta-2,\epsilon_i}(\{r_i<|v|<R_i\})} \leq \frac{1}{i}$$

$$\left\|\widetilde{f}_i\right\|_{C^{2,\alpha}_{\delta,\epsilon_i}(\{r_i<|v|<R_i\})} \leq 2C$$

$$|p_i| = 1 \text{ on } \mathbb{C}^2 - \{0\} \text{ under the scaling } z = |p_i|v$$

where the weight for the weighted Holder spaces here being $\frac{\rho_{\epsilon_i}}{|p_i|}$ and the metric used being g_i. Now we take the limit $i \nearrow \infty$, and we get

$$\begin{aligned}
\widetilde{f_i} &\xrightarrow{C^{2,\frac{\alpha}{2}}_{loc}} \widetilde{f_\infty} \text{ on } \mathbb{C}^2 - \{0\} \\
p_i &\to p_\infty \in S^3(1) \subset \mathbb{C}^2 - \{0\} \\
\left|\widetilde{f_\infty}(p_\infty)\right| &= 1 > 0 \text{ because } \rho_{\epsilon_i}(p_i) \approx |p_i| \\
\Delta_{g_0}\widetilde{f_\infty} &= 0
\end{aligned}$$

Now note that $g_0 := \lim_{i\nearrow\infty} \frac{1}{\epsilon_i^2}\omega_{\epsilon_i}$ is in fact the standard *flat* Kähler metric ω_0 on $\mathbb{C}^2 - \{0\}$. Now since the standard Holder compact embedding $k + \alpha > L + \gamma \Rightarrow C^{k,\alpha} \overset{cpt}{\subset} C^{L,\gamma}$ adapted to the weighted setting is $k + \alpha > L + \gamma, \delta < \beta \Rightarrow C^{k,\alpha}_{\delta} \overset{cpt}{\subset} C^{L,\gamma}_{\beta}$, we have that we may view $\widetilde{f_\infty} \in C^{2,\frac{\alpha}{2}}_{\beta}\big(\mathbb{C}^2 - \{0\}\big)$ for $\beta \in (\delta, 0) \subset (-2, 0)$. Hence because part 1 of Proposition 0.22 gives us that $\Delta_{\omega_0,\beta} : C^{2,\frac{\alpha}{2}}_{\beta}\big(\mathbb{C}^2 - \{0\}\big) \to C^{0,\frac{\alpha}{2}}_{\beta-2}\big(\mathbb{C}^2 - \{0\}\big)$ is a bounded linear isomorphism for $\beta \in (-2, 0)$, we must have that $\widetilde{f_\infty} = 0$, contradicting $\left|\widetilde{f_\infty}(p_\infty)\right| = 1$.

Since we've arrived at a contradiction in all 3 cases, we're done!

■

With these two pieces, we may finally prove

Theorem 0.27. *For $\delta \in (-2, 0)$* ***and*** *$\epsilon > 0$* ***sufficiently small****, the following is a bounded linear isomorphism* ***for both the domain and codomain*** *$C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K}), C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})$* ***restricted to the closed subspace of functions with integral zero, i.e.*** *$\int_{\mathscr{K}} f\omega_\epsilon^2 = 0$:*

$$\Delta_{\omega_\epsilon} : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 \to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0$$

More crucially, we have that the operator norm of the inverse is bounded by a constant ***independent of*** *$\epsilon > 0$:*

$$\left\|\Delta^{-1}_{\omega_\epsilon}\right\| \leq K$$

♦

Proof. The Laplacian in the unweighted Holder spaces is Fredholm of index zero. Now for fixed δ, ϵ, we have that $C^{k,\alpha}_{\delta,\epsilon}(\mathscr{K}), C^{2,\alpha}(\mathscr{K})$ have equivalent norms (they're already the same underlying vector space). Hence $\Delta_{\omega_\epsilon} : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 \to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0$ is also Fredholm of index zero. The improved weighted Schauder estimate $\|f\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})} \leq C\left\|\Delta_{\omega_\epsilon} f\right\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})}$ as well as the integral zero condition gives us that $\ker \Delta_{\omega_\epsilon} = \{0\}$ when $\epsilon > 0$ **is sufficiently small** and $\delta \in (-2, 0)$, hence $\operatorname{ind}\big(\Delta_{\omega_\epsilon}\big) = 0$ implies surjective and hence bounded linear isomorphism, and $\|f\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})} \leq C\big\|\Delta_{\omega_\epsilon} f\big\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})}$ also gives us our $\epsilon > 0$ **independent** bound on the operator norm of the inverse. ∎

§ Section 7: Finishing the Proof §

Recall from section 5 our nonlinear problem:

$$\begin{aligned} \mathcal{F}_\epsilon : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 &\to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0 \\ f &\mapsto \frac{\big(\omega_\epsilon + i\partial\bar{\partial} f\big)^2}{\omega_\epsilon^2} - e^{\phi_\epsilon} \end{aligned}$$

where we solve

$$\begin{aligned} \mathcal{F}_\epsilon(f) &:= \frac{\big(\omega_\epsilon + i\partial\bar{\partial} f\big)^2}{\omega_\epsilon^2} - e^{\phi_\epsilon} \\ &= \underbrace{\Big(1 - e^{\phi_\epsilon}\Big)}_{=\mathcal{F}_\epsilon(0)} \underbrace{-\frac{1}{2}\Delta_{\omega_\epsilon} f}_{=D_0\mathcal{F}_\epsilon(f)} + \underbrace{\frac{i\partial\bar{\partial} f \wedge i\partial\bar{\partial} f}{\omega_\epsilon^2}}_{\text{nonlinearity}} \\ &= 0 \end{aligned}$$

We want to apply the following

Theorem 0.28 (Implicit Function Theorem). *Let X, Y be two Banach spaces. Let $\Phi : X \to Y$ be a C^1 map. Write $\Phi(x) = \Phi(0) + D_0\Phi(x) + \mathsf{N}(x)$ with $\mathsf{N}(x) := \Phi(x) - \Phi(0) - D_0\Phi(x)$ being the nonlinear term. Suppose we have 3 positive constants $L, r_0, N > 0$ such that*

1. *$D_0\Phi : X \to Y$ admits a bounded linear right inverse $R : Y \to X$ such that $\|R\|_{op} \leq L$ ($L > 0$ controls right inverse)*
2. *A bound $\|\mathsf{N}(x) - \mathsf{N}(y)\|_Y \leq N\|x - y\|_X\big(\|x\|_X + \|y\|_X\big), \qquad \forall x, y \in D^X_{r_0}(0)$ ("Lipschitz" bounds on nonlinearity)*
3. *$\|\Phi(0)\|_Y \leq \frac{r}{2L}$ with $r < \min\big\{r_0, \frac{1}{2NL}\big\}$ ("smallness" of $\Phi(0)$)*

Then $\Phi(x) = 0$ has a unique solution $x \in D^X_r(0)$. ♦

Proof. This is essentially the Banach fixed point theorem. We have that $\Phi(x) = 0 \iff \mathcal{A}(x) = x$ with $\mathcal{A}(x) := -R(\Phi(0) + \mathsf{N}(x))$. We want to show that $\mathcal{A}$:

$D_r^X(0) \to D_r^X(0)$ is a contraction mapping. First we have that $\operatorname{im}\mathcal{A} \subset D_r^X(0)$ because for $x \in D_r^X(0)$,

$$\begin{aligned}
\|\mathcal{A}(x)\|_X &= \|R(\Phi(0) + \mathsf{N}(x))\|_X \\
&\leq L(\|\Phi(0)\|_Y + \|\mathsf{N}(x)\|_Y) \\
&\leq L\left(\frac{r}{2L} + Nr^2\right) \\
&< r\left(\frac{1}{2} + \frac{1}{2}\right) \\
&= r
\end{aligned}$$

Next we have that

$$\begin{aligned}
\|\mathcal{A}(x) - \mathcal{A}(y)\|_X &= \|R(\mathsf{N}(x) - \mathsf{N}(y))\|_X \\
&\leq L\|\mathsf{N}(x) - \mathsf{N}(y)\|_Y \\
&\leq LN\|x - y\|_X\big(\|x\|_X + \|y\|_X\big) \\
&\leq 2LNr\|x - y\|_X \\
&< \|x - y\|_X
\end{aligned}$$

Hence we're done, as was to be shown. ■

We've now arrived at the main theorem of this thesis:

Theorem 0.29 (Main Theorem). ***Let*** $\delta \in (-2, 0)$.
Let $\epsilon > 0$ ***be sufficiently small so that:***

(1) ***Each pre-glued Eguchi-Hanson metric*** $\omega_{EH,\epsilon}$ ***from Proposition 0.15 is Kähler.***

(2) $2\epsilon^{\frac{1}{2}} < 1$ ***for the construction of*** $\mathcal{K}$.

(3) $\Delta_{\omega_\epsilon} : C^{2,\alpha}_{\delta,\epsilon}(\mathcal{K})^0 \to C^{0,\alpha}_{\delta-2,\epsilon}(\mathcal{K})^0$ ***for both the domain and codomain*** $C^{2,\alpha}_{\delta,\epsilon}(\mathcal{K}), C^{0,\alpha}_{\delta-2,\epsilon}(\mathcal{K})$ ***restricted to the closed subspace of functions with integral zero, i.e.*** $\int_{\mathcal{K}} f\omega_\epsilon^2 = 0$ ***is a bounded linear isomorphism***

(4) $\epsilon^{1+\frac{\delta}{2}} < \frac{1}{4L^2C_1C_2}$ ***for*** $L, C_1, C_2 > 0$ ***constants which will be defined in the proof.***

Then the equation $\mathcal{F}_\epsilon(f) = 0$ *has a* ***unique*** *solution* f *satisfying:*

- $\int_{\mathcal{K}} f\omega_\epsilon^2 = 0$ ***(integral zero)***

- $\|f\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})} < C_3\epsilon^{3-\frac{\delta}{2}}$ *for* $C_3 > 0$ *chosen so that* $C_3 > 2LC_2$ ***(smallness of norm)***

Hence $\omega_\epsilon + i\partial\bar{\partial}f \in [\omega_\epsilon]$ *is the* ***unique*** *Ricci-flat Kähler metric in the Kähler class of* ω_ϵ*, on* $\mathscr{K}$*.* ♦

Proof. We setup the input needed for the implicit function theorem.

(1) We have that $-\frac{1}{2}\Delta_{\omega_\epsilon} : C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})^0 \to C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})^0$ is a bounded linear isomorphism with $\left\|\frac{1}{2}\Delta_{\omega_\epsilon}\right\| \leq \frac{K}{2} =: L$ and $L > 0$ **independent of** $\epsilon > 0$ from the previous section.

(2) Transferring the unweighted Lipschitz bound on the nonlinearity (which follows from a local coordinate calculation) to the weighted Holder case gives us (by multiplying both sides of the unweighted Lipschitz bound by $\epsilon^{\delta-2}$ and using the definition of the weighted Holder norms), we have that there exists a constant $C_1 > 0$ such that $\|\mathsf{N}(f) - \mathsf{N}(g)\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})} \leq C_1\epsilon^{\delta-2}\|f-g\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})}\left(\|f\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})} + \|g\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})}\right)$, $\forall f, g \in D_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})}(0, r_0)$ with $N := C_1\epsilon^{\delta-2}$ and $r_0 > 0$ to be determined later.

(3) Proposition 0.21 and the Taylor expansion of the exponential function gives us the existence of a constant $C_2 > 0$ such that $\operatorname{supp}\mathcal{F}_\epsilon(0) \subseteq \left\{\epsilon^{\frac{1}{2}} \leq |z| \leq 2\epsilon^{\frac{1}{2}}\right\}$ and $|\mathcal{F}_\epsilon(0)| \leq C_2\epsilon^2$ on its support. By definition of the weighted Holder norm and the shape of ρ_ϵ, we thus have that $\|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})} \leq C_2\epsilon^{2-\frac{\delta-2}{2}} = C_2\epsilon^{3-\frac{\delta}{2}}$.

Set $r_0 := C_3\epsilon^{3-\frac{\delta}{2}}$ with $C_3 > 0$ chosen so that $C_3 > 2LC_2$. We now need to show that $\|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})} \leq \frac{r}{2L}$ with $r < \min\left\{r_0, \frac{1}{2NL}\right\} = \min\left\{C_3\epsilon^{3-\frac{\delta}{2}}, \frac{\epsilon^{2-\delta}}{2LC_1}\right\}$. We have that

$$\|\mathcal{F}_\epsilon(0)\|_{C^{0,\alpha}_{\delta-2,\epsilon}(\mathscr{K})} \leq \frac{2LC_2\epsilon^{3-\frac{\delta}{2}}}{2L}$$

and so we choose our $r := 2LC_2\epsilon^{3-\frac{\delta}{2}}$. Clearly $r < r_0$, and it remains to show that $r < \frac{\epsilon^{2-\delta}}{2LC_1}$. But $\epsilon^{3-\frac{\delta}{2}} = \epsilon^{2-\delta+P}$ where $P := 1+\frac{\delta}{2}$, which is *positive* because $\delta \in (-2, 0)$. Hence since $x > y \Rightarrow c^x < c^y$ for $c \in (0, 1)$ small enough, **making** $\epsilon > 0$ **small enough so that** $\epsilon^{1+\frac{\delta}{2}} < \frac{1}{4L^2C_1C_2}$ forces $r < \frac{\epsilon^{2-\delta}}{2LC_1}$, as was to be shown. ■

Now we have from the smallness of the norm $\|f\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})} < C_3\epsilon^{3-\frac{\delta}{2}}$ of the solution to the nonlinear problem above, the definition of the weighted Holder spaces, and the fact that $\delta \in (-2,0)$ gives $\|f\|_{C^{2,\alpha}_{\delta,\epsilon}(\mathscr{K})} < C_3\epsilon^{3-\frac{\delta}{2}} \Rightarrow \|f\|_{L^\infty(\mathscr{K})} < C_3\epsilon^{3+\frac{\delta}{2}}, \left\|\nabla_{\omega_\epsilon} f\right\|_{L^\infty(\mathscr{K})} < \mathrm{C}_1\epsilon^{2+\frac{\delta}{2}}, \left\|\nabla^2_{\omega_\epsilon} f\right\|_{L^\infty(\mathscr{K})} < \mathrm{C}_2\epsilon^{1+\frac{\delta}{2}}$ for $\mathrm{C}_1, \mathrm{C}_2 > 0$ and all higher order tensor norms defined with respect to ω_ϵ. In particular, this implies that $\left\|i\partial\bar{\partial} f\right\|_{L^\infty(\mathscr{K})} < \mathrm{C}_2\epsilon^{1+\frac{\delta}{2}}$ with the higher order tensor norm defined with respect to ω_ϵ, which due to $\delta \in (-2,0)$ tends to zero as $\epsilon \searrow 0$.

Combining this with the fact that $\left\|\left(\omega_\epsilon + i\partial\bar{\partial} f\right) - \omega_\epsilon\right\|_{L^\infty(\mathscr{K})} = \left\|i\partial\bar{\partial} f\right\|_{L^\infty(\mathscr{K})} < \mathrm{C}_2\epsilon^{1+\frac{\delta}{2}}$ implies the identity map $\mathrm{id}_{\mathscr{K}}$ is a $\mathrm{C}_2\epsilon^{1+\frac{\delta}{2}}$-Gromov-Hausdorff approximation, *Remark* 0.17, and the triangle inequality for the Gromov-Hausdorff metric, we have:

Corollary 0.30. Define $\widetilde{\omega_\epsilon} \coloneqq \omega_\epsilon + i\partial\bar{\partial} f \in [\omega_\epsilon]$ the **unique** Ricci-flat Kähler metric in the Kähler class $[\omega_\epsilon]$ produced from the main theorem.

Pick a sequence $\epsilon_i \searrow 0$.

Then the pair $\left(\mathscr{K}, \widetilde{\omega_{\epsilon_i}}\right) \xrightarrow{GH} \left(\mathbb{T}^4/\{\pm 1\}, \omega_0\right)$ converges *in the Gromov-Hausdorff topology* to the orbifold $\left(\mathbb{T}^4/\{\pm 1\}, \omega_0\right)$ with the singular Kähler metric ω_0 (which is equal to the flat Kähler metric on $\mathbb{T}^4/\{\pm 1\} - \mathcal{S}$). ■

References

[1] Robert Bartnik. "The mass of an asymptotically flat manifold." In: *Communications on Pure and Applied Mathematics* 39.5 (1986), pp. 661–693. DOI: 10.1002/cpa.3160390505 (cit. on pp. 4, 19).

[2] Olivier Biquard and Vincent Minerbe. "A Kummer Construction for Gravitational Instantons." In: *Communications in Mathematical Physics* 308.3 (2011), pp. 773–794. DOI: 10.1007/s00220-011-1366-y (cit. on pp. 3, 8).

[3] Simon Brendle and Nikolaos Kapouleas. "Gluing Eguchi-Hanson Metrics and a Question of Page." In: *Communications on Pure and Applied Mathematics* 70.7 (2017), pp. 1366–1401. DOI: 10.1002/cpa.21678 (cit. on p. 3).

[4] Xiuxiong Chen, Simon Donaldson, and Song Sun. "Kähler-Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities." In: *Journal of the American Mathematical Society* 28.1 (2015), pp. 183–197. DOI: 10.1090/S0894-0347-2014-00799-2 (cit. on p. 8).

[5] Xiuxiong Chen, Simon Donaldson, and Song Sun. "Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than 2π." In: *Journal of the American Mathematical Society* 28.1 (2015), pp. 199–234. DOI: 10.1090/S0894-0347-2014-00800-6 (cit. on p. 8).

[6] Xiuxiong Chen, Simon Donaldson, and Song Sun. "Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof." In: *Journal of the American Mathematical Society* 28.1 (2015), pp. 235–278. DOI: 10.1090/S0894-0347-2014-00801-8 (cit. on p. 8).

[7] Simon Donaldson. "Calabi-Yau metrics on Kummer surfaces as a model glueing problem." In: *Advances in geometric analysis* 21 (2012), pp. 109–118. DOI: 10.48550/arXiv.1007.4218 (cit. on pp. 2, 11).

[8] Lorenzo Foscolo. "ALF gravitational instantons and collapsing Ricci-flat metrics on the $K3$ surface." In: *Journal of Differential Geometry* 112.1 (2019), pp. 79–120. DOI: 10.4310/jdg/1557281007 (cit. on p. 2).

[9] Gary W. Gibbons and Christopher N. Pope. "The positive action conjecture and asymptotically Euclidean metrics in quantum gravity." In: *Communications in Mathematical Physics* 66.3 (1979), pp. 267–290. DOI: 10.1007/BF01197188 (cit. on pp. 2, 11).

[10] Daniel Huybrechts. *Complex Geometry*. Universitext. Springer, 2004. DOI: 10.1007/b137952 (cit. on p. 8).

[11] Claude LeBrun and Michael Singer. "A Kummer-type construction of self-dual 4-manifolds." In: *Mathematische Annalen* 300.1 (1994), pp. 165–180. DOI: 10.1007/BF01450482 (cit. on p. 2).

[12] Cristiano Spotti. *Degenerations of Kähler-Einstein Fano Manifolds*. 2012. arXiv: 1211.5334 [math.DG] (cit. on p. 24).

[13] Gábor Székelyhidi. *An Introduction to Extremal Kähler Metrics*. AMS, 2014 (cit. on p. 13).

[14] Pankaj Topiwala. "A new proof of the existence of Kähler-Einstein metrics on $K3$, I, II." In: *Inventiones mathematicae* 89 (1987), pp. 425–454 (cit. on p. 2).

[15] Thomas Walpuski. *Riemannian Geometry II Lecture Notes*. URL: https://walpu.ski/Teaching/RiemannianGeometry.pdf (cit. on pp. 5, 6).