Markov Chains

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Markov Chains

- Suppose state space, $\mathbb{N}$ with random variables $\{X_n, n = 0, 1, \ldots\}$ giving the state at index n
- Let

$$\begin{aligned} P_{ij} &= P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_1 = i_1 \\ &= P(X_{n+1} = j | X_n = i) \end{aligned}$$

- Process called a **Markov Chain**

Transition Probabilities

- One step transition probabilities

$$p = \begin{bmatrix} P_{00} & P_{01} & \cdots \\ P_{10} & P_{11} & \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

n-Step Transition Probabilities

- Probability in state j after $n+m$ transitions from state i
- Find by summing over all intermediate states, k

$$
\begin{aligned}
P_{ij}^{n+m} &= P[X_{n+m} = j | X_0 = i] \\
&= \sum_{k=0}^{\infty} P[X_{n+m} = j, X_n = k | X_0 = i] \\
&= \sum_{k=0}^{\infty} P[X_{n+m} = j, | X_n = k, X_0 = i] P[X_n = k | X_0 = i] \\
&= \sum_{k=0}^{\infty} P_{ik}^{n} P_{kj}^{m}
\end{aligned}
$$

Transition Matrices

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- Let $\boldsymbol{P}^{(n)}$ be the matrix of n-step transition probabilities
- Then $\boldsymbol{P}^{(n)} = \boldsymbol{P}^{n}$
- Example: Suppose weather is a 2-state Markov chain, so that if it rains today, it rains tomorrow with probability 0.7, and if it is sunny today, then it is sunny tomorrow with probability 0.6.
- If it rains today, what is the probability it rains in two days? Four days?

Example Solution

- 1 day

$$\boldsymbol{P} = \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix}$$

- 2 days

$$\boldsymbol{P}^2 = \begin{bmatrix} .61 & .39 \\ .52 & .48 \end{bmatrix}$$

- 4 days

$$\boldsymbol{P}^4 = \begin{bmatrix} .57 & .43 \\ .57 & .43 \end{bmatrix}$$

Classification of States in a Markov Chain

- State j is *accessible* from state i if $P_{ij}^n > 0$ for some $n \geq 0$.
- If states i and j are accessible from each other, they *communicate*
- A Markov chain is *irreducible* if all states communicate
- Let g_i be the probability that starting in state i, the process will reenter state i
- State i is *recurrent* if $g_i = 1$ and *transient* otherwise.
- State i is *positive recurrent* if starting the i the expected time to return to i is finite
- For a finite state Markov chain, all recurrent states are positive recurrent

Classification of States in a Markov Chain

- If $P_{ii}^n = 0$ for n not divisible by $d, d \in \mathbb{N}$ and d is the largest integer with this property, then state i is *periodic*
- A state with $d = 1$ is *aperiodic*
- Positive recurrent, aperiodic states are *ergodic*

Limiting Probabilities of a Markov Chain

Theorem: Let $lim_{n \to \infty} P_{ij}^{n} = \pi_{j.}$ Then for an irreducible, ergodic Markov chain, π_j exists and is independent of i. Further

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij}, j \geq 0$$

$$\sum_{j=0}^{\infty} \pi_j = 1$$

Example

- Find the long term probabilities for rain, no rain for our previous example
- Solution

$$\begin{aligned}
\pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} \\
\pi_1 &= \pi_0(1 - P_{00}) + \pi_1(1 - P_{10})
\end{aligned}$$

- So

$$\begin{aligned}
\pi_0 &= \frac{P_{10}}{1 + P_{10} - P_{00}} \\
&= 0.57 \\
\pi_1 &= \frac{1 - P_{00}}{1 + P_{10} - P_{00}} \\
&= 0.43
\end{aligned}$$