The Price of Human Nature

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1 INTRODUCTION

Finding the best strategy for network and traffic routing has historically been a problem of great importance combining the theoretical aspects of both game theory and computer science. Seminal work by Roughgarden and Tardos framed the routing problem as a non-cooperative game, in which players' selfish routing decisions increase the social welfare cost, i.e., the overall latency of all users in the network. Our paper begins with this selfish model, presenting a brief overview of the original selfish routing problem and model and the limits on the optimality of these algorithms (termed as the "price of anarchy").

Since the advent of the selfish routing model, more complex (and perhaps more realistic) models have emerged, many of which emphasize the need to take into account the complexity of human behaviors. This paper focuses on three of these recent alternative models, namely models that account for altruistic, risk-averse, and diverse behaviors. We present and clarify the findings of these papers in the context of the original selfish routing paper, and demonstrate how these papers' results can be synthesized into a more general framework addressing optimality of routing with various human behaviors or motivations.

2 BACKGROUND

This section presents a brief history of traffic routing algorithms, defining the terminology and the traffic routing problem.

2.1 The Traffic Routing Problem

Traffic routing problems naturally arise in communication or transportation networks, where users are trying to minimize the latency that they or their data experiences. However, links in the network often becomes *congested* if too many users decide to route their data or cars through that link. Consequently, in these networks, the path each user chooses can affect the travel times of other users. Here, we describe Roughgarden and Tardos' formalization of the problem of minimizing latency as multicommodity flow networks [7, 8], and use Pigou's example network in Figure 1 as a running example. We use this to reason about the deviation from the opimum minimal latency when users make selfish routing decisions.

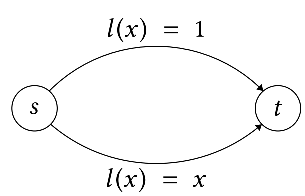


Fig. 1. Pigou's example traffic routing problem, with a demand of $r_{(s,t)} = 1$

The input to a traffic routing problem consists of:

- A network $G = (V, E)$ of $|V|$ destinations (e.g., locations or servers) and $|E|$ links

- A set of k source-destination pairs $S = \{(s_1, t_1), \cdots (s_k, t_k)\}$ representing traffic demands
- A rate r_i of traffic for each $(s_i, t_i) \in S$ representing the demanded amount of traffic from s_i to t_i
- A latency function l that assigns a per-edge function l_e to each edge e describing how adding traffic (i.e., congestion) to e affects the time taken to travel across e. We can also think of l as assigning per-path costs: for any path p in the graph
$$l_p(f) = \sum_{e \in P} l_e(f_e)$$
We assume that l is continuous, nonnegative, and nondecreasing.

In Figure 1, we see a single source-destination input network with an example (linear) cost function with $r_{(s,t)} = 1$.

Solutions correspond to flow assignments to the set of simple paths P_i between s_i and t_i for all i. Note that our solutions assume *nonatomic* entities: the flows we find may not be integral. Intuitively, this means that the demand from one s_i to t_i is generated by an infinite number of entities in the network, which allows us to reason about continuous, rather than discrete, functions. To describe a flow assignment f, we can consider f_p, the flow on a single path $p \in P_i$ (this adds an equal amount of flow f_p to all edges in p), as well as $f_e = \sum_p \sum_{e \in p} f_p$, the flow on edge e (the sum of flow on all paths that use e).

A *feasible* solution given such an input is an assignment of path flows such that the demand from s_i to t_i is met:
$$\forall 1 \leq i \leq k, \sum_{p \in P_i} f_p = r_i$$
An *optimal* (feasible) solution given such an input is the feasible flow assignment f that minimizes the **total weighted latency cost**, or **social welfare latency cost** $C(f)$, where
$$C(f) = \sum_i \sum_{p \in P_i} l_p(f) f_p = \sum_{e \in E} f_e l_e(f_e)$$

Intuitively, we are calculating the latency of each path of a given flow assignment, weighing each path's latency proportional to the amount of flow through the path. More concretely, if we were to let flow represent the routes chosen by (infinitely many) users, $C(f)$ calculates the average latency over all users. Thus, when minimizing $C(f)$, some users may incur more latency so that other users can go faster: the optimal flow is the *socially optimal* solution. Note that there exists an optimal flow f^* minimizing $C(f)$ because we assume l is continuous and the set of feasible flows is compact.

In our running example (Figure 1), a feasible flow is any flow that sends one unit from s to t (divided in any fashion between the top and bottom edges). The optimal flow is the flow that sends half the traffic through the lower edge and half through the upper edge: the users on the lower edge only experience a latency of 1/2, while the users on the upper edge experience a latency of 1, making the total weighted latency cost 3/4.

2.2 Coordination Models and the Price of Anarchy

Before we can create algorithms to solve the traffic routing problem, we must first assume a *coordination model* for our traffic network. There are two clear extremes: (1) centralized control, in which some entity (e.g., an air traffic controller) knows all traffic demands and routes accordingly, and (2) decentralization, i.e., a complete *lack* of coordination between entities in the network. In a centralized setting, there is a clear optimal solution, as shown in the previous section. However, in a decentralized and uncoordinated model, the lack of coordination and the exercise of free will can result in inefficiencies.

The Price of Anarchy (PoA) allows us to measure the inefficiencies of a decentralized model, and was first introduced by Koutsoupias and Papadimitriou in 1999 [5]. We treat the decentralized model as a game in which each individual optimizes for her own individual cost function c, allowing us to compute the achieved flow at the resulting Nash equilibrium (proven to exist if the cost functions are continuous) [1, 4, 10]. The PoA is defined as the ratio between the social welfare latency cost at the optimal flow, and the social welfare latency coast at the flow achieved at equilibrium. (This is similar to how we measured the distance from optimal of an approximation algorithm in a limited computational power model, and of online algorithms in an incomplete information model.)

The set of flows at Nash equilibrium are defined such that for all i source-destination pairs, all the paths from $s_i \rightarrow t_i$ have the minimum-possible cost with respect to c. In other words, in the selfish routing model, the (nonzero) flow paths at Nash equilibrium have equal path costs, and no user decreases her cost by choosing a different path:

$$\forall 0 \leq i \leq k,\ \forall p_1, p_2 \in P_i\ s.t.\ f_{p_1} > 0 \text{ and} f_{p_2} > 0,\ c_{p_1}(f) = c_{p_2}(f)$$

Note that this corresponds exactly to the solutions to the following (convex) program solvable in polynomial time:

$$NE = \min_f \left(\sum_e \int_0^{f_e} c_e(t)dt \right) \text{ subject to feasibility constraints}$$

whereas the flow optimizing the social welfare latency cost corresponds exactly to the (polynomial-time) solutions to the following (convex) program:

$$SW = \min_f \left(\sum_e f_e l_e(f_e) \right) \text{ subject to feasibility constraints}$$

2.3 The Selfish Routing Model

One example of an uncoordinated model is the *selfish routing* model, in which all entities in the network are selfish and choose a route minimizing their individual latency without caring (or knowing) about the effects on other users [8]. The selfish routing model corresponds to flows at a Nash equilibrium where each user optimizes her individual cost function $c^s(f) = l(f)$. Thus, the program optimized at Nash Equilibruim is

$$NE_s = \min_f \left(\sum_e \int_0^{f_e} l_e(t)dt \right) \text{ subject to feasibility constraints}$$

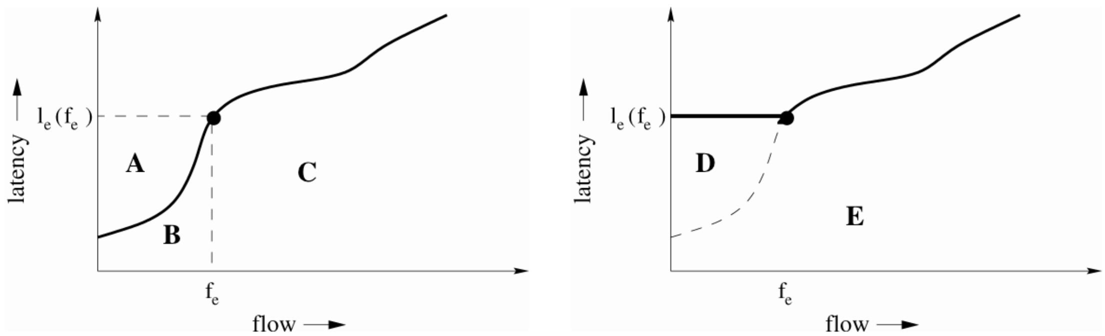
If we revisit our running example in Figure 1, we note that the flow at Nash equilibrium corresponds to a flow that sends the entire unit of traffic through the bottom edge (the 0 flow through the top path has latency 1, and the unit flow through the bottom path will have latency 1). Intuitively, each user routing from s to t will selfishly choose to take the bottom route because she will reason that the bottom route can have latency no worse than the top route. However, by doing so, the bottom route becomes more congested and leads to a total average latency $C(f) = 1$. Thus, in Figure 1, the price of anarchy ρ is $\frac{1}{3/4} = \frac{4}{3}$.

We next describe and present the main results regarding the PoA in this (decentralized) selfish routing model, which will act as a basis to which we will compare traffic routing results in more recently formulated models.

2.4 Main Results

Theorem 1. *If f is a flow at Nash Equlibrium for a given input set (G, r, l) and f^* is a feasible flow for $(G, 2r, l)$, then $C(f) \leq C(f^*)$*

Proof Sketch. In other words, this result suggests that the latency when all users of a network route with only their own interests in mind is utmost the optimal (minimum) latency for the same graph and latency functions with twice as much demand per path.



(a) Graph of latency function ℓ_e and its value at flow value f_e

(b) Graph of latency function $\overline{\ell_e}$

For any Nash Equilibrium solution f for (G, r, ℓ) and any feasible solution f^*, we can draw the figures so that for each edge e in G:

The area $A + B$ represents the latency cost $C(f)$ of the Nash Equilibrium solution f for (G, r, ℓ). The area $B + C$ represents the cost $C(f^*)$ of the feasible solution f^* for $(G, 2r, \ell)$. Since the area $D + E$ is at most area D more than area $B + C$, and area D is less than area $A + B$, we know that $D + E - (B + C) \leq D \leq A + B$. This means that $B + C \geq D + E - (A + B)$. On the other hand $D + E \geq 2(A + B)$. Therefore, we have that $C(f^*) = (B + C) \geq 2(A + B) - (A + B) = A + B = C(f)$. Therefore we have the result that the latency of any Nash equilibrium flow for (G, r, ℓ) is no bigger than represents the latency of any feasible solution for $(G, 2r, \ell)$. The main idea of the proof for this theorem lies in that the subset if the area C is at least as big as the area $A + B$. This is always true because the latency $\bar{\ell}_e$ is a nondecreasing function and thus the graph of latency function gives a shape that looks like a generalized trapezoid (in particular, a right generalized trapezoid lying on the horizontal axis). □

Theorem 2. *If the edge latency functions are linear in that $l_e = a_e f_e + b_e$ for every edge $e \in E$, then the price of anarchy or $\rho(G, R, l) \leq 4/3$.*

Proof Sketch. The latency of any flow f under these edge latency functions is $C(f) = \sum_e a_e f_e^2 + b_e f_e$.

Let's consider two flows f and f^* such that f is at Nash Equlibrium in (G, r, ℓ) and f^* is globally optimal for the same. We first consider optimally routing the first $r/2$ demand across all source-destination pairs. It turns out that $f/2$ is optimal for $(G, r/2, \ell)$ when the edge latency functions are linear. This can be derived from the fact that paths with non-zero flow at a Nash Equlibrium have the same path latency while paths with non-zero flows at the global optimum have the same marginal cost of increasing the flow. Now, if we look at the cost $C(f/2)$ of routing this in terms of the latency of routing the flow f at Nash Equlibrium, we notice that $C(f/2) = \sum_e \frac{1}{4} a_e f_e^2 + \frac{1}{2} b_e f_e \geq \frac{1}{4} C(f)$ from the above cost expression. Thus, in other words, routing the first $r/2$ optimally has a latency that is at least one-fourth of the latency of the Nash Equlibrium flow.

This leaves the remaining $r/2$ that needs to be routed optimally to route f^* fully. To reason about this, let's look at a small δr_i increase in flow from s_i to t_i that already carries x units of flow. For a convex latency function, we expect the increase in latency to be at least $\delta r_i l'(x)$ where l' if the minimum marginal increase in C. If we consider starting at the optimal flow $f/2$ for the $r/2$ demand and increasing the flow on each path by a small

δr_i, the subsequent increase in latency across be all paths can be summed as $\sum_{i=1}^{k} l'(f/2)\delta r_i$. But, for linear edge latency functions, the marginal increase in latency on every edge at $f/2$ is exactly the latency of that edge at f. Thus, $l'_e(f/2) = l_e(f)$. Thus, setting $\delta = 1$ and increasing the rate by $r_i/2$ on every s_i, the overall increase in latency is at least $\frac{1}{2}\sum_{i=1}^{k} l(f)r_i = \frac{1}{2}C(f)$. The last part is by definition of $C(f)$.

In essence, routing the first $r/2$ demand optimally costs at least $C(f)/4$ and the next $r/2$ when augmented, costs at least another $C(f)/2$. In total, the optimal costs at least $\frac{3}{4}C(f)$. In other words, the flow at Nash Equilibrium has cost utmost $\frac{4}{3}C(f^*)$ where f^* is the flow achieving optimal latency. □

3 ALTERNATIVE MODELS

In this section, we present and compare a subset of recent coordination models for the traffic routing problem against the selfish routing model. These models propose a more nuanced (and perhaps more accurate) description of human behavior. XXX TODO more summary once we have more insights?

3.1 Altruism and Spite

The first alternative model we consider is that proposed by Chen and Kempe in 2008 [2], which assumes that users are "not entirely selfish." Chen and Kempe note that social experiments from both economic and psychology have shown humans do not behave rationally in a selfish manner; instead, our behavior is better modeled as either altruistic or malicious (spiteful). Their model proposes a simple way to capture how people make decisions based upon how much latency a particular decision will cost other users; if someone is spiteful, she will want to increase others' latencies, and if she are altruistic, she will want to decrease their latencies.

3.1.1 Formalization. The formal Chen and Kempe model introduces a per-user *altruism* coefficient β and a new individual user cost function c_p^β for all paths p:

$$c_p^\beta(f) = \sum_{e \in p} l_e(f_e) + \beta \sum_{e \in P} f_e l'_e(f_e)$$

where $l_e(\cdot)$ is the latency function from the selfish routing setting, and $l'_e(\cdot)$ is the derivative with respect to f_e.

Note that the first term is exactly the user cost used in the selfish routing model (and thus the two user costs are equivalent when $\beta = 0$). The second term corresponds to the derivative of the social welfare latency cost on p and is weighed by β; we use the derivative, rather than the value, of the social welfare cost on p because each user only controls an infinitesimally small amount of the flow. Thus, if we were to use the value, a single user's choice would have no effect on the social welfare cost! Instead, a user can account for how she will affect the social welfare cost via the rate at which her choice of path affects other users.

If β is negative, a user is spiteful: we know that adding a little more flow to p will increase the social welfare cost of taking p (the derivative l'_e is positive), and since we negate this value, this lowers the user's perceived cost of taking p. Conversely, if β is positive, a user is altruistic: increasing flow increases the social welfare cost on p and also the user's perceived cost of taking p. We assume that β ranges from -1 (extremely spiteful) to 1 (extremely altruistic), where $\beta = 0$ corresponds to selfishness.

All analysis of the model assumes a particular distribution ψ of β for all users. Futhermore, we can compare this model to the selfish model using the PoA as a measure of inefficiency because the atruistic model still achieves Nash equilibrium for any ψ and cost function c_p^β: given any ψ, c_p^β are continuous in the choice of path p, and we can apply the result of Theorem 1 of Mas-Colell [?]. Nash equilibrium is achieved at the flow solutions to the program

$$NE_\beta = \min_f \sum_e \int_0^{f_e} c_e^\beta(t)dt \text{ subject to feasibility constraints}$$

We next present Chen and Kempe's core results about the PoA of arbitrary networks when ψ is uniform (all users have the same β value), and briefly mention their results of non-uniform ψ in parallel-link networks.

3.1.2 Uniformly Distributed Altruism. We first consider the case where ψ is uniformly distributed, such that β and therefore c_p^β is the same for each user. We additionally assume that users tend to be altruistic, i.e., $\beta > 0$.

THEOREM 3. *For any G, demand rates r, and a uniform distribution ψ with $\beta \in (0, 1]$, if l_e is nondecreasing and convex for all e, then the price of anarchy is bounded by*

$$\rho(G, r, l, \psi) \leq \frac{1}{\beta}$$

PROOF SKETCH. Consider the two (convex) functions that we minimize for each of the two objectives NE_β and SW. For simplicity, let $B(f)$ be the function minimized in NE_β; the second is our social welfare cost $C(f)$. We can write these and manipulate them into comparable forms as follows:

$$B(f) = \sum_e \int_0^{f_e} c_e^\beta(t)dt = \sum_e \int_0^{f_e} l_e(t) + \beta t l'e(t)dt \text{ (by definition of } c_e^\beta)$$

$$C(f) = \sum_e f_e l_e(f_e) = \sum_e \int_0^{f_e} (t l_e(t))'dt = \sum_e \int_0^{f_e} l_e(t) + t l'e(t)dt$$

It is clear that for any feasible flow f, $B(f) \leq C(f) \leq \frac{B(f)}{\beta}$ because $\beta \in (0, 1]$. We now let $\hat{f}$ be the flow at Nash Equilibrium and f^* be the flow at optimum social welfare. Because these are the optimal flows for their respective objectives, we know that $C(\hat{f}) \leq \frac{B(\hat{f})}{\beta} \leq \frac{B(f^*)}{\beta} \leq \frac{C(f^*)}{\beta}$, proving that $\rho(G, R, l, \psi) \leq \frac{1}{\beta}$ □

3.1.3 Uniformly Distributed Spite. Chen and Kempe then address the problem of spite: how (uniformly) spiteful can users be before the PoA becomes infinite? It turns out that this depends on the type of latency function! Our analysis begins by reasoning about the PoA of a given class L of latency functions:

$$\rho(G, R, L, \psi)$$

It turns out that the PoA of a class of functions L is lower-bounded by the worst PoA achieved in a two-link, two-node network (such as in Figure 1) with demand rate r and latency function $l_1(x) = l(x)$ and constant latency function $l_2(x) = c^\beta r$, for any choice of $l \in L$.

THEOREM 4. *For any G, demand rates r and uniform distribution ψ of $\beta \in (-1, 1]$,*

$$\rho(G, R, L, \psi) \leq \max_{l \in L} \max_{x, r \geq 0} \frac{r l(r)}{x l(x) + (r - x)(c^\beta(r))}$$

PROOF SKETCH. We give a brief overview of the proof technique here, which proceeds by considering social welfare cost of the flow f^* optimizing C. By unfolding the RHS part of the above goal to get a bound for $x l_e(x)$, we can then apply this bound to each edge with $x = f_e^*$ and $r = \hat{f}_e$, where $\hat{f}$ is the optimizing flow at Nash Equilibrium. With some mathematical manipulation, we can derive a comparison of $C(f^*)$ to $B(\hat{f})$ satisfying the above bound. □

Since we know how to bound $\rho(G, R, L, \psi)$ by the (uniform) value of β, we can now determine at which values of β this lower bound is infinite: how spiteful do users have to be to cause each other infinitely more suffering? The following result shows that if the cost functions are in L_d = polynomials of degree $\leq d$, the PoA is bounded when β is at least $\frac{-1}{d}$ (and is infinite when $\beta < \frac{-1}{d}$).

Theorem 5. *For any G, demand rates r, $l_e \in L_d$, and uniform distribution ψ with $\beta \in (\frac{-1}{d}, 1]$,*

$$\rho(G, R, l, \psi) \leq \left(\left(\frac{1+\beta d}{1+d}\right)^{1/d}\left(\frac{1+\beta d}{1+d} + 1 + \beta d\right) + 1 + \beta d\right)^{-1}$$

Proof Sketch. From Theorem 4, we know that $\rho(G, R, L, \psi)$is bounded above by the worst-case PoA for any cost function $l \in L_d$ in a two-node, two-link network with cost functions $l_1(x) = l(x)$ and $l_2(x) = c^\beta r$. Thus, we only need to consider how bad the PoA in this two-node network can get for any $l \in L_d$.

Let us first consider the price of anarchy to route $r\lambda$ units of flow through the 2-node network for some choice of λ. The solution at Nash Equilibrium will route some $r\lambda$ units of flow from the source node to the destination will put $r\lambda$ flow on the first link with $l_1 = l(x)$, and the rest on the second link with $l_2 = c^\beta(x)$. The solution optimizing social welfare will put all flow on the first link with latency function $l_1 = l(x)$. This gives us a PoA of

$$\left(\frac{\lambda l(r\lambda)}{l(r)} + \left(1-\lambda\right)\left(1 + \frac{\beta r l'(r)}{l(r)}\right)\right)^{-1}$$

WLOG, we can consider latency functions $l(x) = ax^i$ for some $i \leq d$. Choose $\lambda = \left(\frac{1+\beta i}{1+i}\right)^{\frac{1}{i}}$ so that $\frac{l(r\lambda)}{l(r)} = \frac{1+\beta i}{1+i}$ and $\frac{l'(r)}{l(r)} = \frac{i}{r}$.

If we route $r\left(\frac{1+\beta i}{1+i}\right)^{\frac{1}{i}}$ flow across the first link with cost $l_1(x)$ and the rest on the second link with cost $l_2(x)$, then we get (after several manipulation steps) that

$$\max_{l \in L_d} \max_{x, r \geq 0} \frac{rl(r)}{xl(x) + (r-x)(c^\beta(r))} = \left(\left(\frac{1+\beta i}{1+i}\right)^{1/i}\left(\frac{1+\beta i}{1+i} + 1 + \beta i\right) + 1 + \beta i\right)^{-1}$$

This is increasing in i, giving us the worst-case bound when $i = d$:

$$\left(\left(\frac{1+\beta d}{1+d}\right)^{1/d}\left(\frac{1+\beta d}{1+d} + 1 + \beta d\right) + 1 + \beta d\right)^{-1}$$

□

3.1.4 Arbitrarily Distributed Altruism. Chen and Kempe go on to extend their analysis to when users have an arbitrary distribution ψ of altruism (with no spiteful users) in *parallel link networks*. We briefly mention their results here, but direct the reader to the paper for a more detailed proof and explanation. Their main result mirrors that of the PoA with uniform altruism:

Theorem 6. *Given any parallel link network G, demand rates r, altruism density function ψ with average altruism $\bar{\beta}$ and non-negative support, and convex and non-decreasing cost functions l_e, $\rho(G, r, l, \psi) \leq \frac{1}{\bar{\beta}}$*

One interesting observation is that this theorem leads to the following corrollary: because the theorem applies for a distribution in which a rate of $\bar{\beta}$ of users are completely altruistic and 1-$\bar{\beta}$ users are completely selfish, this is equivalent to $\bar{\beta}$ of the population being under centralized (coordinated) control:

Corollary 7. *Given any parallel link network G, demand rates r, altruism density function ψ with average altruism $\bar{\beta}$ and non-negative support, and convex and non-decreasing cost functions l_e, if $\bar{\beta}$ fraction of traffic is controlled by a central authority, then $\rho(G, r, l, \psi) \leq \frac{1}{\bar{\beta}}$*

This is exactly the result proven by Roughgarden in 2004 [6], and provides an interesting connection between models that include altruism and models of centralized/coordinated traffic control.

3.2 risk-aversion

The second model we consider accounts for the tendency of users to pick routes with less variation in latency even if it comes at the cost of some added latency on the paths chosen. This increase in latency can be quantified as the *price of risk-aversion* which is the worst-case ratio of the latency or cost at a risk-averse Nash Equilibrium to that at a risk-neutral Nash equilibrium or one where users are indifferent to variations in the latency itself.

3.2.1 Formalization. The **formal model** introduced in Lianeas et.al [?] defines a risk-aversion coefficient γ that quantifies the users' tendency to prefer paths with less variability. A higher γ means that one is more risk averse. The individual user's costs on each edge $c_e^{\gamma}(x_e)$ now have a deterministic part or $l_e(x_e)$ and a noise modelled by a random variable $\xi_e(x_e)$. The latter is assumed to be independent across edges and has expectation 0 and variance $v_e(x_e)$ for x_e flow allowing us to sum them up over a path. To simplify the analysis, the model also defines κ to bound the variance-to-mean latency ratio. In other words, $v_e(x_e) \leq \kappa l_e(x_e)$. Thus, the individual cost function for each user is of the form

$$c_p^{\gamma}(f) = \sum_{e \in p} l_e(f_e) + \gamma \sum_{e \in p} v_e(f_e)$$

Nash equilibrium is achieved at the flow solutions to the program

$$NE_{\gamma} = \min_{f} \sum_{e} \int_0^{f_e} c_e^{\gamma}(t)dt \text{ subject to feasibility constraints}$$

$c_p^{\gamma}(f)$ is assumed to be non-decreasing. Intuitively, the mean is identical to the original selfish routing individual user cost formulation, while the second term accounts for variance. Minimizing this implies that we want to minimize the variance depending on the value of the risk coefficient itself.

If we let $C(x)$ be the the maximum cost across some set x of flows at the Nash equilibrium for a given problem instance (restricted to some family of inputs and a fixed κ), the **price of risk-aversion** is now defined as the ratio $C(x)/C(z)$. Here $C(z)$ is the cost associated with some risk-neutral Nash equilibrium z. In the following section, we look at bounding this price of risk-aversion. Note that this can be viewed as contributing a multiplicative factor to the price of anarchy in the overall change to the latency or cost of the system. We first prove a more basic result from an older paper on this topic [?] and then proceed to the main result on the price of risk-aversion for a special family of latency functions that are (λ, μ) smooth. Note that additional results can be proved for special classes of graph topologies which can be found in the original paper [?].

3.2.2 Main Results.

Theorem 8. *If a flow f is at a risk averse Nash equlibrium and f^* is any other flow, then $fC(f) \leq f^*C(f)$.*

Proof. By definition, any flow at Nash Equilibrium routes on paths with minimum cost or only sends flow on a given path if its cost is less than the cost of sending the same flow on some other path. Thus, for a fixed demand and fixed path costs determined by $C(f)$, f^* differs from f in atleast moving some ϵ flow from one lower cost path to another higher cost one which increases the total overall cost. Consequently, let's say we have a flow x that is a risk averse Nash Equlibrium for the cost function $c_e = l_e(x_e) + \gamma v_e(x_e)$. If z is a risk neutral Nash Equilibirium, it is still feasible for the risk averse mean-variance cost function, but is not the equilibrium in that scenario. By the above description and costs written as the sum across edges, we have

$$\sum_{e \in E} x_e(l_e(x_e) + \gamma v(x_e)) \leq \sum_{e \in E} z_e(l_e(x_e) + \gamma v_e(x_e))$$

□

Definition 9. A latency function $l(x)$ is (λ, μ)-smooth if for all $x, y \geq 0$

$$yl(x) \leq \lambda yl(y) + \mu xl(x)$$

This is a particular instance of smoothness definitions for functions that allows us to show bounds on the price of risk-aversion for such restricted families of functions.

Theorem 10. *The set of instances with latency functions $l_{e\,e \in E}$ that are $(1, \mu)$-smooth around any risk-averse Nash equilibrium x_e for all $e \in E$ have price of risk-aversion $\leq \dfrac{(1+\gamma\kappa)}{(1-\mu)}$*

Proof. This proof involves separating the edges into two sets A and B where A contains edges whose flow x_e in the risk-averse Nash equilibrium is utmost the flow on the same edge z_e in the risk-neutral Nash equilibrium and B contains the rest of the edges. x continues to be a risk-averse Nash equilibrium while z is a risk-neutral one.

Let's consider the edges in A. We know that by definition, $\sum_{e \in A} l_e(x_e) \leq \sum_{e \in A} l_e(z_e)$. In turn this means

$$\sum_{e \in A}(1+\gamma\kappa)z_e l_e(x_e) \leq \sum_{e \in A}(1+\gamma\kappa)z_e l_e(z_e)$$

.

Let's similarly consider the edges in B. By the definition of $(1, \mu)$-smoothness, we have

$$\sum_{e \in B} z_e l_e(x_e) \leq \sum_{e \in B} z_e l_e(z_e) + \mu x_e l_e(x_e)$$

.

Together, the last two statements mean that (adding some terms two both of them to encompass all the edges in each type of term),

$$\sum_{e \in A}(1+\gamma\kappa)z_e l_e(x_e) + \sum_{e \in B} z_e l_e(x_e) \leq \sum_{e \in E} z_e l_e(z_e) + \sum_{e \in E} \mu x_e l_e(x_e) + \sum_{e \in E}(1+\gamma\kappa)z_e l_e(z_e) = (1+\gamma\kappa)C(z) + \mu C(x)$$

Now, if we are able to show that the total social cost $C(x)$ of the risk averse Nash equilibrium flow is somehow utmost the expression above, we have established our proof by rearranging the terms, because the price of risk-aversion in this case is given by $C(x)/C(z)$.

To establish this, let's take the expression from the Proof of Theorem 8 and use $C(x) = \sum_{e \in E} x_e l_e(x_e)$ as well as separate edges by sets A and B, we have

$$C(x) + \sum_{e \in A} x_e \gamma v_e(x_e) + \sum_{e \in B} x_e \gamma v_e(x_e) \leq \sum_{e \in A} z_e \gamma v_e(x_e) + \sum_{e \in B} z_e \gamma v_e(x_e) + \sum_{e \in E} z_e l_e(x_e)$$

By the definitions of A and B, we can extract the sum of second and third terms from the left hand side and the second term on the right hand size, because the former is larger than the latter and can't contribute to this inequality. If we separate the last term on the right into sets A and B and apply $v_e(x_e) \leq \kappa l_e(x_e)$, we effectively are left with

$$C(x) \leq \sum_{e \in A}(1+\gamma\kappa)z_e l_e(x_e) + \sum_{e \in B} z_e l_e(x_e) \leq (1+\gamma\kappa)C(z) + \mu C(x)$$

which proves exactly what we need. □

3.2.3 Similarities and Extensions. Note that this result is similar to that of the price of anarchy originally derived by Roughgarden and Tardos [8]. If we assume no variation in prices or in other words, set $\kappa = 0$ and consider linear latency functions which by definition are $(1, 1/4)$-smooth [?], we get that the price of risk-aversion is $\leq \dfrac{1}{1-\mu} = 4/3$.

Further, this price of risk-aversion can be lower bounded for a specific case of a recursive Braess graph XXX have we mentioned this before and the gap between the upper and lower bounds can be more neatly quantified. It can also be exactly computed for a series-parallel recursive graph to be $1 + \gamma\kappa$. The details of these proofs can be found in [?].

3.3 Diverse in Interests

The third class of alternative models we consider is a generalization of the altruistic model and the risk-averse model that we have discussed in previous sections. This class is characterized by the diverse selfish behavior by its heterogeneous agents. Each agent pursues their own different selfish goal, resulting in a routing solution of the whole network.

Diverse selfish routing models are useful because they help us understand how we can leverage policies and natural diversity of goals in a network to increase the social welfare and efficiency of the network as a whole. For example, tolls can help increase the social welfare. For example, Beckmann et. al. showed that tolls can help implement the social optimum as an equilibrium, when agents all have the same goal towards a linear combination of time and money [1].

However, there is some ambiguity in measuring the optimality of any outcome of the whole network with diverse selfish behavior, because by definition, the objective function has changed from the objective of selfish routing with no agent heterogeneity and thus only one criterion. There are thus multiple reasonable ways to characterize the social welfare of a diverse routing network. We will discuss the model adopted by Cole, Lianeas and Nikolova and their newly published results in 2018 [3].

3.3.1 Model. We have the same routing network with multiple source-destination pairs and continue with all our previous notations, except that we have included two criteria the players consider in the objective function.

Each agent wants to minimize their own cost c^{ω}, which is a sum of two terms associated with two criteria. Let ℓ_P denote the cost of the first criterion (e.g., the latency) over some path $P = (s_i, t_i)$, and σ_P be the cost of the second criterion, referred to as the *deviation function*. Then given a routing f of the network, the cost of that path is given by $c_p^{\omega} = \ell_P + \omega \cdot \sigma_P = \sum_{e \in P} \ell_e(f_e) + \omega \sum_{e \in P} \sigma_e(f_e)$, where ω is the *diversity parameter*.

Note that the latency function has all the properties as we assumed in previous sections, while the deviation function $\sigma_e(x)$ is assumed to be continuous by not necessarily non-decreasing. However, the function $\ell_e + \omega \cdot \sigma_e$ must be non-decreasing. These assumptions are consistent with our previous risk-averse model in Section 3.2, because if σ_e models the variance, then σ_e could be decreasing in the flow.

Cole et. al. measures the effect of diversity against the resulting flow of a homogeneous agent population of the same size. The homogeneous agent population has the single diversity parameter $\bar{r} = \int r f(r) dr$.

For a discrete distribution of n discrete values $r_1^k, \ldots, r_n^k$, the demand d_k is a vector $d_k = (d_1^k, \ldots, d_n^k)$ where each d_i^k denotes the total demand of commodity k with diversity parameter r_i^k and d^k denote commodity k's total demand $d^k = \sum_{i=1}^{n} d_i^k$. For a heterogeneous equilibrium flow vector g, the *heterogeneous total cost* of commodity k is denoted by $C^{k,ht}(g) = \sum_{j=1,\ldots,n} d_j^k c^{k,r_j^k}(g)$, where $c^{k,r_j^k}(g)$ denotes the common cost at equilibrium g for players of diversity parameter r_j^k in commodity k. The heterogeneous total cost of g is then $C^{ht}(g) = \sum_{k \in K} C^{k,ht}(g)$.

For the corresponding homogeneous equilibrium flow f, i.e. the instance with diversity parameter $\bar{r}^k$, where $\bar{r}^k$ denotes the average diversity parameter for commodity k, players of commodity k share the same cost $c^{\bar{r}^k}(f)$.

Then, the homogeneous total cost of commodity k under f is $C^{k,hm}(f) = d_k c^{\bar{r}^k}(f)$, and the homogeneous total cost of f is $C^{hm}(f) = \sum_{k \in K} C^{k,hm}(f)$.

3.3.2 Results. Let g denote an equilibrium flow for the heterogeneous agent population and f an equilibrium flow for the corresponding homogeneous agent population. Let $C^{ht}(g)$ denote the cost of flow g and $C^{hm}(f)$ the cost of flow f.

A *multi-commodity network* is consistent with all our previous models. We also introduce the definition of a *single-commodity network* as a network whose edges all belong to some single source-destination path as only these edges are going to be used by the equilibria and thus all other edges can be discarded. We present the following main results.

Definition 11. A directed $s - t$ network G is *series-parallel* if it consists of a single edge (s, t), or it is formed by the series or parallel composition of two series-parallel networks with terminals (s_1, t_1) and (s_2, t_2), respectively.

The theorem below essentially states that for single-commodity networks, diversity is always helpful in a single-commodity series parallel network.

Theorem 12. *For any $s - t$ series-parallel network G with a single commodity, we have $C^{ht}(g) \leq C^{hm}(f)$.*

Proof Sketch. The key observation is that since f and g route the same amount of flow from the unique source to the unique sink, there must be a path P where f sends no less flow along than g does. Since the network is series-parallel, for every edge e in P, $f_e \geq g_e$. This is true because a path in a series-parallel network can be broken up into some series-parallel parts in series and some series-parallel parts in parallel, which recursively breaks down to a simple series of edge(s). Hence for any $r \in [0, r_{\max}]$, we have $c_P^r(f) \geq c_P^r(g)$. This simultaneously also means that g could route more flow on P but it doesn't, implying that there is no incentive for g to switch the flow it sends on other paths to path P under the same diversity parameter. Therefore $c^r(g) \leq \sum_{e \in P} \ell_e(g_e) + r \sum_{e \in P} \sigma_e(g_e)$. Then $C^{ht}(g) \leq \sum_{i=1}^{k} d_i (\sum_{e \in P} \ell_e(g_e) + r_i \sum_{e \in P} \sigma_e(g_e)) = \ell_p(g) + \bar{r}\sigma_P(g)$ which is exactly the cost of the homogeneous equilibrium flow f. □

Theorem 13. *For any $s - t$ non-series-parallel network G with a single commodity, there exists cost functions C for which $C^{ht}(g) > C^{hm}(f)$.*

Proof Sketch. If G is not series-parallel then the Braess graph can be embedded in it [9], and there are edge functions such that heterogeneous equilibrium flow has a larger cost than homogeneous equilibrium flow. Detailed example can be found in [3]. □

This theorem essentially states that for single-commodity networks, diversity is always helpful only in a series-parallel network. Together with Theorem 12, we know that the series-parallel structure is a sufficient and necessary condition for diversity to always be helpful.

Now we discuss our main results for multiple commodity network. We use *average-respecting demand* to refer to the property that for any commodity $i, j : \bar{r}^i = \bar{r}^j$.

A multi-commodity network G can be decomposed in subnetworks G_i's that each contains all the vertices and edges of G that belong to a simple $s_i - t_i$ path for commodity i. WLOG, we assume these G_i are acyclic.

Definition 14. A multi-commodity network G is *block-matching* if for every i, G_i is series-parallel, and for every i, j, G_i and G_j are block-matching, respectively.

The next theorem states that for multi-commodity networks, diversity is always helpful on any block-matching network with average-respecting demand.

Theorem 15. *For any k-commodity block-matching network with average-respecting demand, $C^{ht}(g) \leq C^{hm}(f)$.*

Name	Objective	Results
Social Welfare	$SW = \min_f \left(\sum_e f_e l_e(f_e) \right)$	PoA = 1 (optimal by definition)
Selfish	$NE_s = \min_f \left(\sum_e \int_0^{f_e} l_e(t)dt \right)$	TODO
Altruistic	$NE_\beta = \min_f \left(\sum_e \int_0^{f_e} l_e(t) + \beta t l'e(t)dt \right)$	TODO
Risk-averse	$NE_\gamma = \min_f \left(\sum_e \int_0^{f_e} l_e(t) + \gamma v(t)dt \right)$	TODO
Diverse Interests	$NE_\omega = \min_f \left(\sum_e \int_0^{f_e} l_e(t) + \omega \cdot \sigma_e dt \right)$ where l and σ are the costs of two general criterion	TODO

Proof Sketch. The key idea is that for any block representation of a subnetwork $G_i = s_i B_1 v_1 \ldots v_{b_i - 1} B_{b_i} t_i$ for some commodity i, because G is block-matching, for any block B_j connecting v_{j-1} and v_j , any other commodity j either contains block B_j as a block in its block representation or contains none of the edges of B_j. This implies that under any routing of the demand, either all of j's demand goes through B_j or none of it does. So the total traffic routed by f and g are the same from v_{j-1} to v_j. So if restricted to the block, the cost of the heterogeneous equilibrium is less than or equal to that of the homogeneous equilibrium; then the theorem is a result of summing over every block of all commodities. □

Theorem 16. *For any k-commodity network, if diversity helps for every instance on G with average-respecting demand, we have* $C^{ht}(g) \leq C^{hm}(f)$*, then G is a block-matching network.*

Proof Sketch. The proof is by contradiction. First, by our Theorem 13 for single-commodity network, each subnetwork G_i in our multi-commodity network must be a series-parallel network, otherwise we can use the same counterexample as for Theorem 13. Then since we can prove that any two commodities i and j, any block of G_i and any block of G_j either have the same terminals and direction or their terminals has no intersection, we know that G is block-matching. The detailed proof assumes the conditions does not hold and constructs demand and edge functions where diversity hurts to contradict the assumption. See [3]. □

This theorem essentially states that for multi-commodity networks, diversity is always helpful only on any block-matching network with average-respecting demand. Together with Theorem 15, we know that the block-matching structure is a sufficient and necessary condition for diversity to always be helpful in a multi-commodity network.

4 DISCUSSION

4.1 Extensions

Clearly the models we present are a miniscule subset of all potential models for human behavior; furthermore, they are coarse-grained and oversimplistic in comparison with the complexity of the human brain. As understanding of the neurophysiological aspects of human behavior improves, we hope to see a matching evolution in the precision and accuracy of these models for traffic routing as well.

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