The Price of Human Nature

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1 INTRODUCTION

Finding the best strategy for network and traffic routing has historically been a problem of great importance, combining the theoretical aspects of both game theory and computer science. Seminal work by Roughgarden and Tardos framed the routing problem as a non-cooperative game, in which players' selfish routing decisions (decisions made by users bearing only their own interests or latencies in mind) increase the social welfare cost, i.e., the overall latency of all users in the network. Our paper begins by presenting a brief overview of the traffic routing problem, the selfish model, and the limits on the optimality of a routing solutions in the selfish model (termed as the "price of anarchy").

Since the advent of the selfish routing model, more complex (and perhaps more realistic) models have emerged, many of which emphasize the need to take into account the complexity of human behaviors. This paper focuses on three of these recent alternative models, namely models that account for altruistic, risk-averse, and diverse behaviors. We present and clarify the findings of these papers in the context of the original selfish routing paper, and demonstrate how these papers' results can be synthesized into a more general framework addressing optimality of routing with various human behaviors or motivations. In particular, we identify the similarities and differences across these models, their objectives and the corresponding impact on the "price of anarchy".

2 BACKGROUND

This section presents a brief history of traffic routing problem and the selfish routing model, defining the terminology and context in which we describe later results.

2.1 The Traffic Routing Problem

Traffic routing problems naturally arise in communication or transportation networks, where users are trying to minimize the latency that they or their data experiences. Users make these decisions only with their own traffic in mind. But, links in the network often become *congested* if too many users decide to route their data or cars through that link. Consequently, in these networks, the path each user chooses can affect the travel times of other users. Here, we describe Roughgarden and Tardos' formalization of the problem of minimizing latency in terms of multicommodity flow networks [7, 8], and use Pigou's example network in Figure 1 as a running example. In this graph, one path has a constant latency while the other path's latency increases with the number of users using it. We will later use this to reason about how much latency increases from optimum when users make selfish routing decisions. We will also briefly reference Braess's graph (Figure **??**) in which the outer edges have similar latency functions to the Pigou network while the middle edge has a latency of 0.

The input (G, r, ℓ) to a traffic routing problem consists of:

- A network $G = (V, E)$ of $|V|$ destinations (e.g., locations or servers) and $|E|$ links
- A set of k source-destination pairs $S = \{(s_1, t_1), \cdots (s_k, t_k)\}$ representing traffic demands
- A rate r_i of traffic for each $(s_i, t_i) \in S$ representing the traffic demand from s_i to t_i
- A latency function ℓ that assigns a per-edge function ℓ_e to each edge e describing how adding traffic (i.e., congestion) to e affects the time taken to travel across e. We can also think of ℓ as assigning per-path

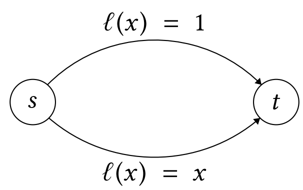


Fig. 1. Pigou's example traffic routing problem, with a demand of $r_{(s,t)} = 1$

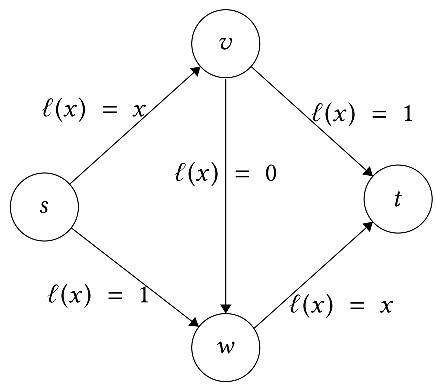


Fig. 2. Braess's graph for the traffic routing problem where the addition of the center edge inadvertently increases the overall latency.

latencies: for any path p in the graph that carries flow f

$$\ell_p(f) = \sum_{e \in P} \ell_e(f_e)$$

We assume that ℓ is continuous, nonnegative, and nondecreasing.

In Figure 1, we see a single source-destination input network with an example (linear) cost function with $r_{(s,t)} = 1$.

Solutions correspond to flow assignments to the set of simple paths P_i between s_i and t_i for all i. Note that our solutions assume *nonatomic* entities: the flows we find may not be integral. Intuitively, this means that the demand from one s_i to t_i is generated by an infinite number of entities in the network, which allows us to reason about continuous, rather than discrete, functions. We can describe a flow assignment f via use its path decomposition (f is made up of flows f_p, the flow on a single path $p \in P_i$, where f_p flow is added to all edges in p); alternatively, we can describe the flow on each edge $f_e = \sum_p \sum_{e \in p} f_p$, (equal to the sum of flow on all paths that use e).

A *feasible* solution given such an input is an assignment of path flows such that the demand from s_i to t_i is met:

$$\forall 1 \leq i \leq k, \sum_{p \in P_i} f_p = r_i$$

An *optimal* (feasible) solution given such an input is the feasible flow assignment f that minimizes the **social welfare latency cost** $C(f)$, where

$$C(f) = \sum_i \sum_{p \in P_i} \ell_p(f) f_p = \sum_{e \in E} f_e \ell_e(f_e)$$

Intuitively, we are calculating the latency of each path of a given flow assignment, weighing each path's latency proportional to the amount of flow through that path. More concretely, if we were to let flow represent the routes chosen by (infinitely many) users, $C(f)$ calculates the average latency over all users. Thus, when minimizing $C(f)$, some users may incur more latency so that other users can go faster: the optimal flow is the *socially optimal* solution. Note that there exists an optimal flow f^* minimizing $C(f)$ because we assume ℓ is continuous and the set of feasible flows is compact.

In our running example (Figure 1), a feasible flow is any flow that sends one unit from s to t (divided in any fashion between the top and bottom edges). The optimal flow is the flow that sends half the traffic through the lower edge and half through the upper edge: the users on the lower edge only experience a latency of $1/2$, while the users on the upper edge experience a latency of 1, making the social welfare cost cost $3/4$.

2.2 Coordination Models and the Price of Anarchy

Before we can create algorithms to solve the traffic routing problem, we must first assume a *coordination model* for our traffic network. There are two clear extremes: (1) centralized control, in which some entity (e.g., an air traffic controller) knows all traffic demands and latencies and routes accordingly, and (2) decentralization, i.e., a complete *lack* of coordination between entities in the network. In a centralized setting, there is a clear optimal solution, as shown in the previous section. However, in a decentralized and uncoordinated model, the lack of coordination and the exercise of free will in accordance to individual motives can result in inefficiencies.

The Price of Anarchy (PoA) allows us to measure the inefficiencies of a decentralized model, and was first introduced by Koutsoupias and Papadimitriou in 1999 [5]. We treat the decentralized model as a game in which each individual optimizes for her own **individual cost function** c, allowing us to compute the achieved flow at the resulting Nash equilibrium (proven to exist if the cost functions are continuous) [1, 4, 10]. The price of anarchy $\rho(G, r, \ell)$ is defined as the ratio between the social welfare latency cost at the flow achieved at Nash equilibrium, and the optimally minimal social welfare latency cost. (This is similar to how we measured the distance from optimal of an approximation algorithm in a limited computational power model, and of online algorithms in an incomplete information model.)

The set of flows at Nash equilibrium are defined such that for all i source-destination pairs, all the paths from $s_i \rightarrow t_i$ have the minimum-possible cost with respect to an individual's cost function c. In other words, the (nonzero) flow paths at Nash equilibrium have equal path costs, and no user could decrease her cost by choosing a different path:

$$\forall 0 \leq i \leq k,\ \forall p_1, p_2 \in P_i\ s.t.\ f_{p_1} > 0 \text{ and} f_{p_2} > 0,\ c_{p_1}(f) = c_{p_2}(f)$$

Note that this corresponds exactly to the solutions to the following (convex) program solvable in polynomial time:

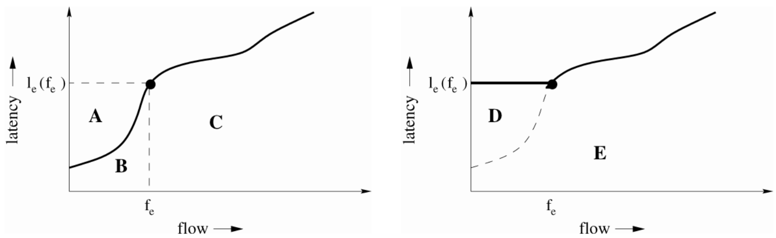
$$NE = \min_f \left(\sum_e \int_0^{f_e} c_e(t)dt \right) \text{ subject to feasibility constraints}$$

whereas the flow optimizing the social welfare latency cost corresponds exactly to the (polynomial-time) solutions to the following (convex) program:

$$SW = \min_f \left(\sum_e f_e \ell_e(f_e) \right) \text{ subject to feasibility constraints}$$

2.3 The Selfish Routing Model

One example of an uncoordinated model is the *selfish routing* model, in which all entities in the network are selfish and choose a route minimizing their individual latency without caring (or knowing) about the effects on



(a) Graph of latency function ℓ_e and its value at flow value f_e

(b) Graph of latency function $\overline{\ell_e}$

Fig. 3. Visualizing latency costs of individual edges

other users [8]. The selfish routing model corresponds to flows at a Nash equilibrium where each user optimizes her individual cost function $c^s(f) = \ell(f)$. Thus, the program optimized at Nash equilibruim is

$$NE^s = \min_f \Big(\sum_e \int_0^{f_e} \ell_e(t)dt \Big) \text{ subject to feasibility constraints}$$

If we revisit our running example in Figure 1, we note that the flow at Nash equilibrium corresponds to a flow that sends the entire unit of traffic through the bottom edge (the 0 flow through the top path has latency 1, and the unit flow through the bottom path will have latency 1). Intuitively, each user routing from s to t will selfishly choose to take the bottom route because she will reason that the bottom route can have latency no worse than the top route. However, when all the users apply this same strategy, the bottom route becomes more congested and leads to a total average latency $C(f) = 1$. Thus, in Figure 1, the price of anarchy ρ is $\frac{1}{3/4} = \frac{4}{3}$. A similar problem plagues Figure **??** too in that the outer paths are optimal and used by selfish users too as long as the middle low latency edge isn't introduced. However, once it is reduced, all users believe they can reduce their latency by using that edge and in essence, everyone's latency increases because they all use a single path through that edge.

We next describe and present the main results regarding the price of anarchy in this (decentralized) selfish routing model, which will act as a basis to which we will compare traffic routing results in more recently formulated models.

2.4 Main Results

Theorem 1. *If f is a flow at Nash equilibrium for a given input set (G, r, ℓ) and f^* is a feasible flow for $(G, 2r, \ell)$, then $C(f) \leq C(f^*)$*

Proof Sketch. This result demonstrates that the latency incurred when users selfishly route r units of flow is at most the optimal (minimum) latency when routing twice as much demand ($2r$).

For any Nash equilibrium solution f for (G, r, ℓ) and any feasible solution f^*, we can draw a figure like Figure **??** for each edge e in G, where A, B, C, D, and E represent the magnitude of the areas of the indicated portions of the graph. $A + B$ represents the latency cost $C(f)$ of the Nash equilibrium solution f for (G, r, ℓ), and $B + C$ represents the cost $C(f^*)$ of the feasible solution f^* for $(G, 2r, \ell)$. D and E together represent the cost of a flow f^* under a modified latency function $\bar{l_e}$ as depicted.

Since $D + E \leq D + B + C$, and $D < A + B$, we know that $D + E - (B + C) \leq D \leq A + B$. This means that $B + C \geq D + E - (A + B)$. On the other hand $D + E \geq 2(A + B)$ (reason below). Therefore, we have that

$C(f^*) = (B + C) \geq 2(A + B) - (A + B) = A + B = C(f)$. This indicates that the latency of any Nash equilibrium flow for (G, r, ℓ) is no bigger than represents the latency of any feasible solution for $(G, 2r, \ell)$.

The main idea of the proof for this theorem lies in that the subset if the area C is at least as big as the area $A + B$. This is always true because the latency $\bar{\ell}_e$ is a nondecreasing function and thus the graph of latency function gives a shape that looks like a generalized trapezoid (in particular, a right generalized trapezoid lying on the horizontal axis). □

Theorem 2. *If the edge latency functions are linear, i.e., $\ell_e = a_e f_e + b_e$ for every edge $e \in E$, then $\rho(G, R, \ell) \leq 4/3$.*

Proof Sketch. The latency of any flow x under these edge latency functions is $C(x) = \sum_e a_e x_e^2 + b_e x_e$.

Let's consider two flows f and f^* such that f is at Nash equilibrium in (G, r, ℓ) and f^* is the flow optimizing the social welfare cost of $G, r, \ell)$. We first consider optimally routing the first $r/2$ demand across all source-destination pairs. It turns out that $f/2$ is optimal for $(G, r/2, \ell)$ when the edge latency functions are linear. This can be derived from the fact that paths with non-zero flow at a Nash equilibrium have the same path latency while paths with non-zero flows at the global optimum have the same marginal cost of increasing the flow. Now, if we look at the cost $C(f/2)$ of routing this in terms of the latency of routing the flow f at Nash equilibrium, we notice that $C(f/2) = \sum_e \frac{1}{4} a_e f_e^2 + \frac{1}{2} b_e f_e \geq \frac{1}{4} C(f)$ from the above cost expression. Thus, in other words, routing the first $r/2$ optimally has a latency that is at least one-fourth of the latency of the Nash equilibrium flow.

This leaves the remaining $r/2$ that needs to be routed optimally to route f^* fully. To reason about this, let's look at a small δr_i increase in flow from s_i to t_i that already carries x units of flow. For a convex latency function, we expect the increase in latency to be at least $\delta r_i \ell'(x)$ where ℓ' if the minimum marginal increase in C. If we consider starting at the optimal flow $f/2$ for the $r/2$ demand and increasing the flow on each path by a small δr_i, the subsequent increase in latency across be all paths can be summed as $\sum_{i=1}^{k} \ell'(f/2) \delta r_i$. But, for linear edge latency functions, the marginal increase in latency on every edge at $f/2$ is exactly the latency of that edge at f. Thus, $\ell_e'(f/2) = \ell_e(f)$. We now know that when we set $\delta = 1$ and increase the rate by $r_i/2$ on every s_i, the overall increase in latency is at least $\frac{1}{2} \sum_{i=1}^{k} \ell(f) r_i = \frac{1}{2} C(f)$. The last part is by definition of $C(f)$.

We have shown that routing the first $r/2$ demand optimally costs at least $C(f)/4$ and the next $r/2$ when augmented, costs at least another $C(f)/2$. In total, the cost $C(f^*)$ associated with the optimal is at least $\frac{3}{4} C(f)$. In other words, the flow at Nash equilibrium has cost utmost $\frac{4}{3} C(f^*)$ where f^* is the flow achieving optimal latency. □

3 ALTERNATIVE MODELS

In this section, we present the models and results of a subset of recent coordination models for the traffic routing problem in the framework described in Section ??. These models propose a more nuanced (and perhaps more accurate) description of human behavior than the selfish model.

3.1 Altruism and Spite

The first alternative model we consider is that proposed by Chen and Kempe in 2008 [2], which assumes that users are "not entirely selfish." Chen and Kempe note that social experiments from both economic and psychology have shown humans do not behave rationally in a selfish manner; instead, our behavior is better modeled as either altruistic or malicious (spiteful). Their model proposes a simple way to capture how people make decisions based upon how much latency a particular decision will cost other users: if someone is spiteful, she will want to increase others' latencies, and if she are altruistic, she will want to decrease their latencies.

3.1.1 Formalization. The formal Chen and Kempe model introduces a per-user *altruism* coefficient β and an individual user cost function of c_p^β for all paths p:

$$c_p^\beta(f) = \sum_{e \in p} \ell_e(f_e) + \beta \sum_{e \in P} f_e \ell_e'(f_e)$$

where $\ell_e(\cdot)$ is the latency function from the selfish routing setting, and $\ell_e'(\cdot)$ is the derivative with respect to f_e.

Note that the first term is exactly the individual user cost from the selfish routing model (and thus the two models are equivalent when $\beta = 0$). The second term corresponds to the derivative of the social welfare latency cost on p and is weighed by β; we use the derivative, rather than the value, of the social welfare cost on p because each user only controls an infinitesimally small amount of the flow: if we were to use the value, a single user's choice would have no effect on the social welfare cost! Instead, a user can account for how she will affect the social welfare cost via the rate at which her choice of path affects other users.

If β is negative, a user is spiteful: we know that adding a little more flow to p will increase the social welfare cost of taking p (the derivative ℓ_e' is positive), and since we negate this value, this lowers the user's perceived cost of taking p. Conversely, if β is positive, a user is altruistic: increasing flow increases the social welfare cost on p and also the user's perceived cost of taking p. We assume that β ranges from -1 (extremely spiteful) to 1 (extremely altruistic), where $\beta = 0$ corresponds to selfishness. All analysis of the model assumes a particular distribution ψ of β for all users.

Note that we can compare this model to the selfish model using the price of anarchy as a measure of inefficiency because the altruistic model still achieves Nash equilibrium for any ψ and cost function c_p^β. (Given any ψ, c_p^β are continuous in the choice of path p and in the distribution of other users' strategies f: Mas-Colell [**?**] showed that any game of infinitely many players with cost functions continuous in the actions of the players and distribution of actions by other players has a Nash equilibrium.)

Nash equilibrium is achieved at the flow solutions to the program

$$NE^\beta = \min_f \sum_e \int_0^{f_e} c_e^\beta(t)dt \text{ subject to feasibility constraints}$$

We next present Chen and Kempe's core results about the price of anarchy of arbitrary networks when ψ is uniform (all users have the same β value), and briefly mention their results of non-uniform ψ in parallel-link networks.

3.1.2 Uniformly Distributed Altruism. We first consider the case where ψ is uniformly distributed, such that β and therefore c_p^β is the same for each user. We additionally assume that users tend to be altruistic, i.e., $\beta > 0$. With these assumptions, we get a nice bound on the price of anarchy:

THEOREM 3. *For any G, demand rates r, and a uniform distribution ψ with $\beta \in (0, 1]$, if ℓ_e is nondecreasing and convex for all e, then the price of anarchy is bounded by*

$$\rho(G, r, \ell, \psi) \leq \frac{1}{\beta}$$

PROOF SKETCH. Consider the two (convex) functions that we minimize for each of the two objectives NE^β and SW. For simplicity, let $B(f)$ be the function minimized in NE^β; the second is simply our social welfare cost $C(f)$.

We can write these and manipulate them into comparable forms as follows:

$$B(f) = \sum_e \int_0^{f_e} \left(c_e^\beta(t)\right)dt = \sum_e \int_0^{f_e} \left(\ell_e(t) + \beta t\ell' e(t)\right)dt \text{ (by definition of } c_e^\beta)$$

$$C(f) = \sum_e f_e \ell_e(f_e) = \sum_e \int_0^{f_e} \left((t\ell_e(t))'\right)dt = \sum_e \int_0^{f_e} \left(\ell_e(t) + tl' e(t)\right)dt$$

It is clear that for any feasible flow f, $B(f) \leq C(f) \leq \frac{B(f)}{\beta}$ because $\beta \in (0,1]$. We now let $\hat{f}$ be the flow at Nash equilibrium and f^* be the flow at optimum social welfare. Because these are the optimal flows for their respective objectives, we know that $C(\hat{f}) \leq \frac{B(\hat{f})}{\beta} \leq \frac{B(f^*)}{\beta} \leq \frac{C(f^*)}{\beta}$, proving that $\rho(G, r, \ell, \psi) \leq \frac{1}{\beta}$ □

3.1.3 Uniformly Distributed Spite. Chen and Kempe then address the problem of spite: how (uniformly) spiteful can users be before ρ becomes infinite? It turns out that this depends on the type of latency function! Our analysis begins by reasoning about the price of anarchy of a given class L of latency functions, $\rho(G, r, L, \psi)$.

We find that the price of anarchy of a class of functions L is lower-bounded by the worst possible price of anarchy (over all $\ell \in L$) achieved in a two-link, two-node network (such as in Figure 1) routing r demand with the following latency functions on its two edges: $\ell_1(x) = \ell(x)$ and $\ell_2(x) = c^\beta r = \ell(r) + \beta r\ell'(r)$.

We refer to this specific two-link problem for a given input (G, r, L, ψ) as T_β, and the maximum price of anarchy of this problem over all $\ell \in L$ as $\rho(T_\beta)$.

Theorem 4. *For any G, demand rates r and uniform distribution ψ of $\beta \in (-1, 1]$,*

$$\rho(G, r, L, \psi) \leq \rho(T_\beta)$$

i.e., the price of anarchy of a class of functions L routing r flow in G is bounded by the price of anarchy routing r flow through the network T_β.

Proof Sketch. We give a brief overview of the proof technique here. Note that by definition, the flow in T_β at Nash equilibrium will route all demand r on the edge with latency function $\ell_1(x) = c(x)$, whereas the socially optimal flow will route some flow x on the edge with latency function $\ell_2 = c^\beta(r)$. This gives us the worst case for $\rho(T_\beta)$ given any $\ell \in L$ as

$$\rho(T_\beta) = \max_{l \in L} \max_{x, r \geq 0} \frac{r\ell(r)}{x\ell(x) + (r - x)(c^\beta(r))}$$

The proof proceeds by considering social welfare cost of the flow f^* optimizing C. By unfolding the definition of $\rho(T_\beta)$ to get a bound for $x\ell_e(x)$ for arbitrary x and r, we can then apply this bound to each edge with $x = f_e^*$ and $r = \hat{f}_e$, where $\hat{f}$ is the optimizing flow at Nash equilibrium.

$$\forall x, r \geq 0,\ x\ell_e(x) \geq \frac{r\ell_e(r)}{\rho(T_\beta)} + (x - r)\ell_e^\beta(r) \implies f_e^* \ell_e(f_e^*) \geq \frac{\hat{f}_e \ell_e(\hat{f}_e)}{\rho(T_\beta)} + (f_e^* - \hat{f}_e)\ell_e^\beta(\hat{f}_e)$$

With some mathematical manipulation, we can derive a comparison of $C(f^*)$ to $B(\hat{f})$ satisfying the above bound:

$$\begin{aligned} C(f^*) = \sum_e f_e^* \ell_e(f_e^*) &\geq \sum_e \frac{\hat{f}_e \ell_e(\hat{f}_e)}{\rho(T_\beta)} + (f_e^* - \hat{f}_e)\ell_e^\beta(\hat{f}_e) \\ &\geq \frac{C(\hat{f})}{\rho(T_\beta)} + \sum_e (f_e^* - \hat{f}_e)\ell_e^\beta(\hat{f}_e) \end{aligned}$$

Note that $\sum_e (f_e^*)\ell_e^\beta(\hat{f}_e) \geq \sum_e \hat{f}_e \ell_e^\beta(\hat{f}_e)$ by definition of $\hat{f}$ being a flow at Nash equilibrium. Thus, we get that

$$C(f^*) \geq \frac{C(\hat{f})}{\rho(T_\beta)} \implies \rho(T_\beta) \geq \frac{C(\hat{f})}{C(f^*)}$$

□

Since we know how to bound $\rho(G, r, L, \psi)$ by the (uniform) value of β, we can now determine at which values of β this lower bound is infinite: how spiteful do users have to be to cause each other infinitely more suffering? The following result shows that if the latency functions are in the class L_d = polynomials of degree $\leq d$, the price of anarchy is bounded when β is at least $\frac{-1}{d}$ (and is infinite when $\beta < \frac{-1}{d}$).

THEOREM 5. *Let L_d be the class of latency functions that are polynomials of degree $\leq d$. For any G, demand rates r, $\ell \in L_d$, and uniform distribution ψ with $\beta \in (\frac{-1}{d}, 1]$,*

$$\rho(G, r, \ell, \psi) \leq \left(\left(\frac{1+\beta d}{1+d}\right)^{1/d}\left(\frac{1+\beta d}{1+d} + 1 + \beta d\right) + 1 + \beta d\right)^{-1}$$

PROOF SKETCH. From Theorem 4, we know that $\rho(G, r, L, \psi)$is bounded above by $\rho(T_\beta)$. Thus, we only need to consider how bad $\rho(T_\beta)$ can get given any $l \in L_d$.

The key observation is that $\exists \lambda \in [0, 1]$ s.t. $c^1(r\lambda) = c^\beta(r)$, where c^1 is the cost function with uniform altruism value $\beta = 1$. This will allow us to reason about the amount of flow $r\lambda$ that the optimum solution will place on the link with latency function $l_1(x) = \ell(x)$, and precisely compute the price of anarchy in terms of β.

Observe that a Nash equilibrium flow in T_β routing r units from the source to the destination will put all flow on the link with cost function $l_1(x) = \ell(x)$. By definition of λ, the solution optimizing social welfare will put $r\lambda$ flow on the first link with latency function $\ell_1(x) = \ell(x)$, and $r(1-\lambda)$ on the second link with latency function $\ell_2(x) = c^\beta(r)$. This gives us a a bound on the price of anarchy:

$$\rho(G, r\lambda, \ell, \psi) \leq \rho(T_\beta) \leq \left(\frac{\lambda\ell(r\lambda)}{\ell(r)} + \left(1-\lambda\right)\left(1 + \frac{\beta r \ell'(r)}{\ell(r)}\right)\right)^{-1}$$

To satisfy our observation, we need to make an appropriate choice of λ. To simplify our calculation of λ, we can, without loss of generality, consider latency functions $\ell(x) = ax^i$ for some $i \leq d$ to represent all functions $\ell \in L_d$. We can then solve $c^1(r\lambda) = c^\beta(r)$ for λ to get $\lambda = \left(\frac{1+\beta i}{1+i}\right)^{\frac{1}{i}}$. With this λ, we get that $\frac{\ell(r\lambda)}{\ell(r)} = \frac{1+\beta i}{1+i}$ and $\frac{\ell'(r)}{\ell(r)} = \frac{i}{r}$. We can plug these values into the bound to get:

$$\rho(G, r, \ell, \psi) = \rho(G, \lambda r, \ell, \beta = 1) \leq \left(\left(\frac{1+\beta i}{1+i}\right)^{1/i}\left(\frac{1+\beta i}{1+i} + 1 + \beta i\right) + 1 + \beta i\right)^{-1}$$

This is increasing in i, giving us the worst-case bound when $i = d$:

$$\left(\left(\frac{1+\beta d}{1+d}\right)^{1/d}\left(\frac{1+\beta d}{1+d} + 1 + \beta d\right) + 1 + \beta d\right)^{-1}$$

□

3.1.4 Arbitrarily Distributed Altruism. Chen and Kempe go on to extend their analysis to when users have an arbitrary distribution ψ of altruism (with no spiteful users) in *parallel link networks*, networks with only two nodes (a source and destination) and parallel edges running between the nodes. We briefly mention their results here, but direct the reader to the paper for a more detailed proof. Their main result mirrors that of uniform altruism:

Theorem 6. *Given any parallel link network G, demand rates r, altruism density function ψ with average altruism $\bar{\beta}$ and non-negative support (no spiteful users), and convex and non-decreasing cost functions ℓ_e, $\rho(G, r, \ell, \psi) \leq \frac{1}{\bar{\beta}}$*

Finally, Chen and Kempe comment on the restriction to $\beta \geq 0$: if even a small fraction of users are spiteful, these users can, in some networks instances, lead to an price of anarchy much greater than 1 regardless of the value of $\bar{\beta}$.

3.2 Risk-aversion

The second model we consider accounts for the tendency of users to pick routes with less variation in latency even if it comes at the cost of some added latency. This increase in latency can be quantified as the *price of risk-aversion*, the worst-case ratio of the latency at a risk-averse Nash equilibrium to that at a risk-neutral Nash equilibrium or one where users are indifferent to variations in the latency itself. The latter is exactly the original problem forumated in Section. 2.3.

3.2.1 Formalization. The formal model introduced in Lianeas et.al [**?**] defines a risk-aversion coefficient γ that quantifies the users' tendency to prefer paths with less variability. A higher γ means that one is more risk-averse. Lianeas et.al define a new individual user cost function, where the cost on each edge includes both the deterministic $\ell_e(x_e)$ and a noise modeled by a random variable $\xi_e(x_e)$. The noise has expectation 0 and variance $v_e(x_e)$ for x_e flow. Furthermore, we assume that it is independent across edges, allowing us to sum the variance and mean of the noise over all edges in a path. To simplify the analysis, the model also defines a bound κ on the variance-to-mean latency ratio: $v_e(x_e) \leq \kappa \ell_e(x_e)$. Thus, the individual cost function for each user on a given path p is of the form

$$c_p^\gamma(f) = \sum_{e \in p} \ell_e(f_e) + \gamma \sum_{e \in p} v_e(f_e)$$

and the resulting Nash equilibrium is achieved at the flow solutions to the program

$$NE^\gamma = \min_f \sum_e \int_0^{f_e} c_e^\gamma(t)dt \text{ subject to feasibility constraints}$$

c_p^γ is assumed to be non-decreasing with increasing inputs. Intuitively, the mean value of $c_p^\gamma(f)$ is identical to the cost given the selfish model's individual user cost formulation, while the second term accounts for variance. Minimizing this implies that we want to minimize the variance depending on the value of the risk coefficient itself.

If we let $C(x)$ be the the maximum cost across some set x of flows at the Nash equilibrium for a given problem instance (restricted to some family of inputs and a fixed κ), the **price of risk-aversion** is now defined as the ratio $C(x)/C(z)$. Here $C(z)$ is the cost associated with some risk-neutral Nash equilibrium z. In the following section, we look at bounding this price of risk-aversion. Bounds on the price of risk-aversion are equivalent to bounding a multiplicative factor of the price of anarchy: the price of risk-aversion tells us how many times worse the price of anarchy can become. We first prove a more basic result from an older paper on this topic [**?**] and then proceed to the main result on the price of risk-aversion for a special family of latency functions that are (λ, μ) smooth. Lianeas et.al also prove additional results for special classes of graph topologies, which we leave for future reading [**?**].

3.2.2 Main Results.

Theorem 7. *If a flow $\hat{f}$ is at a risk-averse Nash equilibrium and f^* is any other flow, then $\hat{f}C(\hat{f}) \leq \hat{f}^* C(f)$.*

Proof. By definition, any flow at Nash equilibrium routes on paths with minimum cost or only sends flow on a given path if its cost is less than the cost of sending the same flow on some other path (with respect to the individual user costs). Thus, any other flow f^* that routes some $\epsilon > 0$ flow differently from $\hat{f}$ will increase the total cost of the flow over all individual users' costs.

Consequently, let flow x be at risk-averse Nash equilibrium for the individual cost function $c_e^\gamma = \ell_e(x_e) + \gamma v_e(x_e)$. Let z be at risk-neutral Nash equilibirium (a feasible flow for the risk-averse mean-variance cost function). Clearly, z is not at Nash equilibrium for c^γ when $\gamma \neq 0$. By the above description and costs written as the sum across edges, we have

$$\sum_{e \in E} x_e(\ell_e(x_e) + \gamma v(x_e)) \leq \sum_{e \in E} z_e(\ell_e(x_e) + \gamma v_e(x_e))$$

□

Definition 8. A latency function $\ell(x)$ is (λ, μ)-smooth if for all $x, y \geq 0$

$$y\ell(x) \leq \lambda y \ell(y) + \mu x \ell(x)$$

This particular smoothness definition defines a class of functions for which we can bound the price of risk-aversion.

THEOREM 9. *The set of instances with latency functions $\ell_{e\, e \in E}$ that are $(1, \mu)$-smooth around any risk-averse Nash equilibrium x_e for all $e \in E$ have price of risk-aversion $\leq \dfrac{(1+\gamma\kappa)}{(1-\mu)}$*

PROOF. This proof involves separating the edges into two sets A and B. A contains edges whose flow x_e at risk-averse Nash equilibrium is at most the flow z_e on the same edge e at risk-neutral Nash equilibrium; B contains the rest of the edges. As before, we let flow x to be at risk-averse Nash equilibrium, and flow z to be at risk-neutral Nash equilibrium.

Let's consider the edges in A. We know that by definition, $\sum_{e \in A} \ell_e(x_e) \leq \sum_{e \in A} \ell_e(z_e)$, implying that

$$\sum_{e \in A} (1+\gamma\kappa) z_e \ell_e(x_e) \leq \sum_{e \in A} (1+\gamma\kappa) z_e \ell_e(z_e)$$

Let's similarly consider the edges in B. By the definition of $(1, \mu)$-smoothness, we have

$$\sum_{e \in B} z_e \ell_e(x_e) \leq \sum_{e \in B} z_e \ell_e(z_e) + \mu x_e \ell_e(x_e)$$

Taken together, with the addition of some terms to encompass all the edges in the graph under each type of term, we can conclude that

$$\sum_{e \in A} (1+\gamma\kappa) z_e \ell_e(x_e) + \sum_{e \in B} z_e \ell_e(x_e) \leq \sum_{e \in E} z_e \ell_e(z_e) + \sum_{e \in E} \mu x_e \ell_e(x_e) + \sum_{e \in E} (1+\gamma\kappa) z_e \ell_e(z_e) = (1+\gamma\kappa) C(z) + \mu C(x)$$

Now, if we are able to show that the total social welfare cost $C(x)$ of the risk-averse Nash equilibrium flow is at most the above expression, we have proven our desired result: we can rearrange the terms to get the price of risk-aversion $C(x)/C(z)$.

If we take the expression from the proof of Theorem 8, use $C(x) = \sum_{e \in E} x_e \ell_e(x_e)$, and consider edges in sets A and B separately, we get

$$C(x) + \sum_{e \in A} x_e \gamma v_e(x_e) + \sum_{e \in B} x_e \gamma v_e(x_e) \leq \sum_{e \in A} z_e \gamma v_e(x_e) + \sum_{e \in B} z_e \gamma v_e(x_e) + \sum_{e \in E} z_e \ell_e(x_e)$$

By the definitions of A and B, we can drop the sum of second and third terms on the LHS and the second term on the RHS because the former is larger than the latter and does not contribute to this inequality. If we separate the last term on the RHS into sets A and B and apply $v_e(x_e) \leq \kappa \ell_e(x_e)$, we effectively are left with

$$C(x) \leq \sum_{e \in A} (1+\gamma\kappa) z_e \ell_e(x_e) + \sum_{e \in B} z_e \ell_e(x_e) \leq (1+\gamma\kappa) C(z) + \mu C(x)$$

which proves exactly what we need. The last inequality is taken from the expression above. □

3.2.3 Similarities and Extensions. Note that the result of Theorem 10 is similar to that of the price of anarchy in the selfish model derived by Roughgarden and Tardos [8] and demonstrated in Section **??**, Theorem **??**. If we assume no variation in prices or in other words, set $\kappa = 0$ and consider linear latency functions which by definition are $(1, 1/4)$-smooth [**?**], we get that the price of risk-aversion is $\leq \frac{1}{1-\mu} = 4/3$.

Furthermore, this price of risk-aversion can be lower bounded for a specific case of a recursive Braess graph (Figure **??**) and the gap between the upper and lower bounds can be more neatly quantified. It can also be exactly computed for a series-parallel recursive graph to be $1 + \gamma\kappa$. The details of these proofs can be found in [**?**].

3.3 Diverse Interests

The third alternative model we consider is a generalization of the altruistic model and the risk-averse model that we have discussed in previous sections. This class is characterized by the diverse selfish behavior of its heterogeneous agents, in which agents pursue potentially different selfish goals (modeled by different individual user costs per agent).

Diverse selfish routing models are useful because they help us understand how we can leverage policies and natural diversity of goals in a network to increase the social welfare and efficiency of the network as a whole. For example, Beckmann et.al [1] showed that tolls can help implement the social optimum at equilibrium when all agents have the same goal that is a linear combination of time and money.

However, there is some ambiguity in measuring the optimality of any outcome of the whole network with diverse selfish behavior, because by definition, the objective function (what we regard as the social welfare cost) is different than in the single criterion in the case of selfish routing with agent homogeneity, where social welfare cost is equivalent to latency. Instead, there are multiple reasonable ways to characterize the social welfare of a diverse routing network. We will discuss the model adopted by Cole, Lianeas and Nikolova and their newly published results in 2018 [3].

3.3.1 Model. We begin with same routing network with multiple source-destination pairs, using all our previous notations. However, there are now two criteria that the players consider in their inidividual user cost function.

Each agent wants to minimize their own cost c^{ω}, which is a sum of two terms associated with two criteria. Let ℓ_P denote the cost of the first criterion (e.g., the latency) over some path $P = (s_i, t_i)$, and σ_P be the cost of the second criterion, referred to as the *deviation function*. Then, given a routing f of the network, the cost of that path is given by $c_p^{\omega} = \ell_P + \omega \cdot \sigma_P = \sum_{e \in P} \ell_e(f_e) + \omega \sum_{e \in P} \sigma_e(f_e)$, where ω is the *diversity parameter*.

Note that the first criterion function (e.g., the latency function) has all the properties as we assumed in previous sections, while the deviation function $\sigma_e(x)$ is assumed to be continuous but not necessarily non-decreasing. However, the function $\ell_e + \omega \cdot \sigma_e$ must be non-decreasing. These assumptions are consistent with our previous risk-averse model in Section 3.2, because if σ_e models the variance, then σ_e could be decreasing in the flow.

Cole et. al. measures the effect of diversity on the resulting flow of a homogeneous agent population of the same size. The homogeneous agent population has the single diversity parameter $\bar{r} = \int rf(r)dr$.

For a discrete distribution of n discrete values $r_1^k, \ldots, r_n^k$, the demand d_k is a vector $d_k = (d_1^k, \ldots, d_n^k)$ where each d_i^k denotes the total demand of commodity k with diversity parameter r_i^k. d^k denotes commodity k's total demand $d^k = \sum_{i=1}^n d_i^k$. For a heterogeneous equilibrium flow vector g, the *heterogeneous total cost* of commodity k is denoted by $C^{k,ht}(g) = \sum_{j=1,\ldots,n} d_j^k c^{k,r_j^k}(g)$, where $c^{k,r_j^k}(g)$ denotes the common cost at equilibrium g for players of diversity parameter r_j^k in commodity k. The heterogeneous total cost of g is then $C^{ht}(g) = \sum_{k \in K} C^{k,ht}(g)$. XXX Vibhaa: It might be helpful to give some intuition on how to think of these r_i^k and d_i^k, like its not clear to me if the n are the different agents in the graph and if so why would there be a different diversity parameter for every agent for every commodity. Even if that is the case, why does the demand vector have different components corresponding to the n different diversity parameters? I'm pretty sure I'm misreading this, but I'm not sure how to interpret these variables is the high level

For the corresponding homogeneous equilibrium flow f, i.e. the instance with diversity parameter $\bar{r}^k$, where $\bar{r}^k$ denotes the average diversity parameter for commodity k, players of commodity k share the same cost $c^{\bar{r}^k}(f)$. In this case, the homogeneous total cost of commodity k under f is $C^{k,hm}(f) = d_k c^{\bar{r}^k}(f)$, and the homogeneous total cost of f is $C^{hm}(f) = \sum_{k \in K} C^{k,hm}(f)$.

3.3.2 Results. Let g denote an equilibrium flow for the heterogeneous agent population and f an equilibrium flow for the corresponding homogeneous agent population. Let $C^{ht}(g)$ denote the cost of flow g and $C^{hm}(f)$ the cost of flow f.

A *multi-commodity network* is consistent with all our previous models. We also introduce the definition of a *single-commodity network* as a network whose edges all belong to some single source-destination path as only these edges are going to be used by the equilibria and thus all other edges can be discarded. We present the following main results.

Definition 10. A directed $s - t$ network G is *series-parallel* if it consists of a single edge (s, t), or it is formed by the series or parallel composition of two series-parallel networks with terminals (s_1, t_1) and (s_2, t_2), respectively.

The theorem below states that for single-commodity networks, diversity is always helpful in a single-commodity series parallel network.

Theorem 11. *For any $s - t$ series-parallel network G with a single commodity, we have $C^{ht}(g) \leq C^{hm}(f)$.*

Proof Sketch. The key observation is that since f and g route the same amount of flow from the unique source to the unique sink, there must be a path P where f sends no less flow along than g does. XXX Vibhaa: This first made sense, but then I was wondering why the opposite isn't true, like why isn't there a path P' where g sends no less flow than f. Can't the opposite theorem also be proved? Since the network is series-parallel, for every edge e in P, $f_e \geq g_e$. This is true because a path in a series-parallel network can be broken up into some series-parallel parts in series and some series-parallel parts in parallel, which recursively breaks down to a simple series of edge(s). Hence for any $r \in [0, r_{\max}]$, we have $c^r_P(f) \geq c^r_P(g)$. This simultaneously also means that g could route more flow on P but it doesn't, implying that there is no incentive for g to switch the flow it sends on other paths to path P under the same diversity parameter. Therefore XXX Vibhaa: Does this denote the Nash equlibrium cost, if so make it clear? $c^r(g) \leq \sum_{e \in P} \ell_e(g_e) + r \sum_{e \in P} \sigma_e(g_e)$. Then $C^{ht}(g) \leq \sum_{i=1}^{k} d_i (\sum_{e \in P} \ell_e(g_e) + r_i \sum_{e \in P} \sigma_e(g_e)) = \ell_p(g) + \bar{r}\sigma_P(g)$ which is exactly the cost of the homogeneous equilibrium flow f XXX Vibhaa: not sure how the last part came come by, maybe I'm missing something trivial. □

Theorem 12. *For any $s - t$ non-series-parallel network G with a single commodity, there exists cost functions C for which $C^{ht}(g) > C^{hm}(f)$.*

Proof Sketch. If G is not series-parallel then the Braess graph (Figure **??**) can be embedded in it [9] XXX Vibhaa: How does embedding a Braess graph help with this proof, might help to give some intuition, and there are edge functions such that heterogeneous equilibrium flow has a larger cost than homogeneous equilibrium flow. Detailed example can be found in [3]. □

This theorem shows that diversity is always helpful in for single-commodity networks if the network is series-parallel. Together with Theorem 12, we know that the series-parallel structure is a sufficient and necessary condition for diversity to always be helpful.

Now we discuss our main results for multiple commodity network. We use *average-respecting demand* to refer to the property that for any commodity $i, j : \bar{r}^i = \bar{r}^j$.

A multi-commodity network G can be decomposed in subnetworks G_i's that each contains all the vertices and edges of G that belong to a simple $s_i - t_i$ path for commodity i. WLOG, we assume these G_i are acyclic.

Definition 13. A multi-commodity network G is *block-matching* if for every i, G_i is series-parallel, and for every i, j, G_i and G_j are block-matching, respectively. XXX Vibhaa: Can we use a picture or an example for this? series-parallel is fairly easy to visualize, but I'm not sure how to visualize this?

The next theorem states that for multi-commodity networks, diversity is always helpful on any block-matching network with average-respecting demand.

Theorem 14. *For any k-commodity block-matching network with average-respecting demand, $C^{ht}(g) \leq C^{hm}(f)$.*

Proof Sketch. Let's consider the block representation of a subnetwork $G_i = s_i B_1 v_1 \ldots v_{b_i - 1} B_{b_i} t_i$ for some commodity i. Because G is block-matching, for any block B_j connecting v_{j-1} and v_j , any other commodity j either contains block B_j as a block in its block representation or contains none of the edges of B_j. This implies that under any routing of the demand, either all of j's demand goes through B_j or none of it does. So the total traffic routed by f and g are the same from v_{j-1} to v_j. So if restricted to the block, the cost of the heterogeneous equilibrium is less than or equal to that of the homogeneous equilibrium; then the theorem is a result of summing over every block of all commodities.

XXX Vibhaa: What are the v_j's here and how do they relate to the block or how do we want to think about them (maybe I'm just still struggling to visualize this? Consequently, how do we go from traffic routed is the same on f and g to cost of heterogenous equlibrium being less than or equal to homogenous in a block □

Theorem 15. *For any k-commodity network, if diversity helps for every instance on G with average-respecting demand, we have $C^{ht}(g) \leq C^{hm}(f)$, then G is a block-matching network.*

Proof Sketch. The proof is by contradiction. First, by our Theorem 13 for single-commodity network, each subnetwork G_i in our multi-commodity network must be a series-parallel network, otherwise we can use the same counterexample as for Theorem 13. Then since we can prove XXX Vibhaa: how can we prove this? that any two commodities i and j, any block of G_i and any block of G_j either have the same terminals and direction or their terminals has no intersection, we know that G is block-matching. The detailed proof assumes the conditions does not hold and constructs demand and edge functions where diversity hurts to contradict the assumption. See [3]. □

This theorem shows that diversity is always helpful in multi-commodity networks if the network is block-matching with average-respecting demand. Together with Theorem 15, we know that the block-matching structure is a sufficient and necessary condition for diversity to always be helpful in a multi-commodity network.

4 DISCUSSION

4.1 Extensions

The models we covered are quite limited and coarse-grained compared to the complexity of the human brain and behavior. The diversity model only explored the realm of having at most two criteria in total in the objective functions of the whole diverse network.

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Name	Objective	Results
Social Welfare	$SW = \min_f \left(\sum_e f_e \ell_e(f_e) \right)$	PoA = 1 (optimal by definition)
Selfish	$NE^s = \min_f \left(\sum_e \int_0^{f_e} \ell_e(t)dt \right)$	TODO
Altruistic	$NE^\beta = \min_f \left(\sum_e \int_0^{f_e} \ell_e(t) + \beta t l' e(t)dt \right)$	TODO
Risk-averse	$NE^\gamma = \min_f \left(\sum_e \int_0^{f_e} \ell_e(t) + \gamma v(t)dt \right)$	TODO
Diverse Interests	$NE^\omega = \min_f \left(\sum_e \int_0^{f_e} \ell_e(t) + \omega \cdot \sigma_e dt \right)$ where ℓ and σ are the costs of two general criterion	TODO

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