Advanced Quantum Theory
Homework 3

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1. Derive an approximation for $\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle$ up to $O(t)$ where $\hat{H} = \frac{1}{2}\hat{p}^2 + U(\hat{x})$.

The Taylor expansion of $e^{-\frac{i}{\hbar}\hat{H}t}$ is:

$$e^{-\frac{i}{\hbar}\hat{H}t} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\hat{H}t\right)^n = 1 - \frac{i}{\hbar}\hat{H}t + O(t^2)$$

Hereafter $= \cdots + O(t^2)$ will be written as $\approx \cdots$. Hence $e^{-\frac{i}{\hbar}\hat{H}t}$ is:

$$e^{-\frac{i}{\hbar}\hat{H}t} \approx 1 - \frac{i}{\hbar}\left(\frac{1}{2}\hat{p}^2 t + U(\hat{x})t\right) \approx \left(1 - \frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t\right)\left(1 - \frac{i}{\hbar}U(\hat{x})t\right) \approx e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t}e^{-\frac{i}{\hbar}U(\hat{x})t}$$

And $\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle$ is:

$$\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle \approx \langle\, p\,|e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t}e^{-\frac{i}{\hbar}U(\hat{x})t}|\, x\,\rangle \approx e^{-\frac{i}{\hbar}U(x)t}\langle\, p\,|e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t}|\, x\,\rangle$$

Insert a resolution of the identity in terms of momentum eigenfunctions:

$$\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle \approx e^{-\frac{i}{\hbar}U(x)t}\int \mathrm{d}p' \langle\, p\,|e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t}|\, p'\,\rangle\langle\, p'\,|\, x\,\rangle$$

$$\approx e^{-\frac{i}{\hbar}U(x)t}\int \mathrm{d}p' e^{-\frac{i}{\hbar}\frac{1}{2}{p'}^2 t}\langle\, p\,|\, p'\,\rangle\langle\, p'\,|\, x\,\rangle$$

We have that $\langle\, p\,|\, p'\,\rangle = \delta(p' - p)$ and $\langle\, p'\,|\, x\,\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}p'x}$, so:

$$\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle \approx e^{-\frac{i}{\hbar}U(x)t}\int \mathrm{d}p' e^{-\frac{i}{\hbar}\frac{1}{2}{p'}^2 t}\delta(p' - p)\frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}p'x}$$

$$\approx e^{-\frac{i}{\hbar}U(x)t}\frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}\frac{1}{2}p^2 t}e^{-\frac{i}{\hbar}px}$$

$$\approx \frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{i}{\hbar}\left(\frac{1}{2}p^2 t + U(x)t + px\right)\right)$$

This is the desired approximation for $\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle$.