Advanced Quantum Theory

Homework 3

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December 4, 2023

1. Derive an approximation for $\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle$ up to $O(t)$ where $\hat{H} = \frac{1}{2}\hat{p}^2 + U(\hat{x})$.

   The Taylor expansion of $e^{-\frac{i}{\hbar}\hat{H}t}$ is:

$$e^{-\frac{i}{\hbar}\hat{H}t} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\hat{H}t\right)^n = 1 - \frac{i}{\hbar}\hat{H}t + O(t^2)$$

   Hereafter $= \cdots + O(t^2)$ will be written as $\approx \cdots$. Hence $e^{-\frac{i}{\hbar}\hat{H}t}$ is:

$$e^{-\frac{i}{\hbar}\hat{H}t} \approx 1 - \frac{i}{\hbar}\left(\frac{1}{2}\hat{p}^2 t + U(\hat{x})t\right) \approx \left(1 - \frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t\right)\left(1 - \frac{i}{\hbar}U(\hat{x})t\right) \approx e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t} e^{-\frac{i}{\hbar}U(\hat{x})t}$$

   And $\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle$ is:

$$\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle \approx \langle\, p\,|e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t}e^{-\frac{i}{\hbar}U(\hat{x})t}|\, x\,\rangle \approx e^{-\frac{i}{\hbar}U(x)t}\langle\, p\,|e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t}|\, x\,\rangle$$

   Insert a resolution of the identity in terms of momentum eigenfunctions:

$$\begin{aligned}
\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle &\approx e^{-\frac{i}{\hbar}U(x)t}\int \mathrm{d}p' \langle\, p\,|e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t}|\, p'\,\rangle\langle\, p'\,|\, x\,\rangle \\
&\approx e^{-\frac{i}{\hbar}U(x)t}\int \mathrm{d}p' e^{-\frac{i}{\hbar}\frac{1}{2}p'^2 t}\langle\, p\,|\, p'\,\rangle\langle\, p'\,|\, x\,\rangle
\end{aligned}$$

   We have that $\langle\, p\,|\, p'\,\rangle = \delta(p'-p)$ and $\langle\, p'\,|\, x\,\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}p'x}$, so:

$$\begin{aligned}
\langle\, p\,|e^{-\frac{i}{\hbar}\hat{H}t}|\, x\,\rangle &\approx e^{-\frac{i}{\hbar}U(x)t}\int \mathrm{d}p' e^{-\frac{i}{\hbar}\frac{1}{2}p'^2 t}\delta(p'-p)\frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}p'x} \\
&\approx e^{-\frac{i}{\hbar}U(x)t}\frac{1}{\sqrt{2\pi\hbar}}e^{-\frac{i}{\hbar}\frac{1}{2}p^2 t}e^{-\frac{i}{\hbar}px} \\
&\approx \frac{1}{\sqrt{2\pi\hbar}}\exp\left(-\frac{i}{\hbar}\left(\frac{1}{2}p^2 t + U(x)t + px\right)\right)
\end{aligned}$$

2. (a) Determine the Feynman diagrams to evaluate $I = \left\langle x_k^2 e^{-\epsilon\sum_{k'} x_{k'}^6}\right\rangle$ up to $O(\epsilon)$, their multiplicities, and their contributions to I.

   The Taylor expansion of $e^{-\epsilon\sum_{k'} x_{k'}^6}$ is:

$$e^{-\epsilon\sum_{k'} x_{k'}^6} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-\epsilon\sum_{k'} x_{k'}^6\right)^n = 1 - \epsilon\sum_{k'} x_{k'}^6 + O(\epsilon^2)$$

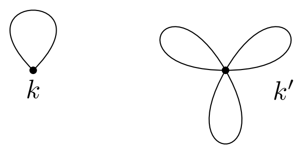

Figure 1: Feynman diagram for $\left\langle x_k^2 x_{k'}^6 \right\rangle$ with contribution $15\left(A_{kk}^{-1}\right)\left(A_{k'k'}^{-1}\right)^3$.

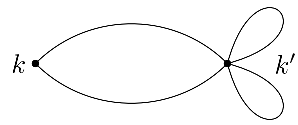

Figure 2: Feynman diagram for $\left\langle x_k^2 x_{k'}^6 \right\rangle$ with contribution $90\left(A_{kk'}^{-1}\right)^2\left(A_{k'k'}^{-1}\right)^2$.

Hereafter $= \cdots + O(\epsilon^2)$ will be written as $\approx \cdots$. Hence I is:

$$I \approx \left\langle x_k^2 \left(1 - \epsilon \sum_{k'} x_{k'}^6\right)\right\rangle \approx \left\langle x_k^2 \right\rangle - \epsilon \left\langle x_k^2 \sum_{k'} x_{k'}^6 \right\rangle \approx \left\langle x_k^2 \right\rangle - \epsilon \sum_{k'} \left\langle x_k^2 x_{k'}^6 \right\rangle$$

We have that, for an average $\left\langle x_k^p x_{k'}^{p'} \right\rangle$, a Feynman diagram with m connections between k and k' has multiplicity:

$$\binom{p}{m}\binom{p'}{m} m!(p-m-1)!!(p'-m-1)!! \tag{1}$$

There are two Feynman diagrams for $\left\langle x_k^2 x_{k'}^6 \right\rangle$ (figures 1 and 2). By equation 1, the multiplicities of the two diagrams are respectively:

$$\binom{2}{0}\binom{6}{0} 0!(2-0-1)!!(6-0-1)!! = 15$$

$$\binom{2}{2}\binom{6}{2} 2!(2-2-1)!!(6-2-1)!! = 90$$

The sum of the multiplicities is $7!! = 105$, as expected. Hence I is:

$$I \approx A_{kk}^{-1} - \epsilon \left(15\left(A_{kk}^{-1}\right)\left(A_{k'k'}^{-1}\right)^3 + 90\left(A_{kk'}^{-1}\right)^2\left(A_{k'k'}^{-1}\right)^2\right)$$