Advanced Quantum Theory
Homework 2

Tim Lawson

November 23, 2023

2.3 Propagator for a free particle

Evaluate the path integral for the propagator of a particle moving freely in one dimension without potential, i.e., a particle with the Lagrangian $L = \frac{1}{2}m\dot{x}^2$.

The action is:

$$S[x] = \int_0^t \frac{1}{2}m\dot{x}(t')^2 \mathrm{d}t' \tag{2.1}$$

Split $x(t')$ into the classical solution $x_{\mathrm{cl}}(t')$ and deviation $\delta x(t')$:

$$x(t') = x_{\mathrm{cl}}(t') + \delta x(t'),\ x(0) = x_0,\ x(t) = x_f,\ \delta x(0) = \delta x(t) = 0 \tag{2.2}$$

Substitute equation 2.2 into equation 2.1. The Taylor expansion is around a stationary point of the action, so the linear term vanishes:

$$\begin{aligned} S[x] &= \int_0^t \frac{1}{2}m\dot{x_{\mathrm{cl}}}(t')^2\mathrm{d}t' + \int_0^t m\dot{x_{\mathrm{cl}}}(t')\dot{\delta x}(t')\mathrm{d}t' + \int_0^t \frac{1}{2}m\dot{\delta x}(t')^2\mathrm{d}t' \\ &= \int_0^t \frac{1}{2}m\dot{x_{\mathrm{cl}}}(t')^2\mathrm{d}t' + \int_0^t \frac{1}{2}m\dot{\delta x}(t')^2\mathrm{d}t' \\ &= S[x_{\mathrm{cl}}] + S[\delta x] \end{aligned} \tag{2.3}$$

Substitute equation 2.3 into the path integral for the propagator and replace the functional integral over x by a functional integral over δx:

$$\begin{aligned} K(x_f, x_0, t) &= \int D[x]\, e^{\frac{i}{\hbar}S[x]} \\ &= \int D[x]\, e^{\frac{i}{\hbar}(S[x_{\mathrm{cl}}]+S[\delta x])} \\ &= e^{\frac{i}{\hbar}S[x_{\mathrm{cl}}]} \int D[\delta x]\, e^{\frac{i}{\hbar}S[\delta x]} \end{aligned} \tag{2.4}$$

First, evaluate the action for the deviation:

$$S[\delta x] = \int_0^t \frac{1}{2}m\dot{\delta x}(t')^2\mathrm{d}t'$$

Discretise the integral and define $\tau = \frac{t}{N}$, $\delta x_i = \delta x(t'_i) = \delta x(i\tau)$:

$$S[\delta x] = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{1}{2}m\left(\frac{\delta x_{i+1} - \delta x_i}{\tau}\right)^2 \tau = \lim_{N\to\infty} \frac{m}{2\tau} \sum_{i=0}^{N-1} (\delta x_{i+1} - \delta x_i)^2$$

The boundary conditions $\delta x(0) = \delta x(t) = 0$ imply $\delta x_0 = \delta x_N = 0$. Rearrange the sum:

$$\begin{aligned}
\sum_{i=0}^{N-1} (\delta x_{i+1} - \delta x_i)^2 &= \sum_{i=0}^{N-1} \delta x_{i+1}^2 - 2\sum_{i=0}^{N-1} \delta x_{i+1}\delta x_i + \sum_{i=0}^{N-1} \delta x_i^2 \\
&= \sum_{i=0}^{N-2} \delta x_{i+1}^2 - 2\sum_{i=1}^{N-2} \delta x_{i+1}\delta x_i + \sum_{i=0}^{N-2} \delta x_i^2 \\
&= 2\sum_{i=1}^{N-1} \delta x_i^2 - 2\sum_{i=1}^{N-2} \delta x_{i+1}\delta x_i
\end{aligned}$$

In terms of the vector $\vec{\delta x} = [\delta x_1 \ \delta x_2 \ \ldots \ \delta x_{N-1}]^\top$ and Kronecker delta δ_{ij}:

$$\sum_{i=0}^{N-1} (\delta x_{i+1} - \delta x_i)^2 = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \left(2\delta_{ij} - (\delta_{i+1,j} + \delta_{i,j+1})\right)(\vec{\delta x})_i(\vec{\delta x})_j$$

$$S[\delta x] = \lim_{N\to\infty} \frac{mN}{2t}\vec{\delta x}\cdot B_{N-1}\vec{\delta x}\,, \quad B_{N-1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

The functional integral in equation 2.4 becomes:

$$\begin{aligned}
\int D[\delta x]\, e^{\frac{i}{\hbar}S[\delta x]} = \lim_{N\to\infty}\left(\frac{mN}{2\pi i\hbar t}\right)^{N/2} &\int \mathrm{d}\delta x_1 \int \mathrm{d}\delta x_2 \cdots \\
&\cdots \int \mathrm{d}\delta x_{N-1} \exp\left(i\frac{mN}{2\hbar t}\vec{\delta x}\cdot B_{N-1}\vec{\delta x}\right)
\end{aligned} \tag{2.5}$$

We have that:

$$\int \mathrm{d}\delta x_1 \int \mathrm{d}\delta x_2 \cdots \int \mathrm{d}\delta x_{\nu-1} \exp\left(i\vec{\delta x}\cdot A\vec{\delta x}\right) = \left(\frac{(i\pi)^\nu}{\det A}\right)^{1/2} \tag{2.6}$$

With $\nu = N-1$, $A = \frac{mN}{2\hbar t}B_{N-1}$:

$$\int D[\delta x]\, e^{\frac{i}{\hbar}S[\delta x]} = \lim_{N\to\infty}\left(\frac{mN}{2\pi i\hbar t}\right)^{N/2}\left(\frac{(i\pi)^{N-1}}{\det\left(\frac{mN}{2\hbar t}B_{N-1}\right)}\right)^{1/2}$$

For a scalar k and $n\times n$ matrix A, $\det kA = k^n \det A$. Hence:

$$\begin{aligned}
\int D[\delta x]\, e^{\frac{i}{\hbar}S[\delta x]} &= \lim_{N\to\infty}\left(\frac{mN}{2\pi i\hbar t}\right)^{N/2}\left(\frac{(i\pi)^{N-1}}{\left(\frac{mN}{2\hbar t}\right)^{N-1}\det B_{N-1}}\right)^{1/2} \\
&= \lim_{N\to\infty}\left(\frac{mN}{2\pi i\hbar t}\right)^{N/2}\left(\frac{2\pi i\hbar t}{mN}\right)^{(N-1)/2} \\
&= \sqrt{\frac{m}{2\pi i\hbar t}}
\end{aligned} \tag{2.7}$$

Next, evaluate the action for the classical solution. The classical solution $x_{\text{cl}}(t')$ is the solution of Lagrange's equation:

$$\frac{\partial L}{\partial x} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} \implies \ddot{x_{\text{cl}}}(t') = 0,\ \dot{x_{\text{cl}}}(t') = \frac{x_f - x_0}{t}$$

The action for the classical solution is:

$$S\left[x_{\text{cl}}\right] = \int_0^t \frac{1}{2} m \dot{x_{\text{cl}}}(t')^2 \mathrm{d}t' = \frac{1}{2} m \frac{(x_f - x_0)^2}{t} \tag{2.8}$$

Substitute equations 2.7 and 2.8 into equation 2.4:

$$K(x_f, x_0, t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(\frac{i}{\hbar}\frac{m}{2t}(x_f - x_0)^2\right) \tag{2.9}$$