Advanced Quantum Theory
Homework 1

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2.1 Path integral in phase space

Show that the propagator of a quantum system can be written as:

$$K(\vec{r}_f, \vec{r}_0, t) = \int D\left[\vec{r}\right] D\left[\vec{p}\right] \exp\left(\frac{i}{\hbar}\int_0^t \left(\vec{p}(t') \cdot \dot{\vec{r}}(t') - H(\vec{r}(t'), \vec{p}(t'))\right) \mathrm{d}t'\right) \tag{2.1.1}$$

We have that:

$$K(\vec{r}_f, \vec{r}_0, t) = \int \mathrm{d}^n r_1 \ldots \int \mathrm{d}^n r_{N-1} \prod_{j=0}^{N-1} \langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{H}\tau} |\, \vec{r}_j\,\rangle \tag{2.1.2}$$

Assume that the Hamiltonian has the form $\hat{H} = \hat{T}(\vec{p}) + \hat{U}(\vec{r})$. Replace the exponential by the first-order Taylor polynomial and vice versa:

$$\begin{aligned}
\langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{H}\tau} |\, \vec{r}_j\,\rangle &\approx \langle\, \vec{r}_{j+1}\,| \left(1 - \frac{i}{\hbar}\hat{H}\tau\right) |\, \vec{r}_j\,\rangle \\
&\approx \langle\, \vec{r}_{j+1}\,| \left(1 - \frac{i}{\hbar}\hat{T}\tau\right)\left(1 - \frac{i}{\hbar}\hat{U}\tau\right) |\, \vec{r}_j\,\rangle \\
&\approx \langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{T}\tau} e^{-\frac{i}{\hbar}\hat{U}\tau} |\, \vec{r}_j\,\rangle
\end{aligned}$$

Apply $\hat{U}$ to the position eigenstate $|\, \vec{r}_j\,\rangle$:

$$\langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{H}\tau} |\, \vec{r}_j\,\rangle \approx \langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{T}\tau} |\, \vec{r}_j\,\rangle e^{-\frac{i}{\hbar}U(\vec{r}_j)\tau}$$

Insert a resolution of the identity in terms of momentum eigenfunctions:

$$\begin{aligned}
\langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{H}\tau} |\, \vec{r}_j\,\rangle &= \langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{T}\tau} \int \mathrm{d}^n p_j |\, \vec{p}_j\,\rangle\langle\, \vec{p}_j\,|\, \vec{r}_j\,\rangle e^{-\frac{i}{\hbar}U(\vec{r}_j)\tau} \\
&= \int \mathrm{d}^n p_j e^{-\frac{i}{\hbar}U(\vec{r}_j)\tau} \langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{T}\tau} |\, \vec{p}_j\,\rangle\langle\, \vec{p}_j\,|\, \vec{r}_j\,\rangle
\end{aligned}$$

Apply $\hat{T}$ to the momentum eigenstate $|\, \vec{p}_j\,\rangle$ and substitute:

$$\begin{aligned}
\langle\, \vec{r}_{j+1}\,| e^{-\frac{i}{\hbar}\hat{H}\tau} |\, \vec{r}_j\,\rangle &= \int \mathrm{d}^n p_j \exp\left(-\frac{i}{\hbar}\left(T(\vec{p}_j) + U(\vec{r}_j)\right)\tau\right) \langle\, \vec{r}_{j+1}\,|\, \vec{p}_j\,\rangle\langle\, \vec{p}_j\,|\, \vec{r}_j\,\rangle \\
&= \int \mathrm{d}^n p_j \exp\left(-\frac{i}{\hbar}H(\vec{r}_j, \vec{p}_j)\tau\right) \langle\, \vec{r}_{j+1}\,|\, \vec{p}_j\,\rangle\langle\, \vec{p}_j\,|\, \vec{r}_j\,\rangle \\
&= \int \mathrm{d}^n p_j \exp\left(-\frac{i}{\hbar}H(\vec{r}_j, \vec{p}_j)\tau\right) \frac{1}{(2\pi\hbar)^{n/2}} e^{\frac{i}{\hbar}\vec{p}_j\cdot\vec{r}_{j+1}} \frac{1}{(2\pi\hbar)^{n/2}} e^{-\frac{i}{\hbar}\vec{p}_j\cdot\vec{r}_j} \\
&= \frac{1}{(2\pi\hbar)^n} \int \mathrm{d}^n p_j \exp\left(\frac{i}{\hbar}\left(\vec{p}_j \cdot \frac{\vec{r}_{j+1} - \vec{r}_j}{\tau} - H(\vec{r}_j, \vec{p}_j)\right)\tau\right)
\end{aligned}$$

The product in equation 2.1.2 is then:

$$\prod_{j=0}^{N-1} \frac{1}{(2\pi\hbar)^n} \int \mathrm{d}^n p_j \exp\left(\frac{i}{\hbar}\left(\vec{p}_j \cdot \frac{\vec{r}_{j+1} - \vec{r}_j}{\tau} - H(\vec{r}_j, \vec{p}_j)\right)\tau\right)$$
$$= \frac{1}{(2\pi\hbar)^{Nn}} \int \mathrm{d}^n p_0 \dots \int \mathrm{d}^n p_{N-1} \exp\left(\frac{i}{\hbar}\sum_{j=0}^{N-1}\left(\vec{p}_j \cdot \frac{\vec{r}_{j+1} - \vec{r}_j}{\tau} - H(\vec{r}_j, \vec{p}_j)\right)\tau\right)$$

Equation 2.1.2 becomes:

$$K(\vec{r}_f, \vec{r}_0, t) = \frac{1}{(2\pi\hbar)^{Nn}} \int \mathrm{d}^n r_1 \dots \int \mathrm{d}^n r_{N-1} \int \mathrm{d}^n p_0 \dots \int \mathrm{d}^n p_{N-1}$$
$$\exp\left(\frac{i}{\hbar}\sum_{j=0}^{N-1}\left(\vec{p}_j \cdot \frac{\vec{r}_{j+1} - \vec{r}_j}{\tau} - H(\vec{r}_j, \vec{p}_j)\right)\tau\right)$$

Replace the sum by an integral in the limit $N \to \infty$, $\tau \to 0$ to obtain:

$$K(\vec{r}_f, \vec{r}_0, t) = \int D[\vec{r}]\, D[\vec{p}] \exp\left(\frac{i}{\hbar}\int_0^t \left(\vec{p}(t') \cdot \dot{\vec{r}}(t') - H(\vec{r}(t'), \vec{p}(t'))\right) \mathrm{d}t'\right)$$

where

$$\int D[\vec{r}]\, D[\vec{p}] \dots = \lim_{N\to\infty} \frac{1}{(2\pi\hbar)^{Nn}} \int \mathrm{d}^n r_1 \dots \int \mathrm{d}^n r_{N-1} \int \mathrm{d}^n p_0 \dots \int \mathrm{d}^n p_{N-1} \dots$$