

# Advanced Quantum Theory

## Homework 2

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### 2.3 Propagator for a free particle

Evaluate the path integral for the propagator of a particle moving freely in one dimension without potential, i.e., a particle with the Lagrangian  $L = \frac{1}{2}m\dot{x}^2$ .

The action is:

$$S[x] = \int_0^t \frac{1}{2}m\dot{x}(t')^2 dt' \quad (1)$$

Split  $x(t')$  into the classical solution  $x_{\text{cl}}(t')$  and deviation  $\delta x(t')$ :

$$x(t') = x_{\text{cl}}(t') + \delta x(t'), \quad x(0) = x_0, \quad x(t) = x_f, \quad \delta x(0) = \delta x(t) = 0 \quad (2)$$

Substitute equation 2 into equation 1. The Taylor expansion is around a stationary point of the action, so the linear term vanishes:

$$\begin{aligned} S[x] &= \int_0^t \frac{1}{2}m\dot{x}_{\text{cl}}(t')^2 dt' + \int_0^t m\dot{x}_{\text{cl}}(t')\dot{\delta x}(t') dt' + \int_0^t \frac{1}{2}m\dot{\delta x}(t')^2 dt' \\ &= \int_0^t \frac{1}{2}m\dot{x}_{\text{cl}}(t')^2 dt' + \int_0^t \frac{1}{2}m\dot{\delta x}(t')^2 dt' \\ &= S[x_{\text{cl}}] + S[\delta x] \end{aligned} \quad (3)$$

Substitute equation 3 into the path integral for the propagator and replace the functional integral over  $x$  by a functional integral over  $\delta x$ :

$$\begin{aligned} K(x_f, x_0, t) &= \int D[x] e^{\frac{i}{\hbar}S[x]} \\ &= \int D[x] e^{\frac{i}{\hbar}(S[x_{\text{cl}}] + S[\delta x])} \\ &= e^{\frac{i}{\hbar}S[x_{\text{cl}}]} \int D[\delta x] e^{\frac{i}{\hbar}S[\delta x]} \end{aligned} \quad (4)$$

First, evaluate the action for the deviation:

$$S[\delta x] = \int_0^t \frac{1}{2}m\dot{\delta x}(t')^2 dt'$$

Discretise the integral and define  $\tau = \frac{t}{N}$ ,  $\delta x_i = \delta x(t'_i) = \delta x(i\tau)$ :

$$S[\delta x] = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \frac{1}{2}m \left( \frac{\delta x_{i+1} - \delta x_i}{\tau} \right)^2 \tau = \lim_{N \rightarrow \infty} \frac{m}{2\tau} \sum_{i=0}^{N-1} (\delta x_{i+1} - \delta x_i)^2$$

The boundary conditions  $\delta x(0) = \delta x(t) = 0$  imply  $\delta x_0 = \delta x_N = 0$ . Rearrange the sum:

$$\begin{aligned} \sum_{i=0}^{N-1} (\delta x_{i+1} - \delta x_i)^2 &= \sum_{i=0}^{N-1} \delta x_{i+1}^2 - 2 \sum_{i=0}^{N-1} \delta x_{i+1} \delta x_i + \sum_{i=0}^{N-1} \delta x_i^2 \\ &= \sum_{i=0}^{N-2} \delta x_{i+1}^2 - 2 \sum_{i=1}^{N-2} \delta x_{i+1} \delta x_i + \sum_{i=0}^{N-2} \delta x_i^2 \\ &= 2 \sum_{i=1}^{N-1} \delta x_i^2 - 2 \sum_{i=1}^{N-2} \delta x_{i+1} \delta x_i \end{aligned}$$

In terms of the vector  $\vec{\delta x} = [\delta x_1 \ \delta x_2 \ \dots \ \delta x_{N-1}]^\top$  and Kronecker delta  $\delta_{ij}$ :

$$\begin{aligned} \sum_{i=0}^{N-1} (\delta x_{i+1} - \delta x_i)^2 &= \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (2\delta_{ij} - (\delta_{i+1,j} + \delta_{i,j+1})) (\vec{\delta x})_i (\vec{\delta x})_j \\ S[\delta x] &= \lim_{N \rightarrow \infty} \frac{mN}{2t} \vec{\delta x} \cdot B_{N-1} \vec{\delta x}, \quad B_{N-1} = \begin{bmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ 0 & -1 & 2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \end{aligned}$$

The functional integral in equation 4 becomes:

$$\begin{aligned} \int D[\delta x] e^{\frac{i}{\hbar} S[\delta x]} &= \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi i \hbar t} \right)^{N/2} \int d\delta x_1 \int d\delta x_2 \dots \\ &\quad \dots \int d\delta x_{N-1} \exp \left( i \frac{mN}{2\hbar t} \vec{\delta x} \cdot B_{N-1} \vec{\delta x} \right) \quad (5) \end{aligned}$$

We have that:

$$\int d\delta x_1 \int d\delta x_2 \dots \int d\delta x_{\nu-1} \exp \left( i \vec{\delta x} \cdot A \vec{\delta x} \right) = \left( \frac{(i\pi)^\nu}{\det A} \right)^{1/2} \quad (6)$$

With  $\nu = N - 1$ ,  $A = \frac{mN}{2\hbar t} B_{N-1}$ :

$$\int D[\delta x] e^{\frac{i}{\hbar} S[\delta x]} = \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi i \hbar t} \right)^{N/2} \left( \frac{(i\pi)^{N-1}}{\det \left( \frac{mN}{2\hbar t} B_{N-1} \right)} \right)^{1/2}$$

For a scalar  $k$  and  $n \times n$  matrix  $A$ ,  $\det kA = k^n \det A$ . Hence:

$$\begin{aligned} \int D[\delta x] e^{\frac{i}{\hbar} S[\delta x]} &= \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi i \hbar t} \right)^{N/2} \left( \frac{(i\pi)^{N-1}}{\left( \frac{mN}{2\hbar t} \right)^{N-1} \det B_{N-1}} \right)^{1/2} \\ &= \lim_{N \rightarrow \infty} \left( \frac{mN}{2\pi i \hbar t} \right)^{N/2} \left( \frac{2\pi i \hbar t}{mN} \right)^{(N-1)/2} \\ &= \sqrt{\frac{m}{2\pi i \hbar t}} \quad (7) \end{aligned}$$

Next, evaluate the action for the classical solution. The classical solution  $x_{\text{cl}}(t')$  is the solution of Lagrange's equation:

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \implies \ddot{x}_{\text{cl}}(t') = 0, \quad \dot{x}_{\text{cl}}(t') = \frac{x_f - x_0}{t}$$

The action for the classical solution is:

$$S[x_{\text{cl}}] = \int_0^t \frac{1}{2} m \dot{x}_{\text{cl}}(t')^2 dt' = \frac{1}{2} m \frac{(x_f - x_0)^2}{t} \quad (8)$$

Substitute equations 7 and 8 into equation 4:

$$K(x_f, x_0, t) = \sqrt{\frac{m}{2\pi i \hbar t}} \exp\left(\frac{i}{\hbar} \frac{m}{2t} (x_f - x_0)^2\right) \quad (9)$$