

# Advanced Quantum Theory

## Homework 1

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### 2.1 Path integral in phase space

Show that the propagator of a quantum system can be written as:

$$K(\vec{r}_f, \vec{r}_0, t) = \int D[\vec{r}] D[\vec{p}] \exp \left( \frac{i}{\hbar} \int_0^t \left( \vec{p}(t') \cdot \dot{\vec{r}}(t') - H(\vec{r}(t'), \vec{p}(t')) \right) dt' \right) \quad (1)$$

We have that:

$$K(\vec{r}_f, \vec{r}_0, t) = \int d^n r_0 \dots \int d^n r_{N-1} \prod_{j=0}^{N-1} \langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{H} \tau} | \vec{r}_j \rangle \quad (2)$$

Assume that the Hamiltonian has the form  $\hat{H} = \hat{T}(\vec{p}) + \hat{U}(\vec{r})$ . Replace the exponential by the first-order Taylor polynomial and vice versa:

$$\begin{aligned} \langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{H} \tau} | \vec{r}_j \rangle &\approx \langle \vec{r}_{j+1} | \left( 1 - \frac{i}{\hbar} \hat{H} \tau \right) | \vec{r}_j \rangle \\ &\approx \langle \vec{r}_{j+1} | \left( 1 - \frac{i}{\hbar} \hat{T} \tau \right) \left( 1 - \frac{i}{\hbar} \hat{U} \tau \right) | \vec{r}_j \rangle \\ [1.5ex] &\approx \langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{T} \tau} e^{-\frac{i}{\hbar} \hat{U} \tau} | \vec{r}_j \rangle \end{aligned}$$

Apply  $\hat{U}$  to the position eigenstate  $|\vec{r}_j\rangle$ :

$$\langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{H} \tau} | \vec{r}_j \rangle \approx \langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{T} \tau} | \vec{r}_j \rangle e^{-\frac{i}{\hbar} U(\vec{r}_j) \tau}$$

Insert a resolution of the identity in terms of momentum eigenfunctions:

$$\begin{aligned} \langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{H} \tau} | \vec{r}_j \rangle &= \langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{T} \tau} \int d^n p_j |\vec{p}_j\rangle \langle \vec{p}_j | \vec{r}_j \rangle e^{-\frac{i}{\hbar} U(\vec{r}_j) \tau} \\ &= \int d^n p_j e^{-\frac{i}{\hbar} U(\vec{r}_j) \tau} \langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{T} \tau} | \vec{p}_j \rangle \langle \vec{p}_j | \vec{r}_j \rangle \end{aligned}$$

Apply  $\hat{T}$  to the momentum eigenstate  $|\vec{p}_j\rangle$  and substitute:

$$\begin{aligned} \langle \vec{r}_{j+1} | e^{-\frac{i}{\hbar} \hat{H} \tau} | \vec{r}_j \rangle &= \int d^n p_j \exp \left( -\frac{i}{\hbar} (T(\vec{p}_j) + U(\vec{r}_j)) \tau \right) \langle \vec{r}_{j+1} | \vec{p}_j \rangle \langle \vec{p}_j | \vec{r}_j \rangle \\ &= \int d^n p_j \exp \left( -\frac{i}{\hbar} H(\vec{r}_j, \vec{p}_j) \tau \right) \langle \vec{r}_{j+1} | \vec{p}_j \rangle \langle \vec{p}_j | \vec{r}_j \rangle \\ &= \int d^n p_j \exp \left( -\frac{i}{\hbar} H(\vec{r}_j, \vec{p}_j) \tau \right) \frac{1}{(2\pi\hbar)^{n/2}} e^{\frac{i}{\hbar} \vec{p}_j \cdot \vec{r}_{j+1}} \frac{1}{(2\pi\hbar)^{n/2}} e^{-\frac{i}{\hbar} \vec{p}_j \cdot \vec{r}_j} \\ [1.5ex] &= \frac{1}{(2\pi\hbar)^n} \int d^n p_j \exp \left( \frac{i}{\hbar} \left( \vec{p}_j \cdot \frac{\vec{r}_{j+1} - \vec{r}_j}{\tau} - H(\vec{r}_j, \vec{p}_j) \right) \tau \right) \end{aligned}$$

The product in equation 2 is then:

$$\begin{aligned} & \prod_{j=0}^{N-1} \frac{1}{(2\pi\hbar)^n} \int d^n p_j \exp \left( \frac{i}{\hbar} \left( \vec{p}_j \cdot \frac{\vec{r}_{j+1} - \vec{r}_j}{\tau} - H(\vec{r}_j, \vec{p}_j) \right) \tau \right) \\ &= \frac{1}{(2\pi\hbar)^{Nn}} \int d^n p_0 \dots \int d^n p_{N-1} \exp \left( \frac{i}{\hbar} \sum_{j=0}^{N-1} \left( \vec{p}_j \cdot \frac{\vec{r}_{j+1} - \vec{r}_j}{\tau} - H(\vec{r}_j, \vec{p}_j) \right) \tau \right) \end{aligned}$$

Equation 2 becomes:

$$\begin{aligned} K(\vec{r}_f, \vec{r}_0, t) &= \frac{1}{(2\pi\hbar)^{Nn}} \int d^n r_0 \dots \int d^n r_{N-1} \int d^n p_0 \dots \int d^n p_{N-1} \\ &\quad \exp \left( \frac{i}{\hbar} \sum_{j=0}^{N-1} \left( \vec{p}_j \cdot \frac{\vec{r}_{j+1} - \vec{r}_j}{\tau} - H(\vec{r}_j, \vec{p}_j) \right) \tau \right) \end{aligned}$$

Replace the sum by an integral in the limit  $N \rightarrow \infty$ ,  $\tau \rightarrow 0$  to obtain:

$$K(\vec{r}_f, \vec{r}_0, t) = \int D[\vec{r}] D[\vec{p}] \exp \left( \frac{i}{\hbar} \int_0^t \left( \vec{p}(t') \cdot \dot{\vec{r}}(t') - H(\vec{r}(t'), \vec{p}(t')) \right) dt' \right)$$

where

$$\int D[\vec{r}] D[\vec{p}] \dots = \lim_{N \rightarrow \infty} \frac{1}{(2\pi\hbar)^{Nn}} \int d^n r_0 \dots \int d^n r_{N-1} \int d^n p_0 \dots \int d^n p_{N-1} \dots$$