

# Advanced Quantum Theory

## Homework 3

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1. Derive an approximation for  $\langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle$  up to  $O(t)$  where  $\hat{H} = \frac{1}{2} \hat{p}^2 + U(\hat{x})$ .

The Taylor expansion of  $e^{-\frac{i}{\hbar} \hat{H} t}$  is:

$$e^{-\frac{i}{\hbar} \hat{H} t} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i}{\hbar} \hat{H} t \right)^n = 1 - \frac{i}{\hbar} \hat{H} t + O(t^2)$$

Hereafter  $= \dots + O(t^2)$  will be written as  $\approx \dots$ . Hence  $e^{-\frac{i}{\hbar} \hat{H} t}$  is:

$$e^{-\frac{i}{\hbar} \hat{H} t} \approx 1 - \frac{i}{\hbar} \left( \frac{1}{2} \hat{p}^2 t + U(\hat{x}) t \right) \approx \left( 1 - \frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t \right) \left( 1 - \frac{i}{\hbar} U(\hat{x}) t \right) \approx e^{-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t} e^{-\frac{i}{\hbar} U(\hat{x}) t}$$

And  $\langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle$  is:

$$\langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle \approx \langle p | e^{-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t} e^{-\frac{i}{\hbar} U(\hat{x}) t} | x \rangle \approx e^{-\frac{i}{\hbar} U(x) t} \langle p | e^{-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t} | x \rangle$$

Insert a resolution of the identity in terms of momentum eigenfunctions:

$$\begin{aligned} \langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle &\approx e^{-\frac{i}{\hbar} U(x) t} \int dp' \langle p | e^{-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t} | p' \rangle \langle p' | x \rangle \\ &\approx e^{-\frac{i}{\hbar} U(x) t} \int dp' e^{-\frac{i}{\hbar} \frac{1}{2} p'^2 t} \langle p | p' \rangle \langle p' | x \rangle \end{aligned}$$

We have that  $\langle p | p' \rangle = \delta(p' - p)$  and  $\langle p' | x \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} p' x}$ , so:

$$\begin{aligned} \langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle &\approx e^{-\frac{i}{\hbar} U(x) t} \int dp' e^{-\frac{i}{\hbar} \frac{1}{2} p'^2 t} \delta(p' - p) \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} p' x} \\ &\approx e^{-\frac{i}{\hbar} U(x) t} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} \frac{1}{2} p^2 t} e^{-\frac{i}{\hbar} p x} \\ &\approx \frac{1}{\sqrt{2\pi\hbar}} \exp \left( -\frac{i}{\hbar} \left( \frac{1}{2} p^2 t + U(x) t + p x \right) \right) \end{aligned}$$

2. (a) Determine the Feynman diagrams to evaluate  $I = \langle x_k^2 e^{-\epsilon \sum_{k'} x_{k'}^6} \rangle$  up to  $O(\epsilon)$ , their multiplicities, and their contributions to  $I$ .

The Taylor expansion of  $e^{-\epsilon \sum_{k'} x_{k'}^6}$  is:

$$e^{-\epsilon \sum_{k'} x_{k'}^6} = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\epsilon \sum_{k'} x_{k'}^6 \right)^n = 1 - \epsilon \sum_{k'} x_{k'}^6 + O(\epsilon^2)$$

Hereafter  $= \dots + O(\epsilon^2)$  will be written as  $\approx \dots$ . Hence  $I$  is:

$$I \approx \left\langle x_k^2 \left( 1 - \epsilon \sum_{k'} x_{k'}^6 \right) \right\rangle \approx \langle x_k^2 \rangle - \epsilon \left\langle x_k^2 \sum_{k'} x_{k'}^6 \right\rangle \approx \langle x_k^2 \rangle - \epsilon \sum_{k'} \langle x_k^2 x_{k'}^6 \rangle$$

We have that, for an average  $\langle x_k^p x_{k'}^{p'} \rangle$ , a Feynman diagram with  $m$  connections between  $p$   $k$  and  $p'$   $k'$  vertices has multiplicity:

$$\binom{p}{m} \binom{p'}{m} m! (p-m-1)!! (p'-m-1)!! \quad (1)$$

There are two Feynman diagrams for  $\langle x_k^2 x_{k'}^6 \rangle$  (figures 1 and 2). By equation 1, the multiplicities of the two diagrams are respectively:

$$\binom{2}{0} \binom{6}{0} 0! (2-0-1)!! (6-0-1)!! = 15$$

$$\binom{2}{2} \binom{6}{2} 2! (2-2-1)!! (6-2-1)!! = 90$$

The sum of the multiplicities is  $7!! = 105$ , as expected. Hence  $I$  is:

$$I \approx (A^{-1})_{kk} - \epsilon \left( 15(A^{-1})_{kk} ((A^{-1})_{k'k'})^3 + 90 ((A^{-1})_{kk})^2 ((A^{-1})_{k'k'})^2 \right)$$



Figure 1: 2a. Feynman diagram for  $\langle x_k^2 x_{k'}^6 \rangle$  of contribution  $15(A^{-1})_{kk} ((A^{-1})_{k'k'})^3$ .

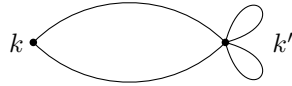


Figure 2: 2a. Feynman diagram for  $\langle x_k^2 x_{k'}^6 \rangle$  of contribution  $90 ((A^{-1})_{kk})^2 ((A^{-1})_{k'k'})^2$ .

2. (b) Given the average  $J = \langle x_k^2 x_{k'}^2 e^{-\epsilon \sum_{k''} x_{k''}^6} \rangle$ , determine the multiplicities of the Feynman diagrams in figures 3, 4 and 5.

For  $m = p = p'$ , equation 1 reduces to  $m!$ ; hence, the diagram in figure 3 has multiplicity  $2! = 2$ . There are  $(6-1)!! = 15$  ways to choose pairs of the 6  $k''$  vertices; hence, the diagram in figure 4 has multiplicity  $2 \times 15 = 30$ . In the diagram in figure 5, there are  $\binom{6}{2}$  ways to choose pairs of the 2  $k$  and 6  $k''$  vertices,  $\binom{4}{2}$  ways to choose pairs of the 2  $k'$  and remaining 4  $k''$  vertices, and  $2!!$  ways to choose pairs of the remaining 2  $k''$  vertices. Hence, it has multiplicity  $\binom{6}{2} \binom{4}{2} 2!! = 180$ .

2. (c) Find a Feynman diagram other than those in figures 3, 4, and 5 for  $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$  in which the two legs of  $k'$  are not connected to each other.



Figure 3: 2b. Feynman diagram for  $\langle x_k^2 x_{k'}^2 \rangle$  of multiplicity 2.

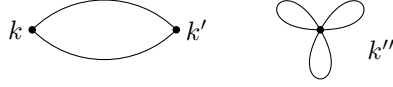


Figure 4: 2b. Feynman diagram for  $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$  of multiplicity 30.

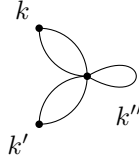


Figure 5: 2b. Feynman diagram for  $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$  of multiplicity 180.

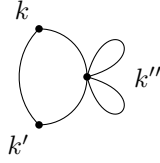


Figure 6: 2c. Feynman diagram for  $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$ .

Figure 6 is such a Feynman diagram.

2. (d) Given the average  $\tilde{J} = \langle x_k^2 x_{k'}^2 e^{-\epsilon \sum_{k''} x_{k''}^m} \rangle$ , determine the multiplicities of the Feynman diagrams analogous to those in figures 3, 4 and 5.

The diagram in figure 3 does not change and has multiplicity 2. There are  $(m-1)!!$  ways to choose pairs of  $m$   $k''$  vertices; hence, the diagram analogous to figure 4 has multiplicity  $2(m-1)!!$ . In the diagram analogous to figure 5, there are  $\binom{m}{2}$  ways to choose pairs of the 2  $k$  and  $m$   $k''$  vertices,  $\binom{m-2}{2}$  ways to choose pairs of the 2  $k'$  and remaining  $(m-2)$   $k''$  vertices, and  $(m-4)!!$  ways to choose pairs of the remaining  $(m-4)$   $k''$  vertices. Hence, it has multiplicity  $\binom{m}{2} \binom{m-2}{2} (m-4)!!$ .