

Advanced Quantum Theory

Homework 3

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1. Derive an approximation for $\langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle$ up to $O(t)$ where $\hat{H} = \frac{1}{2} \hat{p}^2 + U(\hat{x})$.

The Taylor expansion of $e^{-\frac{i}{\hbar} \hat{H} t}$ is:

$$e^{-\frac{i}{\hbar} \hat{H} t} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar} \hat{H} t \right)^n = 1 - \frac{i}{\hbar} \hat{H} t + O(t^2)$$

Hereafter $= \dots + O(t^2)$ will be written as $\approx \dots$. Hence $e^{-\frac{i}{\hbar} \hat{H} t}$ is:

$$e^{-\frac{i}{\hbar} \hat{H} t} \approx 1 - \frac{i}{\hbar} \left(\frac{1}{2} \hat{p}^2 t + U(\hat{x}) t \right) \approx \left(1 - \frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t \right) \left(1 - \frac{i}{\hbar} U(\hat{x}) t \right) \approx e^{-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t} e^{-\frac{i}{\hbar} U(\hat{x}) t}$$

And $\langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle$ is:

$$\langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle \approx \langle p | e^{-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t} e^{-\frac{i}{\hbar} U(\hat{x}) t} | x \rangle \approx e^{-\frac{i}{\hbar} U(x) t} \langle p | e^{-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t} | x \rangle$$

Insert a resolution of the identity in terms of momentum eigenfunctions:

$$\begin{aligned} \langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle &\approx e^{-\frac{i}{\hbar} U(x) t} \int dp' \langle p | e^{-\frac{i}{\hbar} \frac{1}{2} \hat{p}^2 t} | p' \rangle \langle p' | x \rangle \\ &\approx e^{-\frac{i}{\hbar} U(x) t} \int dp' e^{-\frac{i}{\hbar} \frac{1}{2} p'^2 t} \langle p | p' \rangle \langle p' | x \rangle \end{aligned}$$

We have that $\langle p | p' \rangle = \delta(p' - p)$ and $\langle p' | x \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} p' x}$, so:

$$\begin{aligned} \langle p | e^{-\frac{i}{\hbar} \hat{H} t} | x \rangle &\approx e^{-\frac{i}{\hbar} U(x) t} \int dp' e^{-\frac{i}{\hbar} \frac{1}{2} p'^2 t} \delta(p' - p) \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} p' x} \\ &\approx e^{-\frac{i}{\hbar} U(x) t} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar} \frac{1}{2} p^2 t} e^{-\frac{i}{\hbar} p x} \\ &\approx \frac{1}{\sqrt{2\pi\hbar}} \exp \left(-\frac{i}{\hbar} \left(\frac{1}{2} p^2 t + U(x) t + p x \right) \right) \end{aligned}$$

2. (a) Determine the Feynman diagrams to evaluate $I = \langle x_k^2 e^{-\epsilon \sum_{k'} x_{k'}^6} \rangle$ up to $O(\epsilon)$, their multiplicities, and their contributions to I .

The Taylor expansion of $e^{-\epsilon \sum_{k'} x_{k'}^6}$ is:

$$e^{-\epsilon \sum_{k'} x_{k'}^6} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\epsilon \sum_{k'} x_{k'}^6 \right)^n = 1 - \epsilon \sum_{k'} x_{k'}^6 + O(\epsilon^2)$$

Hereafter $= \dots + O(\epsilon^2)$ will be written as $\approx \dots$. Hence I is:

$$I \approx \left\langle x_k^2 \left(1 - \epsilon \sum_{k'} x_{k'}^6 \right) \right\rangle \approx \langle x_k^2 \rangle - \epsilon \left\langle x_k^2 \sum_{k'} x_{k'}^6 \right\rangle \approx \langle x_k^2 \rangle - \epsilon \sum_{k'} \langle x_k^2 x_{k'}^6 \rangle$$

We have that, for an average $\langle x_k^p x_{k'}^{p'} \rangle$, a Feynman diagram with m connections between p k and p' k' vertices has multiplicity:

$$\binom{p}{m} \binom{p'}{m} m! (p - m - 1)!! (p' - m - 1)!! \quad (1)$$

There are two Feynman diagrams for $\langle x_k^2 x_{k'}^6 \rangle$ (figures 1 and 2). By equation 1, the multiplicities of the two diagrams are respectively:

$$\binom{2}{0} \binom{6}{0} 0! (2 - 0 - 1)!! (6 - 0 - 1)!! = 15$$

$$\binom{2}{2} \binom{6}{2} 2! (2 - 2 - 1)!! (6 - 2 - 1)!! = 90$$

The sum of the multiplicities is $7!! = 105$, as expected. Hence I is:

$$I \approx A_{kk}^{-1} - \epsilon \left(15 (A_{kk}^{-1}) (A_{k'k'}^{-1})^3 + 90 (A_{kk'}^{-1})^2 (A_{k'k'}^{-1})^2 \right)$$



Figure 1: 2a. Feynman diagram for $\langle x_k^2 x_{k'}^6 \rangle$ with contribution $15 (A_{kk}^{-1}) (A_{k'k'}^{-1})^3$.

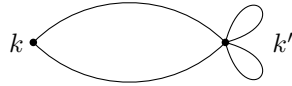


Figure 2: 2a. Feynman diagram for $\langle x_k^2 x_{k'}^6 \rangle$ with contribution $90 (A_{kk'}^{-1})^2 (A_{k'k'}^{-1})^2$.

2. (b) Given the average $J = \langle x_k^2 x_{k'}^2 e^{-\epsilon \sum_{k''} x_{k''}^6} \rangle$, determine the multiplicities of the Feynman diagrams in figures 3, 4 and 5.

For $m = p = p'$, equation 1 reduces to $m!$; hence, the diagram in figure 3 has multiplicity $2! = 2$. There are $(6 - 1)!! = 15$ ways to choose pairs of the 6 k'' vertices; hence, the diagram in figure 4 has multiplicity $2 \times 15 = 30$. In the diagram in figure 5, there are $\binom{6}{2}$ ways to choose pairs of the 2 k and 6 k'' vertices, $\binom{4}{2}$ ways to choose pairs of the 2 k' and remaining 4 k'' vertices, and $2!!$ ways to choose pairs of the remaining 2 k'' vertices. Hence, it has multiplicity $\binom{6}{2} \binom{4}{2} 2!! = 180$.

2. (c) Find a Feynman diagram other than those in figures 3, 4, and 5 for $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$ in which the two legs of k' are not connected to each other.

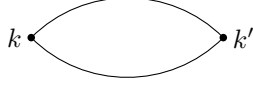


Figure 3: 2b. Feynman diagram for $\langle x_k^2 x_{k'}^2 \rangle$ with multiplicity 2.

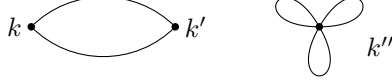


Figure 4: 2b. Feynman diagram for $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$ with multiplicity 15.

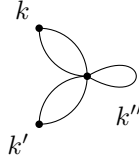


Figure 5: 2b. Feynman diagram for $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$ with multiplicity 90.

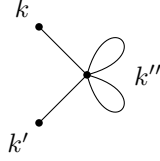


Figure 6: 2c. Feynman diagram for $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$.

See figure 6.

2. (d) Given the average $\tilde{J} = \langle x_k^2 x_{k'}^2 e^{-\epsilon \sum_{k''} x_{k''}^m} \rangle$, determine the multiplicities of the Feynman diagrams analogous to those in figures 3, 4 and 5.

The diagram in figure 3 does not change and has multiplicity 2. There are $(m-1)!!$ ways to choose pairs of m k'' vertices; hence, the diagram analogous to figure 4 has multiplicity $2(m-1)!!$. In the diagram analogous to figure 5, there are $\binom{m}{2}$ ways to choose pairs of the 2 k and m k'' vertices, $\binom{m-2}{2}$ ways to choose pairs of the 2 k' and remaining $m-2$ k'' vertices, and $(m-4)!!$ ways to choose pairs of the remaining $m-4$ k'' vertices. Hence, it has multiplicity $\binom{m}{2} \binom{m-2}{2} (m-4)!!$.