# Advanced Quantum Theory
Homework 2

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## 2.2 Propagator for the harmonic oscillator

Show that the propagator for the harmonic oscillator satisfies the Schrödinger equation:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial {x_f}^2}+\frac{1}{2}m\omega^2 x_f^2\right)K(x_f,x_0,t)=i\hbar\frac{\partial}{\partial t}K(x_f,x_0,t) \tag{2.1}$$

The propagator for the harmonic oscillator is:

$$K(x_f,x_0,t)=\sqrt{\frac{m\omega}{2\pi i\hbar\sin\omega t}}\exp\left(\frac{i}{\hbar}\frac{m\omega}{2\sin\omega t}\left((x_f^2+x_0^2)\cos\omega t-2x_f x_0\right)-i\frac{\pi}{4}\right) \tag{2.2}$$

First, evaluate the left-hand side of equation 2.1. By the chain rule:

$$\frac{\partial^2}{\partial x^2}e^{f(x)}=\left(\frac{\partial^2 f(x)}{\partial x^2}+\left(\frac{\partial f(x)}{\partial x}\right)^2\right)e^{f(x)}$$

Hence:

$$f(x_f)=\frac{i}{\hbar}\frac{m\omega}{2\sin\omega t}\left((x_f^2+x_0^2)\cos\omega t-2x_f x_0\right)-i\frac{\pi}{4}$$

$$\frac{\partial f(x_f)}{\partial x_f}=\frac{i}{\hbar}\frac{m\omega}{\sin\omega t}\left(x_f\cos\omega t-x_0\right)$$

$$\frac{\partial^2 f(x_f)}{\partial {x_f}^2}=\frac{i}{\hbar}\frac{m\omega}{\sin\omega t}\cos\omega t$$

The left-hand side is:

$$\begin{aligned}&\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial {x_f}^2}+\frac{1}{2}m\omega^2 x_f^2\right)K(x_f,x_0,t)\\
&=\left(-\frac{\hbar^2}{2m}\left(\frac{i}{\hbar}m\omega\frac{\cos\omega t}{\sin\omega t}-\frac{1}{\hbar^2}m^2\omega^2\frac{(x_f\cos\omega t-x_0)^2}{\sin^2\omega t}\right)+\frac{1}{2}m\omega^2x_f^2\right)K(x_f,x_0,t)\\
&=\left(-\frac{1}{2}i\hbar\omega\frac{\cos\omega t}{\sin\omega t}+\frac{1}{2}m\omega^2\frac{(x_f\cos\omega t-x_0)^2}{\sin^2\omega t}+\frac{1}{2}m\omega^2x_f^2\right)K(x_f,x_0,t) \qquad (2.3)\end{aligned}$$

Next, evaluate the right-hand side of equation 2.1. Define $g(t)$:

$$K(x_f, x_0, t) = \sqrt{\frac{m\omega}{2\pi\hbar}} e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{\sin\omega t}} e^{\,g(t)}$$

$$\begin{aligned} g(t) &= \frac{i}{\hbar} \frac{m\omega}{2\sin\omega t} \left( (x_f^2 + x_0^2)\cos\omega t - 2x_f x_0 \right) \\ &= \frac{i}{\hbar} m\omega \left( \frac{x_f^2 + x_0^2}{2} \cot\omega t - x_f x_0 \csc\omega t \right) \end{aligned}$$

By the product and chain rules:

$$\frac{\partial}{\partial t} K(x_f, x_0, t) = \sqrt{\frac{m\omega}{2\pi\hbar}} e^{-i\frac{\pi}{4}} \left( \frac{1}{\sqrt{\sin\omega t}} e^{\,g(t)} \cdot \frac{\partial g(t)}{\partial t} + e^{\,g(t)} \cdot \frac{\partial}{\partial t} \frac{1}{\sqrt{\sin\omega t}} \right)$$

Evaluate the derivatives:

$$\frac{\partial}{\partial t} \frac{1}{\sqrt{\sin\omega t}} = -\frac{1}{2} \sin^{-3/2}\omega t \cdot \frac{\partial}{\partial t} \sin\omega t = -\frac{1}{\sqrt{\sin\omega t}} \frac{\omega\cos\omega t}{2\sin\omega t}$$

$$\begin{aligned} \frac{\partial g(t)}{\partial t} &= \frac{i}{\hbar} m\omega \left( -\frac{x_f^2 + x_0^2}{2} \frac{\omega}{\sin^2\omega t} + x_0 x_f \frac{\omega\cos\omega t}{\sin^2\omega t} \right) \\ &= \frac{i}{\hbar} \frac{m\omega^2}{2\sin^2\omega t} \left( 2x_0 x_f \cos\omega t - (x_0^2 + x_f^2) \right) \end{aligned}$$

Hence:

$$\begin{aligned} &\frac{\partial}{\partial t} K(x_f, x_0, t) \\ &= \sqrt{\frac{m\omega}{2\pi\hbar}} e^{-i\frac{\pi}{4}} \left( \frac{1}{\sqrt{\sin\omega t}} e^{\,g(t)} \left( \frac{i}{\hbar} \frac{m\omega^2}{2\sin^2\omega t} \left( 2x_0 x_f \cos\omega t - (x_0^2 + x_f^2) \right) \right) \right. \\ &\qquad\qquad \left. + e^{\,g(t)} \left( -\frac{1}{\sqrt{\sin\omega t}} \frac{\omega\cos\omega t}{2\sin\omega t} \right) \right) \\ &= \sqrt{\frac{m\omega}{2\pi\hbar}} e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{\sin\omega t}} e^{\,g(t)} \left( \frac{i}{\hbar} \frac{m\omega^2}{2\sin^2\omega t} \left( 2x_0 x_f \cos\omega t - (x_0^2 + x_f^2) \right) - \frac{\omega\cos\omega t}{2\sin\omega t} \right) \\ &= \frac{1}{2\sin\omega t} \left( \frac{i}{\hbar} \frac{m\omega^2}{\sin\omega t} \left( 2x_0 x_f \cos\omega t - (x_0^2 + x_f^2) \right) - \omega\cos\omega t \right) K(x_f, x_0, t) \end{aligned}$$

The right-hand side is:

$$\begin{aligned} &i\hbar \frac{\partial}{\partial t} K(x_f, x_0, t) \\ &\quad = \frac{1}{2\sin\omega t} \left( -\frac{m\omega^2}{\sin\omega t} \left( 2x_0 x_f \cos\omega t - (x_0^2 + x_f^2) \right) - i\hbar\omega\cos\omega t \right) K(x_f, x_0, t) \qquad (2.4) \end{aligned}$$

Taking equations 2.3 and 2.4, it remains to show that:

$$\left(-\frac{1}{2}i\hbar\frac{\omega\cos\omega t}{\sin\omega t}+\frac{1}{2}m\frac{\omega^2(x_f\cos\omega t-x_0)^2}{\sin^2\omega t}+\frac{1}{2}m\omega^2x_f^2\right)$$
$$=\frac{1}{2\sin\omega t}\left(-\frac{m\omega^2}{\sin\omega t}\left(2x_0x_f\cos\omega t-(x_0^2+x_f^2)\right)-i\hbar\omega\cos\omega t\right)$$

Multiply both sides by $2\sin^2\omega t$:

$$-i\hbar\omega\cos\omega t\sin\omega t+m\omega^2(x_f\cos\omega t-x_0)^2+m\omega^2x_f^2\sin^2\omega t$$
$$=-m\omega^2\left(2x_0x_f\cos\omega t-(x_0^2+x_f^2)\right)-i\hbar\omega\cos\omega t\sin\omega t$$

Trivially:

$$m\omega^2(x_f\cos\omega t-x_0)^2+m\omega^2x_f^2\sin^2\omega t$$
$$=m\omega^2x_f^2\cos^2\omega t-2m\omega^2x_0x_f\cos\omega t+m\omega^2x_0^2+m\omega^2x_f^2\sin^2\omega t$$
$$=m\omega^2x_f^2\left(\cos^2\omega t+\sin^2\omega t\right)-2m\omega^2x_0x_f\cos\omega t+m\omega^2x_0^2$$
$$=-m\omega^2\left(2x_0x_f\cos\omega t-(x_0^2+x_f^2)\right)$$

I.e., the left-hand side is equal to the right-hand side and the propagator satisfies the Schrödinger equation.