Advanced Quantum Theory

Homework 3

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1. Derive an approximation for $\langle\, p\, | e^{-\frac{i}{\hbar}\hat{H}t} |\, x\,\rangle$ up to $O(t)$ where $\hat{H} = \frac{1}{2}\hat{p}^2 + U(\hat{x})$.

   The Taylor expansion of $e^{-\frac{i}{\hbar}\hat{H}t}$ is:

$$e^{-\frac{i}{\hbar}\hat{H}t} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\hat{H}t\right)^n = 1 - \frac{i}{\hbar}\hat{H}t + O(t^2)$$

   Hereafter $= \cdots + O(t^2)$ will be written as $\approx \cdots$. Hence $e^{-\frac{i}{\hbar}\hat{H}t}$ is:

$$e^{-\frac{i}{\hbar}\hat{H}t} \approx 1 - \frac{i}{\hbar}\left(\frac{1}{2}\hat{p}^2 t + U(\hat{x})t\right) \approx \left(1 - \frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t\right)\left(1 - \frac{i}{\hbar}U(\hat{x})t\right) \approx e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t} e^{-\frac{i}{\hbar}U(\hat{x})t}$$

   And $\langle\, p\, | e^{-\frac{i}{\hbar}\hat{H}t} |\, x\,\rangle$ is:

$$\langle\, p\, | e^{-\frac{i}{\hbar}\hat{H}t} |\, x\,\rangle \approx \langle\, p\, | e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t} e^{-\frac{i}{\hbar}U(\hat{x})t} |\, x\,\rangle \approx e^{-\frac{i}{\hbar}U(x)t} \langle\, p\, | e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t} |\, x\,\rangle$$

   Insert a resolution of the identity in terms of momentum eigenfunctions:

$$\begin{aligned}
\langle\, p\, | e^{-\frac{i}{\hbar}\hat{H}t} |\, x\,\rangle &\approx e^{-\frac{i}{\hbar}U(x)t} \int \mathrm{d}p' \langle\, p\, | e^{-\frac{i}{\hbar}\frac{1}{2}\hat{p}^2 t} |\, p'\,\rangle \langle\, p'\,|\, x\,\rangle \\
&\approx e^{-\frac{i}{\hbar}U(x)t} \int \mathrm{d}p' e^{-\frac{i}{\hbar}\frac{1}{2}{p'}^2 t} \langle\, p\,|\, p'\,\rangle \langle\, p'\,|\, x\,\rangle
\end{aligned}$$

   We have that $\langle\, p\,|\, p'\,\rangle = \delta(p' - p)$ and $\langle\, p'\,|\, x\,\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}p'x}$, so:

$$\begin{aligned}
\langle\, p\, | e^{-\frac{i}{\hbar}\hat{H}t} |\, x\,\rangle &\approx e^{-\frac{i}{\hbar}U(x)t} \int \mathrm{d}p' e^{-\frac{i}{\hbar}\frac{1}{2}{p'}^2 t} \delta(p' - p) \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}p'x} \\
&\approx e^{-\frac{i}{\hbar}U(x)t} \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}\frac{1}{2}p^2 t} e^{-\frac{i}{\hbar}px} \\
&\approx \frac{1}{\sqrt{2\pi\hbar}} \exp\left(-\frac{i}{\hbar}\left(\frac{1}{2}p^2 t + U(x)t + px\right)\right)
\end{aligned}$$

2. (a) Determine the Feynman diagrams to evaluate $I = \left\langle x_k^2 e^{-\epsilon \sum_{k'} x_{k'}^6} \right\rangle$ up to $O(\epsilon)$, their multiplicities, and their contributions to I.

   The Taylor expansion of $e^{-\epsilon \sum_{k'} x_{k'}^6}$ is:

$$e^{-\epsilon \sum_{k'} x_{k'}^6} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\epsilon \sum_{k'} x_{k'}^6\right)^n = 1 - \epsilon \sum_{k'} x_{k'}^6 + O(\epsilon^2)$$

Hereafter $= \cdots + O(\epsilon^2)$ will be written as $\approx \cdots$. Hence I is:

$$I \approx \left\langle x_k^2 \left(1 - \epsilon \sum_{k'} x_{k'}^6\right)\right\rangle \approx \langle x_k^2 \rangle - \epsilon \left\langle x_k^2 \sum_{k'} x_{k'}^6 \right\rangle \approx \langle x_k^2 \rangle - \epsilon \sum_{k'} \langle x_k^2 x_{k'}^6 \rangle$$

We have that, for an average $\left\langle x_k^p x_{k'}^{p'} \right\rangle$, a Feynman diagram with m connections between p k and p' k' vertices has multiplicity:

$$\binom{p}{m}\binom{p'}{m} m!(p-m-1)!!(p'-m-1)!! \tag{1}$$

There are two Feynman diagrams for $\langle x_k^2 x_{k'}^6 \rangle$ (figures 1 and 2). By equation 1, the multiplicities of the two diagrams are respectively:

$$\binom{2}{0}\binom{6}{0} 0!(2-0-1)!!(6-0-1)!! = 15$$

$$\binom{2}{2}\binom{6}{2} 2!(2-2-1)!!(6-2-1)!! = 90$$

The sum of the multiplicities is $7!! = 105$, as expected. Hence I is:

$$I \approx (A^{-1})_{kk} - \epsilon \left(15 (A^{-1})_{kk} \left((A^{-1})_{k'k'}\right)^3 + 90 \left((A^{-1})_{kk'}\right)^2 \left(\left((A^{-1})_{k'k'}\right)^2\right)\right)$$

Figure 1: 2a. Feynman diagram for $\langle x_k^2 x_{k'}^6 \rangle$ of contribution $15(A^{-1})_{kk} \left((A^{-1})_{k'k'}\right)^3$.

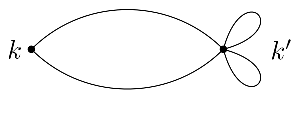

Figure 2: 2a. Feynman diagram for $\langle x_k^2 x_{k'}^6 \rangle$ of contribution $90 \left((A^{-1})_{kk'}\right)^2 \left((A^{-1})_{k'k'}\right)^2$.

2. (b) Given the average $J = \left\langle x_k^2 x_{k'}^2 e^{-\epsilon \sum_{k''} x_{k''}^6} \right\rangle$, determine the multiplicities of the Feynman diagrams in figures 3, 4 and 5.

For $m = p = p'$, equation 1 reduces to $m!$; hence, the diagram in figure 3 has multiplicity $2! = 2$. There are $(6-1)!! = 15$ ways to choose pairs of the 6 k'' vertices; hence, the diagram in figure 4 has multiplicity $2 \times 15 = 30$. In the diagram in figure 5, there are $\binom{6}{2} = 15$ ways to choose pairs of the 2 k and 6 k'' vertices, $\binom{4}{2} = 6$ ways to choose pairs of the 2 k' and remaining 4 k'' vertices, and $(2-1)!! = 1$ ways to choose pairs of the remaining 2 k'' vertices. Hence, it has multiplicity $15 \times 6 \times 1 = 90$.

2. (c) Find a Feynman diagram other than those in figures 3, 4, and 5 for $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$ in which the two legs of k' are not connected to each other.

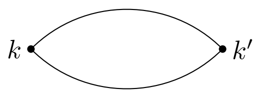

Figure 3: 2b. Feynman diagram for $\langle x_k^2 x_{k'}^2 \rangle$ of multiplicity 2.

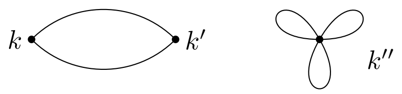

Figure 4: 2b. Feynman diagram for $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$ of multiplicity 30.

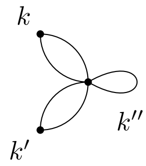

Figure 5: 2b. Feynman diagram for $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$ of multiplicity 90.

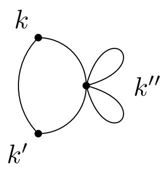

Figure 6: 2c. Feynman diagram for $\langle x_k^2 x_{k'}^2 x_{k''}^6 \rangle$.

Figure 6 is such a Feynman diagram.

2. (d) Given the average $\tilde{J} = \langle x_k^2 x_{k'}^2 e^{-\epsilon \sum_{k''} x_{k''}^m} \rangle$, determine the multiplicities of the Feynman diagrams analogous to those in figures 3, 4 and 5.

The diagram in figure 3 does not change and has multiplicity 2. There are $(m-1)!!$ ways to choose pairs of m k'' vertices; hence, the diagram analogous to figure 4 has multiplicity $2(m-1)!!$. In the diagram analogous to figure 5, there are $\binom{m}{2}$ ways to choose pairs of the 2 k and m k'' vertices, $\binom{m-2}{2}$ ways to choose pairs of the 2 k' and remaining $(m-2)$ k'' vertices, and $(m-4-1)!!$ ways to choose pairs of the remaining $(m-4)$ k'' vertices. Hence, it has multiplicity $\binom{m}{2}\binom{m-2}{2}(m-5)!!$.