Advanced Quantum Theory

Homework 4

Tim Lawson

December 11, 2023

3.3 Wick's theorem

Use Wick's theorem to evaluate the following integrals.

(a) $\int (x+y+z)^2 e^{-10x^2-y^2-6xy-2z^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

We have that:

$$c = \int_{\mathbb{R}_n} \exp\left(-\frac{1}{2}\vec{x}^\top A\vec{x}\right) \mathrm{d}x^n = \left(\frac{(2\pi)^n}{\det A}\right)^{1/2} \tag{1}$$

$$\langle \cdots \rangle = \frac{1}{c}\int_{\mathbb{R}_n} \exp\left(-\frac{1}{2}\vec{x}^\top A\vec{x}\right) \cdots \mathrm{d}x^n \tag{2}$$

$$\langle x_k x_{k'} \rangle = (A^{-1})_{kk'} \tag{3}$$

The exponent is:

$$\begin{aligned}
-10x^2 - y^2 - 6xy - 2z^2 &= -\frac{1}{2}(20x^2 + 2y^2 + 12xy + 4z^2) \\
&= -\frac{1}{2}\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 20 & 6 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= -\frac{1}{2}\vec{x}^\top A\vec{x} \quad \text{where} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad A = \begin{bmatrix} 20 & 6 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}
\end{aligned}$$

Hence, with $\vec{x}$ and A as above, the integral is:

$$\begin{aligned}
&\int_{\mathbb{R}^3} (x+y+z)^2 e^{-10x^2-y^2-6xy-2z^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad = \frac{1}{c}\left\langle (x+y+z)^2 \right\rangle \\
&\quad = \frac{1}{c}\left\langle x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \right\rangle \\
&\quad = \frac{1}{c}\left(\left\langle x^2 \right\rangle + \left\langle y^2 \right\rangle + \left\langle z^2 \right\rangle + 2\langle xy \rangle + 2\langle xz \rangle + 2\langle yz \rangle\right) \\
&\quad = \frac{1}{c}\left((A^{-1})_{11} + (A^{-1})_{22} + (A^{-1})_{33} + 2(A^{-1})_{12} + 2(A^{-1})_{13} + 2(A^{-1})_{23}\right) \qquad (4)
\end{aligned}$$

We also have that:

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A \tag{5}$$

$$\det A = 20(2\times 4 - 0\times 0) - 6(6\times 4 - 0\times 0) + 0(6\times 0 - 2\times 0)$$

$$= 160 - 144 + 0 = 16$$

$$\operatorname{adj} A = \begin{bmatrix} 2\times 4 - 0\times 0 & -(6\times 4 - 0\times 0) & 6\times 0 - 2\times 0 \\ -(6\times 4 - 0\times 0) & 20\times 4 - 0\times 0 & -(20\times 0 - 6\times 0) \\ 6\times 0 - 2\times 0 & -(20\times 0 - 6\times 0) & 20\times 2 - 6\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -24 & 0 \\ -24 & 80 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{16}\begin{bmatrix} 8 & -24 & 0 \\ -24 & 80 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -6 & 0 \\ -6 & 20 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$c = \left(\frac{(2\pi)^3}{16}\right)^{1/2} = \sqrt{\frac{\pi^3}{2}}$$

Finally, the integral is:

$$\int_{\mathbb{R}^3} (x+y+z)^2 e^{-10x^2 - y^2 - 6xy - 2z^2} \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

$$= \frac{1}{2\sqrt{2\pi^3}} \left(2 + 20 + 1 + 2(-6) + 2(0) + 2(0)\right) = \frac{11}{2\sqrt{2\pi^3}}$$

3.7 Feynman diagrams

Use perturbation theory to evaluate the following expressions in terms of integrals over products of factors $iG(t', t'')$ and draw the corresponding Feynman diagrams.

(b) $\left\langle x(t_1)x(t_2)\exp\left(-\epsilon\frac{i}{\hbar}\int_0^t x(t')^6 \mathrm{d}t'\right)\right\rangle$ neglecting terms of order ϵ^2 and higher

The Taylor expansion of the exponential is:

$$\exp\left(-\epsilon\frac{i}{\hbar}\int_0^t x(t')^6 \mathrm{d}t'\right) = 1 - \epsilon\frac{i}{\hbar}\int_0^t x(t')^6 \mathrm{d}t' + O(\epsilon^2)$$

Hereafter $= \cdots + O(\epsilon^2)$ will be written as $\approx \cdots$. Hence:

$$\left\langle x(t_1)x(t_2)\exp\left(-\epsilon\frac{i}{\hbar}\int_0^t x(t')^6 \mathrm{d}t'\right)\right\rangle \approx \langle x(t_1)x(t_2)\rangle - \epsilon\frac{i}{\hbar}\int_0^t \left\langle x(t_1)x(t_2)x(t')^6\right\rangle \mathrm{d}t'$$

There is one Feynman diagram for $\langle x(t_1)x(t_2)\rangle$ (figure 1) and two Feynman diagrams for $\left\langle x(t_1)x(t_2)x(t')^6\right\rangle$ (figures 2 and 3). The diagram in figure 1 has multiplicity 1. There are $(6-1)!!$ ways to choose pairs of the 6 t' vertices; hence, the diagram in figure 2 has multiplicity 15. In the diagram in figure 3, there are $\binom{6}{1}$ ways to choose a pair of the 1 t_1 and 6 t' vertices, $\binom{5}{1}$ ways to choose a pair of the 1 t_2 and remaining 5 t' vertices, and $(4-1)!!$ ways to choose pairs of the remaining 4 t' vertices. Hence, it has multiplicity $6\times 5\times 3 = 90$.

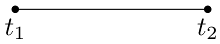

Figure 1: Feynman diagram for $\langle x(t_1)x(t_2)\rangle$ of contribution ...

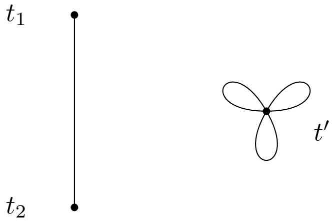

Figure 2: Feynman diagram for $\left\langle x(t_1)x(t_2)x(t')^6\right\rangle$ of contribution ...

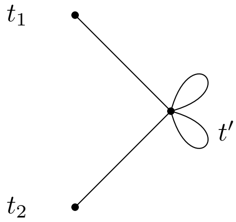

Figure 3: Feynman diagram for $\left\langle x(t_1)x(t_2)x(t')^6\right\rangle$ of contribution ...

We have that:

$$\langle x(t')x(t'')\rangle = iG(t',t'') \quad \text{where} \quad (A^{-1}x)(t') = \int_0^t G(t',t'')x(t'')\mathrm{d}t' \tag{6}$$

Hence:

$$\left\langle x(t_1)x(t_2)\exp\left(-\epsilon\frac{i}{\hbar}\int_0^t x(t')^6\mathrm{d}t'\right)\right\rangle$$

$$\approx \langle x(t_1)x(t_2)\rangle - \epsilon\frac{i}{\hbar}\int_0^t \left\langle x(t_1)x(t_2)x(t')^6\right\rangle \mathrm{d}t'$$

$$\approx iG(t_1,t_2) - \epsilon\frac{i}{\hbar}\int_0^t \left(15iG(t_1,t_2)\left(iG(t',t')\right)^3 + 90iG(t_1,t')iG(t_2,t')\left(iG(t',t')\right)^2\right)\mathrm{d}t'$$

$$\approx iG(t_1,t_2) - \epsilon\frac{i}{\hbar}\int_0^t \left(15G(t_1,t_2)G(t',t')^3 + 90G(t_1,t')G(t_2,t')G(t',t')^2\right)\mathrm{d}t'$$