Advanced Quantum Theory

Homework 4

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December 11, 2023

3.3 Wick's theorem

Use Wick's theorem to evaluate the following integrals.

(a) $\int (x+y+z)^2 e^{-10x^2-y^2-6xy-2z^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$

We have that:

$$c = \int_{\mathbb{R}_n} \exp\left(-\frac{1}{2}\vec{x}^\top A\vec{x}\right) \mathrm{d}x^n = \left(\frac{(2\pi)^n}{\det A}\right)^{1/2} \tag{3.3.1}$$

$$\langle \cdots \rangle = \frac{1}{c}\int_{\mathbb{R}_n} \exp\left(-\frac{1}{2}\vec{x}^\top A\vec{x}\right) \cdots \,\mathrm{d}x^n \tag{3.3.2}$$

$$\langle x_k x_{k'} \rangle = (A^{-1})_{kk'} \tag{3.3.3}$$

The exponent is:

$$\begin{aligned}
-10x^2 - y^2 - 6xy - 2z^2 &= -\frac{1}{2}(20x^2 + 2y^2 + 12xy + 4z^2) \\
&= -\frac{1}{2}\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 20 & 6 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= -\frac{1}{2}\vec{x}^\top A\vec{x} \quad \text{where} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad A = \begin{bmatrix} 20 & 6 & 0 \\ 6 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}
\end{aligned}$$

Hence, with $\vec{x}$ and A as above, the integral is:

$$\begin{aligned}
&\int_{\mathbb{R}^3} (x+y+z)^2 e^{-10x^2-y^2-6xy-2z^2} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&\quad = c\left\langle (x+y+z)^2 \right\rangle \\
&\quad = c\left\langle x^2 + y^2 + z^2 + 2xy + 2xz + 2yz \right\rangle \\
&\quad = c\left(\left\langle x^2 \right\rangle + \left\langle y^2 \right\rangle + \left\langle z^2 \right\rangle + 2\langle xy \rangle + 2\langle xz \rangle + 2\langle yz \rangle\right) \\
&\quad = c\left((A^{-1})_{11} + (A^{-1})_{22} + (A^{-1})_{33} + 2(A^{-1})_{12} + 2(A^{-1})_{13} + 2(A^{-1})_{23}\right)
\end{aligned} \tag{3.3.4}$$

We also have that:

$$A^{-1} = \frac{1}{\det A}\operatorname{adj} A \tag{3.3.5}$$

$$\det A = 20(2\times 4 - 0\times 0) - 6(6\times 4 - 0\times 0) + 0(6\times 0 - 2\times 0)$$

$$= 160 - 144 + 0 = 16$$

$$\operatorname{adj} A = \begin{bmatrix} 2\times 4 - 0\times 0 & -(6\times 4 - 0\times 0) & 6\times 0 - 2\times 0 \\ -(6\times 4 - 0\times 0) & 20\times 4 - 0\times 0 & -(20\times 0 - 6\times 0) \\ 6\times 0 - 2\times 0 & -(20\times 0 - 6\times 0) & 20\times 2 - 6\times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -24 & 0 \\ -24 & 80 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{-1} = \frac{1}{16}\begin{bmatrix} 8 & -24 & 0 \\ -24 & 80 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -6 & 0 \\ -6 & 20 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$c = \left(\frac{(2\pi)^3}{16}\right)^{1/2} = \sqrt{\frac{\pi^3}{2}}$$

Finally, the integral is:

$$\int_{\mathbb{R}^3} (x+y+z)^2 e^{-10x^2 - y^2 - 6xy - 2z^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{c}{4}\left(2 + 20 + 1 + 2(-6) + 2(0) + 2(0)\right) = \frac{11}{4}\sqrt{\frac{\pi^3}{2}}$$

3.7 Feynman diagrams

Use perturbation theory to evaluate the following expressions in terms of integrals over products of factors $iG(t', t'')$, and draw the corresponding Feynman diagrams.

(b) $\left\langle x(t_1)x(t_2)\exp\left(-\epsilon\frac{i}{\hbar}\int_0^t x(t')^6 \mathrm{d}t'\right)\right\rangle$ neglecting terms of order ϵ^2 and higher

The Taylor expansion of the exponential is:

$$\exp\left(-\epsilon\frac{i}{\hbar}\int_0^t x(t')^6 \mathrm{d}t'\right) = 1 - \epsilon\frac{i}{\hbar}\int_0^t x(t')^6 \mathrm{d}t' + O(\epsilon^2)$$

Hereafter $= \cdots + O(\epsilon^2)$ will be written as $\approx \cdots$. Hence:

$$\left\langle x(t_1)x(t_2)\exp\left(-\epsilon\frac{i}{\hbar}\int_0^t x(t')^6 \mathrm{d}t'\right)\right\rangle \approx \langle x(t_1)x(t_2)\rangle - \epsilon\frac{i}{\hbar}\int_0^t \left\langle x(t_1)x(t_2)x(t')^6\right\rangle \mathrm{d}t'$$

There is one Feynman diagram for $\langle x(t_1)x(t_2)\rangle$ (figure 1), which has multiplicity 1.

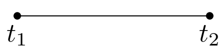


Figure 1: Feynman diagram for $\langle x(t_1)x(t_2)\rangle$ of multiplicity 1.

There are two Feynman diagrams for $\left\langle x(t_1)x(t_2)x(t')^6\right\rangle$ (figures 2 and 3). In the diagram in figure 2, there are $(6-1)!! = 15$ ways to choose pairs of the 6 legs of the t'

vertex; hence, the diagram in figure 2 has multiplicity 15. In the diagram in figure 3, there are $\binom{6}{1} = 6$ ways to choose a pair of the leg of the t_1 vertex and 6 legs of the t' vertex, $\binom{5}{1} = 5$ ways to choose a pair of the leg of the t_2 vertex and remaining 5 legs of the t' vertex, and $(4-1)!! = 3$ ways to choose pairs of the remaining 4 legs of the t' vertex. Hence, it has multiplicity $6 \times 5 \times 3 = 90$. The sum of the multiplicities of the diagrams for $\left\langle x(t_1)x(t_2)x(t')^6 \right\rangle$ is $7!! = 105$, as expected.

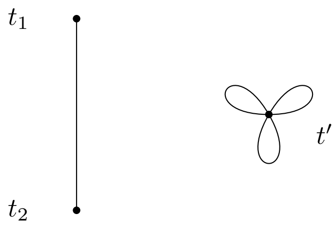


Figure 2: Feynman diagram for $\left\langle x(t_1)x(t_2)x(t')^6 \right\rangle$ of multiplicity 15.

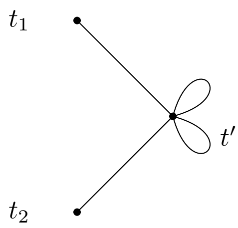


Figure 3: Feynman diagram for $\left\langle x(t_1)x(t_2)x(t')^6 \right\rangle$ of multiplicity 90.

We have that:

$$\langle x(t')x(t'') \rangle = iG(t', t'') \quad \text{where} \quad (A^{-1}x)(t') = \int_0^t G(t', t'')x(t'')\mathrm{d}t' \tag{3.7.1}$$

Hence, the expression is:

$$\begin{aligned}
&\left\langle x(t_1)x(t_2) \exp\left(-\epsilon \frac{i}{\hbar} \int_0^t x(t')^6 \mathrm{d}t'\right) \right\rangle \\
&\approx \langle x(t_1)x(t_2) \rangle - \epsilon \frac{i}{\hbar} \int_0^t \left\langle x(t_1)x(t_2)x(t')^6 \right\rangle \mathrm{d}t' \\
&\approx iG(t_1, t_2) - \epsilon \frac{i}{\hbar} \int_0^t \left(15iG(t_1, t_2)\left(iG(t', t')\right)^3 + 90iG(t_1, t')iG(t_2, t')\left(iG(t', t')\right)^2\right) \mathrm{d}t' \\
&\approx iG(t_1, t_2) - \epsilon \frac{i}{\hbar} \int_0^t \left(15G(t_1, t_2)G(t', t')^3 + 90G(t_1, t')G(t_2, t')G(t', t')^2\right) \mathrm{d}t'
\end{aligned}$$

3.9 Quintic perturbation

(d) Consider a perturbed harmonic oscillator with the Lagrangian:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \epsilon x^n \tag{3.9.1}$$

where $n \geq 3$ is an odd integer. Which are the Feynman diagrams for the leading non-vanishing perturbation to the propagator, and what are their multiplicities?

The propagator is:

$$K_{\text{anh}}(x_f, x_0, t) = K_{\text{harm}}(x_f, x_0, t) \left\langle \exp\left(-\epsilon \frac{i}{\hbar} \int_0^t x(t')^n \, \mathrm{d}t'\right) \right\rangle$$

The Taylor expansion of the exponential is:

$$\exp\left(-\epsilon \frac{i}{\hbar} \int_0^t x(t')^n \, \mathrm{d}t'\right) = 1 - \epsilon \frac{i}{\hbar} \int_0^t x(t')^n \, \mathrm{d}t' \ldots$$
$$\cdots - \frac{1}{2}\epsilon^2 \frac{1}{\hbar^2} \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' x(t')^n x(t'')^n + O(\epsilon^3)$$

Hereafter $= \cdots + O(\epsilon^3)$ will be written as $\approx \cdots$. Hence:

$$\left\langle \exp\left(-\epsilon \frac{i}{\hbar} \int_0^t x(t')^n \, \mathrm{d}t'\right) \right\rangle \approx 1 - \frac{1}{2}\epsilon^2 \frac{1}{\hbar^2} \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' \left\langle x(t')^n x(t'')^n \right\rangle$$

The term of order ϵ vanishes because n is odd and hence $\langle x(t')^n \rangle = 0$.

There are $\frac{1}{2}(n+1)$ Feynman diagrams for $\langle x(t')^n x(t'')^n \rangle$. For the example of $n = 5$, there are 3 diagrams for $m = \{1, 3, 5\}$ connections between the legs of the t' and t'' vertices (figure 4). The multiplicities sum to $(10-1)!! = 945$, as expected.

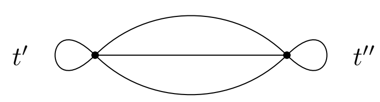
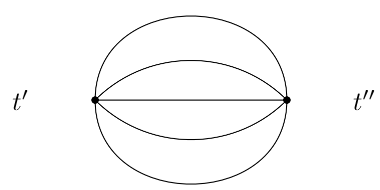
Feynman diagram	Multiplicity
	$1! \times \binom{5}{1}^2 \times ((4-1)!!)^2 = 225$
	$3! \times \binom{5}{3}^2 = 600$
	$5! \times \binom{5}{5}^2 = 120$

Figure 4: Feynman diagrams and their multiplicities for $\left\langle x(t')^5 x(t'')^5 \right\rangle$.

Generalising to odd integers $n \geq 3$, there are $\frac{1}{2}(n+1)$ diagrams for $m = \{1, 3, \ldots, n-2, n\}$ connections between the legs of the t' and t'' vertices. With m

connections, there are $m!\binom{n}{m}^2$ ways to choose m pairs of the legs of the t' and t'' vertices and $(n-m-1)!!$ ways to choose pairs of the remaining legs of each of the t' and t'' vertices. Hence, each of the $\frac{1}{2}(n+1)$ diagrams with m connections has multiplicity:

$$m!\left(\binom{n}{m}(n-m-1)!!\right)^2$$

Define $n = 2j+1$ and $m = 2k+1$ where $j \in \mathbb{N}$, $k \in \{0, 1, \ldots, j-1, j\}$. The average is:

$$\left\langle \exp\left(-\epsilon \frac{i}{\hbar}\int_0^t x(t')^{2j+1}\,\mathrm{d}t'\right)\right\rangle \approx 1 - \frac{1}{2}\epsilon^2 \frac{1}{\hbar^2}\int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' \left\langle x(t')^{2j+1} x(t'')^{2j+1}\right\rangle$$

$$\approx 1 - \frac{1}{2}\epsilon^2 \frac{1}{\hbar^2}\left(\sum_{k=0}^{j}(2k+1)!\left(\binom{2j+1}{2k+1}(2(j-k)-1)\,!!\right)^2 \ldots\right.$$

$$\left.\ldots \int_0^t \mathrm{d}t' \int_0^t \mathrm{d}t'' \;\left(iG(t',t'')\right)^{2k+1}\left(G(t',t')G(t'',t'')\right)^{2(j-k)}\right)$$