Sign Changes in the Prime Number Theorem

Binghamton Arithmetic Seminar

Thomas Morrill[1] Tim Trudgian[2] David Platt[3]

[1]Trine University
[2]University of New South Wales Canberra
[3]University of Bristol
Slides available at `tsmorrill.github.io`.

2021 March 9

Outline

- Prime Number Theorem
- The Riemann Zeta Function
- Alternate Forms of the Prime Number Theorem
- The Problem
- Kaczorowski's Method
- Results
- Future Work

The Prime Number Theorem

Theorem (Hadamard, de la Vallée Poussin, 1896)

Let $\pi(x)$ *denote the number of primes in the interval* $[1, x]$*. Then,*

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log(x)} = 1.$$

Here, we say that $\pi(x)$ and $x/\log(x)$ are *asymptotically equal*, denoted by $\pi(x) \sim x/\log(x)$.

The Riemann Zeta Function

The proof of the PNT was obtained from studying the Riemann zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This function had already garnered interest due to its relationship to the harmonic series. Euler had considered real values of $s > 1$ and obtained the product formula

$$\zeta(s) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - p_n^{-s}} \right),$$

where p_n denotes the nth prime number.

The Riemann Zeta Function

In 1859, Riemann established an analytic continuation of $\zeta(s)$ to $\mathbb{C}\backslash\{1\}$ via the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s),$$

and sketched a proof of the prime number theorem dependent on what we now call the Riemann Hypothesis.

The PNT has been proven unconditionally, but the best known version of the theorem is still only known on assumption of the Riemann Hypothesis.

The Riemann Zeta Function

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s); \quad \zeta(s) = \prod_{n=1}^{\infty}\left(\frac{1}{1-p_n^{-s}}\right).$$

Alternate Forms of the Prime Number Theorem

Theorem (Another Prime Number Theorem)

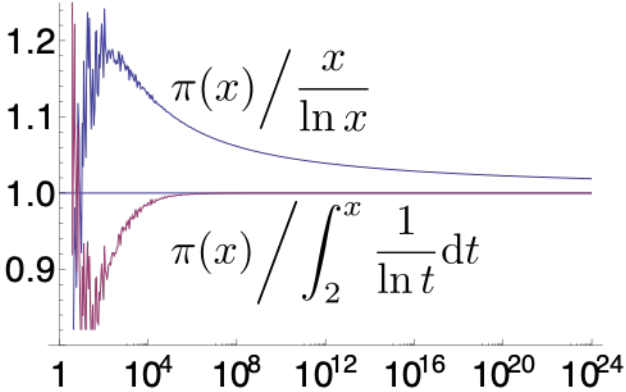
Let li(x) *denote the logarimthic integral function,*

$$\mathrm{li}(x) = \int_0^x \frac{dx}{\log(x)}.$$

Then,

$$\pi(x) \sim \mathrm{li}(x).$$

Alternate Forms of the Prime Number Theorem



Alternate Forms of the Prime Number Theorem

Theorem (Prime Number Theorem, Weighted Count)

Let $\vartheta(x)$ denote the Chebyshev theta function,

$$\vartheta(x) = \sum_{p \leq x} \log(p).$$

Then,

$$\vartheta(x) \sim x.$$

Compare this to

$$\pi(x) = \sum_{p \leq x} 1.$$

Alternate Forms of the Prime Number Theorem

Theorem (Prime Number Theorem, Another Weighted Count)

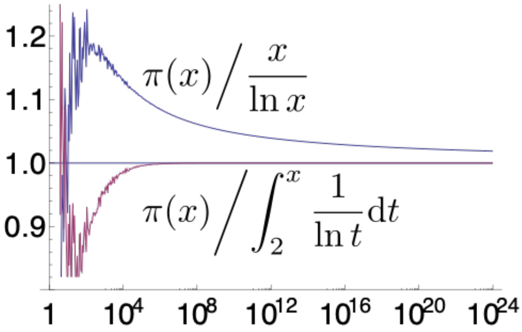
Let $\psi(x)$ denote the Chebyshev psi function,

$$\psi(x) = \sum_{p^k \leq x} \log(p).$$

Then,

$$\psi(x) \sim x.$$

The Problem



Despite the appearance of the graph, Littlewood proved that the quantity $\pi(x) - x/\log(x)$ changes sign infinitely often as $x \to \infty$. (He also proved the same for the other versions of the PNT.)

The Problem

What is the density of sign changes in $\psi(x) - x$ as $x \to \infty$?

Theorem (Pólya, 1969)

Let $V(T)$ denote the number of sign changes of $\psi(x) - x$ in the interval $[1, T]$. Then,

$$\liminf_{T \to \infty} \frac{V(T)}{\log T} \geq \frac{\gamma^*}{\pi},$$

where $\gamma*$ is defined as follows.

The Problem

Write the zeros of $\zeta(s)$ as $\varrho = \beta + i\gamma$, and define

$$\Theta = \sup_{\zeta(\varrho)=0} \beta.$$

If this supremum is achieved, γ^* is the least positive number so that $\Theta + i\gamma^*$ is a zero of $\zeta(s)$. If the supremum is not achieved, then γ^* is ∞ instead.

In practical terms, γ^* is $14.13\ldots$, if the Riemann Hypothesis is true, and at least $3 \cdot 10^{10}$ if the Riemann Hypothesis is false.

The Problem

Theorem (Kaczorowski, 1991)

Let $V(T)$ denote the number of sign changes of $\psi(x) - x$ in the interval $[1, T]$. Then,

$$\liminf_{T\to\infty} \frac{V(T)}{\log T} \geq \frac{\gamma_1}{\pi} + 10^{-250}.$$

.

Kaczorowski's Method

Begin by constructing a function

$$F(z) = e^{-z/2} \sum_{\substack{\varrho=\beta+i\gamma \\ \gamma>0}} \frac{e^{\varrho z}}{\varrho}.$$

Here the summation is taken over zeros ϱ of $\zeta(s)$.

Kaczorowski's Method

Zeroes of $F(z)$ detect sign changes in $V(T)$.

Theorem (Kaczorowski, 1991)

Assume the Riemann Hypothesis is true. Define $\varkappa$ to be

$$\varkappa = \lim_{Y \to 0^+} \lim_{T \to \infty} \#\{z = x + iy : F(z) = 0,\ 0 < x < t,\ y \geq Y\}$$

Then,

$$\liminf_{T \to \infty} \frac{V(T)}{\log T} \geq \frac{\gamma_1}{\pi} + 2\varkappa.$$

.

Kaczorowski's Method

To find such zeroes, we approximate $F(z)$ by

$$F_N(z) = \sum_{n=1}^{N} \frac{e^{i\gamma_n z}}{1/2 + i\gamma}.$$

Note that each summand is periodic in $x = \Re(z)$.

Dirichlet's theorem on Diophantine approximation allows us to approximate the N period lengths simultaneously. Then, Rouché's theorem allows us to associate zeroes of $F(z)$ to zeroes of $F_N(z)$.

Our Method

We use commercial computation to search for zeroes of $F_N(z)$. Once we have some promising candidates, more intensive computation is used to verify the zero and the hypotheses for Dirichlet and Rouché.

These calculations show there is a suitable zero of $F(z)$ in the region

$$14\,685.516\,155\,1 \le \Re(z) \le 14\,685.516\,157\,2$$
$$0.079\,831\,7 \le \Im(z) \le 0.079\,833\,8$$

Theorem (M., Trudgian, Platt 1991)

Let $V(T)$ denote the number of sign changes of $\psi(x) - x$ in the interval $[1, T]$. Then,

$$\liminf_{T\to\infty} \frac{V(T)}{\log T} \geq \frac{\gamma_1}{\pi} + 1.867 \cdot 10^{-30}.$$

.

Future Work

The density of sign changes in the prime number theorem is a topic of active study, one that can include many more advanced techniques.

If we retain Kaczorowski's function $F(z)$, we could possibly take advantage of the theory of holomorphic almost periodic functions.

It is conjectured that $V(T)$ grows on the order of $\sqrt{T}$, but this is out of reach for all currently known methods.

Thank you!