

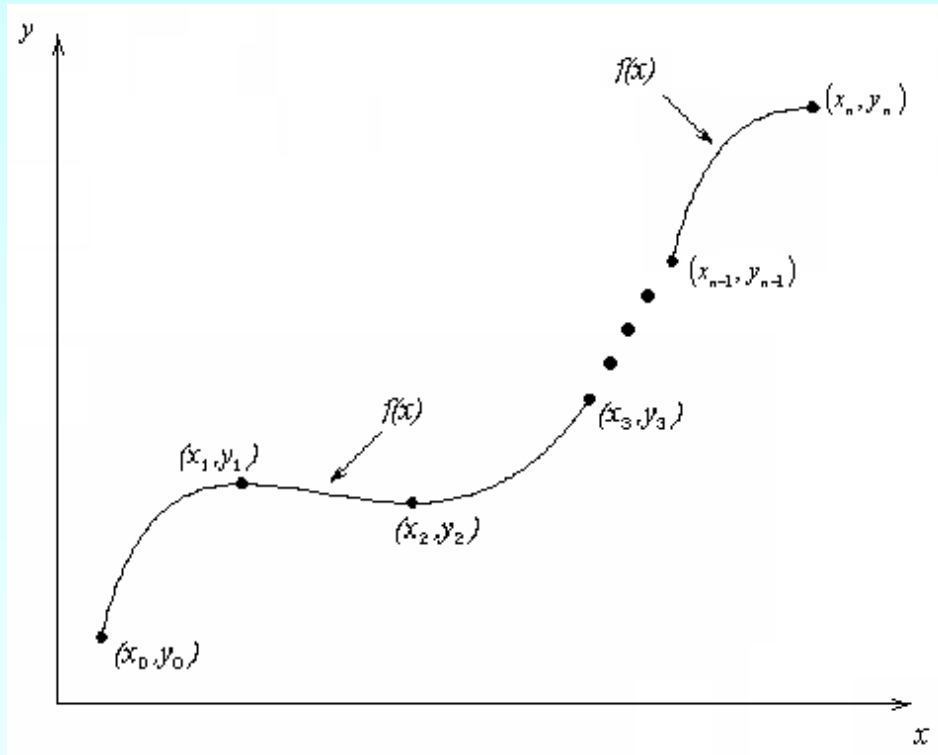
# Interpolation

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SUST

# Direct Method of Interpolation

# What is Interpolation ?

Given  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ , find the value of 'y' at a value of 'x' that is not given.



**Figure 1** Interpolation of discrete.

# Interpolants

Polynomials are the most common choice of interpolants because they are easy to:

- Evaluate
- Differentiate, and
- Integrate

# Direct Method

Given ' $n+1$ ' data points  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ , pass a polynomial of order ' $n$ ' through the data as given below:

$$y = a_0 + a_1x + \dots + a_nx^n.$$

where  $a_0, a_1, \dots, a_n$  are real constants.

- Set up ' $n+1$ ' equations to find ' $n+1$ ' constants.
- To find the value ' $y$ ' at a given value of ' $x$ ', simply substitute the value of ' $x$ ' in the above polynomial.



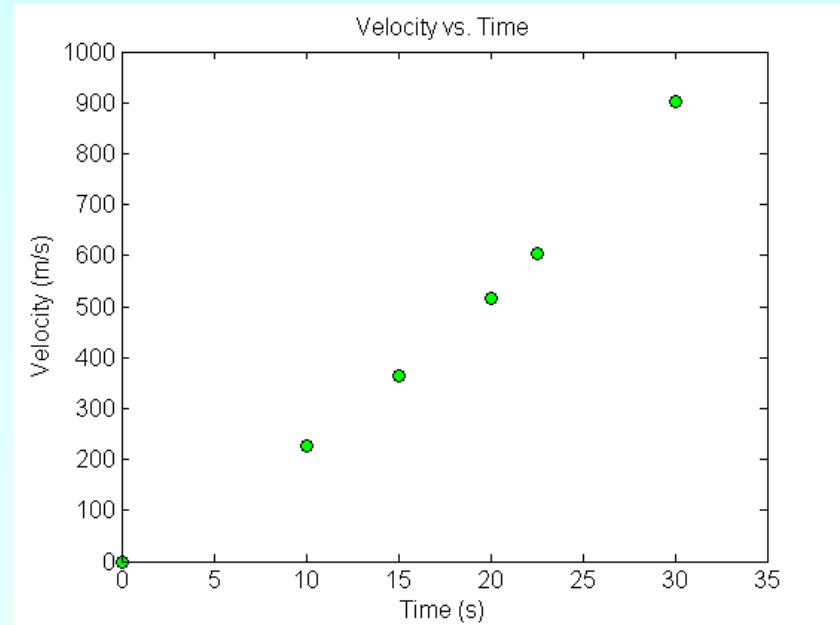
# Example 1

The upward velocity of a rocket is given as a function of time in Table 1.

Find the velocity at  $t=16$  seconds using the direct method for linear interpolation.

**Table 1** Velocity as a function of time.

$t, (s)$	$v(t), (m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



**Figure 2** Velocity vs. time data for the rocket example

# Linear Interpolation

$$v(t) = a_0 + a_1 t$$

$$v(15) = a_0 + a_1(15) = 362.78$$

$$v(20) = a_0 + a_1(20) = 517.35$$

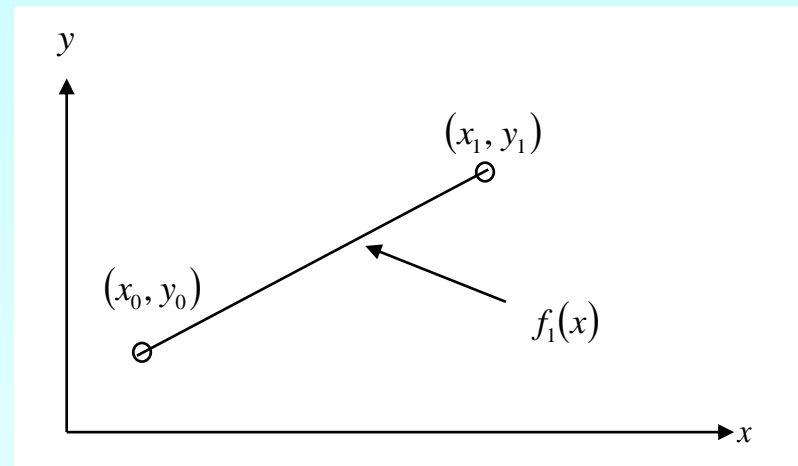
Solving the above two equations gives,

$$a_0 = -100.93 \quad a_1 = 30.914$$

Hence

$$v(t) = -100.93 + 30.914t, \quad 15 \leq t \leq 20.$$

$$v(16) = -100.93 + 30.914(16) = 393.7 \text{ m/s}$$



**Figure 3** Linear interpolation.



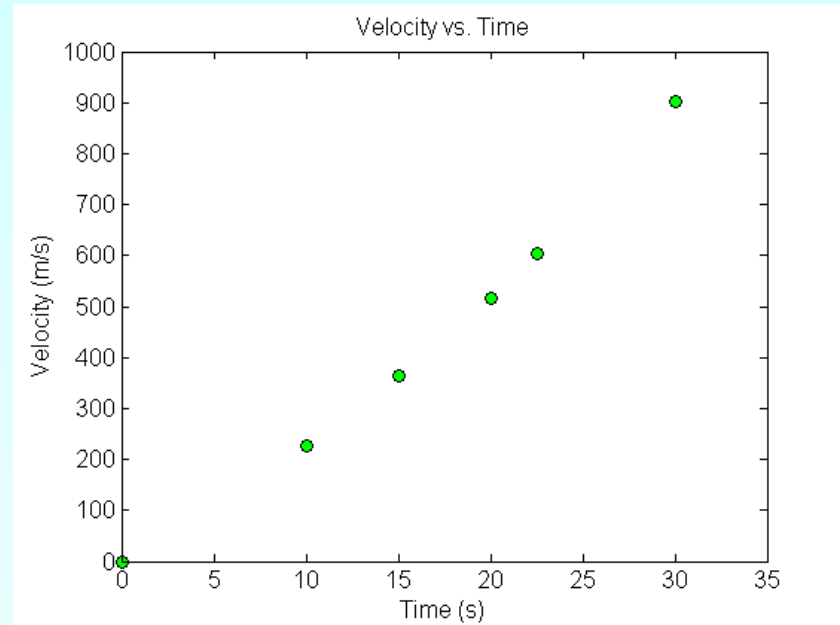
## Example 2

The upward velocity of a rocket is given as a function of time in Table 2.

Find the velocity at  $t=16$  seconds using the direct method for quadratic interpolation.

**Table 2** Velocity as a function of time.

$t, (s)$	$v(t), (m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



**Figure 5** Velocity vs. time data for the rocket example



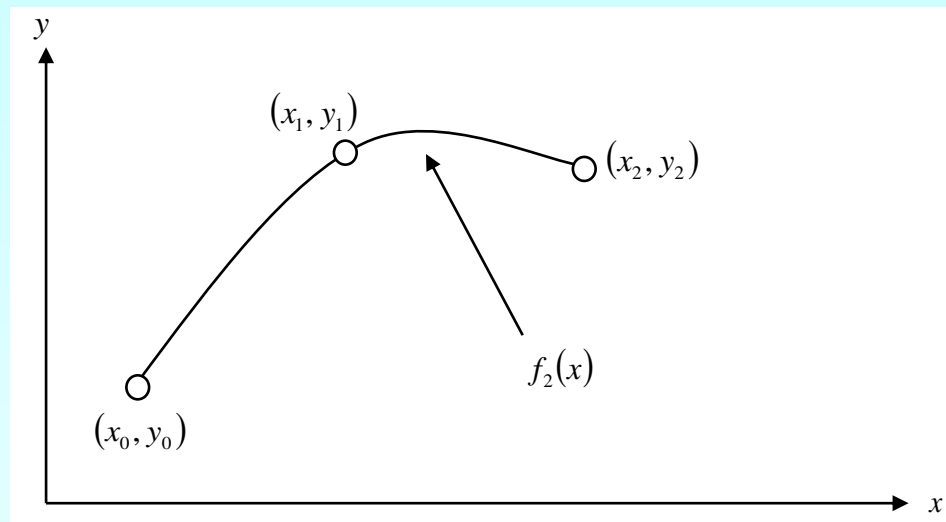
# Quadratic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2$$

$$v(10) = a_0 + a_1(10) + a_2(10)^2 = 227.04$$

$$v(15) = a_0 + a_1(15) + a_2(15)^2 = 362.78$$

$$v(20) = a_0 + a_1(20) + a_2(20)^2 = 517.35$$



**Figure 6** Quadratic interpolation.

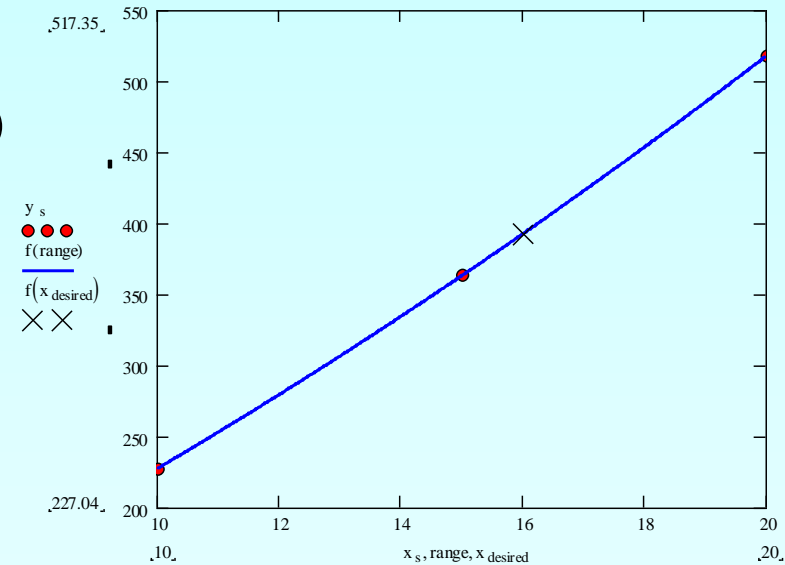
Solving the above three equations gives

$$a_0 = 12.05 \quad a_1 = 17.733 \quad a_2 = 0.3766$$

# Quadratic Interpolation (cont.)

$$v(t) = 12.05 + 17.733t + 0.3766t^2, \quad 10 \leq t \leq 20$$

$$\begin{aligned} v(16) &= 12.05 + 17.733(16) + 0.3766(16)^2 \\ &= 392.19 \text{ m/s} \end{aligned}$$



The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{392.19 - 393.70}{392.19} \right| \times 100 \\ &= 0.38410\% \end{aligned}$$



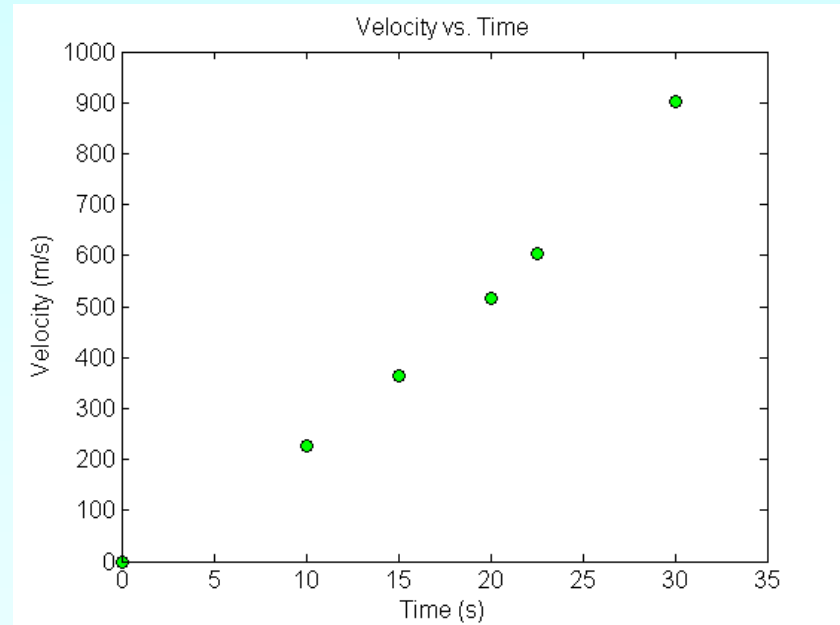
## Example 3

The upward velocity of a rocket is given as a function of time in Table 3.

Find the velocity at  $t=16$  seconds using the direct method for cubic interpolation.

**Table 3** Velocity as a function of time.

$t, (s)$	$v(t), (m/s)$
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67



**Figure 6** Velocity vs. time data for the rocket example

# Cubic Interpolation

$$v(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

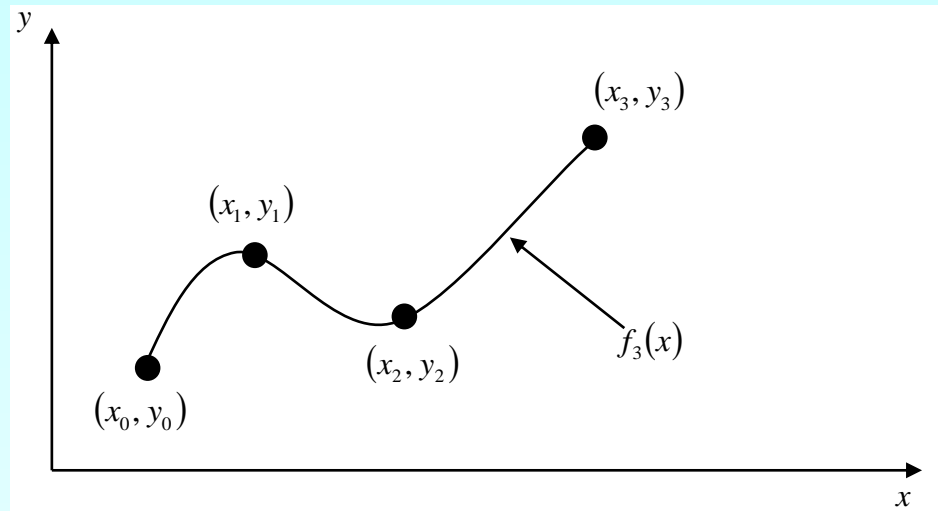
$$v(10) = 227.04 = a_0 + a_1(10) + a_2(10)^2 + a_3(10)^3$$

$$v(15) = 362.78 = a_0 + a_1(15) + a_2(15)^2 + a_3(15)^3$$

$$v(20) = 517.35 = a_0 + a_1(20) + a_2(20)^2 + a_3(20)^3$$

$$v(22.5) = 602.97 = a_0 + a_1(22.5) + a_2(22.5)^2 + a_3(22.5)^3$$

$$a_0 = -4.2540 \quad a_1 = 21.266 \quad a_2 = 0.13204 \quad a_3 = 0.0054347$$

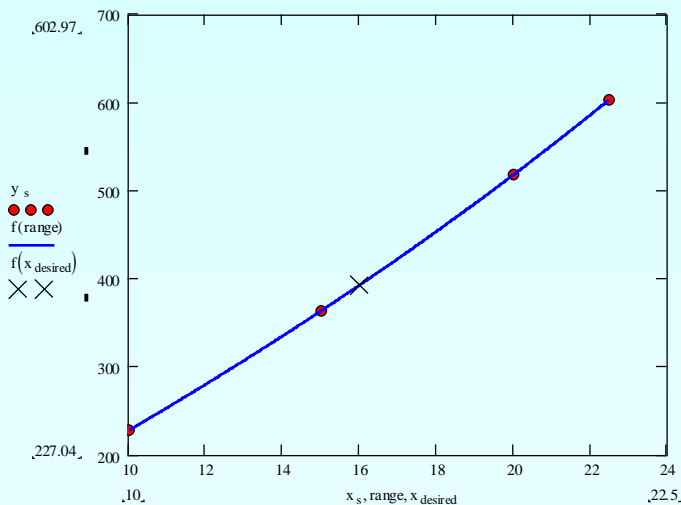


**Figure 7** Cubic interpolation.

# Cubic Interpolation (contd)

$$v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, \quad 10 \leq t \leq 22.5$$

$$\begin{aligned} v(16) &= -4.2540 + 21.266(16) + 0.13204(16)^2 + 0.0054347(16)^3 \\ &= 392.06 \text{ m/s} \end{aligned}$$



The absolute percentage relative approximate error  $|\epsilon_a|$  between second and third order polynomial is

$$\begin{aligned} |\epsilon_a| &= \left| \frac{392.06 - 392.19}{392.06} \right| \times 100 \\ &= 0.033269\% \end{aligned}$$

# Comparison Table

**Table 4** Comparison of different orders of the polynomial.

Order of Polynomial	1	2	3
$v(t = 16) \text{ m/s}$	393.7	392.19	392.06
Absolute Relative Approximate Error	-----	0.38410 %	0.033269 %

# Distance from Velocity Profile

Find the distance covered by the rocket from  $t=11\text{s}$  to  $t=16\text{s}$  ?

$$v(t) = -4.3810 + 21.289t + 0.13064t^2 + 0.0054606t^3, \quad 10 \leq t \leq 22.5$$

$$\begin{aligned} s(16) - s(11) &= \int_{11}^{16} v(t) dt \\ &= \int_{11}^{16} (-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3) dt \\ &= \left[ -4.2540t + 21.266\frac{t^2}{2} + 0.13204\frac{t^3}{3} + 0.0054347\frac{t^4}{4} \right]_{11}^{16} \\ &= 1605 \text{ m} \end{aligned}$$

# Acceleration from Velocity Profile

Find the acceleration of the rocket at  $t=16\text{s}$  given that  $v(t) = -4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3, 10 \leq t \leq 22.5$

$$\begin{aligned} a(t) &= \frac{d}{dt} v(t) \\ &= \frac{d}{dt} (-4.2540 + 21.266t + 0.13204t^2 + 0.0054347t^3) \\ &= 21.289 + 0.26130t + 0.016382t^2, \quad 10 \leq t \leq 22.5 \end{aligned}$$

$$\begin{aligned} a(16) &= 21.266 + 0.26408(16) + 0.016304(16)^2 \\ &= 29.665 \text{ m/s}^2 \end{aligned}$$



# Lagrange Method of Interpolation

# Lagrangian Interpolation

Lagrangian interpolating polynomial is given by

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i)$$

where ‘ $n$ ’ in  $f_n(x)$  stands for the  $n^{th}$  order polynomial that approximates the function  $y = f(x)$  given at  $(n+1)$  data points as  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$ , and

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^n \frac{x - x_j}{x_i - x_j}$$

$L_i(x)$  is a weighting function that includes a product of  $(n-1)$  terms with terms of  $j = i$  omitted.

# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Lagrangian method for linear interpolation.

Table Velocity as a function of time

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

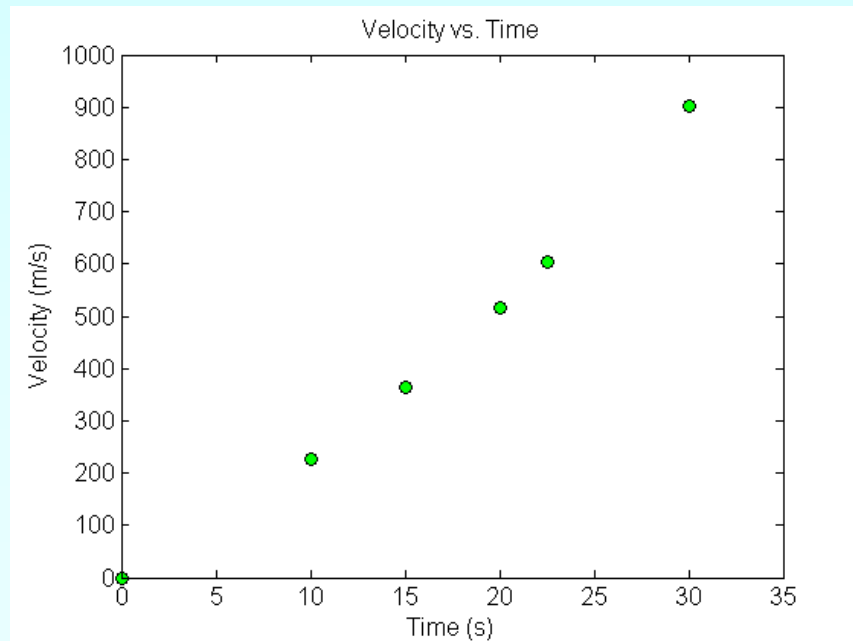


Figure. Velocity vs. time data for the rocket example

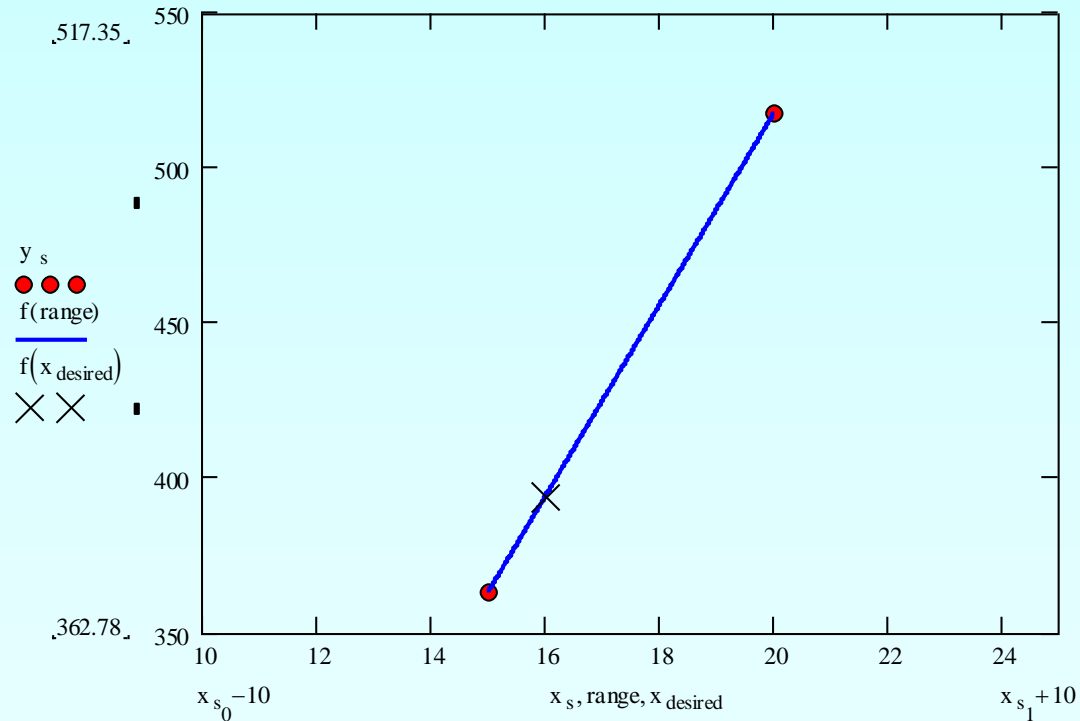
# Linear Interpolation

$$v(t) = \sum_{i=0}^1 L_i(t) v(t_i)$$

$$= L_0(t) v(t_0) + L_1(t) v(t_1)$$

$$t_0 = 15, v(t_0) = 362.78$$

$$t_1 = 20, v(t_1) = 517.35$$



# Linear Interpolation (contd)

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^1 \frac{t - t_j}{t_0 - t_j} = \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^1 \frac{t - t_j}{t_1 - t_j} = \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1} v(t_0) + \frac{t - t_0}{t_1 - t_0} v(t_1) = \frac{t - 20}{15 - 20} (362.78) + \frac{t - 15}{20 - 15} (517.35)$$

$$v(16) = \frac{16 - 20}{15 - 20} (362.78) + \frac{16 - 15}{20 - 15} (517.35)$$

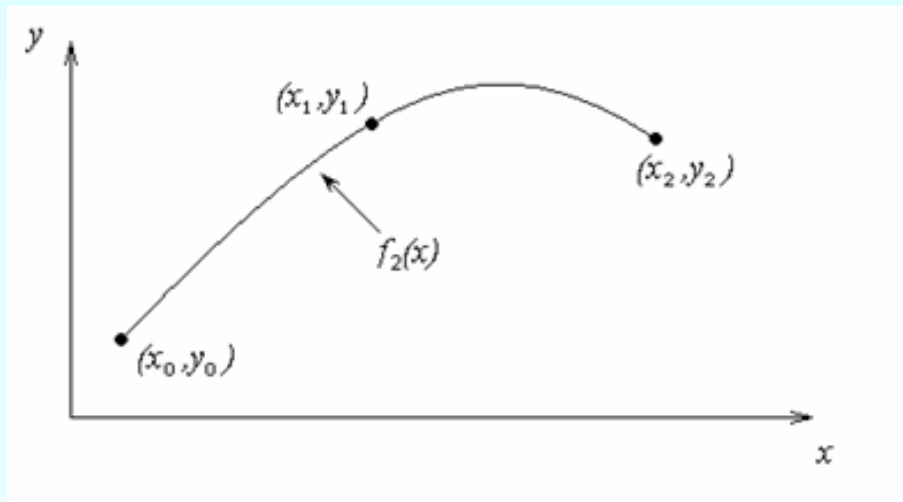
$$= 0.8(362.78) + 0.2(517.35)$$

$$= 393.7 \text{ m/s.}$$

# Quadratic Interpolation

For the second order polynomial interpolation (also called quadratic interpolation), we choose the velocity given by

$$\begin{aligned}v(t) &= \sum_{i=0}^2 L_i(t) v(t_i) \\ &= L_0(t) v(t_0) + L_1(t) v(t_1) + L_2(t) v(t_2)\end{aligned}$$



# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Lagrangian method for quadratic interpolation.

Table Velocity as a function of time

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

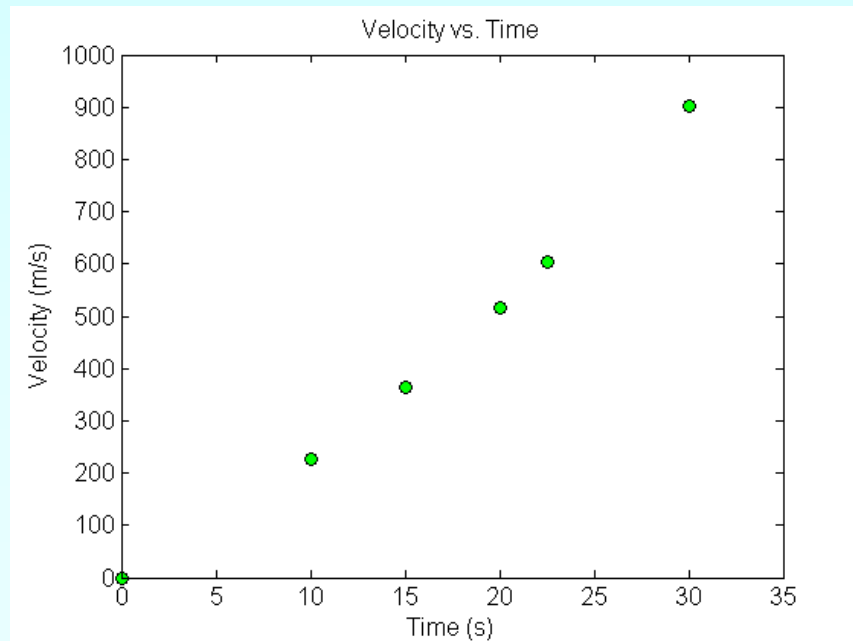


Figure. Velocity vs. time data for the rocket example

# Quadratic Interpolation (contd)

$$t_0 = 10, v(t_0) = 227.04$$

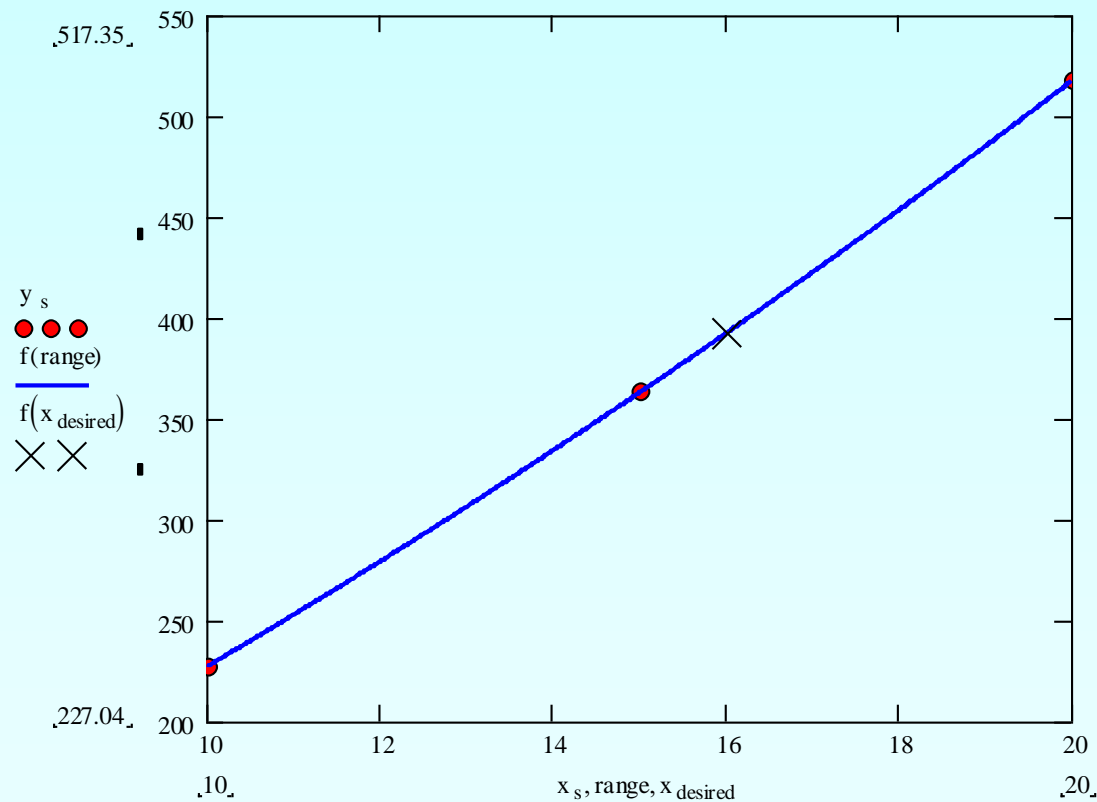
$$t_1 = 15, v(t_1) = 362.78$$

$$t_2 = 20, v(t_2) = 517.35$$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^2 \frac{t - t_j}{t_0 - t_j} = \left( \frac{t - t_1}{t_0 - t_1} \right) \left( \frac{t - t_2}{t_0 - t_2} \right)$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^2 \frac{t - t_j}{t_1 - t_j} = \left( \frac{t - t_0}{t_1 - t_0} \right) \left( \frac{t - t_2}{t_1 - t_2} \right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^2 \frac{t - t_j}{t_2 - t_j} = \left( \frac{t - t_0}{t_2 - t_0} \right) \left( \frac{t - t_1}{t_2 - t_1} \right)$$





## Quadratic Interpolation (contd)

$$\begin{aligned}v(t) &= \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)v(t_0) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)v(t_1) + \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)v(t_2) \\v(16) &= \left(\frac{16-15}{10-15}\right)\left(\frac{16-20}{10-20}\right)(227.04) + \left(\frac{16-10}{15-10}\right)\left(\frac{16-20}{15-20}\right)(362.78) + \left(\frac{16-10}{20-10}\right)\left(\frac{16-15}{20-15}\right)(517.35) \\&= (-0.08)(227.04) + (0.96)(362.78) + (0.12)(527.35) \\&= 392.19 \text{ m/s}\end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

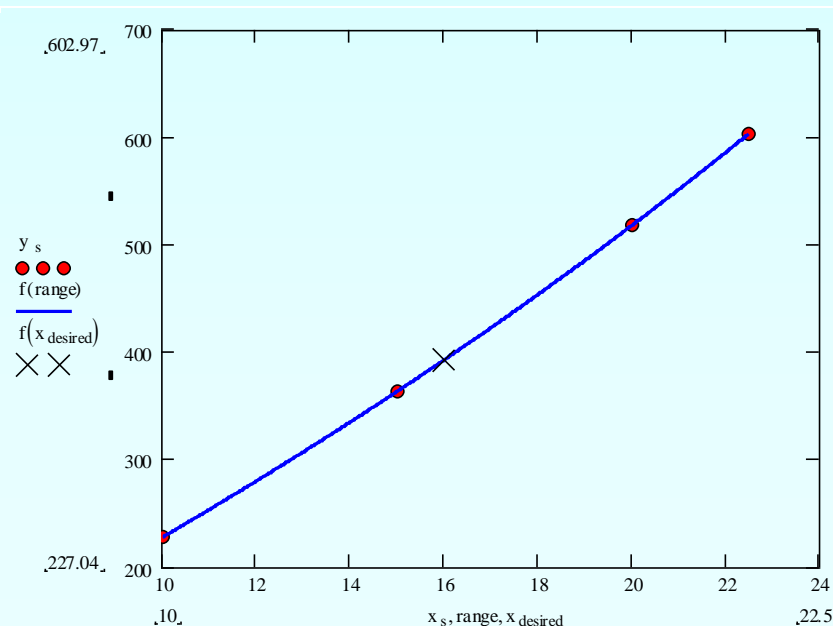
$$\begin{aligned}|\epsilon_a| &= \left| \frac{392.19 - 393.70}{392.19} \right| \times 100 \\&= 0.38410\%\end{aligned}$$

# Cubic Interpolation

For the third order polynomial (also called cubic interpolation), we choose the velocity given by

$$v(t) = \sum_{i=0}^3 L_i(t)v(t_i)$$

$$= L_0(t)v(t_0) + L_1(t)v(t_1) + L_2(t)v(t_2) + L_3(t)v(t_3)$$



# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Lagrangian method for cubic interpolation.

Table Velocity as a function of time

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

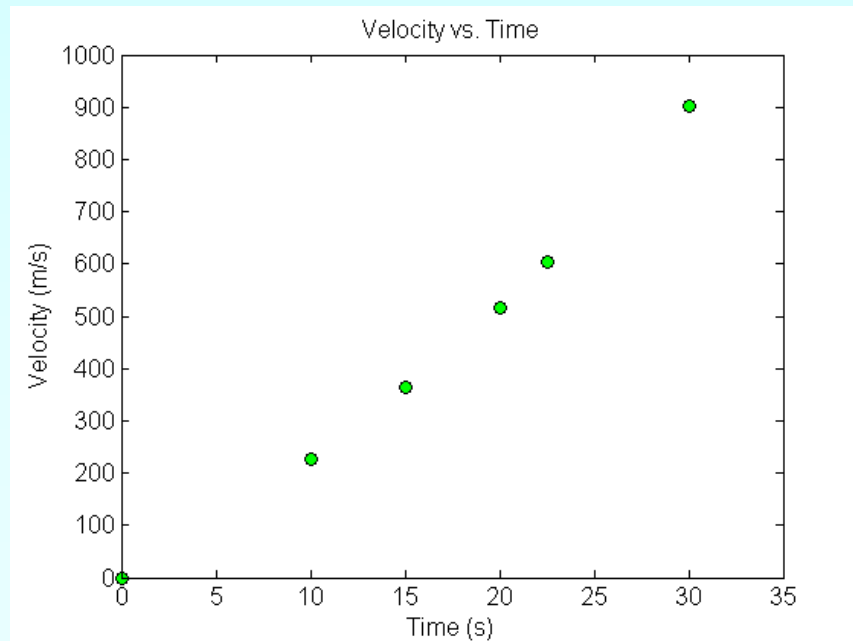


Figure. Velocity vs. time data for the rocket example

# Cubic Interpolation (contd)

$$t_0 = 10, v(t_0) = 227.04 \quad t_1 = 15, v(t_1) = 362.78$$

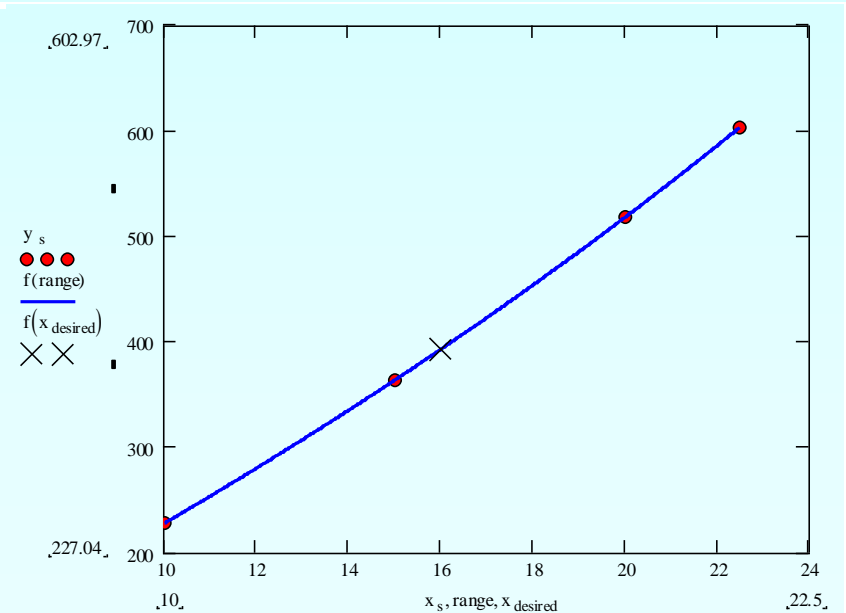
$$t_2 = 20, v(t_2) = 517.35 \quad t_3 = 22.5, v(t_3) = 602.97$$

$$L_0(t) = \prod_{\substack{j=0 \\ j \neq 0}}^3 \frac{t - t_j}{t_0 - t_j} = \left( \frac{t - t_1}{t_0 - t_1} \right) \left( \frac{t - t_2}{t_0 - t_2} \right) \left( \frac{t - t_3}{t_0 - t_3} \right);$$

$$L_1(t) = \prod_{\substack{j=0 \\ j \neq 1}}^3 \frac{t - t_j}{t_1 - t_j} = \left( \frac{t - t_0}{t_1 - t_0} \right) \left( \frac{t - t_2}{t_1 - t_2} \right) \left( \frac{t - t_3}{t_1 - t_3} \right)$$

$$L_2(t) = \prod_{\substack{j=0 \\ j \neq 2}}^3 \frac{t - t_j}{t_2 - t_j} = \left( \frac{t - t_0}{t_2 - t_0} \right) \left( \frac{t - t_1}{t_2 - t_1} \right) \left( \frac{t - t_3}{t_2 - t_3} \right);$$

$$L_3(t) = \prod_{\substack{j=0 \\ j \neq 3}}^3 \frac{t - t_j}{t_3 - t_j} = \left( \frac{t - t_0}{t_3 - t_0} \right) \left( \frac{t - t_1}{t_3 - t_1} \right) \left( \frac{t - t_2}{t_3 - t_2} \right)$$



# Cubic Interpolation (contd)

$$\begin{aligned}v(t) &= \left(\frac{t-t_1}{t_0-t_1}\right)\left(\frac{t-t_2}{t_0-t_2}\right)\left(\frac{t-t_3}{t_0-t_3}\right)v(t_1) + \left(\frac{t-t_0}{t_1-t_0}\right)\left(\frac{t-t_2}{t_1-t_2}\right)\left(\frac{t-t_3}{t_1-t_3}\right)v(t_2) \\&+ \left(\frac{t-t_0}{t_2-t_0}\right)\left(\frac{t-t_1}{t_2-t_1}\right)\left(\frac{t-t_3}{t_2-t_3}\right)v(t_2) + \left(\frac{t-t_1}{t_3-t_1}\right)\left(\frac{t-t_0}{t_3-t_0}\right)\left(\frac{t-t_2}{t_3-t_2}\right)v(t_3) \\v(16) &= \left(\frac{16-15}{10-15}\right)\left(\frac{16-20}{10-20}\right)\left(\frac{16-22.5}{10-22.5}\right)(227.04) + \left(\frac{16-10}{15-10}\right)\left(\frac{16-20}{15-20}\right)\left(\frac{16-22.5}{15-22.5}\right)(362.78) \\&+ \left(\frac{16-10}{20-10}\right)\left(\frac{16-15}{20-15}\right)\left(\frac{16-22.5}{20-22.5}\right)(517.35) + \left(\frac{16-10}{22.5-10}\right)\left(\frac{16-15}{22.5-15}\right)\left(\frac{16-20}{22.5-20}\right)(602.97) \\&= (-0.0416)(227.04) + (0.832)(362.78) + (0.312)(517.35) + (-0.1024)(602.97) \\&= 392.06 \text{ m/s}\end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first and second order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{392.06 - 392.19}{392.06} \right| \times 100 \\&= 0.033269\%\end{aligned}$$

# Comparison Table

<b>Order of Polynomial</b>	<b>1</b>	<b>2</b>	<b>3</b>
$v(t=16)$ m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	-----	0.38410%	0.033269%

# Distance from Velocity Profile

Find the distance covered by the rocket from  $t=11$ s to  $t=16$ s ?

$$v(t) = (t^3 - 57.5t^2 + 1087.5t - 6750)(-0.36326) + (t^3 - 52.5t^2 + 875t - 4500)(1.9348) \\ + (t^3 - 47.5t^2 + 712.5t - 3375)(-4.1388) + (t^3 - 45t^2 + 650t - 3000)(2.5727)$$

$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3, \quad 10 \leq t \leq 22.5$$

$$s(16) - s(11) = \int_{11}^{16} v(t) dt$$

$$\approx \int_{11}^{16} (-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3) dt$$

$$= \left[ -4.245t + 21.265 \frac{t^2}{2} + 0.13195 \frac{t^3}{3} + 0.00544 \frac{t^4}{4} \right]_{11}^{16}$$

$$= 1605 \text{ m}$$

# Acceleration from Velocity Profile

Find the acceleration of the rocket at  $t=16s$  given that

$$v(t) = -4.245 + 21.265t + 0.13195t^2 + 0.00544t^3, \quad 10 \leq t \leq 22.5$$

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}(-4.245 + 21.265t + 0.13195t^2 + 0.00544t^3)$$

$$= 21.265 + 0.26390t + 0.01632t^2$$

$$a(16) = 21.265 + 0.26390(16) + 0.01632(16)^2$$

$$= 29.665 \text{ m/s}^2$$



# Newton's Divided Difference Method of Interpolation

# Newton's Divided Difference Method

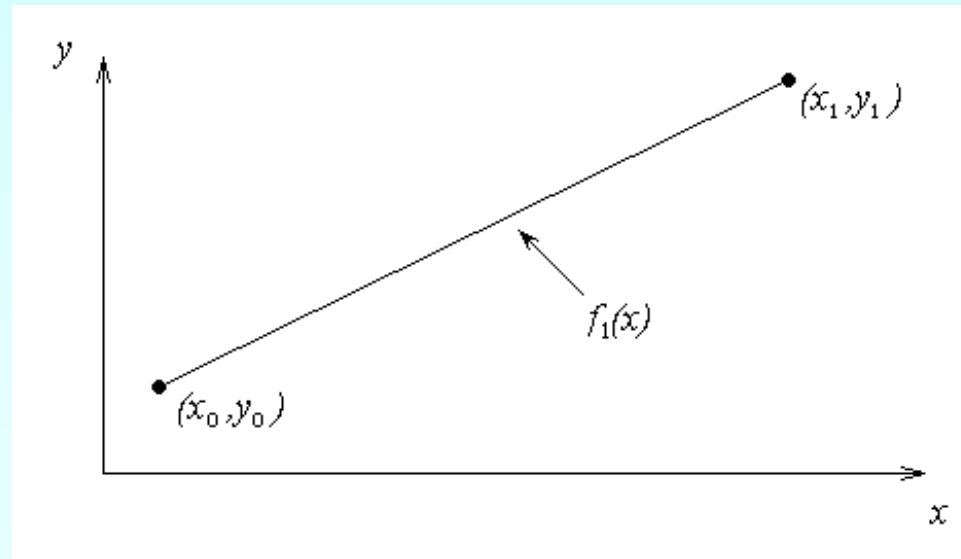
Linear interpolation: Given  $(x_0, y_0)$ ,  $(x_1, y_1)$ , pass a linear interpolant through the data

$$f_1(x) = b_0 + b_1(x - x_0)$$

where

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$



# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Newton Divided Difference method for linear interpolation.

Table. Velocity as a function of time

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

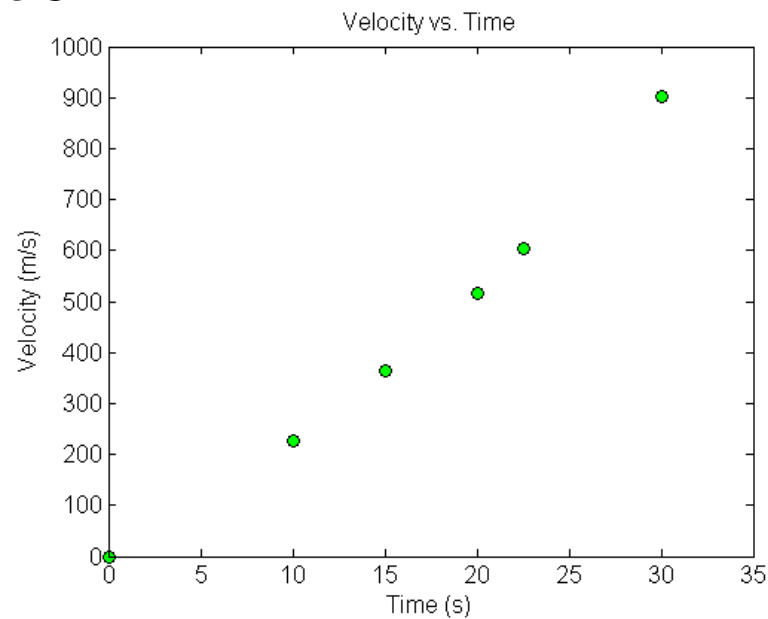


Figure. Velocity vs. time data for the rocket example



# Linear Interpolation

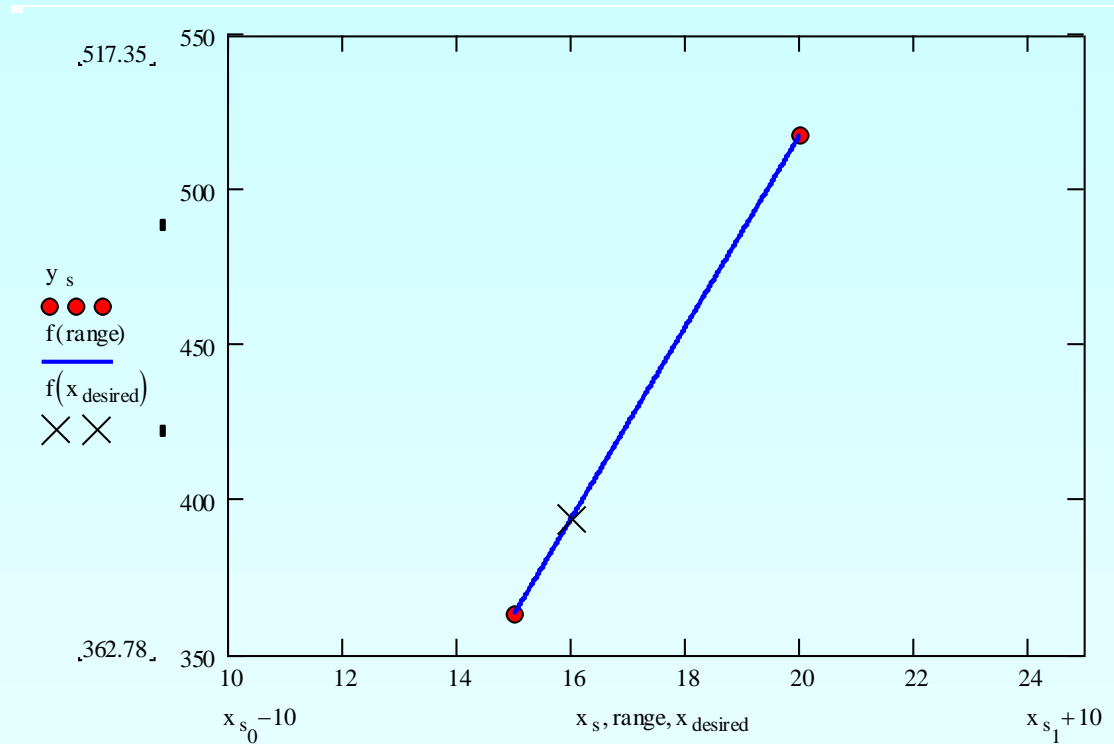
$$v(t) = b_0 + b_1(t - t_0)$$

$$t_0 = 15, v(t_0) = 362.78$$

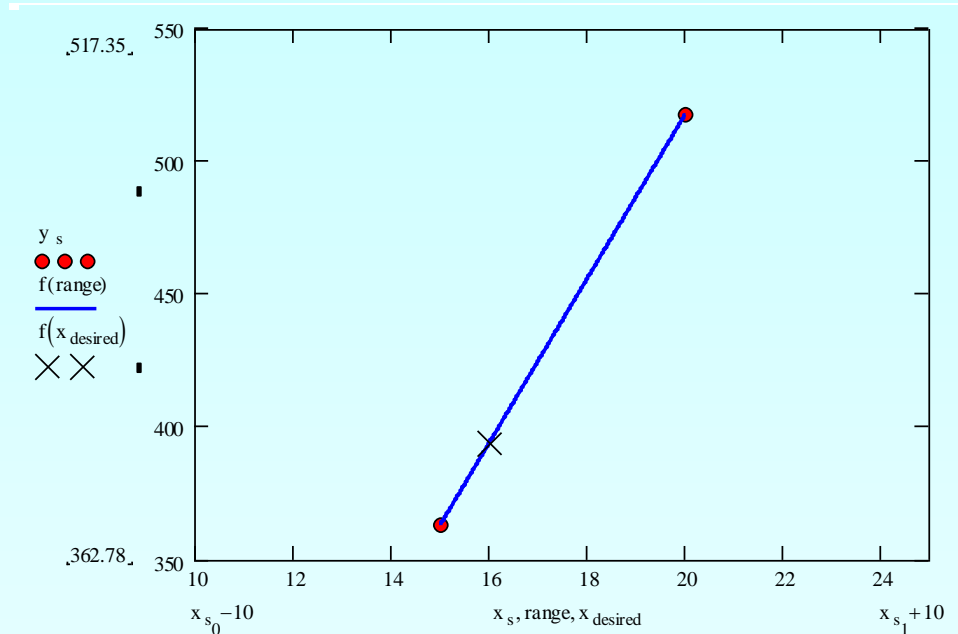
$$t_1 = 20, v(t_1) = 517.35$$

$$b_0 = v(t_0) = 362.78$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = 30.914$$



# Linear Interpolation (contd)



$$v(t) = b_0 + b_1(t - t_0)$$

$$= 362.78 + 30.914(t - 15), 15 \leq t \leq 20$$

At  $t = 16$

$$v(16) = 362.78 + 30.914(16 - 15)$$

$$= 393.69 \text{ m/s}$$

# Quadratic Interpolation

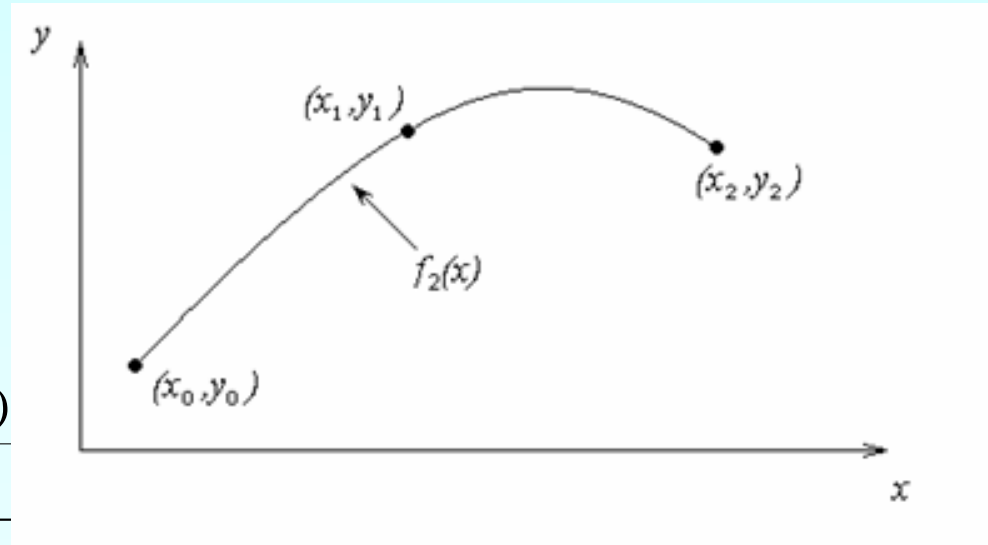
Given  $(x_0, y_0)$ ,  $(x_1, y_1)$ , and  $(x_2, y_2)$ , fit a quadratic interpolant through the data.

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$b_0 = f(x_0)$$

$$b_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$



# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Newton Divided Difference method for quadratic interpolation.

Table. Velocity as a function of time

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

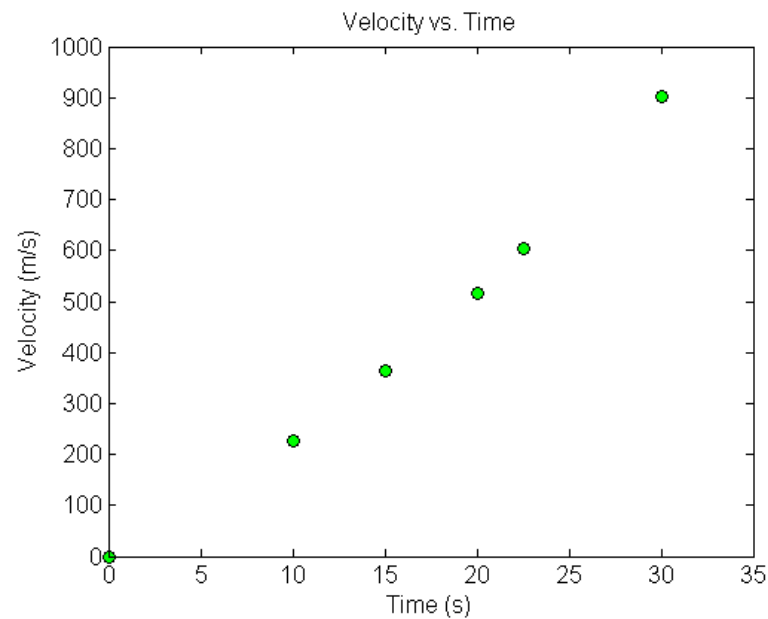
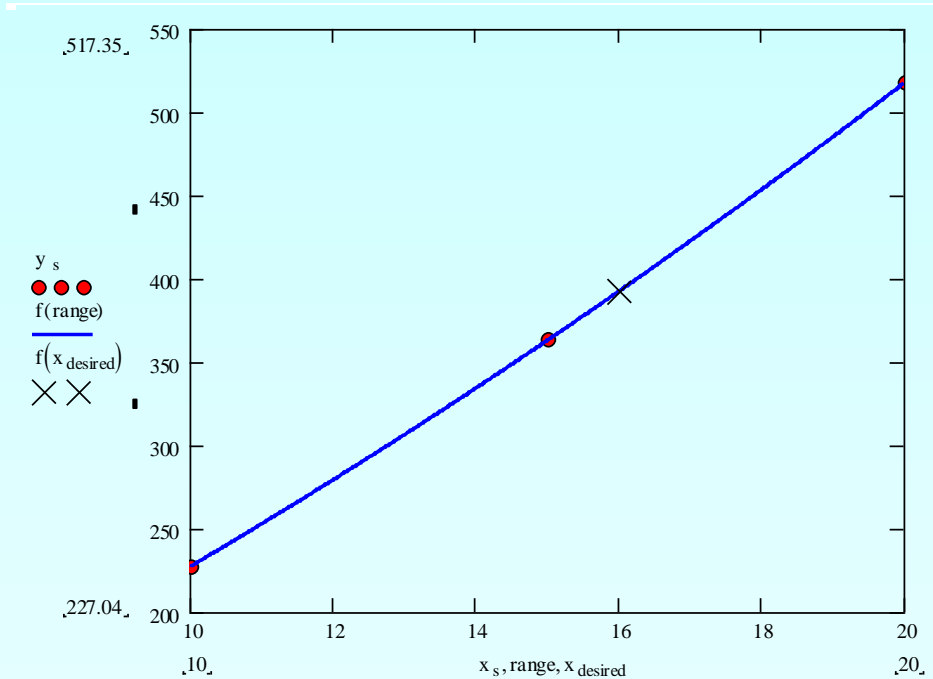


Figure. Velocity vs. time data for the rocket example

# Quadratic Interpolation (contd)



$$t_0 = 10, v(t_0) = 227.04$$

$$t_1 = 15, v(t_1) = 362.78$$

$$t_2 = 20, v(t_2) = 517.35$$



# Quadratic Interpolation (contd)

$$b_0 = v(t_0)$$

$$= 227.04$$

$$b_1 = \frac{v(t_1) - v(t_0)}{t_1 - t_0} = \frac{362.78 - 227.04}{15 - 10}$$

$$= 27.148$$

$$b_2 = \frac{\frac{v(t_2) - v(t_1)}{t_2 - t_1} - \frac{v(t_1) - v(t_0)}{t_1 - t_0}}{t_2 - t_0} = \frac{\frac{517.35 - 362.78}{20 - 15} - \frac{362.78 - 227.04}{15 - 10}}{20 - 10}$$

$$= \frac{30.914 - 27.148}{10}$$

$$= 0.37660$$

# Quadratic Interpolation (contd)

$$\begin{aligned}v(t) &= b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) \\&= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15), \quad 10 \leq t \leq 20\end{aligned}$$

At  $t = 16$ ,

$$v(16) = 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) = 392.19 \text{ m/s}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained between the results from the first order and second order polynomial is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{392.19 - 393.69}{392.19} \right| \times 100 \\&= 0.38502 \%\end{aligned}$$

# General Form

$$f_2(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

where

$$b_0 = f[x_0] = f(x_0)$$

$$b_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$b_2 = f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0} = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

Rewriting

$$f_2(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1)$$

# General Form

Given  $(n + 1)$  data points,  $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1}), (x_n, y_n)$  as

$$f_n(x) = b_0 + b_1(x - x_0) + \dots + b_n(x - x_0)(x - x_1)\dots(x - x_{n-1})$$

where

$$b_0 = f[x_0]$$

$$b_1 = f[x_1, x_0]$$

$$b_2 = f[x_2, x_1, x_0]$$

$$\vdots$$

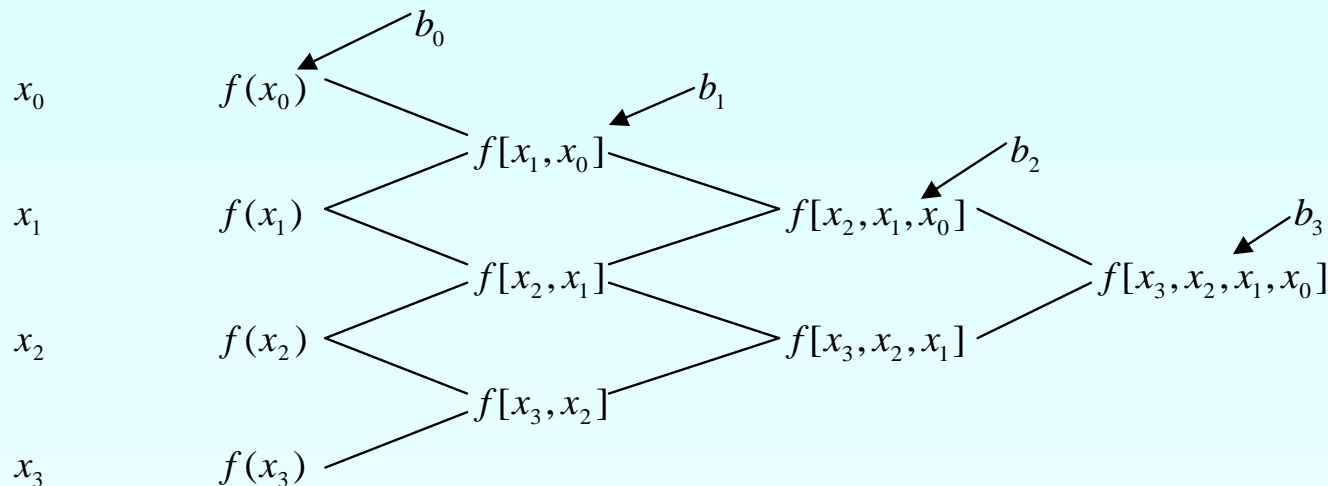
$$b_{n-1} = f[x_{n-1}, x_{n-2}, \dots, x_0]$$

$$b_n = f[x_n, x_{n-1}, \dots, x_0]$$

# General form

The third order polynomial, given  $(x_0, y_0)$ ,  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ , is

$$f_3(x) = f[x_0] + f[x_1, x_0](x - x_0) + f[x_2, x_1, x_0](x - x_0)(x - x_1) + f[x_3, x_2, x_1, x_0](x - x_0)(x - x_1)(x - x_2)$$



# Example

The upward velocity of a rocket is given as a function of time in Table 1. Find the velocity at  $t=16$  seconds using the Newton Divided Difference method for cubic interpolation.

Table. Velocity as a function of time

$t$ (s)	$v(t)$ (m/s)
0	0
10	227.04
15	362.78
20	517.35
22.5	602.97
30	901.67

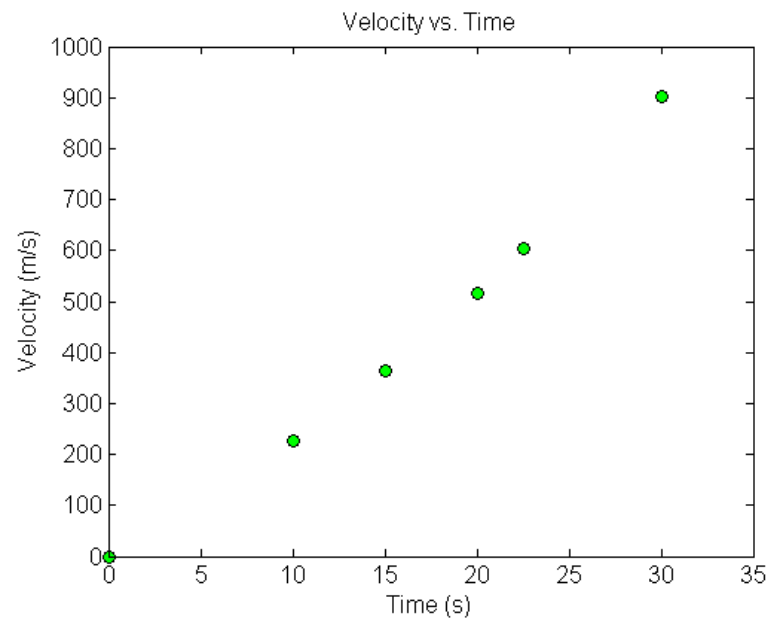


Figure. Velocity vs. time data  
for the rocket example

# Example

The velocity profile is chosen as

$$v(t) = b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2)$$

we need to choose four data points that are closest to  $t = 16$

$$t_0 = 10, \quad v(t_0) = 227.04$$

$$t_1 = 15, \quad v(t_1) = 362.78$$

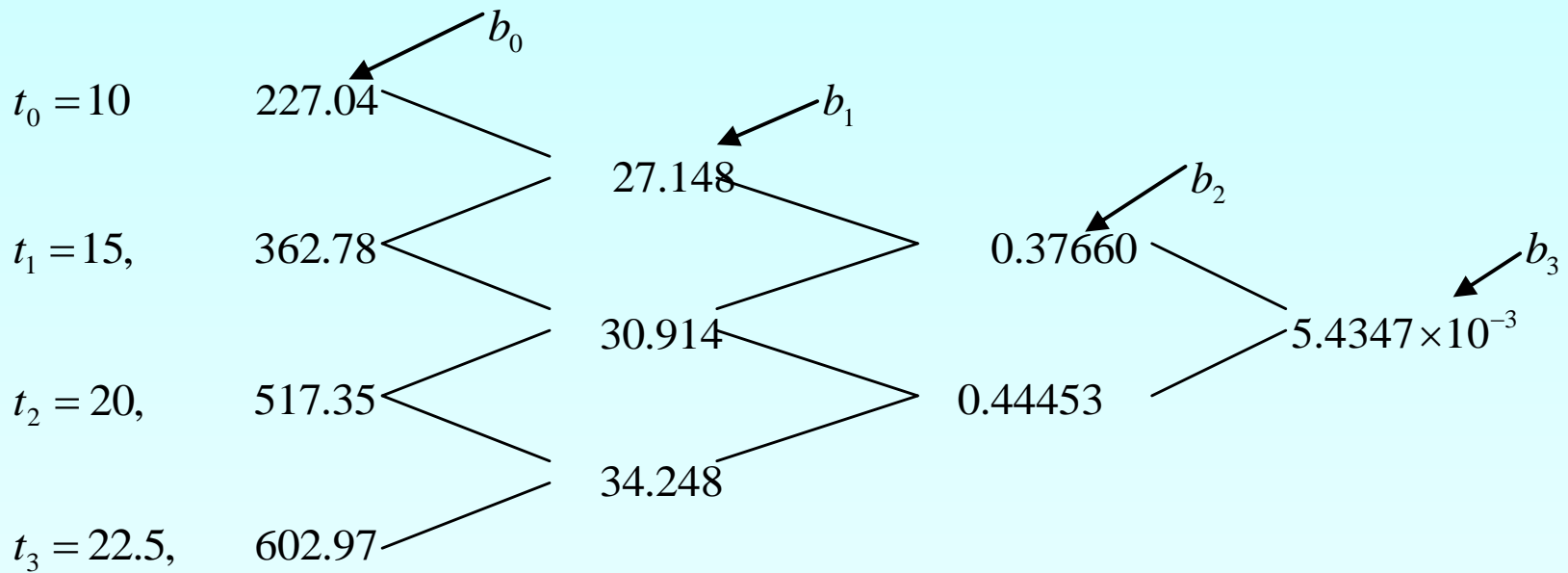
$$t_2 = 20, \quad v(t_2) = 517.35$$

$$t_3 = 22.5, \quad v(t_3) = 602.97$$

The values of the constants are found as:

$$b_0 = 227.04; \quad b_1 = 27.148; \quad b_2 = 0.37660; \quad b_3 = 5.4347 \times 10^{-3}$$

# Example



$$b_0 = 227.04; \quad b_1 = 27.148; \quad b_2 = 0.37660; \quad b_3 = 5.4347 \times 10^{-3}$$



# Example

Hence

$$\begin{aligned}v(t) &= b_0 + b_1(t - t_0) + b_2(t - t_0)(t - t_1) + b_3(t - t_0)(t - t_1)(t - t_2) \\&= 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15) \\&\quad + 5.4347 * 10^{-3}(t - 10)(t - 15)(t - 20)\end{aligned}$$

At  $t = 16$ ,

$$\begin{aligned}v(16) &= 227.04 + 27.148(16 - 10) + 0.37660(16 - 10)(16 - 15) \\&\quad + 5.4347 * 10^{-3}(16 - 10)(16 - 15)(16 - 20) \\&= 392.06 \text{ m/s}\end{aligned}$$

The absolute relative approximate error  $|\epsilon_a|$  obtained is

$$\begin{aligned}|\epsilon_a| &= \left| \frac{392.06 - 392.19}{392.06} \right| \times 100 \\&= 0.033427 \%\end{aligned}$$

# Comparison Table

Order of Polynomial	1	2	3
$v(t=16)$ m/s	393.69	392.19	392.06
Absolute Relative Approximate Error	-----	0.38502 %	0.033427 %

# Distance from Velocity Profile

Find the distance covered by the rocket from  $t=11\text{s}$  to  $t=16\text{s}$  ?

$$v(t) = 227.04 + 27.148(t - 10) + 0.37660(t - 10)(t - 15) + 5.4347 * 10^{-3} (t - 10)(t - 15)(t - 20) \quad 10 \leq t \leq 22.5$$

$$= -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3 \quad 10 \leq t \leq 22.5$$

So

$$\begin{aligned} s(16) - s(11) &= \int_{11}^{16} v(t) dt \\ &= \int_{11}^{16} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3) dt \\ &= \left[ -4.2541t + 21.265 \frac{t^2}{2} + 0.13204 \frac{t^3}{3} + 0.0054347 \frac{t^4}{4} \right]_{11}^{16} \\ &= 1605 \text{ m} \end{aligned}$$

# Acceleration from Velocity Profile

Find the acceleration of the rocket at  $t=16s$  given that

$$v(t) = -4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3$$

$$a(t) = \frac{d}{dt} v(t) = \frac{d}{dt} (-4.2541 + 21.265t + 0.13204t^2 + 0.0054347t^3)$$

$$= 21.265 + 0.26408t + 0.016304t^2$$

$$a(16) = 21.265 + 0.26408(16) + 0.016304(16)^2$$

$$= 29.664 \text{ m/s}^2$$