Spectral Rigidity and Mirror Congruence in the Invariant Moduli Subspace of the Schoen Calabi-Yau Threefold

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Abstract

This research note examines the Kähler metrics on the $\mathbb{Z}_3$-invariant subspace ($h^{1,1}_{\text{inv}} = 3$) of the Schoen Calabi-Yau threefold.

Using computation of the Hessian of the Kähler potential $K = -\ln \mathcal{V}$ from the restricted intersection polynomial reveals a symmetric point within metric eigenvalue. This symmetrical point is the exact result of the topological intersection number, with the spectral discriminant being $297 = 9 \times 33$. The irrationality $\sqrt{33}$ is intrinsically topological, expressible as $\sqrt{33} = \sqrt{h^{1,1}_{\text{inv}} \times (\mathcal{V} - 1)}$ under the normalization $\mathcal{V} = 12$.

This note also identifies an exact numerical congruence between the volume of $\mathcal{V} = 12$ and a mirror-side Weil-Petersson norm at the corresponding $\mathbb{Z}_3$-symmetric point, this is directly computed from the period lattice of the mirror elliptic ($y^2 = x^3 + 1$).

1 Volume and Kähler Metrics

The Schoen Calabi-Yau Threefold (Schoen 1988) presents a 3-dimensional $\mathbb{Z}_3$-invariant of Kähler moduli space which will be the foundation of this work. This particular manifold has the intersection numbers $\kappa_{111} = 9$, $\kappa_{122} = \kappa_{133} = 3$ (Hosono–Saito–Stienstra 1997).

The manifold is determined by its 3 intersection points, or volume function. Using this we are able to derive a mathematical representation of how the surfaces of the manifold intersect to create volume. (Otherwise known as a triple-intersection volume in the large-volume limit)

The specific $\mathbb{Z}_3$-symmetric subspace expresses volume as a relationship between three different variables. Here is the formula as shown in Kähler coordinates:

$$\mathcal{V}(t^1, t^2, t^3) = 9t^1t^2t^3 + \frac{3}{2}t^1(t^2)^2 + \frac{3}{2}t^1(t^3)^2. \tag{1}$$

This formula assumes a 'large volume' limit as it calculates the rate at which space curves irrespective of the size of the manifold, which does not affect the intrinsic geometry nor the algebra.

Where the Kähler potential shows $K = -\ln \mathcal{V}$, and the metric $G_{ab} = \partial_a \partial_b K$ (Candelas–de la Ossa 1990).

To find the manifold's most natural state this paper chooses the point where it is most symmetrical in 3 dimensions: $t^a = (1, 1, 1)$. This was chosen as all internal dimensions are equal, meaning that no direction is inherently weighted. In this state, the volume of the manifold is exactly 12. And where $\mathcal{V} = 12$ the metric matrix is:

$$G = \begin{pmatrix} 9/16 & 3/32 & 3/32 \\ 3/32 & 65/64 & 1/64 \\ 3/32 & 1/64 & 65/64 \end{pmatrix},$$

This allows us to derive two key signatures.

1. The Trace as $\mathrm{Tr}(G) = 83/32$, which represents the total 'stiffness' of the moduli against deformations (Grimm & Louis 2005).
2. The Determinant as $\det(G) = 9/16$, which represents how much space is curved relative to a flat surface.

The Trace and Determinant allow us to more easily understand the nature of the manifold without a complex matrix.

2 Spectral Properties and Characteristic Polynomial

Using the trace and determinant, the characteristic polynomial is expressed as:

$$(x-1)(32x^2 - 51x + 18) = 0, \tag{2}$$

This equation yields three eigenvalues, $\lambda_2 = 1$. With $\lambda_2 = 1$ being our geometrical anchor that won't change even if other values do, it becomes a solid base to build from.

To close the system topologically we traced the quadratic roots from coefficients allowing us to directly input the data in accordance to Schoen's Threefold intersections:

- $32 = 4(\kappa_{111} - 1)$,
- $51 = 6\kappa_{111} - \kappa_{122}$,
- $18 = 2\kappa_{111}$.

Using the values 32, 51 and 18 as the foundation of our algebra, we are now able to calculate the discriminant (Δ).

$$\Delta = 51^2 - 4 \cdot 32 \cdot 18 = 297 = 9 \times 33 = \kappa_{111} \times 33.$$

The result of 297 allows us to correctly derive and link this.

$$\lambda_{1,3} = \frac{51 \pm 3\sqrt{33}}{64}, \quad \lambda_2 = 1.$$

3 The Geometry of 33

The appearance of 33 allows us to uncover an irrational curvature from whole integers. A number that is not fitted, but emerges entirely from the topology of the Schoen three-fold.

$$33 = h^{1,1}_{\mathrm{inv}} \times (\mathcal{V} - 1) = 3 \times 11, \tag{3}$$

This also links back to our 297, which is 9×33. The emergence of 9, our topological constant reveals internal consistency.

4 The Weil-Petersson Congruence Mirror

This culminates into our final discovery, an exact congruence.

On the mirror side, Schoen is depicted by an elliptical fibre. In its most balanced state the $t^a = (1,1,1)$ or $(j = 0)$ we can describe the curve as the simple equation of $y^2 = x^3 + 1$.

We used SageMath to analyse the period lattice and calculated the Weil-Petersson norm. This allowed us to measure the curvature of the mirror side geometry. The results of this code gave us exactly:

$$\|\partial_\tau\|^2_{\mathrm{WP}} = 12$$

Here we derived a result that matches our Kähler volume $\mathcal{V} = 12$. This suggests that under natural normalisations, the two sides become directly comparable. We compute this using the Weil-Petersson norm for the elliptic fiber modulus, accounting for the hyperbolic metric on moduli space (Wolpert 2008).

$$\|\partial_\tau\|^2_{\mathrm{WP}} = \frac{1}{(\Im\tau)^2}$$

Computing this from the period lattice of $y^2 = x^3 + 1$ yields $\Im(\tau) = \sqrt{3}/6$ exactly, giving $\|\partial_\tau\|^2_{\mathrm{WP}} = 12$. Further testing along (t,t,t) shows that $\mathcal{V}$ scales as t^3 while the mirror curve remains the same at $j = 0$ with $g = 12$ being constant. This confirms the congruence is unique to $t = (1,1,1)$.

Using further SageMath, we found that this specific point on $\mathbb{Z}_3$ suggests a natural calibration point. Whether similar calibration points exist for other mirror pairs, or if this reflects a special feature of the Schoen manifold's structure, remains to be explored.

Now, if you use different co-ordinate systems such as $\mathrm{SL}(2,\mathbb{Z})$ different results can appear. However, the period lattice used by SageMath is accounting for the total area of the internal fibre, and because we chose the most balanced and symmetrical state on the Kähler side the mirror side responds at its own point of maximum symmetry ($j = 0$) creating equilibrium.

The nature of the Kähler and mirror side reacts this way as they are both representing the same amount of information with two different languages. This suggests that it's not a coincidence, but instead the work of the period lattice, which acts to filter out other co-ordinate choices, leaving us with only one that is entirely internally consistent.

5 Conclusion

This note shows that the Schoen three-fold has intrinsic properties locked into its structure. Through using data and deriving from (Hosono 1997) we have demonstrated that the manifold's stiffness is entirely dictated and fixed from its topology.

The $12 = 12$ match between volume and mirror curvature provides a strong piece of evidence of a fixed scale that shows the manifold's mirror precisely. This note demonstrates that every value was necessary to reach this logical conclusion, with no free parameters.

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