Poisson processes

Statistical Sciences, UCL

Foundation Fortnight, September 2020

What are Poisson Processes?

- Poisson processes are a fundamental example of a Markov process that are used to model a variety of different phenomena and serve as the basis of more complicated processes.
- Let $N(t)$ be the total number customers that have visited a high street shop up to an arbitrary time t. Poisson processes are often used to model the random process $N(t)$.
- In this context, the actual arrival times of the customers form a Poisson point process.

Preliminaries

- We say that X is Poisson random variable with mean μ if it has probability mass function given by

$$\mathbb{P}(X = n) = \frac{e^{-\mu}\mu^n}{n!} \text{ for } n = 0, 1, 2, \ldots$$

- We say that Z is an exponential random variable with mean μ (rate $\lambda = 1/\mu$) if it is continuous random variable with probability density function given by

$$z \mapsto \tfrac{1}{\mu} e^{-\frac{z}{\mu}} \text{ for } z \geq 0.$$

Exercises: Poisson and exponential random variables

Exercise

Check that the mean of a Poisson random variable with parameter μ does indeed have mean μ. Show that the variance is also μ.

Exercise

Show that the minimal of two independent exponential random variables is again an exponential random variable.

Exercise

What is the memoryless property? Prove that exponential random variables have this property.

Exercises continued

Exercise

Let $n > 0$ and $\mu > 0$. Let W be a Poisson random variable with mean μ. Let $(X_i)_{i=1}^n$ be independent Bernoulli random variables with parameter $p = \lambda/n$. Let $S_n = X_1 + \cdots + X_n$. Show for all nonnegative integers k, we have $\lim_{n\to\infty} \mathbb{P}(S_n = k) = \mathbb{P}(W = k)$.

Exercise

Let $\mu > 0$, $p \in (0,1)$, and W be a Poisson random variable with mean μ. Define the random variable X in the following way: if $W = n$, then we let X be the sum of n independent Bernoulli random variables with parameter p. Show that X is a Poisson random variable with mean $p\lambda$.

Exercise

Show that the sum of independent Poisson random variables is again a Poisson random variable.

Generating a Poisson process as exponential inter-arrivals

There are many way to define a Poisson process. We will start with the following construction which is also useful for simulations.

- Let $(X_i)_{i=1}^{\infty}$ be a sequence of independent exponential random variables all with mean $\mu = 1/\lambda$. (Inter-arrival times)
- Let $T_1 = X_1$. (Time of first arrival)
- Let $T_{n+1} = X_{n+1} + T_n = \sum_{i=1}^{n+1} X_i$ (Arrival times)
- Set $N(t) = \sum_{n=1}^{\infty} \mathbf{1}[T_n \leq t]$ (Total number of arrivals by time t).
- Then we say that N is a **Poisson process on** $[0, \infty)$ **with rate** λ.
- The set of random arrival times $\{T_n : n \in \mathbb{Z}^+\}$ is often referred to as a **Poisson point process on** $[0, \infty)$ **of intensity** λ.

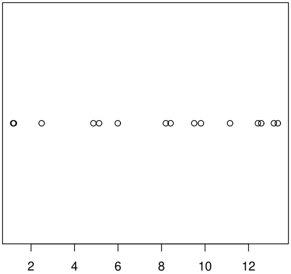
Sample realizations

How would you simulate a Poisson process in R?

- Call k independent exponential random variables of rate $\lambda > 0$ (inter-arrival times).
- Compute the partial sums (arrival times).
- To visualize: plot arrival times to see the resulting Poisson point process

R Code: exponential inter-arrivals

```
inter = rexp(15, 1)
arr = cumsum(inter)
one = rep(1, times = length(arr))
plot(arr, one, yaxt = 'n', ann=FALSE)
```



Experimenting

- Simulate Poisson processes with different intensities and number of arrivals
- Are the points clumped sometimes are evenly spaced?
- Simulate say 50 arrivals with intensity 1. Does it appears as though the arrival time of the last point is always around 50? Explain.
- If someone gave you a (single) sample realization of a Poisson process on a large interval, how would you estimate the intensity as unknown parameter?
- If you are using a Poisson process of intensity λ to model the arrival of customers, what should the *units* be?

A mathematical approach

Let N be a Poisson process on $[0, \infty)$ with rate λ. It is possible to show that:

- For every $t > s > 0$, $N(t) - N(s)$ is a Poisson random variable with mean $\lambda(t - s)$ (Stationary increments distributed as a Poisson)
- For any finite collection of disjoint intervals $(s_1, t_1), \ldots, (s_n, t_n)$ the random variables $N(t_1) - N(s_1), \ldots, N(t_n) - N(s_n)$ are independent.

These two properties also characterize a Poisson process and can be used to define spatial Poisson processes in higher dimensions.
We will prove that our construction using exponential random variables satisfies a weak version of the first property:

Lemma

Let N be a Poisson process constructed with exponential inter-arrival times of rate $\lambda > 0$. Then $N(t)$ is a Poisson random variable with mean $t\lambda$.

Proof of Lemma

Let N be given by exponential inter-arrival times as in Slide 6, so that X_i are the inter-arrival times, and T_k is the time of the k arrival. Observe that

$$\mathbb{P}(N(t) = 0) = \mathbb{P}(X_1 > t) = e^{-\lambda t}.$$

For $k > 0$, we have

$$\begin{aligned}
\mathbb{P}(N(t) = k) &= \mathbb{P}(T_k \leq t, T_{k+1} > t) \\
&= \mathbb{P}(T_k \leq t, T_k + X_{k+1} > t) \\
&= \int_0^t \int_{t-y}^{\infty} g(y) f(x) dx dy,
\end{aligned}$$

where g is the pdf for T_k and f is the pdf for X_{k+1}; here we need to appeal the independence of these two random variables.

Proof continued

Thus

$$\mathbb{P}(N(t) = k) = \int_0^t g(y) e^{-(t-y)\lambda} dy$$

We recall that the sum of independent exponentials has the law of a gamma distribution; specifically

$$g(y) = \frac{\lambda^k}{\Gamma(k)} y^{k-1} e^{-\lambda y}.$$

Thus

$$\begin{aligned}
\mathbb{P}(N(t) = k) &= \frac{\lambda^k e^{-t\lambda}}{\Gamma(k)} \int_0^t y^{k-1} dy \\
&= \frac{\lambda^k e^{-t\lambda} t^k}{k\Gamma(k)} \\
&= \frac{(\lambda t)^k e^{-t\lambda}}{k!},
\end{aligned}$$

as promised.

Exercises

Exercise

Suppose we model the number of customers that arrive at a high street shop on at particular day by a Poisson process of intensity $\lambda > 0$*, where* λ *is measured in customers per hour. We wish to estimate* λ*. The shop keeper has records of how many customers arrive each day for n days given by* $x = (x_1, \ldots, x_n)$ *and opens everyday for* 6 *hours. Find an estimate for* λ*. Carefully justify why this is a reasonable estimate.*

Exercise

Suppose the shop keeper computes the mean the variance for her data and finds that the variance is much smaller than the mean. Would you re-evaluate whether a Poisson process is a good model? Explain.

Exercises continued

Exercise

Suppose the shop is really high-end and on some days has no customers. The shop keeper only keeps track of whether she had has any customers are not; that is, her records $x = (x_1, \ldots, x_n)$ *are a binary sequence. Can you still estimate* λ*?*

Exercise

Suppose that arrivals to a shop are modeled by a Poisson process. Suppose that you are told that there is exactly one arrival in the time interval $[0, 1]$*; let* U *be the time of this arrival. What is distribution of* U*?*

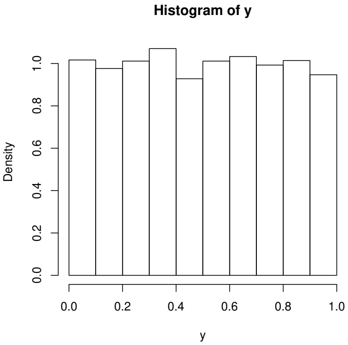
R code: a uniform random variable

We demonstrate the last exercise by simulations in the following way. We simulate Poisson processes, and record the position of the first arrival if it occurs in $[0, 1]$ and the second arrival occurs outside this interval. We plot a histogram of the recorded occurrences

```
record <- function(){
inter = rexp(2,1)
arr = cumsum(inter)
r = 2
if ( (arr[1] < 1) & (arr[2] >1)){
r <- arr[1]
}
r
}
```

R code continued

```
x = replicate(10000, record())
y = setdiff(x, 2)
hist(y, prob=T)
```



A characterization via modelling assumptions

Poisson processes are good models for arrivals only if it is reasonable to assume that one also has the memoryless property that comes with the exponential distribution. In addition, if an arrival process $N(t)$ satisfies the following mild conditions, then it can be shown that it is a Poisson process of intensity λ.

- (Stationarity) The number of arrives in an interval of time depends only on the length of time.) For all $t > s$ we have that $N(t) - N(s)$ has the same distribution as $N(t-s)$.
  (Independent increments) For any finite collection of disjoint intervals $(s_1, t_1), \ldots, (s_n, t_n)$ the random variables $N(t_1) - N(s_1), \ldots, N(t_n) - N(s_n)$ are independent.

Modelling continued

- (Orderliness: two customers do not arrive at the same time)

$$\lim_{h \to 0} \frac{1}{h} \mathbb{P}(N(h) \geq 2) = 0.$$

- (Rate: In a small interval time, the probability that a customer arrives is proportional to λ.)

$$\lim_{h \to 0} \frac{1}{h} \mathbb{P}(N(h) = 1) = \lambda > 0.$$

Modelling continued

We can sketch a proof of our previous lemma which connected Poisson random variables with Poisson processes.

Let $t > 0$, and partition the interval $[0, t]$ into n intervals of size t/n, where n is large. By orderliness condition with stationarity, we can assume that in each interval there is at most one arrival. Let $p = \mathbb{P}(N(t/n) = 1))$. By the independent increments and rate conditions, we have that probability that there are k arrives is given by

$$\mathbb{P}(N(t) = k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n!}{(n-k)!k!} \frac{(\lambda t)^k}{n^k} \left(1 - \frac{\lambda t}{n}\right)^{n-k} + g(n),$$

where $g(n) \to 0$ as $n \to \infty$. Thus the desired limit follows.

Exercises

Exercise

Would a Poisson process be a good model for the number of arrivals to a sandwich shop for an entire day of business? Discuss.

Exercise

How reasonable is the orderliness assumption for arrivals to a restaurant? Discuss.

Exercise

Show that Poisson processes constructed as exponential inter-arrial times satisfy orderliness.

Poisson processes as uniform random variables

Another way to simulate Poisson point processes is given by the following characterization on finite volumes. To simulate a Poisson point process of intensity λ on an interval $[s, t]$:

- Call a Poisson random variable M with mean $\lambda(t - s)$
- If $M = m$, then place m independent random variables in $[s, t]$ that are uniformly distributed.
- If a Poisson point process is desired on $[0, \infty)$, then repeat this procedure independently on each interval $[n, n + 1)$, for every nonnegative integer n.

This characterization also has the advantage that it can easily be generalized to higher dimensions and other spaces.

R code: Poisson point process as uniform random variables

The following code outputs the a sequence of arrivals of a Poisson process of intensity 2 in the interval $[0, 1]$

```
M = rpois(1, 2)
P = runif(M)
sort(P, decreasing=F)
```

Exercise

Using the characterization of Poisson point processes as a Poisson number of uniform random variables, show by simulations that the first arrival is distributed as an exponential random variable.

Spatial Poisson point processes

In more general spaces, it is more natural to consider the point process (the location of the arrivals) rather than the counting process (the total number arrivals). Here we think of the point process Π as a random subset of $\mathbb{R}^d$, and we let $\Pi(A)$ to be the number of Π-point in $A \subset \mathbb{R}^d$.

We say that Π is a **Poisson point process on $\mathbb{R}^d$ with intensity λ** if:

- For every subset $A \subset \mathbb{R}^d$ of finite volume $|A|$, the random variable is $\Pi(A)$ is a Poisson random variable with mean $\lambda|A|$.
- For any finite collection of disjoint sets $A_1, \ldots, A_n$ with finite volume the random variables $\Pi(A_1), \ldots, \Pi(A_n)$ are independent.

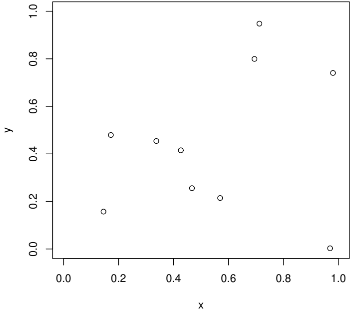
Simulations of spatial Poisson point processes

To simulate a Poisson point process Π of intensity λ on a set $A \subset \mathbb{R}^d$:

- Call a Poisson random variable M with mean $\lambda|A|$
- If $M = m$, then place m independent random variables in A that are uniformly distributed.
- If a Poisson point process is desired on $\mathbb{R}^d$, then consider a partition of $\mathbb{R}^d = \bigcup_{i=1}^{\infty} A_i$ into sets of finite volume and repeat this procedure independently on each set A_i, for every $i \in \mathbb{Z}^+$.

R code: Poisson point process on a square

```
M= rpois(1,10)
x = runif(M)
y = runif(M)
plot(x,y, xlim=c(0,1), ylim=c(0,1))
```



Some other nice properties

Theorem (Superposition)

The sum of two independent Poisson point processes is again a Poisson point process.

Theorem (Colouring)

Given a Poisson point process if we colour the points red or blue independently with probability $p \in (0, 1)$, the resulting blue and red point processes are Poisson and independent.

Theorem (Scaling)

Given a Poisson point process Π on $\mathbb{R}^d$ of intensity λ, the scaled point process $c\Pi$ formed by multiplying each Π-point by $c > 0$, is a Poisson point process of intensity $c^{-1}\lambda$.

Exercises

Exercise

Simulate a Poisson point process on a disc.

Exercise

Show that the law of a Poisson point process is invariant under rotations; that is, if Π is a Poisson point process on a disc, and θ is a rotation, the point process $\theta\Pi$ formed by rotating all the point by θ is still a Poisson point process on the disc.

Exercise

Show that the addition of two independent Poisson point on a disc is a Poisson point process on a disc.

Exercise

Prove the scaling property for Poisson point processes on $[0,\infty)$

Continuous-time Markov chains

Suppose we want to generalize finite state discrete-time Markov chains to allow the possibility of switching states at a random time rather than at unit times. It turns out that if we want to preserve the Markov property, due to the memoryless property of the exponential distribution, the jump times have to be exponential. We will now define a continuous-time Markov chains via a characterization that is useful for simulations.

Let A be a finite set of states. Let π be an initial probability distribution on A. Let P be a transition matrix for A with all zeros in the diagonal so that $p(a,a) = 0$. For each state $a \in A$, we associate a positive number $h_a > 0$ which is referred to the **exponential holding time** at a.

Sampling

We simulate $X \in A^{[0,\infty)}$ a **continuous-time Markov chain** on A with transition matrix p, holding time h, and initial distribution π in the following way:

- Choose an element of $a \in A$ with probability π.
- Call an exponential random variable Y_1 with mean h_a.
- Set $X(t) = a$ for all $t \in [0, Y_1)$.
- Choose a new state $b \in A$ with probability $p(a, b)$.
- Call an exponential random variable Y_2 with mean h_b.
- Set $X(t) = b$ for all $t \in [Y_1, Y_1 + Y_2)$
- Choose a new state $c \in A$ with probability $p(b, c)$.
- Call an exponential random variable Y_3 with mean h_c.
- Repeat....

The times $T_n = Y_1 + \cdots + Y_n$ are referred to as the **jump times**. Thus a continuous-time Markov chain can be visualized as a point process of jump times, where jump times are marked according to the new state that is reached in A at the time of the jump. The discrete-time Markov chain given by $Z_n = X(T_n)$ is sometimes called the **skeleton**, and many of the properties of X are obtained by understanding Z.

R code: for Markov chains

Notice that one can simulate the skeleton first, then required jump times. So the first step in simulating a continuous-time Markov chain is simulating a regular discrete-time Markov chain.
Here we simulate a Markov chain on two states zero and one, started at one.

```
p = matrix(c(0.3,0.1,0.7,0.9), nrow=2, ncol=2)
y=1
update <- function(x){
n = length(x)
if (x[n] ==1){ x <- c(x, rbinom(1,1, p[2,2]))}
if (x[n] ==0){ x <- c(x, rbinom(1,1, p[1,2] ))}
x}
for (i in 1:50000){
y <- update(y)
}
```

Exercises

Exercise

Can you guess what mean(y) will roughly be without implementing the code?

Exercise

Simulate a continuous-time Markov chain.

Summary

Poisson point process can be defined and simulated in various ways, including using exponential inter-arrival times and uniform random variables.

Generalizations of Poisson point processes form the basis of other processes such as continuous-time Markov chains.

The Poisson process is so pervasive that it is an inside joke in mathematics and statistics that every point process will eventually be found out to be the Poisson in disguise.

Grimmett, G. R. and Stirzaker, D. R.
Probability and random processes

Reiss, R.-D.
A course on point processes

Kingman, J. F. C.
Poisson processes