CS 131 Final
December 20th, 2019
Name: ________________
BU ID: ________________
Lab section: ________________

Instructions

- This exam is CLOSED book, notes, and devices.
- The exam consists of 7 questions on the first 13 pages. Write your answers in the space available below each exercise.
- There exist 10 pages of scratch paper attached. **These won't be graded.**
- Please answer all questions on this exam sheet.
- Please read through all questions carefully and be sure that you understand the instructions before working on a problem.
- The exam will be scored out of 100 possible points. There exist 5 extra points.
- Happy holidays!

GOOD LUCK!

1.) ________ (15 points)
2.) ________ (30 points)
3.) ________ (10 points)
4.) ________ (10 points)
5.) ________ (15 points)
6.) ________ (10 points)
7.) ________ (15 points)
Σ: ________ (100 + 5 extra points)

Theorem 1 (Master Theorem). *Let T be a recurrence of the form*

$$T(n) = aT\left(\frac{n}{b}\right) + cn^d,$$

where a, b are positive integers, c is a positive real, and d is a non-negative real. Then:

$$\textit{Case 1: } T(n) \textit{ is } O\left(n^d\right) \textit{ if } a < b^d.$$

$$\textit{Case 2: } T(n) \textit{ is } O\left(n^d \log n\right) \textit{ if } a = b^d.$$

$$\textit{Case 3: } T(n) \textit{ is } O\left(n^{\log_b(a)}\right), \textit{ if } a > b^d.$$

Question 1. Multiple choice and short answers. [15 points]

1) (10 pts) True or false? **Circle** the letter if true, **put an X through it** if false.

(a) (2 pts) $\phi(mn) = \phi(m) \cdot \phi(n)$ for any two integers m, n, where ϕ is the Euler function.

(b) (2 pts)There exist $2^{n(n-1)}$ possible directed graphs on n labeled nodes.

(c) (2 pts) There exists a multiplicative inverse of 2 modulo 100.

(d) (2 pts) $\forall x(P(x) \vee Q(x)) \equiv (\forall x P(x)) \vee (\forall x Q(x))$

(e) (2 pts) $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$

2) (5 pts) For each of the listed recurrences, specify whether Case I, Case II, or Case III of the Master Theorem (provided on back) applies, or state that it does not apply. If it does not apply, explain why. If it applies, give the Θ approximation. No justification needed. Assume T(1) = 1.

(a) [2.5 pts] $T(n) = T(n/2) + e^{2 \log n}$

(b) [2.5 pts] $T(n) = n \cdot T(n/8) + 1$

Question 2. Basics of counting - Short answers (30 pts)
Your answers should be at most few lines. You should not compute factorials, combinations etc. For example, if your answer to a question is 3! or $\binom{4}{2}$ you should leave it as 3! or $\binom{4}{2}$ instead of 6 which is the actual numerical value of both expressions.

(a) (3pts) How many k-digit positive decimal integers exist? Explain your answer in a couple of sentences.

(b) (3pts) In a high-school class of 30 pupils, 12 like mathematics, 14 like physics, 13 like chemistry, 5 like both mathematics and physics, 7 like both physics and chemistry, 4 like mathematics and chemistry. There are also 3 that like all three subjects. How many pupils do not like any of them?

(c) (3pts) How many anagrams can be formed from the word **CHARACTERIZATION**? An anagram is a word having the same letters, each occuring same number of times; the anagram word does not need to have a meaning.

(d) (9pts) In a shop there are k kinds of postcards. We want to send postcards to n friends for the winter holidays. Answer the following questions, with a brief explanation.

(i) (3pts) How many different ways can this be done?

(ii) (3pts) How many different ways can this be done if we want to send them different cards? [Hint: to receive all points consider two cases, $n > k$ and $k \geq n$]

(iii) (3pts) How many different ways can this be done if we want to send two cards to each of them but different persons may get the same card?

(e) (3 pts) How many functions are there from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, n\}$ that are not one-to-one?

(f) (3 pts) How many solutions exist in non-negative integers to the equation $x_1+x_2+\ldots+x_n = m$?

(g) (6 pts) A nurse has to work in a hospital for five days in January (January has 31 days). However, **he is not allowed to work two consecutive days** in the hospital. For example, a valid schedule is January 1st, 5th, 7th, 21st, and 31st but not January 1st, 5th, 6th, 21st, and 31st since the 5th and 6th are two consecutive days. In how many different ways can he choose the five days he will work in the hospital?

Question 3. Quantification logic [10 pts]
Consider the following statement:

> Suppose that there is an integer y such that for all integers x we have $x = y + 1$. Then, for any two integers w and z, $w = z$.

(a) [3 pts] Express this statement in quantificational logic.

(b) [6 pts] Prove that the statement is *true*.

(c) [1 pts] What is the name of the proof technique you used in (b). Use a single word.

Question 4. Pigeonhole principle [10 pts]
Assume that you are at a party of n people ($n \geq 2$). Prove that there are two different people at the party who have exactly the same number of friends at the party. (You may assume that friendship is mutual, e.g. if A is a friend of B, then B is a friend of A).

Question 5. Number theory [15 points]

1. (5 points) Find all values for the natural n for which $n^3 - 1$ is a prime number. [Hint: First factorize $n^3 - 1^3$]

2. For every integer n, $6|n$ if and only if $2|n$ and $3|n$.

   (a) [1 points] Express the above statement using quantificational logic.

(b) [4 points] Prove or disprove the statement.

3. For every integer n, $60|n$ if and only if $6|n$ and $10|n$.

   (a) [1 points] Express the above statement using quantificational logic.

   (b) [4 points] Prove or disprove the statement.

Question 6. Induction [10 pts]
Consider the following sum:

$$1 \cdot 2 + 2 \cdot 3 + \ldots + n \cdot (n+1).$$

(a) [1 pts] Express the sum using the $\sum$ notation.

(b) [7 pts] Prove using induction that the sum is equal to $\frac{n(n+1)(n+3)}{3}$

(c) [2 pts] Fill the the parentheses: The sum is $\Theta(\quad)$.

Question 7. Graph theory [15 points]

i) [5 points] Prove that if a simple undirected graph $G(V, E)$ is self-complementary then $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$.

ii) [10 points] Mr. and Mrs. Smith invited four couples to their home for Thanksgiving. Some guests were friends of Mr. Smith, and some others were friends of Mrs. Smith. When the guests arrived, people who knew each other beforehand shook hands, those who did not know each other just greeted each other. A person never shakes hands with herself/himself, neither two spouses shake hands with each other. After all this took place, the observant Mr. Smith said

> *"How interesting. If you disregard me, there are no two people present who shook hands the same number of times"*

How many times did Mrs. Smith shake hands?

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