CS 131 – Fall 2019, Discussion Worksheet 8
November 6, 2019

Problem 1. (This is problem 2 from midterm 1). One of the students claims he has proven the following new theorem.

Theorem: $\exists x \in \mathbb{R} \, \forall y \in \mathbb{R}(xy^2 = y - x)$

a) Express the theorem in English using one sentence. You may use variable names (x, y) in your sentence (*remainder*: $\mathbb{R}$ is the set of real numbers).

Solution. Exist real number x such that for any real number y product of x and y squared will be equal to y decreased by x.

b) What is wrong with the student's proof that follows?

Proof: Let $x = \frac{y}{y^2+1}$. Then we obtain:

$$y - x = y - \frac{y}{y^2+1} = \frac{y^3}{y^2+1} = \frac{y}{y^2+1}y^2 = xy$$

This proves the theorem. **Q.E.D.**

Solution. The order of quantifiers in the student's proof is wrong. Because student chooses x based on y, he is showing that $\forall y \, \exists x$ that statement holds, but not the theorem.

c) The student's proof in (b) is wrong. But is the claimed theorem correct, i.e., is the statement of the theorem is true? Prove or disprove.

Solution. The theorem is wrong and there are multiple ways to find it.

1) Try a few values of y for $y = 0$ we have that $0 = -x$ which means that $x = 0$. But for $y = 1, x = 0$ the statement become $0 = 1$, which is false.

2) If $x > 0$ the right part fo the equation will be always greater then 0, but left part is not. For $x < 0$ it is vice versa. For $x = 0$ right part is always 0 and left part is not. Thus, such x doesn't exist.

3) If we collect all terms of this equality in left-hand side, we got $xy^2 - y + x$. This is quadratic equation, which could have at maximum two roots and, thus, cannot be true for any y independently from value of x.

Problem 2. Consider the function $f(x) = 2x + 1$. The composition of two functions $f(x)$ and $g(x)$ means $f(g(x))$ or $g(f(x))$. Note that the order of the composition matters. Let's compose $f(x)$ with itself once. We will have $f(f(x)) = f^{(2)}(x) = 2(2x + 1) + 1$. Conjecture a formula for $f^{(n)}(x)$ and prove it by induction.

Solution. Let's write a few compositions of $f(x)$

$$\begin{aligned}
&f(x) = 2x+1\\
&f^{(2)}(x) = 2(2x+1)+1 = 4x+3\\
&f^{(3)}(x) = 4(2x+1)+3 = 8x+7\\
&f^{(4)}(x) = 8(2x+1)+7 = 16x+15\\
&.\\
&.\\
&.\\
&f^{(n)}(x) = 2^n x + 2^n - 1
\end{aligned}$$

Then, we conjecture that the formula is $f^{(n)}(x) = 2^n x + 2^n - 1$. Let's prove it by induction.
Base case: For $n = 1$, $2^1 x + 2^1 - 1 = 2x + 1 = f(x)$. It's true.
Inductive Step: Let $P(k)$ be a predicate that denote $f^{(k)}(x) = 2^k x + 2^k - 1$, and assume that $P(k)$ is true. We want to prove that $P(k+1)$ is true. Let's start with our induction hypothesis

$$\begin{aligned}
f^{(k)}(x) &= 2^k x + 2^k - 1 \quad \text{Induction Hypothesis}\\
f^{(k)+1}(x) &= f(2^k x + 2^k - 1) \quad \text{Compose both sides with } f\\
&= 2^{k+1}x + 2^{k+1} - 2 + 1 = 2^{k+1}x + 2^{k+1} - 1
\end{aligned}$$

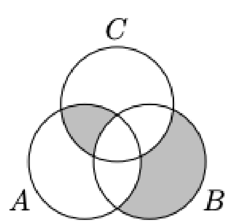
Hence, $P(k+1)$ and the conjectured formula are true.

Problem 3.

a) Write the definition of a symmetric difference between two sets $A \triangle B$ by using logical operators ($\wedge$, $\vee$,etc).

Solution. $A \triangle B = \{x | (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}$.

b) Write an expression for the shaded region by using only the two operations: set difference and symmetric set difference.



Solution. $(A \triangle B) \setminus (A \triangle C)$.

Problem 4. Prove that for any positive integer d, among $d+1$ numbers, there are two that have the same remainder modulo d.

Solution. We will use the Pigeonhole Principle. Let $x_1, x_2, ..., x_{d+1}$ be the given numbers. The holes are the possible remainders $0..(d-1)$ modulo d. The pigeons are the numbers $x_1, ..., x_{d+1}$ which go into holes $(x_1 \mod d), (x_2 \mod d), ..., (x_{d+1} \mod d)$ respectively. There are d holes and

$d+1$ pigeons. So by the Pigeonhole Principle, there is a hole with at least two pigeons. Equivalently, there are two numbers among $x_1, ..., x_{d+1}$ with the same remainder modulo d.

Problem 5. Let A, B, C be sets. Prove that $((A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C) \subseteq (A \cap B)^c \cap C$.

Solution. Let $x \in ((A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)$. We need to prove that $x \in (A \cup B)^c \cap C$.

$$\begin{gathered}
(x \in ((A \cap C) \cup (B \cap C)) \setminus (A \cap B \cap C)) = \\
(x \in (A \cap C) \cup (B \cap C)) \wedge (x \notin A \cap B \cap C) = \\
((x \in A \cap C) \vee (x \in B \cap C)) \wedge \neg(x \in A \cap B \cap C) = \\
((x \in A \wedge x \in C) \vee (x \in B \wedge x \in C)) \wedge \neg(x \in A \wedge x \in B \wedge x \in C) = \\
((x \in A \vee x \in B) \wedge x \in C) \wedge (x \notin A \vee x \notin B \vee x \notin C) = \\
(x \in A \vee x \in B) \wedge x \in C \wedge (x \notin A \vee x \notin B \vee x \notin C)
\end{gathered}$$

So, we know three things:

1. $x \in A \vee x \in B$
2. $x \in C$
3. $x \notin A \vee x \notin B \vee x \notin C$

Because of (2), $x \notin A \vee x \notin B \vee x \notin C = x \notin A \vee x \notin B \vee F = x \notin A \vee x \notin B$ is true.

Our goal is to show $x \in (A \cap B)^c \cap C$. We already showed $x \in C$, therefore we just need to show $x \in (A \cap B)^c$. But $x \notin A \vee x \notin B = \neg(x \in A \wedge x \in B) = \neg(x \in A \cap B) = (x \in (A \cap B)^c)$.