CS 131 – Fall 2019, Discussion Worksheet 9
November 13, 2019

Problem 1.

a) Let $G(n) := gcd(n, n+2)$. Prove or disprove $\forall n \in \mathbb{Z}, (G(n) = 1) \vee (G(n) = 2)$.

Solution. $gcd(n, n+2) = gcd(n, 2)$. If n is even, then $gcd(n, 2) = 2$. And if n is odd, then $gcd(n, 2) = 1$.

b) Let $C(x, y) := \lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil$. Prove or disprove $\forall x, y \in \mathbb{R}, (C(x, y) = 0) \vee (C(x, y) = 1)$.

Solution. Let $n, m \in \mathbb{Z}$, and $0 < \delta, \epsilon < 1$. Any real number that's not an integer can be written as the summation of an integer plus a positive real number less than 1.
Proof by cases:
Case 1: $x \in \mathbb{Z}$ and y isn't an integer:
We have $\lceil x + y \rceil = x + \lceil y \rceil$. Hence, we obtain

$$C(x, y) = \lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = x + \lceil y \rceil - x - \lceil y \rceil = 0$$

Case 2: $y \in \mathbb{Z}$ and x isn't an integer:
We have $\lceil x + y \rceil = \lceil x \rceil + y$. Hence, we obtain

$$C(x, y) = \lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = \lceil x \rceil + y - \lceil x \rceil - y = 0$$

Case 3: x and y aren't integers:
We can write $x = n + \epsilon$ and $y = m + \delta$. Substitute these expressions in φ to obtain

$$\begin{aligned} C(x, y) &= \lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = \lceil n + \epsilon \rceil + \lceil m + \delta \rceil - \lceil n + \epsilon + m + \delta \rceil \\ &= n + 1 + m + 1 - n - m - \lceil \epsilon + \delta \rceil = 2 - \lceil \epsilon + \delta \rceil = \begin{cases} 0 & \text{if } 1 < \epsilon + \delta \leq 2 \\ 1 & \text{if } 0 < \epsilon + \delta \leq 1 \end{cases} \end{aligned}$$

Case 4: x and y are integers: $C(x, y) = \lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = x + y - x - y = 0$

Problem 2.

a) Convert 120 from decimal to binary.

Solution.

$$\begin{aligned} 120 &= 2 \cdot 60 + 0 \\ 60 &= 2 \cdot 30 + 0 \\ 30 &= 2 \cdot 15 + 0 \\ 15 &= 2 \cdot 7 + 1 \\ 7 &= 2 \cdot 3 + 1 \\ 3 &= 2 \cdot 1 + 1 \\ 1 &= 2 \cdot 0 + 1 \end{aligned}$$

Hence, we have 1111000.

b) Convert 317 from octal to binary.

Solution. Take each digit in the octal representation and substitute for it the three binary digits that correspond to it. Hence, for 3 we have 011, 1 we have 001, and 7 we have 111. Therefore, the answer is 011001111.

c) Convert BED from hexadecimal to decimal.

Solution. $BED = 13 \cdot (16^0) + 14 \cdot (16^1) + 11 \cdot (16^2) = 3053$

Problem 3. Compute the following Euler functions: $\phi(10)$, $\phi(13)$, $\phi(30)$, and $\phi(49)$.

Solution. $\phi(10) = \phi(2)\phi(5) = 1 \cdot 4 = 4$ since 2 and 5 are prime numbers. $\phi(13) = 12$ since 13 is a prime number $\phi(p) = p - 1$. $\phi(30) = \phi(5)\phi(6) = 4 \cdot 2 = 8$ since 5 and 6 are coprimes. And $\phi(49) = 42$ since $49 = 7^2$ and 7 is a prime number; hence, we use the formula $\phi(p^k) = p^k - p^{k-1}$.

Problem 4. Prove that for every prime number p, $p + 7$ is composite.

Solution. Note that $p = 2$ is the only even prime number; the rest of the prime numbers are odd. Hence, let's do proof by cases.
Case 1 $p = 2$: Substitute $p = 2$ in $p + 7$ to get

$$p + 7 = 2 + 7 = 9 = 3 \cdot 3$$

It's a composite number.
Case 2 p is odd: We can write $p = 2k + 1$ for some integer k. Now substitute this p in $p + 7$ to get

$$p + 7 = 2k + 1 + 7 = 2k + 8 = 2(k + 4)$$

It's a composite number in all the cases.

Problem 5. Find Bezout's coefficients for $x \cdot 92 + y \cdot 15 = gcd(92, 15)$

Solution. First we need to find the greatest common divisor for 92 and 15.

$$\begin{aligned} 92 &= 6 \cdot 15 + 2 \\ 15 &= 7 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

Hence, $gcd(92, 15) = 1$. Let's rearrange the above equations to be in the form

$$\begin{aligned} 2 &= 92 - 6 \cdot (15) \\ 1 &= 15 - 7 \cdot (2) \end{aligned}$$

Now working the equations backwardly we get

$$1 = 15 - 7 \cdot (2) = 15 - 7 \cdot (92 - 6. \cdot (15)) = 43 \cdot 15 - 7 \cdot 92$$

Hence, $x = -7$ and $y = 43$.