CS 131 – Fall 2019, Discussion Worksheet 2
September 18, 2019

Problem 1. Define the predicate $A(x, y)$ to mean student x admires student y. Define the following scenarios in the class using quantificational logic. Assume the domain of discourse is all BU students.

a) All BU students admire themselves.

Solution. $\forall x A(x, x)$

b) There is a BU student who does not admire themselves.

Solution. $\exists x \neg A(x, x)$

c) All BU students admire some BU student.

Solution. $\forall x \exists y A(x, y)$

d) All BU students admire someone other BU student.

Solution. $\forall x \exists y ((x \neq y) \wedge A(x, y))$

e) There is a BU student that admires no other BU student.

Solution. $\exists x \forall y ((x \neq y) \rightarrow \neg A(x, y))$

Now assume the domain of discourse is all students (not only BU students) and denote $B(x)$ "x is a BU student". Most of the time we will explicitly tell you what the domain of discourse is.

f) Same as part a.

Solution. $\forall x (B(x) \rightarrow A(x, x))$

g) Same as part b.

Solution. $\exists x (B(x) \wedge \neg A(x, x))$

h) All students admire some BU student.

Solution. $\forall x \exists y (B(y) \wedge (A(x, y))$

i) There is a BU student who all students admire.

Solution. $\exists x (B(x) \wedge \forall y A(y, x))$

j) What does this signify?
$\exists x (A(x, x) \wedge \forall y (A(x, y) \rightarrow x = y))$

Solution. There is a student who admires only themselves.

Problem 2. For each of the following formulas, argue whether it is valid or not. If your answer is positive, prove that the formula is true. Otherwise, prove that the negation of the formula is true. Assume the domain of discourse is integers.

a) $\exists x(x < -3)$

Solution. Yes. Choose $x = -4$.

b) $\forall x(x^2 > x)$

Solution. No. $x = 0$ gives a counter-example.

To prove this formally, we can we show that the negation $\exists x(x^2 \leq x)$ is true. Simply choose $x = 0$ as before.

c) $\forall x \forall y(x^2 < y + 1)$

Solution. No. Choose $x = 2$ and $y = 2$ to give a counter-example.

To prove this formally, we can we show that the negation $\exists x \exists y(x^2 \geq y + 1)$ is true. Simply choose $x = y$ and $y = 2$ as before.

d) $\forall x \exists y(x^2 < y + 1)$

Solution. Yes. Start with an arbitrary x. If we then choose $y = x^2$, then we have $x^2 < x^2 + 1$ which is always true.

e) $\exists x \forall y(x^2 < y + 1)$

Solution. No. To prove it, let's prove the negation $\forall x \exists y(x^2 \geq y + 1)$ is true.

Choose any x. We need to show that $\exists y(x^2 \geq y + 1)$ is true. Choose $y = -1$. Clearly $x^2 \geq 0$.

f) $\exists x \exists y(x^2 < y + 1)$

Solution. Yes. Choose $x = y = 0$.

g) $\exists y \forall x(x^2 < y + 1)$

Solution. No. No matter what y we choose, x^2 can grow infinitely large.

As before, let's prove the negation $\forall y \exists x(x^2 \geq y + 1)$ is true. Choose any y. If y is negative, $x = 0$ satisfies $x^2 \geq y + 1$. If y is non-negative, we can choose $x = y + 1$, which always satisfies $x^2 \geq y + 1$.