CS 131 – Fall 2019, Discussion Worksheet 10
November 20, 2019

Problem 1. Prove that 5 divides $n^5 - n$, whenever n is a nonnegative integer. Show two kind of proofs, by remainder cases and by induction.

Solution.

1) Remainder cases: We will analyze remainders for $n^4 mod 5$ and show they will be always equal to 1, thus n^5 and n will have the same remainder and $n^5 - n$ will be divisible by 5:

n	n^2	n^4
0	0	0
1	1	1
2	4	1
3	4	1
4	1	1

2) Let predicate $P(n)$ be $5|n^5 - n$

Basis: $n = 0$, $0 - 0 = 0$, $5|0$, basic case proven.

Inductive step: $P(n) \implies P(n+1)$. For $P(n+1)$ the expression is:

$(n+1)^5 - (n+1) = n^5 + 5n^4 + 10n^3 + 10n^2 + 5n + 1 - n - 1 = (n^5 - n) + 5 * (n^4 + 2n^3 + 2n^2 + n)$

The first term of the sum is divisible by 5 because of inductive assumption, the second term is multiple of 5, thus the sum divisible by 5. Inductive step proven.

Problem 2. How many numbers from 1 to 200 are:

a) Divisible by 13

Solution. $200 \div 13 = 15$

b) Divisible by both 9 and 13

Solution. To be divisible by both 9 and 13, a number need to bi divisible by their product (since these two numbers are relatively prime), $200 \div 9 * 13 = 1$

c) Divisible by 9 but not divisible 11

Solution. To get this quantity, we need to subtract from quantity of numbers divisible by 9 quantity of numbers divisible by both 9 and 11. The answer is 20 (22 - 2)

d) Divisible by 4 but not divisible 8

Solution. There are $200 \div 4 = 50$ numbers divisible by 4, every second is divisible by 8, thus the answer is 25.

e) Divisible by 8 but not divisible 4

Solution. Such numbers do not exist.

Problem 3.

a) Find multiplicative inverse of $7 \mod 9$ using remainders table

Solution.

7 x	0	1	2	3	4
mod 9	0	7	5	3	1

Thus, the answer is 4.

b) Find multiplicative inverse of $7 \mod 39$ using Bezout's coefficients.

Solution. First, we need to find Bezout's coefficients.

Euclid's algorithm	Memorization
$\gcd(39,7) = \gcd(7, 39-5\cdot 7) = \gcd(7,4) =$	4 = 39 - 7· 5
$\gcd(4, 7-4) = \gcd(4,3)$	3 = 7 - 4
$\gcd(4, 4-3) = \gcd(4,1)$	1 = 4 - 3

Now we can find the coefficients itself:

$1 = 4 - 3 = 4 - 1 \cdot (7-4) = -7 + 2 \cdot 4 =$

$= -7 + 2 \cdot (39 - 7 \cdot 5) = 2 \cdot 39 - 11 \cdot 7 =$

Thus, the coefficients we need are 2 and -11. The multiplicative inverse will be $-11 \mod 39 = 28$.

Problem 4. Let n be a positive integer which is not divisible by 2 or 5. Prove that there exists a multiple of n consisting entirely of ones.

Solution. There are infinitely many consisting entirely of ones, thus we will manage to find pair of numbers which have the same remainder being divided by n (since there are finite number of possible remainders we could apply Pigeonhole principle). Let it be to numbers a_1 and a_2 with lengths l_1 and l_2. Let's analyze their difference (assume $l_2 > l_1$):

$$a2 - a1 = \underbrace{11...11}_{l_2 - l_1} \cdot 10^{l_1}$$

Let's call this number k, it will have l_1 zeros in the end (decades where ones from a_1 we subtracted from a_2). We know, that k is divisible by n, because it is a result of two numbers with the same remainder being subtracted one from another. But, 10^{l_1} don't have any common multipliers with n, because n is not divisible by 2 and 5. This, the first part, $\underbrace{11...11}_{l_2 - l_1}$, should be divisible by n, but this number consist only of ones. Thus, statement is proven.