Practice Problems
CS131

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1 Logic

Problem 0. Answer the following questions:

a) Write the truth table for $p \rightarrow q$.

b) Suppose you want to prove a theorem of the form $p \rightarrow q$. Can you connect the direct, contrapositive, and contradiction proof techniques to the rows of this table? Please explain the connection in clear words.

Problem 1. Express the following statements in quantificational logic.

1. Someone did not get an A grade.
2. $A \subseteq B$
3. $A \cap B \subseteq B \backslash C$

Problem 2. Consider the following theorem: *for every real number x, $x^2 \geq 0$*.What is wrong with this proof:

> Suppose not for the sake of contradiction. Then for every real number x, $x^2 < 0$. In particular, plugging in $x = 4$ we would get $16 < 0$, which is clearly false. Contradiction!

Problem 3. Express the following statements in quantificational logic, and prove/disprove each. Be precise about the predicates you use, but also feel free to avoid dividing a predicate when we have already introduced related notation in class (e.g., instead of defining $D(a, b)$ as the predicate for a divides b, use directly $a|b$ in your notation).

1. For all integers a, b, c if a divides b, and b divides c, then a divides c.
2. Let x be an arbitrary real. Then, there exists a real y such that $xy^2 \neq y - x$.
3. For every prime p, $p + 3$ is a composite.

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2 Recursive algorithms

Problem 1. Devise a recursive algorithm for computing $b^n \mod m$ where b, n and m are integers with $m \geq 2, n \geq 0$ and $1 \leq b < m$.

Problem 2. Devise a recursive algorithm for computing Fibonacci numbers.

Problem 3. Describe the binary search algorithm (input, output, how it works, and its running time analysis).

3 Induction

Problem 1. Find a formula for $\sum_{i=1}^{n} i$, and prove it using induction.

Problem 2. Let the "Tribonacci sequence" be defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Prove that $T_n < 2^n$ for all positive integers n.

Problem 3. Consider the function $f(k) = 2k+1$, where k is an integer. Prove that for all integers k and $n \geq 1$, $\underbrace{f(f(\ldots(f(k)))}_{n \text{ compositions of function}} \equiv 1 \mod 2$.

4 Number theory.

Problem 1. Let a, b, c, d be integers. If $(a-c)$ divides $ab+cd$ then $a-c$ also divides $ad+bc$.

Problem 2. Compute the following values of the Euler ϕ-function.

1. $\phi(15)$. Solve the using the definition of the ϕ-function.
2. Let $p_1, \ldots, p_r$ be the primes that appear in the prime factorization of integer n. Then,

$$\phi(n) = n \prod_{i=1}^{r} \frac{p_i - 1}{p_i}.$$

3. $\phi(900)$. Solve this using the previous question.

Problem 3. Prove the following statements:

1. $gcd(n, n+1) = 1$
2. $gcd(2n-1, 2n+1) = 1$
3. $gcd(2n, 2n+2) = 2$
4. $gcd(a, b) = gcd(a, a+b)$
5. $gcd(5a+3b, 13a+8b) = gcd(a, b)$

Problem 4. Prove that $9|4^n + 15n - 1$ for all non-negative integers n.

5 Counting

Problem 1. A total of 1 232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2 092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Problem 2. How many $n \times n$ matrices with entries 0 or 1 have even row and sum columns?

6 Combinatorial proofs

Prove the following identities using combinatorial proofs.

Vandermonde's identity. Let m, n, r be non-negative integers with r not exceeding either m or n. Then,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Christmas stocking identity. Let $n, r \in \mathbb{N}, n > r$. Then,

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}.$$

7 Pigeonhole principle

Problem 1. How many six-digit numbers do you need to choose to ensure that at least two of them have the same last three digits?

1. In the decimal system.
2. In the binary system.
3. In the hex system.

Problem 2. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

Problem 3. Ten points are placed within a unit equilateral triangle. Show that there exists two points with distance at most $\frac{1}{3}$ apart.

8 Graph theory

Problem 0. (a) State and prove the handshaking lemma. (b) Prove Hall's theorem using induction.

Problem 1. Let $G(V, E)$ be a graph with n vertices such that the degree of any vertex is at least $\lceil n/2 \rceil$. Prove that G is connected.

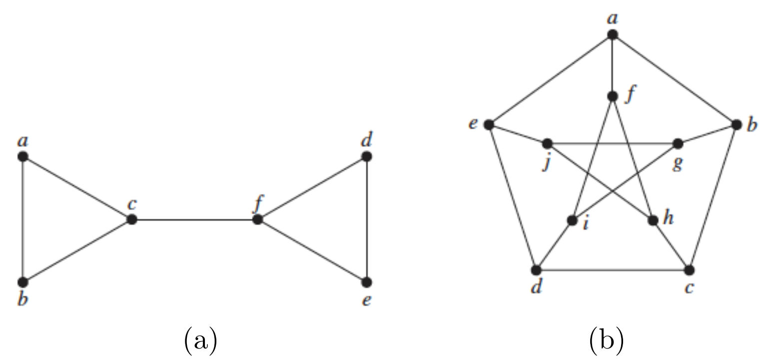

Figure 1: See Problem 5 in Section 8.

Problem 2. Let G be a simple graph on n nodes with k connected components, and let these components have $n_1, \ldots, n_k$ vertices.

1. Evaluate the sum $\sum_{i=1}^{k} n_i$.
2. What is the maximum number of edges in G as a function of $n_1, \ldots, n_k$?
3. What is the maximum number of edges in a disconnected graph G? Characterize the structure of such graphs.

Problem 3. Find the number of perfect matchings for the cycle C_n.

Problem 4. A graph G is called self-complementary if G and its complement $\bar{G}$ are isomorphic. Show that if G is self-complementary graph with n vertices, then $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$.

Problem 5. Determine for each of the graphs in Figure 1 whether there exists a Hamilton Circuit or not.

Problem 6. Answer the following questions.

1. State Ore's theorem. Does it provide a necessary or a sufficient condition for the existence of a Hamilton circuit in a graph?
2. State Dirac's theorem. Prove Dirac's theorem as a simple corollary of Ore's theorem.

Please spend time on as many problems as you can **on your own**! Short answers will be given to you before the final.