CS 131 – Fall 2019, Discussion Worksheet 7
October 30, 2019

Problem 1. The knight in chess can move to a square that is two squares away horizontally and one square vertically, or two squares vertically and one square horizontally.

Suppose we have an $m \times n$ ($m, n \geq 4$) board with a knight in the upper left corner. Prove that the knight can reach any cell on the board.

Solution. The proof is by induction (twice). Let $P(m, n)$ be true if and only if the knight can reach any cell on an $m \times n$ board.

First we will prove $\forall n \geq 4\ P(4, n)$.

In the basis step we need to prove $P(4, 4)$. We can reach any cell on an 4×4 board as follows:

1. $(1,1) \to (2,3)$
2. $(1,1) \to (3,2)$
3. $(2,3) \to (4,2)$
4. $(2,3) \to (4,4)$
5. $(2,3) \to (3,1)$
6. $(4,2) \to (2,1)$
7. $(4,2) \to (3,4)$
8. $(3,1) \to (1,2)$
9. $(2,1) \to (1,3)$
10. $(2,1) \to (3,3)$
11. $(3,4) \to (2,2)$
12. $(2,2) \to (1,4)$
13. $(2,2) \to (4,1)$

Note that the left side of an arrow is reachable as established by a previous line.

Inductive step: suppose $P(4, k)$, where $k \geq 4$. We will prove $P(4, k+1)$. By the inductive hypothesis, we know we can reach any cell (i, j), where $1 \leq i \leq 4$ and $1 \leq j \leq k$. It easy to see how to get to the remaining cells $(i, k+1)$, $1 \leq i \leq 4$, from the previous ones:

1. $(3,k) \to (1,k+1)$
2. $(4,k) \to (2,k+1)$
3. $(1,k) \to (3,k+1)$
4. $(2,k) \to (4,k+1)$

Thus, $P(4, k+1)$ and we have proven $\forall n \geq 4\ P(4, n)$ by induction.

Now, fix an arbitrary $n \geq 4$. We will show that $\forall m \geq 4\ P(m, n)$ by induction.

Our basis case is $P(4, n)$. We know it is true already.

Now assume $P(k, n)$, where $k \geq 4$. We will show that $P(k+1, n)$ is true.

Similarly to the previous proof, by the induction hypothesis, we can reach any cell (i, j), where $1 \leq i \leq k$ and $1 \leq j \leq n$. It is each to see how to reach the last row, $(k+1, j)$ where $1 \leq j \leq n$, from those. Thus, we establish $P(k+1, n)$ and prove $\forall m \geq 4\ P(m, n)$ by induction.

Since n was arbitrary, $P(m, n)$ for all $m, n \geq 4$.

Problem 2. Calculate the following numbers using Euclid's algorithm:

- $\gcd(4, 8)$
- $\gcd(34, 51)$
- $\gcd(112, 75)$

Solution.

- $\gcd(4, 8) = \gcd(4, 8 \mod 4) = \gcd(4, 0) = 4$
- $\gcd(340, 204) = \gcd(340 \mod 204, 204) = \gcd(136, 204) = \gcd(136, 204 \mod 136) = \gcd(136, 68) = \gcd(136 \mod 68, 68) = \gcd(0, 68) = 68$
- $\gcd(112, 75) = \gcd(37, 75) = \gcd(37, 1) = 1$

Problem 3. For given a and b prove or disprove the following claim: there is a natural number N such that for all integers $n \geq N$, n cents can be changed to a cent coins and b cent coins.

a) $a = 6$, $b = 15$

Solution. Notice that both 6 and 15 are divisible by 3. Then, any combination of those must be divisible by 3. Therefore the answer is no: for all N, there is always $n \geq N$ that is not divisible by 3.

b) $a = 143$, $b = 253$

Solution. Same as before: both 143 and 253 are divisible by 11, so the answer is no.

c) $a = 5$, $b = 13$

Solution. The answer is yes. As in the previous lab, we need to find 5 consecutive integers that can be represented as a sum of 5's and 13's. Here they are:

- $50 = 5 \cdot 10$
- $51 = 13 \cdot 2 + 5 \cdot 5$
- $52 = 13 \cdot 4$
- $53 = 13 + 5 \cdot 8$
- $54 = 13 \cdot 3 + 5 \cdot 3$

Problem 4. Let a be a real number.

$\lfloor a \rfloor$ is called the floor of a and is equal to a rounded down to the nearest integer. Technically we define $\lfloor a \rfloor$ to be the biggest element in the set $\{x \in \mathbb{Z} : x \leq a\}$.

$\lceil a \rceil$ is called the ceiling of a and is equal to a rounded up to the nearest integer. Technically we define $\lceil a \rceil$ to be the smallest element in the set $\{x \in \mathbb{Z} : x \geq a\}$.

a) Find $\lfloor 5.3 \rfloor$.

Solution. 5

b) Find $\lfloor 5 \rfloor$.

Solution. 5

c) Find $\lceil 5.3 \rceil$.

Solution. 6

d) Find $\lceil 5 \rceil$.

Solution. 5

e) Prove or disprove: $\lfloor a \rfloor \cdot \lfloor b \rfloor = \lfloor ab \rfloor$.

Solution. This is incorrect. $\left\lfloor \frac{5}{2} \right\rfloor \cdot \lfloor 2 \rfloor = 2 \cdot 2 = 4$, but $\left\lfloor \frac{5}{2} \cdot 2 \right\rfloor = \lfloor 5 \rfloor = 5$.

Problem 5. A number is called *perfect* if it is equal to the sum of its positive divisors other than itself. E.g., $6 = 1 + 2 + 3$. Prove the following claim:

Claim: *If $2^k - 1$ is a prime, then $2^{k-1}(2^k - 1)$ is perfect*

Solution. Since $2^k - 1$ is a prime, factors of $2^{k-1}(2^k - 1)$ are

$$2^0, 2^1, \cdots, 2^{k-1}$$

and

$$(2^k - 1) \cdot 2^0, (2^k - 1) \cdot 2^1, \cdots, (2^k - 1) \cdot 2^{k-2}.$$

The sum of factors are then

$$\sum_{n=0}^{k-1} 2^n + \sum_{m=0}^{k-2} \left((2^k - 1) \cdot 2^m\right) = \sum_{n=0}^{k-1} 2^n + (2^k - 1) \sum_{m=0}^{k-2} 2^m = (2^k - 1) + (2^k - 1)(2^{k-1} - 1),$$

which is equal to the given number, $2^{k-1}(2^k - 1)$.