CS 131 – Fall 2019, Discussion Worksheet 11
December 4, 2019

Problem 1. How many different strings of length 12 containing exactly five a's can be chosen over the following alphabets?

a) The alphabet {a, b}.

Solution. There are $\binom{12}{5}$ ways to select where the five a's will be placed among the 12 possible locations. Once the a's are placed, the unfilled locations are filled with b's. Therefore the number of strings over the alphabet $\{a, b\}$ with exactly five a's is $\binom{12}{5} = 792$.

b) The alphabet {a, b, c}.

Solution. There are $\binom{12}{5}$ ways to select where the five a's will be placed among the 12 possible locations. Once the a's are placed, there are two choices (b or c) for each of the remaining seven unfilled locations. Therefore, there are 2^7 ways to fill the locations that do not have an a. The number of strings over the alphabet {a, b, c} with exactly five a's is $2^7 \cdot \binom{12}{5}$.

Problem 2. Let $f : A \to B$ be a function where $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Count the number of different functions with the given domain, target and additional properties.

a) No additional properties.

Solution. A function is defined in terms of its output for every possible element in the domain. For a given $x \in A$, $f(x)$ can be any element from the target set B. Therefore there are 12 choices for $f(x)$. The output must be chosen for each element x in the domain and the choices are put together by the product rule. There are 8 elements in the domain, so the number of functions whose domain is A and whose target set is B is 12^8.

b) The function is injective.

Solution. A function is defined in terms of its output for every possible element in the domain. Start with $x = 1$ and select the value for $f(1)$. There are 12 possible choices. After $f(1)$ has been chosen, select the value for $f(2)$. There are $12 - 1$ possible choices for $f(2)$ because the output can be any element from the target set except the element that was chosen to be $f(1)$. Select the value of $f(x)$ for each $x \in A$ in such a way that there are no repetitions. Since there are 8 elements selected from a set of size 12 with no repetitions, the total number of ways to select the output values for f is $P(12, 8) = \frac{12!}{4!} = 12 \cdot 11 \times ... \times 5$.

c) The function is surjective.

Solution. Let the elements of the codomain be pigeons and the elements of the domain be holes. There are more pigeons than holes. By the Pigeonhole Principle multiple elements in the domain are mapped to the same element in the codomain which is a contradiction because any function maps every element in the domain to exactly one elements in the codomain. Thus, the answer is 0.

Problem 3. How many ternary strings (digits 0,1, or 2) are there with exactly seven 0's, five 1's and four 2's?

Solution. First, we calculate the number of strings that can be made from $0_1, 0_2, ..., 0_7, 1_1, ..., 1_5, 2_1, ..., 2_4$ (all symbols are considered different). There are 16! options. However, for any string with exactly seven 0's, five 1's and four 2's, we can replace each digit with 0_i's, 1_i's, and 2_i's in $7! \cdot 5! \cdot 4!$ ways.

The answer is $\frac{16!}{7! \cdot 5! \cdot 4!}$.

Problem 4.

a) A leaf is a vertex with degree 0 or 1. Prove that a graph with n vertices and $n-1$ edges (where $n \geq 1$) must contain a leaf.

Solution. Suppose there is a graph with n vertices and $n-1$ edges (where $n \geq 1$) with no leaves. Then the sum of the degrees of its vertices is at least $2n$. The number of edges is half of the sum of the degrees of the vertices. Then there are at least n edges, which is a contradiction.

b) We know that a tree with $n \geq 1$ vertices has exactly $n-1$ edges. In this problem prove by induction that a tree with $n \geq 1$ vertices has at most $n-1$ edges. You can use the fact, which you will prove in the homework, that a tree with $n \geq 1$ edges must contain a leaf.

Solution. Basis step: a tree with 1 vertex must have 0 edges.

Inductive step: take any tree T with $n+1$ vertices, where $n \geq 1$. We know it must have a leaf. Make a new graph T' by removing this leaf from T. Obviously, T' is still connected and doesn't have cycles. So, T' is a tree with n vertices. By the inductive hypothesis, T' has at most $n-1$ edges. Since we removed at most one edge when making T', T has at most n edges.

Problem 5.

a) How many 7-digit integers have digits in non-decreasing order?

Solution. Note that such an integer cannot begin with 0 (because else it's not 7-digit) and therefore cannot contain a 0 (because digits cannot decrease). Note also if I tell you how many times each digit from 1 to 9 is present in the number, you can reconstruct the number uniquely. For example, if I tell you that the number contains two 1s, three 4s, an 8, and a 9, then you know it is 1144489. Thus, I need to count how many way there are to distribute 7 points among 9 digits (or 7 identical balls over 9 bins). This in turn is done by counting the number of $(7+9-1)$-bit strings with exactly 7 zeros and $(7+9-1)$ ones. The answer is $\binom{7+9-1}{7}$.

b) Same question but the integers must contain at least 2 ones.

Solution. We already know that 2 of the balls must be in bin one. Thus we are interested in how many ways there are to distribute the remaining 5 balls among 9 bins. This can be done in $\binom{5+9-1}{5}$ ways.