CS 131 – Fall 2019, Discussion Worksheet 4

October 2, 2019

Problem 1. Show that for all $x \in \mathbb{R}$, it is true that $f(x) > 0$.

$$f(x) = \begin{cases} |x| + 1 & \text{if } x < 5 \\ x^2 - 24 & \text{otherwise} \end{cases}$$

Solution. Proof by cases.

Case 1: if $x < 5$, then $f(x) = |x| + 1$. Since $|x|$ is always non-negative, $|x| + 1 > 0$, so $f(x) > 0$.

Case 2: if $x \geq 5$, then $f(x) = x^2 - 24 \geq 5^2 - 24 = 1$. Therefore $f(x) > 0$ in this case as well.

Since the two cases cover all possibilities for the value of x, and $f(x) > 0$ in both, we can conclude that $f(x) > 0$ is true for all $x \in \mathbb{R}$.

Problem 2. Let $P(x)$ be a predicate and let the domain of discourse be the empty set. Prove by contradiction that $\forall x P(x)$.

Solution. Suppose $\neg\forall x P(x)$. This is equivalent to $\exists x \neg P(x)$. However this is false since no x exists.

Problem 3. Suppose that A and B are sets. Prove $(A \cap B = \emptyset) \to (A \subseteq \bar{B})$.

a) As we have seen in class, explain what is given to you, and what is the goal?

Solution. Given: A and B are sets.
Asked: $(A \cap B = \emptyset) \to (A \subseteq \bar{B})$.

b) Formulate a proof by contradiction strategy. What's given and asked in this strategy?

Solution. Given: A and B are sets, $A \cap B = \emptyset$, and $\neg(A \subseteq \bar{B})$.
Asked: False.

c) Use a proof by contradiction to prove the original statement.

Solution. Recall the following definitions involving sets:

- $A \subseteq B$ iff $\forall x(x \in A \to x \in B)$;
- $\forall x(x \notin \emptyset)$;
- $A \cap B = \{x : x \in A \wedge x \in B\}$.

$$\begin{aligned}
\neg(A \subseteq \bar{B}) &= \neg\forall x \Big((x \in A) \to (x \in \bar{B})\Big) \\
&= \neg\forall x \Big(\neg(x \in A) \vee (x \in \bar{B})\Big) \\
&= \exists x \Big((x \in A) \wedge \neg(x \in \bar{B})\Big) \\
&= \exists x \Big((x \in A) \wedge \neg(x \notin B)\Big) \\
&= \exists x \Big((x \in A) \wedge (x \in B)\Big) \\
&= \exists x \ (x \in A \cap B)
\end{aligned}$$

is true.

But because of the assumption $A \cap B = \emptyset$, we know that $\forall x(x \notin A \cap B) = \neg\exists x(x \in A \cap B)$ which contradicts the above.

d) Formulate a direct proof strategy. What's given and asked in this strategy?

Solution. Given: A and B are sets, and $A \cap B = \emptyset$.
Asked: $A \subseteq \bar{B}$.

e) Use a direct proof to prove the original statement.

Solution. $(A \cap B = \emptyset) = \forall x(x \notin A \cap B)$.
We would like to use the definition of $A \cap B$ but we can only do it for the expression "$x \in A \cap B$". Therefore, we double negate the logical expression to convert "$x \notin A \cap B$" to "$x \in A \cap B$".

$$\begin{aligned}
\forall x(x \notin A \cap B) &= \neg\neg\forall x(x \notin A \cap B) \\
&= \neg\exists x\ (x \in A \cap B) \\
&= \neg\exists x\ \Big((x \in A) \wedge (x \in B)\Big) \\
&= \forall x\ \Big(\neg(x \in A) \vee \neg(x \in B)\Big) \\
&= \forall x\ \Big((x \in A) \to \neg(x \in B)\Big) \\
&= \forall x\ \Big((x \in A) \to (x \in \bar{B})\Big) = A \subseteq \bar{B}
\end{aligned}$$

f) Can we prove $(A \cap B = \emptyset) \to (B \subseteq \bar{A})$? If yes, then specify what's given and asked in your proof strategy, and come up with the simplest proof possible. And if no, then explain why it cannot be proven.

Solution. Yes by using a direct proof and using part (c) or (e).
Given: A and B are sets, and $A \cap B = \emptyset$.
Asked: $B \subseteq \bar{A}$.

Since $A \cap B = \emptyset$, $B \cap A = \emptyset$ is also true. Hence we can apply part (c) or (e) to B and A to conclude $B \subseteq \bar{A}$.

Problem 4. Prove that there is a unique non-negative $\sqrt{2}$, assuming that it exists. I.e., prove that the number y such that $y \geq 0$ and $y^2 = 2$ is unique.

Solution. Here we will prove the uniqueness of $\sqrt{2}$ by contradiction.

Suppose there exist two such non-negative real numbers y_1 and y_2, i.e. $y_1^2 = y_2^2 = 2$ and $y_1 \neq y_2$. First note that both numbers must be strictly positive: $y_1, y_2 > 0$.

Simple algebraic manipulation gives

$$0 = y_1^2 - y_2^2 = (y_1 + y_2)(y_1 - y_2)$$

which implies that either $y_1 + y_2 = 0$ or $y_1 - y_2 = 0$. But from the assumption of $y_1 \neq y_2$, we know that $y_1 - y_2 \neq 0$, and therefore $y_1 + y_2 = 0$ must be the case. Hence, $y_1 = -y_2$. Given that y_1 and y_2 are positive, this yields a contradiction.