CS 131 – Fall 2019, Discussion Worksheet 6
October 16, 2019

Problem 1. Define the geometric sequence: $a_n = x^n$, where $n \in \mathbb{N}$.

Let $x \neq 1$. Prove the formula for the sum of the first n terms of a geometric sequence: for all $n \in \mathbb{N}$,

$$\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}.$$

Solution. Let $P(k)$ denote $\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$. We will prove that $\forall k \in \mathbb{N}\ P(k+1)$ by induction.

Base Case: $n = 0$: $1 = \frac{1-x}{1-x}$.

Inductive Case: Assume $P(k)$ holds for some $k \in \mathbb{N}$, this is our induction hypothesis. This implies $\sum_{i=0}^{k+1} x^i = \sum_{i=0}^{k} x^i + x^{k+1} = \frac{1-x^{k+1}}{1-x} + x^{k+1} = \frac{1-x^{k+1}+x^{k+1}-x^{k+2}}{1-x} = \frac{1-x^{k+2}}{1-x}$.

Therefore, the statement holds true for $n = k + 1$.

Problem 2. Prove that for every $n \geq 1$, it is true that:

$$S_n = \frac{1}{2} + \frac{2}{2^2} + \ldots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$$

Thus, you know that $S_n < 2$.

Solution. Let $P(k)$ denote $\forall k \geq 1 S_k = 2 - \frac{k+2}{2^k}$. We will prove that $\forall k \geq 1\ P(k+1)$ by induction.

Base case: $S_1 = \frac{1}{2} = 2 - \frac{3}{2}$.

Inductive case: Assume $P(k)$ holds for some $k \geq 1$, this is our induction hypothesis. This implies

$$\begin{aligned} S_{k+1} &= S_k + \frac{k+1}{2^{k+1}} \\ &= 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \\ &= 2 - \frac{k+3}{2^{k+1}} \end{aligned}$$

Hence, $P(k+1)$ is true.

Problem 3.

a) Use mathematical induction to prove that $(a^n - b^n)$ is divisible by $(a - b)$.

Solution. Let $P(k)$ denote "$(a^k - b^k)$ is divisible by $(a - b)$". We will prove that $\forall k \geq 1\ P(k+1)$ by induction.

Base Case: $n = 1$, the statement is trivially true because $(a^1 - b^1)$ is divisible by $(a - b)$.

Inductive Case: Assume $P(k)$ holds for some $k \geq 1$, this is our induction hypothesis. This implies

$$\begin{aligned} a^{k+1} - b^{k+1} =& a^{k+1} - b^{k+1} + (ba^k - ba^k) \\ =& \left(a^k \cdot a - a^k \cdot b\right) + \left(b \cdot a^k - b \cdot b^k\right) \\ =& a^k(a-b) + b\left(a^k - b^k\right) \end{aligned}$$

and hence $a^{k+1} - b^{k+1}$ is divisible by $a - b$. Thus the statement holds for all natural numbers $k \geq 1$.

b) Using the above, show that $3^{2n} - 1$ is divisible by 8 for every integer $n \geq 1$.

Solution. $3^{2n} - 1 = 9^n - 1$. Thus, this is special case of the above statement where $a = 9$ and $b = 1$.

Problem 4. Define the sequence $\{a_n\}$ for all $n \in \mathbb{Z}^+$ as follows

$$a_1 = 1 \ , \ a_n = n - a_{n-1}$$

a) Conjecture a closed formula for a_n, and prove it by induction. [Hint: It may help to list some terms of the sequence to guess the formula]

Solution. We have two cases: If n is even, and if n is odd. If n is even, then the formula is $a_n = \frac{n}{2}$. On the other hand, if n is odd, then $a_n = \frac{n+1}{2}$.

Base Case: We have two base cases. One for the even numbers formula, and the other for the odd numbers formula. If $n = 2$, then $a_2 = \frac{2}{2} = 1$ is true. And if $n = 1$, then $a_1 = \frac{1+1}{2} = 1$ is true. Inductive Case: We need to do two inductions because we have two cases. Let $P(k)$ denote "if k is even, then $a_k = \frac{k}{2}$", and let $Q(k)$ denote "if k is odd, then $a_k = \frac{k+1}{2}$". If k is even, then assume $P(k)$ is true. This is our first induction hypothesis. If k is odd, then assume $Q(k)$ is true. This is our second induction hypothesis. If k is even, then $k+1$ is odd and we will have $a_{k+1} = \frac{k+2}{2} = \frac{k}{2}+1$ which proves $Q(k+1)$ is true. And if k is odd, then $k+1$ is even and we will have $a_{k+1} = \frac{k+1}{2}$. which proves $P(k+1)$ is true. Hence, $P(k+1)$ and $Q(k+1)$ are true.

b) Compute the value of the sum $100 - (99 - (98 - (97 - (... - (2 - 1)))))$.

Solution. $a_{100} = 50$