CS 131 – Fall 2019, Discussion Worksheet 3
September 25, 2019

Problem 1. Today's problem focuses on limit of a sequence, which is a key concept of calculus.

Definition: **Sequence** is an infinite collection of objects which are enumerated or could be paired off one-to-one with positive integers. This is the key property of a sequence - each element of it has its own number (positive integer) and by giving a positive integer you can get an element of the sequence. In general, sequence could contain object of any type, but today we will be working with sequences which have real numbers as their elements.

Notation: $\{x_n\}$ - sequence, x_n - element of the sequence which has index n where n is a natural number.

Definition: **Limit:** Number x is a **limit** of sequence x_n for each real number $\epsilon > 0$, there exists a natural number n_0 such that, for every natural number $n \geq N$ inequality $|x_n - x| < \epsilon$ holds.

a) Rewrite the definition with quantifiers.

Solution. n, n_0 are natural numbers, x is a real number, and $\{x_n\}$ is a sequence of natural numbers.

x is a limit of a sequence $\{x_n\} \leftrightarrow \forall \epsilon > 0 \; \exists \; n_0 \; \forall n \geq n_0 \; (|x_n - x| < \epsilon)$

b) Negate the definition.

Solution. n, n_0 are natural numbers, x is a real number, and $\{x_n\}$ is a sequence of natural numbers.

x is not a limit of a sequence $\{x_n\} \leftrightarrow \exists \epsilon > 0 \; \forall n_0 \; \exists n > n_0 \; (|x_n - x| \geq \epsilon)$

c) Translate the negation to English.

Solution. x is not a limit of a sequence $\{x_n\}$ if exists a positive number ϵ, such that for any integer n_0 there exists an integer $n > m$, such that the absolute difference between element number n and x is greater than or equal to ϵ.

d) Prove that sequence $\{x_n\}$, $x_n = \frac{1}{n}$ has 0 as its limit.

Solution.

We will apply direct proof:

Given: n_0, n are natural numbers, ϵ is a real number, $\{x_n\}$ is a sequence of real numbers, $x_n = 1/n$.

Asked: 0 is a limit of $\{x_n\}$.

For any ϵ we can take $n_0 > 1/\epsilon$. For any $n > n_0$, since $n > n_0 > 1/\epsilon$ and they are all positive, $1/n < 1/n_0 < \epsilon$, thus $|1/n - 0| < \epsilon$. QED.

e) Prove that sequence $\{x_n\}$, $x_n = sin(\pi x/2)$ doesn't have a limit.

Solution.

We apply proof by contradiction:

Given: n_0, n are natural numbers, x, ϵ are real numbers, $\{x_n\}$ is a sequence of real numbers, $x_n = sin(\pi x/2)$, x is a limit of $\{x_n\}$.

Asked: Falsehood

Suppose the x is a limit of $\{x_n\}$. Let's choose $\epsilon = 0.5$. For any number n_0 in next 4 following numbers will be values 1 and -1. Thus x should satisfy $|x-1| < 0.5 \wedge |x+1| < 0.5$, which is False. QED.

f) Prove that if sequence $\{x_n\}$ has a limit x, x' ($x' \neq x$) cannot be it's limit.

Solution.

We apply proof by contraposition:

Given: n_1, n_2, n are natural numbers, x, x', ϵ are real numbers, $\{x_n\}$ is a sequence of real numbers, x' is a limit of $\{x_n\}$.

Asked: x is not a limit of $\{x_n\}$.

Let's choose $\epsilon < |x' - x|/2$, for example $\epsilon = |x' - x|/4$. Thus, there exists number n_2 such that $\forall n > n_2\ |x_n - x'| < \epsilon$. But $\forall n > n_2\ |x_n - x| > \epsilon$, thus for this ϵ $\neg\exists n_1\ \forall n > n_1\ |x_n - x| < \epsilon$, thus x is not limit of $\{x_n\}$. QED.