Problem 1. Suppose P is a predicate on one argument and Q is a predicate on two arguments. Evaluate if statements above are equivalent.

a) $\forall x \forall y\ Q(x, y)$ and $\forall y \forall x\ Q(x, y)$

Solution. The statements are equivalent since order of the same quantifiers doesn't matter.

b) $\forall x \exists y\ Q(x, y)$ and $\exists y \forall x\ Q(x, y)$

Solution. The statements are not equivalent. Counterexample is $Q(x, y) = (y > x)$. The left-hand side is true, but the right-hand side is false.

c) $\forall x(P(x) \vee (\exists y Q(x, y)))$ and $\forall x \exists y(P(x) \vee (Q(x, y)))$

Solution. The statements are equivalent since $P(x)$ is independent from y.

Problem 2.

Let $T(x, y)$ means that student x likes cuisine y, where the domain of x consists of all students at BU and the domain of y consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\forall x \forall z \exists y((x \neq z) \to \neg(T(x, y) \wedge T(z, y)))$

Solution. For any two distinct students there exists a cuisine which they don't both like (at least one of them dislikes).

b) $\exists x \exists z \forall y(T(x, y) \leftrightarrow T(z, y))$

Solution. Exists two students which like exactly the same set of cuisines.

c) $\forall x \forall z \exists y(T(x, y) \leftrightarrow T(z, y))$

Solution. For any two students there exists a cuisine they both like or dislike.

Let $S(x)$ be a predicate "x is a student", $F(x)$ be a predicate "x is a faculty member", and $A(x, y)$ the predicate "x has asked y a question", where the domain consists of all people associated with BU. Use quantifiers to express each of these statements.

d) Some student asked every faculty member a question.

Solution. $\exists x S(x) \wedge \forall y(F(y) \to A(x, y))$

e) There is a faculty member who has asked every other faculty member a question.

Solution. $\exists x F(x) \wedge \forall y(F(y) \wedge (x \neq y) \to A(x, y))$

f) some student has never been asked a question by a faculty member

Solution. $\exists x S(x) \wedge \forall y(F(y) \to \neg A(y, x))$

Problem 3.

a) Express the following sums and products by using $\sum$ and/or $\prod$ notations:

i) $\frac{1}{4} + 1 + \frac{9}{4} + 4 + \frac{25}{4} + ...$

Solution. $\sum_{i=1}^{\infty} (\frac{i}{2})^2$

ii) $-8 + 13 - 18 + 23 - 28$

Solution. $\sum_{i=1}^{5} (-1)^i (5i + 3)$

iii) $1 \cdot \frac{1}{3} \cdot \frac{1}{9} \cdot \frac{1}{27} \cdot \frac{1}{81} ...$

Solution. $\prod_{i=0}^{\infty} \frac{1}{3^i}$

b) Evaluate/Compute the following sums and product sums:

i) $\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{i}{j}$

Solution.

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \frac{i}{j} = 1 + \frac{1}{2} + \frac{1}{3} + 2 + 1 + \frac{2}{3} + 3 + \frac{3}{2} + 1 = 11$$

ii) $\prod_{i=0}^{3} \frac{2^{2i}}{3^i}$

Solution.

$$\prod_{i=0}^{3} \frac{2^{2i}}{3^i} = 1 \cdot \frac{2^2}{3} \cdot \frac{2^4}{3^2} \cdot \frac{2^6}{3^3} = \frac{2^{12}}{3^6}$$

iii) $\prod_{i=1}^{3} \prod_{j=1}^{2} ij^2$

Solution.

$$\prod_{i=1}^{3} \prod_{j=1}^{2} ij^2 = (1 \cdot 1) \cdot (1 \cdot 4) \cdot (2 \cdot 1) \cdot (2 \cdot 4) \cdot (3 \cdot 1) \cdot (3 \cdot 4) = 4^3 \cdot 3^2 \cdot 2^2 = 2304$$

Problem 4. Prove that if $x = 1$ or $x = 2$, then $x^2 - 3x + 2 = 0$.

Solution. The logical expression that we are asked to prove is $\varphi := \Big((x = 1) \vee (x = 2)\Big) \rightarrow x^2 - 3x + 2 = 0$. We are going to do a proof by contraposition. The contrapositive of φ is $x^2 - 3x + 2 \neq 0 \rightarrow \Big((x \neq 1) \wedge (x \neq 2)\Big)$. Hence, $x^2 - 3x + 2 \neq 0$ is a given in a proof by contraposition. So we obtain $x^2 - 3x + 2 = (x - 1)(x - 2) \neq 0$. Hence, we reach the conclusion $\Big((x \neq 1) \wedge (x \neq 2)\Big)$.

Problem 5. Prove that there is an integer N such that for all integers $n \geq N$, $2^n < n!$.

Solution. We will prove this for $N = 4$ by induction. Let $P(k)$ be true iff $2^k < k!$.

Basis step: since $16 = 2^4 < 4! = 24$, $P(4)$ is true.

Inductive step: suppose $P(k)$ is true for some integer $k \geq 4$; we will show that $P(k + 1)$ is also true. Since $2^k < k!$, $2^{k+1} < 2^k(k + 1) < k!(k + 1) = (k + 1)!$. Hence, $P(k + 1)$.

By mathematical induction, $P(k)$ is true for all integers $k \geq 4$.