Practice Problems
CS131

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1 Logic

Problem 0. Answer the following questions:

a) Write the truth table for $p \to q$.

b) Suppose you want to prove a theorem of the form $p \to q$. Can you connect the direct, contrapositive, and contradiction proof techniques to the rows of this table? Please explain the connection in clear words.

Answer. Check your notes. I have explained this in detail during a lecture.

Problem 1. Express the following statements in quantificational logic.

1. Someone did not get an A grade.
2. $A \subseteq B$
3. $A \cap B \subseteq B \backslash C$

Answer. 1. $\exists x \neg A(x)$, where $A(x)$ means that student x got an A grade.

2. $\forall (x \in A \to x \in B)$
3. $\forall (x \in A \wedge x \in B \to x \in B \wedge x \notin C)$

Problem 2. Consider the following theorem: *for every real number x, $x^2 \geq 0$.*What is wrong with this proof:

> Suppose not for the sake of contradiction. Then for every real number x, $x^2 < 0$. In particular, plugging in $x = 4$ we would get $16 < 0$, which is clearly false. Contradiction!

Answer. The sentence "Then for every real number x, $x^2 < 0$" is wrong (why? recall that the negation of a universal quantifier results in an existential quantifier).

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Problem 3. Express the following statements in quantificational logic, and prove/disprove each. Be precise about the predicates you use, but also feel free to avoid dividing a predicate when we have already introduced related notation in class (e.g., instead of defining $D(a, b)$ as the predicate for a divides b, use directly $a|b$ in your notation).

1. For all integers a, b, c if a divides b, and b divides c, then a divides c.
2. Let x be an arbitrary real. Then, there exists a real y such that $xy^2 \neq y - x$.
3. For every prime p, $p + 3$ is a composite.

Answer.

1. $\forall a, b, c \in \mathbb{Z}(a|b \wedge b|c \rightarrow a|c)$. True. We have proved it in class (check notes and/or textbook)
2. $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(xy^2 \neq y - x)$. True. When $x = 0$ set $y = 1$. When $x \neq 0$ set $y = 0$ (complete the details of the proof in detail).
3. $\forall x \in \mathbb{Z}(P(x) \rightarrow \neg P(x + 3))$. False. Set $x = 5$. (complete the details of the proof in detail)

2 Recursive algorithms

Problem 1. Devise a recursive algorithm for computing $b^n \mod m$ where b, n and m are integers with $m \geq 2, n \geq 0$ and $1 \leq b < m$.

Answer. See Rosen's book, page 383.

Problem 2. Devise a recursive algorithm for computing Fibonacci numbers.

Answer. See Rosen's book, page 386.

Problem 3. Describe the binary search algorithm (input, output, how it works, and its running time analysis).

Answer. See your notes.

3 Induction

Problem 1. Find a formula for $\sum_{i=1}^{n} i$, and prove it using induction.

Answer. This was the first example we saw. Check the textbook and/or your notes.

Problem 2. Let the "Tribonacci sequence" be defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Prove that $T_n < 2^n$ for all positive integers n.

Answer. For the solutions to Problems 1,2, check the pdf online here `https://faculty.math.illinois.edu/~hildebr/213/inductionsampler.pdf`

Problem 3. Consider the function $f(k) = 2k+1$, where k is an integer. Prove that for all integers k and $n \geq 1$, $\underbrace{f(f(\ldots(f(k)))}_{n \text{ compositions of function}} \equiv 1 \mod 2$.

Proof. Using induction (we have seen this before) we obtain that $\underbrace{f(f(\ldots(f(k)))}_{n \text{ compositions of function}} = 2^n k + 2^n - 1 = 2^n(k+1) - 1$. When $n \geq 1$ this is always odd for all k (why?) $\square$

4 Number theory.

Problem 1. Let a, b, c, d be integers. If $(a-c)$ divides $ab+cd$ then $a-c$ also divides $ad+bc$.

Proof. Notice that $(ab+cd)-(ad+bc) = a(b-d)+c(d-b) = (a-c)(b-d)$. Therefore $ad+bc = (ab+cd)-(a-c)(b-d)$. Since $a-c|(a-c)(b-d)$, and $a-c|ab+cd$, it also divides their linear combinations. We conclude that $a-c|(ab+cd)-(a-c)(b-d) = ad+bc$. $\square$

Problem 2. Compute the following values of the Euler ϕ-function.

1. $\phi(15)$. Solve the using the definition of the ϕ-function.
2. Let $p_1, \ldots, p_r$ be the primes that appear in the prime factorization of integer n. Then,

$$\phi(n) = n \prod_{i=1}^{r} \frac{p_i - 1}{p_i}.$$

3. $\phi(900)$. Solve this using the previous question.

Answer.

1. Among positive numbers less than 15, eliminate multiples of 3 or 5, which are $\{3, 5, 6, 9, 10, 12\}$. The remaining numbers are $\{1, 2, 4, 7, 8, 11, 13, 14\}$ so $\phi(15) = 8$
2. Use the fact that $\phi(p^k) = p^k - p^{k-1}$ when p is a prime, and the multiplicative property of ϕ function (fill in the algebraic details).
3. Observe that $900 = 2^2 \cdot 3^2 \cdot 5^2$. Therefore $\phi(900) = 900\frac{1\cdot2\cdot4}{2\cdot3\cdot5} = 240$.

Problem 3. Prove the following statements:

1. $gcd(n, n+1) = 1$
2. $gcd(2n-1, 2n+1) = 1$
3. $gcd(2n, 2n+2) = 2$
4. $gcd(a, b) = gcd(a, a+b)$
5. $gcd(5a+3b, 13a+8b) = gcd(a, b)$

Proof. We illustrate how to solve one the most complicated one. Make sure you can do the rest on your own. We just apply Euclid's idea.

$$\begin{aligned} gcd(5a+3b, 13a+8b) &= gcd(5a+3b, 13a+8b-2(5a+3b)) = gcd(5a+3b, 3a+2b) = \\ &= gcd(3a+2b, 2a+b) = gcd(2a+b, a+b) = gcd(a+b, a) = gcd(a, b). \end{aligned}$$

□

Problem 4. Prove that $9|4^n + 15n - 1$ for all non-negative integers n.

Proof. We use induction. Let $f(n) = 4^n + 15n - 1$.
Basis: $f(0) = 0$, therefore $9|f(0)$.
Inductive step: Assume that $9|f(n)$, we will prove that 9 also divides $f(n+1)$. Notice

$$f(n+1) = 4^{n+1} + 15n + 15 - 1 = 34^n + 4^n + 15n - 1 + 15 = f(n) + 3(4^n + 5).$$

Since by our IH $9|f(n)$ it suffices to show that $9|3(4^n + 5)$. Equivalently, it suffices to prove that $3|4^n + 5$ for all non-negative integers n. Interestingly, we need to use again induction to prove this claim and finish off the proof. Indeed for $n = 0$, $3|4^0 + 5 = 6$. Suppose that $3|4^n + 5$. Observe that $4^{n+1} + 5 = 3 \cdot 4^n + 4^n + 5$. Since $3|3 \cdot 4^n$, and by IH $3|4^n + 5$. Therefore 3 also divides their sum, i.e., $3|4^{n+1} + 5$. □

5 Counting

Problem 1. A total of 1 232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2 092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

Answer. See Rosen, Section 8.5 (Inclusion-Exclusion), example 4.

Problem 2. How many $n \times n$ matrices with entries 0 or 1 have even row and sum columns?

Proof. The number of even element subsets of $\{1, \ldots, n\}$ is 2^{n-1} (why?). Therefore there are 2^{n-1} ways to choose a row with an even number of ones. The first $(n-1)$ rows can be chosen arbitrarily, in $(2^{n-1})^{n-1} = 2^{(n-1)^2}$ ways. The final row is chosen so that the the condition that the sum of the columns is even (how?). Therefore the total number of ways is $(2^{n-1})^{n-1} = 2^{(n-1)^2}$. □

6 Combinatorial proofs

Prove the following identities using combinatorial proofs.

Vandermonde's identity. Let m, n, r be non-negative integers with r not exceeding either m or n. Then,

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}.$$

Answer. See Rosen's textbook, page 442, Theorem 3.

Christmas stocking identity. Let $n, r \in \mathbb{N}, n > r$. Then,

$$\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}.$$

Answer. See the Wikipedia article for two combinatorial proofs `https://en.wikipedia.org/wiki/Hockey-stick_identity`.

7 Pigeonhole principle

Problem 1. How many six-digit numbers do you need to choose to ensure that at least two of them have the same last three digits?

1. In the decimal system.
2. In the binary system.
3. In the hex system.

Answer. Let b be the base ($b = 10, 2, 16$). Then we need to select $b^3 + 1$ numbers (why?).

Problem 2. During a month with 30 days, a baseball team plays at least one game a day, but no more than 45. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

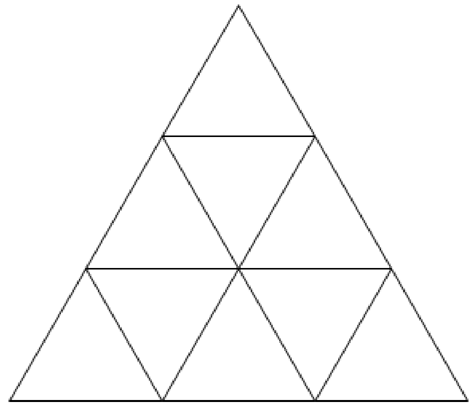
Answer. The solution is in the textbook, see example 10, page 424.

Problem 3. Ten points are placed within a unit equilateral triangle. Show that there exists two points with distance at most $\frac{1}{3}$ apart.

Proof. Consider the partition of the triangle into nine congruent equilateral triangles as shown below:

$\square$

Each of the nine smaller triangles represents a box, with each of the ten points an item to be placed into the boxes. By the pigeonhole principle, at least one of the nine triangles must contain at least two points. Since the maximum distance between any two points in one of these triangles is $\frac{1}{3}$, no two such points can be separated by greater than distance $\frac{1}{3}$.



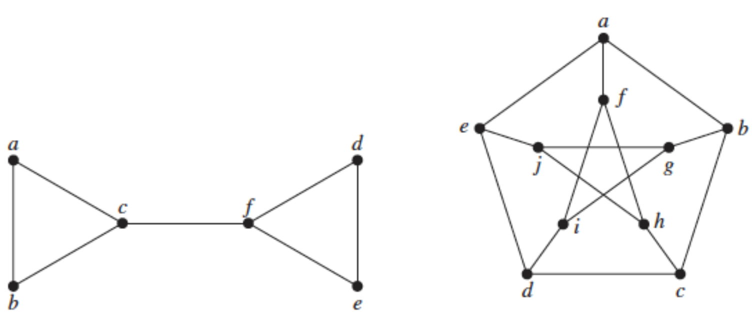
8 Graph theory

Problem 0. (a) State and prove the handshaking lemma. (b) Prove Hall's theorem using induction.

Answer. See the Rosen's textbook for answers to both (a), (b)[1].

Problem 1. Let $G(V, E)$ be a graph with n vertices such that the degree of any vertex is at least $\lceil n/2 \rceil$. Prove that G is connected.

Proof. Suppose G is not connected. This means that there exist at least two connected components, so the smallest component has size less or equal than $\lfloor \frac{n}{2} \rfloor$ (why?). Let's call the smallest component C. The degree of any vertex in the connected component C of G would be less or equal than $|C| - 1 \leq \lfloor \frac{n}{2} \rfloor - 1 < \lceil n/2 \rceil$. Contradiction (why?). □



(a) (b)

Figure 1: See Problem 5 in Section 8.

Problem 2. Let G be a simple graph on n nodes with k connected components, and let these components have $n_1, \ldots, n_k$ vertices.

1. Evaluate the sum $\sum_{i=1}^{k} n_i$.

[1]The proof of Hall's theorem will not be tested in the final but you need to know what the theorem says.

2. What is the maximum number of edges in G as a function of $n_1, \ldots, n_k$?
3. What is the maximum number of edges in a disconnected graph G? Characterize the structure of such graphs.

Answer. 1. $\sum_{i=1}^{k} n_i = n$ (why?)

2. $\sum_{i=1}^{k} \binom{n_i}{2}$. (why?)
3. $\binom{n-1}{2}$ (why?)[2]

Problem 3. Find the number of perfect matchings for the cycle C_n.

Answer. If n is odd, the number of perfect matchings is 0 (why?). If n is even then their number is 2 (why?).

Problem 4. A graph G is called self-complementary if G and its complement $\bar{G}$ are isomorphic. Show that if G is self-complementary graph with n vertices, then $n \equiv 0 \mod 4$ or $n \equiv 1 \mod 4$.

Proof. Since $G, \bar{G}$ are isomorphic they have equal number of edges, i.e., $m_G = m_{\bar{G}}$. Since they are complementary, $m_G + m_{\bar{G}} = \binom{n}{2}$. Therefore $G, \bar{G}$ have $\frac{n(n-1)}{4}$ edges. Since this number has to be an integer, we derive the desired result. □

Problem 5. Determine for each of the graphs in Figure 1 whether there exists a Hamilton Circuit or not.

Proof. (a) A Hamiltonian circuit needs to start and end at the same vertex. Observe that once edge *cf* is used in order to return to the starting vertex you must pass though this edge again. Therefore, there is no Hamiltonian circuit.

(b) This graph is an important graph known as Petersen's graph, and it does not have a Hamiltonian circuit. Observe the graph consists of 10 nodes and 15 edges. It is convenient to categorize these edges in three categories: (i) five edges of the outer pentagon, (ii) five edges of the inner pentagram, and (iii) five crossing edges. A Hamiltonian circuit must return to its starting point, and so must use an even number of crossing edges. Since we need to cross at least one such edge to connect the outer and the inner parts, this means there have to be either 2 or 4 crossing edge.

Case I: If there are two crossing edges, then their ends in both the outer pentagon and the inner pentagram must be joined by paths of length 4. This implies that these ends must be adjacent in both the outer pentagon and the inner pentagram, which is impossible.

Case II: If there are four crossing edges, then we must have three edges in both the outer pentagon and the inner pentagram, including two through each of the points not on chosen crossing edges. There is a unique such configuration, but it consists of two 5-cycles rather than a Hamiltonian circuit. □

[2] This is the number of edges in a graph with one large connected component K_{n-1} (clique on $n-1$ nodes), and an isolated node. Prove that formally that this is the disconnected graph with the maximum possible number of edges. Argue about the following: (i) The number of edges is maximized when each connected component is a complete graph and there are fewer connected components, but at least two (so that the graph is disconnected). (ii) Maximizing $\binom{k}{2} + \binom{n-k}{2}$ for $1 \leq k \leq \frac{n-1}{2}$ happens at $k = 1$.

Problem 6. Answer the following questions.

1. State Ore's theorem. Does it provide a necessary or a sufficient condition for the existence of a Hamilton circuit in a graph?

2. State Dirac's theorem. Prove Dirac's theorem as a simple corollary of Ore's theorem.

Answer. See Rosen, Chapter 10 p. 736.