Mining Large-Scale Networks
Lecture 2

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empirical properties of graphs and networks

Properties of real-world networks

diverse collections of graphs arising from different phenomena

are there typical patterns?

- static networks
  1. heavy tails
  2. clustering coefficients
  3. communities
  4. small diameters
- time-evolving networks
  1. densification
  2. shrinking diameters
- web graph
  1. bow-tie structure
  2. bipartite cliques

Heavy tails

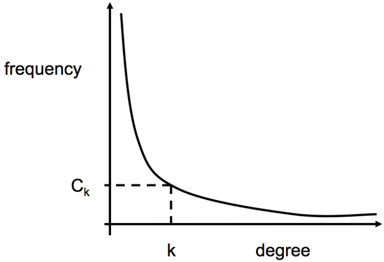
What do the proteins in our bodies, the Internet, a cool collection of atoms and sexual networks have in common? One man thinks he has the answer and it is going to transform the way we view the world.

Scientist 2002

Albert-László Barabási

Degree distribution

- C_k = number of vertices with degree k



- problem : find the probability distribution that fits best the observed data

Power-law degree distribution

- C_k = number of vertices with degree k, then

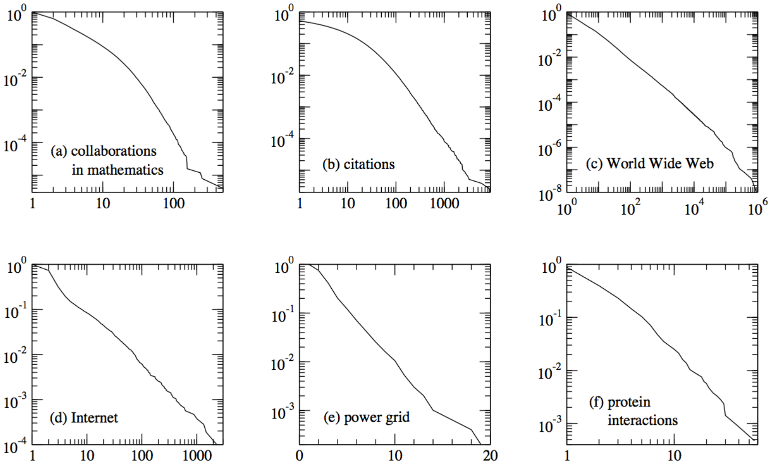
$$C_k = ck^{-\gamma}$$

with $\gamma > 1$, or

$$\ln C_k = \ln c - \gamma \ln k$$

- plotting $\ln C_k$ versus $\ln k$ gives a straight line with slope $-\gamma$
- heavy-tail distribution : there is a non-negligible fraction of nodes that has very high degree (hubs)
- scale free : average is not informative

Power-law degree distribution



power-laws in a wide variety of networks ([Newman, 2003])
sheer contrast with Erdős-Rényi random graphs

Power-law degree distribution

do the degrees follow a power-law distribution?

three problems with the initial studies

- graphs generated with traceroute sampling, which produces power-law distributions, even for regular graphs [Lakhina et al., 2003].
- methodological flaws in determining the exponent see [Clauset et al., 2009] for a proper methodology
- other distributions could potentially fit the data better but were not considered, e.g., lognormal.

disclaimer: we will be referring to these distributions as heavy-tailed, avoiding a specific characterization

Power-law degree distribution

- frequently, we hear about "scale-free networks"
  correct term is networks with scale-free degree distribution

  all networks above have the same degree sequence but structurally are very different (source [Li et al., 2005])

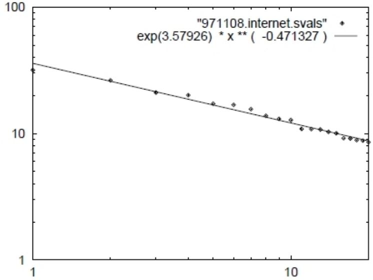
Maximum degree

- for random graphs, the maximum degree is highly concentrated around the average degree z
- for power-law graphs

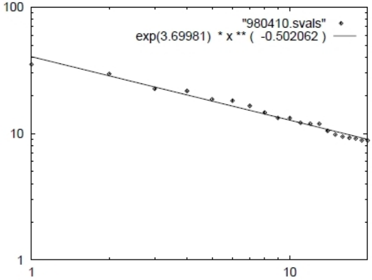
$$d_{\mathrm{max}} \approx n^{1/(\alpha-1)}$$

- hand-waving argument: solve $\quad n \Pr[X \geq d] = \Theta(1)$

Heavy tails, eigenvalues



(a) Int-11-97

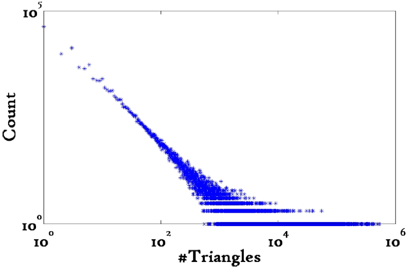


(b) Int-04-98

log-log plot of eigenvalues of the Internet graph in decreasing order

again a power law emerges [Faloutsos et al., 1999]

Heavy tails, triangles



- triangle distribution in `flickr`
- figure shows the count of nodes with k triangles vs. k in log-log scale
- again, heavy tails emerge [Tsourakakis, 2008]

Clustering coefficients

- a proposed measure to capture local clustering is the graph transitivity

$$T(G) = \frac{3 \times \text{number of triangles in the network}}{\text{number of connected triples of vertices}}$$

- captures "transitivity of clustering"
- if u is connected to v and
  v is connected to w, it is also likely that
  u is connected to w

Clustering coefficients

- alternative definition
- local clustering coefficient

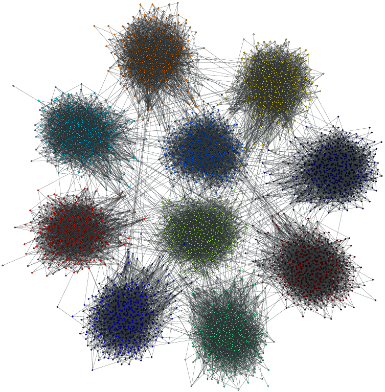
$$C_i = \frac{\text{Number of triangles connected to vertex } i}{\text{Number of triples centered at vertex } i}$$

- global clustering coefficient

$$C(G) = \frac{1}{n} \sum_i C_i$$

Community structure

loose definition of community: a set of vertices densely connected to each other and sparsely connected to the rest of the graph



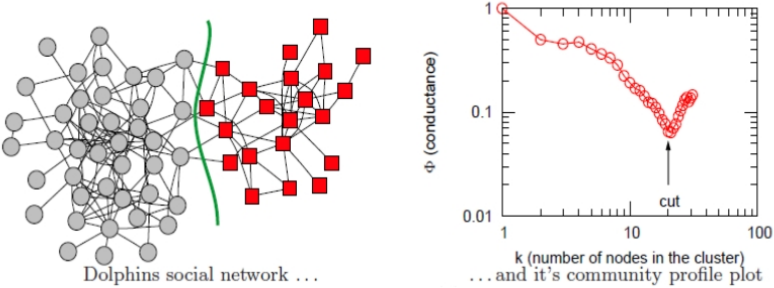
artificial communities:
http://projects.skewed.de/graph-tool/

Community structure

[Leskovec et al., 2009]

- study community structure in an extensive collection of real-world networks
- authors introduce the network community profile plot
- it characterizes the best possible community over a range of scales

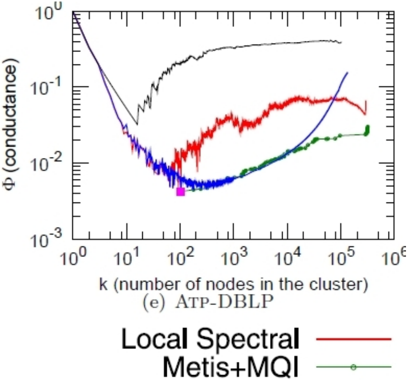
Community structure



`dolphins` network and its NCP
(source [Leskovec et al., 2009])

Community structure

- do large-scale real-world networks have this nice artifical structure? NO!



NCP of a DBLP graph (source [Leskovec et al., 2009])

Community structure

important findings of [Leskovec et al., 2009]

1. up to a certain size k ($k \sim 100$ vertices) there exist good cuts
   - as the size increases so does the quality of the community
2. at the size k we observe the best possible community
   - such communities are typically connected to the remainder with a single edge
3. above the size k the community quality decreases
   - this is because they blend in and gradually disappear

Small-world phenomena

small worlds : graphs with short paths

- Stanley Milgram (1933-1984)
  "The man who shocked the world"
- obedience to authority (1963)
- small-World experiment (1967)

- we live in a small-world
- for criticism on the small-world experiment, see *"Could It Be a Big World After All? What the Milgram Papers in the Yale Archives Reveal About the Original Small World Study"* by Judith Kleinfeld

Small-world experiments

- letters were handed out to people in Nebraska to be sent to a target in Boston
- people were instructed to pass on the letters to someone they knew on first-name basis
- the letters that reached the destination (64 / 296) followed paths of length around 6
- *Six degrees of separation* : (play of John Guare)
- also:
  - the Kevin Bacon game
  - the Erdős number
- small-World project: http://smallworld.columbia.edu/index.html

Small diameter

proposed measures

- diameter : largest shortest-path over all pairs.
- effective diameter : upper bound of the shortest path of 90% of the pairs of vertices.
- average shortest path : average of the shortest paths over all pairs of vertices.
- characteristic path length : median of the shortest paths over all pairs of vertices.
- hop-plots : plot of $|N_h(u)|$, the number of neighbors of u at distance at most h, as a function of h [Faloutsos et al., 1999].

Other properties

- assortativity
- distribution of size of connected components
- distribution of motifs
- ...

Time-evolving networks

J. Leskovec

J. Kleinberg

C. Faloutsos

[Leskovec et al., 2005]

- densification power law:

$$|E_t| \propto |V_t|^{\alpha} \qquad 1 \leq \alpha \leq 2$$

- shrinking diameters: diameter is shrinking over time.

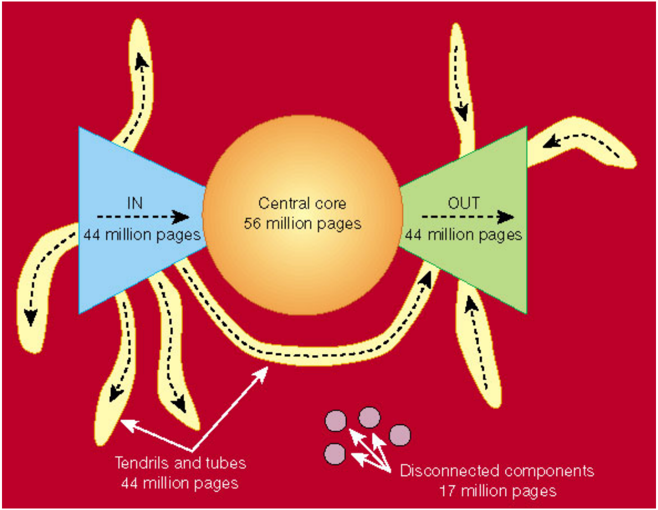
Web graph

- the Web graph is a particularly important real-world network

  > *Few events in the history of computing have wrought as profound an influence on society as the advent and growth of the World Wide Web*
  >
  > [Kleinberg et al., 1999]

- vertices correspond to static web pages
- directed edge (i, j) models a link from page i to page j
- will discuss two structural properties of the web graph:
  1. the bow-tie structure [Broder et al., 2000]
  2. abundance of bipartite cliques [Kleinberg et al., 1999, Kumar et al., 2000]

Web is a bow-tie

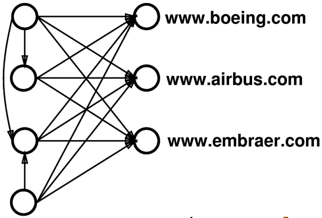


(source [Broder et al., 2000])

Bipartite subgraphs

- websites that are part of the same community frequently do not reference one another

  (competitive reasons, disagreements, ignorance) [Kumar et al., 1999].
- similar websites are *co-cited*
- therefore, web communities are characterized by dense directed bipartite subgraphs



(source [Kleinberg et al., 1999])

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