Feynman–Kac Framework for Parabolic PDEs: Foundations and Application to Option Pricing

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Introduction

This note is devoted to a comprehensive treatment of the Feynman–Kac formula in the context of parabolic partial differential equations and its application to option pricing in the Black–Scholes model based on "Stochastic Calculus: An Introduction Through Theory and Exercises" by P. Baldi (2017). In the first part we recall, without undue detail, the classical results from stochastic calculus and the theory of second-order parabolic PDEs that underpin the probabilistic representation of solutions. Readers already familiar with Brownian motion, Itô's formula, and the existence and uniqueness theory for linear parabolic boundary-value problems may safely skip the Preliminaries and refer back only as needed.

We then fix notation and hypotheses in the Setting and Notation section, where we introduce the diffusion process associated to a uniformly elliptic generator and formulate the backward Cauchy–Dirichlet problem on a bounded spatial domain. Building on Baldi's Theorem 10.4, we prove the Feynman–Kac representation in this bounded setting by applying Itô's formula to a suitably discounted test function, identifying a martingale up to the first exit time, and invoking optional stopping.

The subsequent section based on Baldi's Theorem 10.5 shows how to lift this result to the unbounded whole-space case. By approximating $\mathbb{R}^m$ with an increasing sequence of bounded domains and controlling exit probabilities via moment estimates, we pass to the limit and obtain the Feynman–Kac formula on $\mathbb{R}^m$ under natural growth or nonnegativity conditions on the data.

Having obtained the stochastic representation, we turn to the existence and uniqueness of the classical solution on the unbounded domain. Similar to Baldi's Theorem 10.6, we define the candidate solution by the expectation formula, verify its continuity and differentiability by careful estimates and conditioning arguments, and then employ a stopping-time approach together with the maximum principle to rule out competing solutions in the same growth class.

In the final part of the paper we apply the Feynman–Kac theorem to the multi-asset Black–Scholes market under the risk-neutral measure, we derive the backward pricing PDE by martingale arguments and Itô's formula, and then recover the no-arbitrage price as the discounted expectation of the terminal payoff. In the one-dimensional case this yields the familiar closed-form Black–Scholes formula for European calls and puts, while more complicated payoffs can be handled by Monte Carlo evaluation of the expectation.

1 Preliminaries

1.1 Probability spaces and filtrations

A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where

- Ω is a sample space,
- $\mathcal{F}$ is a σ-algebra of subsets of Ω,
- $\mathbb{P} : \mathcal{F} \to [0, 1]$ is a probability measure with $\mathbb{P}(\Omega) = 1$.

A *filtration* is an increasing family of sub-σ-algebras

$$\mathcal{F}_0 \subseteq \mathcal{F}_{t_1} \subseteq \mathcal{F}_{t_2} \subseteq \cdots \subseteq \mathcal{F}, \qquad 0 \leq t_1 \leq t_2 \leq \cdots .$$

It represents the information available up to time t.
A stochastic process $(X_t)_{t \geq 0}$ is *adapted* to $(\mathcal{F}_t)$ if for each t, X_t is $\mathcal{F}_t$-measurable.
If $Y \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G} \subseteq \mathcal{F}$ is a sub-σ-algebra, then $\mathbb{E}[Y \mid \mathcal{G}]$ is the (a.s.) unique $\mathcal{G}$-measurable random variable satisfying

$$\int_G \mathbb{E}[Y \mid \mathcal{G}] \, d\mathbb{P} = \int_G Y \, d\mathbb{P} \quad \text{for all } G \in \mathcal{G}.$$

1.2 Brownian Motion

A process $(W_t)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is a *standard Brownian motion* (or Wiener process) if

1. $W_0 = 0$ almost surely,
2. *(Independent increments)* for $0 \leq s < t$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$,
3. *(Gaussian increments)* $W_t - W_s \sim N(0, t - s)$,
4. *(Continuity)* $t \mapsto W_t(\omega)$ is almost surely continuous.

Let (W_t) be a standard Brownian motion with its natural filtration $(\mathcal{F}_t)$.

- (W_t) is an $(\mathcal{F}_t)$-martingale (i.e., for all $0 \le s \le t$, $\mathbb{E}\big[W_t \mid \mathcal{F}_s\big] = W_s$).
- For any bounded Borel $f : \mathbb{R} \to \mathbb{R}$, the process

$$M_t \;=\; f(W_t) \;-\; \int_0^t \tfrac{1}{2} f''(W_s)\, ds$$

  is also an $(\mathcal{F}_t)$-martingale.
- (W_t) is a Markov process: for any bounded measurable g and $h > 0$,

$$\mathbb{E}\big[g(W_{t+h}) \mid \mathcal{F}_t\big] = \mathbb{E}\big[g(W_{t+h}) \mid W_t\big] = \int_{\mathbb{R}} g(y)\, \frac{1}{\sqrt{2\pi h}} \exp\Big(-\tfrac{(y-W_t)^2}{2h}\Big)\, dy.$$

  Moreover, the transition density $p_h(y \mid x) = \dfrac{1}{\sqrt{2\pi h}} e^{-(y-x)^2/(2h)}$ satisfies the Chapman–Kolmogorov equations

$$p_{t+s}(z \mid x) \;=\; \int_{\mathbb{R}} p_t(z \mid y)\, p_s(y \mid x)\, dy,$$

  reflecting the semigroup (heat-kernel) property.

1.3 The Itô integral and Itô isometry

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ support an m-dimensional Brownian motion $W = (W^1, \dots, W^m)$.

Denote by $M^2([0,T];\mathbb{R}^m)$ the space of $\mathbb{R}^m$-valued, $(\mathcal{F}_t)$-progressively measurable processes $X = (X_t)_{0 \le t \le T}$ such that

$$\mathbb{E}\Big[\int_0^T |X_t|^2\, dt\Big] \;<\; +\infty.$$

There exists a unique linear map

$$I :\; M^2([0,T];\mathbb{R}^m) \;\longrightarrow\; L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}), \quad X \;\mapsto\; \int_0^T X_s \cdot dW_s,$$

characterised by

$$\int_0^T X_s \cdot dW_s \;=\; \sum_{i=0}^{n-1} X_{t_i} \cdot \big(W_{t_{i+1}} - W_{t_i}\big) \quad \text{for every elementary process } X_t = \sum_{i=0}^{n-1} X_{t_i}\, 1_{[t_i, t_{i+1})}(t),$$

and satisfying the *Itô isometry*

$$\mathbb{E}\Big[\Big|\int_0^T X_s \cdot dW_s\Big|^2\Big] \;=\; \mathbb{E}\Big[\int_0^T |X_s|^2\, ds\Big].$$

Moreover $t \mapsto \int_0^t X_s \cdot dW_s$ is an $(\mathcal{F}_t)$-martingale.

1.4 Itô's formula

Let $X = (X^1, \ldots, X^m)$ be an Itô process satisfying

$$dX_t^i \;=\; b_i(X_t,t)\,dt \;+\; \sum_{j=1}^m \sigma_{ij}(X_t,t)\,dW_t^j, \qquad X_0 = x \in \mathbb{R}^m,$$

with b, σ Lipschitz and of linear growth.

Itô's formula. If $f \in C^{2,1}(\mathbb{R}^m \times [0,T])$ then the process $Y_t = f(X_t,t)$ satisfies

$$dY_t = \Big(\partial_t f + L_t f\Big)(X_t,t)\,dt \;+\; \sum_{i,j=1}^m \partial_{x_i} f(X_t,t)\,\sigma_{ij}(X_t,t)\,dW_t^j,$$

where the *generator* L_t is the second-order operator

$$L_t f(x,t) \;=\; \sum_{i=1}^m b_i(x,t)\,\partial_{x_i} f(x,t) \;+\; \tfrac{1}{2}\sum_{i,j=1}^m a_{ij}(x,t)\,\partial^2_{x_i x_j} f(x,t), \quad a = \sigma\sigma^\top.$$

In particular, if $\partial_t f + L_t f \equiv 0$, then $f(X_t,t)$ is a local martingale.

1.5 Stochastic Differential Equations

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ carry a d-dimensional Brownian motion B_t.

Let $b : \mathbb{R}^m \times [0,T] \to \mathbb{R}^m, \quad \sigma : \mathbb{R}^m \times [0,T] \to \mathbb{R}^{m\times d}$ be measurable. A *(weak) solution* of the SDE

$$dX_t \;=\; b(X_t,t)\,dt \;+\; \sigma(X_t,t)\,dB_t, \qquad X_0 = x \in \mathbb{R}^m,$$

is a filtered probability space supporting a Brownian motion B and an adapted process X satisfying the integral equation. If *for every* Brownian motion on the given space there is an adapted X solving the same equation the SDE adopts a *strong solution*.

Uniqueness in law: any two (weak) solutions with the same initial law induce the same law on $C([0,T];\mathbb{R}^m)$.

Pathwise (strong) uniqueness: if two solutions on the *same* probability space with the *same* Brownian motion and X_0 agree almost surely.

Lipschitz and sublinear growth assumption. There exist constants $L, M > 0$ such that for all $x, y \in \mathbb{R}^m$, $t \in [0,T]$,

$$|b(x,t)| + |\sigma(x,t)| \;\le\; M\,(1+|x|), \quad |b(x,t)-b(y,t)| + |\sigma(x,t)-\sigma(y,t)| \;\le\; L\,|x-y|.$$

Existence and pathwise uniqueness.[1] Under Lipschitz and sublinear growth assumption, for every square-integrable $\mathcal{F}_u$-measurable $X_u^{x,u} = x \in \mathbb{R}^m$ there *exists a unique strong solution* $X_t^{x,u}$ *of the SDE*:

$$X_t^{x,u} \;=\; x \;+\; \int_u^t b(X_t^{x,u},s)\,ds \;+\; \int_u^t \sigma(X_s^{x,u},s)\,dB_s, \quad t\in[u,T],$$

and any two solutions on the same Brownian filtration are indistinguishable.

[1] For detaiils see Baldi Theorem 9.2.

Moment estimates for SDE solutions[2]. Any solution X of the SDE with $X_u \in L^p$ for $p \geq 2$, satisfies

$$\mathbb{E}\Big[\sup_{u\leq s\leq t} |X_s^{x,u}|^p\Big] \;\leq\; C\big(1+\mathbb{E}[|X_u^{x,u}|^p]\big), \quad \mathbb{E}\Big[\sup_{u\leq s\leq t} |X_s^{x,u}-X_u^{x,u}|^p\Big] \;\leq\; C\,(t-u)^{p/2}\big(1+\mathbb{E}[|X_u^{x,u}|^p]\big)$$

L^p–estimates for difference of two solutions[3]**.** If $X^{x,u}$ and $X^{y,v}$ are two solutions started at (x,u) and (y,v), then for each R and large p one shows

$$\mathbb{E}\big|X_s^{x,u}-X_t^{y,v}\big|^p \;\leq\; C\Big(|x-y|^p+|u-v|^{p/2}+|s-t|^{p/2}\Big),$$

for all $\max\{u,v\} \leq s,t \leq T$.

Continuous dependence on initial data[4]**.** Fix (x,s). There is a version $Z^{x,s}(t)$ of the solution started at time s from x so that for a.e. ω the map

$$(x,s,t) \;\mapsto\; Z^{x,s}(t,\omega) \quad 0\leq s\leq t\leq T$$

is continuous.

Extension to locally Lipschitz coefficients.[5] If b,σ are only *locally* Lipschitz with sublinear growth, there is still a unique strong solution up to the (a.s. infinite) explosion time. If they satisfy global sublinear growth, no explosion occurs.

Strong Markov Property. Under Assumption 1.5, the process X_t^x is strong Markov: for any stopping time $\tau \leq T$ and bounded f,

$$\mathbb{E}\big[f(X_{\tau+h}^x)\mid \mathcal{F}_\tau\big] \;=\; \mathbb{E}^{X_\tau^x,\tau}\big[f(X_h)\big].$$

In later chapters we will invoke these properties—continuous paths, moment bounds, and the strong Markov property—when representing PDE solutions probabilistically (Feynman–Kac) and in applications to finance.

1.6 Parabolic Partial Differential Equations and Classical Solutions[6]

Let $D \subset \mathbb{R}^m$ be a bounded, connected open set with C^2 boundary, and fix a time horizon $T > 0$. We consider the second-order differential operator as defined in Section 1.4 and the Cauchy–Dirichlet problem

$$\begin{cases} L_t\,u(x,t) \;+\; \dfrac{\partial u}{\partial t}(x,t) \;-\; c(x,t)\,u(x,t) = f(x,t), & (x,t)\in D\times[0,T),\\ u(x,T)=\varphi(x), & x\in D,\\ u(x,t)=g(x,t), & (x,t)\in\partial D\times[0,T]. \end{cases} \tag{1}$$

Here:

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[2]For details see Baldi Theorem 9.1.

[3]For details see Baldi Proposition 9.1.

[4]For details see Baldi Theorem 9.9.

[5]For details see Baldi Theorem 9.4.

[6]Reference A. Friedman, *Partial Differential Equations of Parabolic Type.*, Prentice–Hall, 1964. (Theorem 4.1). Baldi only states this result in Theorem 10.3.

- The coefficient matrix $a(x,t) = [a_{ij}(x,t)]$ is *uniformly elliptic*: there exists $\lambda > 0$ such that

$$\sum_{i,j=1}^{m} a_{ij}(x,t)\,\xi_i\,\xi_j \;\geq\; \lambda\,|\xi|^2, \quad \forall\,\xi \in \mathbb{R}^m,\;\; (x,t) \in D \times [0,T].$$

- a_{ij} and b_i are Lipschitz in x, uniformly for $t \in [0,T]$, and a_{ij} is continuous in t.
- c, f are Hölder-continuous on $D \times [0,T]$ and $c(x,t) \geq 0$.
- The boundary data $\varphi \in C(\overline{D})$ and $g \in C(\partial D \times [0,T])$ satisfy the compatibility condition $g(x,T) = \varphi(x)$ for $x \in \partial D$.

Under the hypotheses above, there exists a unique function

$$u \;\in\; C^{2,1}\big(D \times [0,T)\big) \,\cap\, C\big(\overline{D} \times [0,T]\big)$$

solving the Cauchy–Dirichlet problem (1). Moreover,

$$\sup_{(x,t)\in\overline{D}\times[0,T]} \big|u(x,t)\big| \;<\; +\infty,$$

and all spatial second derivatives and the time–derivative of u extend continuously up to the parabolic boundary.

Setting and Notation

We consider a time interval $[0, T]$ and a spatial domain $D \subset \mathbb{R}^m$. In the first part, D will be a bounded C^2 domain (open with a sufficiently regular boundary), and in the second part we take $D = \mathbb{R}^m$ (unbounded). Let $a : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ be a matrix-valued function and $b : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ a vector field, representing the diffusion and drift coefficients of an m-dimensional diffusion process. Assume the following:

- **Regularity of coefficients:** $a(x,t) = (a_{ij}(x,t))_{1\le i,j\le m}$ and $b(x,t) = (b_i(x,t))_{1\le i\le m}$ are Lipschitz in x uniformly in t, and a is *uniformly elliptic* (there exists $\lambda > 0$ such that $\xi^\top a(x,t)\,\xi \ge \lambda|\xi|^2$ for all $\xi \in \mathbb{R}^m$ and all (x,t)). These conditions ensure the associated SDE has a unique strong solution and the generator will be a well-behaved second-order operator. We denote by $(X_s)_{s\ge t}$ the diffusion process starting from $X_t = x$ at time t, governed by the Itô SDE:
$$dX_s = b(X_s, s)\,ds + \sigma(X_s, s)\,dW_s, \qquad X_t = x,$$
where $\sigma(x,t)$ is an $m \times d$ matrix with $\sigma\sigma^\top = a$ (for simplicity one can take $d = m$ and σ the positive definite square-root of a). We write $\mathbb{E}^{x,t}[\cdot]$ for expectation with respect to the law of this process given $X_t = x$.

- **The differential operator:** For each fixed time t, let
$$L_t = \frac{1}{2}\sum_{i,j=1}^{m} a_{ij}(x,t)\,\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x,t)\,\frac{\partial}{\partial x_i}$$
be the second-order partial differential operator (infinitesimal generator) associated with the diffusion X. Under joint Lipschitz (x,t) and uniform-ellipticity assumptions, the backward PDE on the bounded cylinder $D \times [0,T]$ admits—by Friedman (1964, Thm 4.1) —a unique classical solution $w \in C^{2,1}(D \times [0,T)) \cap C(\overline{D} \times [0,T])$

- **Continuous source and terminal payoff:** Let $c, f : \mathbb{R}^m \times [0,T] \to \mathbb{R}$,are Hölder continuous on $\overline{D} \times [0,T)$, with $c(x,t) \ge -K$, and $\varphi : \mathbb{R}^m \to \mathbb{R}$, $\varphi \in C(\overline{D})$ continuous function. [7] In financial terminology, $c(x,t)$ will act as a *discount rate* (or cost-of-carry), $f(x,t)$ as a running cost rate, and $\varphi(x)$ as the terminal payoff at time T.

- **Boundary condition (for bounded D):** If D has boundary ∂D, we suppose a continuous boundary payoff $g : \partial D \times [0,T] \to \mathbb{R}$ and assume the compatibility $g(x,T) = \varphi(x) \quad \forall x \in \partial D$. This $g(x,t)$ represents the value of the solution if the underlying process hits the boundary ∂D at time t. In a financial setting, exiting the domain D before time T could correspond to a barrier option knockout event or default, with g as the payoff upon exit.

Under these conditions, consider the following backward parabolic PDE (Kolmogorov backward equation with source term) on $D \times [0,T]$:

$$\begin{cases} L_t w(x,t) + \dfrac{\partial w}{\partial t}(x,t) - c(x,t)\,w(x,t) = f(x,t), & (x,t) \in D \times [0,T), \\ w(x,T) = \varphi(x), & x \in \overline{D}, \\ w(x,t) = g(x,t), & (x,t) \in \partial D \times [0,T]. \end{cases} \tag{2}$$

[7] In the unbounded case we will add polynomial-growth hypotheses on f, φ.

This is a boundary-value problem (terminal condition at $t = T$) with Dirichlet boundary condition on the lateral boundary of the space-time domain. By classical parabolic theory (e.g. Friedman 1964, see Preliminaries Section 1.6), under the above assumptions there is a unique *classical solution* to (2), *i.e.* a function $w : \overline{D} \times [0,T] \to \mathbb{R}$ that is continuous on $\overline{D} \times [0,T]$, continuously differentiable in t and twice continuously differentiable in x on $D \times [0,T)$ (i.e. $w \in C^{2,1}$ locally), and satisfies (2) pointwise. We will prove that this $w(x,t)$ can be represented probabilistically by an expectation involving the diffusion X and the data f, g, φ.

2 Feynman–Kac Theorem on a Bounded Domain

We first handle the case of a bounded spatial domain D. Define the *exit time* (first hitting time of the boundary) for the process X_s starting at (x,t) as

$$\tau_D := \inf\{s \geq t : X_s \notin D\},$$

with the convention that $\inf \emptyset = +\infty$ (so $\tau_D = +\infty$ if the process never exits D). We will apply the stochastic representation before the minimum of τ_D and T.

Theorem 1 (Feynman–Kac Representation Formula on Bounded Domain). *Under the assumptions stated above, the solution $w(x,t)$ of the PDE* (2) *admits the following probabilistic representation for all $(x,t) \in D \times [0,T]$:*

$$\begin{aligned} w(x,t) \;=\; \mathbb{E}^{x,t}\Bigg[& g\big(X_{\tau_D}, \tau_D\big)\, \exp\Big\{ -\int_t^{\tau_D} c\big(X_s, s\big)\, ds \Big\}\, \mathbf{1}_{\{\tau_D < T\}} \\ & +\; \varphi\big(X_T\big)\, \exp\Big\{ -\int_t^{T} c\big(X_s, s\big)\, ds \Big\}\, \mathbf{1}_{\{\tau_D \geq T\}} \\ & -\; \int_t^{\tau_D \wedge T} f\big(X_s, s\big)\, \exp\Big\{ -\int_t^{s} c\big(X_u, u\big)\, du \Big\}\, ds \Bigg], \end{aligned} \tag{3}$$

where the expectation is taken with respect to the law of diffusion process X_s associated to generator L_t starting from $X_t = x$.

Equivalently, $w(x,t)$ is the expected discounted payoff which pays g upon exit (if before T), otherwise φ at T, plus the accumulated running payoff f.

Proof. Fix an initial point $(x,t) \in D \times [0,T]$. The proof of this representation form employs Itô's formula and the martingale properties of the diffusion. The key idea is to show that a certain process defined in terms of $w(X_s, s)$ is a martingale, and then apply optional stopping at the stopping time $\tau_D \wedge T$.

Because w is a classical PDE solution, it is in $C^{2,1}$ and we can apply Itô's formula to $w(X_s, s)$ for $s \in [t, \tau_D \wedge T]$. However, w itself solves a PDE with source term f and a discount c, so to get a martingale we will incorporate an integrating factor and an appropriate correction for f. Consider the process Y_s defined for $s \in [t,\, \tau_D \wedge T]$ by

$$Y_s := \exp\Big\{ -\int_t^s c\big(X_u, u\big)\, du \Big\}\, w\big(X_s, s\big) \;-\; \int_t^s \exp\Big\{ -\int_t^u c\big(X_r, r\big)\, dr \Big\}\, f\big(X_u, u\big)\, du\,. \tag{4}$$

We will show Y_s is a uniformly integrable martingale up to the bounded horizon T and then identify $Y_{\tau_D \wedge T}$, whose expectation will yield (3).

Using Itô's formula on the first term in (4), and noting that the second term is an integral with respect to du, for $s < \tau_D \wedge T$:

$$dY_s = d\Big(e^{-\int_t^s c(X_u,u)du} w(X_s,s)\Big) \;-\; d\Big(\int_t^s e^{-\int_t^u c(X_r,r)dr} f(X_u,u)\,du\Big)$$
$$= e^{-\int_t^s c(X_u,u)du}\Big(dw(X_s,s) - c(X_s,s)\,w(X_s,s)\,ds\Big) \;-\; e^{-\int_t^s c(X_u,u)du}\, f(X_s,s)\,ds\,.$$

Itô's formula applied to $w(X_s,s)$ gives:

$$dw(X_s,s) = \frac{\partial w}{\partial s}(X_s,s)\,ds + (L_s w)(X_s,s)\,ds + \nabla_x w(X_s,s)\cdot\sigma(X_s,s)\,dW_s\,,$$

where $\nabla_x w \cdot \sigma\, dW_s$ denotes the stochastic differential (martingale part). Substituting this into dY_s:

$$dY_s = e^{-\int_t^s c(X_u,u)du}\Big(\frac{\partial w}{\partial s}(X_s,s) + L_s w(X_s,s) - c(X_s,s)w(X_s,s) - f(X_s,s)\Big)ds$$
$$+\; e^{-\int_t^s c(X_u,u)du}\,\nabla_x w(X_s,s)\cdot\sigma(X_s,s)\,dW_s\,.$$

By the PDE (2), the combination in the big parentheses is exactly zero for (X_s,s) as long as $s < \tau_D$ (i.e. $X_s \in D$). Thus on $[t, \tau_D \wedge T)$:

$$dY_s \;=\; e^{-\int_t^s c(X_u,u)du}\,\nabla_x w(X_s,s)\cdot\sigma(X_s,s)\,dW_s\,,$$

which is a stochastic integral with respect to dW_s. This shows that $(Y_s)_{s\in[t,\tau_D\wedge T]}$ is a local martingale.

Now, we would need to justify the integrability. On $[t, \tau_D \wedge T]$, we have

$$Y_s \le \exp\Big\{-\int_t^s c(X_u,u)du\Big\}\,|w(X_s,s)| + \int_t^s \exp\Big\{-\int_t^u c(X_r,r)dr\Big\}\,|f(X_u,u)|\,du\,.$$

Since c is bounded below, the exponential factor $e^{-\int_t^s c(X_u,u)du} \le e^{K(s-t)} \le e^{KT}$ (for $s \le T$). Because w, f are have limited growth by boundary conditions and are continuous on the compact $\overline{D}\times[0,T]$ (w as a classical solution of a well-posed parabolic problem, and f, by assumption), they are bounded. Hence $|Y_s|$ is bounded by a constant: $|Y_s| \le e^{KT}||w||_\infty + Te^{KT}||f||_\infty$ for $s \le \tau_D \wedge T$. In particular Y_s is uniformly integrable up to time T and is a true martingale. Hence, we can use optional stopping at $\tau_D \wedge T$ without issue.

By optional stopping, $\mathbb{E}[Y_{\tau_D\wedge T}] = Y_t = w(x,t)$. On the event $\{\tau_D < T\}$,

$$Y_{\tau_D} = e^{-\int_t^{\tau_D} c\,(X_u,u)\,du}\, g(X_{\tau_D}, \tau_D) \;-\; \int_t^{\tau_D} e^{-\int_t^u c\,(X_r,r)\,dr}\, f(X_u,u)\,du\,.$$

On $\{\tau_D \ge T\}$, using terminal condition,

$$Y_T = e^{-\int_t^T c\,(X_u,u)\,du}\,\varphi(X_T) \;-\; \int_t^T e^{-\int_t^u c\,(X_r,r)dr} f(X_u,u)\,du\,.$$

Writing $Y_{\tau_D\wedge T}$ in a unified way, we get:

$$Y_{\tau_D\wedge T} = e^{-\int_t^{\tau_D\wedge T} c\,(X_u,u)\,du}\, w(X_{\tau_D\wedge T},\, \tau_D\wedge T) - \int_t^{\tau_D\wedge T} e^{-\int_t^u c(X_r,r)\,dr}\, f(X_u,u)\,du$$
$$= e^{-\int_t^{\tau_D} c(X_u,u)\,du}\, g(X_{\tau_D},\tau_D)\,\mathbf{1}_{\{\tau_D<T\}} + e^{-\int_t^T c(X_u,u)\,du}\,\varphi(X_T)\,\mathbf{1}_{\{\tau_D\ge T\}} - \int_t^{\tau_D\wedge T} e^{-\int_t^u c(X_r,r)\,dr}\, f(X_u,u)\,du\,.$$

Finally, putting these together,

$$w(x,t) = \mathbb{E}^{x,t}\big[\, Y_{\tau_D \wedge T} \,\big] = \mathbb{E}^{x,t}\Big[e^{-\int_t^{\tau_D} c(X_u,u)\,du}\, g(X_{\tau_D},\tau_D)\,\mathbf{1}_{\{\tau_D<T\}} + e^{-\int_t^T c(X_u,u)\,du}\,\varphi(X_T)\,\mathbf{1}_{\{\tau_D\geq T\}} - \int_t^{\tau_D\wedge T} e^{-\int_t^u c(X_r,r)\,dr}\, f(X_u,u)\,du \Big],$$

which is exactly (3).

□

In particular, when $g \equiv 0$ on ∂D and $f \equiv 0$, the formula reduces to the simpler representation $w(x,t) = \mathbb{E}^{x,t}[e^{-\int_t^T c(X_u,u)\,du}\,\varphi(X_T)\,\mathbf{1}_{\{\tau_D\geq T\}}]$, meaning the solution is the harmonic extension of the discounted terminal payoff (with absorption at the boundary). This is consistent with the interpretation of w as the value of a derivative that becomes worthless upon leaving D (if $g = 0$). More generally, if $f \equiv 0$ but g is nonzero, we have a pure exit payoff problem (a Dirichlet problem for the homogeneous PDE $L_t w + w_t - cw = 0$). If $c \equiv 0$ as well (no discounting), the formula recovers the representation of a solution to the Dirichlet problem as an expectation of the payoff at the exit time (the Martin–Dynkin representation for harmonic functions).

3 Extension to the Unbounded Domain $\mathbb{R}^m$

We now extend the Feynman–Kac representation to the case $D = \mathbb{R}^m$, i.e. when the spatial domain is the entire space. This situation is typical for many financial applications, for example the Black–Scholes model on $\mathbb{R}$ for logarithm of stock prices where the underlying can, in principle, wander without hitting an absorbing boundary. In such cases, the PDE is posed on an unbounded domain with a terminal condition at T and no explicit boundary conditions; instead growth condition at infinity are imposed to ensure uniqueness of the solution.

The strategy to handle the unbounded domain is to approximate $\mathbb{R}^m$ by an increasing sequence of bounded domains (for example, balls of radius R) with a value of zero or with data w on the boundary, apply the bounded-domain Feynman–Kac formula on each, and then let $R \to \infty$. To justify exchanging the limit with the expectation, we require conditions ensuring integrability and/or monotonicity, for which one of the following should hold:

- **Polynomial growth condition for data:** There exists some $\nu \geq 0$ and constant M such that
$$|\varphi(x)| \leq M\,(1+|x|^\nu), \qquad |f(x,t)| \leq M\,(1+|x|^\nu),$$
for all $x \in \mathbb{R}^m$, $t \in [0,T]$. This is condition of at most polynomial growth on the terminal and running payoff.

- **Non-negativity condition:** $\varphi(x) \geq 0$ and $f(x,t) \geq 0$ for all x, t. In words, the payoffs are non-negative functions.

Either of these conditions, along with the earlier assumptions (a, b are Lipschitz in x (uniformly in t) with sublinear growth, measurable in t, and a uniformly elliptic, c continuous bounded below, φ and f continuous), will guarantee that the expectations we deal with are well-defined and that the limiting procedure $R \to \infty$ is valid (using dominated convergence for the polynomial growth case, or monotone convergence for the non-negative case). Under these assumptions, we will show in Theorem 3 that there is a unique classical solution $w \in C^{2,1}(\mathbb{R}^m \times [0,T])$ of the Cauchy problem:

$$\begin{cases} L_t w(x,t) + \dfrac{\partial w}{\partial t}(x,t) - c(x,t)\, w(x,t) = f(x,t), & x \in \mathbb{R}^m,\ t \in [0,T), \\ w(x,T) = \varphi(x), & x \in \mathbb{R}^m, \end{cases} \tag{5}$$

with the growth condition $|w(x,t)| \le C(1+|x|^{\nu'})$ for some ν' and all (x,t).

Theorem 2. (Feynman–Kac Formula on $\mathbb{R}^m$). *Assume the coefficients a, b to be jointly Lipschitz in (x,t) on $\mathbb{R}^m \times [0,T]$ with a uniformly elliptic, and let $c(x,t)$ be continuous and bounded below (i.e. $c(x,t) \ge -K > -\infty$). Let $f(x,t)$ and $\varphi(x)$ be continuous and assume either the polynomial growth condition or the non-negativity condition. Then if the solution $w(x,t)$ of the Cauchy problem* (5) *exists, it admits the form*

$$w(x,t) \;=\; \mathbb{E}^{x,t}\Bigg[\varphi\big(X_T\big)\, \exp\Big\{-\int_t^T c\big(X_s,s\big)\,ds\Big\} \;-\; \int_t^T f\big(X_s,s\big)\, \exp\Big\{-\int_t^s c\big(X_u,u\big)\,du\Big\}\,ds\Bigg], \tag{6}$$

for all $x \in \mathbb{R}^m$, $t \in [0,T]$.

Equivalently, $w(x,t)$ is the expected discounted (from T to t) terminal payoff $\varphi(X_T)$ of a diffusion process starting at $X_t = x$ plus the accumulated discounted running payoff $f(X_s,s)$.

Proof. The idea of the proof is to repeat the martingale-and-stopping argument from the previous proof on balls of radius R, then show that boundary-term errors vanish as the domain enlarges to $\mathbb{R}^m$ with $R \to \infty$. We will prove the theorem for the polynomial growth on f, φ assumption, and then discuss the case with non-negative f, φ in the remark.

For each $R > 0$, let $D_R := \{x \in \mathbb{R}^m : |x| < R\}$ be the open ball of radius R centered at 0. Consider the stopping time $\tau_R = \inf\{s \ge t : X_s \notin D_R\}$, the exit time from D_R. Since D_R is bounded, we can apply the result of Theorem 1 up to $\tau_R \wedge T$ with an boundary-term at τ_R defined as $w(X_{\tau_R}, \tau_R)$:

$$\begin{aligned} w(x,t) = \mathbb{E}^{x,t}&\Big[w\big(X_{\tau_R},\, \tau_R\big)\, e^{-\int_t^{\tau_R} c(X_u,u)du}\, \mathbf{1}_{\{\tau_R < T\}} \Big] \\ -\; \mathbb{E}^{x,t}&\Big[\int_t^{\tau_R \wedge T} f(X_s,s)\, e^{-\int_t^s c(X_u,u)du}\, ds \Big] \\ +\; \mathbb{E}^{x,t}&\Big[\varphi\big(X_T\big)\, e^{-\int_t^T c(X_u,u)du}\, \mathbf{1}_{\{\tau_R \ge T\}} \Big] \\ =:&\; I_{1,R} - I_{2,R} + I_{3,R}, \end{aligned} \tag{7}$$

for every $R > 0$.

Classical solution w has at most polynomial growth: $|w(y,s)| \le M_1(1+|y|^{\nu'})$ for some ν'. Since c is bounded below by $-K$, on $\{\tau_R < T\}$ we have

$$\big|w(X_{\tau_R}, \tau_R)\, e^{-\int_t^{\tau_R} c(X_u,u)\,du}\big| \;\le\; M_1\big(1+|X_{\tau_R}|^{\nu'}\big)e^{K(\tau_R - t)} \;\le\; M_1\big(1+R^{\nu'}\big)e^{KT}.$$

Therefore

$$I_{1,R}(x,t) \;\le\; M_1(1+R^{\nu'})e^{KT}\, \mathbb{P}^{x,t}\{\tau_R < T\}.$$

We need to show that $\mathbb{P}^{x,t}\{\tau_R < T\}$, decays sufficiently fast as $R \to \infty$. In fact, since the coefficients are Lipschitz, the diffusion has finite moments of all orders (as described in Section 1.5). In addition:

$$\mathbb{E}^{x,t}\Big[\max_{s\in[t,T]} |X_s|^q\Big] \le C_q\big(1+|x|^q\big)\,,$$

for some constant C_q independent of x and t. Then by Markov's inequality,

$$\mathbb{E}^{x,t}[\,\mathbf{1}_{\{\tau_R<T\}}\,] \;=\; \mathbb{P}^{x,t}\{\tau_R < T\} \;=\; \mathbb{P}^{x,t}\Big\{\max_{s\in[t,T]} |X_s| \ge R\Big\} \;\le\; \frac{\mathbb{E}^{x,t}[\max_{s\in[t,T]} |X_s|^q]}{R^q} \;\le\; \frac{C_q(1+|x|^q)}{R^q}\,. \tag{8}$$

Choosing q larger than ν' (for instance $q = \nu' + 2$), we see that $(1 + R^{\nu'})\frac{1}{R^q} \to 0$ as $R \to \infty$ faster than any polynomial. Thus from (3),

$$\lim_{R\to\infty} I_{1,R}(x,t) \;\le\; \lim_{R\to\infty} M_1 e^{KT}(1+R^{\nu'})\frac{C_q(1+|x|^q)}{R^q} = 0\,.$$

We conclude that indeed $I_{1,R}(x,t) \to 0$ as $R \to \infty$.

From (8), $\{\tau_R \wedge T\} \to T$ as $R \to \infty$ in probability. Since the sequence $\tau_R \wedge T \to T$ increases in R, $\{\tau_R \wedge T\} \to T$ almost surely.

Moreover, by hypothesis there are constants M, K, q, ν so that

$$\Big|\varphi(X_T)e^{-\int_t^T c(X_u,u)\,du}\mathbf{1}_{\{\tau_R\ge T\}}\Big| \;\le\; M\,(1+|X_T|^q)\,e^{KT},$$
$$\Big|\int_t^{\tau_R\wedge T} f(X_s,s)\,e^{-\int_t^s c(X_u,u)\,du}\,ds\Big| \;\le\; M\,(T-t)\,e^{KT}\sup_{s\in[t,T]}(1+|X_s|^\nu),$$

and each right-hand side is integrable under our moment-bounds for X. Therefore the dominated convergence theorem gives

$$\lim_{R\to\infty} I_{3,R} = \mathbb{E}^{x,t}\big[\varphi(X_T)\,e^{-\int_t^T c(X_u,u)\,du}\big], \qquad \lim_{R\to\infty} I_{2,R} = \mathbb{E}^{x,t}\Big[\int_t^T f(X_s,s)\,e^{-\int_t^s c(X_u,u)\,du}\,ds\Big].$$

Since we have already shown $I_{1,R} \to 0$, letting $R \to \infty$ in

$$w(x,t) = I_{1,R} - I_{2,R} + I_{3,R}$$

yields

$$w(x,t) = \mathbb{E}^{x,t}\Big[\varphi(X_T)\,e^{-\int_t^T c(X_u,u)\,du} \;-\; \int_t^T f(X_s,s)\,e^{-\int_t^s c(X_u,u)\,du}\,ds\Big],$$

which is exactly the formula (6). This completes the proof.

Remark This argument used the polynomial growth condition. If instead $\varphi \ge 0$ and $f \ge 0$, all three terms in (7) are non-negative and monotone in R: $I_{3,R} = \mathbb{E}\big[\varphi(X_T)e^{-\int_t^T c}_{\{\tau_R\ge T\}}\big]$ increases to $\mathbb{E}[\varphi(X_T)e^{-\int_t^T c}]$; $I_{2,R} = \mathbb{E}\big[\int_t^{\tau_R\wedge T} f(X_s,s)e^{-\int_t^s c}\,ds\big]$ increases to $\mathbb{E}\big[\int_t^T f(X_s,s)e^{-\int_t^s c}\,ds\big]$, and the boundary term $I_{1,R}$ decreases to 0 (by the same exit-probability estimate as before).

Hence, letting $R \to \infty$ and invoking the monotone convergence theorem (Beppo Levi) in place of dominated convergence yields the full Feynman–Kac formula

$$w(x,t) \;=\; \mathbb{E}^{x,t}\Big[\varphi(X_T)e^{-\int_t^T c}\Big] \;-\; \mathbb{E}^{x,t}\Big[\int_t^T f(X_s,s)e^{-\int_t^s c}\,ds\Big],$$

under the sole condition $\varphi, f \ge 0$. □

4 Existence and Uniqueness of Classical Solution

We now *construct* a solution to the PDE (5) using the Feynman–Kac representation, and show it is the unique classical solution satisfying the given growth bounds. Essentially, we will *define* $u(x,t)$ by the right-hand side of (6) and then verify u solves the PDE. The key steps are ensuring u is sufficiently smooth (Lemma 1 will ensure continuity and integrability, and the regularity assumptions on f and c will allow differentiability), and then using an argument similar to the above proof to check the PDE and terminal condition. Lemma 2 will help us prove convergence ar $R \to \infty$. Uniqueness will follow from the representation formula.

Lemma 1 (Continuity of the Feynman–Kac solution). *Suppose the diffusion coefficients $a(x,t), b(x,t)$ are Lipschitz on $[0,T]\times\mathbb{R}^m$, with $a(\cdot)$ uniformly elliptic. Assume $c(x,t)$ is continuous and bounded below by $-K$. Let φ and f be continuous functions of at most polynomial growth. Define the functions u_1 and u_2 on $\mathbb{R}^m \times [0,T]$ by the Feynman–Kac expectations: for each (x,t),*

$$u_1(x,t) := \mathbb{E}^{x,t}\Big[\, \varphi(X_T)\, \exp\Big\{ -\int_t^T c(X_s,s)\,ds\Big\}\,\Big],$$

$$u_2(x,t) := \mathbb{E}^{x,t}\Big[\, \int_t^T f(X_s,s)\, \exp\Big\{ -\int_t^s c(X_u,u)\,du\Big\}\,ds\,\Big],$$

where $X_s := X_s^{x,t}$ denotes the diffusion process starting at $X_t = x$ at time t (with generator L_t). Then $u_1(x,t)$ and $u_2(x,t)$ are well-defined, continuous in (x,t) on $\mathbb{R}^m \times [0,T]$, and have at most polynomial growth.

If, moreover, the diffusion coefficient is bounded, it is sufficient to assume that φ and f have exponential growth (i.e. there exist constants $k_1, k_2 \ge 0$ such that $|\varphi(x)| \le k_1 e^{k_2|x|}$ and $|f(x)| \le k_1 e^{k_2|x|}$ for every $t \in [0,T]$.[8]

In particular, the full candidate solution of Cauchy problem, defined as in (2), given by the Feynman–Kac formula

$$u(x,t) \;:=\; u_1(x,t) \;-\; u_2(x,t) \;=\; \mathbb{E}^{x,t}\Big[\varphi(X_T)e^{-\int_t^T c(X_s,s)ds}\Big] \;-\; \mathbb{E}^{x,t}\Big[\int_t^T f(X_s,s)e^{-\int_t^s c(X_u,u)du}\,ds\Big],$$

is continuous on $\mathbb{R}^m \times [0,T]$.

Proof. By standard moment-bounds for SDEs it is immediate that, since $e^{-\int_t^T c(X_s,s)\,ds} \le e^{KT}$, and because φ and f grow at most like $C(1+|x|^\nu)$ and $X_s^{x,t}$ has bounded moments, $\mathbb{E}\big[|\varphi(X_T)|\,e^{-\int_t^T c\,ds}\big] \le M\,e^{KT}\,C\big(1+|x|^q\big) < \infty$, and similarly $\mathbb{E}\int_t^T |f(X_s,s)|\,e^{-\int_t^s c\,du}\,ds < \infty$, so u_1, u_2 are well-defined and satisfy $|u_i(x,t)| = C'(1+|x|^q)$.

Since $c(x,t) \ge -K$ we have

$$\exp\Big\{ -\int_t^T c(X_s,s)\,ds\Big\} \;\le\; e^{K(T-t)}\,,$$

and by the SDE's uniform moment-bounds (see Lemma 9.1) $\sup_{t\le s\le T} \mathbb{E}\big[|X_s|^p\big] < \infty$ for all p. Hence if φ and f grow at most like $|x|^d$,

$$\mathbb{E}\big[|\varphi(X_T)|e^{-\int c}\big] \;\le\; e^{KT}\,C\big(1+|x|^d\big) < \infty,$$

[8] We omit the proof of this statement since it is not required for our purpose. For details see Baldi Lemma 10.1.

and similarly for u_2, so both expectations are finite and bounded by a constant times $e^{KT}(1+|x|^d)$. Let $(x_n, t_n) \to (x, t)$, and write $X^n_s = X^{x_n,t_n}_s$, $X_s = X^{x,t}_s$.

Under Lipschitz-and-ellipticity assumption for the diffusion coefficients, the solution to SDE is pathwise unique, and we can construct on the same probability space a family of solutions $(X^{x,t})$ which is (a.s.) jointly continuous in the starting point (x, t) and time s, in particular (see sontinuous dependence on initial data in the preliminaries section 1.5):

$$(x_n, t_n) \to (x, t) \quad \Rightarrow \quad \sup_{0\le s\le T} |X^{x_n,t_n}_s - X^{x,t}_s| \xrightarrow{a.s.} 0.$$

For almost every ω the random set

$$K(\omega) \;=\; \big\{(X_s(\omega), s) \colon s \in [0, T]\big\} \;\cup\; \big\{(X^n_s(\omega), s) \colon s \in [0, T],\; n \ge n_0\big\}$$

is compact in $\mathbb{R}^m \times [0, T]$. Moreover, since c is continuous on $\mathbb{R}^m \times [0, T]$, it is uniformly continuous and bounded on $K(\omega)$, so for each ω

$$\left|\int_{t_n}^T c(X^n_s, s)\, ds \; - \int_t^T c(X_s, s)\, ds\right| \le \underbrace{\int_{t_n}^t \big|c(X^n_s, s)\big|\, ds}_{\le M\,|t_n - t|} \; + \; \underbrace{\int_t^T \big|c(X^n_s, s) - c(X_s, s)\big|\, ds}_{\le (T-t)\,\sup_{s\in[t,T]} |c(X^n_s,s) - c(X_s,s)|} \quad .$$

with $M = \sup_{K(\omega)} |c|$. Since $t_n \to t$ and $X^n \to X$ uniformly on $[0, T]$ a.s., the right-hand side goes to 0. Thus $\int_{t_n}^T c(X^n_s, s)\, ds \to \int_t^T c(X_s, s)\, ds$ almost surely, and consequently

$$e^{-\int_{t_n}^T c(X^n_s,s)\,ds} \;\longrightarrow\; e^{-\int_t^T c(X_s,s)\,ds} \quad \text{a.s.}$$

Since each exponential is bounded by e^{KT}, the bounded-convergence theorem applies whenever we need to swap limit and expectation.

The payoff φ may grow polynomially, so $\varphi(X^n_T)$ is unbounded and we cannot directly interchange limit and expectation. We introduce the following truncation scheme to ensure boundedness of the function.

Fix $R > 0$ large enough that $x_n, x \in D_R := \{y \in \mathbb{R}^m : |y| \le R\}$ for all n above some n_0. Define the truncated payoff

$$\varphi_R(y) = \begin{cases} \varphi(y), & |y| \le R, \\ 0, & |y| > R, \end{cases}$$

and write

$$u_1(x_n, t_n) - u_1(x, t) = \underbrace{\mathbb{E}\big[\varphi_R(X^n_T) e^{-C^n}\big] - \mathbb{E}\big[\varphi_R(X_T) e^{-C}\big]}_{=:I_{1,n}} \; + \; \underbrace{\mathbb{E}\big[(\varphi - \varphi_R)(X^n_T) e^{-C^n}\big]}_{=:I_{2,n}} \; - \; \underbrace{\mathbb{E}\big[(\varphi - \varphi_R)(X_T) e^{-C}\big]}_{=:I_{3,n}},$$

where $C^n = \int_{t_n}^T c(X^n_s, s)\, ds$ and $C = \int_t^T c(X_s, s)\, ds$.

To apply bounded-convergence to the term $\varphi_R(X^n_T)e^{-C^n} \to \varphi_R(X_T)e^{-C}$, note that our cutoff $\varphi_R(y)$ is discontinuous only on the sphere $\{|y| = R\}$. But under uniform ellipticity the law of X_T admits a continuous density, so $\mathbb{P}(|X_T| = R) = 0$.

Since φ_R is bounded and continuous, and $e^{-C^n} \le e^{KT}$, the integrand $\varphi_R(X^n_T)e^{-C^n}$ is uniformly bounded by $||\varphi_R||_\infty e^{KT}$ and converges a.s. to $\varphi_R(X_T)e^{-C}$. Hence by the bounded convergence theorem,

$$I_{1,n} \xrightarrow[n\to\infty]{} 0.$$

On $\{|X_T^n| > R\}$ we have $|\varphi - \varphi_R|(X_T^n) = |\varphi(X_T^n)| \le M(1 + |X_T^n|^\nu)$, and $e^{-C^n} \le e^{KT}$. By the L^p-moment bound for SDE solutions (see Section 1.5),

$$\sup_n \mathbb{E}\big[|X_T^n|^q\big] < +\infty \quad \text{for some } q > \nu,$$

so that Markov inequality gives

$$\mathbb{P}\big(|X_T^n| > R\big) \;\le\; \frac{\mathbb{E}[|X_T^n|^q]}{R^q} \xrightarrow[R\to\infty]{} 0,$$

uniformly in n. Hence by Hölder inequality,

$$I_{2,n} \;\le\; e^{KT}\,\mathbb{E}\big[M(1+|X_T^n|^\nu)\,\mathbf{1}_{\{|X_T^n|>R\}}\big] \;\le\; e^{KT} M \left(\mathbb{E}[(1+|X_T^n|^\nu)^2]\right)^{\frac{1}{2}} \mathbb{P}[|X_T^n| > R]^{\frac{1}{2}} \xrightarrow[R\to\infty]{} 0,$$

uniformly in n. The same estimate applies to $I_{3,n}$.

Given any $\delta > 0$, first choose R large enough such that $\sup_n I_{2,n} + I_{3,n} < \delta/2$, then choose N such that for all $n \ge N$, $|I_{1,n}| < \delta/2$. Thus for $n \ge N$,

$$|u_1(x_n, t_n) - u_1(x,t)| \;\le\; |I_{1,n}| + I_{2,n} + I_{3,n} < \delta,$$

showing u_1 is continuous. An identical argument applies to u_2. □

Lemma 2 (Strong Markov property at a stopping time). *Let τ be a t-stopping time (with respect to the natural filtration of X_s) such that $t \le \tau \le T$ almost surely. Then the Feynman–Kac solution components u_1 and u_2 satisfy the following dynamic programming equalities:*

$$u_1(x,t) = \mathbb{E}^{x,t}\Big[\exp\Big\{-\int_t^\tau c(X_s,s)\,ds\Big\}\,u_1\big(X_\tau,\,\tau\big)\Big],$$

$$u_2(x,t) = \mathbb{E}^{x,t}\Big[\int_t^\tau f(X_s,s)\,\exp\Big\{-\int_t^s c(X_u,u)\,du\Big\}\,ds\Big] + \mathbb{E}^{x,t}\Big[\exp\Big\{-\int_t^\tau c(X_s,s)\,ds\Big\}\,u_2\big(X_\tau,\,\tau\big)\Big].$$

In particular, for the full solution $u = u_1 - u_2$, we have

$$u(x,t) \;=\; \mathbb{E}^{x,t}\Big[e^{-\int_t^\tau c(X_u,u)du}\;u(X_\tau,\tau)\Big] \;-\; \mathbb{E}^{x,t}\Big[\int_t^\tau f(X_s,s)\,e^{-\int_t^s c(X_u,u)du}\,ds\Big],$$

which expresses $u(x,t)$ in terms of the solution's value at the stopping time τ (plus the accumulated source contribution up to τ).

Proof. We first consider the case where τ takes only a discrete set of values and the general case will follow by a limiting argument.

Assume τ can take values in a finite or countable set $\{s_1, s_2, \ldots, s_m\} \subset [t,T]$. Let $\mathcal{F}_\tau$ denote the σ-algebra of events up to the stopping time τ. For $u_1(x,t)$, rewrite the exponent in u_1 as $\int_t^T c(X_s,s)\,ds = \int_t^\tau c(X_s,s)\,ds + \int_\tau^T c(X_s,s)\,ds$ with $\int_t^\tau c(X_s,s)\,ds$ $\mathcal{F}_\tau$-measurable. Then we can condition as follows:

$$u_1(x,t) = \mathbb{E}^{x,t}\Big[e^{-\int_t^\tau c(X_s,s)ds}\;\mathbb{E}^{x,t}\big[\varphi(X_T)\,e^{-\int_\tau^T c(X_s,s)ds}\;\big|\;\mathcal{F}_\tau\big]\Big]. \tag{9}$$

Now, under the usual Lipschitz and sublinear growth conditions on the diffusion coefficients, diffusion process satisfies the *strong Markov property.* Thus, conditioned on $\{X_\tau = y,\ \tau = s_k\}$, the process X_s for $s \geq \tau$ behaves like a fresh diffusion started (y, s_k) and is unique in law:

$$\mathbb{E}^{x,t}\big[\,\varphi(X_T)e^{-\int_\tau^T c(X_s,s)ds}\,\big|\,X_\tau = y,\ \tau = s_k\big] = E^{y,s_k}\big[\,\varphi(X_T)e^{-\int_{s_k}^T c(X_s,s)ds}\,\big] = u_1(y, s_k)\ .$$

Since

$$u_1\big(X_\tau,\tau\big) = \sum_{k=1}^m \mathbf{1}_{\{\tau=s_k\}}\,u_1\big(X_{s_k},s_k\big) = \sum_{k=1}^m \mathbf{1}_{\{\tau=s_k\}}\,\mathbb{E}^{x,t}\big[\,\varphi(X_T)e^{-\int_\tau^T c(X_s,s)ds}\,\big|\,X_\tau = y,\ \tau = s_k\big],$$

we have

$$\begin{aligned}\mathbb{E}^{x,t}\Big[\mathbf{1}_{\{c\}}\varphi(X_T)\,e^{-\int_\tau^T c(X_s,s)\,ds}\Big] &= \mathbb{E}^{x,t}\Big[\sum_{k=1}^m \mathbf{1}_{\{c\}\cap\{\tau=s_k\}}\mathbb{E}^{x,t}\Big[\varphi(X_T)\,e^{-\int_\tau^T c(X_s,s)\,ds}\,\Big|\,X_\tau = y,\ \tau = s_k\Big]\Big]\\ &= \mathbb{E}^{x,t}\Big[\sum_{k=1}^m \mathbf{1}_{\{c\}\cap\{\tau=s_k\}}u_1\big(X_{s_k},s_k\big)\Big] = \mathbb{E}^{x,t}\Big[\mathbf{1}_{\{c\}}u_1\big(X_\tau,\tau\big)\Big],\end{aligned}$$

and substituting back to (9) gives the first desired identity

$$u_1(x,t) = \mathbb{E}^{x,t}\big[e^{-\int_t^\tau c(X_s,s)\,ds}\,u_1(X_\tau,\tau)\big].$$

The reasoning for u_2 is similar, but we must take care of the time-integral of f. Once again, split the integral at τ:

$$\int_t^T f(X_s,s)e^{-\int_t^s c(X_u,u)du}\,ds \ = \ \int_t^\tau f(X_s,s)e^{-\int_t^s c(X_u,u)du}\,ds\ +\ \int_\tau^T f(X_s,s)e^{-\int_t^s c(X_u,u)du}\,ds\ .$$

Now take expectation and condition at τ:

$$u_2(x,t) = \mathbb{E}^{x,t}\Big[\int_t^\tau f(X_s,s)\ e^{-\int_t^s c(X_u,u)du}\ ds\Big]\ +\ \mathbb{E}^{x,t}\Big[\mathbb{E}^{X_\tau,\tau}\Big[\int_\tau^T f(X_s,s)\ e^{-\int_t^s c(X_u,u)du}ds\,\Big|\,\mathcal{F}_\tau\Big]\Big]$$

For the second term, observe that on the event $\{\tau = s_k\}$, we have

$$\mathbb{E}^{x,t}\Big[\int_\tau^T fe^{-\int_t^s c(X_u,u)du}ds\,\Big|\,\mathcal{F}_\tau\Big] = e^{-\int_t^\tau c(X_u,u)du}\ \mathbb{E}^{x,t}\Big[\int_\tau^T f(X_s,s)e^{-\int_\tau^s c(X_u,u)du}ds\,\Big|\,\mathcal{F}_\tau\Big]$$

since the factor $e^{-\int_t^{s_k} c(X_u,u)\,du}$ is $\mathcal{F}_\tau$-measurable.

By the Markov property applied at time τ, we get:

$$\mathbb{E}^{x,t}\Big[\int_\tau^T f(X_s,s)e^{-\int_\tau^s c(X_u,u)du}ds\,\Big|\,X_\tau = y,\ \tau = s_k\Big] = E^{y,s_k}\Big[\int_{s_k}^T f(X_s,s)e^{-\int_{s_k}^s c(X_u,u)du}ds\Big] = u_2(y,s_k)\ .$$

Putting these together yields the decomposition

$$u_2(x,t) = \mathbb{E}^{x,t}\Big[\int_t^\tau f(X_s,s)e^{-\int_t^s c(X_u,u)du}\,ds\Big]\ +\ \mathbb{E}^{x,t}\Big[e^{-\int_t^\tau c(X_u,u)du}\ u_2(X_\tau,\tau)\Big],$$

which is exactly the claim for $u_2(x,t)$.

Finally, to remove the assumption of τ taking only discrete values, take a sequence of discrete-valued stopping times τ_n such that $\tau_n \uparrow \tau$ (for instance, let $\tau_n := \frac{\lfloor 2^n\tau\rfloor}{2^n}$ or another variant that takes the last rational time before τ with denominator 2^n). By the a priori bounds on u_1 and u_2 (Lemma 1) together with $0 \leq e^{-\int_t^{\tau_n} c(X_s,s)ds} \leq 1$ and the finite-moment estimates for the diffusion (see 1.5), dominated convergence permits passage of $\mathbb{E}$ to the limit. This concludes the proof for the general τ. $\square$

Proof. Assume first that the stopping time τ takes values in the finite set $\{s_1, \ldots, s_m\} \subset [t, T]$. Denote by $\mathcal{F}_\tau$ the σ–algebra at time τ. For the term defining u_1, observe that

$$u_1(x,t) = \mathbb{E}^{x,t}\big[\varphi(X_T)\, e^{-\int_t^T c(X_s,s)\,ds}\big] = \mathbb{E}^{x,t}\Big[e^{-\int_t^\tau c(X_s,s)\,ds}\; \mathbb{E}^{x,t}\big[\varphi(X_T)\, e^{-\int_\tau^T c(X_s,s)\,ds} \;\mid \mathcal{F}_\tau\big]\Big].$$

On the event $\{\tau = s_k\}$, by the strong Markov property the process restarts afresh from $(X_\tau, \tau) = (y, s_k)$, so

$$\mathbb{E}^{x,t}\big[\varphi(X_T)\, e^{-\int_{s_k}^T c(X_s,s)\,ds} \;\mid \mathcal{F}_{s_k}\big] = \mathbb{E}^{y,s_k}\big[\varphi(X_T)\, e^{-\int_{s_k}^T c(X_s,s)\,ds}\big] = u_1(y, s_k).$$

Hence

$$\mathbb{E}^{x,t}\big[\varphi(X_T)\, e^{-\int_\tau^T c(X_s,s)\,ds} \;\mid \mathcal{F}_\tau\big] = u_1(X_\tau, \tau),$$

and substituting back gives the first desired identity

$$u_1(x,t) = \mathbb{E}^{x,t}\big[e^{-\int_t^\tau c(X_s,s)\,ds}\, u_1(X_\tau, \tau)\big].$$

Turning to u_2, split the time-integral at τ:

$$\int_t^T f(X_s,s)\, e^{-\int_t^s c(X_u,u)\,du}\, ds = \int_t^\tau f(X_s,s)\, e^{-\int_t^s c(X_u,u)\,du}\, ds \;+\; \int_\tau^T f(X_s,s)\, e^{-\int_t^s c(X_u,u)\,du}\, ds.$$

Thus

$$u_2(x,t) = \mathbb{E}^{x,t}\Big[\int_t^\tau f(X_s,s)\, e^{-\int_t^s c(X_u,u)\,du}\, ds\Big] \;+\; \mathbb{E}^{x,t}\Big[\int_\tau^T f(X_s,s)\, e^{-\int_t^s c(X_u,u)\,du}\, ds\Big].$$

On $\{\tau = s_k\}$ the factor

$$e^{-\int_t^{s_k} c(X_u,u)\,du}$$

is $\mathcal{F}_\tau$–measurable and may be pulled outside the inner expectation. Again by the strong Markov property,

$$\mathbb{E}^{x,t}\Big[\int_{s_k}^T f(X_s,s)\, e^{-\int_{s_k}^s c(X_u,u)\,du}\, ds \;\mid\; \mathcal{F}_{s_k}\Big] = \mathbb{E}^{y,s_k}\Big[\int_{s_k}^T f(X_s,s)\, e^{-\int_{s_k}^s c(X_u,u)\,du}\, ds\Big] = u_2(y, s_k).$$

Putting these together yields

$$u_2(x,t) = \mathbb{E}^{x,t}\Big[\int_t^\tau f(X_s,s)\, e^{-\int_t^s c(X_u,u)\,du}\, ds\Big] \;+\; \mathbb{E}^{x,t}\Big[e^{-\int_t^\tau c(X_u,u)\,du}\, u_2(X_\tau, \tau)\Big].$$

Finally, to handle a general $\tau \le T$ one approximates by a decreasing sequence of discrete-valued stopping times $\tau_n \downarrow \tau$. The above identities hold for each τ_n, and under the uniform bounds on u_1 and u_2 (cf. Lemma 1 and the moment estimates for X) one invokes dominated convergence (or uniform integrability) to pass to the limit $n \to \infty$. This completes the proof of Lemma 10.2. $\square$

With these lemmas in hand, we can now prove the main result for the unbounded domain.

Theorem 3 (Existence and uniqueness of the solution on $\mathbb{R}^m$). *Assume $a(x,t)$ is uniformly elliptic and both a and b are Lipschitz in x, uniformly for $t \in [0,T]$. Moreover, c, f are continuous on $\mathbb{R}^m \times [0,T]$, locally Hölder in x*[9]*, and $c(x,t) \geq -K$; φ is continuous on $\mathbb{R}^m$, and φ and f grow at most polynomially in $|x|$.*

Then there exists a unique classical solution $u \in C^{2,1}\big(\mathbb{R}^m \times [0,T)\big) \cap C\big(\mathbb{R}^m \times [0,T]\big)$ of the terminal-value problem

$$\begin{cases} L_t u + \partial_t u - c(x,t)\, u = f(x,t), & x \in \mathbb{R}^m,\ 0 \leq t < T, \\ u(x,T) = \varphi(x), & x \in \mathbb{R}^m, \end{cases}$$

which satisfies the Feynman–Kac representation

$$u(x,t) = \mathbb{E}^{x,t}\Big[\varphi(X_T)\, e^{-\int_t^T c(X_s,s) ds}\Big] \;-\; \mathbb{E}^{x,t}\Big[\int_t^T f(X_s,s)\, e^{-\int_t^s c(X_u,u) du}\, ds\Big].$$

Moreover, u grows at most polynomially in $|x|$, and it is the unique solution in the class of functions with at most polynomial growth.

If, additionally, the diffusion coefficient σ is bounded and φ, f have at most exponential growth, then u remains the unique solution in the class of at most exponential growth.[10]

Proof. Let $(X_s)_{s \geq t}$ be the diffusion with generator L_t, and define

$$u_1(x,t) = \mathbb{E}^{x,t}\Big[\varphi(X_T)\, e^{-\int_t^T c(X_s,s)\, ds}\Big], \qquad u_2(x,t) = \mathbb{E}^{x,t}\Big[\int_t^T f(X_s,s)\, e^{-\int_t^s c(X_r,r)\, dr}\, ds\Big].$$

By Lemma 1, u_1, u_2 are well-defined, continuous on $\mathbb{R}^m \times [0,T]$, and of at most polynomial growth. Set $u = u_1 - u_2$.

Fix any ball $D_R = \{\, |x| < R\}$. On $\partial D_R \times [0,T]$ and at $t = T$, the function u is continuous, and f, c are Hölder in x. By classical parabolic PDE result (see Section 1) Theorem 1, there is a unique $w_R \in C^{2,1}(D_R \times [0,T)) \cap C(\overline{D_R} \times [0,T])$ solving

$$\begin{cases} L_t w_R + \partial_t w_R - c\, w_R = f, & (x,t) \in D_R \times [0,T), \\ w_R(x,T) = \varphi(x), & x \in D_R, \\ w_R(x,t) = u(x,t), & (x,t) \in \partial D_R \times [0,T] \end{cases}$$

that has the following form:

$$\begin{aligned} w_R(x,t) = \mathbb{E}^{x,t}\Big[\, u\big(X_{\tau_R},\, \tau_R\big)\, e^{-\int_t^{\tau_R} c\, du}\, 1_{\{\tau_R < T\}}\Big] & \\ + \mathbb{E}^{x,t}\Big[\, \varphi(X_T)\, e^{-\int_t^T c\, du}\, 1_{\{\tau_R \geq T\}}\Big] \;-\; \mathbb{E}^{x,t}\Big[\int_t^{\tau_R \wedge T} f(X_s,s)\, e^{-\int_t^s c\, du}\, ds\Big] & . \end{aligned} \tag{10}$$

On the other hand Lemma 2 (strong Markov) shows exactly the same identity holds for the candidate function at the stopping time $\tau = \tau_R \wedge T$, $u(x,t)$:

$$\begin{aligned} u(x,t) &= \mathbb{E}^{x,t}\Big[\, u\big(X_{\tau_R \wedge T},\, \tau_R \wedge T\big)\, e^{-\int_t^{\tau_R \wedge T} c(X_u,u)\, du}\Big] \;-\; \mathbb{E}^{x,t}\Big[\int_t^{\tau_R \wedge T} f(X_s,s)\, e^{-\int_t^s c(X_u,u)\, du}\, ds\Big] \\ &= \mathbb{E}^{x,t}\Big[\, u\big(X_{\tau_R},\, \tau_R\big)\, e^{-\int_t^{\tau_R} c(X_u,u)\, du}\, 1_{\{\tau_R < T\}}\Big] \;+\; \mathbb{E}^{x,t}\Big[\, \varphi(X_T)\, e^{-\int_t^T c(X_u,u)\, du}\, 1_{\{\tau_R \geq T\}}\Big] \\ &\qquad - \; \mathbb{E}^{x,t}\Big[\int_t^{\tau_R \wedge T} f(X_s,s)\, e^{-\int_t^s c(X_u,u)\, du}\, ds\Big] = w_R(x,t) \end{aligned}$$

[9] Note that we did not need the Hölder continuity assumption in the previous sections for the proof of the Feynman-Kac representation theorem, but need it for the Lemma 2 to ensure sufficient regularity of the classical PDE solution.

[10] We omit the straightforward adaptation for exponential growth for our purposes. Refer to Baldi Theorem 10.6 for details.

where we used the fact that on the event $\{\tau_R \geq T\}$, $u(X_T, T) = \varphi(X_T)$ by the definition of $u(\cdot)$, which gives $\mathbb{E}^{x,t}\Big[u\big(X_{\tau_R \wedge T},\, \tau_R \wedge T\big)\, e^{-\int_t^{\tau_R\wedge T} c(X_u,u)\,du}\, 1_{\{\tau_R \geq T\}}\Big] = \mathbb{E}^{x,t}\Big[\varphi(X_T)\, e^{-\int_t^T c(X_u,u)\,du}\, 1_{\{\tau_R\geq T\}}\Big]$.

Consequently, the stochastic representation of $u(x,t)$ coincides with that of $w_R(x,t)$ for all $(x,t) \in D_R \times [0,T]$. Letting $R \to \infty$ gives $u \in C^{2,1}_{\text{loc}}(\mathbb{R}^m \times [0,T))$ and $L_t u + u_t - cu = f$ everywhere, with $u(\cdot, T) = \varphi$. which concludes the existence proof.

It is left to prove uniqueness of the solution in the class of functions with at most polynomial growth. Suppose v is another classical solution satisfying the same growth bound, and set $w = v - u$. Then $w \in C^{2,1}(\mathbb{R}^m \times [0,T])$ and solves the homogeneous PDE

$$L_t\, w \;+\; \partial_t w \;-\; c\, w \;=\; 0, \qquad w(\cdot, T) = 0,$$

with $|w(x,t)| \leq C\,(1 + |x|^p)$. By Theorem 2, since $f \equiv 0$, the solution has the following representation

$$w(x,t) \;=\; \mathbb{E}^{x,t}\Big[e^{-\int_t^T c\left(X_s, s\right)\,ds}\; w\big(X_T, T\big)\Big].$$

But $w(\cdot, T) \equiv 0$, so the right-hand side vanishes identically. Hence

$$w(x,t) = 0 \quad \forall\; (x,t) \in \mathbb{R}^m \times [0,T],$$

and uniqueness in the polynomial-growth class follows.

□

5 Application: Pricing European Options in the Black–Scholes Model

We now apply the Feynman–Kac theorem to a fundamental problem in mathematical finance: the pricing of European options in the Black–Scholes model. Consider a market with a risk-free bond and n risky stocks. Let B_t denote the bond (money market account) at time t, accruing interest at a constant rate $r \geq 0$, so $B_t = e^{rt}$ (assuming $B_0 = 1$). Let $\mathbf{S}_t = \big(S_t^1, \ldots, S_t^n\big)$ denote the vector of stock prices, which we model as a stochastic process satisfying the Black–Scholes SDE under the real-world probability measure $\mathbb{P}$:

$$dS_t^i = \mu_i\, S_t^i\, dt + S_t^i \sum_{j=1}^{n} \sigma_{ij}\, dW_t^j, \qquad i = 1, \ldots, n,$$

with constants $\mu_i \in \mathbb{R}$, $\sigma_{ij} \in \mathbb{R}$ and $W_t = (W_t^1, \ldots, W_t^n)$ a standard n-dimensional Brownian motion. Equivalently in vector form,

$$d\mathbf{S}_t = (\mathbf{S}_t)\, \boldsymbol{\mu}\, dt + (\mathbf{S}_t)\, \Sigma\, d\mathbf{W}_t,$$

where $(\mathbf{S}_t)$ is the diagonal matrix with entries S_t^i, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^\top$, and $\Sigma = (\sigma_{ij})_{i,j=1}^n$. This SDE admits the explicit solution

$$S_t^i = S_0^i \exp\Big(\big(\mu_i - \tfrac{1}{2}\sum_{j=1}^{n} \sigma_{ij}^2\big)t + \sum_{j=1}^{n} \sigma_{ij}\, W_t^j\Big),$$

so each component $S_t^i > 0$ for all $t \geq 0$ a.s. (all stock prices stay positive). We impose the usual *no-arbitrage* and *frictionless market* assumptions (no transaction costs or taxes, continuous trading, assets infinitely divisible, etc.).

Self-financing strategy A trading strategy is a pair of adapted processes

$$\left(\boldsymbol{\phi}_t, \psi_t\right)_{0\le t\le T}, \quad \boldsymbol{\phi}_t = (\phi_t^1, \dots, \phi_t^n),$$

where ϕ_t^i and ψ_t represent the number of units held at time t in the i-th stock and in the bond, respectively. The value of the strategy at time t is

$$V_t = \sum_{i=1}^n \phi_t^i \, S_t^i \; + \; \psi_t \, B_t,$$

and we say the strategy is *self-financing* if its value evolves only due to gains or losses from price changes (and not by external cash flows), i.e.

$$dV_t \;=\; \sum_{i=1}^n \phi_t^i \, dS_t^i \; + \; \psi_t \, dB_t, \qquad 0 \le t \le T.$$

Equivalently,

$$\dot{\psi}_t \, B_t + \sum_{i=1}^n \dot{\phi}_t^i \, S_t^i = 0,$$

meaning any change in the bond holding is financed by an opposite change across the stocks.

Arbitrage An arbitrage opportunity is a self-financing strategy

$$\left(\boldsymbol{\phi}_t, \psi_t\right)_{0\le t\le T}, \quad \boldsymbol{\phi}_t = (\phi_t^1, \dots, \phi_t^n),$$

with zero initial cost $V_0 = 0$ such that its terminal value

$$V_T = \sum_{i=1}^n \phi_T^i \, S_T^i + \psi_T \, B_T$$

is almost surely non-negative and $\mathbb{P}(V_T > 0) > 0$. The market satisfies the no-arbitrage condition if no arbitrage opportunity exists.

In an arbitrage-free market, prices of traded assets admit a martingale representation under a suitable change of probability measure. In particular, one can show (see the Fundamental Theorem of Asset Pricing, Baldi Proposition 13.3) that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure under which all discounted asset prices are martingales.

Equivalent martingale (risk-neutral) measure A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ is called an equivalent martingale measure (or risk-neutral measure) if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ (i.e. $\mathbb{Q}$ has the same null sets as $\mathbb{P}$) and under $\mathbb{Q}$ each discounted stock price

$$\widetilde{S}_t^i \;:=\; e^{-rt} S_t^i, \qquad i = 1, \dots, n,$$

is a martingale (with respect to the natural filtration $(\mathcal{F}_t)_{t\ge 0}$). Equivalently, each S_t^i earns the risk-free rate under $\mathbb{Q}$.

By a fundamental result in financial mathematics, an equivalent martingale measure exists if and only if the market has no arbitrage. Moreover, if the market is *complete* (meaning every contingent claim can be replicated by some self-financing strategy using the n stocks and the bond), the equivalent martingale measure is unique. The n-dimensional Black–Scholes model (with n sources of randomness $W^1, \dots, W^n$ and n risky assets $S^1, \dots, S^n$) is a complete market, so we have a unique risk-neutral measure $\mathbb{Q}$. In fact, applying Girsanov's theorem (see Baldi Chapter 12), one can construct $\mathbb{Q}$ so that under $\mathbb{Q}$ the drift of each stock is r. In other words, under $\mathbb{Q}$ the stock dynamics become

$$dS_t^i = r\, S_t^i\, dt + S_t^i \sum_{j=1}^n \sigma_{ij}\, dW_t^{(\mathbb{Q}),j}, \quad i = 1, \dots, n,$$

where $W^{(\mathbb{Q})} = (W^{(\mathbb{Q}),1}, \dots, W^{(\mathbb{Q}),n})$ is a Brownian motion under $\mathbb{Q}$. We will henceforth work under $\mathbb{Q}$ for pricing purposes.

Black–Scholes PDE and Feynman–Kac representation Under the Black–Scholes model assumptions (no arbitrage, frictionless market, etc.), let $V(x,t)$ be the price of a European option with payoff at time T $h(x)$ $\big(x \in (0,\infty)^n\big)$. Assume $V(x,t)$ is sufficiently smooth (of class $C^{2,1}$ in (x,t) on $(0,\infty)^n \times [0,T)$). Then V satisfies the *multi-asset Black–Scholes PDE*: for $0 \le t < T$ and $x = (x_1, \dots, x_n) \in (0,\infty)^n$,

$$\frac{\partial V}{\partial t}(x,t) + \sum_{i=1}^n r\, x_i\, \frac{\partial V}{\partial x_i}(x,t) + \tfrac{1}{2} \sum_{i,j=1}^n (\Sigma\Sigma^\top)_{ij}\, x_i\, x_j\, \frac{\partial^2 V}{\partial x_i\, \partial x_j}(x,t) \;-\; r\, V(x,t) \;=\; 0,$$

with terminal condition $V(x,t) = h(x)$. Moreover, by the Feynman–Kac formula the unique classical solution of this PDE is

$$V(x,t) \;=\; \mathbb{E}^{\mathbb{Q}}_{t,x}\Big[e^{-r\,(T-t)}\, h\big(\mathbf{S}_T\big)\Big],$$

which indeed coincides with the arbitrage-free price of the option.

Derivation of the PDE. We proceed by the martingale/no-arbitrage argument. Since $V(t, \mathbf{S}_t)$ is the option's value at time t, consider the *discounted gain process*

$$M_t \;:=\; e^{-rt}\, V\big(t, \mathbf{S}_t\big).$$

Under the risk-neutral measure $\mathbb{Q}$, all discounted asset prices are martingales, and the option's payoff $h(\mathbf{S}_T)$ is replicated by a self-financing strategy. Hence M_t must be a martingale (a "fair game"), so its drift vanishes.

By Itô's formula (noting that under $\mathbb{Q}$ the stock vector obeys $d\mathbf{S}_t = r\, \mathbf{S}_t\, dt \;+\; (\mathbf{S}_t)\, \Sigma\, d\mathbf{W}_t^{(\mathbb{Q})}$), we get

$$dV\big(t, \mathbf{S}_t\big) = V_t\, dt + \nabla_x V \cdot d\mathbf{S}_t + \tfrac{1}{2} \sum_{i,j=1}^n V_{x_i x_j}\, dS_t^i\, dS_t^j.$$

Since

$$dS_t^i = r\, S_t^i\, dt + \sum_{k=1}^n \sigma_{ik}\, S_t^i\, dW_t^{(\mathbb{Q}),k}, \quad dS_t^i\, dS_t^j = S_t^i\, S_t^j \sum_{k=1}^n \sigma_{ik}\sigma_{jk}\, dt = (\Sigma\Sigma^\top)_{ij}\, x_i\, x_j\, dt,$$

this yields

$$dV = \Big(V_t + \sum_{i=1}^n r\, x_i\, V_{x_i} + \tfrac{1}{2} \sum_{i,j=1}^n (\Sigma\Sigma^\top)_{ij}\, x_i\, x_j\, V_{x_i x_j}\Big)\, dt + \sum_{k=1}^n \Big(\sum_{i=1}^n \sigma_{ik}\, x_i\, V_{x_i}\Big)\, dW_t^{(\mathbb{Q}),k}.$$

Applying Itô's formula to $M_t = e^{-rt}V(t, \mathbf{S}_t)$ gives

$$dM_t = e^{-rt}\,dV - r\,e^{-rt}V\,dt = e^{-rt}\underbrace{\Big[V_t + \sum_i r x_i V_{x_i} + \tfrac{1}{2}\sum_{i,j}(\Sigma\Sigma^\top)_{ij}x_i x_j V_{x_i x_j} - rV\Big]}_{\text{drift term}}dt$$
$$+\, e^{-rt}\sum_k\Big(\sum_i \sigma_{ik}x_i V_{x_i}\Big)dW_t^{(\mathbb{Q}),k}.$$

For M_t to be a martingale, the dt–term must vanish. Setting its coefficient to zero for all $x \in (0,\infty)^n$ and $t < T$ yields the backward PDE

$$V_t(x,t) + \sum_{i=1}^n r\,x_i\,V_{x_i}(x,t) + \tfrac{1}{2}\sum_{i,j=1}^n (\Sigma\Sigma^\top)_{ij}\,x_i\,x_j\,V_{x_i x_j}(x,t) \;-\; r\,V(x,t) \;=\; 0,$$

which is exactly the multi-asset Black–Scholes PDE. The terminal condition is

$$V(x,t) = h(x),$$

since at maturity the option's value equals the payoff with h a continuous function of at most polynomial growth in $|x|$.

Solution via Feynman–Kac. The multi-asset Black–Scholes PDE is a linear second-order backward parabolic equation. To identify its solution, we recognize it as a special case of the Feynman–Kac formula. Indeed, its generator

$$\mathcal{L}f(x) := \sum_{i=1}^n r\,x_i\,\partial_{x_i}f(x) + \tfrac{1}{2}\sum_{i,j=1}^n (\Sigma\Sigma^\top)_{ij}\,x_i\,x_j\,\partial^2_{x_i x_j}f(x)$$

is the infinitesimal generator of the geometric Brownian motion $\mathbf{S}_t$ under $\mathbb{Q}$, and the PDE reads

$$V_t + \mathcal{L}V - r\,V = 0.$$

Observe that in the multi-asset Black–Scholes model the drift and volatility coefficients

$$b(x) = r\,x, \qquad \sigma(x) = \operatorname{diag}(x)\,\Sigma, \quad x \in (0,\infty)^n,$$

are globally Lipschitz in x and grow at most linearly. Although, the diffusion matrix

$$a(x) = \sigma(x)\sigma(x)^\top = \operatorname{diag}(x)\,\Sigma\Sigma^\top\,\operatorname{diag}(x)$$

is positive-definite for each fixed $x \in (0,\infty)^n$, it degenerates as any $x_i \to 0$ and is not uniformly elliptic on $x \in (0,\infty)^n$. To recover uniform ellipticity one performs the change of variables

$$y = \ln x, \quad Y_t = \ln S_t,$$

under which

$$dY_t = \Big(r - \tfrac{1}{2}\operatorname{diag}(\Sigma\Sigma^\top)\Big)\,dt + \Sigma\,dW_t^{(\mathbb{Q})},$$

whose diffusion matrix $\Sigma\Sigma^\top$ is constant, positive-definite, and hence uniformly elliptic on $\mathbb{R}^n$. Applying the Feynman–Kac theorem to

$$u(y,t) := V(e^y, t)$$

then yields the representation

$$V(x,t) = u(\ln x, t) \;=\; \mathbb{E}^{\mathbb{Q}}_{t,x}\big[e^{-r\,(T-t)}\,h(\mathbf{S}_T)\big]. \tag{11}$$

which recovers the risk-neutral pricing formula. By the integrability assumptions on h, the expectation in (11) is finite for all $x \in (0,\infty)^n$, and one can justify differentiating under the expectation to verify that $V \in C^{2,1}$ and indeed satisfies the multi-asset PDE. □

In the particular case of a European call on the i-th stock with strike K (payoff $h(x) = (x_i - K)_+$), the above expectation reduces to the classical one-dimensional Black–Scholes formula since it is possible to compute a closed-form solution. Under $\mathbb{Q}$,

$$\ln S_T^i \sim N\Big(\ln x_i + (r - \tfrac{1}{2}\sigma_i^2)(T-t)\,,\, \sigma_i^2\,(T-t)\Big), \quad \sigma_i^2 \;=\; \sum_{j=1}^{n} \sigma_{ij}^2,$$

so one obtains

$$V(x,t) \;=\; x_i\,\Phi(d_1) \;-\; K\,e^{-r\,(T-t)}\,\Phi(d_2),$$

where

$$d_1 = \frac{\ln(x_i/K) + (r + \frac{1}{2}\sigma_i^2)(T-t)}{\sigma_i\sqrt{T-t}}, \quad d_2 = d_1 - \sigma_i\sqrt{T-t}.$$

For a put on the same asset (payoff $(K - x_i)_+$), one finds

$$V(x,t) = K\,e^{-r\,(T-t)}\,\Phi(-d_2) \;-\; x_i\,\Phi(-d_1).$$

These one-dimensional formulas provide explicit benchmarks even in the multi-asset Black–Scholes framework.

In practice, for more complex payoff functions h, a closed-form evaluation of the expectation in (11) may not be possible. In such cases one typically resorts to numerical methods—most notably Monte Carlo simulation—to approximate V. A detailed discussion of these computational techniques lies beyond the scope of this note.

6 References

Baldi, P. (2017). Stochastic Calculus: An Introduction Through Theory and Exercises. Springer Universitext. (See in particular Ch. 10 for the Feynman–Kac formula and Ch. 13 for finanical applications)