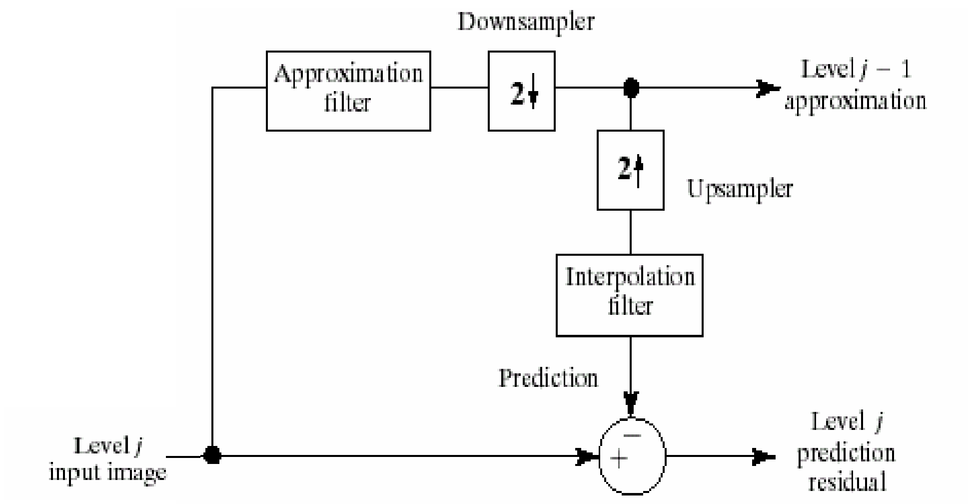
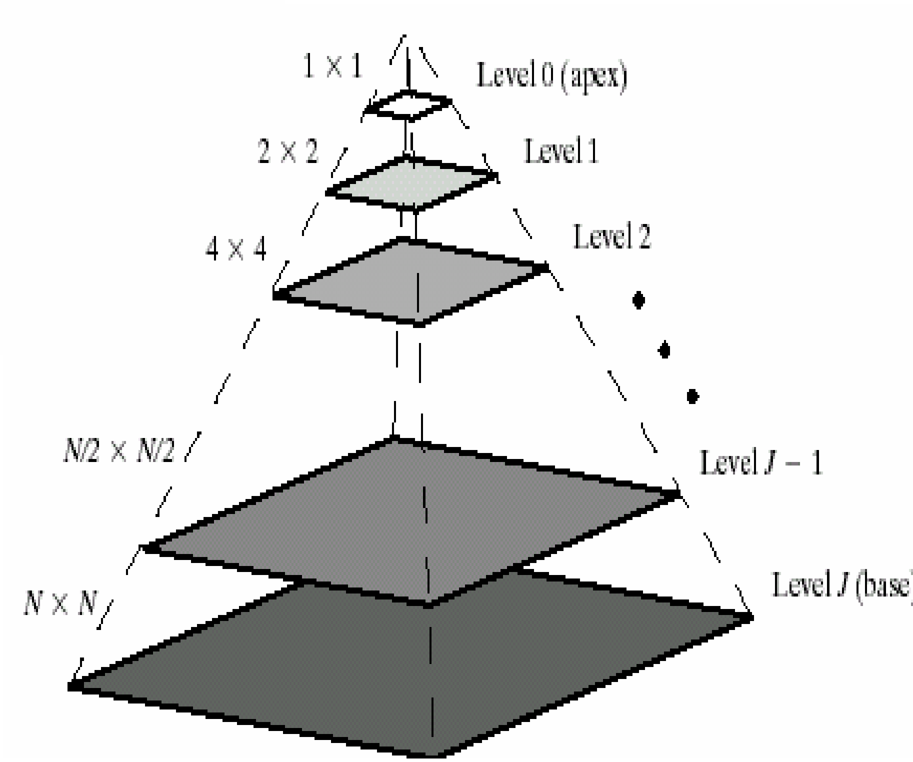
Multiresolution image processing

- Laplacian pyramids
- Some applications of Laplacian pyramids
- Discrete Wavelet Transform (DWT)
- Wavelet theory
- Wavelet image compression

Image pyramids





[Burt, Adelson, 1983]

Image pyramid example

Gaussian pyramid

Laplacian pyramid

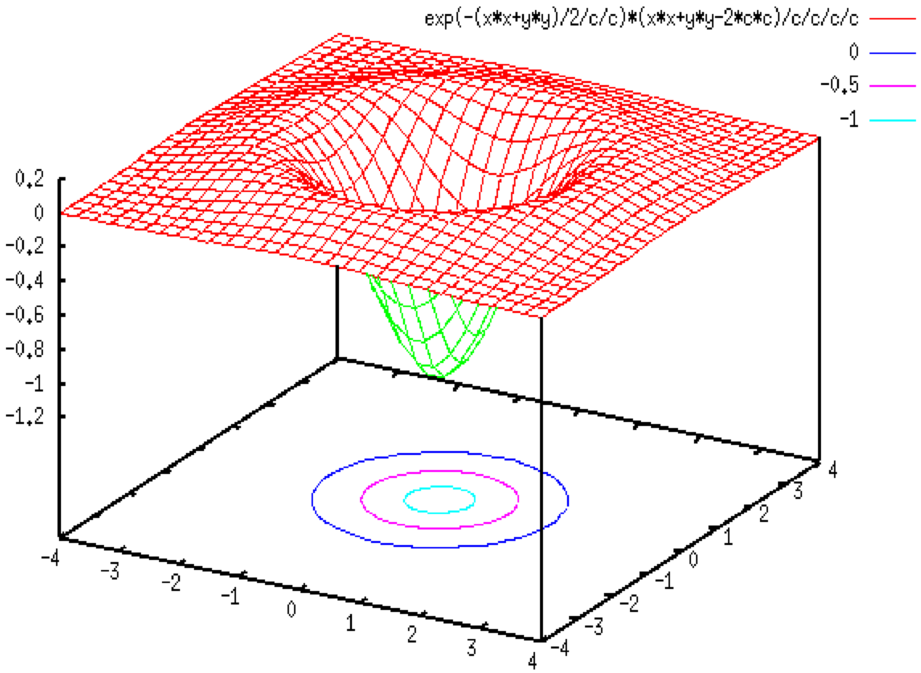
Overcomplete representation

Number of samples in Laplacian or Gaussian pyramid =

$$\left(1+\frac{1}{4}+\frac{1}{4^2}+\ldots+\frac{1}{4^P}\right)\leq\frac{4}{3}\text{ x number of original image samples}$$

LoG vs. DoG

<u>Laplacian of Gaussian</u>



<u>Difference of Gaussians</u>

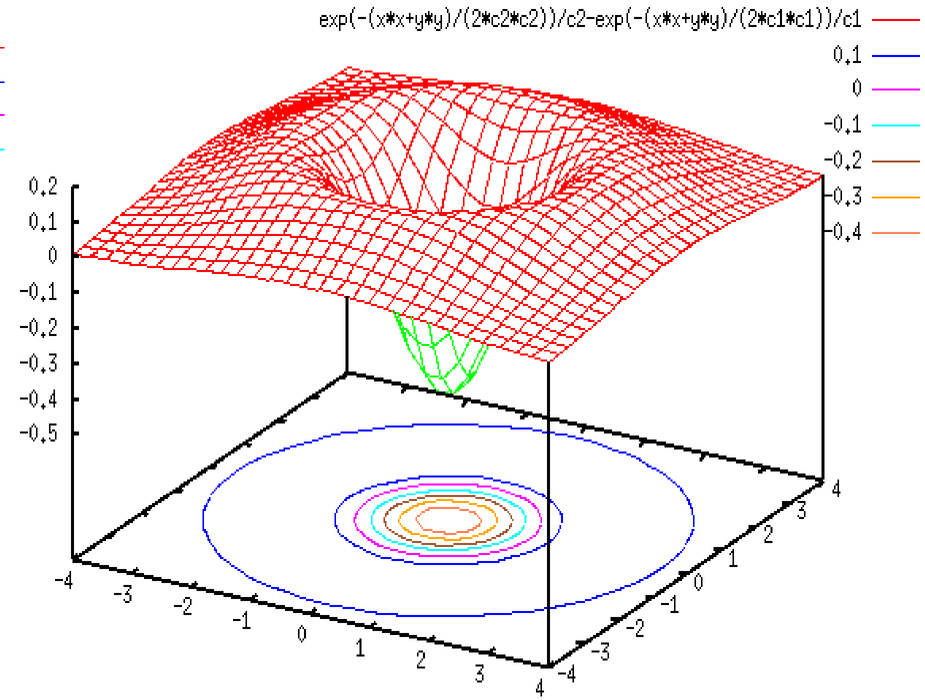
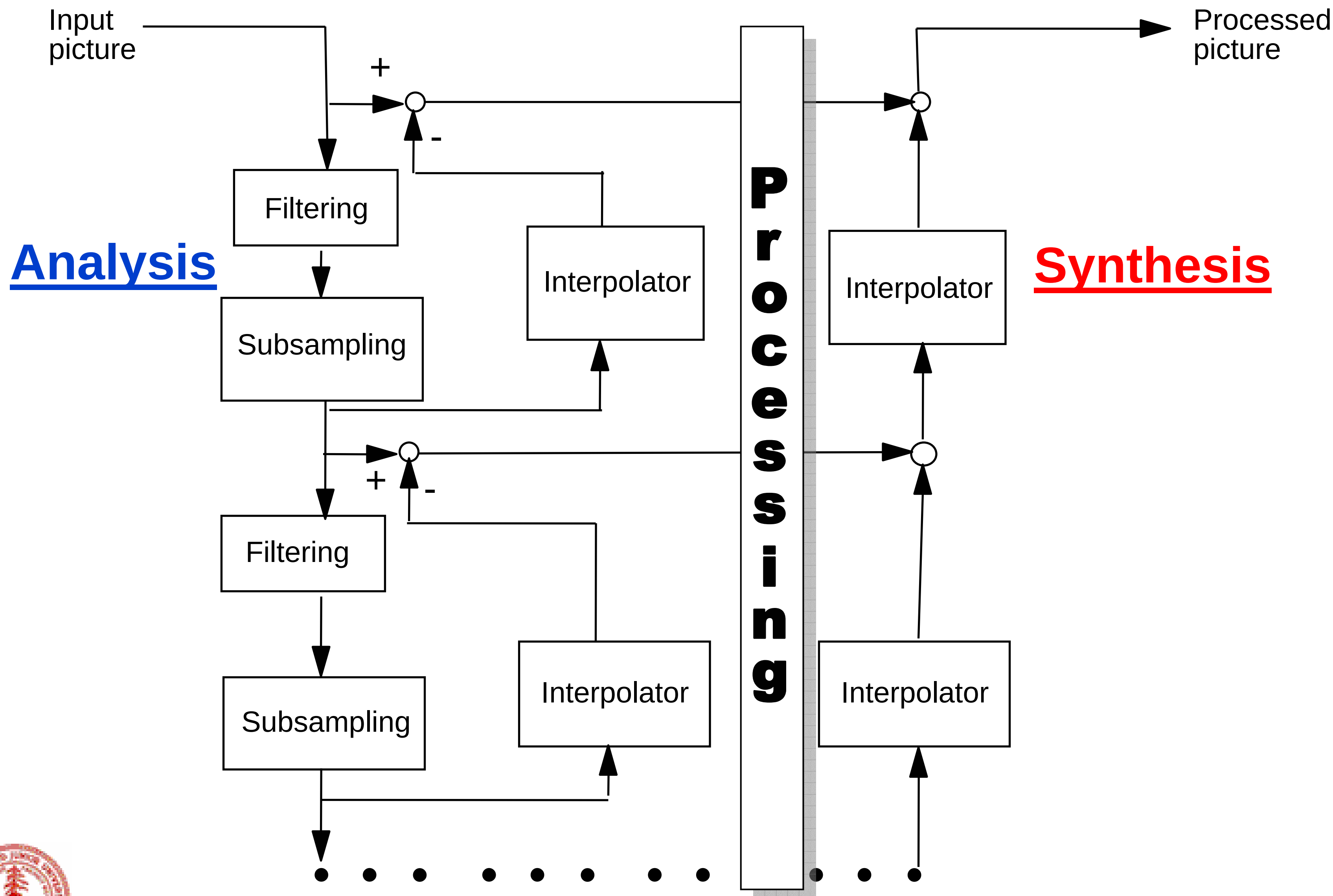
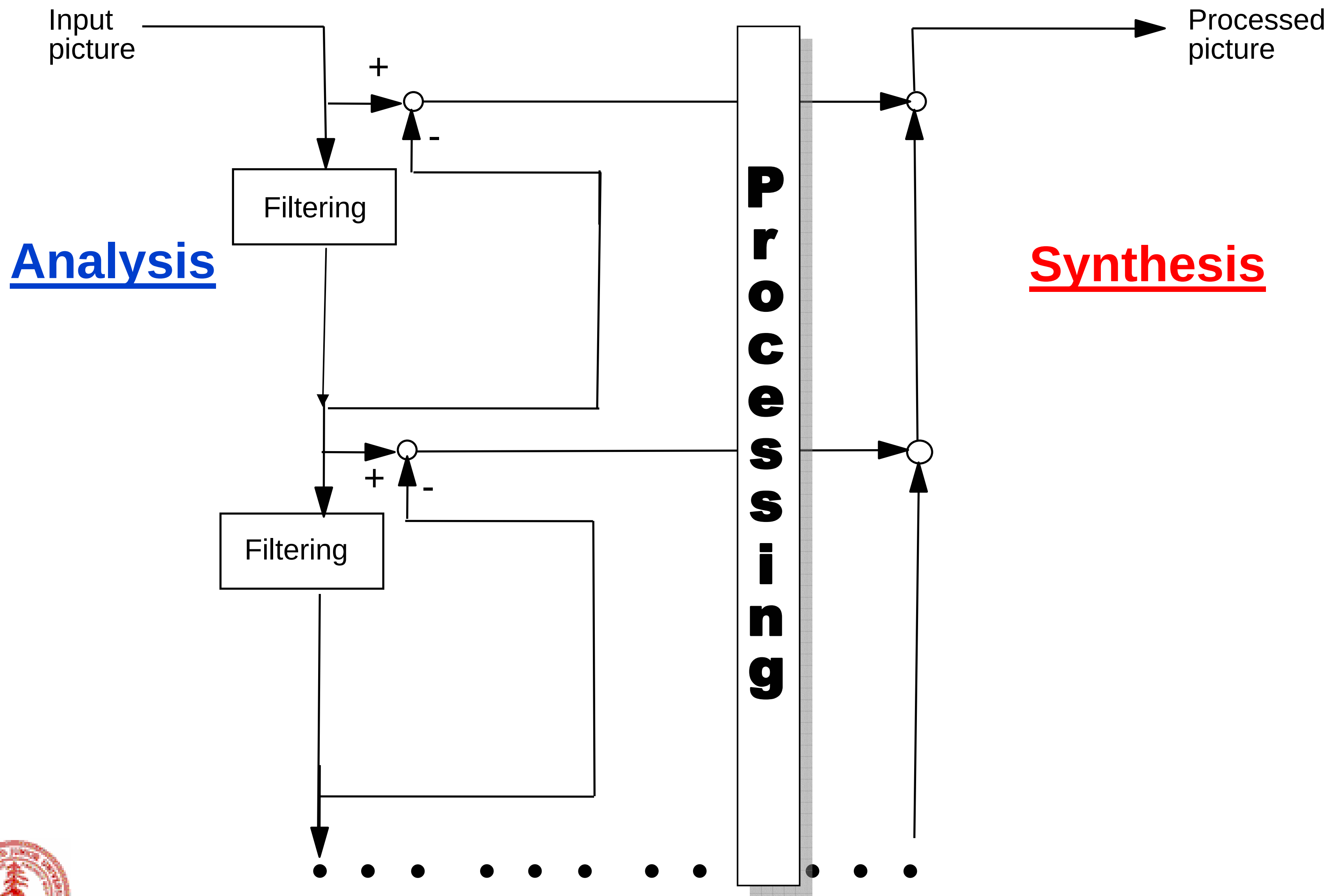


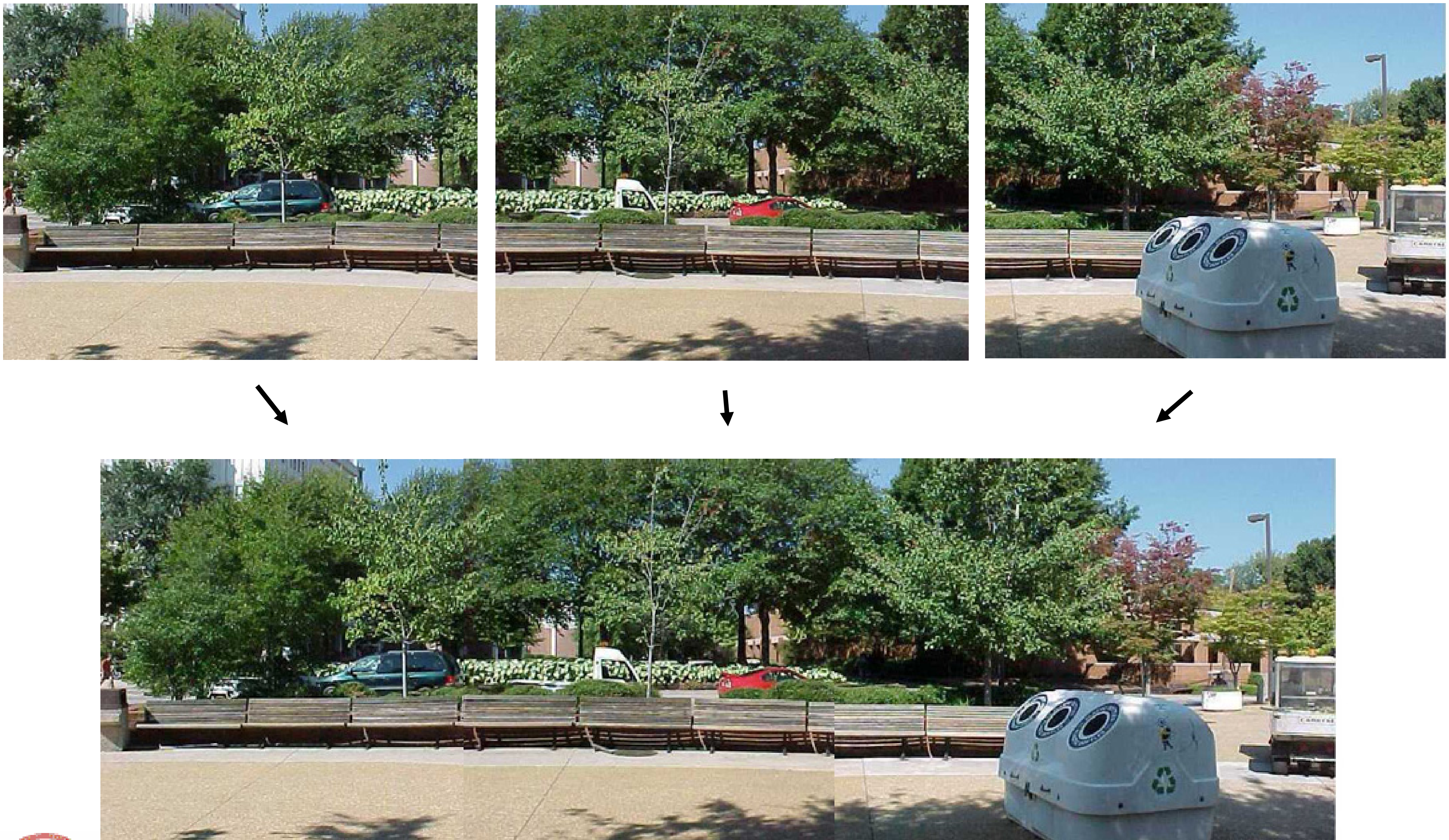
Image processing with Laplacian pyramid



Expanded Laplacian pyramid

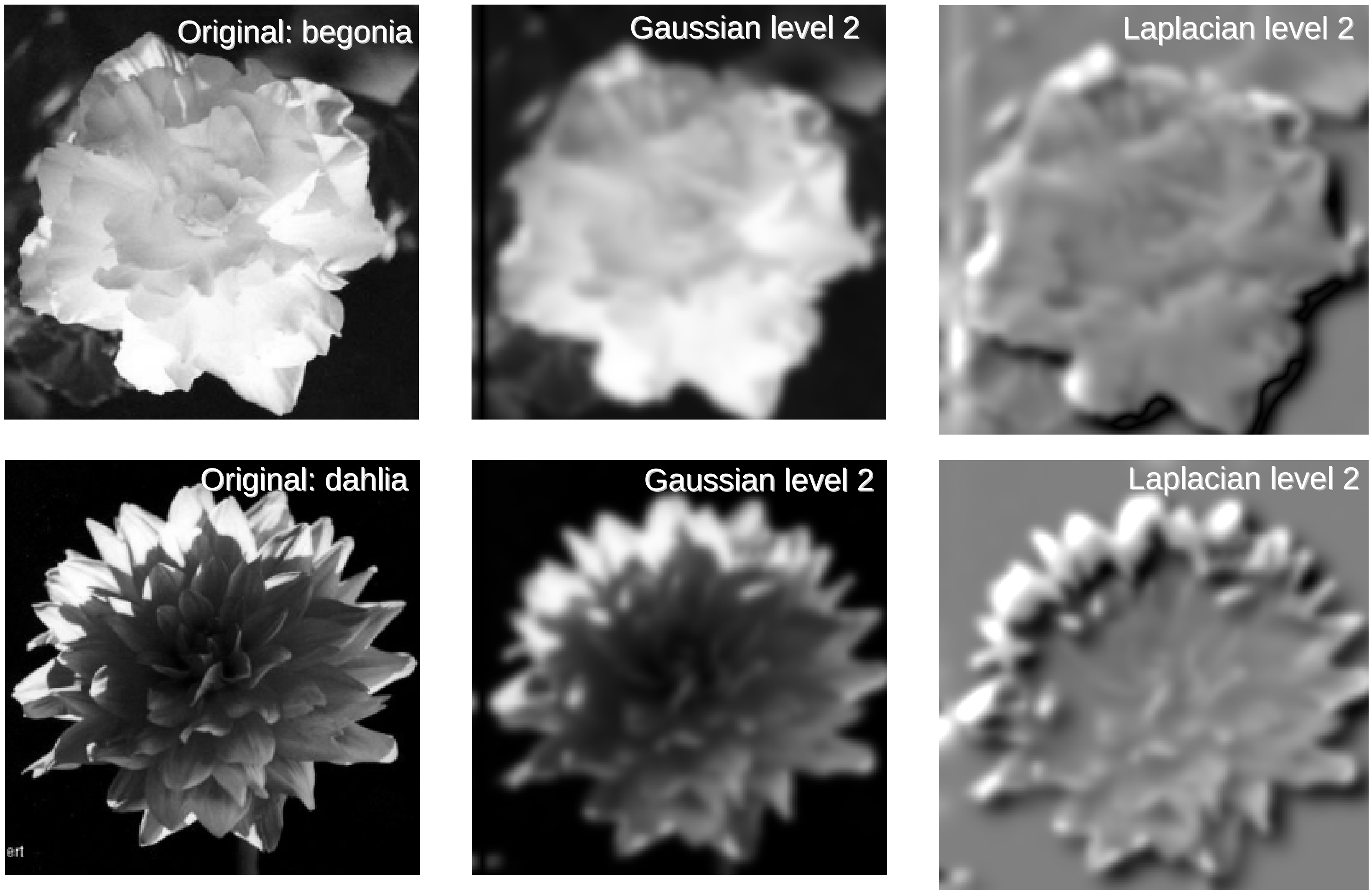


Mosaicing in the image domain



Mosaicing by blending Laplacian pyramids

Expanded Laplacian pyramids



Blending Laplacian pyramids

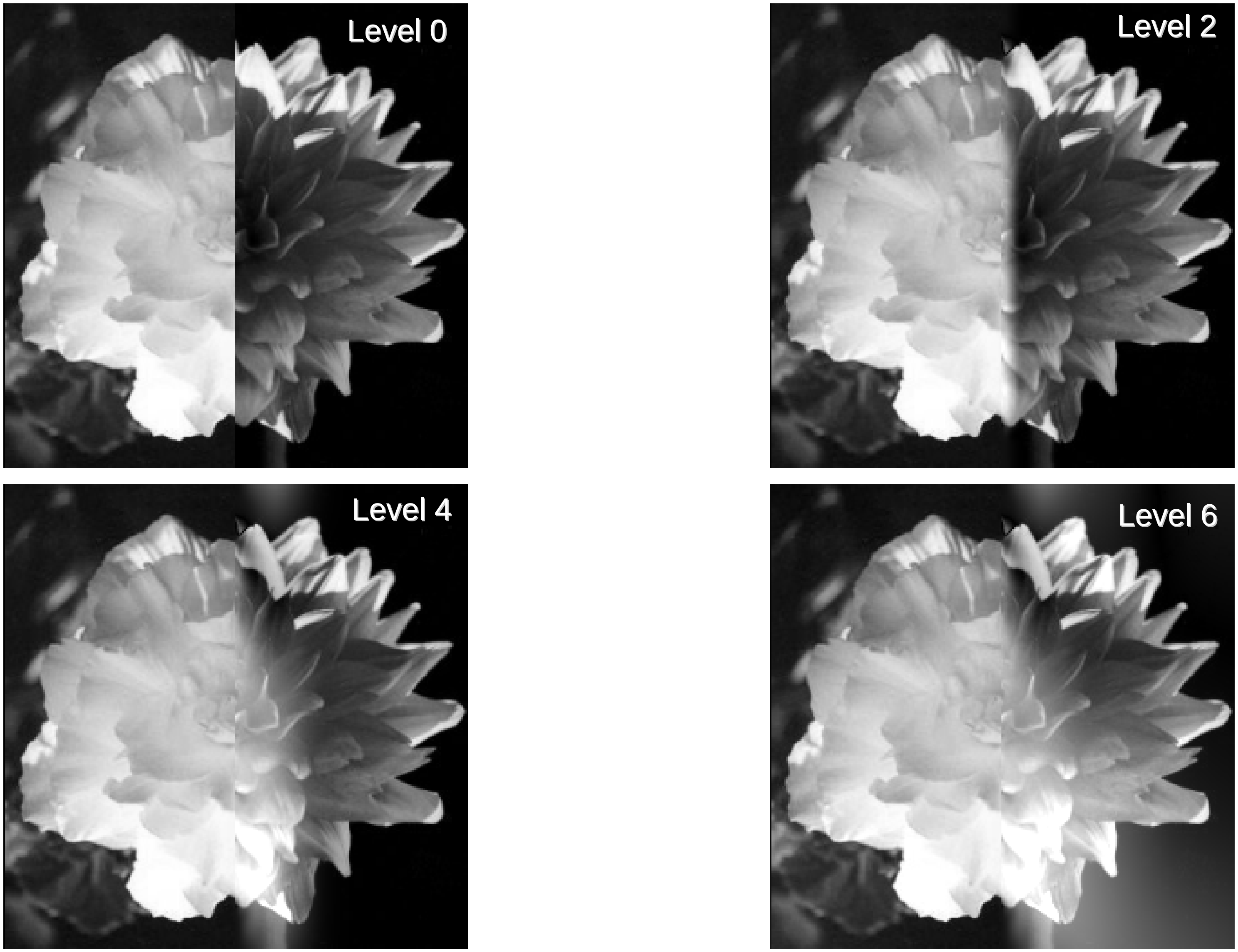
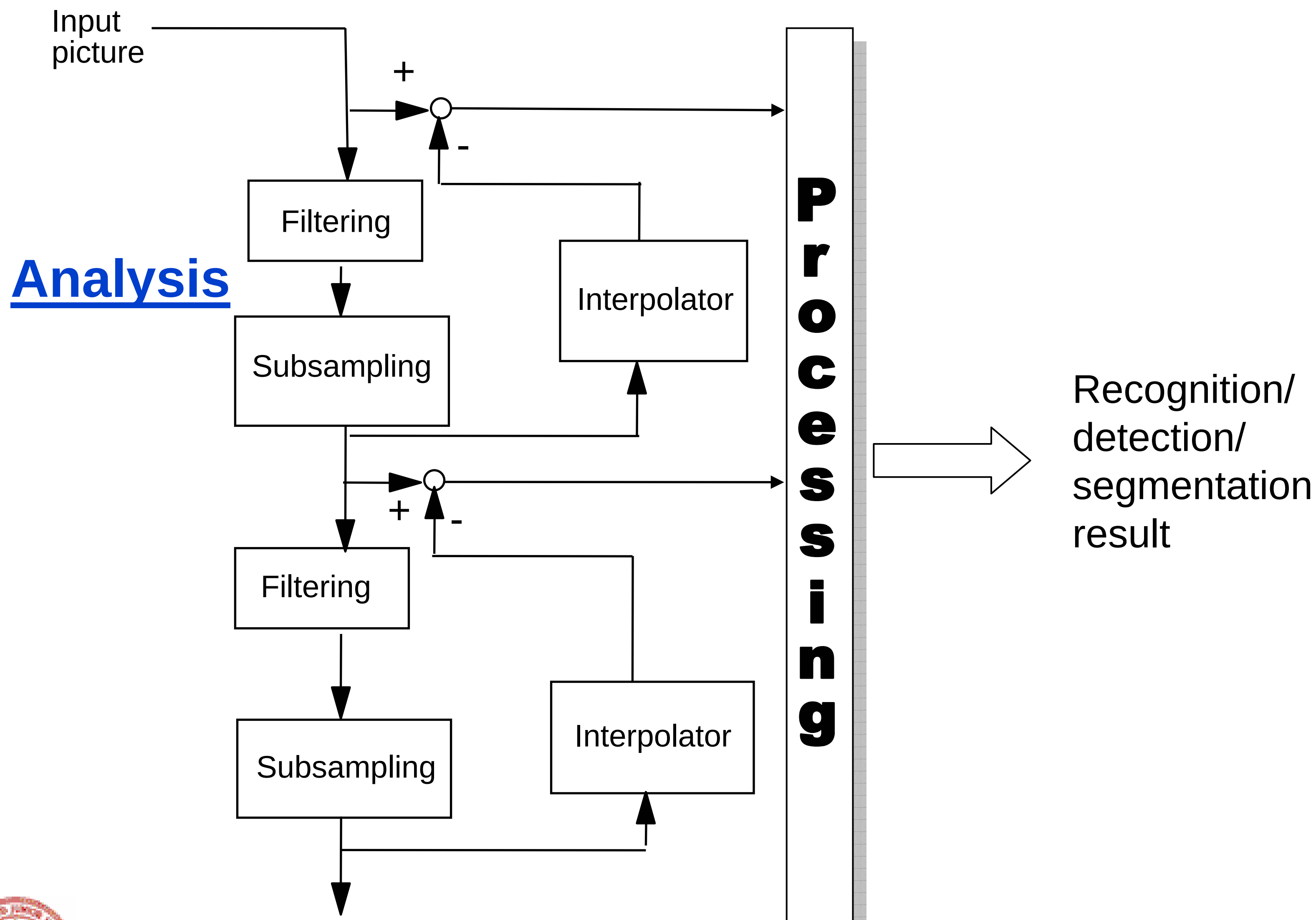
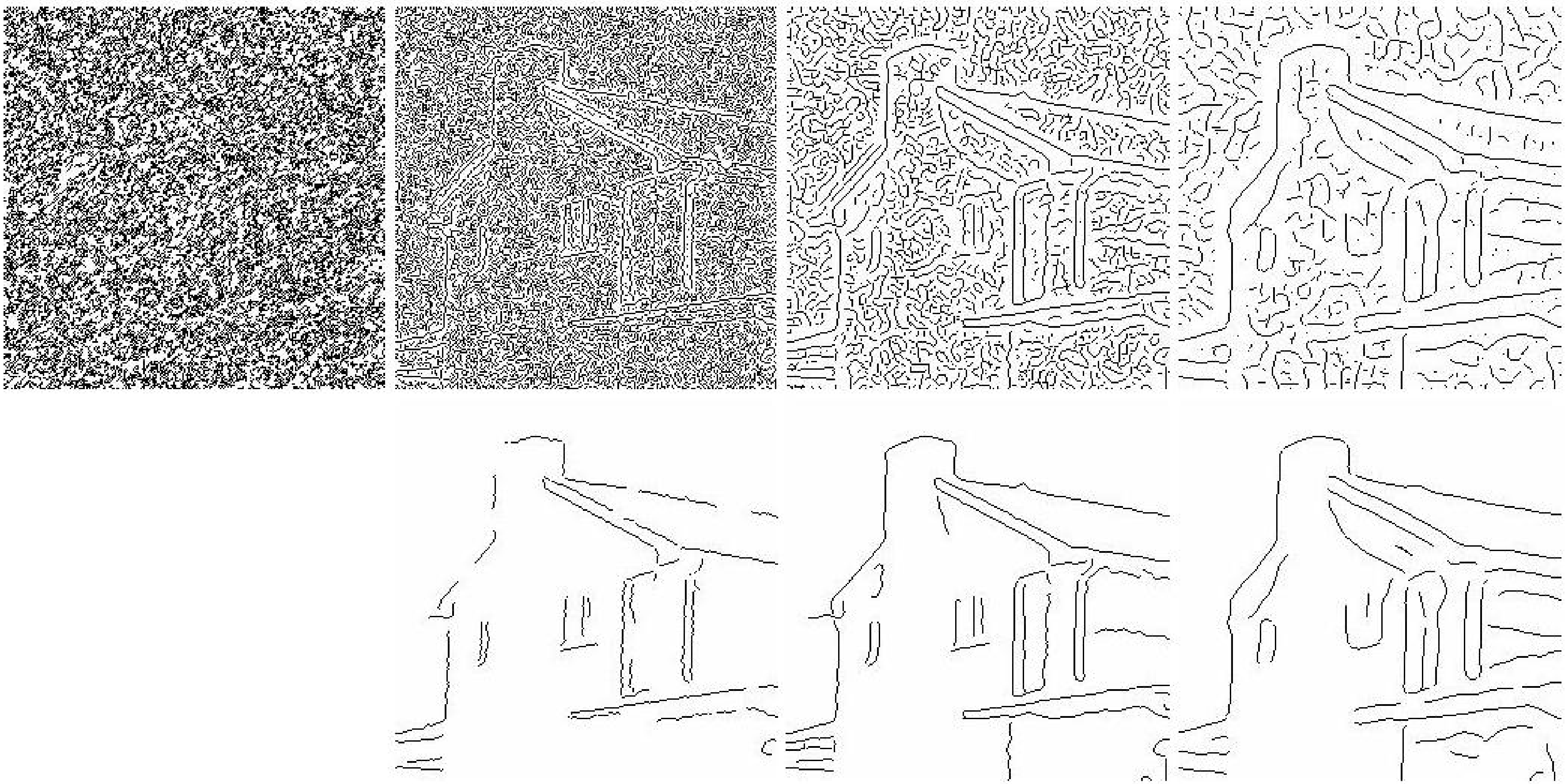


Image analysis with Laplacian pyramid



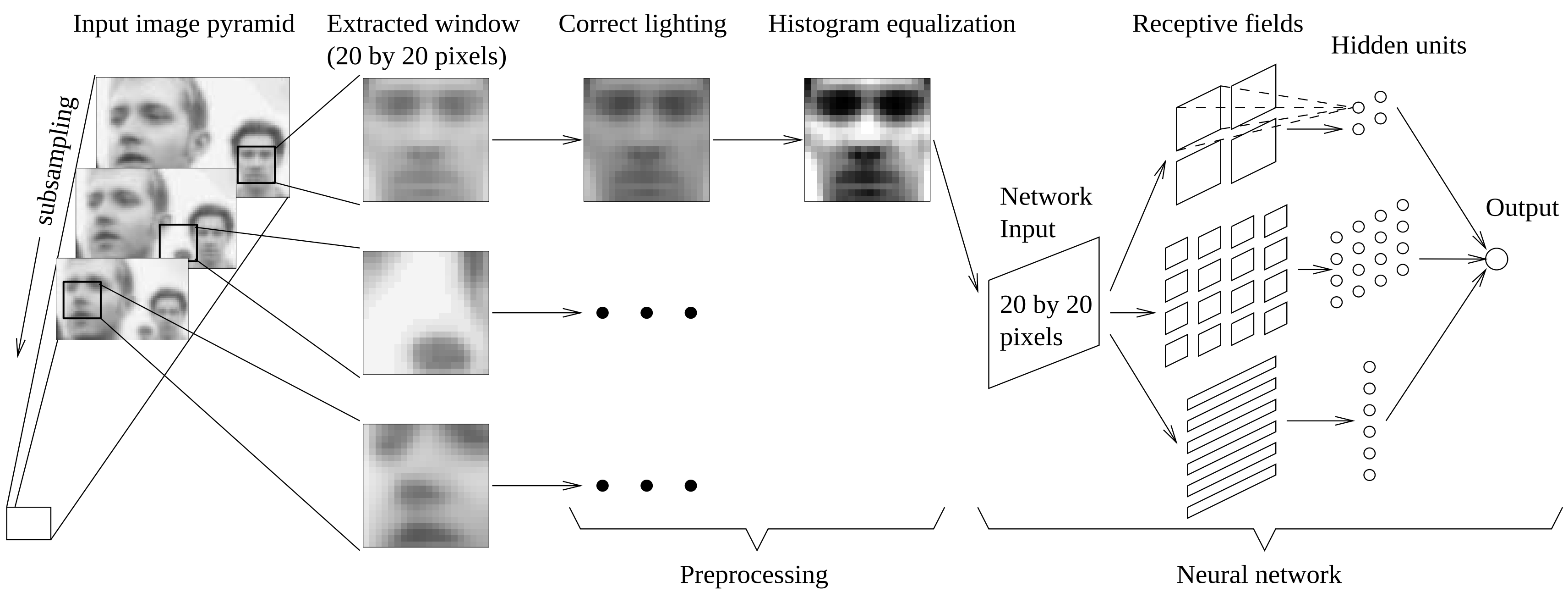
Multiscale edge detection

Zero-crossings of Laplacian images of different scales



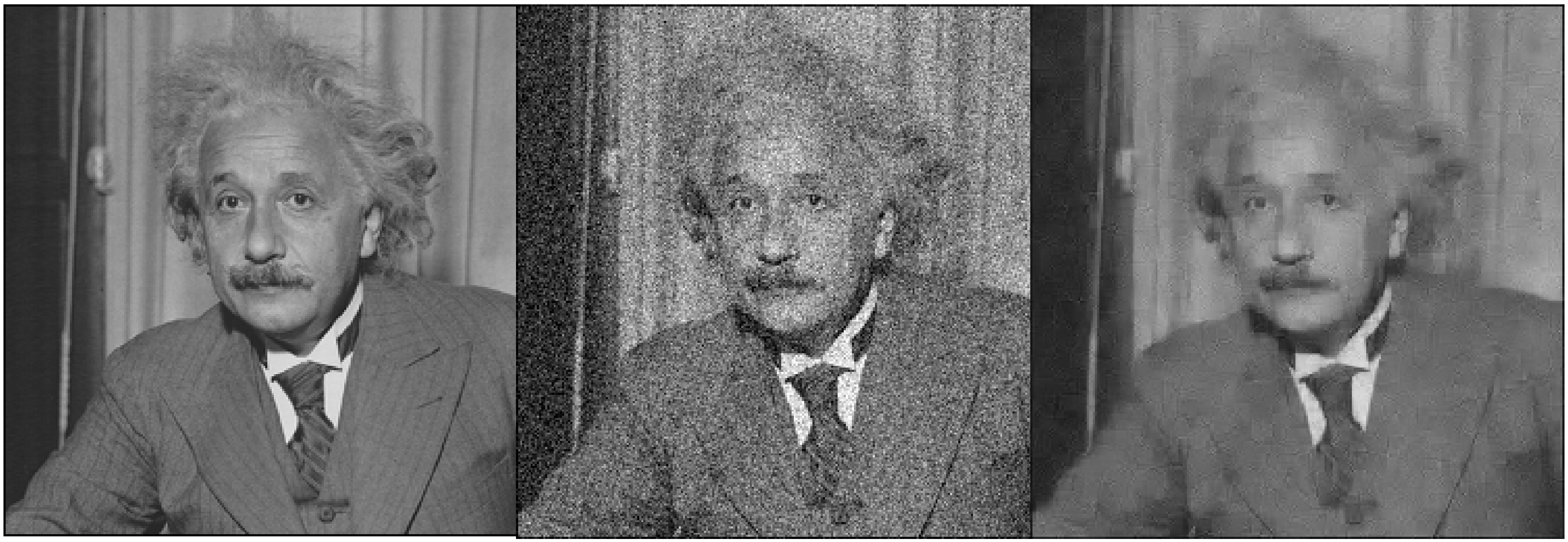
Spurious edges removed

Multiscale face detection



[Rowley, Baluja, Kanade, 1995]

Example: multiresolution noise reduction



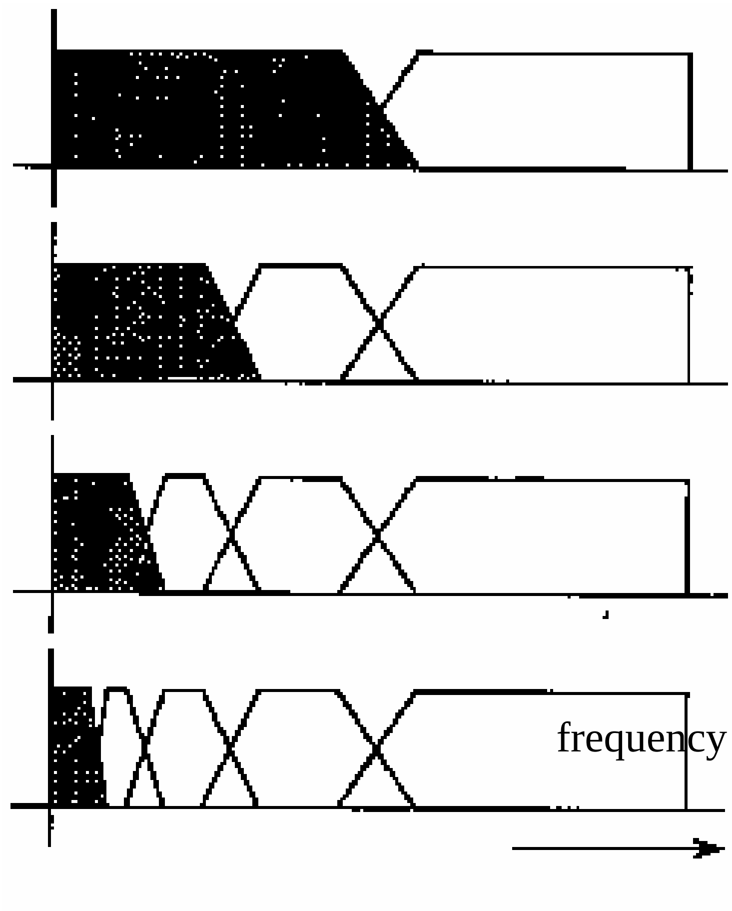
Original

Noisy

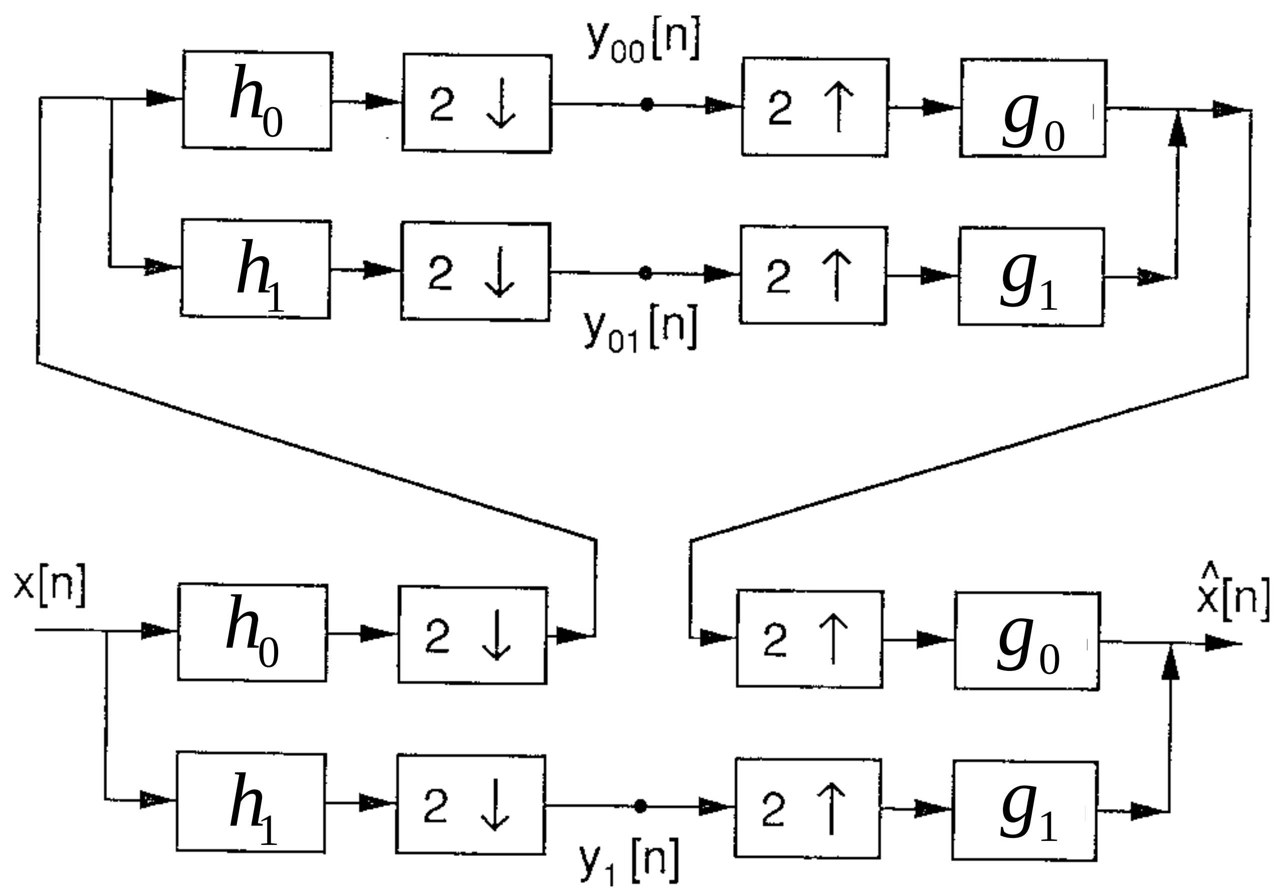
Noise reduction:
3-level Haar transform
no subsampling
w/ soft coring

1-d Discrete Wavelet Transform

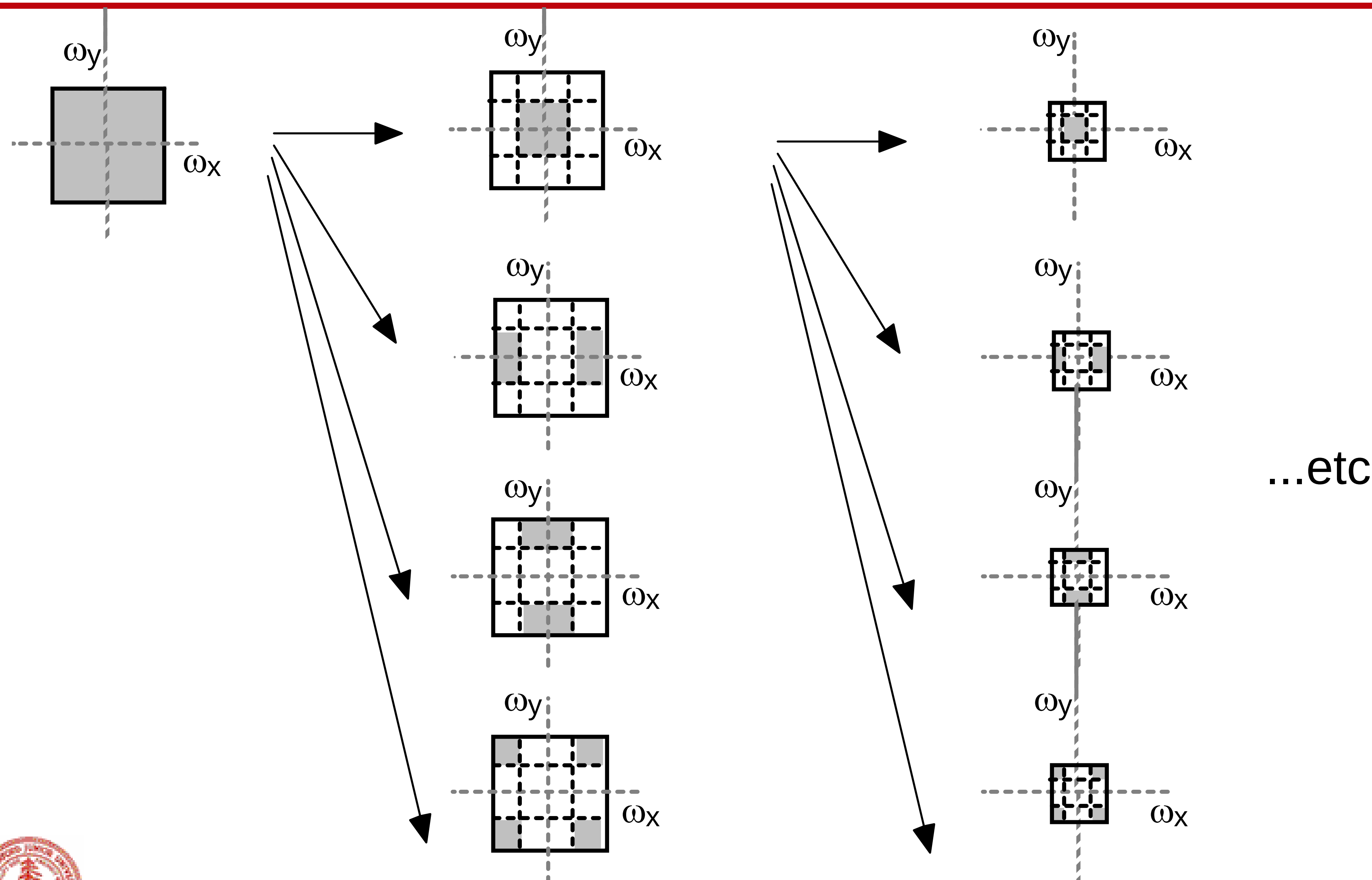
- Recursive application of a two-band filter bank to the lowpass band of the previous stage yields octave band splitting:



Cascaded analysis / synthesis filterbanks



2-d Discrete Wavelet Transform



2-d Discrete Wavelet Transform example

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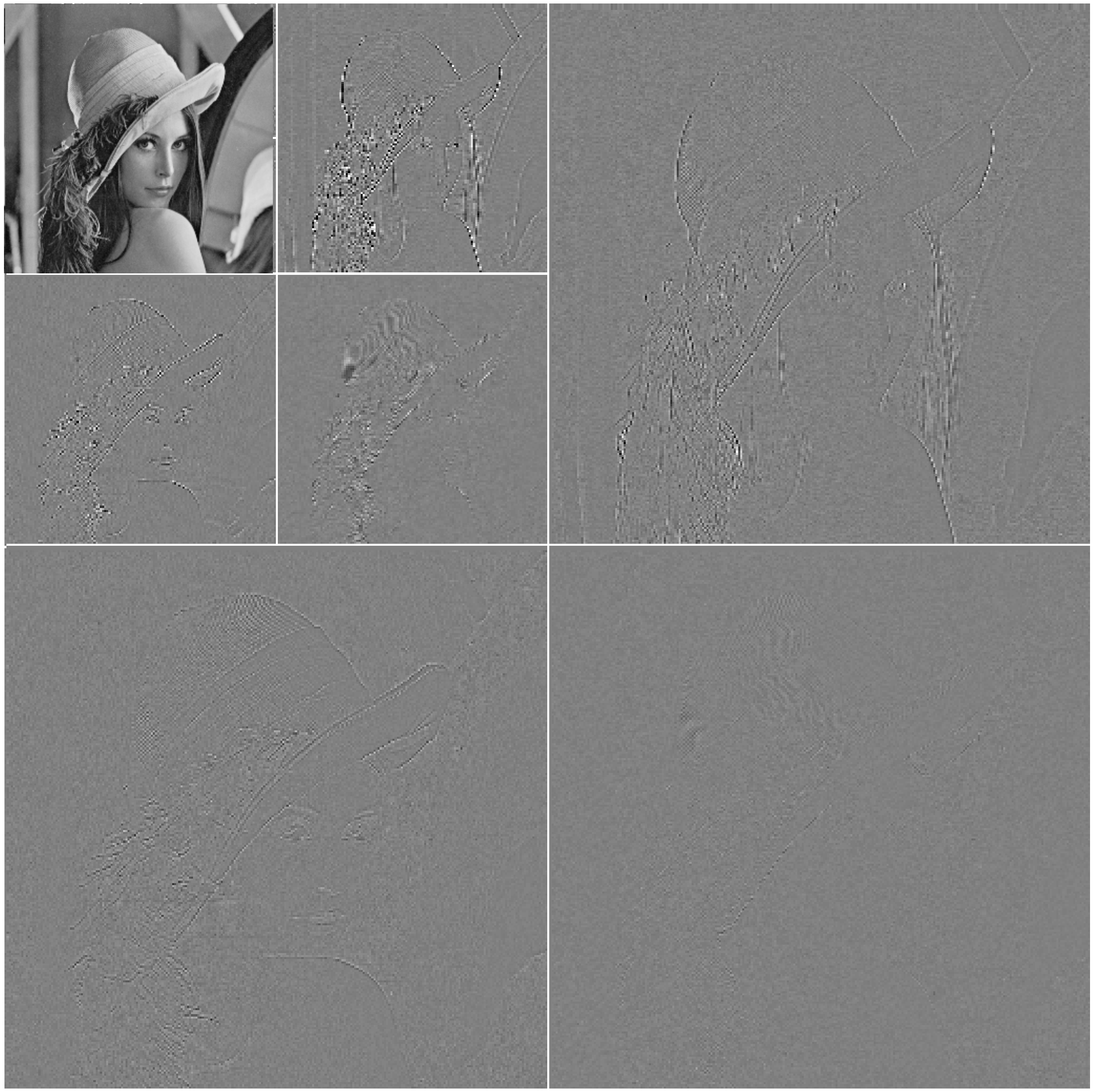
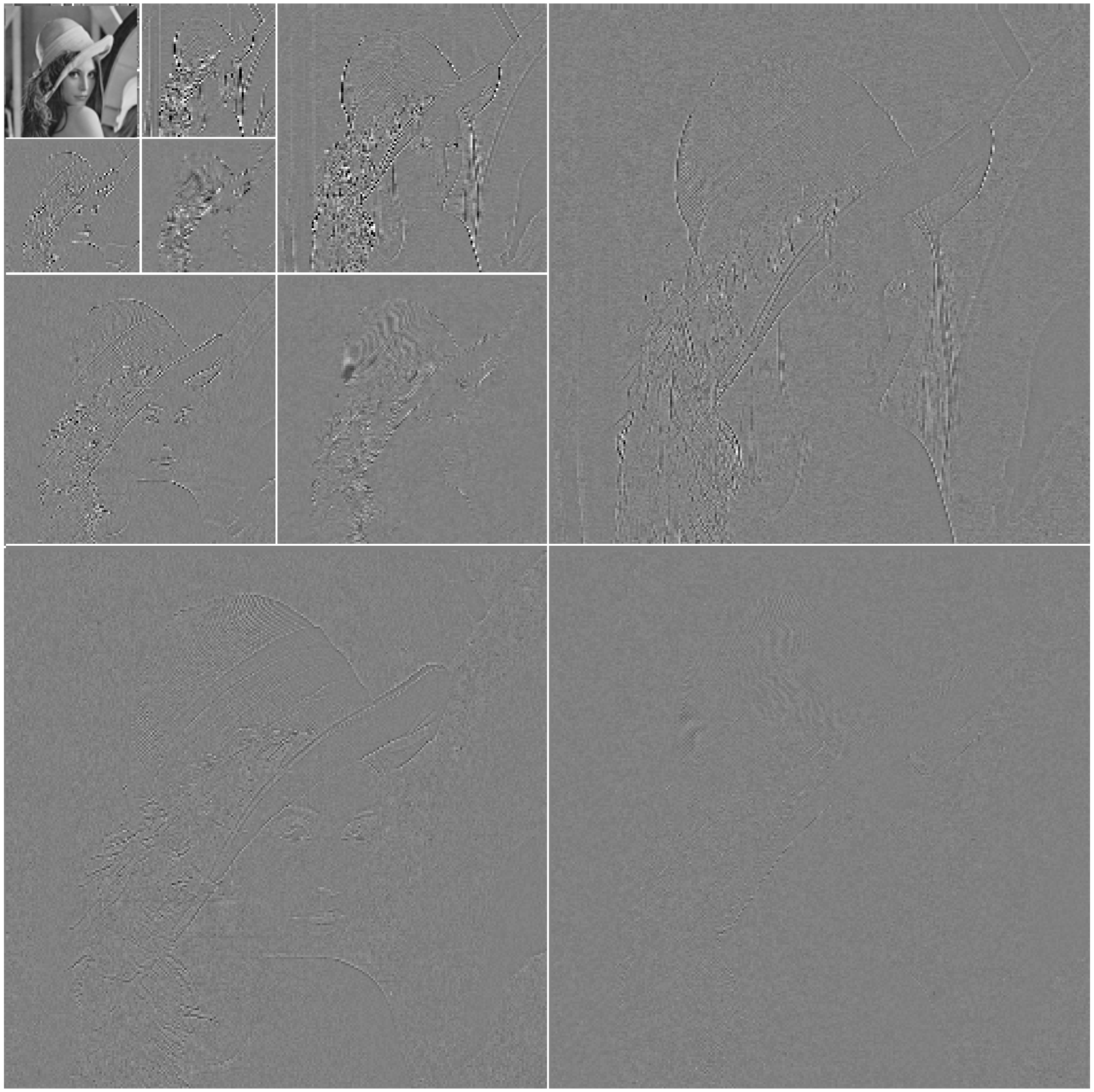
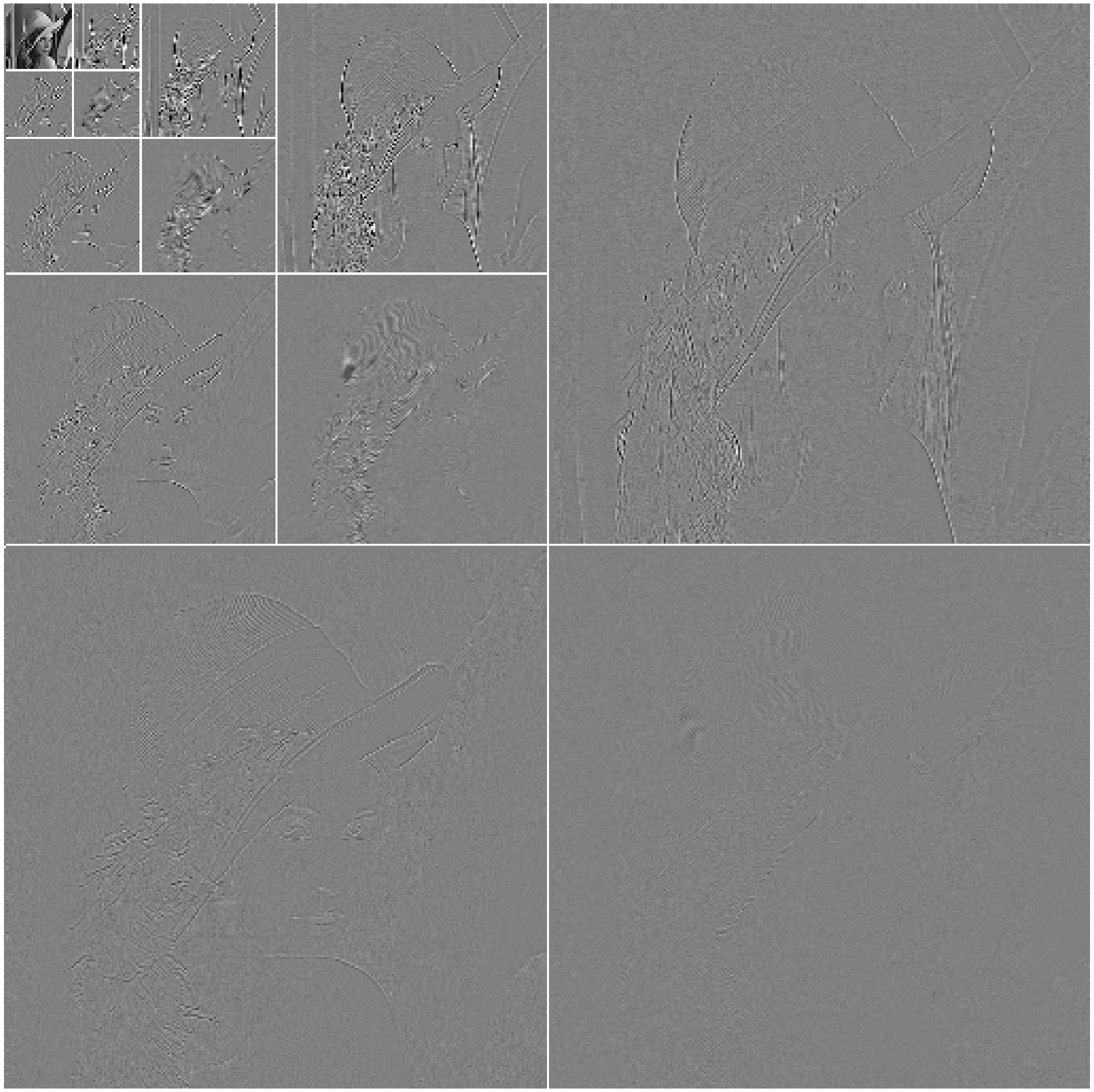


2-d Discrete Wavelet Transform example



2-d Discrete Wavelet Transform example



Review: Z-transform and subsampling

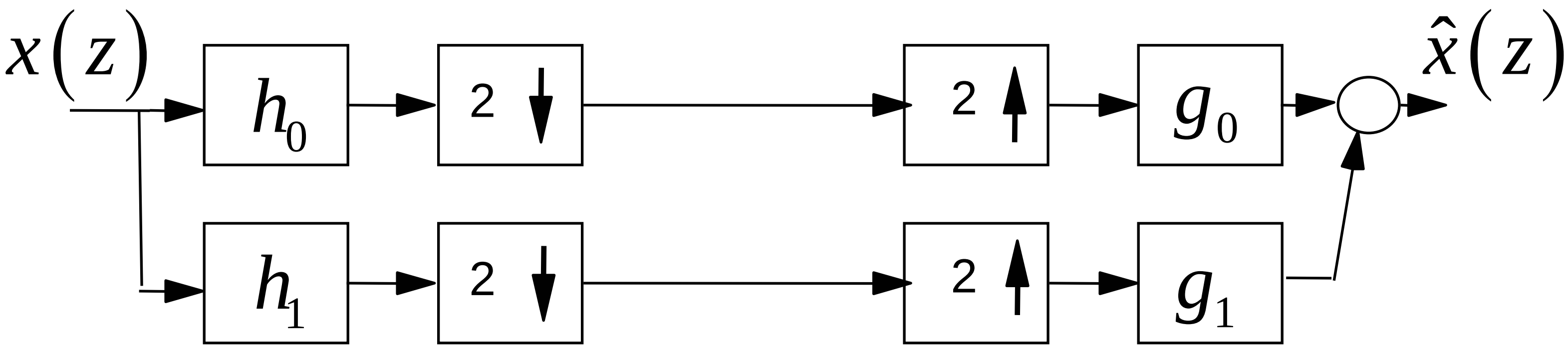
- Generalization of the discrete-time Fourier transform

$$\boxed{x(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \; ; \quad z \in \mathbb{C} \; ; \; r^{-} < |z| < r^{+}}$$

- Fourier transform on unit circle: substitute $z = e^{j\omega}$
- Downsampling and upsampling by factor 2

$$x(z) \longrightarrow \boxed{2\downarrow} \longrightarrow \boxed{2\uparrow} \longrightarrow \frac{1}{2}\left[x(z) + x(-z)\right]$$

Two-channel filterbank



$$\begin{aligned}\hat{x}(z) &= \frac{1}{2} g_0(z)\left[h_0(z)x(z) + h_0(-z)x(-z)\right] + \frac{1}{2} g_1(z)\left[h_1(z)x(z) + h_1(-z)x(-z)\right] \\ &= \frac{1}{2}\left[h_0(z)g_0(z) + h_1(z)g_1(z)\right]x(z) + \underbrace{\frac{1}{2}\left[h_0(-z)g_0(z) + h_1(-z)g_1(z)\right]x(-z)}_{\text{Aliasing}}\end{aligned}$$

- Aliasing cancellation if :

$$\begin{aligned} g_0(z) &= h_1(-z) \\ -g_1(z) &= h_0(-z) \end{aligned}$$

Example: two-channel filter bank with perfect reconstruction

- Impulse responses, analysis filters:

Lowpass

$$\left(\frac{-1}{4},\frac{1}{2},\frac{3}{2},\frac{1}{2},\frac{-1}{4}\right)$$

highpass

$$\left(\frac{1}{4},\frac{-1}{2},\frac{1}{4}\right)$$

- Impulse responses, synthesis filters

Lowpass

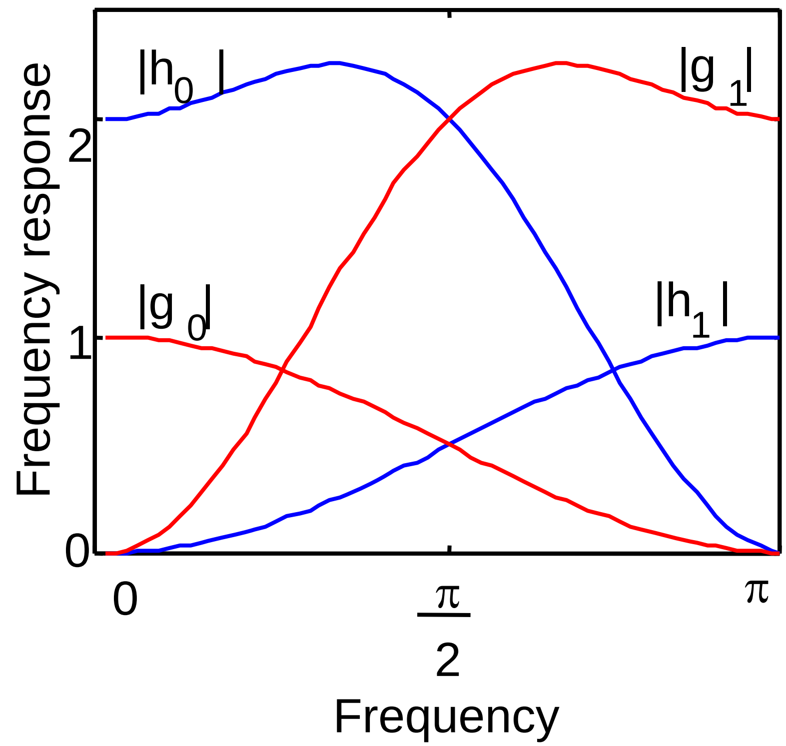
$$\left(\frac{1}{4},\frac{1}{2},\frac{1}{4}\right)$$

highpass

$$\left(\frac{1}{4},\frac{1}{2},\frac{-3}{2},\frac{1}{2},\frac{1}{4}\right)$$

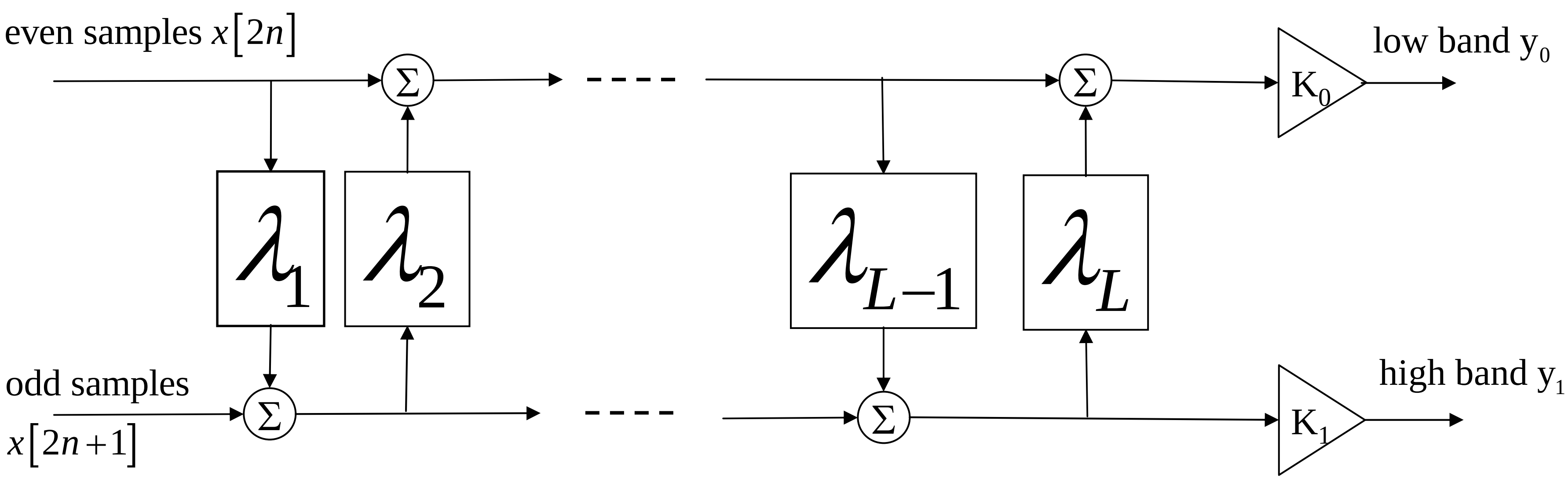
"Biorthogonal 5/3 filters"
"LeGall filters"

- Mandatory in JPEG2000
- Frequency responses:



Lifting

- Analysis filters



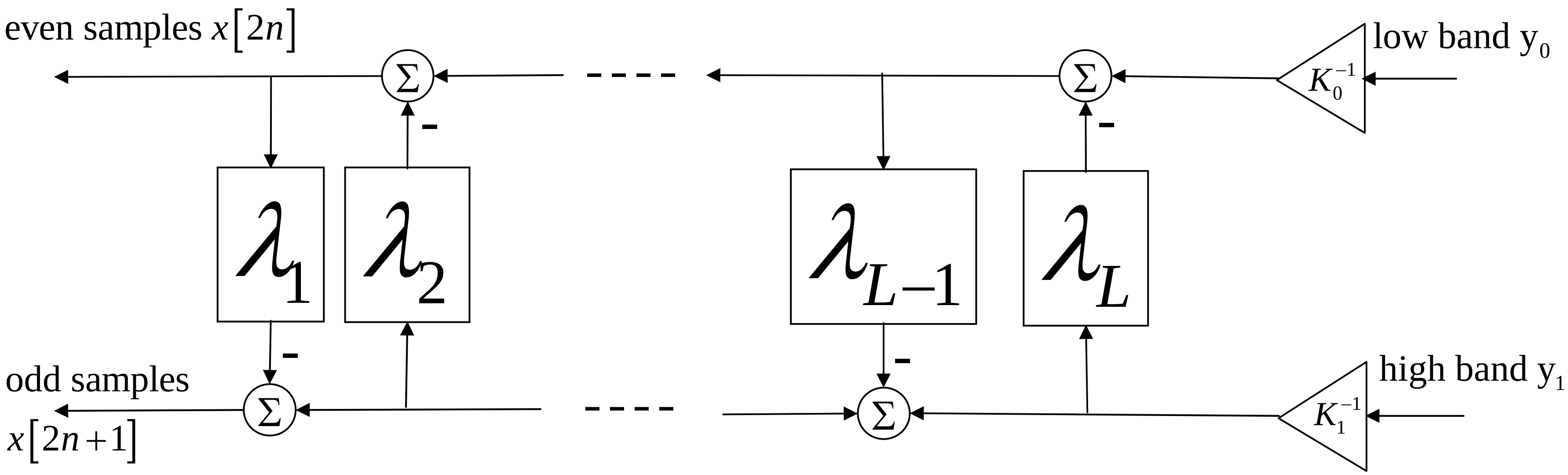
- L "lifting steps"

[Sweldens 1996]

- First step can be interpreted as prediction of odd samples from the even samples

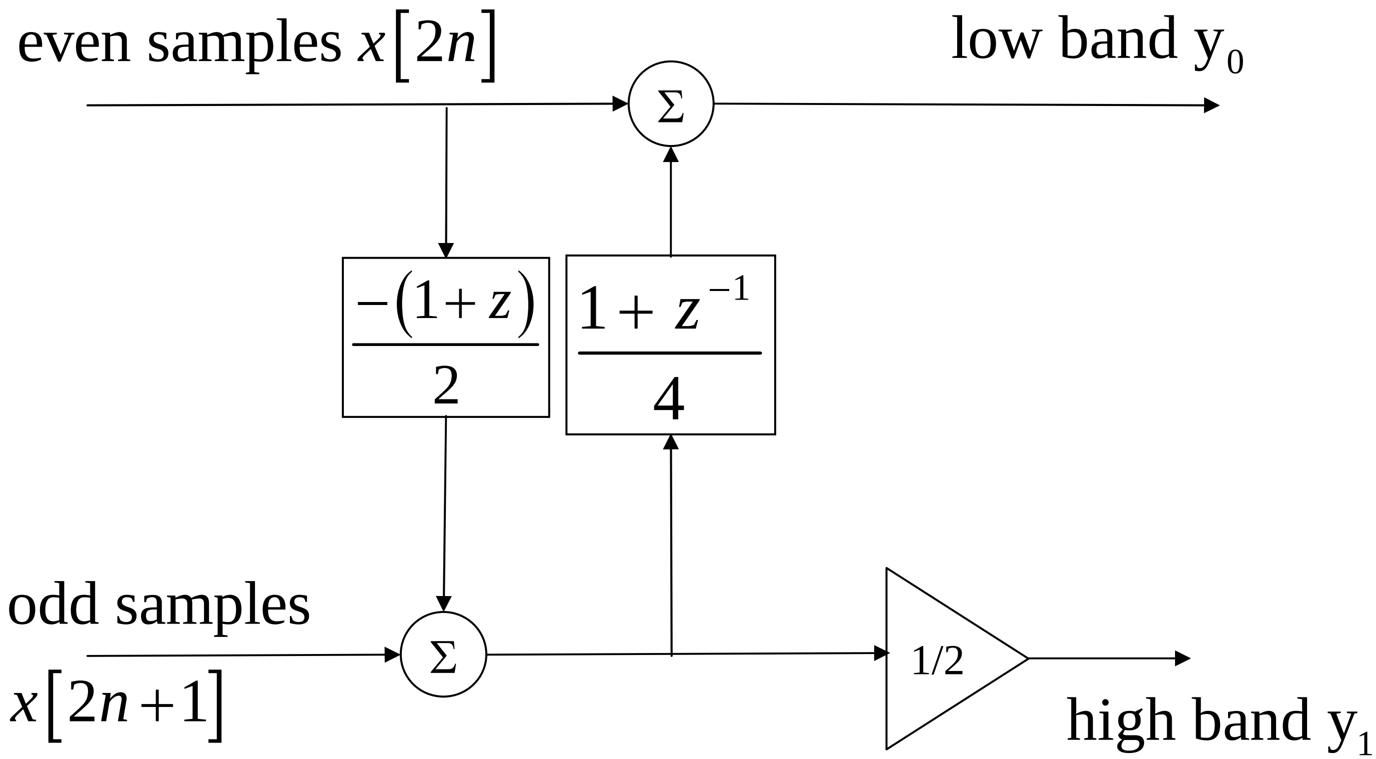
Lifting (cont.)

- Synthesis filters



- Perfect reconstruction (biorthogonality) is directly build into lifting structure
- Powerful for both implementation and filter/wavelet design

Example: lifting implementation of 5/3 filters



Verify by considering response to unit impulse in even and odd input channel.

Conjugate quadrature filters

- Achieve aliasing cancelation by

Prototype filter

$$\boxed{\begin{aligned} h_0(z) &= g_0(z^{-1}) \equiv f(z) \\ h_1(z) &= g_1(z^{-1}) = zf\left(-z^{-1}\right) \end{aligned}}$$

[Smith, Barnwell, 1986]

- Impulse responses

$$\boxed{\begin{aligned} h_0[k] &= g_0[-k] = f[k] \\ h_1[k] &= g_1[-k] = (-1)^{k+1} f\left[-(k+1)\right] \end{aligned}}$$

- With perfect reconstruction: orthonormal subband transform!
- Perfect reconstruction: find power complementary prototype filter

$$\left|F\left(e^{j\omega}\right)\right|^2 + \left|F\left(e^{j(\omega\pm\pi)}\right)\right|^2 = 2$$

Wavelet bases

Consider Hilbert space $\mathcal{L}^2(\mathbb{R})$ of finite-energy functions $\mathbf{x} = x(t)$.

Wavelet basis for $\mathcal{L}^2(\mathbb{R})$: family of linearly independent functions

$$\boxed{\psi_n^{(m)}(t) = \sqrt{2^{-m}}\,\psi\left(2^{-m}t - n\right)}$$

"mother wavelet"

that span $\mathcal{L}^2(\mathbb{R})$. Hence any signal $\mathbf{x} \in \mathcal{L}^2(\mathbb{R})$ can be written as

$$\boxed{\mathbf{x} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y^{(m)}[n]\ \psi_n^{(m)}}$$

Multi-resolution analysis

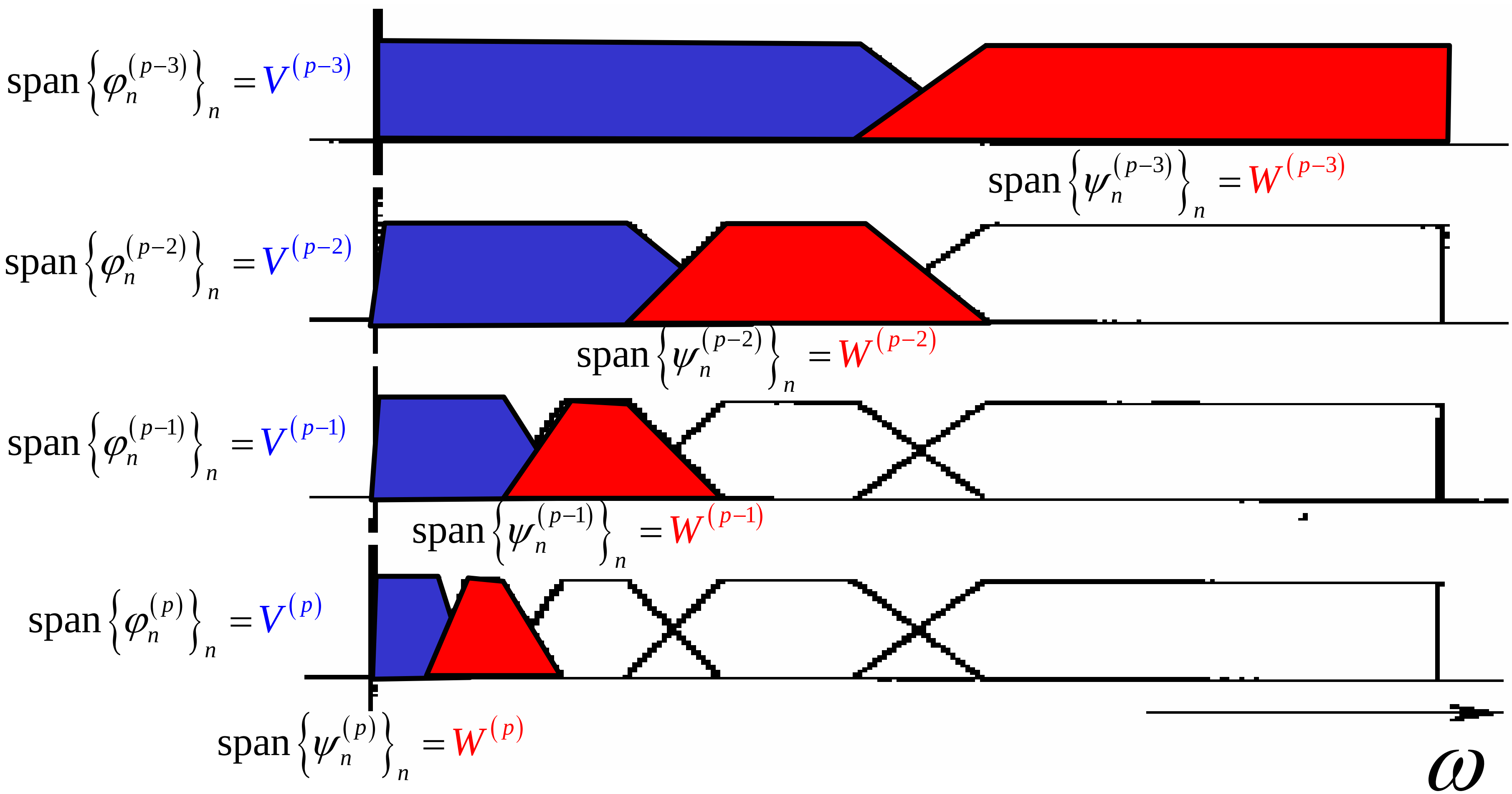
Nested subspaces

$$\ldots \subset V^{(2)} \subset V^{(1)} \subset V^{(0)} \subset V^{(-1)} \subset V^{(-2)} \subset \ldots \subset \mathcal{L}^2(\mathbb{R})$$

Upward completeness	$\bigcup_{m \in \mathbb{Z}} V^{(m)} = \mathcal{L}^2(\mathbb{R})$
Downward completeness	$\bigcap_{m \in \mathbb{Z}} V^{(m)} = \{\mathbf{0}\}$
Self-similarity	$x(t) \in V^{(0)}$ iff $x(2^{-m}t) \in V^{(m)}$
Translation invariance	$x(t) \in V^{(0)}$ iff $x(t-n) \in V^{(0)}$ for all $n \in \mathbb{Z}$

There exists a "scaling function" $\varphi(t)$ with integer translates $\varphi_n(t) = \varphi(t - n)$ such that $\{\varphi_n\}_{n \in \mathbb{Z}}$ forms an orthonormal basis for $V^{(0)}$

Multiresolution Fourier analysis



Relation to subband filters

Since $V^{(0)} \subset V^{(-1)}$, recursive definition of scaling function

$$\varphi(t) = \underbrace{\sum_{n=-\infty}^{\infty} g_0[n]\varphi_n^{(-1)}(t)}_{\substack{\text{linear combination} \\ \text{of scaling functions in } V^{(-1)}}} = \sqrt{2}\sum_{n=-\infty}^{\infty} g_0[n]\varphi(2t-n)$$

Orthonormality

$$\begin{aligned}
\delta[\mathrm{n}] &= \left\langle \varphi_0^{(0)}, \varphi_n^{(0)} \right\rangle \\
&= \int_{-\infty}^{\infty} \left(\sum_i g_0[i]\varphi_i^{(-1)}(t) \sum_j g_0[j]\varphi_{j+2n}^{(-1)}(t) \right) dt \\
&= \sum_{i,j} g_0[i] g_0[j-2n] \left\langle \varphi_i^{(-1)}, \varphi_j^{(-1)} \right\rangle = \underbrace{\sum_i g_0[i] g_0[i-2n]}_{\substack{g_0[\mathrm{k}] \text{ unit norm and orthogonal} \\ \text{to its 2-translates: corresponds} \\ \text{to synthesis lowpass filter of} \\ \text{orthonormal subband transform}}}
\end{aligned}$$

Wavelets from scaling functions

$W^{(p)}$ is orthogonal complement of $V^{(p)}$ in $V^{(p-1)}$

$$W^{(p)} \perp V^{(p)} \quad \text{and} \quad W^{(p)} \cup V^{(p)} = V^{(p-1)}$$

Orthonormal wavelet basis $\left\{\psi_n^{(0)}\right\}$ for $W^{(0)} \subset V^{(-1)}$

$$\psi(t) = \underbrace{\sum_{n=-\infty}^{\infty} g_1[n]\varphi_n^{(-1)}(t)}_{\substack{\text{linear combination} \\ \text{of scaling functions in } V^{(-1)}}} = \sqrt{2}\sum_{n=-\infty}^{\infty} g_1[n]\varphi_n(2t-n)$$

Using conjugate quadrature high-pass synthesis filter

$$g_1[n] = (-1)^{n+1} g_0[-(n-1)]$$

The mutually orthonormal functions, $\left\{\psi_n^{(0)}\right\}_{n\in\mathbb{Z}}$ and $\left\{\varphi_n^{(0)}\right\}_{n\in\mathbb{Z}}$, together span $V^{(-1)}$.

Easy to extend to dilated versions of $\psi(t)$ to construct orthonormal wavelet basis

$$\left\{\psi_n^{(m)}\right\}_{n,m\in\mathbb{Z}} \text{ for } \mathcal{L}^2(\mathbb{R}).$$

Calculating wavelet coefficients for a continuous signal

- Signal synthesis by discrete filter bank

Suppose continuous signal $x^{(0)}(t)=\sum_{n\in\mathbb{Z}} y_0^{(0)}[n]\ \varphi(t-n)=\sum_{n\in\mathbb{Z}} y_0^{(0)}[n]\ \varphi_n^{(0)} \in V^{(0)}$

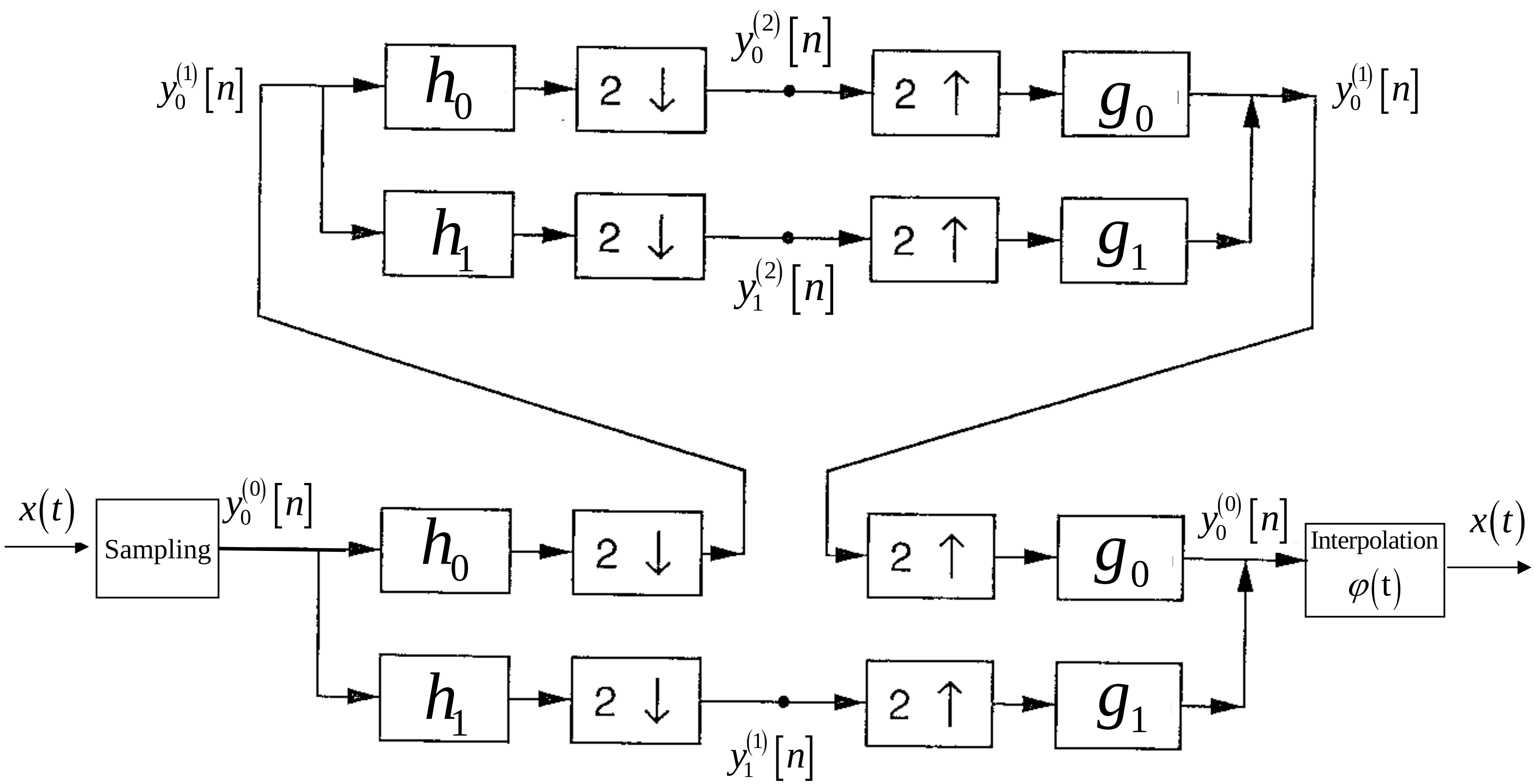
Write as superposition of $x^{(1)}(t)\in V^{(1)}$ and $w^{(1)}(t)\in W^{(1)}$

$$x^{(0)}(t)=\underbrace{\sum_{i\in\mathbb{Z}} y_0^{(1)}[i]\ \varphi_n^{(1)}}_{x^{(1)}(t)\in V^{(1)}}+\underbrace{\sum_{j\in\mathbb{Z}} y_1^{(1)}[j]\ \psi_n^{(1)}}_{w^{(1)}(t)\in W^{(1)}}$$

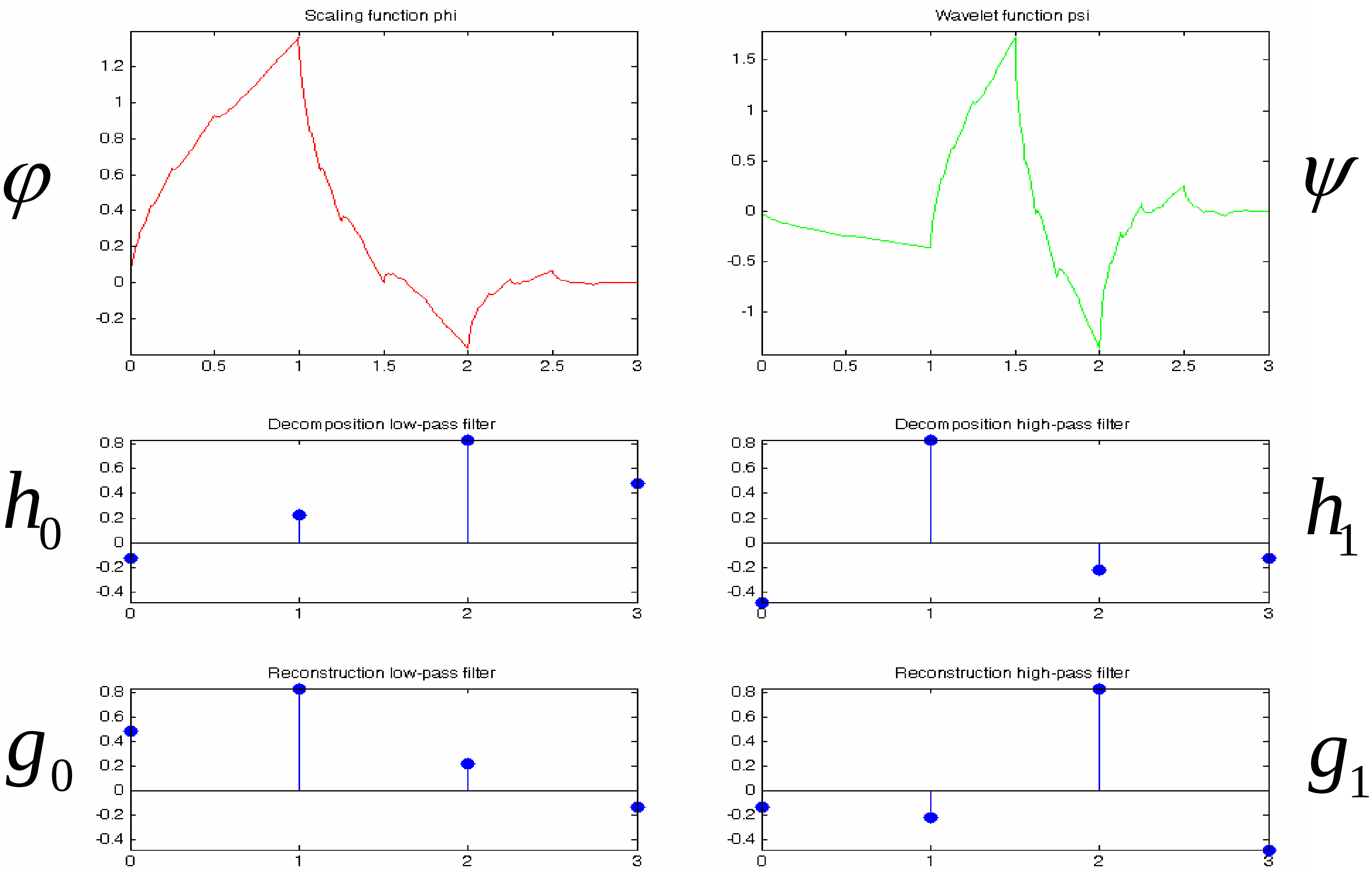
$$=\sum_{n\in\mathbb{Z}} \varphi_n^{(0)} \underbrace{\left(\sum_{i\in\mathbb{Z}} y_0^{(1)}[n]\, g_0[n-2i]+\sum_{j\in\mathbb{Z}} y_1^{(1)}[j]\, g_1[n-2i]\right)}_{y_0^{(0)}[n]}$$

- Signal analysis by analysis filters $h_0[k]$, $h_1[k]$
- Discrete wavelet transform

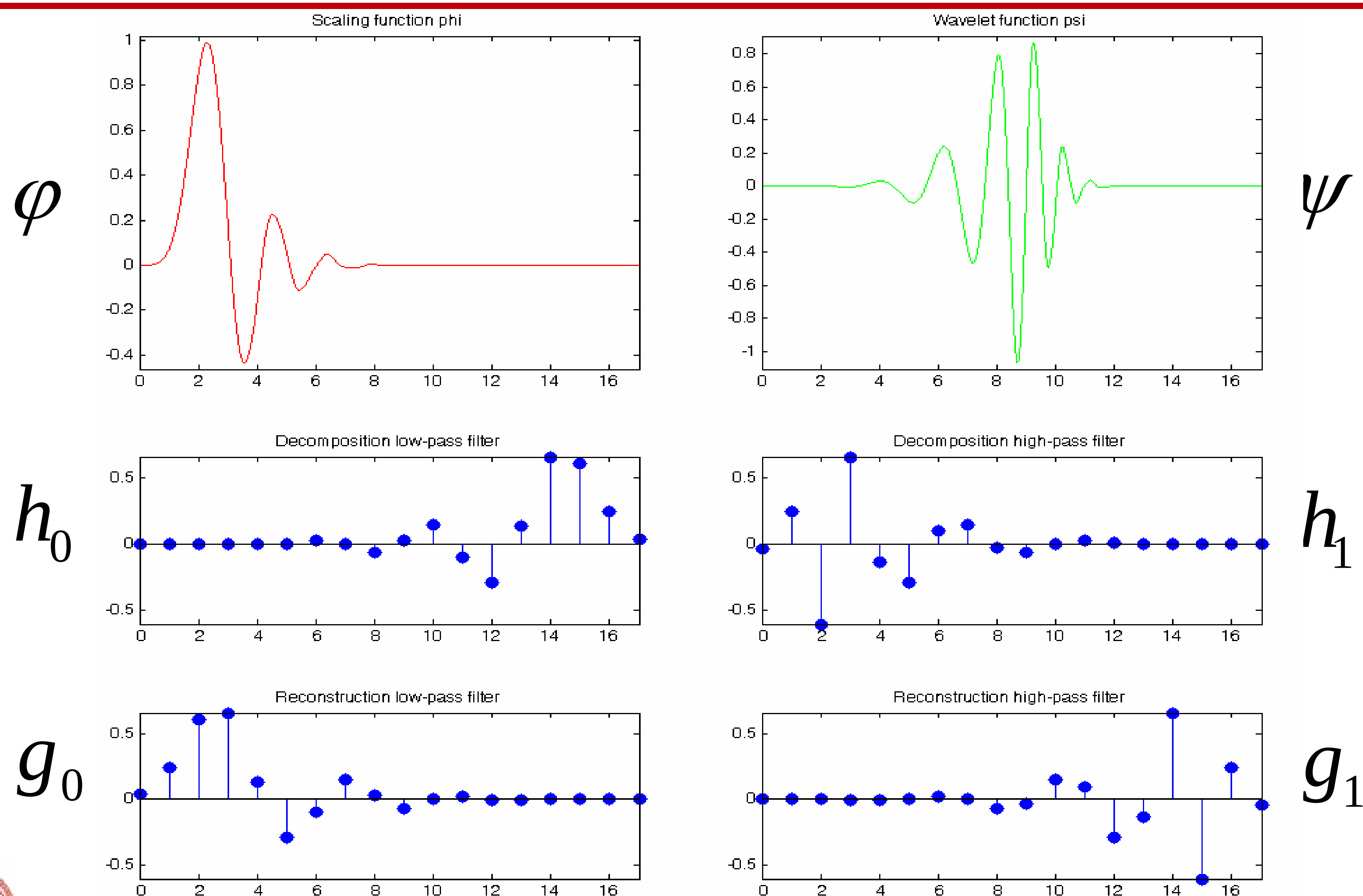
1-d Discrete Wavelet Transform



Example: Daubechies wavelet, order 2



Example: Daubechies wavelet, order 9



Comparison JPEG vs. JPEG2000

Lenna, 256x256 RGB
Baseline JPEG: 4572 bytes

Lenna, 256x256 RGB
JPEG-2000: 4572 bytes

Comparison JPEG vs. JPEG2000

JPEG with optimized Huffman tables
8268 bytes

JPEG2000
8192 bytes