On the Valuation of Options via an Extended Block Exponentially-Fitted Backward Differentiation Formula

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Abstract

We present an Exponentially-Fitted Block Backward Differentiation Formula (EFBBDF) whose coefficients depend on a parameter and the step-size for the valuation of options on a non-dividend-paying stock as well as a dividend-paying stock. The method is provided by a continuous scheme that is defined for all values of the independent variables on the region of interest. In particular, the EFBBDF is formulated from discrete schemes recovered from the continuous scheme. The EFBBDF is applied to solve the Black-Scholes partial differential equation (PDE) after reducing the PDE into a system of ordinary differential equations resulting from the semi-discretization of the PDE via the method of lines. The stability and convergence of the EFBBDF are discussed. It is demonstrated that the American put values are produced by incorporating an additional equation that generates values for the early exercise boundary which are used to ensure that the put option will be optimal. Numerical experiments are performed to validate the performance of the method.

JEL Classification: C63; G10

Keywords: Block Backward Differentiation Formula, Extended block, Options, Black-Scholes partial differential equation.

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1 Introduction

To make informed decisions, financial analysts use models together with certain parameters to determine the theoretical fair values of assets. The computed values are then compared to the market prices of the assets in question. Both computational procedures and mathematical models abound in the financial economics literature for all kinds of assets including derivative securities.

One such celebrated financial models in recent memory is the Black-Scholes (1973) option pricing model. Given the exercise (or strike) price (X), the current stock price (S), the time to maturity (T), the volatility of the stock (σ), and the risk-free rate of interest (r), the model evaluates the theoretical value of a European style option on a non-dividend paying stock. The model is also used to value American style call options; the reason being that it is not optimal to exercise early, an American style call option. However, for American style put options, there exists the possibility of an early exercise. Therefore, a closed-form solution does not exist. Consequently, several analytical approximations and numerical procedures are employed in the valuation of American put options.

In addition to the procedures described in Hull (2012), a great deal of research exists on numerical methods for solving the Black-Scholes differential equation. For example, Khaliq, Voss, and Kazmi (2006) consider the pricing of an American put option as a free boundary problem, and note that the early exercise feature of the American put option transforms the Black-Scholes linear differential equation into a non-linear type. The Khaliq et al study solves the Black Scholes model by adding a small continuous penalty term to the differential equation, and treating the linear penalty term explicitly.

Using the method of lines technique, Jator, Nyonna, and Kerr (2013) solves the Black-Scholes differential equation by forming a stabilized block Adams type method (SBAM). The SBAM is then extended to cover the whole region and used as a single block matrix equation, which is then utilized to value options on stocks paying no dividends.

Applying an extension of the block backward differentiation formula (BBDF) of Akinfenwa, Jator, and Yao (2014), Jator and Nyonna (2014) solves a system that results from a semi-discretization of the Black-Scholes differential equation. This framework is then used to value options on non-dividend paying stocks. Further, the study obtains American put option values by incorporating an additional model that produces values for the early exercise boundary. The Jator and Nyonna paper concludes that the BBDF method is L0-stable and convergent of order 4.

In another study, Huang, Subrahmanyam, and Yu (1996) presents a method for valuing and hedging American style options. The study asserts that a "complicated path integral" implicitly defines the early exercise boundary of an American option, and thus is the reason most research on option pricing focuses on American options on stocks paying no dividends. The Huang et al. (1996) study employs a 'unified framework' that makes use of an analytic formula and the Geske and Johnson (1984) approximation method. This combined framework is then used to price options on dividend paying stocks. Instead of a point-by-point computation of the boundary, the Huang et al. method estimates the early exercise boundary for a few points and uses the Richardson extrapolation to approximate the entire boundary.

From the literature, it is quite obvious most of these studies focused on American put options on non-dividend paying stocks . However, since most American options are written on dividend paying stocks, there is the need for further research to bridge that gap. It is also worth mentioning that the implicit objective of most researchers on this subject is to try and find a method that is simultaneously fast and efficient. Therefore, in this study, we use an exponentially-fitted block backward differentiation formula (EFBBDF) to value an American style put option on a non-dividend paying stock. The result is then extended to value American put options written on dividend paying stocks as well.

1.1 Preliminary Notes

Consider the Black-Scholes model

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1}$$

subject to the initial/boundary conditions

$$V(0,t) = X, \quad V(S,t) = 0 \text{ if } S > 0, \quad V(S,T) = max(X - S, 0)$$

where $V(S,t)$ denotes the value of the option, σ the volatility of the underlying asset, X the exercise price, T the option expiration date, and r the interest rate.

$V(S,T) = max(S-X,0), t = T$ (European call)

$V(S,T) = max(X-S,0), t = T$ (European put)

Thus, consider the Black-Scholes model

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2}$$

subject to the initial/boundary conditions

$V(0,t) = X, \quad V(S,t) = 0$ if $S > 0, \quad V(S,T) = max(X-S,0)$

where $V(S,t)$ denotes the value of the option, σ the volatility of the underlying asset, X the exercise price, T the option expiration date, and r the interest rate.

The method considered in this paper is facilitated by the method of lines approach (Lambert [11], Ramos and Vigo-Aguiar [16], and Cash [2]) which involves seeking a solution in the strip $[a,b] \times [c,d]$, where a,b,c,d are real constants, by first discretizing the variable S with mesh spacings $\Delta S = 1/M$,

$S_m = m\Delta S$, $m = 0,1,\ldots,M$.

We then define $v_m(t) \approx V(S_m,\ t)$, $\mathbf{v}(t) = [V_0(t), V_1(t),\ldots,V_{M-1}(t)]^T$, and replace the partial derivatives $\frac{\partial^2 V(S,\ t)}{\partial S^2}$ and $\frac{\partial V(S,t)}{\partial S}$ occurring in (1) by central difference approximations to obtain

$\frac{\partial^2 V(S_m,\ t)}{\partial S^2} = [v(S_{m+1},t) - 2v(S_m,t) + v(S_{m-1},t)]/(\Delta S)^2$; $\frac{\partial V(S_m,\ t)}{\partial S} = [v(S_{m+1},t) - v(S_{m-1},t)]/2\Delta S$, $m = 1,\ldots,M-1$.

The problem (1) then leads to the resulting semi-discrete problem

$$\frac{dv_i(t)}{dt} = -\frac{1}{2}\sigma^2 S_i^2[v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)]/(\Delta S)^2 - rS_i[v_{i+1}(t) - v_{i-1}(t)]/(\Delta S) + rv_i(t) = 0,$$

which can be written in the form

$$\frac{d\mathbf{v}(t)}{dt} = \mathbf{f}(t,\mathbf{v}), \tag{3}$$

where **f(y, u)=Au+g**, **A** is an $M-1 \times M-1$ matrix arising from the central difference approximations to the derivatives of S, and **g** is a vector of constants. The problem (2) is now a system of ordinary differential equations which is solved by EFBBDF which is L-stable and hence can effectively solve stiff problems.

The rest of the paper proceeds as follows. In section 2, we construct the EFBBDF and give the analysis of the method. In Section 3, the implementation of the method to solve the Black-Scholes model is given. Section 4 is devoted to numerical examples and the conclusion is given in Section 5.

2 The EFBBDF

We define the EFBBDF as

$$\begin{cases} Q_2 V_{n+2} + Q_1 V_{n+1} + Q_0 V_n = h\Upsilon_2 f_{n+2} \\ \overline{Q}_1 V_{n+1} + \overline{Q}_0 V_n = h(\overline{\Upsilon}_1 f_{n+1} + \overline{\Upsilon}_2 f_{n+2}) \end{cases} \quad (4)$$

where $q = wh$, $Q_2 = 1 - e^q + e^{2q} q$, $Q_1 = -1 + e^{2q} - 2e^{2q} q$, $Q_0 = e^q - e^{2q} + e^{2q} q$, $\Upsilon_2 = 1 - 2e^q + e^{2q}$

$$\overline{Q}_1 = e^q q - e^{2q} q,\ \overline{Q}_0 = -e^q q + e^{2q} q, \overline{\Upsilon}_1 = 1 - e^q + e^{2q} q,\ \overline{\Upsilon}_2 = -1 + e^q - e^q q$$

2.1 Main Results

The following block method is formulated from (3):

$$A_1 Y_{\mu+1} = A_0 Y_\mu + h[B_1 F_{\mu+1} + B_0 F_\mu] \quad (5)$$

where $Y_{\mu+1} = (V_{n+1}, V_{n+2})^T$, $Y_\mu = (V_{n-1}, V_n)^T$

$F_\mu = (f_{n+1}, f_{n+2})^T$, $F_{\mu-1} = (f_{n-1}, f_n)^T$

for $\mu = 0, \ldots\ \Gamma$, where $\Gamma = N/2$ is the number of blocks and $n = 0, 2, \ldots, N-2$.

and A_i, B_i, $i = 0, 1$ are 2 by 2 matrices whose entries are given by the coefficients of (3).

Theorem: The global error is reduced by a factor of N/k.

2.2 Construction of method

- We are motivated the mixed collocation method of Coleman and Duxbury (2000), Nguyen et al. (2007), and Ozawa (2005).

- Demand that the coefficients of the EFBBDF are chosen so that the method integrates the scalar form of (2) exactly where the solutions are members of the linear space

  $\langle 1, t, e^{wt} \rangle$.

- Then, we obtain the continuous scheme $Q_2(t)V_{n+2}+Q_1(t)V_{n+1}+Q_0(t)V_n = h\Upsilon_2(t)f_{n+2}$ which is used to provide (3).

2.2.1 Analysis of method

Local truncation error:

$$\begin{cases} \varphi_1[V(t_n);h] = \frac{5}{18}(wV''(t_n) - V'''(t_n))h^3 + O(h^4), \\ \varphi_2[V(t_n);h] = \frac{2}{9}(wV''(t_n) - V'''(t_n))h^3 + O(h^4). \end{cases}$$

Convergence:

The local truncation errors are given by

$$\begin{cases} \varphi_{i+1} = \frac{5}{18}(wV''(t_i+\theta_i) - V'''(t_i+\theta_i))h^3 + O(h^4), \\ \varphi_{i+2} = \frac{2}{9}(wV''(t_i+\theta_i) - V'''(t_i+\theta_i))h^3 + O(h^4), \qquad (6) \\ i = 0, 2, \ldots, N-2, \quad |\theta_i| \le 1. \end{cases}$$

If $\|E\|$ is the norm of maximum global error, it can be shown that $\|E\| = O(h^2)$. Therefore, the EFBBDF is an second-order convergent method.

Stability:

The Linear stability regions is obtained by applying (4) to the test equation $V' = \lambda V$ to give

$$Y_{\mu+1} = M(W;q)Y_\mu \ , \ W = \lambda h, q = wh \tag{7}$$

where the stability matrix $M(W;q))$ is given by

$$M(W;q) = (A_1 - WB_1)^{-1}(A_0 + WB_0)$$

The spectral radius W_{max} of the matrix $M(W;q)$ is a rational function of W given by

$$W_{max} = |\frac{e^q(W - e^q W + q(-1+e^q(1+W)))}{e^{2q}(q-W)(-1+W) - W^2 - e^q}|$$

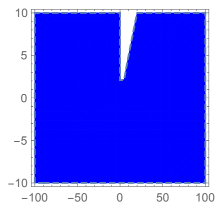
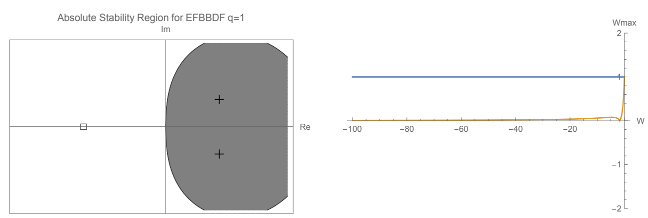


Figure 1: The stability region for the EFBBDF plotted in the $(W$, q)-plane.



(a) A-stability (b) L-stability

Figure 2:

Definition 2.1. *For a fixed q, The method (4) is A-stability if for all* $W \epsilon \mathbb{C}^{-}$,

$$W_{max} < 1$$

In particular, its region of absolute stability contains the left half-plane $\{W \epsilon \mathbb{C} | Re(W) < 0\}$.

Definition 2.2. *For a fixed q, the method (4) is L-stable if it is A-stability and in addition,* $W_{max} \to 0$ *as* $Re(W) \to -\infty$.

3 Implementation

The BNM is implemented using a written code in Mathematica 10.0 enhanced by the features $NSolve[\ \ ]$ for linear problems and $FindRoot[\ \ ]$ for nonlinear problems respectively. In what follows, we summarize how BNM is applied to solve initial value problems (IVPs) in a block-by-block fashion as well as applied to solve boundary value problems (BVPs) via a block unification technique. We summarize the process as follows:

Recall that the system of ODEs is obtained on the partition

$$\pi_M : \{c = S_0 < S_1 < \ldots < S_M = d, \qquad S_m = S_{m-1} + \Delta S\}$$

$\Delta S = \frac{d-c}{M}$ is a constant step-size of the partition of π_M, $m = 1, 2, \ldots, M$, M is a positive integer and m the grid index.

Let the partition

$$\pi_N : \{a = t_0 < t_1 < \ldots < t_N = b, \qquad t_n = t_{n-1} + h\}$$

$h = \Delta t = \frac{b-a}{N}$ is a constant step-size of the partition of π_N, $n = 1, 2, \ldots, N$, N is a positive integer and n the grid index.

Step 1: Use the block extension of (4) to generate from the rectangles $[t_0, t_2] \times [c, d]$, to the rectangle $[t_2, t_4] \times [c, d], \ldots, [t_{N-2}, t_N] \times [c, d]$.

Step 2: Solve the system obtained in step 1 to obtain $\mathbf{V}_n = [V_{0,n}, V_{1,n}, \ldots, V_{M-1,n}]^T, n = 1, 2, \ldots, N$.

Step 3: The solution of (1) is approximated by the solutions in step 2 as $\mathbf{V}$, where $\mathbf{V}(t_n) = [V(S_0, t_n), V(S_1, t_n), \ldots, V(S_{M-1}, t_n)]^T, n = 1, 2, \ldots, N$, where $\mathbf{V}(t_n) = \mathbf{V}_n$.

3.1 IVPs-Block-by-block algorithm

- Step 1: Choose $N, h = (x_N - x_0)/N$, on the partition Q_N.

- Step 2: Using (15), $n = 0, \mu = 1$, solve for the values of $(y_{1/4}, y_{1/2}, y_{3/4}, y_1)^T$ and $(y'_{1/4}, y'_{1/2}, y'_{3/4}, y'_1)^T$ simultaneously on the sub-interval $[x_0, x_1]$, as y_0 and y'_0 are known from the IVP (1).

- Step 3: Next, for $n = 1, \mu = 2$ the values of $(y_{5/4}, y_{3/2}, y_{7/4}, y_1)^T$ and $(y'_{5/4}, y'_{3/2}, y'_{7/4}, y'_2)^T$ are simultaneously obtained over the sub-interval $[x_1, x_2]$, as y_1 and y'_1 are known from the previous block.

- Step 4: The process is continued for $n = 2, \ldots, N-1$ and $\mu = 3, \ldots, N$ to obtain the numerical solution to (1) on the sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]$.

3.2 BVPs-Block unification algorithm

- Step 1: Choose $N, h = (x_N - x_0)/N$, on the partition Q_N.
- Step 2: Using (15), $n = 0, \mu = 1$, generate the variables $(y_{1/4}, y_{1/2}, y_{3/4}, y_1)^T$ and $(y'_{1/4}, y'_{1/2}, y'_{3/4}, y'_1)^T$ on the interval $[x_0, x_1]$ and do not solve yet.
- Step 3: Next, for $n = 1, \mu = 2$ generate the variables $(y_{5/4}, y_{3/2}, y_{7/4}, y_2)^T$ and $(y'_{5/4}, y'_{3/2}, y'_{7/4}, y'_2)^T$ on the sub-interval $[x_1, x_2]$, and do not solve yet.
- Step 4: The process is continued for $n = 2, \ldots, N-1$ and $\mu = 3, \ldots, N$ until all the variables on the sub-intervals $[x_0, x_1], [x_1, x_2], \ldots, [x_{N-1}, x_N]$ are obtained.
- Step 5: Create a single block matrix equation by the unification of all the blocks generated in Step 2 and Step 3 on Q_N.
- Step 6: Solve the single block matrix equation to simultaneously obtain the all solutions for (1) on the entire $[x_0, x_N]$.

4 Numerical Examples

Example 4.1. *As our second test example, we solve the given stiff parabolic equation given in Cash [2].*

$$\frac{\partial V}{\partial t} = \kappa \frac{\partial^2 V}{\partial S^2}, \quad u(0,t) = u(1,t) = 0, \quad u(S,0) = \sin \pi S + \sin \omega \pi S, \ \omega \gg 1.$$

The exact solution $V(S,t) = e^{-\pi^2 \kappa t} \sin \pi S + e^{-\omega^2 \pi^2 \kappa t} \sin \omega \pi S$.
According to Cash [2] as ω increases, equations of the type given in example 4.2 exhibit characteristics similar to model stiff equations. Hence, the methods such as the Crank-Nicolson method which are not L_0-stable are expected to perform poorly. Since the BBDF is L_0-stable it performs very well when applied to this problem. In Table 1, we display the results for $\kappa = 1$ and a range of values for ω.

In order to test for convergence, example 4.1 was solved for various values of $h = \Delta S$ and the results for the global maximum absolute errors ($Err = Max|v_m(t_n) - V(S_m,\ t_n)|$) are reproduced in Table 2. We also give the

$\Delta S = 1/10$, $h = 1/10$		$\Delta S = 1/10$, $h = 1/10$	
ω	EFBBDF	Crank-Nicolson	Cash(abc)
1	2.2×10^{-6}	6.2×10^{-5}	1.5×10^{-5}
2	7.6×10^{-6}	3.8×10^{-5}	7.4×10^{-6}
3	5.7×10^{-6}	9.3×10^{-3}	7.4×10^{-6}
5	1.4×10^{-6}	1.8×10^{-1}	7.4×10^{-6}
10	1.1×10^{-6}	6.1×10^{-1}	7.4×10^{-6}

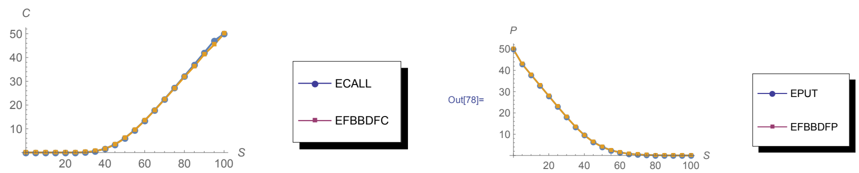
Table 1: A comparison of errors of methods for Example 4.2 at $t = 1$ and $w = 5\pi$.

rate of convergence (ROC) which is calculated using the formula $ROC = \log_2(Err^{2h}/Err^h)$, Err^h is the error obtained using the step size h. In general, the ROC shows that the order of the method is slightly greater than 2. This is expected since the central difference method used for the spatial discretization is of order 2 and hence affects the convergence of the BBDF which is of order 4 with respect to the time variable. In Fig. 7, the solutions obtained using the BBDF are plotted versus S and t and compared with the plots given by the standard finite difference method (FDM). It is obvious from Figure 7 that the BBDF is more accurate as it produces smaller errors.

	EBBDF		FDM	
$\Delta S = h$	Err	ROC	Err	ROC
8	6.47×10^{-2}		1.34×10^{-1}	
16	1.21×10^{-2}	2.48	4.26×10^{-2}	1.65
32	1.39×10^{-3}	3.12	1.18×10^{-2}	1.85
64	1.67×10^{-4}	3.06	3.03×10^{-3}	1.96
128	2.01×10^{-5}	3.02	7.64×10^{-4}	1.99
256	2.57×10^{-6}	3.00	1.91×10^{-4}	2.00

Table 2: A comparison of convergence of methods for Example 4.2 $w = 1$.

Example 4.2. *Consider a five-month European call and put options on a non-dividend-paying stock when the stock price is \$50, the strike price is*



(a) Call (b) Put

Figure 3: Call and Put for example 1

\$50, the risk-free interest rate is 10% per annum, and the volatility is 40% per annum. This example is taken from Hull[6]. In order to compute the call and put options, we use the standard notations to denote $X = 50$, $S = 50$, $r = 0.10$, $\sigma = 0.40$, *and* $T = 0.4167$.

The theoretical solution:

The theoretical solutions for the prices of the European call and put options are given in Hull [6] as follows.

$c = SN(d_1) - Xe^{-r(T-t)}N(d_2)$

$p = Xe^{-r(T-t)}N(-d_2 - SN(-d_1))$

where $d_1 = \frac{\ln(S/X)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$, $\quad d_2 = \frac{\ln(S/X)+(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}$, and $N(x)$ is the cumulative probability distribution function for the standard normal variable.

Acronyms:

The following acronyms are used in the Figures:

- EFBBDFC is Block Backward Differentiation Formula for the call option
- EFBBDFP is Block Backward Differentiation Formula for the put option
- ECALL is the exact solution for the call option
- EPUT is the exact solution for the put option

The call and put options obtained using BBDF and the analytical solution are presented in Figure 3 and Figure 4. It is observed from Figure 3 and Figure 4 that the BBDF fairly approximates the analytical solution.

Example 4.3. *As our third test example, we solve example 1 for the American put by incorporating an additional equation given by (20) that generates*

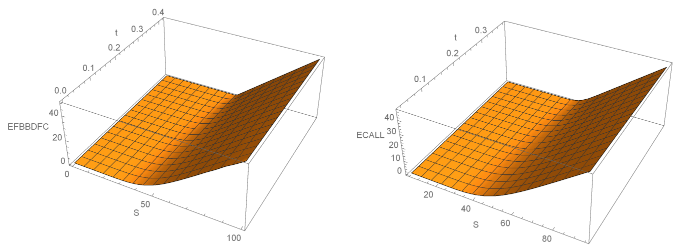


Figure 4: Approximate and exact solutions for the call option for the given Example, $h = 1/12, \Delta S = 20$.

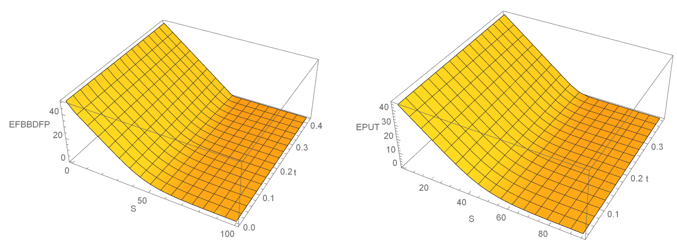


Figure 5: Approximate and exact solutions for the put option for the given Example, $h = 1/12, \Delta S = 20$.

values for the early exercise boundary and hence, ensures that the put option will be optimal. We restate (1) as given in Huang et. al [5].

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

subject to the initial/boundary conditions

$\lim_{S\to\infty} V(S,t) = 0$

$V(S,T) = max(X - S, 0)$

$$\begin{cases} V(S,t) = X - S_t, \\ \\ \frac{\partial V(S,t)}{\partial S} = -1, \quad S = S_t, \end{cases} \qquad (8)$$

where S_t is the free boundary separating the holding and the early exercise regions.

The strike prices were computed by solving the system generated by incorporating (20) into (10). The critical strike price was specified as the lowest price S_t at which the put is exercised early, Carr and Hirsa [4]. In this example, $S_t = Max(X - V(S_t, t)) = \$45$ and the corresponding put value was determined to be \$5. In Figure 8, we plot American-style put values against strike price and time.

5 Conclusion

We have proposed a BBDF for solving the the Black-Scholes partial differential equation. It is shown that the American put values given in Figure 8 are obtained by incorporating an additional equation that generates values for the early exercise boundary which are used to ensure that the put option will be optimal. It is also shown that the method is L_0-stable and convergent of order 4, hence the BBDF is viable candidate for large stiff systems.

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