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APPLICATION OF THE FINITE DIFFERENCE METHOD FOR THE POISSON'S EQUATION IN POLYGONAL DOMAINS

By

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DEDICATION

This work is dedicated to the FOGNE'S family and most especially to my grandparents NGNEHADJI JEAN and MAFONKOU PAULINE for their endless love, supports and encouragements.

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CERTIFICATION

The thesis of **Tinchie Fogne Stevye (SC15P370)** entitled: "**Application of the Finite Difference Method for the Poisson's Equation in Polygonal Domains**", submitted to the Department of Mathematics, Faculty of Science of the University of Buea in partial fulfillment of the requirements for the award of the Master of Science (M.Sc.) Degree in Mathematics has been read, examined and approved by the examination panel composed of:

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Abstract

KEYWORDS: Finite Difference Method(FDM), Elliptic partial differential equation, Dirichlet boundary condition, Basic iterative methods, 5 and 9-point finite difference scheme.

Contents

List of Symbols

$\mathbb{Z}$	The set of integers
$\mathbb{R}^2$	2 – dimensional Euclidean space.
Ω	Given domain in space (bounded connected set in $\mathbb{R}^2$).
$\overline{\Omega}$	Closure of Ω
Ω_i	Sub domain of Ω.
$\partial\Omega$	The boundary of Ω.
n	Outward unit normal vector on $\partial\Omega$ i.e. $n = (n_1, n_2)$.
$\frac{\partial}{\partial n}$	Directional derivative with respect to n.
$C(\overline{\Omega})$	Space of continuous functions defined on $\overline{\Omega}$
$C^k(\overline{\Omega})$	Space of continuous functions whose classical derivatives up the order k belong to $C(\overline{\Omega})$, where k is a nonnegative integer.
$C_0^\infty(\Omega)$	Space of functions in $C^\infty(\overline{\Omega})$ with compact support on Ω.
$\partial^\alpha u$	Derivative of order $|\alpha|$ with respect to the multi-index $\alpha = (\alpha_1, \alpha_2)$, $\alpha_i = 0, 1, \cdots$, $|\alpha| = \alpha_1 + \alpha_2$.

$L^p(\Omega)$	Lebesgue space of measurable functions u defined on Ω for which $\int_\omega \lvert u\rvert^p dx$ is finite, where p$\in [1,\infty[$.
$\lvert\lvert \cdot \rvert\rvert_\infty$	Norm in $L_\infty(\Omega)$.
$H^k(\Omega)$	Sobolev space of functions whose generalized derivatives up to the order k belong to $L^2(\Omega)$.
$\lvert\lvert \cdot \rvert\rvert_V$	Norm in V, where V is a normed space.
$suppv$	Support of a function v.
∇	Gradient operator i.e., $\nabla u(x,y) = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$.
$\nabla \cdot ()$	Divergence operator i.e., $\nabla \cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y}\right) \cdot (iF_1 + jF_2) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$.
$\triangle$	Laplace operator in $\mathbb{R}^2$ i.e. $\triangle := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
$\triangle_h$	Discrete Laplace operator.
Ω_h	Set of mesh points.
h	Discretization parameter.
D^+, D^-, D^0	Difference quotients.
I	Identity matrix.

$a(\cdot, \cdot)$	Bilinear form.
$L(\cdot)$	Linear functional.
L_h	Difference operator.
$(\cdot, \cdot)$	Scalar product in V, if V is Hilbert space.
$f(v)$	Value of the functional $f \in V$ applied to $v \in V$.
$\|\|f\|\|_V$	Norm of the linear functional f.

Chapter ONE

Introduction

Equations relating a function to its derivatives are called partial differential equations.Find a solution of a partial differential equations (PDE) is an important topic in this banch of mathematics and in engineering. (e.g. in Heat transfer, fluid dynamics etc) [7,12].

The solution of a PDE can be found analyticlly or numerically.Find the solution analytically consists to find a function that satisfies the PDE with the given boundary conditions Find the solution numerically consists to find an approximate solution that can approach the analyitic solution without knowing it. The numerical solution is used a lot since for most of PDEs it is very difficult to have the analytical solution.

In practical application, a particular solution for a differential equation requires boundary and / or initial conditions. An appropriate numerical method for the solution of differential problems depends upon the nature of these conditions.

In this thesis, we are concerned with the linear elliptic Poisson problem. Elliptic partial differential equations arise usually from equilibrium or steady-state problems and represents many fields of engineering and science. The most characteristic feature of this class of PDEs is their elliptic regularity, which refers to the phenomenon that their solutions posses an extra-ordinary amount of regularity. This leads to optimal order error estimates for finite difference approximations of boundary value problems contained in this class of PDEs on smooth bounded domains.

We will mainly consider the Poisson equation subject to the homogeneous Dirichlet boundary condition. That is

$$-\triangle u = f \quad \text{in} \quad \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad \text{on} \quad \partial\Omega$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain (a rectangle), $\partial\Omega$ is the boundary of Ω, $f \in C(\Omega)$ and $\triangle u = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ is the Laplace operator, [13,16]. The solution $u(x, y)$, satisfies the boundary conditions on $\partial\Omega$ of the domain Ω on the region

$$\Omega = \{(x, y)|a < x < b, a < y < b\}$$

The solution of the Poisson's Equation depends on the right hand side f and may be solve only for some given functions f. Consequently, numerical simulation must be used in order to aproximate the solution. Although there are several competing algorithms for achieving this goal (Finite Element Method, Boundary Element Method, Finite Volume Method, [4,7,11,15]), one of the simplest and more straightforward of these is called the **Finite Difference Method** (FDM) which we shall confine ourselves to in this work, [11,13,15]. This method in itself has a variety of advantages which make it a good method for approximating PDEs. We concentrate on this method since it is still among the most popular numerical methods for the solution of PDEs because of their simplicity, efficiency, low computational cost, robustness and ease of analysis.
Thus the main objectives of this thesis are of two folds: (1) to study the Finite Difference Method for two dimensional linear elliptic equations in two dimensional polygonal domains and (2) to develop a python software for the numerical simulations of the Poisson problem.

The outline of this work is as follows: In chapter 2, we describe the various mathematical formulations (classical and weak formulations) of the Poisson equation. The Sobolev spaces together with the Lax/Milgram Theorem are then introduced to prove the well-possessedness of the variational formulation of the Poisson equation with the associated boundary condition. In chapter 3, the concepts of Finite Difference analysis in two-dimensional domains are introduced together with the five point and the nine point difference schemes. Finally, Chapter 4 is the description of our software and the numerical experiments with some examples done in Python.

Chapter TWO

Analytic Preliminaries

The purpose of this chapter is to fix some terminologies that will be used throughout this work, and to present some few analytical tools necessary for the treatment of our Dirichlet Poisson problem. It aims at studying the qualitative properties of solution of Poisson equation subject to the homogeneous boundary condition on a rectangle domain.

The chapter is organized as follows. In section 2.1, we present the boundary value problem for the Poisson equation. In section 2.2, the functional spaces and the Sobolev spaces are introduced. In section 2.3, the weak derivatives and the variational formulation of the Poisson equation are presented. Sections 2.4 is concerned with regularity of the weak solution.

2.1 Boundary value problems from the Poisson's Equation

This section is devoted to boundary value problem of the Poisson equation.

2.1.1 Classical formulation of the boundary value problem

Let $\Omega \subset \mathbb{R}^2$ be a polygonal domain with boundary $\partial\Omega$ and consider the following classical Dirichlet boundary value problem. We seek a function $u : \Omega \to \mathbb{R}$ such that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ and

$$\begin{aligned} -\triangle u = f \quad in \quad \Omega \subset \mathbb{R}^2 \\ u = 0 \quad on \quad \partial\Omega \end{aligned} \tag{2.1}$$

where $f \in C(\Omega)$ is a given function, $\triangle$ is the Laplace operator. Equation (2.1) is collectively called the homogeneous Dirichlet problem for the Poisson equation. In the subsequent sections we will want to investigate if this boundary value problem can be solved in the sense of the following definition:

Definition 2.1.1. ***(Jacques Salomon Hadamard's well-possessedness)***
A problem is said to be well-posed if the following three conditions hold

(i) a solution exists

(ii) the solution is unique

(iii) the solution depends continuously on the auxillary data.

Otherwise the problem is ill-posed.

The statement of this definition can be found in ([18])

2.1.2 A real life example modelled by the Poisson problem

We now want to consider a real world problem modelled by the Poisson equation, but before doing so we first introduce the following theorems that will be necessary to achieve this task as seen in ([17]).

Fourier's Law : The Law of Heat Conduction: for many solid materials, the heat flux is a linear function of the temperature gradient, that is:

$$\mathbf{q} = -\kappa \nabla u$$

.

In the above law u is the absolute temperature and $\kappa > 0$ the thermal conductivity, depends on the properties of the material. The minus sign reflects the tendency of heat to flow from hotter to cooler regions.

Theorem 2.1.1 (Gauss' Divergence Theorem)**.** *Suppose V is a subset of $\mathbb{R}^2$ which is compact and has piecewise smooth boundary $\partial V = S$. If F is a continuously differentiable vector field defined on a neighborhood of V, then*

$$\int\int_V (\nabla \cdot F) dV = \int_{\partial V} (F \cdot n) dS$$

<u>Proof</u>

Let $u = u(x)$ be the temperature in a body $\Omega \subset \mathbb{R}^2$ at a point x on the boundary, let

$q = q(x)$ be the heat flux at x, let f be the heat source and let $\omega \subset \Omega$ be a small test volume. By the law of conservation of energy (total energy of an isolated system remains constant ie., it is said to be conserved over time) we have that

$$\int_{\partial\omega} q \cdot n dS - \int_{\omega} f dx = 0,$$

By Fourier's law we have that

$$\int_{\partial\omega} -\kappa \nabla u \cdot n dS = \int_{\omega} f dx.$$

By the Gauss theorem,

$$\int_{\partial\omega} -\kappa \nabla u \cdot n dS = \int_{\omega} \nabla \cdot (-\kappa \nabla u) dx.$$

$\Longrightarrow$

$$-\int_{\omega} \nabla \cdot (\kappa \nabla u) dx = \int_{\omega} f dx.$$

Equation **??** holds for all test volumes $\omega \subset \Omega$. Thus, if u, κ and f are regular enough, we obtain

$$\int_{\omega} (-\nabla \cdot (\kappa \nabla u) - f) dx = 0 \qquad \forall \omega \subset \Omega$$

$\Longrightarrow$

$$-\nabla \cdot (\kappa \nabla u) = f \quad in \quad \Omega$$

If we choose $\kappa = 1$, we obtain the more standard Poisson equation

$$-\triangle u = f \quad in \quad \Omega$$

since the divergence of the gradient equals the Laplacian.

2.2 Some tools of functional analysis and Sobolev spaces

The variational formulation of PDEs is written in the language of functional analysis, therefore we need to introduce basic concepts and notations from functional analysis and Sobolev spaces.

The statements of the following definitions and theorems can be found in [12,17,18].

Definition 2.2.1. *A normed vector space over a field $\mathbb{F}$ is a vector space over the field $\mathbb{F}$ with a norm, a map,*

$$||\cdot|| : V \to \mathbb{R}$$

satisfying the following conditions:

(1.) $||\alpha v|| = |\alpha|||v||, \quad \forall v \in V \quad and \quad \forall \alpha \in \mathbb{F}.$

(2.) $||u + v|| \leq ||u|| + ||v||, \quad \forall u, v \in V$

(3.) $||v|| \geq 0, \quad$ *with equality if and ony if* $\quad v = 0$

Definition 2.2.2. *An inner product space is a vector space with an inner product, a map,*

$$(\cdot, \cdot) : V \times V \to F$$

satisfying the following conditions:

(1.) $(x + y, z) = (x, z) + (y, z), \quad \forall x, y \in V$

(2.) $(\alpha x, z) = \alpha(x, z), \quad \forall x, z \in V, \quad \alpha \in F$

(3.) $(x, z) = \overline{(z, x)}, \quad \forall x, z \in V$

(3.) $(x, x) \geq 0$ *with equality if and ony if* $\quad x = 0 \quad \forall x \in V$

Definition 2.2.3. *A normed vector space is complete if every Cauchy sequenc is convergent.*

Definition 2.2.4. *A Hilbert space is a complete normed inner product space.*

Definition 2.2.5. *Let V be a normed space.*

(i) A mapping $a(\cdot, \cdot) : V \times V \mapsto \mathbb{R}$ is called a bilinear form if it is linear with respect to the first and second arguments respectively. That is if

$$a(\alpha v + \beta w, u) = \alpha a(v, u) + \beta a(w, u),$$

and

$$a(v, \gamma u + \mu z) = \gamma a(v, u) + \mu a(v, z) \quad \forall v, w, u, z \in V \quad \forall \gamma, \alpha, \beta, \mu \in \mathbb{R}..$$

(ii) *A bilinear form* $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ *is said to be bounded if there exist a constant* $C > 0$ *such that* $|a(u,v)| \leq C||u||_V||v||_V, \quad \forall u, v \in V$.

(iii) *A bilinear form* $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ *is said to be V-elliptic if there exists* $C_a > 0$ *such that* $a(v,v) \geq C_a||v||_V^2 \quad \forall v \in V$.

(iv) *A bilinear form* $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ *is said to be positive if* $a(v,v) \geq 0$ *for all* $v \in V$.

(v) *A bilinear form* $a(\cdot,\cdot) : V \times V \mapsto \mathbb{R}$ *is said to be strictly positive if* $a(v,v) > 0$ *for all* $0 \neq v \in V$.

Definition 2.2.6. *Given a normed space* V *with norm* $||\cdot||_V$. *A mapping* $L(\cdot) : V \to \mathbb{R}$ *is called a linear form or linear functional if* $L(\alpha v + \beta w) = \alpha L(v) + \beta L(w) \quad \forall v, w \in V$ *and* $\alpha, \beta \in \mathbb{R}$.

Definition 2.2.7. *A linear form* $L(\cdot) : V \to \mathbb{R}$ *is called bounded if there exists a positive constant* C *such that* $||L(v)||_{\mathbb{R}} \leq C||v||_V$.

The following theorem is an existence and uniqueness theorem for elliptic problems.

Theorem 2.2.1 (Lax-Milgram Theorem). *Let* V *be a Hilbert space endowed with inner product* $(\cdot,\cdot)$ *and norm* $||\cdot||$, $a(\cdot,\cdot) : V \times V \to \mathbb{R}$ *a bounded V-elliptic bilinear form and* $L(\cdot)$ *a continuous linear form. Then there exists a unique solution to the problem: Find* $u \in V$ *such that*

$$a(u,v) = L(v) \quad \textit{for all} \quad v \in V$$

Moreover the following inequality holds:

$$||u|| \leq \frac{1}{\alpha}||L||_V$$

Definition 2.2.8. *Let* Ω *be an open set in* $\mathbb{R}^2$ *and* p *be a positive real number. The space* $L^1(\Omega)$ *is the set of measurable and integrable functions on* Ω. *We note that*

$$||u||_{L^1(\Omega)} = \int_\Omega |u(x)|dx$$

We define now for $1 \leq p < \infty$ the space

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R}, u \text{ measurable and } \quad |u|^p \in L^1(\Omega)\},$$

that we endow with norm

$$||u||_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p dx \Big)^{\frac{1}{p}}$$

Definition 2.2.9. *Let Ω be an open set in $\mathbb{R}^2$, then the space of locally integrable function is given as*

$$L^1_{loc}(\Omega) = \{u \in L^1(K) : \forall K \subset \Omega \quad compact\}$$

The statement of definition 2.2.8 and 2.2.9 can be found in ([2])

2.2.1 Weak derivatives

Of special importance in the development of variational formulation is the notion of weak(distributional) derivatives.
The statements of the following definitions can be found in ([12,18])

Definition 2.2.10. *Denote $\overline{A}$ the closure of a subset $A \subset V$. For $u \in C(\overline{\Omega})$ the support of u is defined by*

$$suppu := \overline{\{x \in \Omega : u(x) \neq 0\}}$$

Definition 2.2.11. *A vector $\alpha = (\alpha_1, \alpha_2)$ where $\alpha_i \geq 0$ for all i and $\alpha_i \in \mathbb{Z}$ is called a multi index and is of order $|\alpha| = \sum_{i=1}^{2} \alpha_i$*

Definition 2.2.12. *Consider the multi-index $\alpha := (\alpha_1, \dots, \alpha_n)$ where each α_i is a non-negative integer. We introduce*

$$\partial^\alpha u := \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots x_n^{\alpha_n}}$$

for the derivative of order $|\alpha|$ with respect to the multi-index α.

Definition 2.2.13. *A locally integrable function v is called an α-th weak derivative of $u \in L^1_{loc}(\Omega)$ if it satisfies*

$$\int_\Omega \phi v dx = (-1)^{|\alpha|} \int_\Omega u \partial^\alpha \phi dx \quad for\ all \quad \phi \in C_0^\infty(\Omega) \tag{2.2}$$

Definition 2.2.14. *A function $f : \Omega \to \mathbb{R}$ is called smooth if all its partial derivatives (of any orders) exist and are continuous on Ω. The set of all smooth functions on Ω is denoted by $C^\infty(\Omega)$*

Definition 2.2.15. *A function $f : \Omega \to \mathbb{R}$ is called a test function (on Ω) if f is smooth and have a compact support in Ω. The set of all test functions on Ω is denoted by $C_0^\infty(\Omega)$.*

2.2.2 Sobolev Spaces

Sobolev spaces which play an important role in the variational formulation of partial differential equations are built on the function space $L^p(\Omega)$ introduced earlier. The main related results are outlined in this section.
The statements of the following definition and theorem can be found in ([2,17])

Let again Ω be an open connected subset of $\mathbb{R}^2$

Definition 2.2.16. *Let k be a non-negative integer and $p \in [1, \infty)$. Then the Sobolev space denoted by $W^{k,p}(\Omega)$ is defined as*

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p, \quad \forall \alpha : |\alpha| \leq k\}$$

with norm

$$||u||_{k,p} = (\sum_{|\alpha| \leq k} \int_\Omega |\partial^\alpha u|^p dx)^{\frac{1}{p}} \quad if \quad 1 \leq p < \infty$$

When p = 2 then we have $W^{k,2}(\Omega) = H^k$, where H^k is defined as

$$H^k(\Omega) = \{u \in L^2(\Omega) : \partial^\alpha u \in L^2, \quad \forall \alpha : |\alpha| \leq k\}$$

$H^k(\Omega)$ is a Hilbert space equipped with the following:

(1) **norm**;

$$\|u\|_k := \left(\sum_{|\alpha| \leq k} \int |\partial^\alpha u|^2 dx \right)^{\frac{1}{2}}, \quad \forall u \in H^k(\Omega) \tag{2.3}$$

(2) **semi norm**;

$$|u|_k := \left(\sum_{|\alpha|=k} \int |\partial^\alpha u|^2 dx \right)^{\frac{1}{2}}, \quad \forall u \in H^k(\Omega) \tag{2.4}$$

(3) **scalar product**;

$$(u,v)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int \partial^\alpha u \partial^\alpha v dx, \quad \forall u, v \in H^k(\Omega) \tag{2.5}$$

Definition 2.2.17. *We denote by $H_0^k(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$ in the $H^k(\Omega)$-norm.*

Theorem 2.2.2. ***(Trace theorem)***
Assume Ω is bounded and $\partial\Omega$ is C^1.Then there exists a bounded linear operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ such that

(i) $\gamma u = u|_{\partial\Omega}$ *if* $u \in H^1(\Omega) \cap C(\overline{\Omega})$

(ii) $||\gamma u||_{L^2(\Omega)} \leq c||u||_1$, *for each $u \in H^1(\Omega)$, with the constant c depending on Ω.*
γ is called the trace of u on $\partial\Omega$.

With the above notion of the trace of a function, let's introduce the following characterization

Theorem 2.2.3. *Assume Ω is bounded and $\partial\Omega$ is C^1. Let $u \in H^1(\Omega)$. Then*

$$u \in H_0^1(\Omega) \iff \gamma u = 0 \quad on \quad \partial\Omega$$

Theorem 2.2.4. ***(Poincaré-Friedrich inequality)***
Let Ω be a bounded domain with C^1- boundary, then there exists a constant $C > 0$ depending on Ω such that $||u||_{L^2(\Omega)} \leq C|u|_{H^1(\Omega)} \quad \forall u \in H_0^1(\Omega)$

2.3 Variational formulation of the boundary value problem

Many physical problems formulated in the classical sence recquire another formulation (called variational or weak formulation) in order to prove the existence and uniqueness of the solution.

2.3.1 Derivation of the weak problem

Let's recall the classical formulation of our Poisson problem (2.1)

$$-\triangle u = f \quad in \quad \Omega \subset \mathbb{R}^2$$
$$u = 0 \quad on \quad \partial\Omega$$

Now let us replace the classical representation by a weak formulation, see ([7,17,18]) Guided by the notion of weak derivative, we start by multiplying (2.1) with a test function $v \in C_0^\infty(\Omega)$ and Integrate the result over Ω:

$$-\int_\Omega v\triangle u dx = \int_\Omega f v dx$$

The objective here is to obtain a problem in the form given in the Lax-Milgram theorem. So we apply the second Green's formula (integration by parts) defined as follows:

$$\int_\Omega v\triangle u dx + \int_\Omega \nabla v \cdot \nabla u dx = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu} ds$$

to the integrand with second order derivatives to have

$$-\int_\Omega (\triangle u) v dx = \int_\Omega \nabla u \cdot \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v ds$$

where $\frac{\partial u}{\partial n}$ is the derivative of u in the outward normal direction on the boundary. The test function v is required to vanish on the parts of the boundary where u is known, which in the present problem implies that $v = 0$ on the whole boundary $\partial\Omega$. It follows that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \tag{2.6}$$

Equation (2.6) holds for all v in the function space $H_0^1(\Omega)$. The trial function u also lies in the function space $H_0^1(\Omega)$. We refer to equation (2.6) as the weak form of the original BVP (2.1)

Thus the weak form is stated as follows :

$$\text{find} \quad u \in H_0^1(\Omega) \qquad \text{such that} \tag{2.7}$$
$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega)$$

In the language of linear forms, equation (2.7) becomes

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v dx$$

and

$$L(v) = \int_\Omega f v dx$$

The weak formulation of the problem (2.1) reads:

$$\begin{aligned} &\text{find} \quad u \in H_0^1(\Omega) \qquad \text{such that} \\ &a(u,v) = L(v) \quad \forall v \in H_0^1(\Omega) \end{aligned} \tag{2.8}$$

2.3.2 Existence and uniqueness of weak solution

The existence and uniqueness of the solution of problem (2.8) will be achieved through the Lax/Milgram theorem (Theorem 2.2.1)

Theorem 2.3.1. *Let $a(u,v)$ and $L(v)$ be given as in (2.8), then problem (2.8) has a unique weak solution.*

Proof of existence of the weak solution of the Poisson equation

Proof. We check that problem (2.8) satisfies the hypotheses of the Lax-Milgram Theorem.

(1.) **Continuity of the bilinear form** $a(u,v)$

since for all $u, v \in H_0^1(\Omega)$, then by the Cauchy Schwartz inequality we have

$$\begin{aligned} |a(u,v)| &= \left| \sum_{i=1}^{2} \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx \right| \leq \sum_{i=1}^{2} \int_\Omega \left| \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \right| dx \leq \sum_{i=1}^{2} \int_\Omega \left| \frac{\partial u}{\partial x_i} \right| \cdot \left| \frac{\partial v}{\partial x_i} \right| \\ &\leq \sum_{i=1}^{2} ((\int_\Omega |\frac{\partial u}{\partial x_i}|^2)^{\frac{1}{2}} (\int_\Omega |\frac{\partial v}{\partial x_i}|^2)^{\frac{1}{2}}) \leq \sum_{i=1}^{2} (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}} (\int_\Omega |\nabla v|^2 dx)^{\frac{1}{2}} \\ &\leq M ||\nabla u||_{L^2(\Omega)} ||\nabla v||_{L^2(\Omega)} \leq M ||u||_{H^1(\Omega)} ||v||_{H^1(\Omega)} \end{aligned} \tag{2.9}$$

Hence $a(u,v)$ is continuous.

(2.) **ellipticity of the bilinear form** $a(u,v)$

Proving the $H_0^1(\Omega)$-ellipticity of our variational problem is an immediate consequence the Poincaré inequality:

$$a(u,u) = \int_\Omega \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_i} dx = \int_\Omega (\frac{\partial u}{\partial x_i})^2 dx = |u|^2_{H^1(\Omega)} \geq \frac{1}{\alpha} ||u||^2_{H^1(\Omega)}$$

Hence $a(u,v)$ is $H_0^1(\Omega)$ elliptic.

(3.) **Continuity of the linear form** $L(v)$

$$|Lv| = \left|\int_\Omega fvdx\right| \leq \int_\Omega |fv|dx \leq \big(\int_\Omega |f|^2\big)^{\frac{1}{2}}dx\big(\int_\Omega |v|^2\big)^{\frac{1}{2}}dx \leq M||f||_{L^2(\Omega)}||v||_{H^1(\Omega)}$$

where M=1. Hence L is continuous.

Therefore by the Lax-Milgram Theorem, problem (2.8) has a unique weak solution $u \in H_0^1(\Omega)$. Consequently, we have the following theorem

Theorem 2.3.2. *For $f \in L^2(\Omega)$, problem (2.8) has a unique weak solution $u \in H_0^1(\Omega)$. Additionally, there exist a constant C_Ω such that*

$$||u||_{H^1(\Omega)} \leq C||f||_{L^2(\Omega)} \qquad \textit{(see point 4 below)}$$

(4.) **Continuous dependence**

by ellipticity,

$$||u||^2_{H^1(\Omega)} \leq \frac{1}{\alpha}a(u,u) \leq \frac{1}{\alpha}|a(u,u)| = \frac{1}{\alpha}|L(u)| = \frac{1}{\alpha}\left|\int_\Omega fudx\right|$$

By the continuity of the linear form,

$$\frac{1}{\alpha}\left|\int_\Omega fudx\right| \leq \frac{1}{\alpha}||f||_{L^2(\Omega)}||u||_{H^1(\Omega)}$$

$$\implies ||u||_{H^1(\Omega)} \leq \frac{1}{\alpha}||f||_{L^2(\Omega)} \implies ||u||_{H^1(\Omega)} \leq C||f||_{L^2(\Omega)}$$

□

Chapter THREE

The Finite Difference Method

Of the many different approaches for solving partial differential equations numerically (finite differences, finite element, spectral, finite volume, collocation method, see, for example, ([13, 14, 16]), we shall study the finite difference method (FDM). The Finite Difference Method (FDM) is a numerical method for solving differential equations by approximating derivatives with finite difference equations. It is nothing more than a direct conversion of differential equations from continuous functions and operators into their discretely-sampled counterparts, [14,16,17,22,23]. This converts the entire problem into a system of linear equations that is readily solved via system solvers. Thus in this realm we say that the FDM is a discretization method. The purposes of this chapter are of three folds : (1) approximating the derivatives of a known function by finite difference formula, (2) generating and solving the system matrix and (3) discuss error analysis and convergence.

This chapter is organized as follows. In section 3.1, the fundamentals of the Finite Difference Method are introduced. In section 3.2, we introduce the 5-point and the 9-point Finite Difference for the Laplacian . In section 3.3, we discuss consistency, stability, convergence and error estimates of the Finite Difference Method.

3.1 Fundamentals of Finite Difference Method

Before tackling how to replace the derivatives of differential equations with finite difference (FD) approximations, we first consider the more basic question of how we can approximate the derivatives of a known function by finite difference formula based only on values of the function itself at discrete points. This provides a basis for the later development of the FDM for solving our PDE.

3.1.1 Finite Difference Schemes

Today, in the context of numerical methods, the term "finite difference" is often taken as synonyms with finite difference approximation. To shorten the writing of formulas for difference quotients, let us define the following *difference operators* (difference schemes) where the discretization (incremental) step size is h> 0.

But before this, we first introduce the main mathematical tool for constructing these difference schemes which is the Taylor's theorem, [3].

Theorem 3.1.1 (Taylor's Theorem)**.** *Let $u \in C^4[a,b]$. Let $h > 0$ be a small real number and let $x \in (a,b)$ be fixed and chosen such that $x \pm h \in [a,b]$ then we have*

$$u(x+h) = u(x) + u^{'}(x)\frac{h}{1!} + u^{''}(x)\frac{h^2}{2!} + u^{'''}(x)\frac{h^3}{3!} + u^{iv}(x+\theta h)\frac{h^4}{4!}, \quad 0<\theta<1$$

$$u(x-h) = u(x) - u^{'}(x)\frac{h}{1!} + u^{''}(x)\frac{h^2}{2!} - u^{'''}(x)\frac{h^3}{3!} + u^{iv}(x+\theta h)\frac{h^4}{4!}, \quad 0<\theta<1$$

Let us first examine the first derivative $u^{'}(x)$ of $u(x)$ at a point x using the nearby function value $u(x \pm h)$, where h is called the step size. There are three commonly used formulas, see for example [14]:

Definition 3.1.1. *Consider the function $u(x) \in C^4(\Omega)$ and suppose we want to approximate $u'(x)$ by a FD approximation based on values of u at a finite number of points near x, then*

(i) ***The forward (right-sided) difference quotient****. This is an expression of the form*

$$(D^{+}u)(x) := \frac{u(x+h) - u(x)}{h} \approx u^{'}(x)$$

Depending on the application, the spacing h may be variable or constant. When omitted, h is taken to be 1. Geometrically, $D^{+}u(x)$ is the slope of the line interpolating u at points x and $x+h$ and it is termed a one-sided approximation to u' since u is evaluated only at values greater or equal to x.

(ii) ***The Backward(left-sided) difference quotient****.*

The backward difference quotient uses the function values at x and $x-h$, instead of the values at $x+h$ and x. It has the form

$$(D^{-}u)(x) := \frac{u(x) - u(x-h)}{h} \approx u^{'}(x)$$

(iii) ***The Centered(or symmetric) difference quotient****. This is an expression of the form*

$$(D^0u)(x) := \frac{u(x+h) - u(x-h)}{2h} = \frac{(D^+u)(x) - (D^-u)(x)}{2} \approx u^{'}(x)$$

This is the slope of the line interpolating u at $x-h$ and $x+h$ and is simply the average of the two one-sided above and also give a better approximation with error proportional to h^2 i.e., $O(h^2)$ and hence is much smaller than the error in a first order approximation when h is small, [14,23].

We can apply the finite difference operators twice to get finite difference formulas for approximating the second order derivative $u^{''}(x)$. For example, the central finite difference formula for approximating $u^{''}(x)$ can be obtained from

$$D^+D^-u^{''}(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = u^{''}(x) + O(h^2) \tag{3.1}$$

If we use the same difference operator D^+ or D^- twice, then we get the corresponding one-sided finite difference formula for approximating $u^{''}(x)$

$$D^+D^+u(x) = \frac{u(x) - 2u(x+h) + u(x+2h)}{h^2} = u^{''}(x) + O(h)$$

and

$$D^-D^-u(x) = \frac{u(x) - 2u(x-h) + u(x-2h)}{h^2} = u^{''}(x) + O(h)$$

which are only first order accurate.

3.1.2 Grid Functions and Meshing

When a finite difference method (FDM) is used to treat numerically a partial differential equation, the differentiable solution is approximated by some grid function, i.e., by a function that is defined only at a finite number of so-called grid points that lie in the underlying domain and its boundary where it is required to find the approximate values of the solution at the interior grid points. Each derivative that appear in the partial differential equation has to be replaced by a suitable difference scheme of function values at the chosen grid points.

As an introduction to finite difference methods, we consider the following example. We are interested in finding an approximation to a sufficiently smooth function u that for a given f satisfies Poisson's equation in the unit square and vanishes on its boundary:

$$-\triangle u = f \quad \Omega := (0,1)^2 \subset \mathbb{R}^2, \tag{3.2}$$
$$u = 0 \quad \text{on} \quad \partial\Omega.$$

Finite difference methods provide values $u_{i,j}$ that approximate the desired function values $u(x_{i,j})$ at a finite number of points, i.e., at the grid points $\{x_{i,j}\}$. For this purpose we use the set of points

$$\overline{\Omega}_h = \{(ih, jk)^T \in \overline{\Omega}, \quad i = 0, 1, \ldots, n, j = 0, 1, \ldots, m\}$$

as the set of all grid points i.e., we chose an equidistant grid that is defined by the points of intersection obtained when one translates the coordinate axes through consecutive equidistant steps with step size h:=$\frac{1}{n}$ and k:=$\frac{1}{m}$. Here $n, m \in \mathbb{N}$ denotes the number of shifted grid lines in each coordinate direction. We distinguish between those grid points lying in the domain Ω and those at the boundary $\partial\Omega$ by setting

$$\Omega_h := \overline{\Omega}_h \cap \Omega \quad \text{and} \quad \partial\Omega_h := \overline{\partial\Omega}_h \cap \partial\Omega \tag{3.3}$$

Unlike the continuous problem, whose solution u is defined on all of $\overline{\Omega}$, the discretization leads to a *discrete solution* $u_h : \overline{\Omega}_h \to \mathbb{R}$ that is defined only at a finite number of grid points. Such mappings $\overline{\Omega}_h \to \mathbb{R}$ are called *grid functions*.

3.2 Difference Stencil.

Each difference operator appearing in a finite difference method is often characterized by its *difference stencil*, which is also called a *difference star*. For any grid point, this describes the neighboring nodes that are included in the discrete operator. For the purpose of this work we are going to consider the **5-point stencil** and the **9-point stencil** for the Laplacian, [5,14].

3.2.1 The 5-point stencil for the Laplacian

Let's consider again the Dirichlet Poisson problem on the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$. We use a uniform Cartesian grid consisting of grid points $u(x+nh, y+mk)$ where $\triangle x$ = h and $\triangle y$ = k. A section of such a grid is shown in figure 3.2(a).

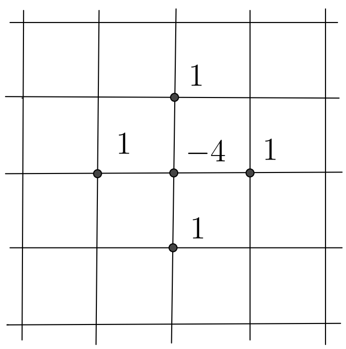
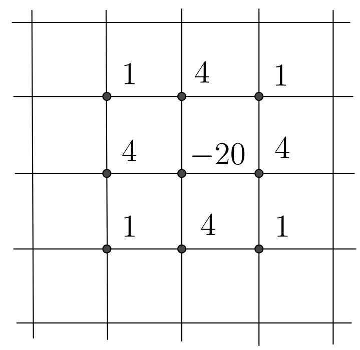


Figure 3.2 (a) : *The* 5 − *point stencil for the Laplacian*



(b) : *The* 9 − *point stencil for the Laplacian*

In order to discretized this Dirichlet Poisson equation, the domain is covered by a square grid of size $h_x \times h_y$ as illustrated in following steps.

Step 1: Generate a grid. For example, a uniform Cartesian grid can be generated as

$$x_i = a + ih_x, \quad i = 0, 1, 2, \ldots, n, \quad h_x = \frac{b-a}{n}$$

$$y_j = c + jh_y, \quad j = 0, 1, 2, \ldots, m, \quad h_y = \frac{d-c}{m}$$

We denote by U a grid function whose value $U_{i,j}$ at a typical point (x_i, y_j) in domain Ω is intended to approximate the exact solution $u(x_i, y_j)$.

The solution at the boundary nodes (points) is known from the boundary condition and the solution at the internal grid points are to be approximated.

Step 2: At a typical internal grid point (x_i, y_j) we approximate the second partial derivative of u by second order central difference, which is second order accurate since the remainder term is $O(h^2)$ see section 3.1 of chapter 3.

we replace both the second derivative of $u(x, y)$ with respect to x and with respect to y

with centered finite differences, which gives

$$\frac{u(x-h,y)-2u(x,y)+u(x+h,y)}{h^2}+\frac{u(x,y-k)-2u(x,y)+u(x,y+k)}{k^2}=f(x,y) \tag{3.4}$$

For simplicity of notation we consider the case h = k. We can then re-write equation (3.4) as

$$\frac{u(x-h,y)+u(x+h,y)+u(x,y-k)+u(x,y+k)-4u(x,y)}{h^2}=f(x,y) \tag{3.5}$$

This finite difference scheme can be represented by the *5-point stencil* shown in figure 3.2(a). We thus have a linear system of $(m-1)^2$ unknowns. The difference equations at points near the boundary will of course involve the known boundary values, which can be moved to the right-hand side.

If we collect all of these equations into a matrix equation, we will have an $(m-1)^2 \times (m-1)^2$ matrix which is sparse, i.e., most of the elements are zero. Since most of the equations involve at most 5 unknowns (less near the boundary), each row the matrix has almost 5 non zeros and at-least $(m-1)^2-5$ elements that are zeros. The exact structure of the matrix depends on the order in which we order the unknowns and write down the equations, but no ordering is ideal.

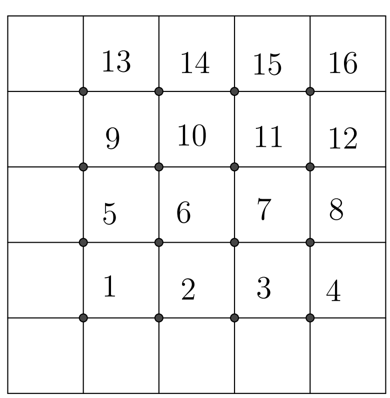


Figure 3.3 : The natural rowwise order of unknowns and equations on a 4×4 grid

We will always use the *lexicographic ordering (row wise ordering)*, where we take the unknowns along the bottom row, $u_{11}, u_{21}, u_{31}, \ldots, u_{m1}$, followed by the unknowns in the second row, $u_{12}, u_{22}, u_{33}, \ldots, u_{m2}$, and so on, as illustrated in figure 3.3.

This gives a matrix equation where A has the form

$$A = \frac{1}{h^2}\begin{bmatrix} T & -I & & & 0 \\ -I & T & -I & & \\ & -I & & & -I \\ & & & & \\ 0 & & & -I & T \end{bmatrix} \tag{3.6}$$

which is an $(m-1)\times(m-1)$ block tridiagonal matrix in which each block T or I is itself an $(m-1)\times(m-1)$ matrix,

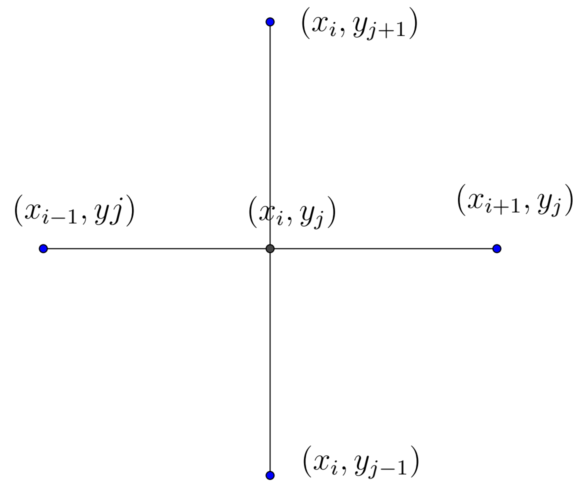
$$T := \begin{bmatrix} 4 & -1 & & & 0 \\ -1 & 4 & -1 & & \\ & -1 & & & -1 \\ & & & & \\ 0 & & & -1 & 4 \end{bmatrix}$$

and I is the $(m-1)\times(m-1)$ identity matrix. The -1 values in the I matrices are separated from the diagonal by $m-2$ zeros, since these coefficients correspond to grid points lying above or below the central point in the stencil and hence are in the next or previous row of unknowns.

Considering equation (3.5), it can therefore be arranged into

$$4U_{i,j} - U_{i-1,j} - U_{i,j-1} - U_{i+1,j} - U_{i,j+1} = h^2 f_{i,j} \tag{3.7}$$

The replacement of equation (3.7) is depicted as in the five-point stencil:



We are then required to solve the linear system of algebraic equations (3.7) to get the approximate values for the solution at all grid points.

The linear system of equations will be transformed into a matrix-vector form :

$$AU = F \tag{3.8}$$

where, from 2D Poisson equations, the unknowns $U_{i,j}$ are a 2D array which we will order into a 1D array.

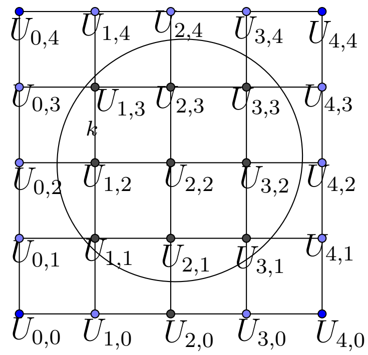
Since $h_x = h_y = h$, let's suppose that n=m=4. Hence, there are 9 equations and 9 unknowns. The equations are ordered in the same way as the unknowns so that each row of the matrix of coefficients representing the left of equation (3.7) will contain almost 5 non-zero entries with the coefficient 4 appearing on the diagonal.

When equation (3.7) is applied at points adjacent to the boundary, one or more of the neighboring values will be known from the boundary condition and the corresponding term will be moved to the right side of the equations. For example, when i = j = 1:

$$4U_{1,1} - U_{1,2} - U_{2,1} = h^2 f_{1,1} + U_{1,0} + U_{0,1}$$

The values $U_{1,0}$ and $U_{0,1}$ are known from the boundary condition, hence they are on the right side of the equations. Then the first row of the matrix will contain only three non-zero entries.

Inner grid points are to be approximated



$$U_1 = \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \end{bmatrix} \qquad U_2 = \begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \end{bmatrix} \qquad U_3 = \begin{bmatrix} U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix}$$

Arranging U_1, U_2, U_3 into a vector gives $U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$

Working on the first row of the inner grid points gives us

$$i = 1, j = 1 : 4U_{1,1} - U_{1,2} - U_{2,1} = h^2 f_{1,1} + U_{1,0} + U_{0,1}$$

$$i = 2, j = 1 : 4U_{2,1} - U_{1,1} - U_{3,1} - U_{2,2} = h^2 f_{2,1} + U_{2,0}$$

$$i = 3, j = 1 : 4U_{3,1} - U_{2,1} - U_{3,1} = h^2 f_{3,1} + U_{3,0} + U_{4,1}$$

Arranging in matrix-vector form yields

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \end{bmatrix} = h^2 \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \end{bmatrix} + \begin{bmatrix} U_{1,0} + U_{0,1} \\ U_{2,0} \\ U_{3,0} + U_{4,1} \end{bmatrix}$$

The second row of the inner grid points gives us

$$i = 1, j = 2 : 4U_{1,2} - U_{2,2} - U_{1,1} - U_{1,3} = h^2 f_{1,2} + U_{0,2}$$

$$i = 2, j = 2 : 4U_{2,2} - U_{1,2} - U_{3,2} - U_{2,1} - U_{2,3} = h^2 f_{2,2}$$

$$i = 3, j = 2 : 4U_{3,2} - U_{2,2} - U_{3,1} - U_{3,3} = h^2 f_{3,2} + U_{4,2}$$

Arranging in matrix-vector form yields

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \end{bmatrix} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \end{bmatrix} + \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = h^2 \begin{bmatrix} f_{1,2} \\ f_{2,2} \\ f_{3,2} \end{bmatrix} + \begin{bmatrix} U_{0,2} \\ 0 \\ U_{4,2} \end{bmatrix}$$

The third row gives

$$i = 1, j = 3 : 4U_{1,3} - U_{2,3} - U_{1,2} = h^2 f_{1,3} + U_{0,3} + U_{1,4}$$

$$i = 2, j = 3 : 4U_{2,3} - U_{1,3} - U_{3,3} - U_{2,2} = h^2 f_{2,3} + U_{2,4}$$

$$i = 3, j = 3 : 4U_{3,3} - U_{2,3} - U_{3,2} = h^2 f_{3,3} + U_{4,3} + U_{3,4}$$

Arranging in matrix-vector form yields

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \end{bmatrix} + \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = h^2 \begin{bmatrix} f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \begin{bmatrix} U_{0,3} + U_{1,4} \\ U_{2,4} \\ U_{4,3} + U_{3,4} \end{bmatrix}$$

Combining all of these gives the following system of linear equations

$$\begin{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = h^2 \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{3,2} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \begin{bmatrix} U_{1,0} + U_{0,1} \\ U_{2,0} \\ U_{3,0} + U_{4,1} \\ U_{0,2} \\ 0 \\ U_{4,2} \\ U_{1,4} + U_{0,3} \\ U_{2,4} \\ U_{3,4} + U_{4,3} \end{bmatrix}$$

3.2.2 The 9-point stencil for the Laplacian

Above we have used the 5-point stencil for the Laplacian which we will denote by $\triangle_5 u(x, y)$, where this denotes the left-hand side of the equation (3.5). Another possible approximation is the 9-point stencil for the Laplacian which is defined by adding the contributions from the "corners" of the 5-point stencil for the Laplacian : $u(x+h, y+k), u(x+h, y-k), u(x-h, y+k)$ and $u(x-h, y-k)$, see [1].

In order to refine our Poisson equation , we replace both the derivative of $u(x, y)$ with respect to x and with respect to y with centered finite differences. This gives the general form of the 9-point stencil

$$\begin{aligned}\triangle_{9,\alpha} u(x,y) = (2\alpha-4)u(x,y) + (1-\alpha)[u(x-h,y) + u(x+h,y) + u(x,y-k) + u(y,y+k)] \\ + \frac{\alpha}{2}[u(x-h,y-k) + u(x+h,y-k) + u(x+h,y+k) + u(x-h,y+k)]\end{aligned} \tag{3.9}$$

or in stencil form by

$$\triangle_{9,\alpha} u(x,y) = \left\{(1-\alpha) \begin{array}{|c|c|c|} \hline 0 & 1 & 0 \\ \hline 1 & -4 & 1 \\ \hline 0 & 1 & 0 \\ \hline \end{array} + \frac{\alpha}{2} \begin{array}{|c|c|c|} \hline 1 & 0 & 1 \\ \hline 0 & -4 & 0 \\ \hline 1 & 0 & 1 \\ \hline \end{array} \right\} u(x,y)$$

The standard 5-point stencil formula is obtained with $\alpha = 0$. The only value of α which yields a higher order of accuracy for the Poisson equation (or Laplace equation) is $\alpha = \frac{1}{3}$; this give the optimal 9-point discrete Laplacian.

Substituting this value of α in equation (3.9) gives

$$\begin{aligned}\triangle_{9,\frac{1}{3}} u(x,y) = -\frac{10}{3h^2}u(x,y) + \frac{2}{3h^2}[u(x-h,y) + u(x+h,y) + u(x,y-k) + u(y,y+k)] \\ + \frac{1}{6h^2}[u(x-h,y-k) + u(x+h,y-k) + u(x+h,y+k) + u(x-h,y+k)]\end{aligned} \tag{3.10}$$

$$\begin{aligned}\triangle_{9,\frac{1}{3}} u(x,y) = \frac{4u(x-h,y) + 4u(x+h,y) + 4u(x,y-k) + 4u(x,y+k) + u(x-h,y-k)}{6h^2} \\ + \frac{u(x-h,y+k) + u(x+h,y-k) + u(x+h,y+k) - 20u(x,y)}{6h^2}\end{aligned} \tag{3.11}$$

This finite difference scheme can be represented by the 9-point stencil shown in Figure 3.2(b). We thus have a linear system of $(m-1)^2$ unknowns. The difference equations at points near the boundary will of course involve the known boundary values, which can be moved to the right-hand side.

If we collect all of these equations into a matrix equation, we will have an $(m-1)^2 \times (m-1)^2$ matrix which is very sparse, i.e., most of the elements are zero. Since most of the equations involve at most 9 unknowns (less near the boundary), each row of the matrix has at most 9 non zeros elements. The exact structure of the matrix depends on the order in which we order the unknowns and write down the equations, see Figure 3.3

This gives a matrix equation where A has the form

$$A = \frac{1}{6h^2}\begin{bmatrix} T & -T_1 & & & 0 \\ -T_1 & T & -T_1 & & \\ & -T_1 & & & -T_1 \\ & & & & \\ 0 & & & -T_1 & T \end{bmatrix} \tag{3.12}$$

which is an $(m-1)\times(m-1)$ block tridiagonal matrix in which each block T or T_1 is itself an $(m-1)\times(m-1)$ matrix,

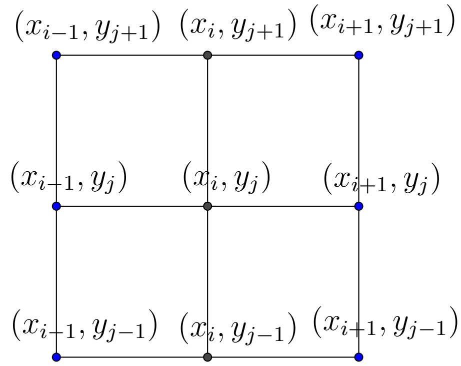
$$\left[T := \begin{bmatrix} 20 & -4 & & & 0 \\ -4 & 20 & -4 & & \\ & -4 & & & -4 \\ & & & & \\ 0 & & & -4 & 20 \end{bmatrix}\right] \text{ and } \left[T_1 := \begin{bmatrix} 4 & -1 & & & 0 \\ -1 & 4 & -1 & & \\ & -1 & & & -1 \\ & & & & \\ 0 & & & -1 & 4 \end{bmatrix}\right]$$

The -4 values in the T_1 matrices are separated from the diagonal by $m-2$ zeros, since these coefficients correspond to grid points lying above or below the central point in the stencil and hence are in the next or previous row of unknowns.

Considering equation (3.11), it can therefore be arranged into

$$20U_{i,j} - 4U_{i-1,j} - 4U_{i+1,j} - 4U_{i,j-1} - 4U_{i,j+1} - U_{i-1,j-1} - U_{i-1,j+1} - U_{i+1,j-1} - U_{i+,j+1} = 6h^2 f_{i,j} \tag{3.13}$$

The replacement of equation (3.13) is depicted as in the 9-point stencil:



We are the required to solve the linear system of algebraic equations (3.13) to get the approximate values for the solution at all grid points.

The linear system of equations will transform into a matrix-vector form :

$$AU = F \tag{3.14}$$

where, from 2D Poisson equations, the unknowns $U_{i,j}$ are a 2D array which we will order into a 1D array.

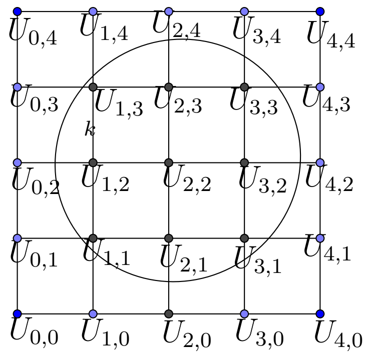
Since $h_x = h_y = h$, let's suppose that n=m=4. Hence, there are 9 equations and 9 unknowns.The equations are ordered in the same way as the unknowns so that each row of the matrix of coefficients representing the left of equation (3.11) will contain at most 5 non-zero entries with the coefficient 4 appearing on the diagonal.

When equation (3.13) is applied at points adjacent to the boundary, one or more of the neighboring values will be known from the boundary condition and the corresponding term will be moved to the right side of the equations. For example, when i = j = 1:

$$20U_{1,1} - 4U_{1,2} - 4U_{2,1} - U_{2,2} = h^2 f_{1,1} + 4U_{1,0} + 4U_{0,1} + U_{0,0} + U_{2,0} + U_{0,2}$$

The values $U_{1,0}$ and $U_{0,1}$ are known from the boundary condition, hence they are on the right side of the equations. Then the first row of the matrix will contain only three non-zero entries.

Inner grid points are to be approximated



$$U_1 = \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \end{bmatrix} \qquad U_2 = \begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \end{bmatrix} \qquad U_3 = \begin{bmatrix} U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix}$$

Arranging U_1, U_2, U_3 into a vector gives $U = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$

Working on the first row of the inner grid points gives us

$$i = 1, j = 1 : 20U_{1,1} - 4U_{1,2} - 4U_{2,1} - U_{2,2} = h^2 f_{1,1} + 4U_{1,0} + 4U_{0,1} + U_{0,0} + U_{2,0} + U_{0,2}$$

$$i = 2, j = 1 : 20U_{2,1} - 4U_{1,1} - 4U_{3,1} - 4U_{2,2} - U_{1,2} - U_{3,2} = h^2 f_{2,1} + 4U_{2,0} + U_{1,0} + U_{3,0}$$

$$i = 3, j = 1 : 20U_{3,1} - 4U_{2,1} - 4U_{3,2} - U_{2,2} = h^2 f_{3,1} + 4U_{3,0} + 4U_{4,1} + U_{2,0} + U_{4,0} + U_{4,2}$$

Arranging in matrix-vector form yields

$$\begin{bmatrix} 20 & -4 & 0 \\ -4 & 20 & -4 \\ 0 & -4 & 20 \end{bmatrix} \begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \end{bmatrix} + \begin{bmatrix} -4 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -1 & -4 \end{bmatrix} \begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \end{bmatrix} = h^2 \begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \end{bmatrix} + \begin{bmatrix} 4U_{1,0} + 4U_{0,1} + U_{0,0} + U_{2,0} + U_{0,2} \\ U_{2,0} + U_{1,0} + U_{3,0} \\ 4U_{3,0} + 4U_{4,1} + U_{2,0} + U_{4,0} + U_{4,2} \end{bmatrix}$$

The second row of the inner grid points gives us

$$i = 1, j = 2 : 20U_{1,2} - 4U_{2,2} - 4U_{1,1} - 4U_{1,3} - U_{2,1} - U_{2,3} = h^2 f_{1,2} + 4U_{0,2} + U_{0,1} + U_{0,3}$$

$$i = 2, j = 2 : 20U_{2,2} - 4U_{1,2} - 4U_{3,2} - 4U_{2,1} - 4U_{2,3} - U_{1,1} - U_{3,1} - U_{1,3} - U_{3,3} = h^2 f_{2,2}$$

$$i = 3, j = 2 : 20U_{3,2} - 4U_{2,2} - 4U_{3,1} - 4U_{3,3} - U_{2,1} - U_{2,3} = h^2 f_{3,2} + 4U_{4,2} + U_{4,1} + U_{4,3}$$

Arranging in matrix-vector form yields

$$\begin{bmatrix} -4 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -1 & -4 \end{bmatrix}\begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \end{bmatrix} + \begin{bmatrix} 20 & -4 & 0 \\ -4 & 20 & -4 \\ 0 & -4 & 20 \end{bmatrix}\begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \end{bmatrix} + \begin{bmatrix} -4 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -1 & -4 \end{bmatrix}\begin{bmatrix} U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = h^2\begin{bmatrix} f_{1,2} \\ f_{2,2} \\ f_{3,2} \end{bmatrix} + \begin{bmatrix} 4U_{0,2} + U_{0,1} + U_{0,3} \\ 0 \\ 4U_{4,2} + U_{4,1} + U_{4,3} \end{bmatrix}$$

The third row gives

$$i = 1, j = 3 : 20U_{1,3} - 4U_{2,3} - 4U_{1,2} - U_{2,2} = h^2 f_{1,3} + 4U_{0,3} + 4U_{1,4} + U_{0,2} + U_{0,4} + U_{2,4}$$

$$i = 2, j = 3 : 20U_{2,3} - 4U_{1,3} - 4U_{3,3} - 4U_{2,2} - U_{1,2} - U_{3,2} = h^2 f_{2,3} + 4U_{2,4} + U_{1,4} + U_{3,4}$$

$$i = 3, j = 3 : 20U_{3,3} - 4U_{2,3} - 4U_{3,2} - U_{2,2} = h^2 f_{3,3} + 4U_{4,3} + 4U_{3,4} + U_{4,2} + U_{4,4} + U_{2,4}$$

Arranging in matrix-vector form yields

$$\begin{bmatrix} -4 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -1 & -4 \end{bmatrix}\begin{bmatrix} U_{1,2} \\ U_{2,2} \\ U_{3,2} \end{bmatrix} + \begin{bmatrix} 20 & -4 & 0 \\ -4 & 20 & -4 \\ 0 & -4 & 20 \end{bmatrix}\begin{bmatrix} U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = h^2\begin{bmatrix} f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} + \begin{bmatrix} 4U_{0,3} + 4U_{1,4} + U_{0,2} + U_{0,4} + U_{2,4} \\ 4U_{2,4} + U_{1,4} + U_{3,4} \\ 4U_{4,3} + 4U_{3,4} + U_{4,2} + U_{4,4} + U_{2,4} \end{bmatrix}$$

Combining all of these gives the following system of linear equations

$$\begin{bmatrix} \begin{bmatrix} 20 & -4 & 0 \\ -4 & 20 & -4 \\ 0 & -4 & 20 \end{bmatrix} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} & \begin{bmatrix} 20 & -4 & 0 \\ -4 & 20 & -4 \\ 0 & -4 & 20 \end{bmatrix} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} & \begin{bmatrix} 20 & -4 & 0 \\ -4 & 20 & -4 \\ 0 & -4 & 20 \end{bmatrix} \end{bmatrix}\begin{bmatrix} U_{1,1} \\ U_{2,1} \\ U_{3,1} \\ U_{1,2} \\ U_{2,2} \\ U_{3,2} \\ U_{1,3} \\ U_{2,3} \\ U_{3,3} \end{bmatrix} = h^2\begin{bmatrix} f_{1,1} \\ f_{2,1} \\ f_{3,1} \\ f_{1,2} \\ f_{2,2} \\ f_{3,2} \\ f_{1,3} \\ f_{2,3} \\ f_{3,3} \end{bmatrix} +$$

$$\begin{bmatrix} 4U_{1,0}+4U_{0,1}+U_{0,0}+U_{2,0}+U_{0,2} \\ U_{2,0}+U_{1,0}+U_{3,0} \\ 4U_{3,0}+4U_{4,1}+U_{2,0}+U_{4,0}+U_{4,2}4U_{0,2}+U_{0,1}+U_{0,3} \\ 0 \\ 4U_{4,2}+U_{4,1}+U_{4,3}4U_{0,3}+4U_{1,4}+U_{0,2}+U_{0,4}+U_{2,4} \\ 4U_{2,4}+U_{1,4}+U_{3,4} \\ 4U_{4,3}+4U_{3,4}+U_{4,2}+U_{4,4}+U_{2,4} \end{bmatrix}$$

The matrices obtained in both the 5 and the 9-point finite difference are symmetric positive definite, strictly diagonally dominant and moreover satisfy the properties of an M-matrix. Additionally, these matrices are sparse and banded. For a banded matrix, all non-zero elements lie in a band of width $2r+1$ along the main diagonal. Suppose we consider a 9×9 matrix, then in the case of the 5-point finite difference, the band width of its matrix is given when $r = 1$ while the band width of the matrix of the 9-point finite difference is given when $r = 4$.

Given such large sparse matrices, direct methods such as the **inversion method, Cramer's rule, Gauss elimination, LU decomposition, QR decomposition, Cholesky decomposition** etc tend out not to be the best in solving such matrices. However, when direct methods are no longer competitive, iterative methods are introduced. Iterative methods become necessary when the system to be solve is too large and even when it is large and sparse. Some of such methods include **Jacobi iteration, Gauss Seidel iteration, conjugate gradient and SOR iteration.** For the purpose of this work, we considered some direct methods such as the **LU decomposition, QR decomposition and Cholesky decomposition** and some iterative methods such the **Jacobi iteration, Gauss Seidel iteration and conjugate gradient.**

3.3 Consistency, stability, convergence, and error estimates of finite difference methods.

When we use a finite difference method to solve a differential equation, we need to know how accurate the approximate solution is compared to the true solution.

Global error

Let $\mathbf{U} = [U_1, U_2, \dots U_n]^T$ be the solution of the finite difference scheme, and let $\mathbf{u} = [u(x_1), u(x_2), \dots u(x_n)]$ be the exact solution at grid points, $x_1, x_2, \dots, x_n$. The global error vector is defined as $\mathbf{E} = \mathbf{U} - \mathbf{u}$. Naturally, we wish to give a smallest upper bound for the error vector. Usually we can use different norms such as the **maximum norm or the infinity norm**, **the 1-norm** and the **2-norm**. But for the purpose of this work we shall dwell mainly on the maximum norm $||\mathbf{E}||_\infty = \max_i\{|e_i|\}$, see for example [13,14,22]. If $||E|| \leq Ch^p, p > 0$, we call the finite difference method p-**th order accurate.** Naturally, we wish to have reasonably high order method while keeping the computational cost low.

Definition 3.3.1. *A finite difference method is called* ***convergent*** *if* $\lim_{h\to 0} ||\boldsymbol{E}|| = 0$.

The statement of this definition can be found in ([13,14]).

Local truncation error

The intuitive definition of the local truncation error is the differential equation and the finite difference equation at grid points.

In general, any given partial differential equation problem, including its boundary conditions, can be expressed as an abstract operator

$$Fu = f \tag{3.15}$$

with appropriately chosen function spaces U and V, a mapping F:U$\rightarrow V$, and $f \in V$. The related discrete problem can be stated analogously as

$$F_h u_h = f_h \tag{3.16}$$

with F_h:$\mathrm{U}_h \to V_h$, $f_h \in V_h$, and discrete spaces U_h, V_h.
The statements of the following definition can be found in [13,14]

Definition 3.3.2. *The local truncation error is defined as*

$$T = F_h u - f_h u$$

The local truncation error only depends on the solution in the finite difference stencil but do not depends on the solution globally. The local truncation errors measure how well the finite difference discretization approximates the differential equation.

Definition 3.3.3. *A finite difference scheme is called* ***consistent*** *if*

$$\lim_{h\to 0} T = \lim_{h\to 0}(F_h u - f_h u) = 0$$

If $|T| \leq Ch^p$, $p > 0$, *then we say the discretization is p-th order accurate.*

Stability

The consistency alone can not guarantee the convergence of a finite difference. We need another condition to determine whether a finite difference method converges or not. Such a condition is called the **stability** of the finite difference method.

For the model problem, we have

$$Au = F + T$$

where A is the coefficient matrix of the finite difference equations, F is the modified source term by taking into account the boundary condition, and T is the vector of local truncation error at the grid points where the solution is unknown.

Definition 3.3.4. *Suppose a finite difference method for a linear boundary value problem gives a sequence of linear equations of the form* $A_h U_h = F_h$ *where* h *is the mesh width. We say that the method is stable if* $(A_h)^{-1}$ *exists for all* h *sufficiently small (for* $h < h_o$*, say) and if there is a constant* C*, independent of* h*, such that*

$$||(A)^{-1}|| \leq C, \quad \textit{for all} \quad h < h_o,$$

The statement of this definition can be found in [13,14].

Convergence

From the definition of the consistency and the stability, we have the following theorem.

Theorem 3.3.1. *A consistent and stable finite difference method is convergent.*

That is

$$consistency + stability \implies convergence. \tag{3.17}$$

The statement and prove of this can be found in [22, p41].

Statement (3.19) is sometimes called the *fundamental theorem of finite difference methods*. In fact, as our above analysis indicates, this can generally be strengthened to say that

$$O(h^h) \quad \text{local truncation error} + \text{stability} \implies O(h^h) \quad \text{global error}. \tag{3.18}$$

Chapter FOUR

Numerical Experiments

In this chapter we are going to numerically illustrate the concept of the 5-point and the 9-point Finite Difference with a specific example of the Poisson equation with homogeneous boundary condition. This chapter is outlined as follows. In section 4.1 we describe the software used to carry out the numerical experiment and give a brief description of the functions used in the developed python program. In section 4.2 we illustrate the numerical computation of a specific Poisson equation. Section 4.3 is devoted to results and interpretation.

4.1 Different functions used in the software

Python is a high-level and powerful modern computer programming language. It bears similarities with Mat lab and Fortran, but it is much more powerful than both. Python allows you to use variables without declaring them, define classes at your convenience and relies on indentation as a control structure. It is designed to be highly readable, uses English keywords frequently and has fewer syntactical constructions than other languages. Python was developed by **Guido van Rossum** in the late 1980s with the following features: simplicity, easy to learn, portability, free and open source, interpretable, embeddable etc. Python codes are very similar to MAT LAB codes, has a rich set of modules for scientific computing and an extensive support for graphics.

All the python functions in build in our developed software that implements the finite difference method for our homogeneous Poisson problem are presented as follows:

The introductory part of the program request the user to chose between the 5 and the 9 point finite difference method. It is not defined as a module.

def InputDomain(): Requests the user to input the coordinates of the desired domain. The coordinates can be given manually or taken from a file.

def SmallerRecs(): Performs and plot the initial (first) discretization of the given domain.

def NumPartition(): Requests the user to specify the number of partitions desired for the partitioning of the domain

def TotalgridPoints(): Collects and stores the coordinates of the entire mesh grid including the boundary points.

def MatrixInitialize(): Initializes a matrix of size $(n+1) \times (m+1)$ whose entries are all zeros.

def ModifyDiagon(): Modifies the diagonal elements of the initialized matrix.

def ModifyUnderDia(): Modifies the under and upper diagonal elements of the initialized matrix with respect to the stencil considered.

def SourceFunx(): Requests the user to enter the true solution, the source function and the boundary function. The true solution and the source term are evaluated at the interior nodes whereas the boundary function is evaluated at the boundary nodes. The graph of the true solution is then plotted.

def SystemMatrix(): Returns the entire system matrix. That is, matrix consisting of all the nodes of the domain including the boundary nodes.

def ReducedMatrix(): Returns the reduced matrix. That is, matrix consisting of only the interior nodes.

def Correction(): Modifies the right hand side with the boundary condition if the boundary condition is not homogeneous.

def AppSolution(): Computes the approximate solution, the error, the convergence rate and plots the graph of the approximate solution. The error is computed using the infinity norm.

def timestapm(): Returns the run time of the program.

4.2 Computational examples

Example 1: Using the 5-point Finite Difference

Suppose that the domain is $\Omega = (a,b)\times(c,d) = (-1,1)\times(-1,1)$ and consider the 2D Poisson equation given by

$$-\triangle u = (-3y^2-4xy^3-x^2y^4+4+8xy+2x^2y^2+y^4-3x^2-4yx^3-y^2x^4+x^4)e^{xy} \quad for \quad (x,y)\in\Omega \tag{4.1}$$

with boundary condition

$$u(x,y)|_{\partial\Omega} = 0 \tag{4.2}$$

where the true (exact) solution u(x,y) is given as

$$u(x,y) = (x^2-1)(y^2-1)e^{xy} \tag{4.3}$$

Let us solve Equation 4.1 in Ω with Dirichlet boundary conditions. Following the description given in chapter 3 (section 3.2) on the experimental Poisson equation

$$-\triangle u = (-3y^2-4xy^3-x^2y^4+4+8xy+2x^2y^2+y^4-3x^2-4yx^3-y^2x^4+x^4)e^{xy} \quad in \quad \Omega \tag{4.4}$$

with boundary condition

$$u(x,y)|_{\partial\Omega} = 0 \tag{4.5}$$

gives the system of equations

$$AU = F \tag{4.6}$$

where

$$A = \begin{pmatrix} 16 & -4 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 16 & -4 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 16 & 0 & 0 & -4 & 0 & 0 & 0 \\ -4 & 0 & 0 & 16 & -4 & 0 & -4 & 0 & 0 \\ 0 & -4 & 0 & -4 & 16 & -4 & 0 & -4 & 0 \\ 0 & 0 & -4 & 0 & -4 & 16 & 0 & 0 & -4 \\ 0 & 0 & 0 & -4 & 0 & 0 & 16 & -4 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & -4 & 16 & -4 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & -4 & 16 \end{pmatrix}, \qquad F = \begin{pmatrix} 5.42 \\ 3.31 \\ 0.95 \\ 3.31 \\ 4.0 \\ 3.31 \\ 0.95 \\ 3.31 \\ 5.42 \end{pmatrix} \tag{4.7}$$

F is the evaluated source term plus the boundary nodes.

Example 1: Using the 9-point Finite Difference

$$A = \begin{pmatrix} 13.3 & -2.7 & 0 & -2.7 & -0.7 & 0 & 0 & 0 & 0 \\ -2.7 & 13.3 & -2.7 & -0.7 & -2.7 & -0.7 & 0 & 0 & 0 \\ 0 & -2.7 & 13.3 & 0 & -0.7 & -2.7 & 0 & 0 & 0 \\ -2.7 & -0.7 & 0 & 13.3 & -2.7 & 0 & -2.7 & -0.7 & 0 \\ -0.7 & -2.7 & -0.7 & -2.7 & 13.3 & -2.7 & -0.7 & -2.7 & -0.7 \\ 0 & -0.7 & -2.7 & 0 & -2.7 & 13.3 & 0 & -0.7 & -2.7 \\ 0 & 0 & 0 & -2.7 & -0.7 & 0 & 13.3 & -2.7 & 0 \\ 0 & 0 & 0 & -0.7 & -2.7 & -0.7 & -2.7 & 13.3 & -2.7 \\ 0 & 0 & 0 & 0 & -0.7 & -2.7 & 0 & -2.7 & 13.3 \end{pmatrix}, \quad F = \begin{pmatrix} 5.42 \\ 3.31 \\ 0.95 \\ 3.31 \\ 4.0 \\ 3.31 \\ 0.95 \\ 3.31 \\ 5.42 \end{pmatrix} \tag{4.8}$$

F is the evaluated source term plus the boundary nodes.

Example 2: Using the 5-point Finite Difference

Suppose that the domain is $\Omega = (a, b) \times (c, d) = (-1, 1) \times (-1, 1)$ and consider the 2D Poisson equation given by

$$\begin{aligned} -\triangle u = &-8x^2(y^2-1)^2 \sin x \cos y - 8x(x^2-1)(y^2-1)^2 \cos x \cos y - 8y^2(x^2-1)^2 \sin x \cos y + \\ &8y(x^2-1)^2(y^2-1) \sin x \sin y + 2(x^2-1)^2(y^2-1)^2 \sin x \cos y - 4(x^2-1)^2(y^2-1) \sin x \cos y - \\ &4(x^2-1)(y^2-1)^2 \sin x \cos y \end{aligned} \tag{4.9}$$

with boundary condition

$$u(x, y)|_{\partial\Omega} = 0 \tag{4.10}$$

where the true (exact) solution u(x,y) is given as

$$u(x, y) = (x^2-1)^2(y^2-1)^2 \sin x \cos y \tag{4.11}$$

Let us solve Equation 4.9 is to be solved in Ω subject to the Dirichlet boundary condition. Following the description given in chapter 3 (section 3.2) on this experimental Poisson equation with boundary condition

$$u(x,y)|_{\partial\Omega} = 0 \tag{4.12}$$

we have the following system of equations

$$AU = F \tag{4.13}$$

where

$$A = \begin{pmatrix} 16 & -4 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 16 & -4 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & -4 & 16 & 0 & 0 & -4 & 0 & 0 & 0 \\ -4 & 0 & 0 & 16 & -4 & 0 & -4 & 0 & 0 \\ 0 & -4 & 0 & -4 & 16 & -4 & 0 & -4 & 0 \\ 0 & 0 & -4 & 0 & -4 & 16 & 0 & 0 & -4 \\ 0 & 0 & 0 & -4 & 0 & 0 & 16 & -4 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & -4 & 16 & -4 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & -4 & 16 \end{pmatrix}, \qquad F = \begin{pmatrix} -1.65 \\ 0.0 \\ 1.65 \\ -4.73 \\ 0.0 \\ 4.73 \\ -1.65 \\ 0.0 \\ 1.65 \end{pmatrix} \tag{4.14}$$

F is the evaluated source term plus the boundary nodes.

Example 2: Using the 9-point Finite Difference

$$A = \begin{pmatrix} 13.3 & -2.7 & 0 & -2.7 & -0.7 & 0 & 0 & 0 & 0 \\ -2.7 & 13.3 & -2.7 & -0.7 & -2.7 & -0.7 & 0 & 0 & 0 \\ 0 & -2.7 & 13.3 & 0 & -0.7 & -2.7 & 0 & 0 & 0 \\ -2.7 & -0.7 & 0 & 13.3 & -2.7 & 0 & -2.7 & -0.7 & 0 \\ -0.7 & -2.7 & -0.7 & -2.7 & 13.3 & -2.7 & -0.7 & -2.7 & -0.7 \\ 0 & -0.7 & -2.7 & 0 & -2.7 & 13.3 & 0 & -0.7 & -2.7 \\ 0 & 0 & 0 & -2.7 & -0.7 & 0 & 13.3 & -2.7 & 0 \\ 0 & 0 & 0 & -0.7 & -2.7 & -0.7 & -2.7 & 13.3 & -2.7 \\ 0 & 0 & 0 & 0 & -0.7 & -2.7 & 0 & -2.7 & 13.3 \end{pmatrix}, \qquad F = \begin{pmatrix} -1.65 \\ 0.0 \\ 1.65 \\ -4.73 \\ 0.0 \\ 4.73 \\ -1.65 \\ 0.0 \\ 1.65 \end{pmatrix} \tag{4.15}$$

F is the evaluated source term plus the boundary nodes.

Solving equations (4.7),(4.8) and equations (4.14),(4.15) using iterative methods for system of equations gives the results in the following tables. We use the maximum (infinity) norm to calculate the error between the exact solution (u) at the interior grid points and the approximate solution (u_h) obtained by solving the above linear system. We expect this error to decrease as we increase the number of refinements.

4.3 Results and Interpretation

# of subintervals,n	Step size,h	# of nodes	$\|\|u - u_h\|\|_\infty$	Rate,α
4	0.5	25	$1.47006314552\times10^{-2}$	
8	0.25	81	$3.87406969501\times10^{-3}$	1.92
16	0.125	289	$9.71298868515\times10^{-4}$	1.99
32	0.0625	1089	$2.44431588331\times10^{-4}$	2.0
64	0.03125	4225	$6.1117249915\times10^{-05}$	2.0
128	0.015625	16641	$1.52798449494\times10^{-05}$	2.0

Table 1: Error Analysis of the 5-point Finite Difference of example 1.

Where $||u - u_h||_\infty = \sup_{x \in \Omega} |u(x) - u_h(y)|$

# of subintervals,n	Step size,h	# of nodes	$\|\|u - u_h\|\|_\infty$	Rate,α
4	0.5	25	$4.54144221912 \times 10^{-2}$	
8	0.25	81	$1.11843305513 \times 10^{-2}$	2.0
16	0.125	289	$2.90980862588 \times 10^{-3}$	2.0
32	0.0625	1089	$7.29237444039 \times 10^{-4}$	2.0
64	0.03125	4225	$1.82430551479 \times 10^{-4}$	2.0

Table 2: Error Analysis of the 9-point Finite Difference of example 1.

# of subintervals,n	Step size,h	# of nodes	$\|\|u - u_h\|\|_\infty$	Rate,α
4	0.5	25	$1.27173244643 \times 10^{-1}$	
8	0.25	81	$2.96834523051 \times 10^{-2}$	2.0
16	0.125	289	$7.30252810185 \times 10^{-3}$	2.0
32	0.0625	1089	$1.81843774534 \times 10^{-3}$	2.0
64	0.03125	4225	$4.55832430482 \times 10^{-4}$	2.0

Table 3: Error Analysis of the 5-point Finite Difference of example 2.

# of subintervals,n	Step size,h	# of nodes	$\|\|u - u_h\|\|_\infty$	Rate,α
4	0.5	25	$1.69787926651 \times 10^{-1}$	
8	0.25	81	$3.82419715336 \times 10^{-2}$	2.0
16	0.125	289	$9.3030885368 \times 10^{-3}$	2.0
32	0.0625	1089	$2.3097703688 \times 10^{-3}$	2.0
64	0.03125	4225	$5.76444485371 \times 10^{-4}$	2.0

Table 4: Error Analysis of the 9-point Finite Difference of example 2.

Interpretation: Table 1,2,3 and 4 present the convergence history of the two computational examples considered. We start with a second refinement that leads to 4 elements with 9 interior nodes and continue the refinements up to 3969 interior points. It can be seen as expected that the convergence rates of both schemes are $\alpha = 2$ as their respective errors tend to zero with decreasing step size, h. Hence, we conclude that our approximate solution is a good approximation of the true solution with respect to the maximum norm.

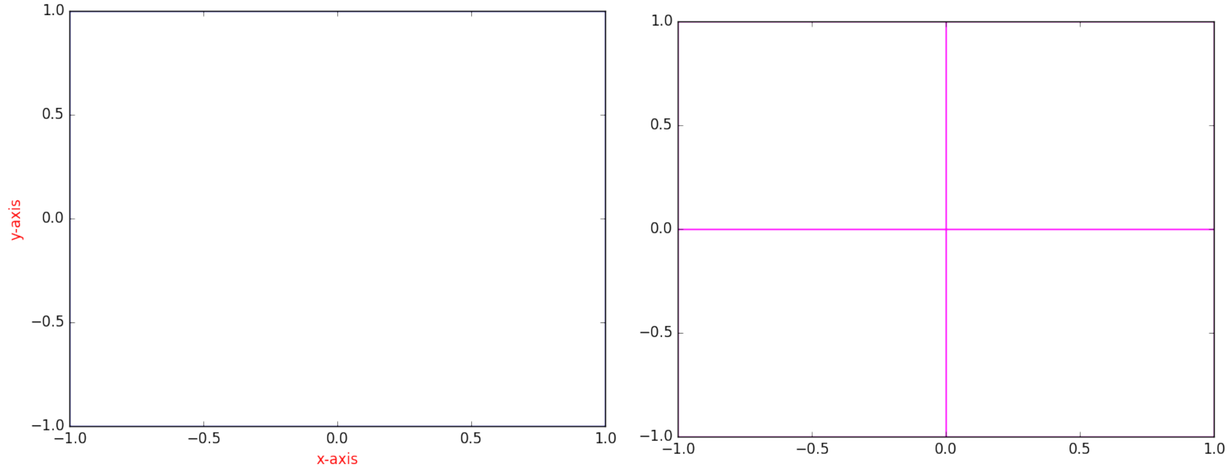
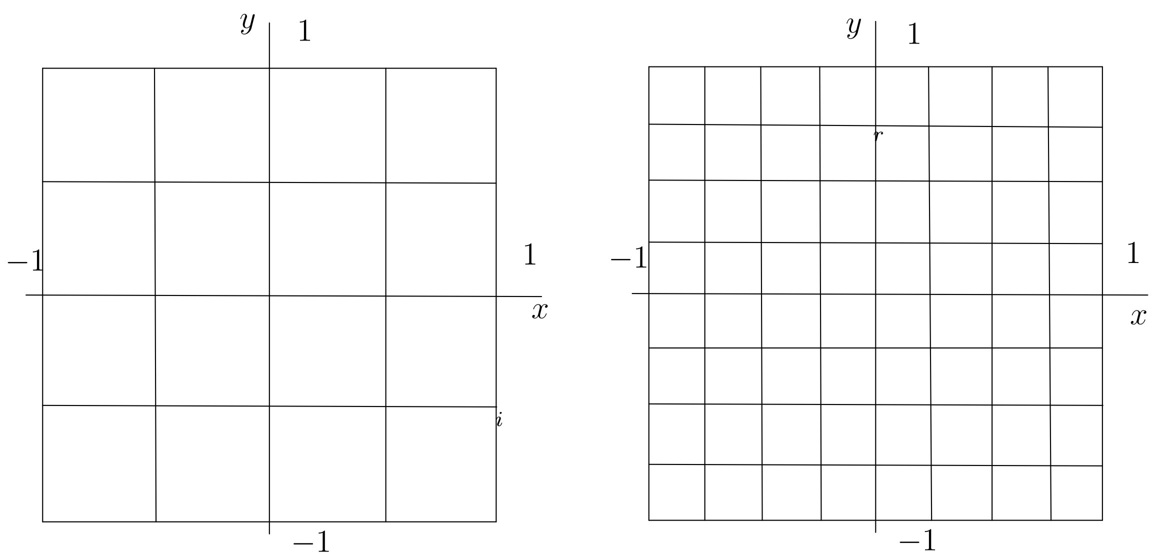


Figure 4.1: a) 2D plot of the square domain   b) Initial refinement of the domain



Second refinement   *Third refinement*

Figure (4.2) represent 3D surface graphs of the true solutions of the considered computational examples. The different colors indicate the variation of heat distribution across the domain and the values on the right indicate the color scale ranging from 0.1 to 0.9 for the first graph and from -0.16 to 0.16.

Figures (4.3) to (4.8) present the graphical representations of the various approximates solutions with different step sizes. It can be seen that these solutions approach the graph of the true solutions as we refine the domain more and more. Hence, with a very very smaller step size (much more refined domain), these graphs will eventually tend to be like those of the true solutions.

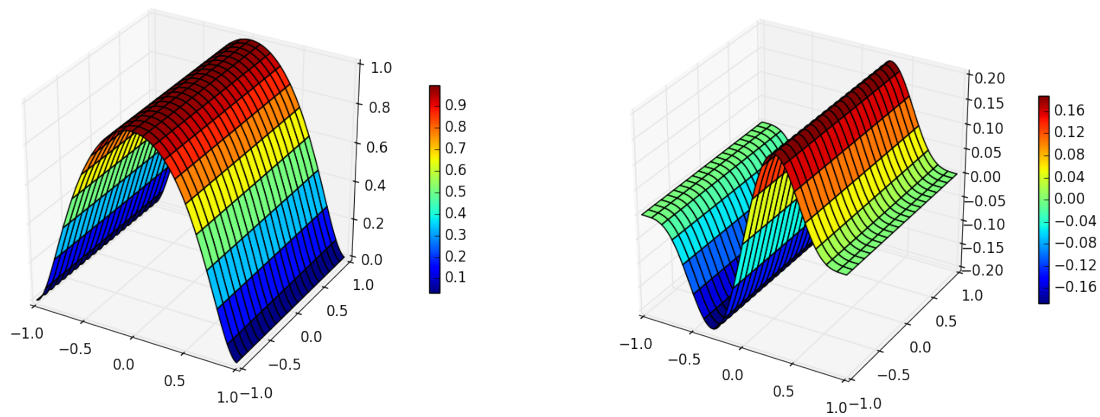


Figure 4.2: a) 3D plot of the true solution of example 1 b) 3D plot of the true solution of example 2

The convergence curves below (Figures 4.9 and 4.10) all consist of one straight line that corresponds to the relationship between $log2(error)$ and -$log2(h)$, where h is the step size. The horizontal axis corresponds to -$log2(h)$ and the vertical axis represents $log2(error)$. The rate of change of each of the straight lines corresponds to the rate of convergence of the said method. This convergence rate could also be computed as

$$log2\left(\frac{\text{previous error}}{\text{next error}}\right) \tag{4.16}$$

which may or may not directly give the desired convergence rate. However, in case the desired convergence rate is not obtained directly, expression (4.16) will eventually lead to the desired convergence rate with more and more refinement of the domain i.e., a much decrease in the step size, $h = \frac{b-a}{2^n}$.

Figure 4.3: a) Graph of the 5-point approximate solution for example 1 at the second partition. b) Graph of the 5-point approximate solution for example 1 at the third partition.

Figure 4.4: a) Graph of the 5-point approximate solution for example 1 at the fourth partition. b) Graph of the 5-point approximate solution for example 1 at the fifth partition.

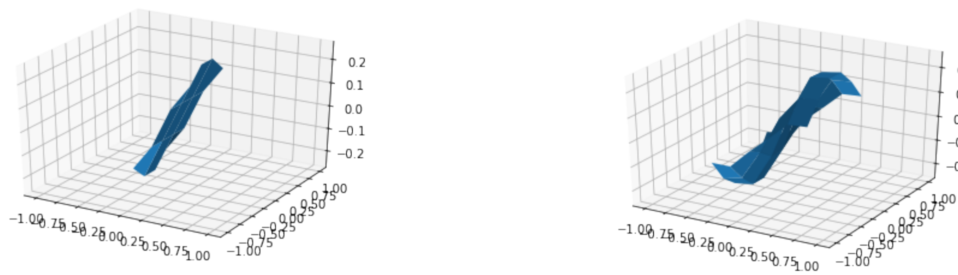


Figure 4.5: a) Graph of the 5-point approximate solution for example 2 at the second partition. b) Graph of the 5-point approximate solution for example 2 at the third partition.

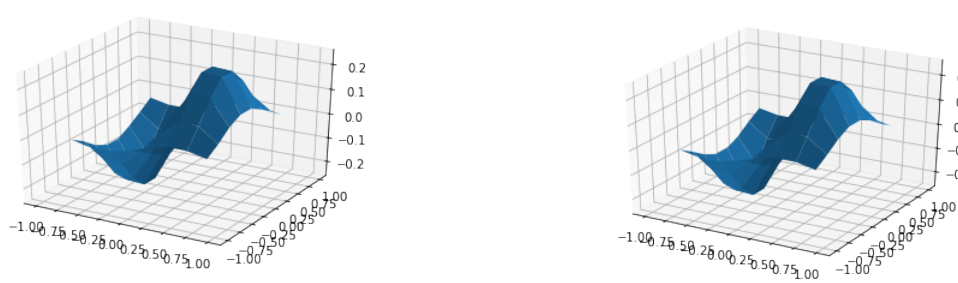


Figure 4.6: a) Graph of the 5-point approximate solution for example 2 at the fourth partition. b) Graph of the 5-point approximate solution for example 2 at the fifth partition.

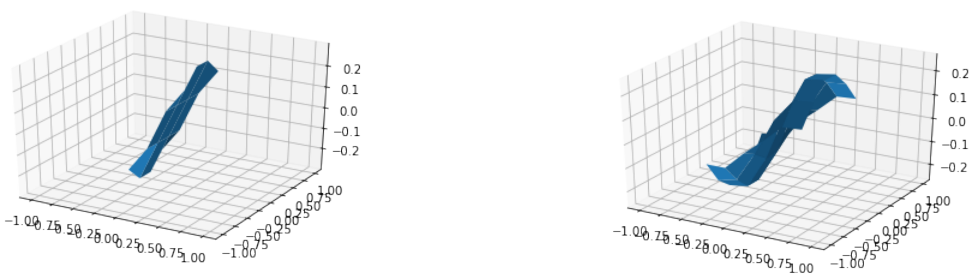


Figure 4.7: a) Graph of the 9-point approximate solution for example 2 at the second partition. b) Graph of the 9-point approximate solution for example 2 at the third partition.

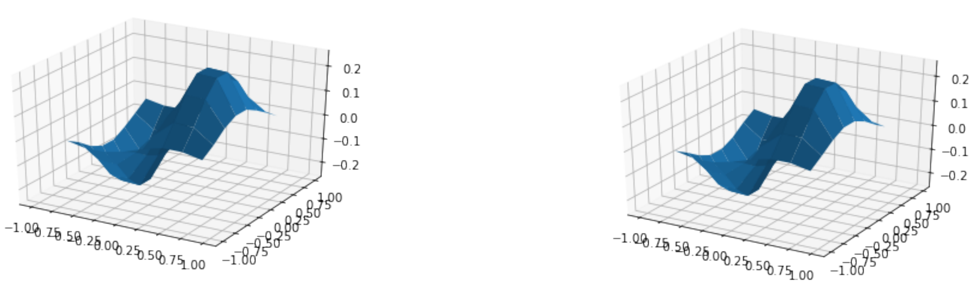


Figure 4.8: a) Graph of the 9-point approximate solution for example 2 at the fourth partition. b) Graph of the 9-point approximate solution for example 2 at the fifth partition.

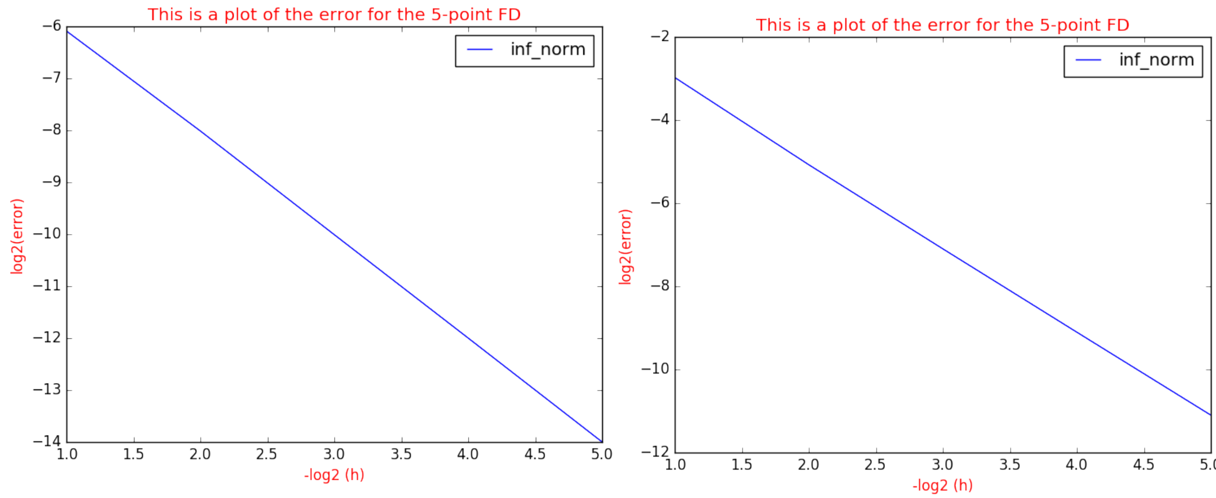


Figure 4.9: a) Graph of the 5-point error against the step size for example 1. b) Graph of the 5-point error against the step size for example 2.

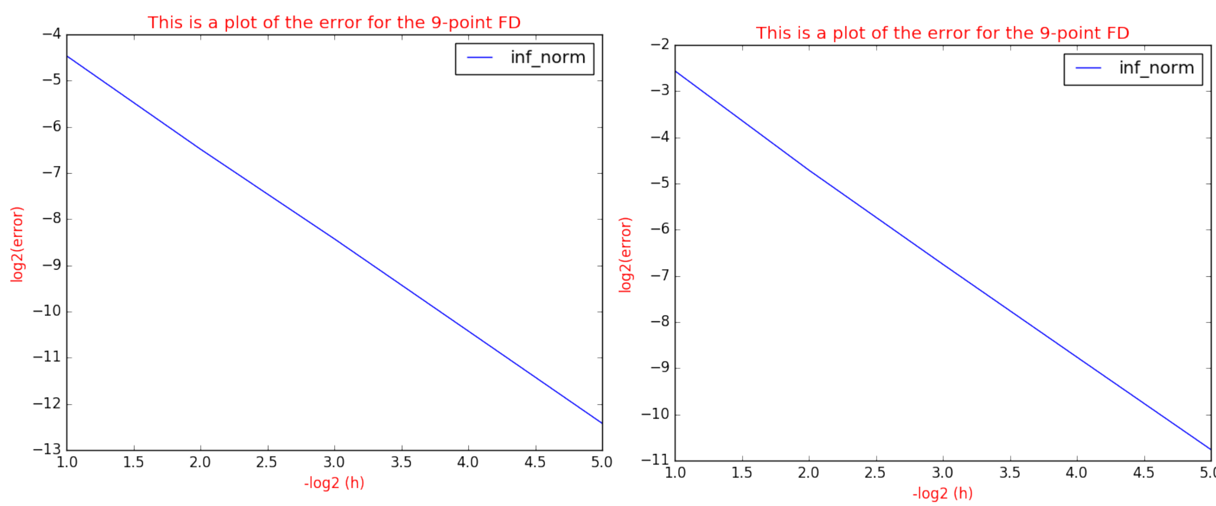


Figure 4.10: a) Graph of the 9-point error against the step size for example 1. b) Graph of the 9-point error against the step size for example 2.

Conclusion

In this thesis, we emphasized on the numerical solution of the linear elliptic problem for the Poisson equation using the 5 and 9 points finite difference discretization. The generated linear system is then solved with basic iterative system solvers such as; Jacobi, Gauss-Seidel and conjugate gradient methods. The structures of the matrices generated from the 5 point and 9 point finite difference discretization are sparse, banded and symmetric positive definite.

Two practical experimental problems were solved for various values of the step size,h in order to illustrate the basis of the Finite Difference Method and the basic system solvers. The analysis of the results show that the approximated solution converges to the exact solution as the step size $h \to 0$. We would definitely have much more better results with the use of more powerful computers. We also observe that the 5 and the 9 point finite difference converge with the same order, $\alpha = 2$.

Finally, although these results are interesting, as future perspectives, we will want to consider not only the homogeneous Dirichlet boundary condition but also the homogeneous and non-homogeneous mixed boundary value problem for the Poisson equation on polygonal domains constituted of piecewise squares or rectangles.

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